

ENDOFINITE MODULES

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Vladimir Lukiyanov
School of Mathematics

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The University of Manchester

Vladimir Lukiyanov
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This is an exposition of the various results which relate modules which are of finite length when considered over the endomorphism rings (endofinite modules) and the Ziegler spectrum, including various properties of isolation and decomposition properties of products. Use is made of the locally finitely presented functor category, the corresponding endocategory is used both from the model theoretic and category theoretic perspectives. A large part of this exposition concerns results on certain pure-injective modules which are precisely the totally transcendental modules from model theory.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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We begin with an overview of the following five chapters and the logical connections that link them together. In short, the order is as follows. The first two chapters contains sketches of the category theoretic (Chapter 1) and model theoretic (Chapter 2) preliminaries. The next two chapters are concerned with the technical properties associated to chain conditions of the endocategory (Chapters 3 and 4). The endocategory is a useful tool which can be seen from either the model theoretic or functorial perspective. The final Chapter 5 explores a class of modules known as the endofinite modules. These satisfy strong properties and hence have stronger, more amenable, versions of the technical results of Chapter 4.

In Chapter 1 we begin by setting up the categories of functors $(\text{mod } R, \mathbf{Ab})$ and $\text{fp}(\text{mod } R, \mathbf{Ab})$ (Section 1.1) for a ring R . We work with these throughout, and go on to consider localisation by certain classes of objects in $\text{fp}(\text{mod } R, \mathbf{Ab})$ (Section 1.2). This localisation corresponds to considering what happens “modulo a complete theory of modules closed under products”, it also has connections with certain approximations (Section 1.3). These approximations will be used, briefly, in Chapter 4 when considering product complete modules. The tools used here are primarily category-theoretic, relying on the good localisation properties of the Grothendieck abelian category $(\text{mod } R, \mathbf{Ab})$.

In Chapter 2 we consider $\text{fp}(\text{mod } R, \mathbf{Ab})$ from a different perspective, known as model theory. We begin by defining pp-formulas (Section 2.1), then pure injective modules (Section 2.2), and finally pp-pairs (first part of Section 2.3). The pp-pairs fit into an abelian category which is isomorphic to $\text{fp}(\text{mod } R, \mathbf{Ab})$. Definable subcategories of $\text{Mod } R$, Serre subcategories of $\text{fp}(\text{mod } R, \mathbf{Ab})$, theories of R -modules, and others, are all linked by a certain topological space known as the Ziegler spectrum (second part of Section 2.3). The tools here, though essentially equivalent to the ones of Chapter 1, come from a branch of mathematical logic known as model theory. Sometimes this perspective is useful (for example at the beginning of Chapter 4), and at other times it is complimentary to the functorial one (for example in the definition of the endocategory in Chapter 3).

As mentioned above, Chapter 3 concentrates on the development of a certain endocategory of a module. This category is best approached from both category and model theoretic perspectives (Section 3.1) and is equivalent to the finitely presented objects of a localisation of $(\text{mod } R^{\text{op}}, \mathbf{Ab})$, where we localise at a localising subcategory which corresponds to the complete theory of M (or all the pp-pairs closed on M). Thus these are “localised pp-pairs”. We consider chain conditions of this endocategory, with the main result a decomposition (end of Section 3.2) arising from the general theory of locally noetherian categories. The last part of this chapter (Section 3.3) deals with a duality on the Ziegler spectrum.

In Chapter 4 we consider properties of a certain class of modules, known as the Σ -pure injective modules. Each such module has a chain condition on the endocategory, this is shown at the beginning (Section 4.1) using model theoretic tools. We then consider the definable subcategory which is generated by a Σ -pure injective module (Section 4.2), which leads directly to a discussion of the Ziegler-closed subsets related to Σ -pure injective modules (Section 4.3). At the end of this discussion we obtain a decomposition. In the last part of Chapter 4 (that is in Section 4.4) we consider a certain technical class of modules, the product complete modules.

The final Chapter 5 deals with endofinite modules. Results from Chapter 4 are strengthened (the brief Section 5.1) and we then consider the endofinite points of the Ziegler spectrum (Section 5.2), this latter part uses the aforementioned technical results of Section 4.4. Finally we consider endofinite modules in the Ziegler spectrum.

Chapter 1

Functorial methods

1.1 Finitely presented functors

In this section we recall the basic properties of finitely presented objects in the category $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ of covariant additive functors from the category of finitely presented modules (over a fixed ring R) to the category of abelian groups. This category, much studied for its links with model theory, forms the basis for the methods which we consider in Chapter 1 and beyond.

Throughout we will make extensive use of results from the theory of categories, with particular emphasis on abelian categories. Modern references for category theory are [4] and [22], alternatively one may consult the more brief [32] (Chapter II) and the older [20]. Prerequisites on abelian categories can be found in [32] (Chapter IV, V). We will, by convention, assume that all functors are additive and covariant.

Let \mathcal{C} be a preadditive category in which the class of objects, considered up to isomorphism, is a set (in such case it is said that \mathcal{C} is skeletally small). It is known that the associated category of covariant functors, denoted $(\mathcal{C}, \mathbf{Ab})$, is a Grothendieck abelian category (see [32], Chapter V), which is both complete and cocomplete, inheriting these properties from the category of abelian groups \mathbf{Ab} . We denote by $\text{fp}(\mathcal{C}, \mathbf{Ab})$ the full subcategory of $(\mathcal{C}, \mathbf{Ab})$ which consists of finitely presented functors, that is those functors $F : \mathcal{C} \rightarrow \mathbf{Ab}$ such that the induced representable functor,

$(F, -) : (\mathcal{C}, \mathbf{Ab}) \longrightarrow \mathbf{Ab}$, commutes with filtered colimits¹. Equivalently a functor F is finitely presented if there exists an exact sequence of functors

$$\coprod_{i=0}^m (Y_i, -) \xrightarrow{f_*} \coprod_{j=0}^n (X_j, -) \longrightarrow F \longrightarrow 0,$$

where the arrow f_* arises via the Yoneda embedding from maps $f_{i,j} : X_j \rightarrow Y_i$ in \mathcal{C} (see [9], Satz 7.6). Note that $(X_j, -)$ and $(Y_i, -)$ are finitely generated projectives in $(\mathcal{C}, \mathbf{Ab})$.

Example 1.1. Let R be a ring. Then we can consider R as a single object preadditive category \mathbf{R} , which is (obviously) small. To do this we equip \mathbf{R} with an endomorphism $\cdot r$ corresponding to left multiplication by each element $r \in R$, and define composition by $(\cdot r_1)(\cdot r_2) = \cdot(r_2 r_1)$. Then, by considering $\mathcal{C} = \mathbf{R}^{\text{op}}$, where the composition is now $(\cdot r_1)(\cdot r_2) = \cdot(r_1 r_2)$, we obtain that $(\mathbf{R}^{\text{op}}, \mathbf{Ab})$ is equivalent to the category of right R -modules $\text{Mod } R$. Further the category $\text{fp}(\mathbf{R}^{\text{op}}, \mathbf{Ab})$ consists precisely of the finitely presented right R -modules. We denote the category $\text{fp}(\mathbf{R}^{\text{op}}, \mathbf{Ab})$ by $\text{mod } R$. Note also that each finitely generated projective $\coprod_{i=1}^m (R, -)$ corresponds to the right finitely generated projective R -module R^m . In what follows we drop the \cdot when using the category \mathbf{R} .

Example 1.2. Let $\mathcal{C} = \text{mod } R^{\text{op}}$ be the category of left finitely presented R -modules (as above). This category is skeletally small. To see this note that every finitely presented module is, up to isomorphism, determined by finitely many generators and relations. The categories $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ and $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ will interest us throughout this chapter.

Because $\text{mod } R^{\text{op}}$ is an additive (though not necessarily abelian) category, each object $F \in \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ will have a simpler presentation, $(Y, -) \xrightarrow{f_*} (X, -) \longrightarrow F \longrightarrow 0$, with $f : X \rightarrow Y$ an arrow in $\text{mod } R^{\text{op}}$. This is because an additive category has finite products. We also note that there are two embeddings into $(\text{mod } R^{\text{op}}, \mathbf{Ab})$,

¹We use the term *filtered colimit* to mean a colimit over a filtered category. Without further comment we make the convention to treat these the same as direct limits. This does not cause problems in the situations we consider, though the two are formally different, see [27] Appendix E, p.714. The reason for this is twofold (i) colimit is the right name for this object, avoiding possible confusion, and (ii) it is easier to see that filtered colimits commute with finite limits.

firstly the *Yoneda embedding*

$$\begin{aligned} (\text{Mod } R^{\text{op}})^{\text{op}} &\longrightarrow (\text{mod } R^{\text{op}}, \mathbf{Ab}) \\ M &\longmapsto (M, -), \end{aligned}$$

and secondly the *tensor embedding*

$$\begin{aligned} \text{Mod } R &\longrightarrow (\text{mod } R^{\text{op}}, \mathbf{Ab}) \\ N &\longrightarrow N \otimes -. \end{aligned}$$

Both of these embeddings are readily seen to be full and faithful.

We will now concentrate on the properties of $(\text{mod } R^{\text{op}}, \mathbf{Ab})$, for a fixed ring R . It is worth noting that many properties of $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ are known to apply in a wider context (see [27], Chapter 10; and the monograph [28]).

We begin by considering the categorical properties of $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$, a subcategory which plays a particularly important role in the structure of $(\text{mod } R^{\text{op}}, \mathbf{Ab})$. Note first that $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ has the structure of a full abelian subcategory of $(\text{mod } R^{\text{op}}, \mathbf{Ab})$. To see this one shows that every finitely generated subfunctor² of a finitely presented functor is also finitely presented ([27], Corollary 10.2.3). Note secondly that $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ is the free abelian category on \mathbf{R} ([17], Lemma 1.2), that is on the ring R considered as a preadditive category. This means that any additive functor $\mathbf{R} \rightarrow \mathcal{A}$, into an abelian category \mathcal{A} , factors uniquely (up to natural equivalence) as the composition of an exact functor and the full and faithful inclusion $\mathbf{R} \hookrightarrow \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$. The latter is defined by considering that endomorphisms of $R_R \otimes -$ can be identified with (endo)morphisms in \mathbf{R} ; to be explicit we define this last functor as follows. Firstly, the object \mathbf{R} maps to $R_R \otimes -$, secondly each endomorphism $r : \mathbf{R} \rightarrow \mathbf{R}$ maps to the endomorphism $r \otimes - : R_R \otimes - \rightarrow R_R \otimes -$ (see Example 1.1 for a definition of \mathbf{R}). Faithfulness is immediate and fullness is easily verified.

²A finitely generated functor in $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ is a functor with an explicit presentation $(X, -) \rightarrow F \rightarrow 0$.

The above “freeness” of $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ implies that there must be an equivalence of categories

$$\begin{aligned} \text{Mod } R = (\mathbf{R}^{\text{op}}, \mathbf{Ab}) &\longrightarrow \text{Ex}(\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})^{\text{op}}, \mathbf{Ab}) \\ M &\longmapsto E_M = (\text{mod } R^{\text{op}}, \mathbf{Ab})(-, M \otimes -), \end{aligned}$$

where the expression on the bottom right is evaluated in the first argument ([17], Lemma 1.4). The discussion above also implies that there must be a duality

$$\begin{aligned} \text{fp}(\text{mod } R, \mathbf{Ab})^{\text{op}} &\xrightarrow{d} \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab}) \\ F &\longmapsto dF(M) = (\text{mod } R^{\text{op}}, \mathbf{Ab})(F, M \otimes -), \end{aligned}$$

which in particular implies ([17], Lemma 1.6) that a functor $F : \text{mod } R^{\text{op}} \rightarrow \mathbf{Ab}$ is finitely presented if and only if it is of the form $\ker f \otimes -$, for some morphism f in $\text{mod } R$.

We now consider an example of a particularly amenable ring.

Example 1.3. Let $R = k[X]$ be the k -algebra of polynomials in one variable X , for some algebraically closed field k . Thus with the infinite k -basis $\{1\} \cup \{X^i\}_{0 < i < \aleph_0}$. This algebra is alternatively described as the path algebra of the quiver

$$1 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \alpha,$$

where we recall that the category $\text{Mod } k[X]$ is equivalent to the category of representations of this quiver. Recall that a representation of this quiver³ consists of a pair, (M, f) , with $M \in \mathbf{Vect}_k$ and $f : M \rightarrow M$ a morphism in the category of vector spaces over k ; that is in the category \mathbf{Vect}_k .

While $k[X]$ is finitely generated over its centre (that is, over itself), the said centre fails to be artinian (e.g. $(X) \supsetneq (X^2) \supsetneq \dots$), so $k[X]$ is not an artin algebra. Yet $k[X]$ is a PID and so $\text{mod } k[X]$ consists of finite sums of finitely generated indecomposable $k[X]$ -modules (see e.g. [19], Section III.7). These indecomposables are: (i) $k[X]$ itself, which is of infinite k -dimension and (ii) $P_{\lambda,n} = k[X]/(X - \lambda)^n k[X]$ for some $\lambda \in k$, $n \in \mathbb{Z}_{\geq 1}$, which are of finite k -dimension. Note that the finitely generated

³See, for example, [1], Chapter II.

modules are precisely the finitely presented ones, for $k[X]$ is noetherian.

Let us mention three finitely generated functors in $(\text{mod } R, \mathbf{Ab})$. We also mention the correspondence with pp-formulas, the reader is referred to Section 2.1 for further details of this correspondence.

- Let $\mathcal{A}_{\lambda,n}$ be defined as

$$(P_{\lambda,n}, -) \longrightarrow \mathcal{A}_{\lambda,n} \longrightarrow 0,$$

where the morphism is the epimorphic restriction of the morphism induced by $1_{\lambda,n} : k[X] \rightarrow P_{\lambda,n}$, via the Yoneda lemma. That is $\mathcal{A}_{\lambda,n}$ is the image of $(1_{\lambda,n}, -) : (P_{\lambda,n}, -) \rightarrow (k[X], -)$. Then for any $k[X]$ -module M and any $m \in \mathcal{A}_{\lambda,n}M$ there is an $f : P_{\lambda,n} \rightarrow M$ in the image such that, if we evaluate the expression $m(X - \lambda)^n = f(1_{\lambda,n})(X - \lambda)^n = f(0)$ in M , we conclude that m is annihilated by $(X - \lambda)^n$. Conversely, if $m \in M$ is annihilated by $(X - \lambda)^n$, we can form (a well defined) morphism from $P_{\lambda,n}$ to M by specifying that $1_{\lambda,n}$ maps to m and extending linearly. This corresponds to the annihilator of $(X - \lambda)^n$, that is the pp-formula (see Section 2.1) $\phi(x_1)$ written

$$x_1(X - \lambda)^n = 0,$$

where x_1 is the free variable.

- Let $\mathcal{D}_{\lambda,n}$ be defined as

$$(k[X], -) \longrightarrow \mathcal{D}_{\lambda,n} \longrightarrow 0,$$

where $\mathcal{D}_{\lambda,n}$ is the image of the map corresponding to $(X - \lambda)^n : k[X] \rightarrow k[X]$ (we map 1 to $(X - \lambda)^n$). So, as before, we obtain that $m \in \mathcal{D}_{\lambda,n}M$ satisfies $m = f((X - \lambda)^n) = f(1)(X - \lambda)^n$ for some $f : k[X] \rightarrow M$. Hence $f(1)$ is a quotient of m by $(X - \lambda)^n$ in M , and vice-versa. This corresponds to divisibility by $(X - \lambda)^n$, that is a pp-formula (see Section 2.1) $\psi(x_1)$ written as

$$\exists x_2 \ x_2(X - \lambda)^n = x_1,$$

where x_1 is the free variable.

- Let $\mathcal{D}_{\lambda,n,s}$ be defined as

$$(P_{\lambda,s} \oplus k[X], -) \longrightarrow \mathcal{D}_{\lambda,n,s} \longrightarrow 0,$$

where the morphism corresponds to $(1, (X - \lambda)^{n-s}) : k[X] \rightarrow P_{\lambda,s} \oplus k[X]$. If $m = f((1, (X - \lambda)^{n-s}))$ then $m(X - \lambda)^s = f((0, (X - \lambda)^n)) = f((0, 1))(X - \lambda)^n$. On the other hand given an $m, m' \in M$ such that

$$m'(X - \lambda)^n = m(X - \lambda)^s$$

we produce a well-defined morphism $f : P_{\lambda,s} \oplus k[X] \rightarrow M$ by specifying

$$\begin{aligned} (1, X - \lambda)^{n-s} &\longmapsto m \\ (0, 1) &\longmapsto m', \end{aligned}$$

and extending linearly. The condition here corresponds to slightly more elaborate divisibility, the pp-formula (see Section 2.1) $\eta(x_1)$ written

$$\exists x_2 \ x_2(X - \lambda)^s = x_1(X - \lambda)^n,$$

for $s < n \in \mathbb{Z}_{\geq 1}$.

1.2 Localisation

Localisation is a broad theme throughout mathematics, often studied in rings, and also somewhat akin to sheaffication. A general formulation of the theory of localisation in categories is well known and particularly well studied in the setting of locally finitely presented categories. The category $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ is an example of this class of categories. We review the basic notation of localisation in this section, however the reader unfamiliar with localisation of categories is referred to the various sources mentioned here. We note that, however, a certain intuition for the elements of localisation which we require may be gained by combining Theorem 1.6 and some knowledge of model theory (see Chapter 2).

It is known that the category $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ is an example of a locally finitely presented category (see [5], Chapter 5). This implies that $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ is the closure

of $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ under all filtered colimits, in fact one has ([5], Theorem 5.3.7) the following equivalence

$$(\text{mod } R^{\text{op}}, \mathbf{Ab}) \longrightarrow \text{Lex}((\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab}))^{\text{op}}, \mathbf{Ab}),$$

where above is induced by the restriction of the Yoneda embedding. Objects $F : \text{mod } R^{\text{op}} \rightarrow \mathbf{Ab}$ which satisfy the stronger property that $(-, F)$ is exact (and not only left exact) on $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ are called *fp-injective*. One can show directly ([17], Lemma 1.3) that, up to isomorphism, the fp-injective objects are functors of the form $M \otimes -$, for some R -module M . Given this, and as $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ is abelian, it is often said that $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ is locally coherent.

