

PURITY RELATIVE TO CLASSES OF FINITELY PRESENTED MODULES

A THESIS SUBMITTED TO THE UNIVERSITY OF MANCHESTER
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
IN THE FACULTY OF ENGINEERING AND PHYSICAL SCIENCES

2013

Akeel Ramadan Mehdi

School of Mathematics

Contents

Abstract	4
Declaration	5
Copyright Statement	6
Publications	7
Acknowledgements	8
List of Notation	9
1 Introduction	13
2 Background	18
2.1 Homological algebra	18
2.1.1 Injective and projective modules	20
2.1.2 $\text{Ext}_R^n(-, -)$ and $\text{Tor}_n^R(-, -)$	21
2.1.3 Direct limits of modules	22
2.2 Pure exact sequences and derived notions	22
2.2.1 Pure exact sequences	23
2.2.2 Pure-injective and pure-projective modules	25
2.2.3 \mathcal{S} -purity	26
2.3 (Pre)-Covers and (Pre)-envelopes	28
2.4 Ziegler spectrum and full support topology	29
2.4.1 Definable subclasses	29
2.4.2 Ziegler spectrum and full support topology	30

3	Purities	31
3.1	Purities	32
3.2	Comparing purities	40
3.3	(m, n) -Purity over semiperfect rings	46
3.4	Purity over finite-dimensional algebras	50
3.5	Rings whose indecomposable modules are S -pure-projective	61
4	Definability of the classes $(\varinjlim S)$ and $(\text{Prod } S^*)$	65
4.1	Introduction	65
4.2	Almost dual pairs	66
4.2.1	Definition, examples and some properties	66
4.2.2	Natural bijections induced by almost dual pairs	71
4.2.3	Almost dual pairs induced by classes of finitely presented modules	75
4.3	Definable classes induced by almost dual pairs	85
4.4	Definability of the class $(\varinjlim S)$	90
4.4.1	Definability of the class $(\varinjlim S)$ by using duality of modules	90
4.4.2	Definability of the class $(\varinjlim S)$ in terms of $(\varinjlim S)^\ominus$	96
4.4.3	Definability of the class $(\varinjlim S)$ using duality between preenvelopes and precovers	99
4.5	Definability of the class $\text{Prod } S^*$	102
4.5.1	Definability of $\text{Prod } S^*$ in terms of preenvelopes and precovers	102
4.5.2	Definability of S -Pinj in terms of the functors Axt_S^1 and Pext_S^1	109
5	S-Cotorsion pairs	116
5.1	Cotorsion pairs, covers and envelopes relative to the proper class \mathcal{P}_S	117
5.2	S -Cotorsion pairs induced by S' -Pinj	120
5.3	When is $(\mathcal{F}, \mathcal{F}^{\perp_S})$ a perfect S -cotorsion pair?	130
5.4	Some examples and applications	142
	Bibliography	150

The University of Manchester

Akeel Ramadan Mehdi

Doctor of Philosophy

Purity Relative to Classes of Finitely Presented Modules

July 2, 2013

Any set of finitely presented left modules defines a relative purity for left modules and also a purity for right modules. Purities defined by various classes are compared and investigated, especially in the contexts of modules over semiperfect rings and over tame hereditary, and more general, finite-dimensional algebras. Connections between the indecomposable relatively pure-injective modules and closure in the full support topology (a refinement of the Ziegler spectrum) are described.

Duality between left and right modules is used to define the concept of a class of left modules and a class of right modules forming an almost dual pair. Definability of such classes is investigated, especially in the case that one class is the closure of a set of finitely presented modules under direct limits. Elementary duality plays an important role here.

Given a set of finitely presented modules, the corresponding proper class of relatively pure-exact sequences can be used to define a relative notion of cotorsion pair, which we investigate.

The results of this thesis unify and extend a wide range of results in the literature.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

- i.** The author of this thesis (including any appendices and/or schedules to this thesis) owns certain copyright or related rights in it (the “Copyright”) and s/he has given The University of Manchester certain rights to use such Copyright, including for administrative purposes.
- ii.** Copies of this thesis, either in full or in extracts and whether in hard or electronic copy, may be made **only** in accordance with the Copyright, Designs and Patents Act 1988 (as amended) and regulations issued under it or, where appropriate, in accordance with licensing agreements which the University has from time to time. This page must form part of any such copies made.
- iii.** The ownership of certain Copyright, patents, designs, trade marks and other intellectual property (the “Intellectual Property”) and any reproductions of copyright works in the thesis, for example graphs and tables (“Reproductions”), which may be described in this thesis, may not be owned by the author and may be owned by third parties. Such Intellectual Property and Reproductions cannot and must not be made available for use without the prior written permission of the owner(s) of the relevant Intellectual Property and/or Reproductions.
- iv.** Further information on the conditions under which disclosure, publication and commercialisation of this thesis, the Copyright and any Intellectual Property and/or Reproductions described in it may take place is available in the University IP Policy (see <http://documents.manchester.ac.uk/DocuInfo.aspx?DocID=487>), in any relevant Thesis restriction declarations deposited in the University Library, The University Library’s regulations (see <http://www.manchester.ac.uk/library/aboutus/regulations>) and in The University’s Policy on Presentation of Theses.

Publications

- 1- Chapter three of this thesis will be published in the Journal of Algebra and Its Applications, under the title: Purity relative to classes of finitely presented modules [44].
- 2- Some results from chapter four will be a part of the preprint "Almost dual pairs and definable classes of modules" [45].

Acknowledgements

First of all, all praise and thanks go to Allah for his mercy and blessing.

I wish to express my sincere thanks to my supervisor Professor Mike Prest, for his valuable help, suggestions, guidance, encouragement, patience, endless support and excellent supervision during my studies and preparation of this thesis.

It is also my pleasure to thank the Department of Mathematics at The University of Manchester for providing an excellent research environment and my great thanks to all my officemates for all their help.

I am grateful to the Iraqi Ministry of Higher Education and Scientific Research for sponsoring and giving me the opportunity to study for a PhD at the University of Manchester.

I am thankful to my wife for her support, love, understanding and encouragement over the years. My children, Noor Al-Deen and Malk, who have been a very strong source of inspiration and who have given me much happiness.

Finally, I would like to dedicate this work to the memory of my mother.

List of Notation

All rings in this thesis are associative with unity, all modules are unital, all classes of modules are closed under isomorphism and all matrices are matrices with finitely many rows and finitely many columns.

Notation	Description
R	associative ring with unity
${}_R M$ (resp. M_R)	a left (resp right) R -module M
$R\text{-Mod}$ (resp. $\text{Mod-}R$)	the class of left (resp. right) R -modules
$R\text{-mod}$ (resp. $\text{mod-}R$)	the class of finitely presented left (resp. right) R -modules
\mathbb{Z}^+	the positive integers
ω	the first infinite ordinal number
$M \simeq N$	M is isomorphic to N
$\text{Hom}_R(M, N)$	the group of R -homomorphisms from M to N
$\text{End}_R(M)$	the endomorphism ring of the module M
1_M	the identity homomorphism $1_M : M \rightarrow M$
$f\bar{a}$	$(f(a_1), f(a_2), \dots, f(a_n)) \in B^n$ where $f : A \rightarrow B$ is a homomorphism and $\bar{a} = (a_1, a_2, \dots, a_n) \in A^n$
M^*	the character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$
X^\diamond	$\text{Hom}_R(X, R_R)$ where X is a right R -module
M^*	the E -dual module $\text{Hom}_K(M, E)$, where K is a commutative ring and E is an injective cogenerator for $K\text{-Mod}$, p. 23
\mathcal{F}^*	the class $\{M^* \mid M \in \mathcal{F}\}$
δ_M	the canonical monomorphism $\delta_M : M \rightarrow M^{**}$, p. 23
$\ker(\alpha)$ (resp. $\text{im}(\alpha)$)	the kernel (resp. image) of the homomorphism α
$\text{coker}(\alpha)$	the cokernel of the homomorphism α

Notation	Description
$M_{n \times m}(R)$	the set of $n \times m$ matrices over R
$H_{n \times m}$	an $n \times m$ matrix H over R
$0_{1 \times 1}$	the 1×1 zero matrix over R
ρ_H (resp. λ_H)	the homomorphism $\rho_H : {}_R R^n \rightarrow {}_R R^m$ (resp. $\lambda_H : R_R^m \rightarrow R_R^n$) determined by right (resp. left) multiplication by $H_{n \times m}$
$L_H (H \in M_{n \times m}(R))$	the (m, n) -presented left R -module $R^m / \text{im}(\rho_H)$
$D_H (H \in M_{n \times m}(R))$	the (n, m) -presented right R -module $R^n / \text{im}(\lambda_H)$
$L_{\mathcal{H}}$ (resp. $D_{\mathcal{H}}$)	the class $\{L_H \mid H \in \mathcal{H}\}$ (resp. $\{D_H \mid H \in \mathcal{H}\}$)
$PE(M)$	pure-injective hull of M , p. 25
$J(M)$	Jacobson radical of M
$\leq T \geq$	the left ideal generated by the subset T of a ring R
$\text{gen}(M)$	the minimal number of generators of M
$\text{rel}(M)$	the minimal number of relations on any generating set of M
$\prod_{i \in I} M_i$	the direct product of the modules M_i
$\bigoplus_{i \in I} M_i$	the direct sum of the modules M_i
$\lim_{\rightarrow} M_i$	the direct limit of the directed system of modules M_i
$\lim_{\rightarrow} \mathcal{F}$	the class of direct limits of direct systems of modules in \mathcal{F}
${}_R \text{Inj}$ (resp. Inj_R)	the class of injective left (resp. right) R -modules
$\text{Prod } \mathcal{F}$	the class of modules that are direct summands of direct products of modules from a class \mathcal{F}
$\text{Add } \mathcal{F}$ (resp. $\text{add } \mathcal{F}$)	the class of modules that are direct summands of (resp. finite) direct sums of modules from a class \mathcal{F}
$R\text{-inj}$	the class of indecomposable injective left R -modules
${}_R \text{pinj}$	the class of indecomposable pure-injective left R -modules
${}_R \text{Pinj}$ (resp. Pinj_R)	the class of pure-injective left (resp. right) R -modules, p. 25
${}_R \mathcal{F} \text{flat}$ (resp. $\mathcal{F} \text{lat}_R$)	the class of flat left (resp. right) R -modules
${}_R \mathcal{APure}$	the class of absolutely pure left R -modules, p. 24
\mathcal{APure}_R	the class of absolutely pure right R -modules
${}_R \text{Pproj}$ (resp. ${}_R \text{Proj}$)	the class of pure-projective (resp. projective) left R -modules

Notation	Description
$S\text{-Pinj}$ (resp. $S\text{-Pproj}$)	the class of S -pure-injective (resp. projective) left R -modules where $S \subseteq R\text{-mod}$, p. 27
${}_R\mathcal{RD}\text{-Inj}$	the class of RD -injective left R -modules, p. 27
${}_R\mathcal{RD}\text{-Proj}$	the class of RD -projective left R -modules, p. 27
${}_R\mathcal{RD}\text{-Flat}$	the class of RD -flat left R -modules, p. 79
$\mathcal{RD}\text{-Coflat}_R$	the class of RD -coflat right R -modules, p. 94
$M \otimes_R N$	the tensor product over R of M_R and ${}_R N$
$\mathcal{D}(S)$	the elementary dual of the definable class S , p. 29
$D(M)$	an Auslander-Bridger dual of M , p. 43
$\text{Pinj } \mathcal{F}$	the class of pure-injective modules in \mathcal{F}
\mathcal{F}^+	the class of modules that are pure submodules of modules in \mathcal{F}
\mathcal{F}^\ominus	the class $\{M \mid M^* \in \mathcal{F}\}$
$R\text{-ind}$	the class of finitely presented indecomposable left R -modules
$\text{ind}(M)$	the class of (isomorphism types of) indecomposable direct summands of M , where $M \in R\text{-mod}$
$\text{ind}(S)$	the class $\bigcup_{M \in S} \text{ind}(M)$, where $S \subseteq R\text{-mod}$
$\text{fsc}(T)$ (resp. \overline{T})	the closure of the class T of modules in the full support (resp. Ziegler) topology, p. 56
$\langle \mathcal{X} \rangle$	the definable subcategory generated by the class \mathcal{X} of modules, p. 29
$S[\infty]$ (resp. \hat{S})	the Prüfer (resp. adic) R -module corresponding to a simple regular R -module S , p. 55
${}_R\mathbf{I}$ (resp. ${}_R\mathbf{P}$, resp. ${}_R\mathbf{R}$)	the class of indecomposable preinjective (resp. preprojective, resp. regular) left R -modules, p. 79
T_S	the class $\{M \mid M \text{ is an indecomposable regular left } R\text{-module with } \text{Hom}_R(M, S) \neq 0\}$ where S is a simple regular left R -module, p. 58

Notation	Description
\mathcal{P}_S	the class of S -pure short exact sequences of left R -modules where $S \subseteq R\text{-mod}$, p.27
$\text{Ext}_R^n(-, -)$	the n th cohomology group derived from $\text{Hom}_R(-, -)$ using right (Inj_R) -resolutions
$\text{Tor}_1^R(-, -)$	the 1st homology group derived from the tensor product over R using left (Proj_R) -resolutions
$\text{Axt}_R^n(-, -)$	the n th cohomology group derived from $\text{Hom}_R(-, -)$ using right (\mathcal{APure}_R) -resolutions, p.108
$\text{Pext}_S^n(-, -)$	the n th cohomology group derived from $\text{Hom}_R(-, -)$ using right $(S\text{-Pinj})$ -resolutions where $S \subseteq \text{mod-}R$, p.109
$\text{Tor}_n^S(-, -)$	the n th homology group derived from the tensor product over R using left $(S\text{-Pproj})$ -resolutions, p.144
$\text{Axt}_S^n(-, -)$	the n th cohomology group derived from $\text{Hom}_R(-, -)$ using right $((S\text{-Pinj})^+)$ -resolutions, p.109
${}^\perp \mathcal{F}$	the class $\{M \mid \text{Ext}_R^1(M, \mathcal{F}) = 0\}$
\mathcal{F}^\perp	the class $\{M \mid \text{Ext}_R^1(\mathcal{F}, M) = 0\}$
${}^\perp_S \mathcal{F}$	the class $\{M \mid \text{Pext}_S^1(M, \mathcal{F}) = 0\}$
\mathcal{F}^\perp_S	the class $\{M \mid \text{Pext}_S^1(\mathcal{F}, M) = 0\}$

Chapter 1

Introduction

Cohn in [13] introduced the concept of purity for modules over general rings. Many relative versions of purity have been considered since then (see for example, [1], [14], [39], [56], [67] and [79]). More generally, let S be a class of left R -modules. Following Warfield [75], an exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ of left R -modules is said to be S -pure if the sequence $0 \rightarrow \text{Hom}_R(M, A) \rightarrow \text{Hom}_R(M, B) \rightarrow \text{Hom}_R(M, C) \rightarrow 0$ is exact, for all $M \in S$. A module M is said to be S -pure-injective (resp. S -pure-projective), if M is injective (resp. projective) relative to every S -pure exact sequence of modules.

The aim of this thesis is to investigate S -purities which are determined by classes S of finitely presented modules.

Chapter 2 contains background material used throughout the thesis.

In chapter 3, the new material begins. First, we present a number of characterizations and properties of S -pure-exact sequences and of the associated classes of relatively projective and relatively injective modules.

In [56] purity and S -purity are compared. In particular, it is proved that S -purity and purity are equivalent if and only if S -pure-injectivity and pure-injectivity are equivalent if and only if $R\text{-mod} \subseteq \text{add}(S \cup \{R\})$ [56, Theorem 2.5, p. 2136]. Extending this result, we compare S -purity and T -purity for arbitrary classes S and T of finitely presented left R -modules. We show that there is a natural ordering on purities for left modules given by T -purity \leq S -purity if every T -pure exact sequence is S -pure exact. In Theorem 3.2.1 we give various characterizations of this relation, for example this condition is equivalent to $S \subseteq \text{add}(T \cup \{R\})$. We also show the relation between the purity for left modules which is determined by S and the purity for right modules determined by S ; this is said most directly

in terms of the matrices presenting the modules in S .

Al-Kawarit and Cauchot [1] gave conditions in the context of commutative rings under which purities determined by matrices of certain sizes are different. We obtain related results over semiperfect rings, specifically in Theorem 3.3.6 we give a generalization of [1, Theorem 3.5(1), p. 3888] in which we prove the following. Suppose that (n, m) and (r, s) are any two pairs of positive integers such that $n \neq r$ and that one of the following two conditions is satisfied: (a) R is semiperfect and there exists an ideal I of R with $\text{gen}(I_R) = \max\{n, r\}$ and $I \subseteq e_j R$ for some local idempotent e_j ; (b) R is Krull-Schmidt and there exists a right ideal I of R with $\text{gen}(I) = \max\{n, r\}$ and $I \subseteq e_j R$ for some local idempotent e_j . Then: (1) (m, n) -purity and (s, r) -purity of short exact sequences of left R -modules are not equivalent; (2) (n, m) -purity and (r, s) -purity of short exact sequences of right R -modules are not equivalent.

In section 3.4, we study purity over finite-dimensional algebras and we consider in detail the question: what are conditions under which purities determined by matrices of certain sizes are different over such algebras?. Firstly, we compare purities over the Kronecker algebra over an algebraically closed field k . In Proposition 3.4.4 we prove that if R is a finite-dimensional algebra over a field k and it is not of finite representation type, then for every $r \in \mathbb{Z}^+$, there is $n > r$ such that (\mathfrak{K}_0, n) -purity \neq (\mathfrak{K}_0, r) -purity for left R -modules. In Theorem 3.4.6 we give a description of the S -pure-injective modules in terms of the type-definable category generated by τS where τ is Auslander-Reiten translate. Let \mathcal{H} be a set of matrices over a tame hereditary finite-dimensional algebra R over a field k . Conditions under which the generic module is $L_{\mathcal{H}}$ -pure-injective are given in Proposition 3.4.10. Also, over such algebra, a characterization of when the S -adic module (where S is a simple regular left R -module) belongs to the full support topology closure of a fixed class of indecomposable regular modules is given in Theorem 3.4.19. Finally, we give a complete description of the full support topology closure of any class of indecomposable finite-dimensional modules over a tame hereditary finite-dimensional algebra R over a field k . Also, this result gives a complete description of the indecomposable S -pure-injective modules over such algebra.

In the last section of chapter 3 we give a condition on a left R -module M for every S -pure submodule of M to be a direct summand and prove that such a module is a direct sum of indecomposable submodules. As a corollary of this result we give a number of characterizations of rings whose indecomposable R -modules are S -pure-projective.