For locally coherent Grothendieck categories there exists a well developed theory of localisation. One may consult [27] (Chapter 11) or [15] (9. Locally Coherent Categories). A more brief treatment can be found in [17] (Appendix A). Here we review only the basic details.

Definition 1.4. Let \mathcal{C} be an abelian category. A full subcategory, \mathcal{S} of \mathcal{C} , is called a *Serre subcategory* if, for every short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$, the object M is in \mathcal{S} if and only if both M' and M'' are in \mathcal{S} . If \mathcal{T} is a Serre subcategory which is furthermore closed under arbitrary coproducts then it is called a *localising subcategory*.

We define the quotient category \mathcal{C}/\mathcal{S} with associated quotient functor $q : \mathcal{C} \rightarrow \mathcal{C}/\mathcal{S}$ as the universal pair of an abelian category and an exact functor from \mathcal{C} . The universal property is that q makes objects of \mathcal{S} isomorphic to zero. It can be proven that such a pair exists.

Definition 1.5 ([27], p.431 and Proposition 11.1.12, p.437). Let \mathcal{C} be a Grothendieck abelian category and suppose that \mathcal{T} is a full subcategory of \mathcal{C} which is closed in \mathcal{C} under taking epimorphic images, extensions, arbitrary direct sums and subobjects. Recall that an extension in \mathcal{C} of $A, C \in \mathcal{T}$ is any B for which there exists an exact sequence in \mathcal{C} of the form $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$.

For a fixed such \mathcal{T} define \mathcal{F} to be the class of objects

$$\{D \in \mathcal{C} : (T, D) = 0 \text{ for all } T \in \mathcal{T}\}.$$

The pair $(\mathcal{T}, \mathcal{F})$ is then called a *hereditary torsion theory on \mathcal{C}* , further \mathcal{T} is called the *torsion class*, while \mathcal{F} is called the *torsion-free class*. A hereditary torsion theory is said to be of *finite type* if \mathcal{F} is closed under filtered colimits.

If \mathcal{G} is a Grothendieck abelian category then, for a localising subcategory \mathcal{T} , the quotient \mathcal{G}/\mathcal{T} is again Grothendieck abelian, so in particular it has injective envelopes. In this case \mathcal{T} is always the torsion class of a hereditary torsion theory on \mathcal{G} (see e.g. [32], Chapter VI); in fact this can be seen to imply that, up to equivalence, the category \mathcal{G}/\mathcal{T} is the full subcategory of \mathcal{T} -closed objects

$$\{X \in \mathcal{G} : (\mathcal{T}, X) = 0 \text{ and } \text{Ext}^1(\mathcal{T}, X) = 0\}.$$

Recall that $\text{Ext}^1(\mathcal{T}, X) = 0$ means that, for any $T \in \mathcal{T}$, the only extension of X by T is the trivial split extension.

In the context of $(\text{mod } R^{\text{op}}, \mathbf{Ab})$, a localising subcategory \mathcal{T} is said to be of finite type, if it is the closure, under filtered colimits, of some Serre subcategory \mathcal{S} of $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$; we write this as $\mathcal{T} = \text{colim} \mathcal{S}$. In particular this means that \mathcal{S} -closed and \mathcal{T} -closed objects coincide, thus in these circumstances the following statements can be deduced (see [17], Propositions A.4 to A.8):

- (i). The full subcategory of finitely presented objects in the quotient, $\text{fp}((\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T})$, is equivalent to $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{S}$. Here fp stands for the finitely presented objects of a category.
- (ii). The full subcategory of injective (resp. fp -injective) objects in the quotient $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}$ is equivalent to the intersection of the torsion-free class corresponding to the torsion theory of which \mathcal{T} is the torsion class i.e. the set

$$\mathcal{F} = \{X \in (\text{mod } R^{\text{op}}, \mathbf{Ab}) : (S, X) = 0 \text{ for all } S \in \mathcal{S}\}$$

and the class of injective (resp. fp -injective) objects in $(\text{mod } R^{\text{op}}, \mathbf{Ab})$.

The above has an immediate connection to model theory (see Chapter 2). We will need a few definitions. First, we say that a full subcategory, \mathcal{X} of $\text{Mod } R$, is a *definable subcategory* if for some $\{f_i : X_i \rightarrow Y_i\}_{i \in I} = \Phi \subseteq \text{Mor}(\text{mod } R^{\text{op}})$ the following holds,

$$M \in \mathcal{X} \iff (Y_i, M) \rightarrow (X_i, M) \text{ is surjective for all } i \in I.$$

Localisation implies an immediate consequence of this definition.

Theorem 1.6 ([17], Theorem 2.1; [27], Theorem 3.4.7 and Section 3.4.1). *Let \mathcal{X} be a full subcategory of $\text{Mod } R$, then the following are equivalent.*

- (i). \mathcal{X} is a definable subcategory of $\text{Mod } R$.
- (ii). \mathcal{X} is closed under taking (a) filtered colimits (b) arbitrary⁴ products and (c) pure submodules.
- (iii). There exists a Serre subcategory $\mathcal{S}_{\mathcal{X}}$ of $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ such that

$$M \in \mathcal{X} \iff (\mathcal{S}_{\mathcal{X}}, M \otimes -) = 0 \text{ in } (\text{mod } R^{\text{op}}, \mathbf{Ab}).$$

- (iv). There exists a hereditary torsion theory of finite type $(\mathcal{T}_{\mathcal{X}}, \mathcal{F}_{\mathcal{X}})$ on $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ such that

$$M \in \mathcal{X} \iff (M \otimes -) \in \mathcal{F}_{\mathcal{X}}.$$

In the functorial proof of the above Theorem it is possible to extract that, if we are given the data of (i), the following hold

$$\mathcal{T}_{\mathcal{X}} = \{F \in (\text{mod } R^{\text{op}}, \mathbf{Ab}) : (F, M \otimes -) = 0 \text{ for all } M \in \mathcal{X}\}$$

$$\mathcal{S}_{\mathcal{X}} = \text{Serre subcategory generated by } \{\ker f_i \otimes -\}_{i \in I} = \mathcal{T}_{\mathcal{X}} \cap \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab}),$$

for this last part see [17] (p.17).

1.3 Approximations

Approximations make an appearance in Section 4.4 and their use there is limited, thus a reader not interested in connections between endofinite modules and approximations

⁴Recall that this means the usual set indexed products, not just finite products.

may wish to skip this Section, at least on first reading. For us there are two types of relevant approximation in $(\text{mod } R^{\text{op}}, \mathbf{Ab})$. The first of these are the almost split morphisms.

Definition 1.7 ([27], p.235). Let $f : M' \rightarrow M$ be a morphism in an additive category \mathcal{C} . We say that f is *left almost split* if f is not a split monomorphism (that is there is no retraction $r : M \rightarrow M'$ such that $rf = 1_{M'}$), and if every monomorphism $g : M' \rightarrow L$ which does not split in \mathcal{C} factors through f .

Let us make three useful observations; taken together these are just a reformulation of the above definition.

- (i). The ring $\text{End}_R(M')$ is local. To see this let $k, l \in \text{End}_R(M')$ be two non-units, then they both factor through f as $k = k'f$ and $l = l'f$. If $k + l$ were a unit then we would obtain a splitting of f , hence it is a non-unit and so the ring is local.
- (ii). The unique maximal ideal, $\text{radEnd}_R(M')$ of $\text{End}_R(M')$, is the image of the induced map

$$(M, M') \xrightarrow{f_*} (M', M') = \text{End}_R(M').$$

This identification is clear, for if kf were a unit, for some $k \in (M, M')$, we would obtain a splitting of f . The reverse inclusion follows as in (i).

- (iii). If, for a morphism $g : M' \rightarrow L$, the image of the induced map

$$(L, M') \xrightarrow{(g_*)_{M'}} (M', M') = \text{End}_R(M')$$

is contained in $\text{radEnd}_R(M')$, then g factors through f .

There are two further remarks to be made when the category \mathcal{C} is $\text{mod } R$ or $\text{Mod } R$.

- (iv). The well known “splitting lemma” (e.g. [24], Corollary 7.4) says that a short exact sequence of the form

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is split, that is $M = M' \oplus M''$, if and only if f is a split monomorphism (if and only if g is a split epimorphism, defined dually).

- (v). If $h : M \rightarrow N$ is a split monomorphism with retraction r then, for any $x \in N$, one obtains $x - fr(x) \in \ker r$ and, as $\text{im } f \cap \ker r = 0$, it is then clear that $N = \text{im } h \oplus \ker r$.

We say that a full subcategory \mathcal{D} of $\text{mod } R$ has left almost split maps if every non-injective indecomposable object M in \mathcal{D} has a left almost split map $f : M \rightarrow D$ in \mathcal{D} (in particular with $D \in \mathcal{D}$). Dually we define right almost split maps and categories with right almost split maps (see [8] and [2]).

Let k be a commutative Artinian ring and let R be a ring equipped with a central action of k , that is we equip R with a ring morphism $f : k \rightarrow R$ such that $\text{im}(f) \subset Z(R)$, where $Z(R)$ denotes the center of R . Further assume that R is finitely generated as a k -module. Say that a ring R satisfying these conditions is an *artin algebra*. Note that immediately, by looking at R over $Z(R)$, we see that R must be both left and right Artinian. It is known that in this case the category $\text{mod } R$ has both left and right almost split maps. In fact using projective resolutions in the functor category ([8], Theorem 1.3) one obtains almost split sequences, that is non-split short exact sequences in which the monomorphism is left almost split and the epimorphism is right almost split. One further well known property is that in such case $\text{mod } R$ can be equipped with a well defined duality $D : (\text{mod } R)^{\text{op}} \rightarrow \text{mod } R^{\text{op}}$, extending the duality defined on \mathbf{Vect}_k ([3], II.Theorem 3.3).

The simplest examples of almost split sequences appear in a subcategory of $\text{mod } k[X]$. Note that this ring is not an artin algebra.

Example 1.8. Let $R = k[X]$ (see Example 1.3) and denote by $\text{fd } k[X]$ the smallest full subcategory of $\text{Mod } k[X]$ containing the objects $P_{\lambda,n}$ for all $\lambda \in k$ and $n \geq 1$. Denote by $b_{\lambda,n}^i$ the i th basis element of $P_{\lambda,n}$, that is the coset of $(X - \lambda)^i$. In terms of this basis there are two fundamental types of morphisms in $\text{fd } k[X]$. The first is

the monomorphism

$$P_{\lambda,n} \xrightarrow{\mathbf{m}_n} P_{\lambda,n+1}$$

$$1_{\lambda,n} \mapsto b_{\lambda,n}^1,$$

induced by multiplication. The second is the epimorphism

$$P_{\lambda,n+1} \xrightarrow{\mathbf{p}_n} P_{\lambda,n}$$

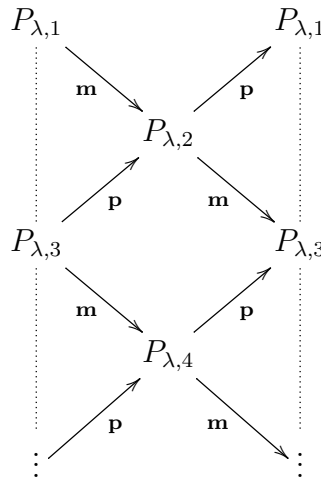
$$b_{\lambda,n}^{m-1} \mapsto 0,$$

which has a one dimensional kernel. It is usually convenient to drop the superscripts for \mathbf{m} and \mathbf{p} , in that notation it is clear that “ $\mathbf{m}\mathbf{p} = \mathbf{p}\mathbf{m}$ ” as endomorphisms of any $P_{\lambda,n}$. By noting that every morphism $f : P_{\lambda,n} \rightarrow P_{\mu,m}$ in $\text{fd } k[X]$ is determined by its value at $1_{\lambda,n}$ one can obtain (see e.g. [31], Lemma XIX.2.2) that, for $m \geq n$, $\dim(P_{\lambda,n}, P_{\lambda,m}) = n$ and the basis consists of $\mathbf{m}^{m-n}, \mathbf{m}^{m-n+1}\mathbf{p}, \dots, \mathbf{m}^{m-1}\mathbf{p}^{n-1}$, similarly for $m \leq n$.

In fact this (and the fact that the only morphism $P_{\lambda,n} \rightarrow P_{\gamma,m}$ for $\lambda \neq \gamma$ is the 0 morphism) clearly implies that there are almost split exact sequences in $\text{fd } k[X]$ of the form

$$0 \longrightarrow P_{\lambda,n} \xrightarrow{(\mathbf{p}_{n-1}, \mathbf{m}_n)} P_{\lambda,n-1} \oplus P_{\lambda,n+1} \xrightarrow{(\mathbf{m}_{n-1}, -\mathbf{p}_n)} P_{\lambda,n} \longrightarrow 0,$$

for $n \geq 1$, though we do have to set (by slight abuse of notation) $\mathbf{p}_0 = \mathbf{m}_0 = 0$. From this we can easily extract a description of the irreducible morphisms in $\text{fd } k[X]$ as precisely the \mathbf{p} and \mathbf{m} for suitable indices, so for each λ we will have a homogeneous tube



and taking the union of all of these tubes we obtain a picture (an analogue of the Auslander-Reiten quiver) of the indecomposable modules and the irreducible morphisms in $\text{fd } k[X]$.

The second type of approximation is the following.

Definition 1.9 ([27], Section 3.4.4; [18], p.3). Let \mathcal{D} be a class of objects in an abelian category \mathcal{A} . Then a morphism $f : M \rightarrow D$ is a *left \mathcal{D} -approximation of M* if $D \in \mathcal{D}$ and for every $D' \in \mathcal{D}$ there is an exact sequence $(D, D') \xrightarrow{(f*)_{D'}} (M, D') \rightarrow 0$. That is for every $g : M \rightarrow D'$, there is a $h : D \rightarrow D'$, such that $hf = g$. Dually one defines a *right \mathcal{D} approximation*.

If every object in \mathcal{A} has a left \mathcal{D} -approximation then the subcategory \mathcal{D} is called *covariantly finite in \mathcal{A}* . Dually if every object in \mathcal{A} has a right \mathcal{D} -approximation then the subcategory is called *contravariantly finite in \mathcal{A}* .

Let R be an artin algebra. Then one can directly see that a covariantly finite full subcategory \mathcal{D} of $\text{mod } R$ has right almost split maps, and dually for contravariantly finite subcategories and left almost split maps. For any ring R we have the following connection with definable categories.

Theorem 1.10 ([27], Corollary 3.4.37; [17], Proposition 3.11). *Let \mathcal{D} be a full additive subcategory of $\text{mod } R$ and let \mathcal{X} be the closure of \mathcal{D} under filtered colimits in $\text{Mod } R$. Then \mathcal{X} is a definable subcategory if and only if \mathcal{D} is covariantly finite in $\text{mod } R$.*

It is an observation (just using the definitions, see e.g. [29], Lemma 3.1) that, considering left approximations by $\mathcal{D} \subseteq \text{Mod } R$, it is irrespective if we further assume that \mathcal{D} is closed under direct summands. If we denote by $\text{Sumd } \mathcal{D}$ the full additive subcategory generated by all the direct summands of objects of \mathcal{D} ([29], Theorem 3.3), then \mathcal{D} is covariantly finite in $\text{Mod } R$ if and only if $\text{Sumd } \mathcal{D}$ is covariantly finite in $\text{Mod } R$. For brevity we will denote by $\text{Prod } \mathcal{D}$ the class of modules isomorphic to products of modules in \mathcal{D} and by $\text{Sumd}^{\text{II}} \mathcal{D}$ the class of all modules isomorphic to direct summands of products of modules in \mathcal{D} ; similarly $\text{Sumd}^{\text{II}} \mathcal{D}$ for coproducts. These will be prefixed with full subcategory when we wish to consider this and not just the class of modules.

Theorem 1.11 ([29], Theorem 3.5). *Let \mathcal{D} be a class of objects of $\text{Mod } R$. Then the following hold.*

- (i). *$\text{Prod } \mathcal{D}$ and $\text{Sumd}^{\text{II}} \mathcal{D}$ are covariantly finite in $\text{Mod } R$.*
- (ii). *$\text{Coproduct } \mathcal{D}$ is covariantly finite in $\text{Mod } R$ if and only if $\text{Sumd}^{\text{II}} \mathcal{D}$ is covariantly finite in $\text{Mod } R$ if and only if $\text{Sumd}^{\text{II}} \mathcal{D}$ is closed under products.*
- (iii). *If \mathcal{D} is closed under pure submodules then \mathcal{D} is covariantly finite in $\text{Mod } R$ if and only if $\text{Sumd } \mathcal{D}$ is closed under products.*

The third part uses that if \mathcal{D} is closed under pure submodules then any $f : M \rightarrow D$ with $D \in \mathcal{D}$ will factor through some product of modules in some $\mathcal{D}_M \subseteq \mathcal{D}$ depending on M ; this follows as in [27] (Proposition 3.4.42), and in particular shows that every definable subcategory is covariantly finite in $\text{Mod } R$.

Chapter 2

Model theoretic methods

2.1 Pp-formulas and invariants

The functorial approach to $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ has a corresponding model theoretic one, applicable by means of a duality. References which constitute general overviews of model theory are [7], [6] (Chapter 8) and [21]. Two comprehensive references for the model theory of modules are [26] and [13]. In this chapter we will review, very briefly, the elements of model theory which play a role in Chapters 3, 4 and 5.

In model theory a *language* \mathcal{L} is determined by a tuple $(F_{\mathcal{L}}, R_{\mathcal{L}}, C_{\mathcal{L}})$ of *function symbols*, *relation symbols* and *constant symbols* ([7], Definition 1.1.1). A *function symbol* or a *relation symbol* is a pair (S, n) of a symbol S and an $n \in \mathbb{Z}_{\geq 1}$ called its arity, while a constant symbol is just a symbol. We assume that all the symbols used are distinct. All the languages we consider will have $R_{\mathcal{L}} = \emptyset$, thus we now fix a language \mathcal{L} determined by (F, C) , with function symbols $F = \{(f_i, n_i)\}_{i \in I}$ and constant symbols $C = \{c_j\}_{j \in J}$.

A *first order formula* in \mathcal{L} , the language determined by (F, C) , is a special kind of finite string built out of the symbols of \mathcal{L} , variables x_0, x_1, x_2, \dots , logical connectives \neg, \wedge, \vee and quantifiers \exists, \forall . It is clear that not all finite strings are meaningful formulas (e.g. $\forall \forall \neg x_0$ is clearly nonsensical) and the set of first order formulas is constructed inductively as follows.

First we define a set of *terms* inductively ([7], Definition 1.1.4). Every variable

and every constant symbol is a *term*. If (f, n) is a function symbol and t_1, \dots, t_n are terms, then $f(t_1, \dots, t_n)$ is also a term. Now we can define the set of first order formulas inductively ([7], p.10).

- (i). If t_1 and t_2 are terms then $t_1 = t_2$ is a first order formula.
- (ii). If ϕ is a first order formula then so is $\neg\phi$.
- (iii). If ϕ_1 and ϕ_2 are first order formulas then so are $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$.
- (iv). If ϕ is a first order formula and x_i is a variable then $\forall x_i \phi$ and $\exists x_i \phi$ are first order formulas.

We will assume that the reader is familiar with the usual meanings of the quantifiers \exists (there exists), \forall (for all) and the logical connectives \vee (or), \wedge (and) and \neg (not). The logical connective \rightarrow (implies) used as $\phi \rightarrow \psi$ is shorthand for $\neg\phi \vee \psi$ ([7], Remark 1.1.7). By a formula we will always mean a first order formula.

We can now define an appropriate language for studying R -modules. For this fix a ring R and define *the language of right R -modules*, denoted by \mathcal{L}_R , to be the language with constant symbol 0 , function symbol $+$ of arity two and a set of function symbols $\{r\}_{r \in R}$ of arity one. We will write the function symbol r of arity one on the right as $\cdot r$ (e.g. $r(t)$ is written $t \cdot r$ for a term t). Also we will write the function symbol $+$ of arity two in infix notation rather than prefix notation (e.g. $+(t_1, t_2)$ is written $t_1 + t_2$ for two terms t_1 and t_2). We will also use brackets to make formulas more readable.

Fix an R -module M . Then M has tacitly associated to it an element $0^M \in M$, a function $+^M : M^2 \rightarrow M$, and functions $\cdot r^M : M \rightarrow M$ for each $r \in R$, given by the action of R on M . This is said to endow M with a \mathcal{L}_R -structure (see [7], Definition 1.1.2), wherein we can then make sense of \mathcal{L}_R -formulas in M . Note that we will drop the superscripts where no confusion can occur.

We say that a variable x_i occurs *freely* (or is *free*) in a formula ϕ if it is not inside a $\exists x_i$ or $\forall x_i$ quantifier. If a variable does not occur freely then it is *bound*. For example in the formula

$$\exists x_2 (x_1 = x_2 \cdot r)$$

x_1 is free while x_2 is bound. While in the formula

$$\forall x_1 \forall x_2 ((x_1 + x_2) \cdot r = x_1 \cdot r + x_2 \cdot r)$$

both x_1 and x_2 are bound. We say that a formula is an \mathcal{L}_R -sentence if all variables appearing in the formula are bound.

Let ϕ be a \mathcal{L}_R -formula in which the free variables are x_1, \dots, x_n (note that we allow the case when there are no free variables, in which the inductive definition below still holds using (vi) and (v)). We then write $\phi(x_1, \dots, x_n)$. If (m_1, \dots, m_n) is a tuple of elements of an R -module M , then we define $M \models \phi[m_1, \dots, m_n]$ inductively ([7], Definition 1.1.6).

(i). If ϕ is $t_1 = t_2$ for two terms t_1 and t_2 , then we need to first interpret the terms.

A term t in M (whose variables are contained in x_1, \dots, x_n) is interpreted as representing a function $t^M : M^n \rightarrow M$ using $+^M$, r^M and 0^M , where the inputs are the free variables x_1, \dots, x_n . For example if the term t is

$$(x_1 \cdot r_1 + x_2 \cdot r_2 + 0) \cdot r_3,$$

then it defines the function

$$t^M(m_1, \dots, m_n) = (m_1 \cdot r_1^M + m_2 \cdot r_2^M + 0^M) \cdot r_3^M.$$

We then say that $M \models (t_1 = t_2)[m_1, \dots, m_n]$ if $t_1^M(m_1, \dots, m_n) = t_2^M(m_1, \dots, m_n)$ is true in the module M .

- (ii). If ϕ is $\psi_1 \vee \psi_2$ then $M \models \phi[m_1, \dots, m_n]$ if $M \models \psi_1[m_1, \dots, m_n]$ or $M \models \psi_2[m_1, \dots, m_n]$.
- (iii). If ϕ is $\psi_1 \wedge \psi_2$ then $M \models \phi[m_1, \dots, m_n]$ if $M \models \psi_1[m_1, \dots, m_n]$ and $M \models \psi_2[m_1, \dots, m_n]$.
- (iv). If ϕ is $\neg\psi$ then $M \models \phi[m_1, \dots, m_n]$ if it is not the case that $M \models \psi[m_1, \dots, m_n]$.
- (v). If ϕ is $\exists x_j \psi(x_1, \dots, x_n, x_j)$ then $M \models \phi[m_1, \dots, m_n]$ if there exists some $m \in M$ such that $M \models \psi[m_1, \dots, m_n, m]$.

- (vi). If ϕ is $\forall x_j \psi(x_1, \dots, x_n, x_j)$ then $M \models \phi[m_1, \dots, m_n]$ if for every $m \in M$ it is the case that $M \models \psi[m_1, \dots, m_n, m]$.

When $M \models \phi[m_1, \dots, m_n]$ holds it is said that M *satisfies* ϕ at (m_1, \dots, m_n) . For example the first order formula $\phi(x_1)$ written as $\exists x_2 (x_1 = x_2 \cdot r)$ is satisfied in a module M at $m_1 \in M$ if there exists some $m_2 \in M$ that satisfies the equation $m_1 = m_2 \cdot r$ in M . For further details see [7] (Chapter 1, Structures and Theories).

The size of the language is then $|R| + \aleph_0$. Here $|R|$ denotes the cardinality of the ring; note that for an infinite ring the size of the ring determines the size of the language, for all the other elements of the language have a fixed cardinality of \aleph_0 . Using first order formulas we can write the *the axioms of R -modules* or *the theory of R -modules*. For example the formula

$$\forall x_1 \forall x_2 ((x_1 + x_2) \cdot r = x_1 \cdot r + x_2 \cdot r)$$

is one of the axioms. Many other, quite often complex, first-order statements about modules can be formulated, though as we shall see they all reduce to a simpler form.

We define a *theory of modules* to be a consistent (or satisfiable) set of \mathcal{L}_R -sentences which contains the axioms of R -modules (it is important to note that the axioms of module theory are all \mathcal{L}_R -sentences). Such a theory is called *complete* if it is a maximal consistent set of sentences, in particular if for every sentence ϕ the theory contains either ϕ or $\neg\phi$, and *incomplete* otherwise. As an example, the theory of R -modules is incomplete¹, but the (first order) theory of any R -module M , denoted $\text{Th}(M) = \{\phi : M \models \phi\}$, is complete (it has a model M , and M either has a first order property or it has the negated property). Here we use the usual model-theoretic notation, if a module M satisfies a sentence ϕ , we write $M \models \phi$.

A well known and central result of the model theory of modules ([13], Chapter 6; [26], Corollary 2.13) states that, modulo a complete theory of modules T , any \mathcal{L}_R -formula is equivalent to a (finite) boolean combination of simpler formulas. These are the positive primitive formulas (pp for short).

¹This has the following more obvious reformulation. There exist some \mathcal{L}_R -sentence which is be valid in one R -module and invalid in another.

Definition 2.1 ([26], p.14). A formula $\phi(x_1, \dots, x_n)$, in the language \mathcal{L}_R , is a *pp-formula* if it is of the form

$$\exists x_{n+1} \dots x_m \bigwedge_{i=1}^k \sum_{j=1}^m x_j \cdot r_{j,i} = 0.$$

Here $\{r_{j,i}\}_{i=1,j=1}^{k,m}$ are the unary function symbols corresponding to elements of R , but written in a more natural way, the x_{n+1} to x_m are said to be the bound variables and the x_1 to x_n are said to be the free variables. Note that $\bigwedge_{i=1}^k$ denotes repeated \wedge and that $\exists x_{n+1} \dots x_m$ denotes $\exists x_{n+1} \exists x_{n+2} \dots \exists x_m$. As before if a pp-formula $\phi(x_1, \dots, x_n)$ holds at the tuple (m_1, \dots, m_n) of elements of M then we write $M \models \phi[m_1, \dots, m_n]$. This specifies at which elements of the module M the formula holds, in a sense this asserts that if these elements replace the free variables, the resulting expression is then true in M .

We write pp_n^R to denote the set of all pp-formulas in n free variables, and remark that this set has the natural structure of a lattice (see Section 2.2).

In the above Definition 2.1 we have already considered that a tuple of elements from a module can either satisfy or not satisfy a given \mathcal{L}_R -formula with free variables (the tuple must have the same length as the number of free variables). If we are given a pp-formula $\phi(x_1, \dots, x_n)$ and an R -module M , then there is a subset of M^n that is a “solution set” of ϕ in M . This is the subset $\phi(M)$ of M^n , defined by ϕ , and consists of all elements $(m_1, \dots, m_n) \in M^n$ such that $M \models \phi[m_1, \dots, m_n]$. Note that this is an abelian subgroup of M^n , but is not necessarily an R -submodule of M^n . There are two well defined module structures on $\phi(M)$. It is an $\text{End}_R(M)^{\text{op}}$ -submodule and a $Z(R)$ -submodule of M^n , both of the actions being restrictions of the corresponding actions on M^n . Thence it is natural to think of $\phi(M)$ as an $\text{End}_R(M)$ - $Z(R)$ -bimodule. If ϕ has only one free variable x_1 , then $\phi(M)$ is said to be a *subgroup of finite definition of M* or a *pp-definable subgroup of M* .

The result quoted before the definition implies that, when studying complete theories of modules, it is possible (and more convenient) to work with pp-formulas instead of general \mathcal{L}_R -formulas. In fact, as a consequence of the same result (see [26],

Corollary 2.15), it can be shown that, in the theory of R -modules, any \mathcal{L}_R -sentence is equivalent to a boolean combination of certain invariant conditions. These invariant conditions are defined as follows.

Definition 2.2 ([26], p.34). Let $\phi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ be pp-formulas in n free variables. Let M be an R -module. Then $\text{Inv}(M, \phi, \psi) := |\phi(M)/\phi(M) \cap \psi(M)|$ is called an *invariant*. Here $|X|$ stands for the cardinality of the set X .

One can show that, for each $k \in \mathbb{Z}_{\geq 0}$ and every invariant, there exists an \mathcal{L}_R -sentence, which we denote by $\text{Inv}(-, \phi, \psi) > k$, which has the property that

$$M \models \text{Inv}(-, \phi, \psi) > k \iff \text{Inv}(M, \phi, \psi) > k.$$

The set of such sentences, for all $k \in \mathbb{Z}_{\geq 0}$ and all invariants, along with their negations, is the set of *invariant conditions*. In words the sentence $\text{Inv}(-, \phi, \psi) > k$ says that there exist $k + 1$ tuples which satisfy ϕ and which lie in differing cosets of ψ . Since the language \mathcal{L}_R is finitary, that is we permit only finite formulas, sentences like this cannot deal with infinite cardinalities (see also Definition 2.5).

It can further be shown ([26], Corollary 2.19) that any complete theory of modules can (equivalently) be axiomatised by boolean combinations of invariant sentences, and this can be further reduced to conditions of the form $\text{Inv}(-, \phi, \psi) = k$ or $\text{Inv}(-, \phi, \psi) > k$, for ϕ and ψ pp-formulas in one free variable.

We can now start to apply some of the category theory developed in Chapter 1. Fix a pp-formula ϕ and note that evaluation at a module is functorial, we write this as

$$F_\phi : \text{mod } R \longrightarrow \mathbf{Ab}$$

$$M \longrightarrow \phi(M),$$

where a map $f : M \rightarrow N$ induces a corresponding, and well-defined, restriction $F_\phi f : \phi(M) \rightarrow \phi(N)$. We note that $F_\phi \in (\text{mod } R, \mathbf{Ab})$. In fact we could equally consider F_ϕ defined on all of $\text{Mod } R$, or we could extend it as defined above by viewing an R -module as a filtered colimit of finitely presented modules; this is equivalent as it can be seen that $\phi(\text{colim}_{\rightarrow i \in I} M_i) = \text{colim}_{\rightarrow i \in I} \phi(M_i)$, over any filtered category I .

If ϕ has n free variables then the functor F_ϕ can be shown ([27], Corollary 1.2.19, p.37) to fit into a diagram $(C_\phi, -) \rightarrow F_\phi \rightarrow (R^n, -)$, in which the first map is an epimorphism and the last map is a monomorphism of functors, while the module C_ϕ lies in $\text{mod } R$ (for the above diagram itself lies in $(\text{mod } R, \mathbf{Ab})$). By the Yoneda lemma the composition of the morphisms in this diagram corresponds to some map $\bar{c} : R^n \rightarrow C_\phi$ which takes (r_1, \dots, r_n) to $\sum_{i=1}^n c_i r_i$. We now define such a finitely presented module C_ϕ and map $\bar{c} : R^n \rightarrow C_\phi$ associated to a pp-formula $\phi(x_1, \dots, x_n)$.

Definition 2.3. Let $\phi(x_1, \dots, x_n)$ be the pp-formula

$$\exists x_{n+1} \dots x_m \bigwedge_{i=1}^k \sum_{j=1}^m x_j \cdot r_{j,i} = 0$$

in the free variables x_1, \dots, x_n and bound variables x_{n+1}, \dots, x_m .

Fix a set of generators $\{g_1, g_2, \dots, g_m\}$. For each $1 \leq i \leq k$ the linear part of the formula above is

$$\sum_{j=1}^m x_j r_{j,i} = 0,$$

corresponding to which we introduce a relation $R_i = \sum_{j=1}^m g_j r_{j,i}$ in the generators.

We then let C_ϕ be the finitely presented module with generators $\{g_1, g_2, \dots, g_m\}$ and relations $\{R_i\}_{1 \leq i \leq k}$. The map $\bar{c} : R^n \rightarrow C_\phi$ takes the l th generator of R^n to g_l for $1 \leq l \leq n$ and is extended linearly.

Example 2.4. Consider a very simple pp-formula, $\phi(x_1)$, of the form

$$\exists x_2 (x_1 r_{1,1} + x_2 r_{1,2} = 0)$$

as in Definition 2.1. To simplify notation we set $r_{1,1} = r_1$ and $r_{1,2} = r_2$ (recall that both are elements of the ring R , or more precisely both represent elements of the ring R). Thus we can write the formula $\phi(x_1)$ in the more natural form $\exists x_2 (x_1 r_1 = x_2 r_2)$. A free realisation C_ϕ will have generators g_1 and g_2 and have the relation $r_1 g_1 - r_2 g_2$. The map $c : R \rightarrow C_\phi$ corresponds to mapping 1 to g_1 .

Turning to model theory this (C_ϕ, \bar{c}) corresponds to a certain free realisation of the pp-formula ϕ . A *free realisation* is a pair (C_ϕ, \bar{c}) where C_ϕ is a finitely presented

module and \bar{c} is a tuple of elements which satisfies $\text{pp}^{C_\phi}(\bar{c}) = \langle \phi \rangle$, where these two sets are defined as follows:

$$\begin{aligned} \text{pp}^{C_\phi}(\bar{c}) &= \{\psi \in \text{pp}_n^R : C_\phi \models \psi[c_1, \dots, c_n]\} \\ \langle \phi \rangle &= \{\psi \in \text{pp}_n^R : \psi(M) \supseteq \phi(M) \text{ for all modules } M\}. \end{aligned}$$

In fact the above sets are pp-types (see the beginning of Section 2.2 for the relevant definitions). Note that a free realisation need not be minimal, it need merely satisfy the above model theoretic property (for a discussion see [27], pp.22-24). The (C_ϕ, \bar{c}) we constructed in the above Definition 2.3 is a (model theoretic) free realisation of ϕ , for a proof of this see [27] (Section 1.2.2).

The existence of such a free realisation is the essential link between the model theoretic and functorial approaches. This means that an n -pointed object of $\text{mod } R$, say (C_ϕ, \bar{c}) , corresponds to some pp-formula ϕ , and as noted above, vice versa. We also note the projective resolution of F_ϕ ([27], 10.2.29, p.444) is seen to be

$$0 \longrightarrow (C_\phi/(\bar{c})) \longrightarrow (C_\phi, -) \longrightarrow F_\phi \longrightarrow 0,$$

where for $\bar{c} = (c_1, \dots, c_n)$ the submodule (\bar{c}) is generated by all the c_i so that $C_\phi/(\bar{c})$ is the same as C_ϕ but with extra relations $\{c_i = 0\}_{i=1}^n$.

To consider F_ϕ in the category $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ we apply duality (see Section 1.1) to the exact sequence $0 \rightarrow F_\phi \rightarrow (R^n, -)$, to get an exact sequence $(R^n \otimes -) \rightarrow dF_\phi \rightarrow 0$, and taking the kernel of this we obtain a subfunctor of $(R^n, -)$. This is then of the form $F_{D\phi}$ for some pp-formula $D\phi$ in the language of R^{op} -modules. One can show that this yields a lattice duality $D : \text{pp}_n^R \rightarrow \text{pp}_n^{R^{\text{op}}}$, an order reversing lattice morphism between the lattice of pp-formulas for R and R^{op} (see the beginning of Section 2.2 for the relevant definitions).