In chapter 4, we consider definability of the classes $\varinjlim S$ (the class of direct limits of direct systems of modules in S) and $\text{Prod} S^*$ (the class of modules that are direct summands of direct products of modules from the class S^* of modules dual to those in S), where $S = \text{add} S \subseteq R\text{-mod}$. First, we introduce the concept of almost dual pairs. An almost dual pair over R is a pair $(\mathcal{F}, \mathcal{G})$, where \mathcal{F} (resp. \mathcal{G}) is a class of left (resp. right) R -modules such for any left R -module M , $M \in \mathcal{F}$ if and only if the E -dual module $M^* = \text{Hom}_K(M, E) \in \mathcal{G}$, and \mathcal{G} is closed under direct summands and direct products. We show that the definition of an almost dual pair is independent of the duality used and give some examples and properties of these concepts. Some natural bijections induced by almost dual pairs are given. In Theorem 4.2.13 we show that there are natural bijections between the following: classes of pure-injective right R -modules closed under direct summands and products; almost dual pairs $(\mathcal{F}, \mathcal{G})$ in which \mathcal{G} is a class of pure-injective right R -modules; almost dual pairs $(\mathcal{F}, \mathcal{G})$ in which \mathcal{G} is closed under pure submodules and pure-injective hulls. Let $S = \text{add} S \subseteq R\text{-mod}$. Several authors (see, e.g. [36], [4] and [5]) studied properties of $\varinjlim S$. Holm [26] considered the class $\text{Prod} S^*$ and he proved in [26, Theorem 1.4, p. 545] that a module M is in $\varinjlim S$ if and only if $M^* = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is in $\text{Prod} S^*$. Thus $(\varinjlim S, \text{Prod} S^*)$ is an almost dual pair. Let $S, T \subseteq R\text{-mod}$. As a generalization of Holm's theorem, we introduce in Theorem 4.2.24 types of almost dual pair defined in terms of conditions on S and T which cover many well known examples of almost dual pairs. In Proposition 4.2.34, we compare two purities by using the class $\varinjlim(\text{add} S)$.

In section 4.3, we get the first main aim of this chapter, which is to obtain new results on definable classes by using duality of modules, where a subclass \mathcal{X} of $R\text{-Mod}$ is said to be definable if it is closed under direct products, direct limits and pure submodules. Throughout this paragraph let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. In Proposition 4.3.1, we show that the class \mathcal{F} is definable if and only if $\mathcal{F}^{**} \subseteq \mathcal{F}$. Also, in Theorem 4.3.2, we get several equivalent characterizations of definability of \mathcal{G}^+ in terms of dual modules where $\mathcal{G}^+ = \{M \mid M \text{ is a pure submodule of a module in } \mathcal{G}\}$. For example, we prove that \mathcal{G}^+ is definable if and only if $\mathcal{G}^* \subseteq \mathcal{F}$ if and only if $(\mathcal{G}^+)^{**} \subseteq \mathcal{G}^+$. In Proposition 4.3.7, we give the relation between definability of \mathcal{F} and definability of \mathcal{G}^+ . For example, we show that if \mathcal{G} is closed under pure-injective hulls, then (1) \mathcal{G}^+ is definable \Leftrightarrow (2) \mathcal{F} is definable and if $M \in \mathcal{G}$ then M is a pure submodule of N^* for some $N^* \in \mathcal{G} \Leftrightarrow$ (3) \mathcal{F} is definable and $\text{Prod} \mathcal{F}^* = \text{Pinj} \mathcal{G}$ where $\text{Pinj} \mathcal{G}$ is the class of pure-injective modules in \mathcal{G} . Proposition 4.3.8

shows that the class \mathcal{G} is definable if and only if $(\mathcal{G}, \mathcal{F})$ is an almost dual pair. At the end of this section we show that if \mathcal{G}^+ is definable, then $\mathcal{D}(\mathcal{G}^+) = \mathcal{F}$ and $\mathcal{D}(\mathcal{F}) = \mathcal{G}^+$, where \mathcal{D} is the elementary dual and this is a generalization of Theorem 4.4(2) in [61, p. 17].

Let $S \subseteq R\text{-mod}$. In the literature, many characterizations of definability of $\varinjlim(\text{add}S)$ have been given but without using the duality of modules (see, e.g. [15, Theorem 4.2, p. 1665] and [32, Proposition 3.11, p. 27]). The second main aim of chapter 4 is to obtain new results on definability of the class $\varinjlim(\text{add}S)$ by using duality of modules and to show that the converse of theorem of Holm, [26, Theorem 5.6, p. 556], (see Theorem 4.4.3) is true too. We achieve this aim in Theorem 4.4.1 by taking $(\mathcal{F}, \mathcal{G})$ to be the almost dual pair $(\varinjlim(\text{add}S), \text{Prod}S^*)$ and applying some results in section 4.3. In this theorem, we prove for example that $\varinjlim(\text{add}S)$ is definable if and only if $(\text{Prod}S^*)^+$ is definable if and only if $(\text{Prod}S^*)^* \subseteq \varinjlim(\text{add}S)$ if and only if $(\text{Prod}S^*)^+$ is closed under pure homomorphic images if and only if $\mathcal{D}((\text{Prod}S^*)^+) = \varinjlim(\text{add}S) = \langle S \rangle$, where $\langle S \rangle$ is the definable subcategory generated by S . Theorem 4.4.1 is a generalization of [71, Theorem 3.2, p. 325], [80, 1.6, p. 383] and [12, Theorem 1, p. 176]. In Theorem 4.4.11, we give equivalent characterizations of definability of any class of modules \mathcal{F} in terms of the class \mathcal{F}^\ominus , where $\mathcal{F}^\ominus = \{M \mid M^* \in \mathcal{F}\}$ and as a corollary of this result we get equivalent characterizations of definability of $\varinjlim(\text{add}S)$ in terms of the class $(\varinjlim(\text{add}S))^\ominus$. Also, Proposition 4.4.15 gives an equivalent characterization of definability of $\varinjlim(\text{add}S)$ in terms of the class $(\varinjlim(\text{add}S))^\ominus$. Let $S \subseteq R\text{-mod}$ be such that $S = \text{add}S$. In Corollary 4.4.23, we give an equivalent characterization of definability of the class $\varinjlim S$ in terms of the duality between $(\text{Prod}S^*)^+$ -preenvelopes (or $(\text{Prod}S^*)$ -preenvelopes) and $(\varinjlim S)$ -precovers.

We end this chapter by considering definability of the class $\text{Prod}S^*$, where $S \subseteq R\text{-mod}$. Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair such that $\mathcal{G} \subseteq \text{Pinj}_R$ (in particular, $\mathcal{G} = \text{Prod}S^*$ where $S \subseteq R\text{-mod}$). Several characterizations of definability of \mathcal{G} are given in Theorem 4.5.1 and Theorem 4.5.4. For example, we prove that (1) \mathcal{G} is definable \Leftrightarrow (2) \mathcal{G} is closed under direct sums \Leftrightarrow (3) $\mathcal{G} = \mathcal{G}^+$ \Leftrightarrow (4) $(\mathcal{G}, \mathcal{F})$ is an almost dual pair \Leftrightarrow (5) $M \in \mathcal{G}$ if and only if $M^{**} \in \mathcal{G}$ \Leftrightarrow (6) $M \in \mathcal{G}$ if and only if $M^* \in \mathcal{F}$. Theorem 4.5.1 (resp. Theorem 4.5.4) is a generalization of [19, Theorem 5.4.1, p. 120], [43, Theorem 3, p. 564] and [26, Theorem 1.3] (resp. [12, Theorem 2, p. 176]). In Corollary 4.5.5, we achieve another aim in this chapter, that is, to prove Holm's theorem, [26, Theorem 1.5, p. 545], (see Theorem 4.1.1) without his assumptions (${}_R R \in S$ and $\varinjlim(\text{add}S)$ is definable). In Theorem 4.5.10, we characterize

the definability of the class $\text{Prod}S^*$ in terms of preenvelopes and precovers. We prove for example that $\text{Prod}S^*$ is definable if and only if a pure monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a $(\text{Prod}S^*)$ -preenvelope of M if and only if $\varphi^* : X^* \rightarrow M^*$ is a $(\varinjlim(\text{add}S))$ -precover of M^* in $R\text{-Mod}$. The final main purpose in this chapter is to generalize theorem of Pinzon [50, Theorem 5.4, p. 51] (see Theorem 4.5.11) from definability of the class Inj_R (which is equivalent to R being right noetherian) to definability of the class $S\text{-Pinj}$, where $S \subseteq \text{mod-}R$. This is in Theorem 4.5.16.

In chapter 5, we investigate the notion of cotorsion pairs relative to the proper class of S -pure short exact sequences of left R -modules (in short, S -cotorsion pairs), where $S \subseteq R\text{-mod}$. They are defined as complete orthogonal classes with respect to the functor Pext_S^1 which is defined in subsection 4.5.2. This notion was considered by Hovey in [29] for any proper class of short exact sequences. In section 5.2, we induce S -cotorsion pairs by using S' -Pinj, where $S, S' \subseteq R\text{-mod}$. For example in Theorem 5.2.7, we consider S -cotorsion pairs induced by $S\text{-Pinj}$ and we give equivalent characterizations of a ring over which every module in $(S\text{-Pinj})^+$ is S -pure-projective. In Theorem 5.2.10, we give equivalent characterizations of definability of the class $S\text{-Pinj}$ in terms of S -cotorsion pairs and $(S\text{-Pinj})^+$ -preenvelopes. We prove for example that $S\text{-Pinj}$ is definable if and only if $({}^{\perp S}((S\text{-Pinj})^+), S\text{-Pinj})$ is an S -cotorsion pair if and only if every $(S\text{-Pinj})$ -preenvelope of a left R -module M is a $(S\text{-Pinj})^+$ -preenvelope of M . Proposition 5.2.11 provides characterizations of the equivalence of two purities by using relative cotorsion pairs. We prove for example that S -purity = S' -purity for short exact sequences of left R -modules if and only if $(R\text{-Mod}, S\text{-Pinj})$ is an S' -cotorsion pair, where $S, S' \subseteq R\text{-mod}$. The main purpose of section 5.3 is to generalize the results of Holm and Jørgensen [27, Theorem 3.4, p. 697] (see Theorem 5.3.1) and [28, Theorem 3.1(c), p. 629] (see Theorem 5.3.2) to the case of S -cotorsion pairs. This is in Theorem 5.3.11 and Corollary 5.3.12 respectively. In the final section we provide some examples and applications of some results in this chapter. For example, in Example 5.4.2 we show that if $S = \text{add}S \subseteq R\text{-mod}$ contains $\{{}_R R\}$, then by using Corollary 4.2.25 and Corollary 5.3.12 we have that $(\varinjlim S, (\varinjlim S)^{\perp S})$ is a perfect S -cotorsion pair. In particular, $(\varinjlim S)^{\perp S}$ is enveloping in $R\text{-Mod}$.

Chapter 2

Background

In this chapter we give the background material used throughout the thesis. For other basic definitions, results and notations, we refer the reader to [6], [7], [19], [52], [62] and [79] as background references.

2.1 Homological algebra

Background material for this section can be found in [79].

A sequence $\cdots \rightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots$ (infinite or finite) of left R -modules connected by R -homomorphisms is said to be **exact at M_i** if $\ker(f_i) = \text{im}(f_{i-1})$, and is **exact** if it is exact at each M_i (except at the ends).

An exact sequence of left R -modules of the form $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is called a **short exact sequence** or an **extension** of C by A .

Recall that a monomorphism $f \in \text{Hom}_R(A, B)$ is said to be a **split monomorphism** if there exists $h \in \text{Hom}_R(B, A)$ such that $hf = 1_A$. An epimorphism $g \in \text{Hom}_R(B, C)$ is said to be a **split epimorphism** if there exists $h \in \text{Hom}_R(C, B)$ such that $gh = 1_C$.

Lemma 2.1.1 [79, 8.3, p. 58] For a short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ of left R -modules the following statements are equivalent:

- (1) f is a split monomorphism;
- (2) g is a split epimorphism;
- (3) $\text{im}(f)(= \ker(g))$ is a direct summand in B .

Recall that a short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ of left R -modules which satisfies

the equivalent conditions of Lemma 2.1.1 is said to be **split**.

Lemma 2.1.2 [62, Proposition 2.28, p. 52] If an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of left R -modules is split, then $B \simeq A \oplus C$.

Lemma 2.1.3 Given a commutative diagram with exact rows,

$$\begin{array}{ccccccc} M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ f \downarrow & & g \downarrow & & h \downarrow & & \\ M' & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0, \end{array}$$

there exists a unique R -homomorphism $h : L \rightarrow L'$ making the augmented diagram commute. Moreover, h is an isomorphism if f and g are isomorphisms.

PROOF: See [62, Proposition 2.70, p. 89]. □

Lemma 2.1.4 (Homotopy Lemma) Given a commutative diagram with exact rows,

$$\begin{array}{ccccccc} M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \longrightarrow & 0 \\ \varphi_1 \downarrow & & \varphi_2 \downarrow & & \varphi_3 \downarrow & & \\ 0 \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & . \end{array}$$

Then the following statements are equivalent:

- (1) there exists $\alpha : M_3 \rightarrow N_2$ with $g_2 \alpha = \varphi_3$;
- (2) there exists $\beta : M_2 \rightarrow N_1$ with $\beta f_1 = \varphi_1$.

PROOF: See [79, Lemma 7.16(4), p. 53]. □

Lemma 2.1.5 (see [72, Exercise 5, p. 44]) Let $\{M_i\}_{i \in I}$ be a class of left R -modules and let N be any left R -module. Then:

- (1) $\text{Hom}_R(N, \prod_{i \in I} M_i) \simeq \prod_{i \in I} \text{Hom}_R(N, M_i)$
- (2) $\text{Hom}_R(\bigoplus_{i \in I} M_i, N) \simeq \prod_{i \in I} \text{Hom}_R(M_i, N)$

Lemma 2.1.6 (see, e.g. [62, Theorem 2.75, p. 92]) Let R and K be rings, A a right R -module, B an (R, K) -bimodule, and C a right K -module. Then:

$$\text{Hom}_K(A \otimes_R B, C) \simeq \text{Hom}_R(A, \text{Hom}_K(B, C)).$$

Lemma 2.1.7 (see [22, Lemma 6.1, p. 33]) In the tensor product $A \otimes_R B$ of the R -modules A, B , a relation $\sum_{i=1}^n (a_i \otimes b_i) = 0$ ($a_i \in A, b_i \in B$) holds if and only if there exist elements $c_j \in B$ ($j = 1, \dots, m$) and $r_{ij} \in R$ ($i = 1, \dots, n; j = 1, \dots, m$) such that $b_i = \sum_{j=1}^m r_{ij} c_j$ for all i and $\sum_{i=1}^n a_i r_{ij} = 0$ for all j .

2.1.1 Injective and projective modules

Let $\Sigma : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of left R -modules. A left R -module M is said to be **injective relative to** Σ if the sequence $0 \rightarrow \text{Hom}_R(C, M) \rightarrow \text{Hom}_R(B, M) \rightarrow \text{Hom}_R(A, M) \rightarrow 0$ is exact. Dually, a module M is said to be **projective relative to** Σ if the sequence $0 \rightarrow \text{Hom}_R(M, A) \rightarrow \text{Hom}_R(M, B) \rightarrow \text{Hom}_R(M, C) \rightarrow 0$ is exact.

A left R -module M is said to be **injective** (resp. **projective**) if it is injective (resp. projective) relative to every short exact sequence of left R -modules. We will use ${}_R\text{Inj}$ (resp. ${}_R\text{Proj}$) to denote the class of injective (resp. projective) left R -modules.

A module is said to be **finitely presented** if it is the factor module of a free module of rank n modulo a m -generated submodule, for some $n, m \in \mathbb{Z}^+$.

We will use $R\text{-Mod}$ (resp. $\text{Mod-}R$) to denote the class of left (resp. right) R -modules. Also, we use $R\text{-mod}$ (resp. $\text{mod-}R$) to denote the class of finitely presented left (resp. right) R -modules.

Lemma 2.1.8 (see, e.g. [72, Lemma 13.2, p. 42]) A right R -module M is finitely presented if and only if $\varphi : M \otimes_R \left(\prod_{i \in I} B_i \right) \rightarrow \prod_{i \in I} (M \otimes_R B_i)$ is an isomorphism for every family $\{B_i\}_{i \in I}$ of left R -modules, where φ is defined by $\varphi(a \otimes (b_i)_{i \in I}) = (a \otimes b_i)_{i \in I}$ for every $a \in M$ and $(b_i)_{i \in I} \in \prod_{i \in I} B_i$.

Lemma 2.1.9 (see, e.g. [19, Theorem 3.2.11, p. 78]) Let R and K be rings, A a finitely presented left R -module, B an (R, K) -bimodule, and E an injective right K -module. Then: $\text{Hom}_K(B, E) \otimes_R A \simeq \text{Hom}_K(\text{Hom}_R(A, B), E)$.

Recall (see, e.g. [19, 3.2.7, p. 77]) that an injective R -module E is said to be an **injective cogenerator** for $R\text{-Mod}$ if $\text{Hom}_R(M, E) \neq 0$ for any R -module $M \neq 0$.

Lemma 2.1.10 (see, e.g. [19, Lemma 3.2.8, p. 77]) Let R and K be rings and E be an injective cogenerator for K -modules. Then a sequence $0 \rightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \rightarrow 0$ of (R, K) -bimodules

is exact if and only if the sequence $0 \rightarrow \text{Hom}_K(C, E) \xrightarrow{\psi^*} \text{Hom}_K(B, E) \xrightarrow{\varphi^*} \text{Hom}_K(A, E) \rightarrow 0$ of right R -modules is exact.

An **injective resolution** of a left R -module M is an exact sequence $\Sigma : 0 \rightarrow M \xrightarrow{\alpha_{-1}} I_0 \xrightarrow{\alpha_0} I_1 \xrightarrow{\alpha_1} I_2 \rightarrow \dots$ in which each I_n is injective. Dually, a **projective resolution** of a left R -module M is an exact sequence $\Sigma : \dots \rightarrow P_2 \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \rightarrow M \rightarrow 0$ in which each P_n is projective.

2.1.2 $\text{Ext}_R^n(-, -)$ and $\text{Tor}_n^R(-, -)$

By a **chain complex** Σ of R -modules we mean a sequence

$$\Sigma : \dots \rightarrow C_2 \xrightarrow{\alpha_2} C_1 \xrightarrow{\alpha_1} C_0 \xrightarrow{\alpha_0} C_{-1} \xrightarrow{\alpha_{-1}} C_{-2} \xrightarrow{\alpha_{-2}} C_{-3} \rightarrow \dots$$

of R -modules and R -homomorphisms such that $\alpha_{n-1} \circ \alpha_n = 0$, for all $n \in \mathbb{Z}$.

Similarly, by a **cochain complex** Σ of R -modules we mean a sequence

$$\Sigma : \dots \rightarrow C_{-2} \xrightarrow{\alpha_{-2}} C_{-1} \xrightarrow{\alpha_{-1}} C_0 \xrightarrow{\alpha_0} C_1 \xrightarrow{\alpha_1} C_2 \xrightarrow{\alpha_2} C_3 \rightarrow \dots$$

of R -modules and R -homomorphisms such that $\alpha_n \circ \alpha_{n-1} = 0$, for all $n \in \mathbb{Z}$.

Let $M, N \in \text{Mod-}R$ and let $n \geq 0$. We can define $\text{Ext}_R^n(N, M)$ as follows: take an injective resolution of M : ($\Sigma : 0 \rightarrow M \xrightarrow{\alpha} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \rightarrow \dots$). Let Σ_0 be the deleted injective resolution of M (i.e., the cochain complex $\Sigma_0 : 0 \rightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \rightarrow \dots$).