We can also apply the above to get a more category-theoretic reformulation of pp-definable subgroups. The above implies that pp-definable subgroups occur as either (i) the kernel of a morphism $M \rightarrow M \otimes_R N$ sending $m \mapsto m \otimes n$ for some fixed n and N (which determine the pp-formula via free realisation), or (ii) as the cokernel of a morphism $(N, -) \rightarrow (R, -)$ determined by $n : R \rightarrow N$, which picks out the element n . This last part can be seen by directly applying the duality d .

We also want to consider formulas with parameters in some fixed R -module M . To define a pp-formula with parameters from M we can proceed via the usual route. That is, enrich the language \mathcal{L}_R with constants for each element of M , to get a new bigger language, say $\mathcal{L}_R(M)$, and then define pp-formulas in that language. The following is an equivalent and more amenable definition.

Definition 2.5. A *pp-definable coset*, or a subgroup defined by a *pp-formula with parameters from M* , is a coset $(m_1, \dots, m_n) + \phi(M)$ for some pp-formula $\phi(x_1, \dots, x_n)$ and some $(m_1, \dots, m_n) \in M^n$. A formula with parameters which defines the coset, in the simple case $m + \phi(M)$ where ϕ has one free variable, is $\exists y (x_1 = m + y \wedge \phi(y))$, suitably rearranged into the standard format of a pp-formula.

There is a corresponding notion of a definable subset of an R -module N , for a formula ϕ with parameters from another R -module M . First, recall that a monomorphism $f : M \rightarrow N$ in $\text{Mod } R$ is a *pure monomorphism* if the induced morphism $M \otimes - \xrightarrow{f \otimes -} N \otimes -$ is a monomorphism in $(\text{mod } R^{\text{op}}, \mathbf{Ab})$. This condition is equivalent to $\psi(\text{im } f) = \text{im } f \cap \psi(N)$ for every pp-formula ψ without parameters. If we now assume that $M \rightarrow N$ is a pure monomorphism then we can meaningfully (because the part with the parameters then remains unchanged in N) evaluate the formula ϕ with parameters from M on N , and we write this as $\phi(N, \bar{m})$, where \bar{m} is the parameter. This means that we can put an order on the poset of all pp-definable cosets, which we denote by pp_n^M .

2.2 Pp-types and pure injective modules

Denote by pp_n the poset of all pp-formulas in n free variables. The ordering $\phi \geq \psi$ is defined to hold when the axioms of R -modules prove $(\phi \rightarrow \psi)$. This ordering becomes strict if we conjunct with $\neg(\psi \rightarrow \phi)$. The poset pp_n has the structure of a lattice in which \wedge corresponds to the logical connective \wedge and \vee is given by addition of pp-formulas. Addition of pp-formulas is defined in the following way. If $\phi(x_1)$ and

$\phi(x_1)$ are in pp_1 , then the formula $(\phi + \psi)(x_1)$ is

$$\exists y_1, y_2 (y_1 + y_2 = x_1) \wedge \phi(y_1) \wedge \psi(y_2),$$

where we assume that the symbols y_1, y_2 are distinct from any variables in ϕ or ψ . Similarly for pp_n .

A filter on pp_n is called a *pp-type* without parameters. Note that at given pp-type is finitely satisfiable, that is every finite subset of pp-formulas in a pp-type is satisfied in some R -module. To see this take any finite subset of pp-formulas in a pp-type and find a common intersection by the property of the pp-type being a filter on pp_n (this will at worst contain just 0). It is in fact not difficult to construct, for any pp-type p in n free variables, using a reduced product (a weakened form of ultraproduct) an R -module L and a tuple $\bar{l} = (l_1, \dots, l_n) \in L^n$ such that $\text{pp}^L(\bar{l}) = p$ ([27], Theorem 3.3.6). Here $\text{pp}^L(\bar{l})$ denotes the pp-type consisting of all pp-formulas ϕ , in n free variables, such that $L \models \phi[\bar{l}]$. We can also assume that pp-types are closed under conjunction and implication (this will not change the definable subsets), and we also note that, since a pp-type without parameters is (at worst) determined by some set of logical and non-logical symbols of cardinality at most \aleph_0 , and some subset of R , then we can strictly bound their number by $2^{\max(\aleph_0, |R|)}$. Finally, as with pp-formulas, we can evaluate a pp-type p at an R -module M ; this is just the intersection of the corresponding pp-definable subgroups, and we write $p(M) = \bigcap_{\phi \in p} \phi(M)$.

Functorially, a pp-type p without parameters corresponds (via duality) to the additive subfunctor $F_{Dp} = \sum_{\phi \in p} F_{D\phi}$ of $R^n \otimes -$ in $(\text{mod } R^{\text{op}}, \mathbf{Ab})$. In fact any additive subfunctor of $R^n \otimes -$ is equal to F_{Dp} for some pp-type p (see [27], Proposition 12.2.1, p.470).

If we view p as $\text{pp}^M(\bar{m}) = \{\phi : \bar{m} \in \phi(M)\}$ for some R -module M and $\bar{m} \in M^n$, then it can be proven that F_{Dp} is also the kernel of the morphism

$$\begin{aligned} R^n \otimes - &\xrightarrow{\bar{m} \otimes -} M \otimes - \\ (r_1, r_2, \dots, r_n) &\longmapsto \sum_{i=1}^n r_i m_i, \end{aligned}$$

where all the objects are considered in $(\text{mod } R^{\text{op}}, \mathbf{Ab})$.

As before we want to define pp-types which allow parameters from some R -module. Define *pp-types with parameters from M* as filters on the poset pp_n^M of pp-definable cosets of M (see Section 2.1). As before every finite subset of a pp-type with parameters from M is satisfied in M , so that a pp-type with parameters is a consistent set of pp-formulas modulo the complete theory $\text{Th}(M)$. In fact, given a set of parameters $A \subset M$, we can define the pp- n -type with parameters of some $\bar{m} \in M^n$ over A as

$$\text{pp}^M(\bar{m}/A) = \{\bar{a} + \phi(M) : m \in \bar{a} + \phi(M)\},$$

where \bar{a} is a tuple of elements of A . This captures what is true in M of \bar{m} using A as the parameters. The set of all pp- n -types over A is denoted by $S_n^+(A)$ (see [26], p.24).

If p is a pp-type with parameters from M , then for a pure embedding $M \hookrightarrow N$, we set $p(N)$ to be the intersection of all the subsets $\phi(N, \bar{m})$ as $\phi(x_1, \dots, x_n, \bar{m})$ runs over p . As before there is a set of pp- n -types with parameters from M ; the size of this set is bounded by $2^{\max(\aleph_0, |R|, |M|)}$.

An R -module M is said to be *algebraically compact* if every pp-1-type (equivalently every pp- n -type) with parameters from M has a solution in M . Further, a module M is said to be *pure injective* if $M \otimes -$ is an injective object of $(\text{mod } R^{\text{op}}, \mathbf{Ab})$. Note that $M \otimes -$ is always an fp-injective object of $(\text{mod } R^{\text{op}}, \mathbf{Ab})$, thus what we require here is a stronger property. The following is well known.

Theorem 2.6 ([27], Theorem 4.3.11, Lemma 4.2.1). *An R -module M is pure injective if and only if it is algebraically compact.*

Thus an injective module is algebraically compact, for it is clearly pure injective, because it is equivalent to say that a pure injective module is injective with respect to pure monomorphisms (and an injective module is injective with respect to *all* monomorphisms). Using the tensor embedding combined with the fact that $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ has injective envelopes, we see that every R -module M must have a pure injective envelope $H(M)$. In fact we can associate pure injectives to pp-types of elements of modules in the following way. If for some R -module M we have the

pp-type p of \bar{m} , that is $p = \text{pp}^M(\bar{m})$, then $H(p)$ is a direct summand of $H(M)$ minimal with respect to containing \bar{m} . This can be shown to be uniquely determined up to isomorphism and independent of the chosen realisation (M, \bar{m}) of p ([27], Section 4.3.5). This has the following important consequence.

Lemma 2.7 ([27], Corollary 4.3.37). *Let N be an indecomposable pure injective R -module, then there is a pp-type p (i.e. the pp-type of a non-zero element of N) such that $N = H(p)$, hence there is only a set (of isomorphism classes) of indecomposable pure injective R -modules.*

2.3 Pp-pairs and the Ziegler spectrum

Let T be a complete theory of R -modules, that is $T = \text{Th}(M)$ for some R -module M . We say that T is *closed under products* if its class of models is closed under arbitrary products; or equivalently if $T = T^{\aleph_0} := \text{Th}(\prod_{\aleph_0} M)$. If T is a theory which is closed under products and under pure submodules then by, [26] (Corollary 2.31), modulo the theory of R -modules, the set T has the form $\{\text{Inv}(-, \phi_i, \psi_i) = 1\}_{i \in I}$, for some set of pp-formulas $\{\phi_i(x_1), \psi_i(x_1)\}_{i \in I}$ in one free variable x_1 . If we now take the smallest full subcategory, say \mathcal{X}_T of $\text{Mod } R$, which contains all the models of T , then this \mathcal{X}_T will be a definable subcategory of $\text{Mod } R$. In fact every definable subcategory is of this form (using, for example, Theorem 2.12 in [13]). The above discussion motivates the following definition.

Definition 2.8. Let $\phi \geq \psi$ be two pp-formulas in pp_n . The pair ϕ/ψ is called a *pp-pair*, and we note the following:

- (i). A pp-pair ϕ/ψ is said to be *closed* on an R -module M if $\phi(M) = \psi(M)$.
- (ii). A pp-pair ϕ/ψ is said to be *open* on an R -module M if $\phi(M) > \psi(M)$.

Also notice that the conditions above are exclusive.

Given a pp-pair ϕ/ψ and free realisations (C_ϕ, \bar{c}) and (C_ψ, \bar{c}') , we can consider

these realisations as elements of F_ϕ and F_ψ respectively². Then the pp-pair is completely determined by a map $f : (C_\phi, \bar{c}) \rightarrow (C_\psi, \bar{c}')$ in the larger category of elements of F_ϕ . Using this observation it is not difficult to see that to every pp-pair there corresponds a finitely presented functor $F_{\phi/\psi} \in \text{fp}(\text{mod } R, \mathbf{Ab})$, or alternatively a finitely presented functor $F_{D\psi/D\phi} \in \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$, via duality. This correspondence can be made into a categorical equivalence by defining appropriate arrows.

Definition 2.9 ([27], pp.91-92). The category $\mathbb{L}_R^{\text{eq}+}$ is the category with pp-pairs as objects and arrows defined as follows. Given two pp-pairs ϕ/ψ and ϕ'/ψ' , in free variables \bar{x} and \bar{x}' respectively, with these sets disjoint, an arrow $\eta : \phi/\psi \rightarrow \phi'/\psi'$ is determined by a pp-formula $\eta(\bar{x}, \bar{x}')$ which satisfies for any R -module M :

- (i). If $M \models \eta[\bar{m}, \bar{m}']$ and $\bar{m} \in \phi(M)$ then $\bar{m}' \in \phi'(M)$.
- (ii). If $M \models \eta[\bar{m}, \bar{m}']$ and $\bar{m} \in \psi(M)$ then $\bar{m}' \in \psi'(M)$.
- (iii). If $\bar{m} \in \phi(M)$ then there exists an \bar{m}' such that $M \models \eta[\bar{m}, \bar{m}']$.

Equipped with these arrows the category $\mathbb{L}_R^{\text{eq}+}$ is abelian ([27], Proposition 3.2.10).

To make the above more transparent, consider the pp-formulas $\phi(x_1), \psi(x_1)$, and $\phi'(x_2), \psi'(x_2)$ and let $\eta(x_1, x_2)$ be a map $\phi/\psi \rightarrow \phi'/\psi'$ in $\mathbb{L}_R^{\text{eq}+}$. Using the explicit set theoretic definition of a function ([6], Definition 7.7) we have that in any module M there is a function

$$\begin{aligned} \phi(M)/\psi(M) &\xrightarrow{\eta_M} \phi'(M)/\psi'(M) \\ [m + \psi(M)] &\longmapsto [m' + \psi'(M)], \end{aligned}$$

defined by the condition that $M \models \eta[m, m']$. By additivity of pp-formulas $M \models \eta[m'', m + m']$ if $M \models \eta[m''', m]$ and $M \models \eta[m''', m']$ with $m''' + m'''' = m''$, so η_M is in fact an arrow in \mathbf{Ab} . One can check that this defines a natural transformation $\eta : F_{\phi/\psi} \rightarrow F_{\phi'/\psi'}$ in $(\text{mod } R, \mathbf{Ab})$, and vice versa, so in fact ([27], Theorem 10.2.30) $\mathbb{L}_R^{\text{eq}+}$ and $\text{fp}(\text{mod } R, \mathbf{Ab})$ are equivalent categories.

²An element (A, \bar{a}) of a functor $F : \mathcal{C} \rightarrow \mathbf{Ab}$ consists of an object A of \mathcal{C} and a set of elements \bar{a} of FA of fixed finite length. An arrow $(A, \bar{a}) \rightarrow (B, \bar{b})$ between elements of F is a morphism $FA \rightarrow FB$ which carries \bar{a} to \bar{b} .

From the definition of pp-pairs we have that definable subcategories correspond to subcategories of $\text{Mod } R$ which consists precisely of those R -modules on which a fixed set of pp-pairs is closed. Using this it is then possible to define a topology on the set of indecomposable pure-injective R -modules. This is called the Ziegler topology and by convention we write Zg_R for both the underlying set and the topological space on that set. A subset of Zg_R is *closed* or *Ziegler closed* if it is of the form $\mathcal{X} \cap \text{Zg}_R$ for some definable subcategory \mathcal{X} . The basic Ziegler closed sets are sets of the form $[\phi/\psi] = \{N \in \text{Zg}_R : \phi(N) = \psi(N)\}$ for some pp-pair ϕ/ψ . One can check that this defines a topology ([17], Proposition 2.10). We also write (ϕ/ψ) for the complement of $[\phi/\psi]$, and both give, respectively, the base of open and closed sets of the Ziegler topology. This space carries a significant amount of the model theoretic information about $\text{Mod } R$. The open neighborhoods of a point, for example, are described as follows (see the reference for the relevant definitions).

Theorem 2.10 ([27], Theorem 5.1.21). *Let N be an R -module with $N \in \text{Zg}_R$. Then there is an irreducible pp-type p such that $N = H(p)$ and for any such p the sets (ϕ, ψ) with $\phi \in p$ and $\psi \notin p$ form a basis of open neighborhoods of N in Zg_R .*

There are also various obvious bijections between closed subsets of Zg_R and e.g. (i) Serre subcategories of $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ (ii) theories of R -modules closed under products (iii) definable subcategories of $\text{Mod } R$.

Theorem 2.11 ([17], Proposition A.9; [26], Corollary 4.36). *Let M be an R -module and let \mathcal{X} be a definable subcategory of $\text{Mod } R$. Then $M \in \mathcal{X}$ if and only if M is a pure submodule of a product $\{N_i\}_{i \in I}$ with $N_i \in \mathcal{X} \cap \text{Zg}_R$. In particular a definable subcategory is determined by its intersection with Zg_R .*

Chapter 3

Endocategory of a module

3.1 Basic properties

Let M be an R -module. Throughout this chapter we will denote by $\Gamma = \Gamma_M = \text{End}_R(M)^{\text{op}}$ the opposite of the endomorphism ring of M , and it is an easy observation that M can be considered as an object of $\text{Mod } \Gamma$. In this section we develop a number of alternative descriptions of the endocategory of a module (Definition 3.1 and Theorem 3.2 below), which is the technical tool we will study throughout Chapter 3. We consider both the covariant and contravariant constructions of the endocategory.

Definition 3.1 (Endocategory). Consider the additive and exact evaluation functor

$$\begin{aligned} \mathbb{L}_R^{\text{eq}+} &\xrightarrow{E_M^{\text{eq}+}} \text{Mod } \Gamma \\ \phi/\psi &\longmapsto \phi(M)/\psi(M), \end{aligned}$$

where endomorphisms of M act on $\phi(M)/\psi(M)$ in the obvious way. We denote by $\mathcal{S}^{\text{eq}+} = \ker E_M^{\text{eq}+}$ the full subcategory of $\mathbb{L}_R^{\text{eq}+}$ consisting of all objects that become isomorphic to 0 under $E_M^{\text{eq}+}$. Equivalently this is the full subcategory consisting of all the pp-pairs closed on M . Let $\text{M}^{\text{eq}+}$ denote the quotient $\mathbb{L}_R^{\text{eq}+}/\mathcal{S}_M$ by the Serre subcategory \mathcal{S}_M and consider the functor

$$\begin{aligned} \text{M}^{\text{eq}+} &\longrightarrow \text{Mod } \Gamma \\ q(\phi/\psi) &\longmapsto \phi(M)/\psi(M), \end{aligned}$$

which is induced by the universal property of the quotient functor $q : \mathbb{L}_R^{\text{eq}^+} \rightarrow \mathbb{M}^{\text{eq}^+}$ (for localisation see Section 1.2, in particular the paragraph after Definition 1.4). Denote by \mathcal{E}_M the image of this functor in $\text{Mod } \Gamma$ and call this category the *endocategory* of M . This is an abelian category, in fact the evaluation functor induces an equivalence between \mathbb{M}^{eq^+} and \mathcal{E}_M .