Applying the functor $\text{Hom}_R(N, -)$ on Σ_0 , we obtain a cochain complex:

$$0 \rightarrow \text{Hom}_R(N, I_0) \xrightarrow{d_0^{\otimes}} \text{Hom}_R(N, I_1) \xrightarrow{d_1^{\otimes}} \text{Hom}_R(N, I_2) \rightarrow \dots,$$

where $d_n^{\otimes} : \text{Hom}_R(N, I_n) \rightarrow \text{Hom}_R(N, I_{n+1})$ is defined by $d_n^{\otimes}(f) = d_n f$, for all $f \in \text{Hom}_R(N, I_n)$.

Define $\text{Ext}_R^n(N, M)$ to be the cohomology group:

$$\text{Ext}_R^n(N, M) = H^n(\text{Hom}_R(N, \Sigma_0)) = (\ker d_n^{\otimes}) / (\text{im } d_{n-1}^{\otimes}), \quad n = 0, 1, 2, \dots, \text{ where } d_{-1}^{\otimes} = 0.$$

Also we can define $\text{Tor}_n^R(-, -)$ as follows: let $M \in R\text{-Mod}$, let $N \in \text{Mod-}R$ and let $n \in \mathbb{Z}^+$. Let $\Sigma : \dots \rightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\alpha} N \rightarrow 0$ be a projective resolution of N and let Σ_0 be the deleted projective resolution of N (i.e., the chain complex $\Sigma_0 : \dots \rightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0$). Then we have a chain complex: $\dots \rightarrow F_2 \otimes_R M \xrightarrow{d_2 \otimes_R 1_M} F_1 \otimes_R M \xrightarrow{d_1 \otimes_R 1_M} F_0 \otimes_R M \rightarrow 0$.

Define $\text{Tor}_n^R(N, M)$ to be the homology group:

$$\text{Tor}_n^R(N, M) = H_n(\Sigma_0 \otimes_R M) = \ker(d_n \otimes_R 1_M) / \text{im}(d_{n+1} \otimes_R 1_M), \quad n = 1, 2, \dots$$

2.1.3 Direct limits of modules

A partially ordered set (I, \leq) is said to be **directed** if it is non-empty and for any two elements $i, j \in I$, there exists (at least one) $k \in I$ with $i \leq k$ and $j \leq k$.

Let (I, \leq) be a directed set. A **direct system** of left R -modules $(M_i, f_{ij})_I$ consists of

- (1) a family of left R -modules $\{M_i\}_{i \in I}$ and
- (2) a family of R -homomorphisms $f_{ij} : M_j \rightarrow M_i$ for all pairs (i, j) with $j \leq i$, satisfying $f_{ii} = 1_{M_i}$ and $f_{kj}f_{ji} = f_{ki}$ for $i \leq j \leq k$.

A **direct system of homomorphisms** from $(M_i, f_{ij})_I$ into an R -module M is a family of homomorphisms $\{\alpha_i : M_i \rightarrow M\}_{i \in I}$ with $\alpha_j f_{ji} = \alpha_i$ whenever $i \leq j$.

Definition 2.1.11 (see, e.g. [79, 24.1, p. 197]) The **direct limit** of a direct system $(M_i, f_{ij})_I$ of left R -modules is a direct system of homomorphisms $\{g_i : M_i \rightarrow M\}_I$ where $M \in R\text{-Mod}$ is such that if $\{h_i : M_i \rightarrow N\}_I$ is another direct system of homomorphisms with $N \in R\text{-Mod}$, then there is a unique R -homomorphism $f : M \rightarrow N$ such that $f g_i = h_i$ for all $i \in I$.

If $\{g'_i : M_i \rightarrow M'\}_I$ is another direct limit of $(M_i, f_{ij})_I$, then by definition there is a homomorphism $h : M \rightarrow M'$ which is easily seen to be an isomorphism with $h g_i = g'_i$ for $i \in I$. Hence M is uniquely determined up to isomorphism. The direct limit $\{g_i : M_i \rightarrow M\}_I$ is denoted by $\varinjlim M_i$.

Let $\mathcal{F} \subseteq R\text{-Mod}$. We will use $\varinjlim \mathcal{F}$ to denote the class of direct limits of direct systems of modules in \mathcal{F} .

Lemma 2.1.12 (see [65, Lemma 2, p. 454]) Every module is a direct limit of finitely presented modules.

2.2 Pure exact sequences and derived notions

Background material for this section can be found in [52] and [79].

The concept of purity plays an important role in our analysis of a module category. It goes back to Cohn [13] and has been further developed by various mathematicians. In this section we collect the basic facts about this concept.

2.2.1 Pure exact sequences

A short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ of left R -modules is said to be **pure** if the sequence $0 \rightarrow \text{Hom}_R(M, A) \rightarrow \text{Hom}_R(M, B) \rightarrow \text{Hom}_R(M, C) \rightarrow 0$ is exact, for every finitely presented left R -module M ; in this case f is said to be a **pure monomorphism** and g is said to be a **pure epimorphism**. A submodule A of a left R -module B is a **pure submodule** of B if the canonical exact sequence $0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} B/A \rightarrow 0$ is pure.

Theorem 2.2.1 (see [20, Theorem 1.27, p. 15]) Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of left R -modules. The following statements are equivalent.

- (1) The sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is pure.
- (2) For every finitely presented right R -module M the induced sequence of abelian groups $0 \rightarrow M \otimes_R A \rightarrow M \otimes_R B \rightarrow M \otimes_R C \rightarrow 0$ is exact.
- (3) For every right R -module M the induced sequence of abelian groups $0 \rightarrow M \otimes_R A \rightarrow M \otimes_R B \rightarrow M \otimes_R C \rightarrow 0$ is exact.

Lemma 2.2.2 (see [52, Lemma 2.1.2, p. 44]) (1) If A is a direct summand of B , then the embedding of A into B is pure.

- (2) Any direct product of pure monomorphisms is a pure monomorphism.

In this work, the notation M^* is used to denote the E -dual of an R -module M (i.e., $M^* = \text{Hom}_K(M, E)$), where R is an algebra over a commutative ring K and E is an injective cogenerator for $K\text{-Mod}$. If we apply $*$ over all left R -modules, then we say that $*$ is the duality on $R\text{-Mod}$ induced by ${}_K E$. If R is an algebra over a field k , then M^* will denote the k -dual, $\text{Hom}_k(M, k)$. Also, we use M^* to denote the character module $M^* = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ (i.e., the \mathbb{Q}/\mathbb{Z} -dual of M).

Lemma 2.2.3 (see [20, 1.28, p. 16]) An exact sequence of right R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is pure if and only if the exact sequence of left R -modules $0 \rightarrow C^* \rightarrow B^* \rightarrow A^* \rightarrow 0$ is split.

For every left (or right) R -module M , there is a canonical monomorphism $\delta_M : M \rightarrow M^{**}$ defined by $(\delta_M(a))(f) = f(a)$ for every $a \in M, f \in M^*$.

Lemma 2.2.4 (see, e.g. [20, 1.30, p. 17]) Let M be a left R -module. Then the canonical monomorphism $\delta_M : M \rightarrow M^{**}$ is pure.

Lemma 2.2.5 (see [36, Proposition 2.1, p. 736]) Let M be a left R -module and let $\mathcal{F} \subseteq R\text{-mod}$ such that $\mathcal{F} = \text{add } \mathcal{F}$. Then $M \in \varinjlim \mathcal{F}$ if and only if there is a pure epimorphism $\alpha : \bigoplus_{i \in I} F_i \rightarrow M$ with modules F_i from \mathcal{F} .

For the following lemma see [79, 33.9, p. 280].

Lemma 2.2.6 (1) For every family $\{M_i\}_{i \in I}$ of left R -modules, the canonical embedding $f : \bigoplus_{i \in I} M_i \rightarrow \prod_{i \in I} M_i$ is pure.

(2) For every direct system $(M_i, f_{ij})_I$ of left R -modules, the canonical epimorphism $\eta : \bigoplus_{i \in I} M_i \rightarrow \varinjlim M_i$ is pure.

Recall that a left R -module M is said to be **absolutely pure** if every embedding $M \rightarrow N$ in $R\text{-Mod}$ is pure. Let ${}_R\mathcal{APure}$ (resp. \mathcal{APure}_R) denote the class of absolutely pure left (resp. right) R -modules.

Lemma 2.2.7 (see [52, Propositions 2.3.1 and 2.3.2]) For any left R -module M the following statements are equivalent:

- (1) M is absolutely pure;
- (2) $\text{Ext}_R^1(N, M) = 0$ for every finitely presented left R -module N ;
- (3) M is a pure submodule of an injective left R -module.

A left R -module M is said to be **flat** if, for every exact sequence of right R -modules $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$, the sequence $0 \rightarrow A \otimes_R M \xrightarrow{f \otimes 1_M} B \otimes_R M \xrightarrow{g \otimes 1_M} C \otimes_R M \rightarrow 0$ is exact. We will use ${}_R\mathcal{Flat}$ (resp. \mathcal{Flat}_R) to denote the class of flat left (resp. right) R -modules.

Lemma 2.2.8 (see [19, Theorems 2.1.8 and 3.2.10]) For any left R -module M the following statements are equivalent:

- (1) M is flat;
- (2) $\text{Tor}_1^R(N, M) = 0$ for every right R -module N ;
- (3) $\text{Tor}_1^R(N, M) = 0$ for every finitely presented right R -module N .

Lemma 2.2.9 (see [52, Proposition 2.3.14, p. 62]) A left R -module M is flat if and only if every exact sequence $0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$ of left R -modules is pure.

Lemma 2.2.10 (see [19, Theorem 3.1.9, p. 70]) A right R -module M is injective if and only if $\text{Ext}_R^1(N, M) = 0$ for all right R -modules N .

Theorem 2.2.11 (see [34, Theorem, p. 239]) A left R -module M is flat if and only if M^* is injective.

2.2.2 Pure-injective and pure-projective modules

A left R -module M is said to be **pure-injective** (resp. **pure-projective**) if it is injective (resp. projective) relative to every pure exact sequence of left R -modules. We will use ${}_R\text{Pinj}$ (resp. ${}_R\text{Pproj}$) to denote the class of pure-injective (resp. pure-projective) left R -modules.

Proposition 2.2.12 (see, e.g. [52, Proposition 4.3.29, p. 151]) If M is a left R -module, then $M^* = \text{Hom}_K(M, E)$ is a pure-injective right R -module.

Definition 2.2.13 (see, e.g. [52, 4.3.16, p. 146]) Let M be a left R -module. The **pure-injective hull** of M is a pure-injective module $PE(M)$ and a pure-embedding $j : M \rightarrow PE(M)$ such that if fj is a pure embedding for some $f \in \text{Hom}_R(PE(M), N)$ then f also must be a pure embedding.

Theorem 2.2.14 (see, [52, Theorem 4.3.18, p. 147]) Every left R -module M has a pure-injective hull $j : M \rightarrow PE(M)$ which is unique to isomorphism over M : if $g : M \rightarrow N$ is any second pure injective hull of M , there exists an isomorphism $f : PE(M) \rightarrow N$ such that $fj = g$.

Recall that a left R -module M is said to be **Σ -pure-injective** if the direct sum $M^{(I)}$ of copies of M is pure-injective.

A ring R is said to be left **pure-semisimple** if every left R -module is pure-injective, in which case every left R -module is, by the definition, Σ -pure-injective.

All classes of modules in this thesis will be assumed to be closed under isomorphism.

Definition 2.2.15 A class \mathcal{F} of left R -modules is said to be **closed under**:

(1) direct summands (resp. pure submodules) if, given $M \in \mathcal{F}$, then every direct summand (resp. pure submodule) of M is in \mathcal{F} .

(2) direct products (direct sums) if, whenever $\{M_i\}_{i \in I}$ is a subclass of \mathcal{F} , then $\prod_{i \in I} M_i \in \mathcal{F}$ (resp. $\bigoplus_{i \in I} M_i \in \mathcal{F}$).

(3) direct limits if, whenever $(M_i, f_{ij})_I$ is a direct system of modules in \mathcal{F} , then $\lim_{\rightarrow} M_i \in \mathcal{F}$.

(4) pure quotient modules (or pure homomorphic images) if, given $M \in \mathcal{F}$, then every pure homomorphic image of M is in \mathcal{F} .

(5) extensions (resp. pure extensions) if, whenever $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a (resp. pure) short exact sequence in which L and N are in \mathcal{F} , then $M \in \mathcal{F}$.

(6) pure-injective hulls if the pure-injective hull of every module in \mathcal{F} is also in \mathcal{F} .

2.2.3 S -purity

Let S be a class of left R -modules. Following Warfield [75], an exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ of left R -modules is said to be S -**pure** if the sequence $0 \rightarrow \text{Hom}_R(M, A) \rightarrow \text{Hom}_R(M, B) \rightarrow \text{Hom}_R(M, C) \rightarrow 0$ is exact, for all $M \in S$; in this case f is said to be an S -**pure monomorphism** and g is said to be an S -**pure epimorphism**. Note that S -pure = $S \cup \{R\}$ -pure. If $S = R\text{-mod}$ then a short exact sequence of modules is S -pure if and only if it is pure.

Lemma 2.2.16 (see [79, 33.2, p. 275]) Let $S \subseteq R\text{-mod}$ and let $f : K \rightarrow L$, $g : L \rightarrow N$ be R -homomorphisms.

- (1) (i) If f and g are S -pure epimorphisms, then gf is also a S -pure epimorphism.
(ii) If gf is a S -pure epimorphism, then g is a S -pure epimorphism.
- (2) (i) If f and g are S -pure monomorphisms, then gf is a S -pure monomorphism.
(ii) If gf is a S -pure monomorphism, then f is a S -pure monomorphism.

Let \mathcal{P} be a class of short exact sequences of left R -modules. If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ belongs to \mathcal{P} , then f is called a \mathcal{P} -**monomorphism** and g a \mathcal{P} -**epimorphism**.

Recall (see e.g., [37, p. 367]) that a class \mathcal{P} of short exact sequences of left R -modules is said to be a **proper class** if it satisfies the following axioms.

- (P1) \mathcal{P} is closed under isomorphisms.
- (P2) \mathcal{P} contains all split short exact sequences of left R -modules.
- (P3) For every pair of R -homomorphisms $f : K \rightarrow L$ and $g : L \rightarrow N$ then:
 - (1) If f and g are \mathcal{P} -epimorphisms, then gf is also a \mathcal{P} -epimorphism.
 - (2) If gf is a \mathcal{P} -epimorphism, then g is a \mathcal{P} -epimorphism.

- (3) If f and g are \mathcal{P} -monomorphisms, then gf is a \mathcal{P} -monomorphism.
- (4) If gf is a \mathcal{P} -monomorphism, then f is a \mathcal{P} -monomorphism.

Let $S \subseteq R\text{-mod}$. We will use \mathcal{P}_S to denote the class of S -pure short exact sequences of left R -modules. Since \mathcal{P}_S is closed under isomorphisms and contains all split short exact sequences of left R -modules it follows from Lemma 2.2.16 that \mathcal{P}_S is a proper class.

Lemma 2.2.17 (see [79, 33.4(2), p. 277]) Let $S \subseteq R\text{-mod}$. If the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{f_1} & L_1 \\ f_2 \downarrow & & \downarrow g_1 \\ L_2 & \xrightarrow{g_2} & N \end{array}$$

is a pushout and f_1 is a S -pure monomorphism, then g_2 is also a S -pure monomorphism.

A module M is said to be **S -pure-injective** (resp. **S -pure-projective**), if M is injective (resp. projective) relative to every S -pure exact sequence of modules. We denote by $S\text{-Pinj}$ (resp. $S\text{-Pproj}$) the class of S -pure-injective (resp. S -pure-projective) left R -modules.

The class $S\text{-Pinj}$ (resp. $S\text{-Pproj}$) is closed under direct summands and direct products (resp. direct sums), see for example [79, p. 278-279].

If S is the class of $(1, 1)$ -presented left R -modules, then any S -pure exact sequence is called **RD -pure**. A left R -module M is **RD -injective** (resp. **RD -projective**) if M is injective (resp. projective) relative to every RD -pure exact sequence of left R -modules. We will use $\mathcal{RD}\text{-Inj}_R$ to denote the class of RD -injective right R -modules. Also, we will use ${}_R\mathcal{RD}\text{-Proj}$ (resp. $\mathcal{RD}\text{-Proj}_R$) to denote the class of RD -projective left (resp. right) R -modules.

Theorem 2.2.18 (see [79, 33.6, p. 278]) Let $S \subseteq R\text{-mod}$. Then a left R -module M is S -pure-projective if and only if every S -pure exact sequence $0 \rightarrow L \rightarrow N \rightarrow M \rightarrow 0$ of left R -modules splits.

Theorem 2.2.19 (see [79, 33.7, p. 279]) Let $S \subseteq R\text{-mod}$. Then a left R -module M is S -pure-injective if and only if every S -pure exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ of left R -modules splits.

2.3 (Pre)-Covers and (Pre)-envelopes

Background material on (Pre)-Covers and (Pre)-envelopes can be found in [19] and [81].

Definition 2.3.1 (see, e.g. [19, 5.1.1, p. 105]) Let $\mathcal{F} \subseteq R\text{-Mod}$ and let $M \in R\text{-Mod}$. An \mathcal{F} -**precover** of M is an R -homomorphism $f : N \rightarrow M$ with $N \in \mathcal{F}$ such that for each R -homomorphism $g : F \rightarrow M$ with $F \in \mathcal{F}$ there is a homomorphism $h : F \rightarrow N$ such that $fh = g$. This can be expressed by the diagram

$$\begin{array}{ccc} & & F \\ & \nearrow h & \downarrow g \\ N & \xrightarrow{f} & M \end{array}$$

The homomorphism $f : N \rightarrow M$ is said to be right minimal if every $h \in \text{End}_R(N)$ such that $fh = f$ is an automorphism. An \mathcal{F} -precover $f : N \rightarrow M$ of a left R -module M is said to be an \mathcal{F} -**cover** of M if f is right minimal. We say that a class \mathcal{F} is **precovering** (resp. **covering**) if every module in $R\text{-Mod}$ has an \mathcal{F} -precover (resp. \mathcal{F} -cover).

Theorem 2.3.2 (see [27, Theorem 2.5, p. 696]) Let \mathcal{F} be a class of left R -modules which is closed under pure quotient modules. Then the following statements are equivalent.

- (1) \mathcal{F} is closed under direct sums.
- (2) \mathcal{F} is precovering.
- (3) \mathcal{F} is covering.

Definition 2.3.3 (see, e.g. [19, 6.1.1., p. 129]) Let $\mathcal{F} \subseteq R\text{-Mod}$ and let $M \in R\text{-Mod}$. An \mathcal{F} -**preenvelope** of M is an R -homomorphism $f : M \rightarrow N$ with $N \in \mathcal{F}$ such that for each R -homomorphism $g : M \rightarrow F$ with $F \in \mathcal{F}$ there is a homomorphism $h : N \rightarrow F$ such that $hf = g$. This can be expressed by the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow g & & \nearrow h \\ & & F \end{array}$$

The homomorphism $f : M \rightarrow N$ is said to be left minimal if every $h \in \text{End}_R(N)$ such that $hf = f$ is an automorphism. An \mathcal{F} -preenvelope $f : M \rightarrow N$ of a left R -module M is said to be an \mathcal{F} -**envelope** of M if f is left minimal. We say that a class \mathcal{F} is **preenveloping** (resp. **enveloping**) if every module in $R\text{-Mod}$ has an \mathcal{F} -preenvelope (resp. \mathcal{F} -envelope).