The above is the natural and covariant model theoretic construction. Alternatively one can use the equivalent dual construction used by Krause [14], for this consider the additive and exact evaluation functor

$$\begin{aligned} \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab}) &\xrightarrow{E_M} (\text{Mod } \Gamma)^{\text{op}} \\ F &\longmapsto dF(M) = (F, M \otimes -), \end{aligned}$$

where Γ acts on $dF(M)$ in the obvious manner, via the equivalence between endomorphisms of M in $\text{Mod } R$ and endomorphisms of $M \otimes -$ in $(\text{mod } R^{\text{op}}, \mathbf{Ab})$. Then the endocategory \mathcal{E}_M is the image of the functor $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\ker E_M \rightarrow (\text{Mod } \Gamma)^{\text{op}}$, induced by the universal property of the quotient functor; with an equivalence between $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\ker E_M$ and $(\mathcal{E}_M)^{\text{op}}$ induced by evaluation.

Theorem 3.2 ([17], Lemma 6.1; [14], Proposition 2.3). *The endocategory \mathcal{E}_M has the following properties.*

- (i). $M \in \mathcal{E}_M$ is an object of $\text{Mod } \Gamma$.
- (ii). $N \hookrightarrow M$ is a subobject in \mathcal{E}_M if and only if N is isomorphic to a pp-definable subgroup of M .
- (iii). An object L of $\text{Mod } \Gamma$ is in \mathcal{E}_M if and only if there exists an exact sequence

$$(Y, M) \xrightarrow{(f_*)_M} (X, M) \longrightarrow L \longrightarrow 0,$$

with $f : X \rightarrow Y$ a morphism in $\text{mod } R$ i.e. if it is the cokernel of the component of f_* at M .

- (iv). Every object in \mathcal{E}_M is a subquotient of M^n .

Proof. For (i) it is clear that the evaluation of the pp-pair $(x_1 = x_1)(M)/(x_1 = 0)(M)$ will be isomorphic to M . Further any monomorphism $\eta : \phi/\psi \rightarrow (x = x)/(x = 0)$ in $M^{\text{eq}+}$, when evaluated at M , will produce the monomorphism $\eta_M : \phi(M)/\psi(M) \rightarrow M$. Then η_M identifies with the inclusion of $\phi(M)$ into M , hence $\psi(M) = 0$, and this shows (ii). The parts (iii) and (iv) follow by using that $\mathbb{L}_R^{\text{eq}+}$ is equivalent to the category $\text{fp}(\text{mod } R, \mathbf{Ab})$. \square

3.2 Chain conditions

An object X , in an abelian category \mathcal{C} , is said to be *noetherian* (resp. *artinian*) if its modular lattice of subobjects has the ascending chain condition (resp. descending chain condition). We will say that the category \mathcal{C} is noetherian (resp. artinian) if every object in \mathcal{C} is noetherian (resp. artinian). Further if \mathcal{D} is a locally finitely presented Grothendieck category then we will say that \mathcal{D} is *locally noetherian* (resp. *locally artinian*) if there is a set of noetherian (resp. artinian) generators ([32], V.4).

We continue with the assumptions and notation of the previous section. Additionally we denote by \mathcal{T}_M the full subcategory of $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ consisting of all the filtered colimits of $\mathcal{S}_M = \ker E_M \subseteq \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ in $(\text{mod } R^{\text{op}}, \mathbf{Ab})$, and recall that by Section 1.1 the subcategory \mathcal{T}_M is localising of finite type.

Proposition 3.3. *If \mathcal{E}_M is noetherian (resp. artinian) then the category $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ is locally artinian (resp. locally noetherian).*

Proof. Consider the following equivalences

$$\text{fp}((\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M) \simeq \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{S}_M \simeq (\mathcal{E}_M)^{\text{op}}$$

and note the contravariance of the final equivalence. Here fp denotes the finitely presented objects of a category. \square

We will be interested in the case when \mathcal{E}_M is artinian. In this setting the above implies that $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ will be locally noetherian. Properties of locally noetherian categories are well known (see e.g. [30]) and we begin with three technical Propositions.

Proposition 3.4 (Baer’s Criterion; [17], Lemma A.10). *Let F be an object in a locally finitely presented Grothendieck category \mathcal{C} . Then F is injective if and only if $\text{Ext}^1(G, F) = 0$ for every finitely generated object $G \in \mathcal{C}$.*

Proof. For the non-trivial direction first recall that an object is injective if and only if every essential extension is an isomorphism ([32], Proposition 2.4). If F is not injective then we can take the injective envelope $E(F)$ and obtain a non-zero cokernel

$$F \xrightarrow{f} E(F) \xrightarrow{p} \text{coker } f = E(F)/\text{im } f;$$

we then use that \mathcal{C} is locally finitely presented to contradict this possibility. Thence let $\text{coker } f = \varinjlim_{i \in I} C_i$, for some $C_i \in \text{fp}(\mathcal{C})$ and I a filtered category. For any C_i consider the canonical morphism $g_i : C_i \rightarrow \text{coker } f$ and note that $\text{im } g_i = C_i/\ker g_i$ is still a finitely generated functor (though may no longer be finitely presented). This is a subobject of $\text{coker } f$, hence we can form a pullback:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{f} & E(F) & \xrightarrow{p} & \text{coker } f & \longrightarrow & 0 \\ & & \uparrow 1_F & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F & \longrightarrow & P & \longrightarrow & \text{im } g_i & \longrightarrow & 0 \end{array}$$

In the above diagram the bottom row is exact ([32], Proposition 5.3) and all the columns are monic. By assumption $\text{Ext}^1(\text{im } g_i, F) = 0$, so the bottom sequence splits, being equivalent to the trivial extension. In particular we obtain that $P = F \oplus \text{im } g_i$, hence f factors as a composition of two monics, both of which must then be essential ([32], Lemma 2.1), hence $E(\text{im } g_i) = 0$ and so $\text{im } g_i = 0$. In particular $\varinjlim_{i \in I} C_i = 0$. This is a contradiction and hence F is injective. \square

Proposition 3.5 ([17], Proposition A.1). *If \mathcal{C} is a locally noetherian Grothendieck category then every fp-injective object is injective. Furthermore every filtered colimit of injective objects is injective.*

Proof. In this case every finitely generated object is a quotient of a noetherian object, hence is noetherian, hence by Proposition 3.4 every fp-injective object is injective.

The second part follows by considering that if $\varinjlim_{i \in I} M_i$ is a filtered colimit of fp-injective objects then for any $X \in \text{fp } \mathcal{C}$ we will have

$$(X, \varinjlim_{i \in I} M_i) = \varinjlim_{i \in I} (X, M_i).$$

As \mathbf{Ab} is a Grothendieck category, the filtered colimit will be right exact. This implies the result. \square

Proposition 3.6 ([32], Proposition V.4.5; [17], Proposition A.1). *If \mathcal{C} is a locally noetherian Grothendieck category then every injective object is a coproduct of indecomposable injective objects (and these have local endomorphism rings).*

Proof. The case $M = 0$ is trivial, so assume that $M \neq 0$ is an injective object in \mathcal{C} . Fix any non-zero finitely generated subobject $X \hookrightarrow M$ and then consider the set Λ of all injective subobjects of M which do not contain X . This is non-empty. Any chain in Λ has an upper bound by Proposition 3.5 so there is a maximal element A in Λ . Thence $M = A \oplus A'$. The object A' must be indecomposable, for otherwise we can adjoin one of its non-zero direct summands to A and contradict the maximality of A . Hence M has an indecomposable direct summand.

Now let Γ be the set of all families of independent indecomposable direct summands of M . By independent we mean that the intersection of any one of them with the sum of the others is zero so that their sum in M is just their coproduct.

This set is non-empty, as remarked above, and clearly any chain has an upper bound. This implies that there is a maximal object in Γ . This object is a family $\{M_i\}_{i \in I}$ of indecomposable direct summands of M ; the coproduct of this family $\coprod_{i \in I} M_i$ must then be M . \square

3.3 Duality

Recall, from Section 1.1, that there is duality between $\text{fp}(\text{mod } R, \mathbf{Ab})$ and $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$.

This duality takes the form of a functor denoted by d . We now consider the effect of this duality on the Ziegler spectrum.

The duality d induces a bijection between the closed subsets of Zg_R and $Zg_{R^{op}}$ as follows ([14], Proposition 4.1). To begin with let \mathbf{U} be a closed set of Zg_R and let $\mathcal{S}_{\mathbf{U}}$ be the Serre subcategory of $\text{fp}(\text{mod } R^{op}, \mathbf{Ab})$ corresponding to \mathbf{U} . Then define $\mathcal{S}_{D\mathbf{U}}$ to be the Serre subcategory of $\text{fp}(\text{mod } R, \mathbf{Ab})$ consisting of all the duals of objects in $\mathcal{S}_{\mathbf{U}}$ and let $D\mathbf{U}$ be the closed subset of $Zg_{R^{op}}$ corresponding to $\mathcal{S}_{D\mathbf{U}}$. It is clear that D is an bijection between the closed subsets, but note that this is not a bijection between points of Zg_R , e.g. as Zg_R is in general not separated.

To further describe the above duality, let us recall some definitions from general topology:

- A point $M \in X \subseteq Zg_R$, where X is closed, is said to be *isolated in X* or *X -isolated* if the subset $\{M\}$ is open in the induced topology on X i.e. if there is a pp-pair such that $(\phi/\psi) \cap X = \{M\}$.
- A point $G \in X \subseteq Zg_R$, where X is again closed, is said to be a *generic point of X* if its closure $\overline{\{G\}}$ is all of X ¹.

The complexity of the information in a closed set U of Zg_R is often measured using a Cantor-Bendixson analysis of U , which is defined by using isolated points. We start at 0 with U and at each successor ordinal remove all isolated points (as defined above) and endow the remaining set with the induced topology, taking the intersection at a limit ordinal (see [27], Section 5.3.6). If there is an ordinal α at which the process arrives at the empty set, there need not be one in every case, then we note the minimum such α . The whole space Zg_R is compact² and hence, if defined, this α will be a successor ordinal. To get the *Cantor-Bendixson (CB) rank* we subtract 1 from α (which is defined since α a successor ordinal).

We will be concerned with the relation between isolated points and duality. To begin with we need a definition.

¹One may remark that isolated points find importance in model theory (e.g. [21], pp.141-142) and generic points are important in algebraic geometry.

²Note that Zg_R is *not* in general Hausdorff (see e.g. [27], Theorem 8.2.89) hence sometimes it is said to be *quasi-compact* (e.g. [17]).

Definition 3.7 ([14], p.426). An R -module $M \in \mathbf{Zg}_R$ is called *reflexive* provided that the following conditions are satisfied:

- (i). M is isolated in its closure $\text{supp } M = \overline{\{M\}}$.
- (ii). There is an isolated point DM in $D\overline{\{M\}}$, which is then uniquely determined by M , called the *reflection of M* .

We will write $\text{Rf } R$ for the set of reflexive points in \mathbf{Zg}_R and endow this set with the induced topology. Equivalently a reflexive point M is the generic point of $X = \overline{\{M\}}$ such that DM is the generic point of DX .

We can now state the key property of $\text{Rf } R$.

Proposition 3.8 ([14], Corollary 4.7). *Reflection is an idempotent homeomorphism between $\text{Rf } R$ and $\text{Rf } R^{\text{op}}$ which induces, for every $M \in \text{Rf } R$, an equivalence*

$$\begin{aligned} \mathcal{E}_M &\xrightarrow{d_M} \mathcal{E}_{DM}^{\text{op}} \\ (X, M) &\longmapsto X \otimes DM. \end{aligned}$$

Furthermore we have the following. Since every object of \mathcal{E}_M is (Theorem 3.2), up to isomorphism, a cokernel of $(Y, M) \xrightarrow{(f_*)_M} (X, M)$ for some map $f : X \rightarrow Y$ in $\text{mod } R$, then d_M maps this to the kernel of $X \otimes DM \xrightarrow{f \otimes 1_{DM}} Y \otimes DM$. This is clearly an object of \mathcal{E}_{DM} .

Proof. By definition \mathcal{E}_M and $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{S}_M$ are equivalent. Hence the result follows by considering the equivalence $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{S}_M \rightarrow \text{fp}(\text{mod } R, \mathbf{Ab})^{\text{op}}/\mathcal{S}_M$ induced by d . □

Let X be a closed subset of the Ziegler spectrum \mathbf{Zg}_R and let N be a point of X . Then the point N is said to be *X -neg-isolated* if, in the category $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{J}_{DX}$, the functor $N \otimes -$ is the injective envelope of a simple functor. Here the localising subcategory \mathcal{J}_{DX} corresponds to the torsion part of the hereditary torsion theory of finite type corresponding to the definable subcategory of the closed set $DX \subset \mathbf{Zg}_{R^{\text{op}}}$ (see Theorem 1.6).

Theorem 3.9 ([27], Theorem 5.3.1). *Let X be a closed subset of the Ziegler spectrum Zg_R and N a point of X . Assume that N is isolated in X . Then the functor $N \otimes -$, considered in the category $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_{DX}$, is the injective envelope of a simple functor.*

Proof. Let us sketch this. Choose a pp-pair ϕ/ψ such that $(\phi/\psi) \cap X = \{N\}$. That is we choose the pp-pair such that $(\phi/\psi)(N) = (F_{D\psi/D\phi}, N \otimes -)$ is non-zero only when evaluated at N in X . Choose a non-zero f in $(F_{D\psi/D\phi}, N \otimes -)$. Now find a subfunctor $G \hookrightarrow F_{D\psi/D\phi}$ maximal with respect to containing $\ker f$ but not identifying with $F_{D\psi/D\phi}$ in the localised category $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_{DX}$; this exists since $F_{D\psi/D\phi}$ is finitely generated³. Then $F_{D\psi/D\phi}/G \in \mathcal{F}_{DX}$ and picking any larger G' will break this. Thus we obtain a simple object in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_{DX}$, it is then not difficult to see that $N \otimes -$ is its injective envelope. \square

Thus an X -isolated point is also X -neg-isolated. We can now mention the relationship with reflexive points of the Ziegler spectrum, though we need some more terminology. For any R -module M the pp-pair ϕ/ψ is said to be M -minimal if $\phi(M) > \psi(M)$ and there are no pp-definable subgroups in the interval (in the lattice of pp-definable subgroups of M).

Definition 3.10. Let X be a closed subset of Zg_R . A pp-pair ϕ/ψ is said to be an X -minimal pair if for some M , with $\text{supp } M = X$, the pp-pair ϕ/ψ is M -minimal. Also we will say that $F \in \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ is an X -minimal functor if its localised image in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_{DX}$ is simple.

By definition, if $F = F_{D\psi/D\phi}$ is X -minimal, then ϕ/ψ will be an X -minimal pair, and vice versa.

Theorem 3.11 ([27], Theorem 5.4.12). *Let N be a point of the Ziegler spectrum Zg_R contained in a closed subset X . If N is isolated by an X -minimal pair ϕ/ψ , then N is a reflexive point of the Ziegler spectrum Zg_R . Furthermore, the reflection DN is isolated in DX by the DX -minimal pair $D\psi/D\phi$.*

³This is true just as for modules, see e.g. [32] Lemma I.6.8.

Proof. First note that a module $N \in X \subseteq \mathbf{Zg}_R$ is isolated by an X -minimal pair ϕ/ψ if and only if $\{N\} \cap X = (\phi/\psi)$. This holds if and only if $N \otimes -$ is the injective envelope of $F_{D\phi/D\psi}$ whose localisation at \mathcal{T}_{DX} is simple ([27], Theorem 5.3.2).

From the fact that D is a lattice duality it is clear that $D\psi/D\phi$ is a DX -minimal pair. Hence there is a unique point which is DX -isolated by $D\psi/D\phi$, say DN . This is then the generic point of $D\text{supp } N = D\overline{\{N\}}$ by the bijection between closed sets of \mathbf{Zg}_R and $\mathbf{Zg}_{R^{\text{op}}}$. \square

We can construct points isolated by minimal pairs from the endocategory using the following result. This result and its corollary will be of use in the next chapter. One should note that the corollary provides the link between properties of reflexive points of the Ziegler spectrum and properties of the endocategory.

Lemma 3.12 ([27], Proposition 5.3.10). *Let M be a point of the Ziegler spectrum \mathbf{Zg}_R , that is let M be an indecomposable pure injective. Then the pp-pair ϕ/ψ is M -minimal if and only if $(F_{D\psi/D\phi}, M \otimes -)$ is a simple object in \mathcal{E}_M if and only if it is simple in $\text{Mod } \Gamma$.*

Proof. As before we write Γ for $\text{End}_R(M)^{\text{op}}$. If $(F_{D\psi/D\phi}, M \otimes -) = \phi(M)/\psi(M)$ is simple, in either \mathcal{E}_M or $\text{Mod } \Gamma$, then any pp-formula ψ' with $\psi < \psi' < \phi$ in the lattice of pp-formulas would correspond to a proper Γ -submodule $\phi(M)/\psi'(M)$ of $\phi(M)/\psi(M)$. This is a contradiction and hence ϕ/ψ is M -minimal.