Lemma 2.3.4 (see [32, 3.15, p. 29]) Let \mathcal{F} be a class of pure-injective left R -modules which is closed under direct summands. Then the following statements are equivalent.

- (1) \mathcal{F} is closed under direct products.
- (2) \mathcal{F} is preenveloping.
- (3) \mathcal{F} is enveloping.

Lemma 2.3.5 (see [57, 3.5(c), p. 904]) Let \mathcal{F} be a class of left R -modules closed under pure submodules. Then \mathcal{F} is preenveloping if and only if \mathcal{F} is closed under direct products.

2.4 Ziegler spectrum and full support topology

2.4.1 Definable subclasses

See for instance [52, Section 3.4.1] for details about definable subclasses.

Definition 2.4.1 (see [52, 3.4.7, p. 107]) A subclass \mathcal{X} of $R\text{-Mod}$ is said to be **definable** if it is closed under direct products, direct limits and pure submodules.

Theorem 2.4.2 (see [52, 3.4.8, p. 109]) Every definable subcategory of $R\text{-Mod}$ is closed in $R\text{-Mod}$ under pure-injective hulls and images of pure epimorphisms.

Definition 2.4.3 (see [52, p. 109]) Let $\mathcal{X} \subseteq R\text{-Mod}$. The **definable subcategory generated by \mathcal{X}** is the intersection of all definable subcategories containing \mathcal{X} . We use $\langle \mathcal{X} \rangle$ to denote the definable subcategory generated by the class \mathcal{X} .

Let S be a definable class of modules. We will use $\mathcal{D}(S)$ to denote the elementary dual of S . The original definition of the elementary dual uses model theory [24] but $\mathcal{D}(S)$ can be defined to be the class of pure submodules of the form M^* where $M \in S$.

Lemma 2.4.4 (see [52, 3.4.18, p. 113]) If S is any definable subcategory of $R\text{-Mod}$ then $\mathcal{D}(S)$ is definable and $\mathcal{D}^2(S) = S$.

Lemma 2.4.5 (see [52, 3.4.20, p. 113]) Let S be a definable subcategory of $\text{Mod-}R$. If $M \in S$, then $M^* \in \mathcal{D}(S)$.

Lemma 2.4.6 (see [52, 3.4.21, p. 113]) For any left R -module M , the double dual M^{**} belongs to the definable subcategory generated by M : $M^{**} \in \langle M \rangle$.

2.4.2 Ziegler spectrum and full support topology

Background material on the Ziegler spectrum and full support topology can be found in [52, Chapter 5].

A non-zero left R -module M is called **indecomposable** if 0 and M are the only direct summands of M . Let ${}_R\text{pinj}$ be the set of isomorphism classes of indecomposable pure-injective left R -modules. The **Ziegler spectrum** of a ring R is a topological space which was introduced by Ziegler in [86], the underlying set of which is the set ${}_R\text{pinj}$. The closed sets of this topology are exactly those of the form $\mathcal{X} \cap {}_R\text{pinj}$ with $\mathcal{X} \subseteq R\text{-Mod}$ a definable subcategory.

Theorem 2.4.7 [52, Theorem 5.1.1, p.211] The closed sets of the Ziegler topology are exactly those of the form $\mathcal{X} \cap {}_R\text{pinj}$ with $\mathcal{X} \subseteq R\text{-Mod}$ a definable subcategory.

Recall the following from [52, 5.3.63, p. 264].

Definition 2.4.8 A subclass \mathcal{X} of $R\text{-Mod}$ is said to be **type-definable** if it is closed under direct products, pure submodules and pure-injective hulls.

Burke in [10] introduced a topology on the set ${}_R\text{pinj}$, which Burke called the **full support topology**. The closed sets of this topology are exactly those of the form $\mathcal{X} \cap {}_R\text{pinj}$ with $\mathcal{X} \subseteq R\text{-Mod}$ a type-definable subcategory.

Theorem 2.4.9 [52, Theorem 5.3.64, p. 265] The closed sets of the full support topology are exactly those of the form $\mathcal{X} \cap {}_R\text{pinj}$ with $\mathcal{X} \subseteq R\text{-Mod}$ a type-definable subcategory.

Chapter 3

Purities

Purity for modules over general rings was defined in [13] and many relative versions of purity have been considered since then. We consider those purities which, like the original one, are determined by classes S of finitely presented modules. We present a number of characterizations of S -pure exact sequences and of the associated classes of relatively projective and relatively injective modules. We also show the relation between the purity for left modules which is determined by S and the purity for right modules determined by S ; this is said most directly in terms of the matrices presenting the modules in S .

Al-Kawarit and Cauchot [1] gave conditions in the context of commutative rings under which purities determined by matrices of certain sizes are different. We obtain related results over semiperfect rings and we also consider this question in detail over finite-dimensional algebras.

Over finite-dimensional algebras we give a description of the S -pure-injective modules in terms of the type-definable category generated by τS where τ is Auslander-Reiten translate and, in the case of tame hereditary algebras, using results from [52] and [59], we give a complete description of these modules.

Finally we give a number of characterizations of rings whose indecomposable modules are S -pure-projective.

Let $n, m \in \mathbb{Z}^+$. We use the notation $M_{n \times m}(R)$ for the set of all $n \times m$ matrices over R . **All matrices in this thesis are matrices with finitely many rows and finitely many columns.** An R -module M is said to be (n, m) -**presented** if it is the factor module of the module R^n modulo an m -generated submodule. Let H be an $n \times m$ matrix over R . Then right (resp. left)

multiplication by H determines a homomorphism $\rho_H : {}_R R^n \rightarrow {}_R R^m$ (resp. $\lambda_H : R_R^m \rightarrow R_R^n$). Thus H determines the (m, n) -presented left R -module $R^m/\text{im}(\rho_H)$; we will denote it by L_H . Also, H determines the (n, m) -presented right R -module $R^n/\text{im}(\lambda_H)$; we will denote it by D_H . Let \mathcal{H} be a set of matrices over a ring R ; we will denote by $L_{\mathcal{H}}$ the class of left R -modules $\{L_H \mid H \in \mathcal{H}\}$ and by $D_{\mathcal{H}}$ the class of right R -modules $\{D_H \mid H \in \mathcal{H}\}$. In view of Proposition 3.1.2 below we may, where convenient, interpret L_{\emptyset} as $\{R_R\}$ and D_{\emptyset} as $\{R_R\}$.

Throughout this chapter we will use the following conventions.

If S and T are classes of finitely presented left R -modules, we will denote by \mathcal{H} and \mathcal{K} sets of matrices over R such that $L_{\mathcal{H}}$ -purity = S -purity and $L_{\mathcal{K}}$ -purity = T -purity.

3.1 Purities

The following theorem collects together and extends results from the literature (in particular see [20] and [79]).

Theorem 3.1.1 Let R be an algebra over a commutative ring K and let E be an injective cogenerator for K -modules. Let S be a class of finitely presented left R -modules and let $\Sigma : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be an exact sequence of left R -modules. Then the following statements are equivalent.

- (1) Σ is an S -pure exact sequence of left R -modules.
- (2) For any two positive integers n, m , for any $n \times m$ matrix H in \mathcal{H} and for all $\bar{c} \in C^m$, if $H\bar{c} = 0$, then there is $\bar{b} \in B^m$ with $g\bar{b} = \bar{c}$ and $H\bar{b} = 0$.
- (3) For any two positive integers n, m , for any $n \times m$ matrix H in \mathcal{H} and for all $\bar{a} \in A^n$, if the matrix equation $H\bar{x} = f\bar{a}$ has a solution in B^m then the equation $H\bar{x} = \bar{a}$ has a solution in A^m .
- (4) The sequence $0 \rightarrow M \otimes_R A \rightarrow M \otimes_R B \rightarrow M \otimes_R C \rightarrow 0$ is exact, for all $M \in D_{\mathcal{H}}$.
- (5) For any two positive integers n, m and for any $n \times m$ matrix H in \mathcal{H} , for every commutative diagram of left R -modules

$$\begin{array}{ccc}
 R^n & \xrightarrow{\rho_H} & R^m \\
 \alpha \downarrow & & \downarrow \\
 0 \longrightarrow & A & \xrightarrow{f} B
 \end{array}$$

there exists a homomorphism $\beta : R^m \rightarrow A$ such that $\alpha = \beta\rho_H$.

(6) The dual exact sequence of right R -modules $0 \rightarrow C^* \rightarrow B^* \rightarrow A^* \rightarrow 0$ is $D_{\mathcal{H}}$ -pure, where $M^* = \text{Hom}_K(M, E)$.

PROOF: (1) \Rightarrow (2) Let n, m be any two positive integers and let H be any $n \times m$ matrix over R with $H \in \mathcal{H}$. Let $\bar{c} \in C^m$ such that $H\bar{c} = 0$. Let $e_i (i = 1, \dots, m)$ be canonical generators of ${}_R R^m$ and let $\pi : {}_R R^m \rightarrow L_H$ be the canonical epimorphism. Then L_H is generated by

l_1, \dots, l_m , where $l_i = \pi(e_i)$ and $H\bar{l} = 0$, where $\bar{l} = \begin{bmatrix} l_1 \\ \vdots \\ l_m \end{bmatrix}$. Define $h : L_H \rightarrow C$ as follows:

let $x \in L_H$; thus $x = \sum_{i=1}^m s_i l_i$, for some $s_i \in R$. Define $h(x) = \sum_{i=1}^m s_i c_i$. It is clear that h is a well-defined R -homomorphism. By hypothesis, there is an R -homomorphism $\alpha : L_H \rightarrow B$ such that $g\alpha = h$. Set $\bar{b} = \alpha\bar{l}$. Since $H\bar{l} = 0$, thus $H\bar{b} = H(\alpha\bar{l}) = \alpha H\bar{l} = \alpha 0 = 0$. Also, $g\bar{b} = g\alpha\bar{l} = h\bar{l} = \bar{c}$.

(2) \Rightarrow (3) Let n, m be any two positive integers and let H be any $n \times m$ matrix over R with $H \in \mathcal{H}$. Let $\bar{a} \in A^n$ such that the matrix equation $H\bar{x} = f\bar{a}$ has a solution in B^m . Thus there is $\bar{b} \in B^m$ such that $H\bar{b} = f\bar{a}$ and hence $H(g\bar{b}) = gH\bar{b} = gf\bar{a} = 0$. By hypothesis, there is $\bar{b}' \in B^m$ such that $H\bar{b}' = 0$ and $g\bar{b}' = g\bar{b}$. Let $\bar{b}'' = \bar{b} - \bar{b}'$. Then $g\bar{b}'' = g\bar{b} - g\bar{b}' = 0$ and hence $\bar{b}'' \in (\ker(g))^m = (\text{im}(f))^m$. Thus there is $\bar{a}'' \in A^m$ such that $f\bar{a}'' = \bar{b}''$. Hence $fH\bar{a}'' = H(f\bar{a}'') = H\bar{b}'' = H\bar{b} - H\bar{b}' = H\bar{b} = f\bar{a}$. Since f is an R -monomorphism, $H\bar{a}'' = \bar{a}$. Therefore, the matrix equation $H\bar{x} = \bar{a}$ has a solution in A^m .

(3) \Rightarrow (1) Let $L \in L_{\mathcal{H}}$, thus there is $n \times m$ matrix $H \in \mathcal{H}$ for some $n, m \in \mathbb{Z}^+$ such that $L = L_H$. Let $e_i (i = 1, \dots, m)$ be canonical generators of ${}_R R^m$ and let $\pi : {}_R R^m \rightarrow L_H$ be the canonical epimorphism. Then L is generated by l_1, \dots, l_m , where $l_i = \pi(e_i)$ and $H\bar{l} = 0$,

where $\bar{l} = \begin{bmatrix} l_1 \\ \vdots \\ l_m \end{bmatrix}$. Let $h : L \rightarrow C$ be any R -homomorphism and set $\bar{c} = h\bar{l}$. Then $H\bar{c} =$

$H(h\bar{l}) = hH\bar{l} = 0$. Since g is an R -epimorphism, there is $\bar{b} \in B^m$ such that $g\bar{b} = \bar{c}$. Then $gH\bar{b} = H(g\bar{b}) = H\bar{c} = 0$ and hence $H\bar{b} \in (\ker(g))^n = (\text{im}(f))^n$. Thus there is $\bar{a} \in A^n$ such that $H\bar{b} = f\bar{a}$ and hence the matrix equation $H\bar{x} = f\bar{a}$ has a solution in B^m . By hypothesis, the matrix equation $H\bar{x} = \bar{a}$ has a solution in A^m and hence there is $\bar{a}' \in A^m$ such that $H\bar{a}' = \bar{a}$. Set $\bar{b}' = \bar{b} - f\bar{a}'$. Then $H\bar{b}' = H\bar{b} - H(f\bar{a}') = f\bar{a} - fH\bar{a}' = f\bar{a} - f\bar{a} = 0$. Define $\alpha : L \rightarrow B$

as follows: let $x \in L$; thus $x = \sum_{i=1}^m s_i l_i$, for some $s_i \in R$. Define $\alpha(x) = \sum_{i=1}^m s_i b'_i$. It is clear that α is a well-defined R -homomorphism and $g\bar{b}' = g\bar{b} - gf\bar{a}' = g\bar{b} = \bar{c}$. Let $x \in L$; thus $x = \sum_{i=1}^m s_i l_i$, for some $s_i \in R$ and $(g\alpha)(x) = (g\alpha)\left(\sum_{i=1}^m s_i l_i\right) = g\left(\alpha\left(\sum_{i=1}^m s_i l_i\right)\right) = g\left(\sum_{i=1}^m s_i b'_i\right) = \sum_{i=1}^m s_i c_i = \sum_{i=1}^m s_i h(l_i) = h\left(\sum_{i=1}^m s_i l_i\right) = h(x)$. Therefore, the sequence $0 \rightarrow \text{Hom}_R(L, A) \rightarrow \text{Hom}_R(L, B) \rightarrow \text{Hom}_R(L, C) \rightarrow 0$ is exact and hence Σ is an S -pure exact sequence of left R -modules.

(3) \Rightarrow (4) Let $H \in \mathcal{H}$ and let $x \in D_H \otimes_R A$ be such that $(1_{D_H} \otimes f)(x) = 0$. Let $e_i (i = 1, \dots, n)$ be canonical generators of R_R^n and let $\pi : R_R^n \rightarrow D_H$ be the canonical epimorphism. Then D_H is generated by d_1, \dots, d_n , where $d_i = \pi(e_i)$ and defining relations are $\sum_{i=1}^n d_i r_{ij} = 0$, for all $j = 1, \dots, m$. Thus $x = \sum_{i=1}^n d_i \otimes a_i$, for $a_i \in A$ and hence $\sum_{i=1}^n d_i \otimes f(a_i) = (1_{D_H} \otimes f)\left(\sum_{i=1}^n d_i \otimes a_i\right) = (1_{D_H} \otimes f)(x) = 0$. By [62, Lemma 3.68, p. 147], there exist elements $b_j \in B (j = 1, \dots, m)$ such that $f(a_i) = \sum_{j=1}^m r_{ij} b_j$, $(i = 1, \dots, n)$ and hence the matrix equation $H\bar{x} = f\bar{a}$ has a solution in B^m . By hypothesis, the matrix equation $H\bar{x} = \bar{a}$ has a solution in A^m and hence there are elements $c_j \in A (j = 1, \dots, m)$ such that $\sum_{j=1}^m r_{ij} c_j = a_i$, for all $i = 1, \dots, n$. Thus $x = \sum_{i=1}^n d_i \otimes a_i = \sum_{i=1}^n \left(d_i \otimes \sum_{j=1}^m r_{ij} c_j\right) = \sum_{j=1}^m \left(\sum_{i=1}^n d_i r_{ij}\right) \otimes c_j = \sum_{j=1}^m 0 \otimes c_j = 0$ and hence $1_{D_H} \otimes f$ is a \mathbb{Z} -monomorphism. Therefore, the sequence $0 \rightarrow M \otimes_R A \rightarrow M \otimes_R B \rightarrow M \otimes_R C \rightarrow 0$ is exact, for all $M \in D_{\mathcal{H}}$.

(4) \Rightarrow (3) Let n, m be any two positive integers and let $H = (r_{ij})_{ij}$ be any $n \times m$ matrix over R with $H \in \mathcal{H}$. Let $\bar{a} \in A^n$ such that the matrix equation $H\bar{x} = f\bar{a}$ has a solution in B^m . Thus there exists $\bar{b} \in B^m$ such that $H\bar{b} = f\bar{a}$ and hence $\sum_{j=1}^m r_{ij} b_j = f(a_i)$, for all

$i = 1, \dots, n$, where $\bar{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\bar{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$. Let $e_i (i = 1, \dots, n)$ be canonical generators

of R_R^n and let $\pi : R_R^n \rightarrow D_H$ be the canonical epimorphism. Then D_H is generated by d_1, \dots, d_n , where $d_i = \pi(e_i)$ and defining relations are $\sum_{i=1}^n d_i r_{ij} = 0$, for all $j = 1, \dots, m$. By Lemma 2.1.7, $\sum_{i=1}^n (d_i \otimes f(a_i)) = 0$ and hence $(1_{D_H} \otimes f)\left(\sum_{i=1}^n (d_i \otimes a_i)\right) = 0$. By (4), $1_{D_H} \otimes f$ is a \mathbb{Z} -monomorphism and hence $\sum_{i=1}^n d_i \otimes a_i = 0$. By [62, Lemma 3.68, p. 147], there exist elements $h_j \in A (j = 1, \dots, m)$ such that $a_i = \sum_{j=1}^m r_{ij} h_j$, $(i = 1, \dots, n)$. Therefore, the matrix equation $H\bar{x} = \bar{a}$ has a solution in A^m .

(1) \Leftrightarrow (5) This follows by Lemma 2.1.4.

(1) \Rightarrow (6) Let $H \in \mathcal{H}$. By hypothesis, the sequence $0 \rightarrow \text{Hom}_R(L_H, A) \rightarrow \text{Hom}_R(L_H, B) \rightarrow \text{Hom}_R(L_H, C) \rightarrow 0$ of K -modules is exact. Since E is an injective K -module, the sequence $0 \rightarrow \text{Hom}_K(\text{Hom}_R(L_H, C), E) \rightarrow \text{Hom}_K(\text{Hom}_R(L_H, B), E) \rightarrow \text{Hom}_K(\text{Hom}_R(L_H, A), E) \rightarrow 0$ is exact. Since E is an injective K -module and L_H is a finitely presented left R -module it follows from Lemma 2.1.9 that $\text{Hom}_K(M, E) \otimes_R L_H \simeq \text{Hom}_K(\text{Hom}_R(L_H, M), E)$, for any left R -module M and hence the sequence $0 \rightarrow \text{Hom}_K(C, E) \otimes_R L_H \rightarrow \text{Hom}_K(B, E) \otimes_R L_H \rightarrow \text{Hom}_K(A, E) \otimes_R L_H \rightarrow 0$ is exact. Thus the dual exact sequence of right R -modules $0 \rightarrow C^* \rightarrow B^* \rightarrow A^* \rightarrow 0$ is $D_{\mathcal{H}}$ -pure.

(6) \Rightarrow (1) This is proved by reversing the argument of (1) \Rightarrow (6). \square

Note that there is a right hand version of Theorem 3.1.1 (and of many of the results we state) which is obtained by making obvious modifications to the statement and proof.

We use $\text{Add}(T)$ (resp. $\text{add}(T)$) to denote the class of modules that are direct summands of direct sums (resp. finite direct sums) of modules from T . Also, we use $\text{Prod}(T)$ to denote the class of modules that are direct summands of direct products of modules from T .

Let T be a class of finitely presented left R -modules. Note that if $S \subseteq T$ then every T -pure exact sequence of left R -modules is S -pure, so S -pure-injective implies T -pure-injective and S -pure-projective implies T -pure-projective.

Proposition 3.1.2 (as [74, Proposition 1, p. 700]) Let S be a class of finitely presented left R -modules and let M be a left R -module. Then:

(1) There exists an S -pure exact sequence of left R -modules $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ with F being a direct sum of copies of modules in $S \cup \{R_R\}$.

(2) $\text{Add}(S \cup \{R_R\})$ is the class of S -pure-projective left R -modules.

Corollary 3.1.3 Let S be a class of finitely presented left R -modules. Then for any left R -module N there is an S -pure monomorphism $\alpha : N \rightarrow F^*$ such that F is a direct sum of copies of modules in $D_{\mathcal{H}} \cup \{R_R\}$.

PROOF: Let N be any left R -module. By the right hand version of Proposition 3.1.2, there is a $D_{\mathcal{H}}$ -pure exact sequence of right R -modules $0 \rightarrow G \xrightarrow{f} F \xrightarrow{g} N^* \rightarrow 0$ where F is a direct sum of copies of modules in $D_{\mathcal{H}} \cup \{R_R\}$. By the right hand version of Theorem 3.1.1, the

dual exact sequence of left R -modules $0 \rightarrow N^{**} \xrightarrow{g^*} F^* \xrightarrow{f^*} G^* \rightarrow 0$ is $L_{\mathcal{H}}$ -pure. The canonical monomorphism $\delta_N : N \rightarrow N^{**}$ is pure (by Lemma 2.2.4) and hence it is $L_{\mathcal{H}}$ -pure. Since a composition of $L_{\mathcal{H}}$ -pure monomorphisms clearly is $L_{\mathcal{H}}$ -pure, $g^* \delta_N : N \rightarrow F^*$ is an $L_{\mathcal{H}}$ -pure monomorphism. \square

Let S be a class of left (or right) R -modules. We use S^* to denote the class $\{M^* \mid M \in S\}$.

Theorem 3.1.4 (as [75, Theorem 1]) Let S be a class of finitely presented left R -modules. Then $\text{Prod}((D_{\mathcal{H}} \cup \{R_R\})^*)$ is the class of S -pure-injective left R -modules.

PROOF: Let M be any S -pure-injective left R -module. By Corollary 3.1.3, there exists an S -pure, hence split, monomorphism $\alpha : M \rightarrow F^*$ where $F = \bigoplus_{i \in I} F_i$ with $F_i \in D_{\mathcal{H}} \cup \{R_R\}$. Since $F^* = (\bigoplus_{i \in I} F_i)^* \simeq \prod_{i \in I} F_i^*$ it follows that $M \in \text{Prod}((D_{\mathcal{H}} \cup \{R_R\})^*)$.

Conversely, let $H \in \mathcal{H}$ and let $\Sigma : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be any $L_{\mathcal{H}}$ -pure exact sequence of left R -modules. By Theorem 3.1.1, the sequence $D_H \otimes_R \Sigma : 0 \rightarrow D_H \otimes_R A \rightarrow D_H \otimes_R B \rightarrow D_H \otimes_R C \rightarrow 0$ is exact. Since E is an injective K -module, the sequence $0 \rightarrow \text{Hom}_K(D_H \otimes_R C, E) \rightarrow \text{Hom}_K(D_H \otimes_R B, E) \rightarrow \text{Hom}_K(D_H \otimes_R A, E) \rightarrow 0$ is exact. This is isomorphic to the sequence $0 \rightarrow \text{Hom}_R(C, \text{Hom}_K(D_H, E)) \rightarrow \text{Hom}_R(B, \text{Hom}_K(D_H, E)) \rightarrow \text{Hom}_R(A, \text{Hom}_K(D_H, E)) \rightarrow 0$. That is, the sequence $0 \rightarrow \text{Hom}_R(C, (D_H)^*) \rightarrow \text{Hom}_R(B, (D_H)^*) \rightarrow \text{Hom}_R(A, (D_H)^*) \rightarrow 0$ is exact. Therefore, $(D_H)^*$ is $L_{\mathcal{H}}$ -pure-injective. Since the dual of any flat right R -module is an injective left R -module (see for instance [19, Theorem 3.2.9, p. 77]), $(R_R)^*$ is injective and thus each module in $(D_{\mathcal{H}} \cup \{R_R\})^*$ is S -pure-injective. It follows that every module in $\text{Prod}((D_{\mathcal{H}} \cup \{R_R\})^*)$ is S -pure-injective. \square

Remark 3.1.5 By using Theorem 3.1.4, we deduce that F^* in Corollary 3.1.3 is S -pure-injective.

Proposition 3.1.6 Let S be a class of finitely presented left R -modules and let $\Sigma : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be any exact sequence of left R -modules. Then the following statements are equivalent.

- (1) Σ is S -pure.
- (2) Every S -pure-injective left R -module is injective relative to Σ .

(3) $(D_H)^*$ is injective relative to Σ , for all $H \in \mathcal{H}$.

(4) Every S -pure-projective left R -module is projective relative to Σ .

PROOF: (1) \Rightarrow (2) and (1) \Rightarrow (4) are obvious and (2) \Rightarrow (3) is immediate from Theorem 3.1.4.

(3) \Rightarrow (1) Let $H \in \mathcal{H}$. By hypothesis, the sequence

$$0 \rightarrow \text{Hom}_R(C, \text{Hom}_K(D_H, E)) \rightarrow \text{Hom}_R(B, \text{Hom}_K(D_H, E)) \rightarrow \text{Hom}_R(A, \text{Hom}_K(D_H, E)) \rightarrow 0,$$

equivalently, the sequence

$$0 \rightarrow \text{Hom}_K(D_H \otimes_R C, E) \rightarrow \text{Hom}_K(D_H \otimes_R B, E) \rightarrow \text{Hom}_K(D_H \otimes_R A, E) \rightarrow 0$$

is exact. Since E is an injective cogenerator for K -modules it follows from Lemma 2.1.10 that the sequence $0 \rightarrow D_H \otimes_R A \rightarrow D_H \otimes_R B \rightarrow D_H \otimes_R C \rightarrow 0$ is exact. Thus Σ is S -pure.

(4) \Rightarrow (1) This is immediate from Proposition 3.1.2, and the definition of S -pure exact sequence. \square

Proposition 3.1.7 (as [67, Propositions 2.4 and 3.5]) Let S be a class of finitely presented left R -modules. Then for a left R -module M :

(1) M is S -pure-projective if and only if it is projective relative to every S -pure exact sequence $0 \rightarrow K \rightarrow G \rightarrow F \rightarrow 0$ of left R -modules where G is S -pure-injective;

(2) M is S -pure-injective if and only if M is injective relative to every S -pure exact sequence $0 \rightarrow K \rightarrow P \rightarrow L \rightarrow 0$ of left R -modules where P is S -pure-projective.

PROOF: (1) (\Rightarrow) is obvious.

(\Leftarrow) Let $0 \rightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \rightarrow 0$ be any S -pure exact sequence of left R -modules. By Corollary 3.1.3 and Theorem 3.1.4, there is an S -pure exact sequence $0 \rightarrow B \xrightarrow{\lambda} G \xrightarrow{\rho} N \rightarrow 0$

of left R -modules where G is S -pure-injective. We have the following pushout diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C \longrightarrow 0 \\
 & & \parallel & & \downarrow \lambda & & \downarrow \varphi \\
 0 & \longrightarrow & A & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & D \longrightarrow 0 \\
 & & & & \downarrow \rho & & \downarrow \delta \\
 & & & & N & = & N \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Since μ and λ are S -pure R -monomorphisms so is $\lambda\mu$. Since $\alpha = \lambda\mu$, the exact sequence $0 \rightarrow A \xrightarrow{\alpha} G \xrightarrow{\beta} D \rightarrow 0$ is S -pure. Let $\psi \in \text{Hom}_R(M, C)$. By hypothesis, there is $\gamma \in \text{Hom}_R(M, G)$ such that $\beta\gamma = \varphi\psi$. We have $\rho\gamma = \delta\beta\gamma = \delta\varphi\psi = 0$ so $\text{im}(\gamma) \subseteq \ker(\rho) = \text{im}(\lambda)$ and hence $\gamma = \lambda\gamma'$ for some $\gamma' \in \text{Hom}_R(M, B)$. Then we have $\varphi\nu\gamma' = \beta\lambda\gamma' = \beta\gamma = \varphi\psi$. Since φ is a monomorphism, $\nu\gamma' = \psi$. Hence M is S -pure-projective.

(2) The proof is dual to that of (1). □

Theorem 3 in [68, p.507] is a special case of the following corollary if we take $S = R\text{-mod}$.

Corollary 3.1.8 Let S be a class of finitely presented left R -modules. Then the following statements are equivalent.

- (1) For every S -pure exact sequence $0 \rightarrow N \rightarrow M \rightarrow K \rightarrow 0$ of left R -modules, if M is S -pure-projective, then N is S -pure-projective.
- (2) For every S -pure exact sequence $0 \rightarrow N \rightarrow M \rightarrow K \rightarrow 0$ of left R -modules, if M is S -pure-injective, then K is S -pure-injective.

PROOF: (1) \Rightarrow (2) Let $0 \rightarrow N \xrightarrow{\nu} M \xrightarrow{\mu} K \rightarrow 0$ be any S -pure exact sequence of left R -modules where M is S -pure-injective. Let $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ be any S -pure exact sequence of left R -modules where B is S -pure-projective. By hypothesis, A is S -pure-projective. Let $f : A \rightarrow K$ be any R -homomorphism. Thus there is an R -homomorphism $g : A \rightarrow M$ such that $\mu g = f$. Since M is S -pure-injective, there is an R -homomorphism

$h : B \rightarrow M$ such that $h\alpha = g$. Put $\lambda = \mu h$, thus $\lambda\alpha = (\mu h)\alpha = \mu(h\alpha) = \mu g = f$. Hence K is injective relative to every S -pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ where B is S -pure-projective. By Proposition 3.1.7, K is S -pure-injective.

(2) \Rightarrow (1) Let $0 \rightarrow N \xrightarrow{\nu} M \xrightarrow{\mu} K \rightarrow 0$ be any S -pure exact sequence of left R -modules where M is S -pure-projective. Let $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ be any S -pure exact sequence of left R -modules where B is S -pure-injective. By hypothesis, C is S -pure-injective. Let $f : N \rightarrow C$ be any R -homomorphism. Thus there is an R -homomorphism $g : M \rightarrow C$ such that $g\nu = f$. Since M is S -pure-projective, there is an R -homomorphism $h : M \rightarrow B$ such that $\beta h = g$. Put $\lambda = h\nu$, thus $\beta\lambda = \beta h\nu = g\nu = f$. Hence N is projective relative to every S -pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of left R -modules where B is S -pure-injective. By Proposition 3.1.7, N is S -pure-projective. \square

Examples 3.1.9 (1) Recall that a ring R is said to be **von Neumann regular** if for every a in R there exists b in R such that $a = aba$. Let R be an infinite product of fields and let $S \subseteq R\text{-mod}$. By [47, Example 1.34(1), p. 19], R is a commutative von Neumann regular self-injective ring that is not semisimple. By [49, Corollary, p. 650], there is an ideal I of R which is not projective. Since every short exact sequence of modules over a von Neumann regular ring is pure (see, e.g. [79, 37.2, p. 314]) it follows that every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of R -modules is S -pure and hence every S -pure-projective R -module is projective. Thus I is an S -pure submodule of ${}_R R$ and it is not S -pure-projective. Hence $S\text{-Pproj}$ is not closed under S -pure submodules. By Corollary 3.1.8, $S\text{-Pinj}$ is not closed under S -pure quotient modules.

(2) Let R be a Dedekind domain (recall that an integral domain R is said to be a **Dedekind domain** if every ideal of R is projective, for example the ring of integers \mathbb{Z}). By [25, Proposition 25, p. 257], every submodule of a pure-projective R -module is pure-projective. By taking $S = R\text{-mod}$ and applying Corollary 3.1.8 we have that ${}_R\text{Pinj}$ is closed under pure quotient modules, and in fact this is Proposition 27 in [25, p. 258].

3.2 Comparing purities

There is a natural ordering on purities for left modules given by T -purity \leq S -purity if every T -pure exact sequence is S -pure. In Theorem 3.2.1 we characterize this relation in various ways and from this and Corollary 3.2.3 it follows that this partial order (indeed, as follows directly from Theorem 3.2.1(4)), lattice for left modules is naturally isomorphic to that for right modules.

Theorem 3.2.1 Let S and T be classes of finitely presented left R -modules. Then the following statements are equivalent.

- (1) Every T -pure short exact sequence of left R -modules is S -pure.
- (2) Every T -pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of left R -modules where B is T -pure-injective is S -pure.
- (3) Every S -pure-projective left R -module is T -pure-projective.
- (4) $S \subseteq \text{add}(T \cup \{R_R\})$.
- (5) Every T -pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of left R -modules where B is T -pure-projective is S -pure.
- (6) Every S -pure-injective left R -module is T -pure-injective.
- (7) $(D_{\mathcal{H}})^* \subseteq \text{Prod}((D_{\mathcal{K}} \cup R_R)^*)$.
- (8) The corresponding assertions for right modules.

PROOF: (1) \Rightarrow (2) and (1) \Rightarrow (5) are obvious.

(2) \Rightarrow (3) Let M be any S -pure-projective left R -module and let $\Sigma : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be any T -pure exact sequence of left R -modules where B is T -pure-injective. By hypothesis, Σ is S -pure and hence the sequence $0 \rightarrow \text{Hom}_R(M, A) \rightarrow \text{Hom}_R(M, B) \rightarrow \text{Hom}_R(M, C) \rightarrow 0$ is exact. Thus M is projective relative to every T -pure exact sequence $\Sigma : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of left R -modules where B is T -pure-injective. By Proposition 3.1.7, M is T -pure-projective.

(3) \Rightarrow (4) This follows by Proposition 3.1.2.

(4) \Rightarrow (1) Let $\Sigma : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be any T -pure exact sequence of left R -modules and let $M \in S$. By assumption and Proposition 3.1.2, M is T -pure-projective. Thus the sequence $0 \rightarrow \text{Hom}_R(M, A) \rightarrow \text{Hom}_R(M, B) \rightarrow \text{Hom}_R(M, C) \rightarrow 0$ is exact. Therefore Σ is

S -pure.

(5) \Rightarrow (6) Let M be any S -pure-injective left R -module and let $\Sigma : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be any T -pure exact sequence of left R -modules where B is T -pure-projective. By hypothesis, Σ is S -pure and hence the sequence $0 \rightarrow \text{Hom}_R(C, M) \rightarrow \text{Hom}_R(B, M) \rightarrow \text{Hom}_R(A, M) \rightarrow 0$ is exact. It follows by Proposition 3.1.7 that M is T -pure-injective.

(6) \Rightarrow (7) Let $M \in (D_{\mathcal{H}})^*$, thus M is an S -pure-injective left R -module (by Theorem 3.1.4). By hypothesis, M is T -pure-injective so by Theorem 3.1.4 we have that $M \in \text{Prod}((D_{\mathcal{K}} \cup R_R)^*)$.

(7) \Rightarrow (1) Let $\Sigma : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be any T -pure exact sequence of left R -modules. Let $H \in \mathcal{H}$, thus by hypothesis, $(D_H)^* \in \text{Prod}((D_{\mathcal{K}} \cup R_R)^*)$, hence $(D_H)^*$ is T -pure-injective, in particular $(D_H)^*$ is injective relative to Σ . By Proposition 3.1.6, Σ is S -pure.

(1) \Rightarrow (8) Let $\Sigma : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be any $D_{\mathcal{K}}$ -pure exact sequence of right R -modules. By the right hand version of Theorem 3.1.1, the exact sequence of left R -modules $\Sigma^* : 0 \rightarrow C^* \rightarrow B^* \rightarrow A^* \rightarrow 0$ is T -pure. By hypothesis, Σ^* is S -pure and hence by Theorem 3.1.1 again, Σ is $D_{\mathcal{H}}$ -pure.

(8) \Rightarrow (1) This follows by right/left symmetry. \square

Recall (see [54, p. 353]) that a ring R is said to be an **indiscrete ring** if the only closed sets of the Ziegler spectrum over R are the trivial closed sets, i.e., \emptyset and the whole space.

Remark 3.2.2 As an application of Theorem 3.2.1 we have that if R is an indiscrete ring but is not von Neumann regular (see for example [54, 2.2, p. 357]) then, although the Ziegler topology is trivial, the full support topology is not. To see that the full support topology is not trivial, note that not every embedding between R -modules is pure. Therefore an $\{R_R\}$ -pure exact (that is, exact) sequence need not be pure exact ($= (R\text{-mod})$ -pure exact). Hence $(R\text{-mod})$ -pure-injective ($=$ pure-injective) does not imply $\{R_R\}$ -pure-injective and so by Theorem 3.2.1 we have that $\text{Prod}((R_R)^*)$ is properly contained in the collection $\text{Prod}((\text{mod-}R)^*)$ of all pure-injective left R -modules. Since the class of modules which are pure in a module of $\text{Prod}((\text{mod-}R)^*)$ is $R\text{-Mod}$, hence is a definable category, it has an **elementary cogenerator**, that is a pure-injective module M such that every R -module purely embeds in a direct product of copies of M [52, Corollary 5.3.52, p. 259]. Furthermore, by

[52, Corollary 5.3.50, p. 259], M can be taken to be a product of indecomposable pure-injectives. If every indecomposable pure-injective were in $\text{Prod}((R_R)^*)$, then M would be in $\text{Prod}((R_R)^*)$ and hence we would have $\text{Prod}((R_R)^*) = \text{Prod}((\text{mod-}R)^*)$, contradiction.

The following corollary is immediately obtained from Theorem 3.2.1.

Corollary 3.2.3 Let S and T be classes of finitely presented left R -modules. Then the following statements are equivalent.