On the other hand if ϕ/ψ is M -minimal, then the functor $F_{D\psi/D\phi}$ is simple in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$. As $((F_{D\psi/D\phi})_{\mathcal{T}_M}, M \otimes -) = (F_{D\psi/D\phi}, M \otimes -) \neq 0$ then, by assumption on the pp-pair ϕ/ψ being open on M , there is then a non-zero morphism $f : (F_{D\psi/D\phi})_{\mathcal{T}_M} \rightarrow M \otimes -$. This f must be a monomorphism which identifies $(F_{D\psi/D\phi})_{\mathcal{T}_M}$ with $\text{soc}(M \otimes -)$, for $M \otimes -$ is the injective envelope of $(F_{D\psi/D\phi})_{\mathcal{T}_M}$ and the socle is the intersection of the essential subobjects of $M \otimes -$. Using this it is not difficult to show that $((F_{D\psi/D\phi})_{\mathcal{T}_M}, M \otimes -)$ is cyclic over $\Gamma/\text{rad } \Gamma$ and hence simple. To see this last part note that $M \otimes -$ has a simple essential socle in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ and that the endomorphism ring of $M \otimes -$ is isomorphic to $\Gamma = \text{End}_R(M)^{\text{op}}$. Hence, by the above, any morphism $(F_{D\psi/D\phi})_{\mathcal{T}_M} \rightarrow M \otimes -$ will correspond to a morphism

onto $\text{soc}(M \otimes -)$ followed by some endomorphism of $\text{soc}(M \otimes -)$. Hence as a $\Gamma/\text{rad}\Gamma$ -module $((F_{D\psi/D\phi})_{\mathcal{T}_M}, M \otimes -)$ will be isomorphic to $f(\Gamma/\text{rad}\Gamma)$. \square

Corollary 3.13. *Let M be an indecomposable pure-injective R -module and suppose that \mathcal{E}_M is an artinian category, then M is reflexive.*

Proof. As \mathcal{E}_M is artinian it contains a simple object. By the above Lemma 3.12 there is an M -minimal pair which isolates M in $\text{supp } M$, hence M is reflexive by Theorem 3.11. \square

Chapter 4

Σ -pure injective modules

4.1 Basic properties

An R -module M is said to be Σ -pure injective if the coproduct $\coprod_{i \in I} M$ is a pure injective R -module for every set I . It is equivalent to define a Σ -pure injective R -module as one in which the set of pp-definable subgroups (or subgroups of finite definition, this terminology is equivalent) of M has the descending chain condition. We begin by proving this equivalence. Then we will consider further model theoretic properties of Σ -pure injective modules, these will then be applied (in Section 4.2) to extract information on Σ -pure injective points of the Ziegler spectrum.

Theorem 4.1 ([27], Theorem 4.4.5; [26] Theorem 2.11; [13], Theorem 8.1). *An R -module M is Σ -pure injective if and only if it has the descending chain condition on pp-definable subgroups.*

Proof. If M has dcc on pp-definable subgroups then so does $\coprod_{i \in I} M$ for any I , because any pp-formula ϕ satisfies $\phi(\coprod_{i \in I} M) = \coprod_{i \in I} \phi(M)$. The dcc clearly extends to cosets. Thus $\coprod_{i \in I} M$ is algebraically compact as it will realise every pp-1-type with parameters from M hence, by Theorem 2.6, it will be pure-injective. Thus M will be a Σ -pure injective R -module.

On the other hand assume that M is Σ -pure injective and fix a descending chain of

pp-definable subgroups

$$\phi_0(M) \supsetneq \phi_1(M) \supsetneq \phi_2(M) \supsetneq \dots,$$

for some pp-formulas $\{\phi_i(x_1)\}_{i < \aleph_0}$. Fix an infinite cardinal κ large enough such that $\kappa \geq |M|$ and $\kappa^{\aleph_0} > \kappa$ are satisfied. Since for any cardinal κ we have $\kappa < \kappa^{\text{cf } \kappa}$, where $\text{cf } \kappa$ denotes the cofinality, in our case we may as well just assume that $\text{cf } \kappa > \aleph_0$. This means that in the minimal case we need more than \aleph_0 subsets of cardinality less than κ in order that their union be κ . Now form a descending chain

$$\phi_0\left(\coprod_{\kappa} M\right) \supsetneq \phi_1\left(\coprod_{\kappa} M\right) \supsetneq \phi_2\left(\coprod_{\kappa} M\right) \supsetneq \dots,$$

and proceed to choose, for each $0 < i < \kappa$ and $0 < j < \aleph_0$, an element

$$a_{i,j} \in \phi_{j-1}\left(\coprod_{\kappa} M\right) \setminus \phi_j\left(\coprod_{\kappa} M\right),$$

such that $a_{i,j} = a_{i',j'}$ if and only if $(i,j) = (i',j')$. We can do this because of the size of the coproduct. For technical reasons we also set $a_{0,j} = 0$ for all j and $a_{0,1} = 0$.

For any natural number $0 < n < \aleph_0$ consider the function $\eta : \{1, \dots, n\} \rightarrow \kappa$ (i.e. $\eta \in \kappa^{\aleph_0}$). Then define $a_\eta = \sum \{a_{i,\eta(i)}\}_{i \in \text{dom } \eta}$. What we have constructed is a tree, defined by starting with $\phi_0(\coprod_{\kappa} M)$, then branching out into κ -many branches $\{a_{i,1} + \phi_1(\coprod_{\kappa} M)\}_{i < \kappa}$ at the first level. Then at each node, for example if we fix the i , we form another κ -branching $\{a_{i,0} + a_{i',1} + \phi_2(\coprod_{\kappa} M)\}_{i' < \kappa}$. We do this all the way up to \aleph_0 . This gives a κ -branching tree of depth \aleph_0 , in which there are κ^{\aleph_0} branches, each corresponding to a nested chain of cosets. But each branch (that is, an infinite path going up the tree) clearly corresponds to a pp-type with parameters from $\coprod_{\kappa} M$. As $\coprod_{\kappa} M$ is pure injective, hence algebraically compact, we know that each branch will have non-empty intersection. This intersection is precisely the set of elements of $\coprod_{\kappa} M$ realising the pp-type, and as each branch is disjoint from another (that is they are mutually contradictory as types) we will certainly have $|\coprod_{\kappa} M| \geq \kappa^{\aleph_0}$. This is a contradiction. For we have $|\coprod_{\kappa} M| \leq \kappa$ and κ is infinite. Hence we see that there is no properly descending infinite chain of pp-definable subgroups for M . \square

A finite sum of Σ -pure injective modules must be Σ -pure injective, we see this by projecting chains of pp-definable subgroups (or subgroups of finite definition) onto

the factors of a finite sum. Either this observation, or a modification of the proof above, in which we use pp- n -types instead of pp-1-types, shows that we may infer the descending chain condition on subgroups of finite definition for M^n .

A further result can be obtained along similar lines.

Lemma 4.2 ([27], Theorem 4.4.8). *An R -module M is Σ -pure injective if and only if $\coprod_{\aleph_0} M$ is pure injective.*

Proof. One direction is evident. So assume that $\coprod_{\aleph_0} M$ is pure injective and choose a descending chain of pp-definable subgroups for M

$$\phi_0(M) \supsetneq \phi_1(M) \supsetneq \phi_2(M) \supsetneq \dots,$$

for some pp-formulas $\{\phi_i(x_1)\}_{i < \aleph_0}$. For each $0 < i < \aleph_0$ and $j = 1$ choose

$$a_{i,1} \in \phi_{i-1}\left(\coprod_{\aleph_0} M\right) \setminus \phi_i\left(\coprod_{\aleph_0} M\right),$$

such that $a_{i,1} = a_{i',1}$ if and only if $(i, 1) = (i', 1)$. Also set $a_{i,0} = 0$ for each i and set $a_{0,1} = 0$. Note that we can do this because of the property of the coproduct (in particular because it contains \aleph_0 copies of M). As in the above proof form a 2-branching tree of depth \aleph_0 . Then note that each branch gives a pp-1-type with parameters from $\coprod_{\aleph_0} M$, so is realised in $\coprod_{\aleph_0} M$ by assumption.

Then fix an infinite branch, say

$$\phi_0\left(\coprod_{\aleph_0} M\right) \supseteq x_1 + \phi_1\left(\coprod_{\aleph_0} M\right) \supseteq x_2 + \phi_2\left(\coprod_{\aleph_0} M\right) \supseteq \dots,$$

where x_n represents the value at the branching. Pick some x in the intersection, i.e. one that realises the pp-type in $\coprod_{\aleph_0} M$ and fix some $x_{n'}$ with $n' > n$ such that the element added at the n' stage is not zero. Then x realises the pp-formula defining the coset $x_{n'} + \phi_{n'}(\coprod_{\aleph_0} M)$, so in particular $x - x_{n'} \in \phi_{n'}(\coprod_{\aleph_0} M)$. But projecting to the element of $\coprod_{\aleph_0} M$, which corresponds to what is the (non-zero) added element in $x_{n'}$ at the n' stage, we obtain that this element is contained in $\phi_{n'}(\coprod_{\aleph_0} M)$. A contradiction to the way it was chosen. \square

It is also true that if a pure injective R -module M has cardinality $< 2^{\aleph_0}$, then it must be Σ -pure injective. For otherwise we can form a descending chain of subgroups of finite definition for $\coprod_{\aleph_0} M$, hence realise 2^{\aleph_0} distinct pp-types in $\coprod_{\aleph_0} M$, and obtain a contradiction by cardinal arithmetic. For $|\coprod_{\aleph_0} M| < \aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0}$. Yet another slight modification permits one to prove that, if R is a k -algebra over a field k , then any pure injective R -module of dimension $< 2^{\aleph_0}$ (over k) must be Σ -pure injective ([27], Corollary 4.4.10).

Example 4.3 ([26], Section 2.Z; [25], Section 2.4). Let $R = k[X]$. For background and the definition of \mathbf{m} and \mathbf{p} , which we need here, the reader is referred to Example 1.3 and Example 1.8. For this ring there are a number of Σ -pure injective modules in \mathbf{Zg}_R . Obviously each λ -finite point $P_{\lambda,n}$ is Σ -pure injective of CB-rank 0. For a fixed $\lambda \in k$ the Ziegler closure of $\{P_{\lambda,n}\}_{n < \aleph_0}$ contains three pure-injective modules $\{P_{\lambda,\infty}, \overline{P_{(\lambda)}}$, $k(X)\}$ none of which are finitely presented. Firstly the λ -Prüfer module $P_{\lambda,\infty}$ is a pure injective module obtained by taking the filtered colimit along \mathbf{m} in the tube of finitely presented modules

$$\dots \longrightarrow P_{\lambda,n} \xrightarrow{\mathbf{m}} P_{\lambda,n+1} \longrightarrow \dots,$$

note that this module evidently has a countable basis, hence is Σ -pure injective. It is in fact easy to see that it has dcc but not acc on subgroups of finite definition. The λ -adic module $\overline{P_{(\lambda)}}$ obtained by taking the cofiltered limit along \mathbf{p} is however not Σ -pure injective, one can easily obtain a properly descending chain of subgroups of finite definition. Note also that both of these points have CB-rank 1. The third point $k(X)$ is pure-injective and has a countable basis over k and is hence Σ -pure injective, in fact it clearly has both acc and dcc on subgroups of finite definition. This last point has CB-rank 2.

There is a key property of Σ -pure injective modules which underlies the proof of Theorem 4.1 and this property has its roots in model theory. The model theoretic statement of this property is as follows: the complete theory $\text{Th}(M)$ of a Σ -pure injective module M is totally transcendental. Recall that the complete theory is the

set of all \mathcal{L}_R -sentences satisfied by M (see Section 2.1), while the definition of totally transcendental is more technical. A complete theory $T = \text{Th}(M)$ of an R -module M is said to be *totally transcendental* if, considered as a reduct to any language of cardinality at most \aleph_0 (so we reduce R to some countable subring) it is \aleph_0 -stable. This latter condition \aleph_0 -stability itself means that the number of (pp-)1-types over any subset $|A| < \aleph_0$ of any model is the minimum possible i.e. it is less than or equal to \aleph_0 . The set of all 1-types over A is usually denoted by $S_1^+(A)$. We will not concern ourselves with the technicalities required to give a complete treatment of these definitions, the reader is referred to [26] (Section 3.1) for a thorough development.

Another curious fact from model theory is that the size of the theory has certain consequences. For example it is an easy consequence ([26], Theorem 3.12) of the downward Lowenheim-Skolem Theorem that if a theory of modules T is of size $< 2^{\aleph_0}$ and all models of T are pure-injective then T is necessarily totally transcendental (so every model of T is then Σ -pure injective).

We end this section by considering cyclic $\text{End}_R(M)^{\text{op}}$ -subgroups of a Σ -pure injective module M .

Lemma 4.4 ([27], Corollary 4.4.7; [26], Theorem 3.1). *Let M be a Σ -pure injective module. Then for $\bar{m} = (m_1, \dots, m_n) \in M^n$ and $p = \text{pp}^M(\bar{m})$ there is a pp-formula ϕ such that*

$$\phi(M) = p(M) = \text{End}_R(M)\bar{m}.$$

Proof. By the discussion which directly followed Theorem 4.1 we can infer that M^n has dcc on pp-definable subgroups (that is on subgroups of finite definition). For the first equality, if $p(M) \neq \phi(M)$ for all $\phi \in p$, then we can construct a descending chain

$$\phi_0(M) \supsetneq \phi_0(M) \cap \phi_1(M) \supsetneq \phi_0(M) \cap \phi_1(M) \cap \phi_2(M) \supsetneq \dots,$$

choosing, for $\phi_i(M) \supsetneq p(M)$, a formula ϕ_{i+1} in p incomparable to ϕ_i in the lattice of pp-definable subgroups of M^n . This chain does not stop by the assumption that $p(M) \neq \phi(M)$ for any ϕ . But this would contradict the assumption that M is Σ -pure injective, so we infer that $p(M) = \phi(M)$ for some pp-formula $\phi \in p$.

For the second equality, if $m' \in p(M)$ then $p_1 = \text{pp}^M(\overline{m'}) \supseteq \text{pp}^M(\overline{m}) = p_2$, hence there is a corresponding morphism

$$f' : (R^n \otimes -)/F_{Dp_1} \rightarrow (R^n \otimes -)/F_{Dp_2},$$

which is just the morphism from $\text{im}(\overline{m'} \otimes -)$ to $\text{im}(\overline{m} \otimes -)$. Thus we obtain a diagram

$$\begin{array}{ccc} \text{im}(\overline{m'} \otimes -) & \hookrightarrow & M \otimes - \\ \downarrow f' & & \downarrow f \\ \text{im}(\overline{m} \otimes -) & \hookrightarrow & M \otimes - \end{array}$$

from which, by injectivity of $M \otimes -$, we obtain an arrow f . By the fullness of the embedding this morphism corresponds to some $f : M \rightarrow M$. This clearly takes (each component of) the tuple $\overline{m'}$ to \overline{m} . Thus we obtain the second equality, which combined with the first equality gives the result. \square

4.2 Definable subcategory generated by a Σ -pure injective module

In this section we consider how a Σ -pure injective module decomposes as a coproduct of indecomposables. The result which we will use in later Sections of this Chapter is Corollary 4.8. After this crucial Corollary we consider the related question of how products of Σ -pure injective modules decompose, and consider a result from the model theory of Σ -pure injective modules. This is contained in Theorem 4.12. For the convenience of the reader we note that this latter result is not crucial and uses a considerable amount of model theory, thus it may be skipped on first reading.

There are a number of straightforward consequences to Theorem 4.1, in particular when we use the technical results of Section 3.2 and the result of Corollary 3.13, we obtain the following two Corollaries.

Corollary 4.5 ([14], Corollary 2.8; [10], Lemma 1). *An R -module M is Σ -pure injective if and only if \mathcal{E}_M is artinian if and only if $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ is locally noetherian.*

Corollary 4.6. *An indecomposable Σ -pure injective module is a reflexive point of the Ziegler spectrum.*

We now study the definable subcategory generated by a Σ -pure injective module. Recall that a definable subcategory (see Theorem 1.6) is generated by an R -module M if it corresponds to the closure of $\{M\}$ under isomorphisms, arbitrary products, filtered colimits and pure submodules. We denote this definable subcategory by \mathcal{X}_M and note that it is the smallest definable subcategory containing M . We also note that the definable subcategory generated by M is determined by the Serre subcategory of $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$ which consists of the duals of the pp-pairs closed on M .

Proposition 4.7 ([27] Proposition 4.4.12). *Let M be a Σ -pure injective R -module, then the definable subcategory \mathcal{X}_M which is generated by M consists of Σ -pure injective modules.*

Proof. Note that if every module in \mathcal{X}_M is pure injective, then as \mathcal{X}_M is closed under filtered colimits and hence coproducts (or the use of [26], Corollary 2.24) implies that every module in \mathcal{X}_M will be Σ -pure injective.

Let $M' \in \mathcal{X}_M$ be arbitrary and denote by \mathcal{S}_M the full subcategory of the duals of the pp-pairs closed on M , that is $\{F \in \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab}) : dF(M) = (F, M \otimes -) = 0\}$. Clearly if $dG(M') \neq 0$ for some $G \in \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})$, then $dG(M) \neq 0$, so a pp-pair open on M' must also be open on M . Now consider a properly descending chain of subobjects of M' in $\mathcal{E}_{M'}$ of the form

$$\phi_0(M') \supsetneq \phi_1(M') \supsetneq \phi_2(M') \supsetneq \dots,$$

for some pp-formulas $\{\phi_i\}_{i < \aleph_0}$. The localised pp-pair $\phi_{i-1}/\phi_i \wedge \phi_{i-1}$ is open on M' , hence it is open on M , and this contradicts that \mathcal{E}_M is artinian. This in turn contradicts that M is Σ -pure injective. \square

Corollary 4.8 ([27], Theorem 4.4.19; [10], Lemma 3). *A Σ -pure injective R -module M has a decomposition $M = \coprod_{i \in I} M_i$, for some indecomposable Σ -pure injective modules M_i with local endomorphism rings.*

Proof. Combine the above result with Corollary 4.5 and Proposition 3.6. \square

We now consider another form of decomposition result for Σ -pure injective modules, which will be one further way of showing that a module is Σ -pure injective. In its original application it was used to show that the theory of a Σ -pure injective module is totally transcendental and is the result of Garavaglia in [10] (Lemma 4). There are a number of proofs of this, including algebraic proofs which involve various tools such as p-functors (see [12], Theorem 10), we will follow the proof of [10] after a few preliminaries.

To begin with we define a certain class of modules called superstable modules. We then show that superstable modules occurs in the definable subcategory generated by a Σ -pure injective module.

Definition 4.9. An R -module M is called *superstable* if for any descending chain of pp-definable subgroups (or subgroups of finite definition) of M , say

$$\phi_0(M) \supseteq \phi_1(M) \supseteq \phi_2(M) \supseteq \dots,$$

there exists an $n < \aleph_0$, such that the index of $\phi_{m+1}(M)$ in $\phi_m(M)$ (or in the notation of invariants $\text{Inv}(M, \phi_{m+1}, \phi_m)$) is finite for every $m \geq n$.