- (1) T -purity = S -purity for short exact sequences of left R -modules.
- (2) S -pure-projectivity = T -pure-projectivity for left R -modules.
- (3) $\text{add}(S \cup \{R_R\}) = \text{add}(T \cup \{R_R\})$.
- (4) S -pure-injectivity = T -pure-injectivity for left R -modules.
- (5) $\text{Prod}(\{(D_H)^* \mid H \in \mathcal{H} \cup \{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \}\}) = \text{Prod}(\{(D_K)^* \mid K \in \mathcal{K} \cup \{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \}\})$.
- (6) The corresponding assertions on the right.

A short exact sequence (Σ) of left (resp. right) R -modules is called (m, n) -**pure** if it remains exact when tensored with any (m, n) -presented right (resp. left) R -module. A left R -module M is said to be (m, n) -**pure-projective** (resp. (m, n) -**pure-injective**) if it is projective (resp. injective) relative to every (m, n) -pure exact sequence of left R -modules. A short exact sequence (Σ) of left (or right) R -modules is called (\mathfrak{K}_0, n) -**pure** exact (resp. (m, \mathfrak{K}_0) -**pure** exact) if, for each positive integer m (resp. n) (Σ) is (m, n) -pure [1]. Observe that the (m, n) -pure exact sequences of left R -modules are exactly the $L_{\mathcal{H}}$ -pure exact sequences, where $\mathcal{H} = M_{m \times n}(R)$, and the (n, m) -pure exact sequences of right R -modules are exactly the $D_{\mathcal{H}}$ -pure exact sequences of right modules. Also, (\mathfrak{K}_0, n) -pure exact sequences of left R -modules are exactly the $L_{\mathcal{H}}$ -pure exact sequences, where $\mathcal{H} = \bigcup_{m \in \mathbb{Z}^+} M_{m \times n}(R)$ and then the (n, \mathfrak{K}_0) -pure exact sequences of right R -modules are exactly the $D_{\mathcal{H}}$ -pure exact sequences. Note that for left modules (n, m) -presented implies (m, n) -pure-projective, whereas for right modules (n, m) -presented implies (n, m) -pure-projective. For all $n, m, s, t \in \mathbb{Z}^+$ with $n \geq s$ and $m \geq t$ we have from the following corollary that every (m, n) -pure exact sequence of left R -modules is (t, s) -pure.

Corollary 3.2.4 Let $n, m, s, t \in \mathbb{Z}^+$. Then the following statements are equivalent.

- (1) Every (m, n) -pure short exact sequence of left R -modules is (s, t) -pure.

- (2) Every (n, m) -pure short exact sequence of right R -modules is (t, s) -pure.
- (3) Every (s, t) -pure-projective (resp. (s, t) -pure-injective) left R -module is (m, n) -pure-projective (resp. (m, n) -pure-injective).
- (4) Every (t, s) -presented left R -module is in $\text{add}(\{M \mid M \text{ is an } (n, m)\text{-presented left } R\text{-module}\})$.
- (5) Every (s, t) -presented right R -module is in $\text{add}(\{M \mid M \text{ is an } (m, n)\text{-presented right } R\text{-module}\})$.

PROOF: Take $S = L_{\mathcal{H}}$ and $T = L_{\mathcal{K}}$ where $\mathcal{H} = M_{s \times t}(R)$ and $\mathcal{K} = M_{m \times n}(R)$ and apply Theorem 3.2.1. \square

Examples 3.2.5 (1) It is a result of Stafford [69, Proposition 4.4, p. 256] that every finitely generated indecomposable module over a simple noetherian ring of Krull dimension n is generated by $2n + 1$ elements and hence $(2n + 1, 2n + 1)$ -purity = purity in the usual sense.

(2) If a ring R satisfies the conditions of Corollary 3.2.4 for $n = m = 1$ and for all $s, t \in \mathbb{Z}^+$ then R is said to be an **RD-ring** (see e.g. [56]).

Let M be a finitely presented left (or right) R -module. We denote by $\text{gen}(M)$ its minimal number of generators and by $\text{rel}(M)$ the minimal number of relations on these generators. Therefore there is an exact sequence $R^{\text{rel}(M)} \rightarrow R^{\text{gen}(M)} \rightarrow M \rightarrow 0$.

Let R be any ring and M be any finitely presented right R -module. An **Auslander-Bridger dual** of M is denoted by $D(M)$ and defined as follows. Choose an exact sequence $Q \xrightarrow{\phi} P \rightarrow M \rightarrow 0$ in which P and Q are finitely generated projective right R -modules. Define $D(M)$ to be the cokernel of the homomorphism $\phi^\diamond : P^\diamond \rightarrow Q^\diamond$ where $X^\diamond = \text{Hom}_R(X, R_R)$, for any right R -module X and ϕ^\diamond is defined by $\phi^\diamond(\alpha) = \alpha\phi$ for all $\alpha \in P^\diamond$ [77]. Although $D(M)$ depends on the choice of exact sequence, if $D'(M)$ is another such dual of M then $D(M) \oplus A \simeq D'(M) \oplus B$ for some finitely generated projective modules, A, B .

A commutative ring R is said to be a **valuation ring** if the ideals of R are linearly ordered by inclusion. In the following corollary, we classify the commutative local rings over which

$(n, 1)$ -purity and $(n, n + 1)$ -purity are equivalent, where $n \in \mathbb{Z}^+$.

Corollary 3.2.6 Let R be a commutative local ring. Then the following statements are equivalent.

- (1) R is a valuation ring.
- (2) $(1, 1)$ -purity = $(1, 2)$ -purity.
- (3) $(n, 1)$ -purity = $(n, n + 1)$ -purity for some $n \in \mathbb{Z}^+$.

PROOF: (1) \Rightarrow (2) Since every finitely presented module over a valuation ring is a direct sum of $(1, 1)$ -presented modules (see [76, Theorem 1, p. 168]) it follows from Corollary 3.2.3 that $(1, 1)$ -purity = $(1, 2)$ -purity.

(2) \Rightarrow (3) This is obvious.

(3) \Rightarrow (1) Assume that R is not a valuation ring, thus there are elements $a, b \in R$ such that $Ra \not\subseteq Rb$ and $Rb \not\subseteq Ra$. Let A be the ideal generated by a and b . By [1, Proposition 3.3, p. 3886], there is a finitely presented module M with $\text{gen}(M) = n$, $\text{rel}(M) = n + 1$ and $\text{End}_R(M)$ a local ring. Thus $D(M)$, an Auslander-Bridger dual of M , is an indecomposable finitely presented module with $\text{gen}(D(M)) = \text{rel}(M) = n + 1$ and $\text{rel}(D(M)) = \text{gen}(M) = n$ and hence $D(M)$ is an $(n + 1, n)$ -presented R -module. By hypothesis and Corollary 3.2.3 we have that $D(M)$ is a direct summand of a direct sum of $(1, n)$ -presented modules. It follows that $D(M)$ is cyclic and this contradicts that $\text{gen}(D(M)) \geq 2$. Thus R is a valuation ring. \square

Proposition 3.2.7 Let S and T be classes of finitely presented left R -modules. Consider the following statements:

- (1) Every S -pure short exact sequence of left R -modules is T -pure.
- (2) Each indecomposable direct summand of a module in T is in $\text{add}(S \cup \{ {}_R R \})$.
- (3) Each indecomposable direct summand of a module in T is a direct summand of a module in $S \cup \{ {}_R R \}$.

Then (1) implies (2) and

(a) If each indecomposable direct summand of a module in T has local endomorphism ring then (2) implies (3).

(b) If each module in T is a direct sum of indecomposable modules then (3) implies (1).

PROOF: (1) \Rightarrow (2) This follows by Theorem 3.2.1.

(a) Assume that each indecomposable direct summand M of a module in T has local endomorphism ring. Thus by hypothesis, $M \in \text{add}(S \cup \{R\})$. Suppose that M is a direct summand of $\bigoplus_{i \in I} F_i$ where $F_i \in S \cup \{R\}$, for all $i \in I$ and I is a finite set and let B be a submodule of $\bigoplus_{i \in I} F_i$ such that $M \oplus B = \bigoplus_{i \in I} F_i$. Since $\text{End}_R(M)$ is local we have (see, e.g., [20, Theorem 2.8, p. 37]) that M has the finite exchange property. So (see, e.g., [20, Lemma 2.7, p. 37]) there is an index $j \in I$ and a direct sum decomposition $F_j = B_j \oplus C_j$ of F_j with $M \simeq C_j$. Hence M is a direct summand of a module in $S \cup \{R\}$.

(b) Assume that each module in T is a direct sum of indecomposable modules. Let Σ be any S -pure short exact sequence of left R -modules and let $M \in T$. By assumption and (3), $M = \bigoplus_{i \in I} M_i$ with M_i a direct summand of a module in $S \cup \{R\}$, for all $i \in I$. Thus M is an S -pure-projective module and hence M is projective relative to Σ . Therefore Σ is T -pure. \square

A ring R is said to be **Krull-Schmidt** if every finitely presented left (or right) R -module is a direct sum of modules with local endomorphism rings (see [20, p. 97]).

Corollary 3.2.8 Let R be a left Krull-Schmidt ring and let n, m be positive integers. Then the following statements are equivalent.

- (1) (m, n) -purity = (\aleph_0, n) -purity for short exact sequences of left R -modules.
- (2) For each $s \in \mathbb{Z}^+$, each indecomposable (n, s) -presented left R -module is a direct summand of an (n, m) -presented left R -module.
- (3) (n, m) -purity = (n, \aleph_0) -purity for short exact sequences of right R -modules.
- (4) For each $s \in \mathbb{Z}^+$, each indecomposable (s, n) -presented right R -module is a direct summand of an (m, n) -presented right R -module.

PROOF: Put $S = L_{\mathcal{H}}$ and $T = L_{\mathcal{K}}$, where $\mathcal{H} = M_{m \times n}(R)$ and $\mathcal{K} = \bigcup_{t \in \mathbb{Z}^+} M_{t \times n}(R)$. Since R is Krull-Schmidt, each indecomposable direct summand of a module in T has local endomorphism ring and each module in T is a direct sum of indecomposable modules. Hence the result follows on applying Proposition 3.2.7 and Corollary 3.2.4. \square

3.3 (m, n) -Purity over semiperfect rings

Recall, from before Corollary 3.2.6 the definitions of $\text{gen}(M)$ and $\text{rel}(M)$.

Remark 3.3.1 Let M be a finitely presented left R -module and let N be a direct summand of M . Then it is easy to see that $\text{gen}(N) \leq \text{gen}(M)$ and $\text{rel}(N) \leq \text{rel}(M) + \text{gen}(M)$.

Proposition 3.3.2 Let H be any matrix over a ring R such that $\text{End}_R(L_H)$ is local and L_H is not projective. Set $\mathcal{H} = \bigcup \{M_{r \times q}(R) \mid q < \text{gen}(L_H) \text{ or } r + q < \text{rel}(L_H)\}$. Then L_H is an L_H -pure-projective left R -module which is not $L_{\mathcal{H}}$ -pure-projective and hence not $L_{\mathcal{G}}$ -pure-projective for any $\mathcal{G} \subseteq \mathcal{H}$. In particular L_H -purity and $L_{\mathcal{H}}$ -purity are not equivalent.

PROOF: By Proposition 3.1.2, L_H is L_H -pure-projective and, if L_H is $L_{\mathcal{H}}$ -pure-projective, then $L_H \in \text{Add}(L_{\mathcal{H}} \cup \{R\})$. Since $\text{End}_R(L_H)$ is local, L_H is, as in Proposition 3.2.7, a direct summand of a module in $L_{\mathcal{H}} \cup \{R\}$. Thus either L_H is a direct summand of L_G , where $G \in \mathcal{H}$ or L_H is projective. If L_H is a direct summand of L_G , by Remark 3.3.1, $\text{gen}(L_H) \leq \text{gen}(L_G) \leq q$ and $\text{rel}(L_H) \leq \text{rel}(L_G) + \text{gen}(L_G) \leq r + q$ and this contradicts $G \in \mathcal{H}$. \square

Note that if M is a left R -module, I is a left ideal of R and $\alpha \in \text{End}_R(M)$ then there is an induced homomorphism $\bar{\alpha} : M/IM \rightarrow M/IM$ which is an isomorphism if α is an isomorphism.

Let R be a ring and let $J(R)$ be its Jacobson radical. Recall that R is **semiperfect** if $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$. Say that an idempotent $e \in R$ is **local** if eRe is a local ring. We have (e.g., [79, 42.6, p. 375]) that R is semiperfect if and only if $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$, for local orthogonal idempotents e_i .

Lemma 3.3.3 Let $m \in \mathbb{Z}^+$. Suppose that one of the following two conditions is satisfied.

(1) The ring R is semiperfect and I is a nonzero ideal with $\text{gen}(I_R) = m$ and $I \subseteq e_jR$ for some local idempotent e_j of R .

(2) The ring R is Krull-Schmidt and I is a nonzero right ideal with $\text{gen}(I) = m$ and $I \subseteq e_jR$ for some local idempotent e_j of R .

Then $\text{End}_R(e_jR/I)$ is a local ring.

PROOF: Let $P = e_j R$.

In case (1): Since $\text{End}_R(e_j R) \simeq e_j R e_j$ it follows that $\text{End}_R(P)$ is a local ring. Let $\alpha \in \text{End}_R(P/I)$ and consider the following diagram:

$$\begin{array}{ccc} P & \xrightarrow{\pi} & P/I \\ \alpha' \downarrow & & \downarrow \alpha \\ P & \xrightarrow{\pi} & P/I \end{array}$$

where π is the natural epimorphism. By projectivity of P , there exists an R -homomorphism $\alpha' : P \rightarrow P$ such that $\pi \alpha' = \alpha \pi$ and $\alpha'(I) \subseteq I$. Since $\text{End}_R(P)$ is a local ring, either α' or $1_P - \alpha'$ is an isomorphism. The inverse of that isomorphism will, as noted above, induce an isomorphism on $P/I = P/PI$ which will be an inverse of α or $1_{(P/I)} - \alpha$, as appropriate. Hence $\text{End}_R(e_j R/I)$ is a local ring.

In case (2): Since $e_j R$ is a local right R -module, every homomorphic image of $e_j R$ is indecomposable [73, Proposition 4.1, p. 246]. Hence $e_j R/I$ is indecomposable. Since R is Krull-Schmidt, $\text{End}_R(e_j R/I)$ is a local ring. \square

Lemma 3.3.4 Let $m \in \mathbb{Z}^+$ and let M be any $(1, m)$ -presented right R -module. Then $D(M)$ is a (n, m) -pure-projective left R -module, for all $n \in \mathbb{Z}^+$.

PROOF: Applying $\text{Hom}_R(-, R_R)$ to a presentation $R_R^m \xrightarrow{\lambda_H} R_R^1 \rightarrow M \rightarrow 0$ of M gives the presentation ${}_R R^1 \xrightarrow{\rho_H} {}_R R^m \rightarrow D(M) \rightarrow 0$ of $D(M)$. Thus $D(M)$ is $(m, 1)$ -presented, hence $(1, m)$ -pure-projective, hence (n, m) -pure-projective for all $n \geq 1$. \square

Proposition 3.3.5 Let $m \in \mathbb{Z}^+$. Suppose that one of the following two conditions is satisfied.

(1) The ring R is semiperfect and I is a nonzero ideal with $\text{gen}(I_R) = m + 1$ and $I \subseteq e_j R$ for some local idempotent e_j of R .

(2) The ring R is Krull-Schmidt and I is a nonzero right ideal with $\text{gen}(I) = m + 1$ and $I \subseteq e_j R$ for some local idempotent e_j of R .

Then $D(e_j R/I)$ is not an $L_{\mathcal{H}}$ -pure-projective left R -module, where $\mathcal{H} = \bigcup_{s \times t} \{M_s(R) \mid s, t \in \mathbb{Z}^+ \text{ with } t < m + 1\}$.

PROOF: By Lemma 3.3.3, $\text{End}_R(e_jR/I)$ is a local ring and hence $\text{End}_R(D(e_jR/I))$ is local [77, Theorem 2.4, p.196]. Dualising a projective presentation of e_jR/I , it follows easily that $\text{gen}(D(e_jR/I)) = m + 1$ and $\text{rel}(D(e_jR/I)) = 1$. Since e_jR/I is not projective, neither is $D(e_jR/I)$ so, by Proposition 3.3.2, $D(e_jR/I)$ is not $L_{\mathcal{H}}$ -pure-projective. \square

Al-Kawarit's and Couchot's theorem [1, Theorem 3.5(1), p.3888] is a special case of the following theorem if we take the ring R to be commutative.

Theorem 3.3.6 Let (n, m) and (r, s) be any two pairs of positive integers such that $n \neq r$. Suppose that one of the following two conditions is satisfied:

(a) R is semiperfect and there exists an ideal I of R with $\text{gen}(I_R) = \max\{n, r\}$ and $I \subseteq e_jR$ for some local idempotent e_j

(b) R is Krull-Schmidt and there exists a right ideal I of R with $\text{gen}(I) = \max\{n, r\}$ and $I \subseteq e_jR$ for some local idempotent e_j .

Then:

(1) (m, n) -purity and (s, r) -purity of short exact sequences of left R -modules are not equivalent;

(2) (n, m) -purity and (r, s) -purity of short exact sequences of right R -modules are not equivalent.

PROOF: (1) Without loss of generality, we can assume that $n < r$. By Lemma 3.3.4 and Proposition 3.3.5, $D(e_jR/I)$ is (s, r) -pure-projective and not (m, n) -pure-projective. Thus (m, n) -pure-projectivity and (s, r) -pure-projectivity of left R -modules are not equivalent and hence by Corollary 3.2.4, (m, n) -purity and (s, r) -purity for left R -modules are not equivalent.

(2) By (1) and Corollary 3.2.4. \square

Examples 3.3.7 By applying Theorem 3.3.6 in the following examples, we have that $(m, 1)$ -purity and $(n, 2)$ -purity of left R -modules are not equivalent for any $n, m \in \mathbb{Z}^+$.

(1) Let $R = Z_p[X, Y] / \langle \{X, Y\}^3 \rangle$, where p is a prime number and X, Y are noncommuting indeterminates. By [66, Theorem 13, p. 402], R is a local noncommutative ring with $J(R) = \langle \bar{X}, \bar{Y} \rangle$. Let $I = J(R)$, thus I is a two-sided ideal of R and $\text{gen}(I) = 2$.

(2) Let $R = k\tilde{A}_1$ be the Kronecker algebra over a field k . That is, R is the path algebra of the quiver $1 \bullet \begin{matrix} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{matrix} \bullet 2$. Then $R_R = e_1R \oplus e_2R$, where e_1R (resp. e_2R) is a projective right R -module of dimension vector $(0, 1)$ (resp. $(1, 2)$). Let $I_R = J(e_2R)$; since $I_R = \alpha R \oplus \beta R$ it follows that $\text{gen}(I_R) = 2$.

Corollary 3.3.8 Let R be a local ring, let I be a finitely generated ideal of R and set $\text{gen}(I_R) = r$. Then for all $n < r$ and for all m, s :

- (1) (m, n) -purity and (s, r) -purity for left R -modules are not equivalent.
- (2) (n, m) -purity and (r, s) -purity for right R -modules are not equivalent.

PROOF: Since R is local it is semiperfect and 1 is a local idempotent. By Theorem 3.3.6, the result holds. □

Let M be a finitely generated left module over a semiperfect ring R . Warfield in [77] defined $\text{Gen}(M)$ to be the number of summands in a decomposition of M/JM as a direct sum of simple modules where $J = J(R)$. If M is a finitely presented left module over a semiperfect ring R , and $f : P \rightarrow M$ a projective cover, with $K = \ker(f)$, then Warfield defined $\text{Rel}(M)$ by $\text{Rel}(M) = \text{Gen}(K)$. If M is a left R -module and $x \in M$, we say x is a **local element** if Rx is a local module. The number of elements in any minimal generating set of local elements of M is exactly $\text{Gen}(M)$ [77, Lemma 1.11]. One may prove results, similar to those above, for these notions.

Proposition 3.3.9 Let H be a matrix over a semiperfect ring R such that L_H is not projective and $\text{End}_R(L_H)$ is a local ring and let $\mathcal{H} = \{K \mid K \text{ is a matrix with } \text{Gen}(L_H) > \text{Gen}(L_K) \text{ or } \text{Rel}(L_H) > \text{Rel}(L_K)\}$. Then L_H is not $L_{\mathcal{H}}$ -pure-projective.

PROOF: Assume that L_H is $L_{\mathcal{H}}$ -pure-projective, thus by Proposition 3.1.2, $L_H \in \text{add}(L_{\mathcal{H}} \cup \{R\})$. Since $\text{End}_R(L_H)$ is a local ring, L_H is as in Proposition 3.2.7, a direct summand of a module in $L_{\mathcal{H}} \cup \{R\}$. Thus either L_H is a direct summand of L_D , where

$D \in \mathcal{H}$ or L_H is a direct summand of ${}_R R$. Since L_H is not projective, L_H is a direct summand of L_D , thus by [77, Lemma 1.10, p. 192], $\text{Gen}(L_H) \leq \text{Gen}(L_D)$ and $\text{Rel}(L_H) \leq \text{Rel}(L_D)$ and this contradicts $D \in \mathcal{H}$. Therefore, L_H is not $L_{\mathcal{H}}$ -pure-projective. \square

Remark 3.3.10 Since, if K is an $r \times q$ matrix, we have $q\text{Gen}(R) \geq \text{Gen}(R)\text{gen}(L_K) \geq \text{Gen}(L_K)$ and similarly for relations, if H is as in Proposition 3.3.9 then L_H is not $L_{\mathcal{H}_i}$ -pure-projective for any of the sets of matrices:

$$\mathcal{H}_1 = \{K \mid \text{Gen}(L_H) > \text{Gen}(R)\text{gen}(L_K) \text{ or } \text{Rel}(L_H) > \text{Gen}(R)\text{rel}(L_K)\};$$

$$\mathcal{H}_2 = \{K_{r \times q} \mid r, q \in \mathbb{Z}^+ \text{ such that } \text{Gen}(L_H) > q\text{Gen}(R) \text{ or } \text{Rel}(L_H) > \text{Rel}(L_K)\};$$

$$\mathcal{H}_3 = \bigcup_{r \times q} \{M(R) \mid r, q \in \mathbb{Z}^+ \text{ such that } \text{Gen}(L_H) > q\text{Gen}(R) \text{ or } \text{Rel}(L_H) > r\text{Gen}(R)\}.$$

3.4 Purity over finite-dimensional algebras

In this section we assume some knowledge of the representation theory of finite-dimensional algebras, for which see [6], [7] for example. Let R be a Krull-Schmidt ring and let M be any finitely presented left R -module. We will use $\text{ind}(M)$ to denote the class of (isomorphism types of) indecomposable direct summands of M . If S is a class of finitely presented left R -modules, we define $\text{ind}(S) = \bigcup_{M \in S} \text{ind}(M)$.

Proposition 3.4.1 Let R be a Krull-Schmidt ring and let S be a class of finitely presented left R -modules. Then the following statements are equivalent for a left R -module M .

- (1) M is S -pure-projective.
- (2) M is $\text{ind}(S)$ -pure-projective.
- (3) M is isomorphic to a direct sum of modules in $\text{ind}(S \cup \{{}_R R\})$.

PROOF: (1) \Rightarrow (2) Assume that M is S -pure-projective. By Proposition 3.1.2(2), $M \in \text{Add}(S \cup \{{}_R R\})$. Since R is a Krull-Schmidt ring, each element in $S \cup \{{}_R R\}$ is a direct sum of modules in $\text{ind}(S \cup \{{}_R R\})$ and hence $M \in \text{Add}(\text{ind}(S) \cup \text{ind}(\{{}_R R\}))$. By Proposition 3.1.2(2), M is $\{\text{ind}(S) \cup \text{ind}(\{{}_R R\})\}$ -pure-projective and hence it is $\text{ind}(S)$ -pure-projective.

(2) \Rightarrow (3) Assume that M is $\text{ind}(S)$ -pure-projective. By Proposition 3.1.2(2), $M \in \text{Add}(\text{ind}(S) \cup \{{}_R R\})$. Since R is a Krull-Schmidt ring, ${}_R R$ is a direct sum of modules in

$\text{ind}(\{R\})$. Thus M is isomorphic to a direct summand of a direct sum of copies of modules in $\text{ind}(S \cup \{R\})$. Since R is a Krull-Schmidt, each element in $\text{ind}(S \cup \{R\})$ is a finitely presented left R -module with local endomorphism ring. By [20, Corollary 2.55, p. 67], M is isomorphic to a direct sum of modules in $\text{ind}(S \cup \{R\})$.

(3) \Rightarrow (1) Since each element in $S \cup \{R\}$ is S -pure-projective, each element in $\text{ind}(S \cup \{R\})$ is S -pure-projective. Since the class S -Pproj is closed under direct sums, M is S -pure-projective. \square

This equivalence certainly needs an assumption such as Krull-Schmidt. For instance Puninski [55, Proposition 6.2, p. 324] gives an example of a pure-projective over a uniserial (hence semiperfect) ring which is not a direct sum of indecomposable modules.

The following corollary is immediate from Proposition 3.4.1 and Theorem 3.2.1.

Corollary 3.4.2 Let R be a Krull-Schmidt ring and let S and T be two classes of finitely presented left R -modules. Then T -purity implies S -purity if and only if $\text{ind}(S) \subseteq \text{ind}(T \cup \{R\})$.

Let $R = k\tilde{A}_1$ be the Kronecker algebra over an algebraically closed field k (for the Kronecker algebra see [6], [7] for example). Left R -modules may be viewed as representations of the quiver $1 \bullet \begin{matrix} \alpha \\ \xleftarrow{\quad} \\ \beta \\ \xrightarrow{\quad} \end{matrix} \bullet 2$. The preinjective and preprojective indecomposable finite-dimensional left R -modules are up to isomorphism uniquely determined by their dimension vectors. For $n \in \mathbb{N}$ we will denote by I_n (resp. P_n) the finite-dimensional indecomposable preinjective (resp. preprojective) left R -module with dimension vector $(n, n+1)$ (resp. $(n+1, n)$). Also, for $n \in \mathbb{Z}^+$ we will use $R_{\lambda, n}$ to denote the finite-dimensional indecomposable regular left R -module with dimension vector (n, n) and parameter $\lambda \in k \cup \{\infty\}$ where $R_{\lambda, 1}$ is the module $k \begin{matrix} 1 \\ \xleftarrow{\quad} \\ \lambda \\ \xrightarrow{\quad} \end{matrix} k$ for $\lambda \in k$ and $R_{\infty, 1} = k \begin{matrix} 0 \\ \xleftarrow{\quad} \\ 1 \\ \xrightarrow{\quad} \end{matrix} k$.

Example 3.4.3 Let $R = k\tilde{A}_1$ be the Kronecker algebra over an algebraically closed field k . Let $n \in \mathbb{Z}^+$ and let $S_1 = \{P_i \mid i \leq n\}$, $S_2 = \{I_i \mid i \leq n-1\}$, $S_3 = \{R_{\lambda, i} \mid i \leq n \text{ and } \lambda \in k \cup \{\infty\}\}$ and $S_4 = \{R_{\lambda, 1} \mid \lambda \in k \cup \{\infty\}\} \cup \{P_0, P_1\}$. Then:

(i) $S_1 \cup S_2 \cup S_3$ -purity = (\aleph_0, n) -purity = $(2n+1, n)$ -purity, for short exact sequences of left R -modules.

(ii) S_4 -purity = $(1, 1)$ -purity for left R -modules.

(iii) $(1, 1)$ -purity is not equivalent to (\aleph_0, n) -purity for left R -modules.

PROOF: (i) Let $\mathcal{H} = \bigcup_{m \in \mathbb{Z}^+, m \times n} M(R)$. It follows directly from the description of the finite-dimensional indecomposable modules and Remark 3.3.1 that $\text{ind}(L_{\mathcal{H}}) = S_1 \cup S_2 \cup S_3$. Thus, by Proposition 3.4.1 we have that $S_1 \cup S_2 \cup S_3$ -purity = $L_{\mathcal{H}}$ -purity = (\aleph_0, n) -purity for left R -modules.

Let $M \in S_1 \cup S_2 \cup S_3$. It can be checked that, if $M \in S_1$ then $\text{rel}(M) \leq 2n - 1$, if $M \in S_2$ then $\text{rel}(M) \leq 2n + 1$ and if $M \in S_3$ then $\text{rel}(M) \leq 2n$ and hence $\text{rel}(M) \leq 2n + 1$ in all cases. Since $\text{gen}(M) \leq n$, each module in $S_1 \cup S_2 \cup S_3$ is $(n, 2n + 1)$ -presented. Thus $(2n + 1, n)$ -purity = $S_1 \cup S_2 \cup S_3$ -purity = (\aleph_0, n) -purity.

(ii) Let $\lambda \in k \cup \{\infty\}$ and let $M = R_{\lambda,1} \oplus P_0$. Since the sequence ${}_R R \xrightarrow{(\alpha + \lambda\beta) \times -} {}_R R \rightarrow M \rightarrow 0$ is exact, M is $(1, 1)$ -presented and hence $R_{\lambda,1}$ is a direct summand of a $(1, 1)$ -presented module. Thus every module in S_4 is a direct summand of a $(1, 1)$ -presented module. Conversely, let N be any indecomposable direct summand of a $(1, 1)$ -presented left R -module, thus $\text{gen}(N) = 1$ and $\text{rel}(N) \leq 2$ (by Remark 3.3.1) and hence either $N = P_0$ or $N = P_1$ or $N = R_{\lambda,1}$ for some $\lambda \in k \cup \{\infty\}$. Thus N is a direct summand of a module in $S_4 \cup \{{}_R R\}$. By Proposition 3.2.7, S_4 -purity = $(1, 1)$ -purity.

(iii) Assume that $(1, 1)$ -purity = (\aleph_0, n) -purity for some $n \in \mathbb{Z}^+$. Thus, by (i) and (ii) above we have that S_4 -purity = $S_1 \cup S_2 \cup S_3$ -purity. This contradicts Corollary 3.4.2, because $I_0 \in S_1 \cup S_2 \cup S_3$ and $I_0 \notin S_4$. \square

Proposition 3.4.4 Let R be a finite-dimensional algebra over a field k . If R is not of finite representation type, then for every $r \in \mathbb{Z}^+$, there is $n > r$ such that (\aleph_0, n) -purity \neq (\aleph_0, r) -purity for left R -modules.

PROOF: Suppose that R is not of finite representation type. Assume that there is $r \in \mathbb{Z}^+$ such that for all $n > r$ then (\aleph_0, n) -purity = (\aleph_0, r) -purity for left R -modules. Since R is a finite-dimensional algebra and it is not of finite representation type it follows from [7, Corollary 1.5, p. 194] that there is a finitely generated indecomposable left R -module M such that $\text{gen}(M) \geq r + 1$. By assumption, $(\aleph_0, \text{gen}(M))$ -purity = (\aleph_0, r) -purity for left R -modules and hence by Corollary 3.4.2, $M \in \text{ind}(\{(r, s)$ -presented left R -modules $| s \in \mathbb{Z}^+\})$,

which is a contradiction. \square

Proposition 3.4.5 Let R be a finite-dimensional algebra over a field k and let \mathcal{H} be a set of matrices over R . Then a left R -module M is $L_{\mathcal{H}}$ -pure-injective if and only if M is a direct summand of a direct product of modules in $\text{ind}(\{(D_H)^* \mid H \in \mathcal{H} \cup \{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \}\})$.

PROOF: (\Rightarrow) Assume that M is an $L_{\mathcal{H}}$ -pure-injective left R -module. By Theorem 3.1.4, $M \in \text{Prod}(\{(D_H)^* \mid H \in \mathcal{H} \cup \{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \}\})$. Since R is an artinian ring and $(D_H)^*$ is a finitely presented left R -module, for all $H \in \mathcal{H} \cup \{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \}$ it follows that $(D_H)^*$ is a finite direct sum of modules in $\text{ind}(\{(D_H)^* \mid H \in \mathcal{H} \cup \{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \}\})$. Thus M is a direct summand of a direct product of modules in $\text{ind}(\{(D_H)^* \mid H \in \mathcal{H} \cup \{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \}\})$.

(\Leftarrow) By Theorem 3.1.4, each module in $\text{ind}(\{(D_H)^* \mid H \in \mathcal{H} \cup \{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \}\})$ is an $L_{\mathcal{H}}$ -pure-injective left R -module. Since the class of $L_{\mathcal{H}}$ -pure-injective left R -modules is closed under direct products and direct summands, M is $L_{\mathcal{H}}$ -pure-injective. \square

The **Auslander-Reiten translate** of a module M is given by the formula $\tau M = (D(M))^*$ where $D(M)$ is the Auslander-Bridger dual (= transpose) of M obtained from a minimal projective resolution of M .

We now describe these modules in terms of $\text{ind}(\{L_H \mid H \in \mathcal{H} \cup \{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \}\})$.

Theorem 3.4.6 Let R be a finite-dimensional algebra over a field k and let S be a set of indecomposable finite-dimensional modules. Then the S -pure-injective left R -modules are exactly the direct summands of direct products of modules in $\tau S \cup R\text{-inj}$, where τ is the Auslander-Reiten translate and $R\text{-inj}$ denotes the set of indecomposable injective left R -modules.

PROOF: Since $\tau L_H = (D_H)^*$, this follows from Proposition 3.4.5. \square

Corollary 3.4.7 Let R be a finite-dimensional algebra over a field k , and let \mathcal{H} be a set of matrices over R . If $\text{ind}(\{(D_H)^* \mid H \in \mathcal{H} \cup \{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \}\})$ is finite then it is the set of indecomposable $L_{\mathcal{H}}$ -pure-injective left R -modules and every $L_{\mathcal{H}}$ -pure-injective module is a direct sum

of copies of these modules.

PROOF: Let $T = \text{ind}\{(D_H)^* \mid H \in \mathcal{H} \cup \left\{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \right\}\}$. Assume that T is a finite set and let $T = \{N_i\}_{i \in I}$ where I is a finite index set. Assume that M is an indecomposable $L_{\mathcal{H}}$ -pure-injective left R -module. By Proposition 3.4.5, $M \in \text{Prod} T$ and hence M is a direct summand of $\bigoplus_{j \in J} N_j^{I_j}$ where $J \subseteq I$. For each $j \in J$, since R is an artin algebra and N_j is a finitely generated R -module, N_j is of finite length over its endomorphism ring. Since N_j is indecomposable it follows from a theorem of Garavaglia, see [52, Theorem 4.4.28, p. 180], that $N_j^{I_j} = N_j^{(J_j)}$ for some set J_j and hence M is a direct summand of $\bigoplus_{j \in J} N_j^{(J_j)}$. Since M is indecomposable pure-injective, M is a direct summand of N_l for some $l \in I$, since indecomposable pure-injectives have local endomorphism ring (see [52, Theorem 4.3.43, p. 157]). Since N_l is indecomposable, $M = N_l$. Hence $M \in \text{ind}\{(D_H)^* \mid H \in \mathcal{H} \cup \left\{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \right\}\}$. The converse is true by applying Proposition 3.4.5. Thus $\text{ind}\{(D_H)^* \mid H \in \mathcal{H} \cup \left\{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \right\}\}$ is the set of indecomposable $L_{\mathcal{H}}$ -pure-injective left R -modules.

Now we will prove that every $L_{\mathcal{H}}$ -pure-injective module is a direct sum of copies of modules in T . Let N be any $L_{\mathcal{H}}$ -pure-injective module. By a similar argument to that above we can prove that N is a direct summand of $\bigoplus_{j \in J} N_j^{(J_j)}$, for some sets J_j , with $J \subseteq I$. For each $j \in J$, since R is a finite-dimensional algebra and N_j is an indecomposable finitely generated R -module it follows that N_j has local endomorphism ring. By [20, Corollary 2.55, p. 67], N is a direct sum of copies of modules in T . \square

A class T of pure-injective modules closed under direct products and direct summands is definable if and only if each direct sum of modules in T is pure-injective, that is if and only if each element in T is Σ -pure-injective (see, for example, [52, Proposition 4.4.12, p. 176]). In this case every module in T is a direct sum of indecomposable modules (see, for example, [52, Theorem 4.4.19, p. 178]).

Corollary 3.4.8 Let R be a finite-dimensional algebra over a field k and let \mathcal{H} be a set of matrices over R . Then $L_{\mathcal{H}}\text{-Pinj}$ is a definable subclass of $R\text{-Mod}$ if and only if each direct sum of modules in $\text{ind}(\{(D_H)^* \mid H \in \mathcal{H}\})$ is pure-injective.

PROOF: Let $T = \text{ind}(\{(D_H)^* \mid H \in \mathcal{H}\})$ and $T' = T \cup R\text{-inj}$.

One direction follows from the remarks above and Proposition 3.4.5.

(\Leftarrow) By hypothesis, each direct sum of modules in T is pure-injective. Since R is a left noetherian ring, each direct sum of modules in $R\text{-inj}$ is injective. Thus each direct sum of modules in T' is pure-injective and hence is Σ -pure-injective. Let $M \in L_{\mathcal{H}}\text{-Pinj}$. By Proposition 3.4.5, there exists a subfamily $\{M_i\}_{i \in I}$ of T' such that M is a direct summand of $\prod_{i \in I} M_i$. By the proof above, $\bigoplus_{i \in I} M_i$ is Σ -pure-injective. Since $\prod_{i \in I} M_i$ is in the definable subcategory generated by $\bigoplus_{i \in I} M_i$ it follows from [52, Proposition 4.4.12, p. 176] that $\prod_{i \in I} M_i$ is Σ -pure-injective. It follows that M is Σ -pure-injective and hence each element in $L_{\mathcal{H}}\text{-Pinj}$ is Σ -pure-injective. Therefore $L_{\mathcal{H}}\text{-Pinj}$ is a definable subclass of $R\text{-Mod}$. \square

Let R be a finite-dimensional algebra over a field k . Recall that a connected component Γ of the Auslander-Reiten quiver is said to be **preprojective** (resp. **preinjective**) if Γ consists of indecomposable projective (resp. injective) modules, P (resp. I), together with all their Auslander-Reiten translates, $\tau^{-n}P$ (resp. $\tau^n I$), where $n \geq 0$. Recall that a connected component Γ of the Auslander-Reiten quiver of a hereditary algebra is said to be **regular** if Γ contains neither projective nor injective modules. An indecomposable R -module is said to be preprojective (resp. preinjective, resp. regular) if it belongs to a preprojective (resp. preinjective, resp. regular) component of the Auslander-Reiten quiver.