Remark 4.10. In model theory the above is usually defined in terms of types and stability (see [26], Theorem 3.1). An R -module M is then said to be superstable if there exists a cardinal κ_0 such that its complete theory $\text{Th}(M)$ is κ -stable for all $\kappa \geq \kappa_0$. While not all modules are superstable, there is at least one cardinal η such that it is η -stable. For more information on this connection with model theory see [26].

In the following Lemma we keep the notation of this Section. Thus \mathcal{X}_M denotes the definable subcategory generated by the R -module M .

Lemma 4.11 ([33] Corollary 2.2; [26], Corollary 3.3). *Every module in the definable subcategory \mathcal{X}_M is superstable if and only if M is Σ -pure injective.*

Proof. If M is a Σ -pure injective then every module in \mathcal{X}_M is Σ -pure injective, hence totally transcendental, so stable for all $\kappa \geq |\text{Th}(M)|$, hence superstable. On the other hand, if every module in \mathcal{X}_M is superstable, fix a descending chain of pp-definable subgroups of M

$$\phi_0(M) \supseteq \phi_1(M) \supseteq \phi_2(M) \supseteq \dots,$$

and note that for any set I , with e.g. $|I| = \alpha$, the descending sequence

$$\phi_0\left(\prod_{\alpha} M\right) \supseteq \phi_1\left(\prod_{\alpha} M\right) \supseteq \phi_2\left(\prod_{\alpha} M\right) \supseteq \dots$$

will have finite indices for all $n \geq m$. This means that

$$\left| \frac{\phi_n\left(\prod_{\alpha} M\right)}{\phi_{n+1}\left(\prod_{\alpha} M\right)} \right| = \prod_{\alpha} \left| \frac{\phi_n(M)}{\phi_{n+1}(M)} \right| < \aleph_0,$$

so one of the following holds:

- (i). The index of the descending sequence for M never becomes finite.
- (ii). The index becomes 1 at some point.

But note that the first possibility would contradict the superstability of M , hence we infer that M is Σ -pure injective. \square

Theorem 4.12 ([10], Lemma 4; [26], Theorem 11.4). *An R -module M is Σ -pure injective if and only if there exists some cardinal κ such that every product $\prod_I M$ is a coproduct $\coprod_{j \in J} M_j$ of some submodules M_j of M , each of cardinality less than κ .*

Proof. If M is Σ -pure injective then, by Corollary 4.8, it is a coproduct of indecomposable Σ -pure injective submodules, by [33] (Corollary 4.2) each of these has cardinality at most $2^{|R| + \aleph_0}$.

For the converse, to show that a module satisfying the condition is Σ -pure injective, it will suffice, by the above Lemma 4.11, to show that all modules in \mathcal{X}_M are superstable. Without loss of generality fix a module $N \in \mathcal{X}_M$ and a cardinal $\lambda \geq \max(\kappa, 2^{\aleph_0 + |R|})$. Let X be a subset of N with cardinality $|X| \leq \lambda$. We proceed to count the number of pp-types over X and we want to obtain that is is bounded by λ .

For each $x \in X \subseteq N \subseteq H(N)$ define a morphism in $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ as

$$(x \otimes -) : R \otimes - \rightarrow H(N) \otimes -,$$

defined by sending $1_R \mapsto x$. Then let $H(X) \otimes -$ be the injective envelope of $\sum_{x \in X} \text{im}(x \otimes -)$ and note that $H(X)$ is a well defined summand of $H(N)$ ([27], Proposition 4.3.33). Now note that $H(N) \in \mathcal{X}_M$ (this follows functorially or by a construction, e.g. [27], Theorem 4.3.21) and we have a decomposition $H(N) = H(X) \coprod N'$. As M is an elementary cogenerator (see the beginning of Section 4.3 for a definition) of \mathcal{X}_M , by Corollary 4.18, it follows that $H(N)$ embeds in some product of M , hence it must be a direct summand of a module of the form $\coprod_{k \in K} M_k$ for some $|M_k| < \kappa$. Consider the composition of pure embeddings

$$X \longrightarrow H(X) \xrightarrow{q} \coprod_{k \in K} M_k, \quad (4.1)$$

where note that $X \subset \coprod_{k' \in K'} M_{k'}$ with $K' \subset K$ and $|K'| \leq |X| + \aleph_0$. This follows as each element is contained in only a finite number of summands.

The pp-type in M of each $x \in X$ is, by definition, $\text{pp}^M(x) = \text{pp}^{H(X)}(x) = \text{pp}^{\coprod_{k' \in K'} M_{k'}}(x)$ e.g. via the functorial definition of a pp-type of an element (see Section 2.2), so the composition of the hull and the projection onto $\coprod_{k' \in K'} M_{k'}$ in (4.1) is pure. By the definition of $H(X)$ above it follows that

$$|H(X)| \leq \left| \coprod_{k' \in K'} M_{k'} \right| = \kappa \cdot (|X| + \aleph_0) = \kappa \cdot |X|,$$

though for this we must make the additional assumption that κ is infinite.

For the final step let $n, n' \in N'$ be such that $n \neq n'$. Assume that they both realise the same pp-type p over 0, so $n, n' \in p(N)$. If we take any $y \in H(X)$ then the tuples (y, n) and (y, n') must realise the same pp-type over X ; we deduce this by additivity of pp-formulas as follows. Let $\phi(v, \bar{x})$ be a pp-formula with parameters \bar{x} from X , then $H(N) \models \phi[(y, n), \bar{x}]$ is equivalent to $H(X) \models \phi[y, \bar{x}]$ and $N' \models \phi[n, 0]$ holding which, in turn, is equivalent to $H(X) \models \phi[y, \bar{x}]$ and $N' \models \phi[n', 0]$ and hence $H(N) \models \phi[(y, n'), \bar{x}]$; all of this is by additivity of pp-formulas.

So we get a bound on the number of pp-types with parameters from X (recall that $|X| \leq \lambda$) that are realised in $H(N)$, namely that this is less than or equal to $|H(X)| \cdot 2^{\aleph_0 + |R|}$, which we know is less than or equal to λ , so N will be superstable. Thus all the modules in the definable subcategory \mathcal{X}_M will be superstable, hence M will be Σ -pure injective. \square

4.3 Σ -pure injective modules in the Ziegler spectrum

Let L be a pure injective R -module with the property that every $N \in \mathcal{X}_M$ embeds purely in $\prod_{i \in I} L$ for some set I . Call such a module L an *elementary cogenerator* for \mathcal{X}_M . We are going to show that a Σ -pure injective module M is an elementary cogenerator for \mathcal{X}_M , this will allow us to show that such an M contains a copy of every neg-isolated copy of its support as a direct summand.

Proposition 4.13. *The module M is an elementary cogenerator for \mathcal{X}_M if and only if $M \otimes -$ is an injective cogenerator for $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$.*

Proof. For notation see Theorem 1.6. If $M \otimes -$ is an injective cogenerator then, translating back to $\text{Mod } R$, it is clear that M is an elementary cogenerator. On the other hand if M is an elementary cogenerator then the injective envelope $E(S)$ of every simple object S in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ embeds in $\prod_{i \in I} M \otimes -$. Because $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ is in particular locally finitely generated an object which contains the injective envelope of every simple object must be an injective cogenerator. \square

Remark 4.14. In the more general case (which, for simplicity, we do not consider) the above can be adapted to show that every definable subcategory \mathcal{X} has an elementary cogenerator. For, it is not difficult to see that by taking $E = E(\prod_{i \in I} E(S_i))$, where S_i runs over all the simple objects of $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_{\mathcal{X}}$, we get an injective cogenerator and as we have noted earlier $E = L \otimes -$ for some R -module L . Translating back to $\text{Mod } R$ we obtain that L an elementary cogenerator of \mathcal{X} .

Theorem 4.15 ([27], Corollary 5.3.54; [17], Proposition 6.6). *Let M be a Σ -pure injective R -module, then M is an elementary cogenerator for \mathcal{X}_M .*

Proof. To prove that $M \otimes -$ is an elementary cogenerator for $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ it will suffice to show that $(F, M \otimes -) \neq 0$ for every finitely generated $F \in (\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$. By Corollary 4.5 we know that $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ is locally noetherian so every

finitely generated object is finitely presented (and hence noetherian). As the category $\text{fp}((\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M)$ is equivalent to the category $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{S}_M$, we need to show that there is a non-zero morphism from any $F \in \text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{S}_M$ to $M \otimes -$. Furthermore as $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{S}_M$ is a full subcategory it will suffice to show the existence of this morphism in $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{S}_M$. But this follows from definition of $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{S}_M$. \square

As an immediate consequence of the above we summarise those properties of the definable subcategory generated by a Σ -pure injective module, which follow from the results of this Section.

Corollary 4.16. *The definable subcategory \mathcal{X}_M generated by a Σ -pure injective module M consists of Σ -pure injective modules each of which:*

- (i). *Embeds purely into some $\prod_{i \in I} M$.*
- (ii). *Is a coproduct of indecomposable Σ -pure injective modules, each of which has a local endomorphism ring.*

There is final general result about elementary cogenerators which we require. For the relevant definition and discussion concerning isolation and neg-isolation see the paragraphs which precede Theorem 3.9.

Theorem 4.17 ([27], Theorem 5.3.50). *An R -module M is an elementary cogenerator if and only if M contains, as a direct summand, every neg-isolated point of $\text{supp } M = \mathcal{X}_M \cap \text{Zg}_R$.*

Proof. Recall that N is a neg-isolated point of $\mathcal{X}_M \cap \text{Zg}_R$ if and only if $N \otimes -$ is the injective envelope of a simple functor in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$. If $M \otimes -$ is an injective cogenerator of $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ then it will certainly contain the injective envelope of every simple object as a direct summand. The converse follows from the fact that $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ is (in particular) a locally finitely generated Grothendieck category. For suppose that $M \otimes -$ is an injective object in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_{D(\text{supp } M)}$ which contains the injective envelopes of all the simple objects. Then it suffices to

check that there is a non-zero morphism from F to $M \otimes -$ for every finitely generated functor F . To see such a morphism consider a maximal subfunctor F' of F ; then the morphism $F \rightarrow F/F' \rightarrow M \otimes -$ is non-zero because F/F' is simple. \square

From Corollary 4.8 we know that, if M is Σ -pure injective, then $\text{supp } M = \text{Zg}_R \cap \mathcal{X}_M$ is just the closure, in Zg_R , of all the direct summands M_i such that $M = \coprod_{i \in I} M_i$. For completeness note the following obvious consequence of the above Theorem 4.17.

Corollary 4.18. *A Σ -pure injective R -module M contains a copy of every negated isolated point of its support as a direct summand.*

4.4 Product complete modules

Krause and Saorin in [18] introduce a certain class of R -modules called *product complete* modules. A product complete module is a module which is Σ -pure injective and contains, as a direct summand, *every* point of its support $\mathcal{X}_M \cap \text{Zg}_R$ (compare with Corollary 4.18). This class is strictly smaller than the class of Σ -pure injective modules and yet strictly larger than the class of endofinite modules (a module is endofinite if it finite length over its endomorphism ring, see Chapter 5), as shown by the following example.

Example 4.19. Consider the Σ -pure injective $k[X]$ -module $P_{\lambda, \infty}$ with support $\mathcal{X}_{P_{\lambda, \infty}} \cap \text{Zg}_R$ containing $k(X)$. Clearly $k(X)$ does not occur as a direct summand of $P_{\lambda, \infty}$, hence $P_{\lambda, \infty}$ is not product complete. However $P_{\lambda, \infty} \coprod k(X)$ is product complete, yet as its summand $P_{\lambda, \infty}$ is not endofinite (since it is not product complete), it is not endofinite itself (for this last part see Theorem 5.8).

By the previous results on Σ -pure injective modules we can obtain a more amenable description of product complete modules. This also serves to explain their name.

Theorem 4.20 ([18], Proposition 3.7). *A Σ -pure injective R -module M is product complete if and only if, for every set J , the module $\coprod_{j \in J} M$ is a direct summand of $\coprod_{h \in H} M$, for some set H .*

Proof. We know that $M = \coprod_{i \in I} M_i$ with M_i indecomposable Σ -pure injective modules. By Corollary 4.8 we also know that the closed set $\text{supp } M$ is the closure of $\{M_i\}_{i \in I}$; it then follows by the description of \mathcal{X}_M in Corollary 4.18 that $\text{supp } M$ consists of the indecomposable direct summands of products of M . Clearly if M contains every point of $\text{supp } M$ as a direct summand then $\prod_{j \in J} M$ will be a direct summand of $\prod_{h \in H} M$. On the other hand if $N \in \text{supp } M$ then N is a summand of e.g. $\prod_{j \in J} M$ and if this is a summand of $\prod_{h \in H} M$ then, by the Krull-Schmidt-Remark-Azumaya Theorem (see [23], Section 4.8, p.193; also [32] V.5.5), we must have that N is a direct summand of M . \square

Using Theorem 1.11 the following can then be obtained.

Lemma 4.21 ([18], Theorem 3.1). *Let M be an R -module. Then the following are equivalent:*

- (i). *M is product complete.*
- (ii). *Every $\prod_{j \in J} M$ is a coproduct of copies of indecomposable direct summands of M .*
- (iii). *Every R -module has a left $\text{Sumd}^{\text{II}} M$ approximation, i.e. $\text{Sumd}^{\text{II}} M$ is covariantly finite in $\text{Mod } R$.*
- (iv). *Every R -module has a minimal left $\text{Sumd}^{\text{II}} M$ approximation.*

Proof. The part (i) implies (ii) follows by arguments of Theorem 4.20. Also we know that (ii) implies (iii) by the third part of Theorem 1.11, for $\text{Sumd}^{\text{II}} M$ is closed under products and consists of pure-injective modules. The part (iii) implies (i) follows by the same result. Also note that (iii) and (iv) are equivalent by the use of [18] (Proposition 1.2). \square

Chapter 5

Endofinite modules

5.1 Basic properties

Let M be an R -module. As in Chapter 3 we will work with modules over $\Gamma = \Gamma_M = \text{End}_R(M)^{\text{op}}$, the opposite of the endomorphism ring of M . If an R -module M has a particular property as an object of $\text{Mod } \Gamma$, then we will prefix the name of this property by *endo* and say that M satisfies this *endo*-prefixed property. So, for example, if M has finite length in $\text{Mod } \Gamma$, then we will say that M is *endofinite*. We will denote by $\varepsilon\ell(M)$ the *endolength* of M , that is the length of M as an object of $\text{Mod } \Gamma$.

If M is an endofinite R -module, then it is in particular both endoartinian and endonoetherian, and so as pp-definable subgroups are Γ -submodules of M , it follows from Theorem 4.1 that M is Σ -pure injective. We can prove more.

Proposition 5.1 ([27], 4.4.25). *For an endofinite R -module M every Γ -submodule is a pp-definable subgroup.*

Proof. By Lemma 4.4 every cyclic Γ -submodule (i.e. one of the form $\text{End}_R(M)x = \Gamma x$, for some $x \in M$) is definable by some pp-formula. Since M is, in particular, endonoetherian it follows that every Γ -submodule N of M is finitely generated ([32], V.4), hence a sum

$$N = \text{End}_R(M)x_1 + \text{End}_R(M)x_2 + \dots + \text{End}_R(M)x_n,$$

which may not be direct but which consists of cyclic Γ -submodules. By considering addition of pp-formulas (see Section 2.2) it is evident that a finite sum of pp-definable subgroups is a pp-definable subgroup. Thus N is pp-definable and the result follows. \square

Either by the above Proposition 5.1 (or directly, by projecting composition chains) we note that any coproduct, $\coprod_{i \in I} M$, of copies of an endofinite R -module M is endofinite of the same endolength. Of course even the finite coproduct of *different* endofinite modules is endofinite, but the endolength of this coproduct may not be the sum of their endolengths. This happens when, for example, there are repetitions amongst the indecomposable endofinite summands. This is a question to which will return to in Section 5.2.

We now collect some straightforward consequences of the methods which were used in Section 3.2 and Section 4.3, the reader may wish to briefly review these sections again. However the relevant results to compare with are cited in the proofs of the following Corollary 5.2 and Corollary 5.3.

For us the relevant consequence of Proposition 5.1 is the following. We know that the endolength of an endofinite module M equals the length of the lattice of pp-definable subgroups of M (see Section 2.1 for a definition). From this we conclude that \mathcal{E}_M consists of finite length objects. We now recall that an abelian category \mathcal{C} is said to be a *length category* if every object $X \in \mathcal{C}$ is of finite length i.e. the modular lattice of subobjects of X has finite length (see [32], Chapter III). Similarly we will say that \mathcal{C} is *locally finite* if it has a generating set of finite length objects.

Corollary 5.2 ([17], Lemma 6.13). *An R -module M is endofinite if and only if \mathcal{E}_M is a length category, if and only if $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ is locally finite.*

Proof. This follows from Proposition 3.3 and the definition of \mathcal{E}_M . Compare with Corollary 4.5. \square

Corollary 5.3 ([27], Proposition 4.4.27). *The definable subcategory \mathcal{X}_M generated by an endofinite module M has the following properties:*

- (i). Every $N \in \mathcal{X}_M$ is endofinite with endolength bounded by $\varepsilon\ell(M)$.
- (ii). Every $N \in \mathcal{X}_M$ embeds purely as $N \hookrightarrow \prod_{i \in I} M$ for some set I .
- (iii). Every $N \in \mathcal{X}_M$ is a coproduct of endofinite modules with local endomorphism rings.

Proof. See Section 4.2 and compare with Corollary 4.18 (parts (ii) and (iii) of the current result follow directly from this result and part (i) of the current result). \square

5.2 Indecomposable endofinite modules

As promised earlier, we now come to indecomposable endofinite modules, that is those endofinite modules that occur as points of the Ziegler spectrum. The aim of this Section 5.2 is to obtain a strong decomposition result (that is, to strengthen and understand in more detail part (iii) of Corollary 5.3) for endofinite modules. This is much stronger than the one we obtained for Σ -pure injective modules in Chapter 4.

Theorem 5.4 ([17], Theorem 6.14). *Let M be a non-zero R -module. Then the following properties are equivalent:*

- (i). M is an indecomposable endofinite module.
- (ii). The objects $\coprod_I M$, for I any set, form a definable subcategory of $\text{Mod } R$.
- (iii). M is indecomposable and every $\prod_I M$ is also some $\prod_J M$.

Proof. For (i) implies (ii), note that, by Corollary 5.2, the category $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ is locally finite, so it is in particular locally noetherian. From Theorem 1.6 and Corollary 4.18 we know that every object X , in the definable subcategory generated by M , is correspondingly an injective object $X \otimes -$ in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$. Hence $M \otimes -$ is a coproduct of such. Every indecomposable injective object in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ is then, by [32](Proposition V.2.8), an injective envelope of a simple subobject, as every object in $\text{fp}(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ has a composition series. Now fix a simple object S in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$. Evidently we have $(S, M \otimes -) \neq 0$ by the definition of \mathcal{T}_M .

There is a morphism $S \hookrightarrow M \otimes -$, so S is a subobject of $M \otimes -$ and hence $M \otimes -$ is the unique injective envelope of S (see [32], Proposition V.5.8). It follows that $M \otimes -$ is the unique indecomposable injective object in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$. Thus every injective object in $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ is a coproduct $\coprod_I (M \otimes -)$. This now implies (ii) when translated back to $\text{Mod } R$. It is easy to see that (ii) implies (iii). For (iii) implies (i) note that, by Theorem 4.1 (iii), M must be Σ -pure injective, that is $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$ must be locally noetherian. Also it follows that $M \otimes -$ is the injective cogenerator for $(\text{mod } R^{\text{op}}, \mathbf{Ab})/\mathcal{T}_M$, hence the unique indecomposable injective object. It follows that $(\text{mod } R^{\text{op}}, \mathbf{Ab})$ is locally finite, thus M is an indecomposable endofinite module. \square

There are a number of important consequences of the above Theorem 5.4. These consequences concern the definable subcategory of an indecomposable endofinite R -module (as in part (i) of the above Theorem 5.4).

Corollary 5.5. *Let M be an indecomposable endofinite R -module, that is an endofinite point of the Ziegler spectrum Zg_R . Then every $N \in \mathcal{X}_M$ is a coproduct of the form $\coprod_{i \in I} M$ for some index set I .*

Proof. This follows from the above Theorem 5.4 and Corollary 5.3. \square

Corollary 5.6. *Let M be an indecomposable R -module, that is a point of the Ziegler spectrum Zg_R . Then M is endofinite if and only if M is product complete.*

Proof. This follows from the above Theorem 5.4 and Lemma 4.21. \square

In fact (with some work) one can use Theorem 5.4 and the above two Corollaries to obtain a very satisfactory decomposition for endofinite modules in terms of indecomposable endofinite modules. We begin by setting up some (mostly standard) notation. Let M be an R -module and let I be a set, then:

- (i). We denote by M^I the object $\prod_{i \in I} M$.
- (ii). We denote by $M^{(I)}$ the object $\coprod_{i \in I} M$.

Furthermore we extend this notation to cardinals (which are sets), so for example M^κ will denote κ -many products of M .

Lemma 5.7. *Let M be an endofinite R -module with decomposition $M = \coprod_{i=1}^n M_i^{\kappa_i}$, for some cardinals κ_i , and some pairwise non-isomorphic indecomposable modules M_i with local endomorphism rings. Then $\varepsilon\ell(M) = \sum_{i=1}^n \varepsilon\ell(M_i)$.*

Proof. It is obvious that $\varepsilon\ell(M) \leq \sum_{i=1}^n \varepsilon\ell(M_i)$, so we just need to show that $\varepsilon\ell(M) \geq \sum_{i=1}^n \varepsilon\ell(M_i)$. Recall that the Jacobson radical $\text{radEnd}_R(M)$ is ([32], Proposition VIII.1.5) precisely the largest ideal consisting of $f : M \rightarrow M$ each of whose components in the decomposition of M is a non-isomorphism. Since any map $M_i \rightarrow M_j$, with $i \neq j$, is in $\text{radEnd}_R(M)$ then by the properties of the coproduct we will have

$$\frac{\text{End}_R(M)}{\text{radEnd}_R(M)} = \prod_{i=1}^n \frac{\text{End}_R(M_i)}{\text{radEnd}_R(M_i)},$$

which is semisimple for it is a product of division rings. Construct the following sequence of $\text{End}_R(M)^{\text{op}}$ -subgroups of M

$$M \supset (\text{radEnd}_R(M)^{\text{op}})M \supset (\text{radEnd}_R(M)^{\text{op}})^2M \supset \dots,$$

where each module is considered over the ring $\text{End}_R(M)/\text{radEnd}_R(M)$. Then, for $i \geq 0$, the semisimple factor module $(\text{radEnd}_R(M)^{\text{op}})^i M / (\text{radEnd}_R(M)^{\text{op}})^{i+1} M$ is a module over $\text{End}_R(M)/\text{radEnd}_R(M)$. Without loss of generality assume $\kappa_i = 1$ for all i (see comments after Proposition 5.1), then using the above sequence it is clear that the sum of the lengths of M_i over $\text{End}_R(M_i)/\text{radEnd}_R(M_i)$ is bounded by the length of M over $\text{End}_R(M)/\text{radEnd}_R(M)$, i.e. by the endolength of M . This finishes the proof. \square

Theorem 5.8 ([17], Proposition 6.15; [27], Lemma 4.4.29). *An R -module M is endofinite if and only if there are indecomposable pairwise non-isomorphic endofinite modules M_1, \dots, M_n such that*

$$M \cong \prod_{i=1}^n M_i^{(\kappa_i)},$$

for some cardinals $\kappa_1, \dots, \kappa_n$.

Proof. Let M be endofinite, then M is Σ -pure injective, so therefore decomposes as some $M = \coprod_{i \in I} M^{(\kappa_i)}$ (though I may be infinite), where the M_i are indecomposable pairwise non-isomorphic and Σ -pure injective. From Lemma 5.7 we deduce that I must be finite and each $\varepsilon\ell(M_i) \leq \varepsilon\ell(M)$, hence M is a coproduct of powers of only finitely many of these, all endofinite. On the other hand if $M = \coprod_{i=1}^n M_i^{(\kappa_i)}$ then as $\varepsilon\ell(M_i^{\kappa_i}) = \varepsilon\ell(M_i)$ we have $\varepsilon\ell(M) \leq \sum_{i=1}^n \varepsilon\ell(M_i)$. So M must be endofinite. \square

We have already noted in Corollary 5.6 that an indecomposable module is endofinite if and only if it is product complete. It is clear that if M_i is product complete then so is $M_i^{(\kappa)}$. It is also clear that finite coproducts of product complete modules are product complete. Hence every direct summand of an endofinite module is product complete. There is a converse to this and here we follow the proof of Krause and Saorin which makes use of the Ziegler spectrum.

Theorem 5.9 ([18], Theorem 4.1). *An R -module M is endofinite if and only if every direct summand of M is product complete.*

Proof. For the non-trivial direction (note the above paragraph) we suppose that every direct summand of M is product complete, so in particular M itself is product complete. Then $\text{supp } M$ consists precisely of the indecomposable direct summands of M . Any arbitrary subset $X \subset \text{supp } M$ is closed, hence every point will be isolated, but by compactness of $\text{supp } M$ there can only be finitely many points in $\text{supp } M$. Thus as M is a coproduct of copies the modules in its support, it then follows by Corollary 5.6 and Theorem 5.8 that M is endofinite. \square

5.3 Endofinite modules in the Ziegler spectrum

If M is an endofinite R -module with decomposition

$$M = \coprod_{i=1}^n M_i^{(\kappa_i)},$$

where each M_i endofinite and indecomposable, then by Theorem 5.8 the indecomposable pure injective R -modules in the definable subcategory \mathcal{X}_M , generated by M , are precisely the M_i .

Lemma 5.10 ([11], p.544). *For every $n < \aleph_0$ the modules M such that $\varepsilon\ell(M) \leq n$ form a closed subset of the Ziegler spectrum.*

Proof. Fix an n . Then for any descending chain of pp-formulas

$$\phi_0 \geq \phi_1 \geq \dots \geq \phi_{n+1},$$

we will have the basic Ziegler-open sets

$$(\phi_i/\phi_{i+1}) = \{N \in \mathbf{Zg}_R : \phi_i(N) > \phi_{i+1}(N)\},$$

whose intersection is $\bigcap_{i=1}^n (\phi_i/\phi_{i+1})$. Then the evaluation functor

$$E_M^{\text{eq}+} : \mathbb{L}_R^{\text{eq}+} \rightarrow \text{Mod } \Gamma,$$

by Proposition 5.1, will induce a surjection which sends a finite chain of pp-formulas to a finite chain of pp-definable subgroups. It follows that every point of endolength $> n$ will be contained in some intersection of this form. Take (the possibly infinite) union of all these (always finite) intersections of open sets, then the complement of this union, say el_n , is precisely the closed set consisting of all modules of endolength $\leq n$. □

In the example of $R = k[X]$ the structure of the endofinite modules in \mathbf{Zg}_R is particularly simple. To make this observation formal, notice that one class consists of the λ -finite modules $P_{\lambda,n}$ for $\lambda \in k$ and $n \geq 1$, which all have CB-rank 0. The only other endofinite module is $k(X)$, which has CB-rank 2. In fact Theorem 5.8 states that all endofinite $k[X]$ -modules are isomorphic to some module of the form $P_{\lambda_1, n_1}^{(\kappa_1)} \amalg \dots \amalg P_{\lambda_n, n_n}^{(\kappa_n)} \amalg k(X)^{(\kappa_{n+1})}$.

Let R be a ring and denote by $\text{Endf } R$ the class of indecomposable endofinite R -modules. Also denote by $\text{endf } R$ the class of finitely presented indecomposable endofinite R -modules. By definition $\text{Endf } R \setminus \text{endf } R$ is the class of *generic modules*.

Let $M \in \text{Endf } R$, then \mathcal{E}_M contains a unique simple Γ -module S (for by Corollary 4.18 the ring $\text{End}_R(M)$ is local) and this corresponds to the minimal pp-pair which isolates M in its support. Thus, following on from Section 3.3, we can identify

$(M, E(S))$ in $\text{Mod } \Gamma$ as a sum of the copies of the indecomposable module DM . If, furthermore, $M \in \text{endf } R$ then $DM = (M, E(S))$ ([27], Corollary 5.4.15).

In [16] Krause specifies that a ring R is *right dualizing* if for every $M \in \text{endf } R$ the dual satisfies $DM \in \text{endf } R^{\text{op}}$. The ring R is said to be *dualizing* if it is both left and right dualizing. The primary example of such rings are the artin algebras (in fact noetherian suffices here), see Section 1.3 for the relevant definitions.

Theorem 5.11 ([16], Theorem 4.7). *Let R be a dualizing ring and $M \in \text{Endf } R$, then the following are equivalent:*

- (i). $M \in \text{endf } R$.
- (ii). M is isolated in Zg_R .
- (iii). M occurs as a source of a left almost split map in $\text{Mod } R$.
- (iv). M is isomorphic to DN for some $N \in \text{endf } R^{\text{op}}$.

Using results of Herzog and Crawley-Boevey, the following can then be proven; note that this means that for artin algebras of infinite representation type (where there are $\geq \aleph_0$ non-isomorphic indecomposable modules in $\text{mod } R$) there will exist a point in Zg_R which will not be finitely presented.

Theorem 5.12 ([16], Theorem 5.6). *Let R be a right dualizing ring, then there exists a generic module (i.e. a module in $\text{Endf } R$ but not in $\text{endf } R$) provided either of the following holds: either (i) there is a non-zero R -module having no finitely presented indecomposable endofinite direct summand and no dense chain of pp-definable subgroups or (ii) there exists an infinite number of finitely presented indecomposable R -modules of endolength at most n for some $n \geq 0$.*

Proof. For part (i) see the reference. For part (ii) assume otherwise, then if there does not exist a non-isolated (hence of infinite length) module in Zg_R we obtain a contradiction by the compactness of Zg_R . To see this suppose that the finitely presented points are precisely the isolated ones. If these are all the points in Zg_R then there would have to be finitely many of them. \square

Bibliography

- [1] Ibrahim Assem, Andrzej Skowronski, and Daniel Simson. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*. London Mathematical Society Student Texts 65. Cambridge University Press, 2006.
- [2] M Auslander and Sverre O Smalø. Almost split sequences in subcategories. *Journal of Algebra*, 69(2):426 – 454, 1981.
- [3] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation Theory of Artin Algebras*. Cambridge Studies in Advanced Mathematics 36. Cambridge University Press, 1997.
- [4] Francis Borceux. *Handbook of Categorical Algebra I: Basic category theory*. Encyclopedia of Mathematics and its Applications 50. Cambridge University Press, 1994.
- [5] Francis Borceux. *Handbook of Categorical Algebra II: Categories and structures*. Encyclopedia of Mathematics and its Applications 51. Cambridge University Press, 1994.
- [6] R. Cori and D. Lascar. *Mathematical Logic, Part I (Ch. 1 to 4) and Part II (Ch. 5 to 8)*. Oxford University Press, 2000, 2001.
- [7] David Marker. *Model Theory: An Introduction*. Graduate Texts in Mathematics 217. Springer-Verlag, 2002.

- [8] Peter Gabriel. Auslander-Reiten sequences and representation-finite algebras. In Vlastimil Dlab and Peter Gabriel, editors, *Representation Theory I*, volume 831 of *Lecture Notes in Mathematics*, pages 1–71. Springer-Verlag, 1980.
- [9] Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien*, volume 221 of *Lecture Notes in Mathematics*. Springer-Verlag, 1971.
- [10] Steven Garavaglia. Decomposition of Totally Transcendental Modules. *The Journal of Symbolic Logic*, 45(1):pp. 155–164, 1980.
- [11] I Herzog. The Ziegler Spectrum of a Locally Coherent Grothendieck Category. *Proceedings of the London Mathematical Society*, 74(03):503–558, 1997.
- [12] Birge Huisgen-Zimmermann. Purity, Algebraic compactness, Direct sum decompositions, and Representation type. In H. Krause and C. M. Ringel, editors, *Infinite Length Modules*, Trends in Mathematics, pages 331 – 367. Birkhauser, 2000.
- [13] Christian Jensen and Helmut Lenzing. *Model Theoretic Algebra with particular emphasis on Fields, Rings and Modules*. Algebra, Logic and Applications 2. Gordon and Breach, 1989.
- [14] Henning Krause. The Endocategory of a Module. In R. Bautista, R. Martinez-Villa, and J.A. de la Pena, editors, *Representation Theory of Algebras*, CMS Conference Proceeding 18, pages 419–432, 1996.
- [15] Henning Krause. Functors on Locally Finitely Presented Additive Categories. *Colloquium Mathematicum*, 75(01):105–132, 1998.
- [16] Henning Krause. Generic Modules over Artin Algebras. *Proc. London Math. Soc.*, 76(3):276–306, 1998.
- [17] Henning Krause. *The Spectrum of a Module Category*. Memoirs of AMS 707. American Mathematical Society, 2001.

- [18] Henning Krause and Manuel Saorin. On Minimal Approximations of Modules. In E. Green and B. Huisgen-Zimmermann, editors, *Conference on Representations of Algebras, Seattle 1997*, Contemporary Mathematics 229, pages 227–236. American Mathematical Society (AMS), 1998.
- [19] Serge Lang. *Algebra, Revised Third Edition*. Graduate Texts in Mathematics 211. Springer-Verlag, 2004.
- [20] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, 2nd edition, 1998.
- [21] Annalisa Marcja and Carlo Toffalori. *A guide to Classical and Modern Model Theory*. Trends in Logic 19. Springer-Verlag, 2003.
- [22] Colin McLarty. *Elementary Categories, Elementary Toposes*. Oxford Logic Guides 21. Oxford Science Publications, 1995.
- [23] Bodo Pareigis. *Categories and Functors*. Pure and Applied Mathematics, Vol. 39. Academic Press, 1970.
- [24] Nicolae Popescu. *Abelian Categories with Applications to Rings and Modules*. London Mathematical Society Monographs. Academic Press, 1973.
- [25] Mike Prest. Model Theory and Representation Type of Algebras. In F.R. Drake and J.K. Truss, editors, *Logic Colloquium '86, Proceedings of the Colloquium held in Hull*, volume 124 of *Studies in Logic and the Foundations of Mathematics*, pages 219 – 260. Elsevier, 1987.
- [26] Mike Prest. *Model Theory and Modules*. London Mathematical Society Lecture Note Series 130. Cambridge University Press, 1988.
- [27] Mike Prest. *Purity, Spectra and Localisation*. Encyclopedia of Mathematics. Cambridge University Press, 2009.
- [28] Mike Prest. *Definable Additive Categories: Purity and Model Theory*. Memoirs of AMS 987. American Mathematical Society, 2011.

- [29] Juan Rada and Manuel Saorin. Rings Characterized by (Pre)envelopes and (Pre)covers of their Modules. *Comm. in Algebra*, 26(3):pp. 899–912, 1998.
- [30] Jan-Erik Roos. Locally Noetherian Categories and Generalized Strictly Linearly Compact Rings. In *Category Theory, Homology Theory and their Applications II*, Lecture Notes in Mathematics 92, pages 197–277. Springer-Verlag, 1969.
- [31] Daniel Simson and Andrzej Skowronski. *Elements of the Representation Theory of Associative Algebras: Volume 3, Representation-infinite Tilted Algebras*. London Mathematical Society Student Texts 72. Cambridge University Press, 2007.
- [32] Bo Stenström. *Rings of Quotients, An Introduction to Methods of Ring Theory*. Die Grundlehren der mathematischen Wissenschaften 217. Springer-Verlag, 1975.
- [33] M Ziegler. Model Theory of Modules. *Ann. Pure Appl. Logic*, 26(2):149213–558, 1984.