Every finite-dimensional module is Σ -pure-injective and by [36, Theorem 4.6, p. 750] every direct sum of preinjective modules is Σ -pure-injective. The equivalence of (1) and (2) in the next result therefore follows from the description of the Σ -pure-injective modules in [53, Theorem 2.1, p. 847] and the equivalence with (3) follows since the duality $\text{Hom}_k(-, k)$ interchanges preprojective and preinjective modules and sends regular modules to regular modules.

Proposition 3.4.9 Let R be a tame hereditary finite-dimensional algebra over a field k and let \mathcal{H} be a set of matrices over R . Then the following statements are equivalent.

- (1) $L_{\mathcal{H}}\text{-Pinj}$ is a definable subclass of $R\text{-Mod}$.
- (2) The set of preprojective or regular modules in $\text{ind}(\{(D_H)^* \mid H \in \mathcal{H}\})$ is finite.
- (3) The set of preinjective or regular modules in $\text{ind}(\{D_H \mid H \in \mathcal{H}\})$ is finite.

From now on, for simplicity of some statements, we assume that R is connected.

Let R be a tame hereditary finite-dimensional algebra over a field k and let S be a **simple regular** left R -module (that is, a module which is simple in the category of regular modules). The infinite-dimensional indecomposable pure-injective modules over a tame hereditary finite-dimensional algebra have been classified: for each simple regular module S there is a Prüfer module $S^{[\infty]}$ and an adic module \hat{S} ; there is also the generic module (which is of finite length over its endomorphism ring). See e.g. [59, p. 106] for the definitions of these modules.

Proposition 3.4.10 Let R be a tame hereditary finite-dimensional algebra over a field k . Let \mathcal{H} be a set of matrices over R such that $L_{\mathcal{H}}\text{-Pinj}$ is definable. Then the following statements are equivalent.

- (1) The generic module is $L_{\mathcal{H}}$ -pure-injective.
- (2) The set of preinjective left R -modules in $\text{ind}(\{(D_H)^* \mid H \in \mathcal{H}\})$ is infinite.
- (3) All but at most $n(R) - 2$ Prüfer modules are $L_{\mathcal{H}}$ -pure-injective, where $n(R)$ is the number of isomorphism classes of simple R -modules.
- (4) At least one Prüfer R -module is $L_{\mathcal{H}}$ -pure-injective.

PROOF: (1) \Rightarrow (2) Let $T = \text{ind}(\{(D_H)^* \mid H \in \mathcal{H}\})$. Assume that the set of preinjective left R -modules in T is finite. Since $L_{\mathcal{H}}\text{-Pinj}$ is definable it follows from Proposition 3.4.9 that T is finite. By Corollary 3.4.7, the generic module cannot be $L_{\mathcal{H}}$ -pure-injective.

(2) \Rightarrow (3) Let X be the class of all indecomposable $L_{\mathcal{H}}$ -pure-injective modules. Since $L_{\mathcal{H}}\text{-Pinj}$ is definable it follows from Theorem 2.4.7 that X is a closed set of the Ziegler topology. Since X contains infinitely many non-isomorphic preinjective modules, by [59, Corollary, p. 113], all but at most $n(R) - 2$ Prüfer modules belong to X , where $n(R)$ is the number of isomorphism classes of simple R -modules.

(3) \Rightarrow (4) This is obvious.

(4) \Rightarrow (1) Assume that there is a Prüfer module which is $L_{\mathcal{H}}$ -pure-injective. As noted in (3), X is a closed set of the Ziegler topology and by hypothesis, it contains at least one module which is not of finite length. By [59, Theorem, p. 106], the generic module belongs to X . □

Remark 3.4.11 If R is the Kronecker algebra over a field k then condition (3) above becomes: (3) Every Prüfer module is $L_{\mathcal{H}}$ -pure-injective.

Let $T \subseteq R\text{-ind}$, the class of all finitely presented indecomposable left R -modules. If R is a finite-dimensional algebra then every finitely generated module is pure-injective (see, e.g. [52, Remark 4.5.34, p. 201]) so T will be a subset of ${}_R\text{pinj}$. We use $\text{fsc}(T)$ (resp. \overline{T}) to denote the closure of T in the full support topology (resp. the Ziegler topology).

Lemma 3.4.12 Let R be a finite-dimensional algebra over a field k and let $T \subseteq R\text{-ind}$. Then $\text{fsc}(T) = (\text{Prod}(T)) \cap {}_R\text{pinj}$.

PROOF: Let $M \in (\text{Prod}(T))^+$, thus M is a pure submodule of N for some $N \in \text{Prod}(T)$ and hence by [22, Exercise 2.3, p. 433] we have that $PE(M)$ is a direct summand of $PE(N)$. Since $\text{Prod}(T) \subseteq {}_R\text{Pinj}$ it follows that $N \simeq PE(N)$ and hence $PE(M) \in \text{Prod}T \subseteq (\text{Prod}T)^+$. Thus $(\text{Prod}T)^+$ is closed under pure-injective hulls.

Since $(\text{Prod}(T))^+$ is closed under pure submodules and direct products it follows that $(\text{Prod}(T))^+$ is a type-definable subclass of $R\text{-Mod}$. By Theorem 2.4.9, $(\text{Prod}(T))^+ \cap {}_R\text{pinj}$ is a closed set of the full support topology. Since $(\text{Prod}(T)) \cap {}_R\text{pinj} = (\text{Prod}(T))^+ \cap {}_R\text{pinj}$ it follows that $(\text{Prod}(T)) \cap {}_R\text{pinj}$ is a closed set of the full support topology. Since $T \subseteq (\text{Prod}(T)) \cap {}_R\text{pinj}$ it follows that $\text{fsc}(T) \subseteq \text{fsc}((\text{Prod}(T)) \cap {}_R\text{pinj}) = (\text{Prod}(T)) \cap {}_R\text{pinj}$.

Conversely, let $M \in (\text{Prod}(T)) \cap {}_R\text{pinj}$. Thus $M \in {}_R\text{pinj}$ and M is a direct summand of $\prod_{i \in I} M_i$ where $M_i \in T$. Since $\text{fsc}(T)$ is a closed set of the full support topology it follows from Theorem 2.4.9 that $\text{fsc}(T) = \mathcal{X} \cap {}_R\text{pinj}$ for some type-definable subclass \mathcal{X} of $R\text{-Mod}$. Thus $T \subseteq \mathcal{X}$ and hence $M_i \in \mathcal{X}$ for all $i \in I$. Since \mathcal{X} is a type-definable subclass, $M \in \mathcal{X}$ and hence $M \in \mathcal{X} \cap {}_R\text{pinj} = \text{fsc}(T)$. Thus $(\text{Prod}(T)) \cap {}_R\text{pinj} \subseteq \text{fsc}(T)$ and hence $\text{fsc}(T) = (\text{Prod}(T)) \cap {}_R\text{pinj}$. \square

Lemma 3.4.13 Let R be a finite-dimensional algebra over a field k and let $T \subseteq R\text{-ind}$. If $\text{Prod}(T)$ is definable then $\overline{T} = \text{fsc}(T)$.

PROOF: Suppose that $\text{Prod}(T)$ is definable. It is clear that $\text{fsc}(T) \subseteq \overline{T}$. Since $T \subseteq \text{Prod}(T)$ it follows that $\langle T \rangle \subseteq \text{Prod}(T)$, where $\langle T \rangle$ is the definable subcategory generated by T . Thus $\overline{T} \subseteq \text{fsc}(T)$ and hence $\overline{T} = \text{fsc}(T)$. \square

Remark 3.4.14 Let T be a class of pure-injective left R -modules and let $S \subseteq T$. If $\text{Prod}(T)$ is a definable subclass of $R\text{-Mod}$ then so is $\text{Prod}(S)$.

Corollary 3.4.15 Let R be a tame hereditary finite-dimensional algebra over a field k and let I_1 be a class of indecomposable preinjective left R -modules. Then $\text{fsc}(I_1) = \overline{I_1}$.

PROOF: Since a hereditary algebra R is tame if and only if the class $\text{Prod}({}_R I)$ is definable, where ${}_R I$ is the class of all indecomposable preinjective left R -modules (by [83, Theorem 3.2, p. 351]) it follows that $\text{Prod}({}_R I)$ is definable. Since $I_1 \subseteq {}_R I$ it follows from Remark 3.4.14 that $\text{Prod}(I_1)$ is definable. By Lemma 3.4.13, $\text{fsc}(I_1) = \overline{I_1}$. \square

Remark 3.4.16 Let R be a finite-dimensional algebra over a field k and let $T \subseteq R\text{-ind}$. Then T is the class of all indecomposable finite-dimensional left R -modules in $\text{Prod}(T)$. This follows from [52, Corollary 5.3.33, p. 250].

The following fact is known; it can be found stated in [60, p. 47]. We include a proof here.

Proposition 3.4.17 Let R be a tame hereditary finite-dimensional algebra over a field k and let P_1 be a class of indecomposable preprojective left R -modules. Then $\text{fsc}(P_1) = P_1$.

PROOF: Let $M \in \text{fsc}(P_1)$. Thus M is a direct summand of $\prod_{i \in I} P_i$ where $P_i \in P_1$. Choose a non-zero element $a \in M$, so $a_j \neq 0$ for some $j \in I$, where a_j is the j th component in a . Define $\alpha : M \rightarrow P_j$ by $\alpha = \pi_j i$ where $i : M \rightarrow \prod_{i \in I} P_i$ is the inclusion and $\pi_j : \prod_{i \in I} P_i \rightarrow P_j$ is the projection. Since $\alpha(a) = a_j \neq 0$ it follows that $\text{Hom}_R(M, {}_R P) \neq 0$, where ${}_R P$ is the class of all indecomposable preprojective left R -modules. By [16, Lemma 1, p. 46], M has a preprojective direct summand, and hence M is finite-dimensional and therefore we have from Remark 3.4.16 that $M \in P_1$. \square

Lemma 3.4.18 Let R be a tame hereditary finite-dimensional algebra over a field k and let R_1 be a class of indecomposable regular left R -modules. Then:

- (1) The generic module does not belong to $\text{fsc}(R_1)$.
- (2) There is no Prüfer R -module in $\text{fsc}(R_1)$.

PROOF: (1) Assume that the generic module $G \in \text{fsc}(\mathbf{R}_1)$, thus $G \in \text{Prod}(\mathbf{R}_1)$. As in the proof of Proposition 3.4.17 it follows that $\text{Hom}_R(G, \mathbf{R}_1) \neq 0$, contradicting [60, p. 46]. Therefore $G \notin \text{fsc}(\mathbf{R}_1)$.

(2) Assume that there is a Prüfer module M such that $M \in \text{fsc}(\mathbf{R}_1)$. By [59, Proposition 3, p. 110], the generic module G is a direct summand of M^I for some I so $G \in \text{Prod}(\mathbf{R}_1)$ and this contradicts (1) above. Thus there is no Prüfer module in $\text{fsc}(\mathbf{R}_1)$. \square

We use T_S to denote the class $T_S = \{M \mid M \text{ is an indecomposable regular left } R\text{-module with } \text{Hom}_R(M, S) \neq 0\}$.

Theorem 3.4.19 Let R be a tame hereditary finite-dimensional algebra over a field k . Let \mathbf{R}_1 be a class of indecomposable regular left R -modules and let S be a simple regular left R -module. Then the following statements are equivalent.

- (1) $\hat{S} \in \text{fsc}(\mathbf{R}_1)$.
- (2) $\mathbf{R}_1 \cap T_S$ is infinite.
- (3) $\hat{S} \in \text{fsc}(\mathbf{R}_1 \cap T_S)$.

PROOF: (1) \Rightarrow (2) Suppose that $\hat{S} \in \text{fsc}(\mathbf{R}_1)$. Assume that $\mathbf{R}_1 \cap T_S$ is finite. Let $D = \{M \mid \text{Hom}_R(M, S) = 0\}$. By [16, Examples, p. 42], D is a definable subclass of $R\text{-Mod}$ and hence $C = D \cap {}_R\text{pinj}$ is a closed set in the Ziegler topology. Since $\mathbf{R}_1 \cap T_S$ is a finite class of finite-dimensional indecomposable modules it follows from [16, 2.5] that $\mathbf{R}_1 \cap T_S$ is a closed set in the Ziegler topology and hence $C \cup (\mathbf{R}_1 \cap T_S)$ is. Thus $C \cup (\mathbf{R}_1 \cap T_S)$ is a closed set in the full support topology. Since $\mathbf{R}_1 \subseteq C \cup (\mathbf{R}_1 \cap T_S)$ it follows that $\text{fsc}(\mathbf{R}_1) \subseteq \text{fsc}(C \cup (\mathbf{R}_1 \cap T_S)) = C \cup (\mathbf{R}_1 \cap T_S)$. Since $\text{Hom}_R(\hat{S}, S) \neq 0$ it follows that $\hat{S} \notin C \cup (\mathbf{R}_1 \cap T_S)$ and hence $\hat{S} \notin \text{fsc}(\mathbf{R}_1)$ and this contradicts the hypothesis. Thus $\mathbf{R}_1 \cap T_S$ is infinite.

(2) \Rightarrow (3) Suppose that $\mathbf{R}_1 \cap T_S$ is infinite, thus $(\mathbf{R}_1 \cap T_S)^*$ is an infinite class of regular right R -modules. Let $X \in (\mathbf{R}_1 \cap T_S)^*$, thus $X = M^*$ for some $M \in \mathbf{R}_1 \cap T_S$. Hence $\text{Hom}_R(M, S) \neq 0$. Thus $\text{Hom}_R(S^*, X) \neq 0$ for all $X \in (\mathbf{R}_1 \cap T_S)^*$. By [59, Proposition 1, p. 107], $S^*[\infty]$ is the direct limit of a chain of monomorphisms $X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \dots$ with $X_i \in (\mathbf{R}_1 \cap T_S)^*$. Therefore, by Lemma 2.2.6(2), there is a pure exact sequence $0 \rightarrow N \rightarrow \bigoplus_{i < \omega} X_i \rightarrow S^*[\infty] \rightarrow 0$. Therefore the exact sequence $0 \rightarrow (S^*[\infty])^* \rightarrow (\bigoplus_{i < \omega} X_i)^* \rightarrow N^* \rightarrow 0$ is split. By [16, Examples, p. 44], $\hat{S} = (S^*[\infty])^*$ and hence \hat{S} is a direct summand of $\prod_{i < \omega} X_i^*$.

Thus $\hat{S} \in \text{Prod}(\mathbf{R}_1 \cap T_S)$ and this implies that $\hat{S} \in \text{fsc}(\mathbf{R}_1 \cap T_S)$.

(3) \Rightarrow (1) This is obvious. \square

Corollary 3.4.20 Let R be a tame hereditary finite-dimensional algebra over a field k and let \mathbf{R}_1 be a class of indecomposable regular left R -modules. Then $\text{fsc}(\mathbf{R}_1) = \mathbf{R}_1 \cup \{\hat{S} \mid \mathbf{R}_1 \cap T_S \text{ is infinite}\}$.

PROOF: Let $M \in \{\hat{S} \mid \mathbf{R}_1 \cap T_S \text{ is infinite}\}$. Thus $M = \hat{S}_i$ for some simple regular module S_i with $\mathbf{R}_1 \cap T_{S_i}$ is infinite and hence by Theorem 3.4.19 we have that $M \in \text{fsc}(\mathbf{R}_1)$. It follows that $\mathbf{R}_1 \cup \{\hat{S} \mid \mathbf{R}_1 \cap T_S \text{ is infinite}\} \subseteq \text{fsc}(\mathbf{R}_1)$. Conversely, let $M \in \text{fsc}(\mathbf{R}_1)$. If M is finite-dimensional then $M \in \mathbf{R}_1$, by Remark 3.4.16. If M is infinite-dimensional then M is neither Prüfer nor the generic module, by Lemma 3.4.18. Hence $M = \hat{S}_j$ for some simple regular module S_j . By Theorem 3.4.19, $\mathbf{R}_1 \cap T_{S_j}$ is infinite and hence $M \in \{\hat{S} \mid \mathbf{R}_1 \cap T_S \text{ is infinite}\}$. Thus $\text{fsc}(\mathbf{R}_1) \subseteq \mathbf{R}_1 \cup \{\hat{S} \mid \mathbf{R}_1 \cap T_S \text{ is infinite}\}$ and hence $\text{fsc}(\mathbf{R}_1) = \mathbf{R}_1 \cup \{\hat{S} \mid \mathbf{R}_1 \cap T_S \text{ is infinite}\}$. \square

In the following corollary we give a complete description of the closure of any subclass of R -ind in the full support topology and hence, by Theorem 3.4.6, a description of the indecomposable S -pure-injective modules for any purity defined by a class S of finitely presented modules.

Corollary 3.4.21 Let R be a tame hereditary finite-dimensional algebra over a field k . Let I_1 (resp. P_1 , resp. \mathbf{R}_1) be a class of indecomposable preinjective (resp. preprojective, resp. regular) left R -modules. Then $\text{fsc}(I_1 \cup P_1 \cup \mathbf{R}_1) = \overline{I_1} \cup P_1 \cup \mathbf{R}_1 \cup \{\hat{S} \mid \mathbf{R}_1 \cap T_S \text{ is infinite}\}$.

PROOF: This follows from Corollary 3.4.15, Proposition 3.4.17 and Corollary 3.4.20. \square

3.5 Rings whose indecomposable modules are S -pure-projective

Let T be a set. A family F of subsets of T is said to be **directed** if for any $U, V \in F$, there exists $W \in F$ such that $U \subseteq W$ and $V \subseteq W$.

By using Theorem 3.1.1, we can prove the following lemma.

Lemma 3.5.1 Let S be a class of finitely presented left R -modules and let $\{N_i\}_{i \in I}$ be any directed family of S -pure submodules of a left R -module M . Then $N = \bigcup_{i \in I} N_i$ is an S -pure submodule of M .

Let N be a submodule of a left R -module M and let T be a set of submodules of M . We will use $N(T)$ to denote the submodule $N(T) = N + \sum_{A \in T} A$.

Lemma 3.5.2 Let S be a class of finitely presented left R -modules, let N be a submodule of a left R -module M and let T be a set of submodules of M . If $N(F)$ is an S -pure submodule of M for all finite subsets F of T , then $N(T)$ is an S -pure submodule of M .

PROOF: Assume that $N(F)$ is an S -pure submodule of M for all finite subsets F of T . Let $\{F_i\}_{i \in I}$ be the family of all finite subsets of T and let $D = \{N(F_i)\}_{i \in I}$. Let $N(F_i), N(F_j) \in D$, thus F_i and F_j are finite subsets of T . Let $W = F_i \cup F_j$, thus W is a finite subset of T . Since F_i and F_j are subsets of W it follows that $N(F_i)$ and $N(F_j)$ are subsets of $N(W)$ and $N(W) \in D$. Thus D is a direct family of S -pure submodules of M and hence by Lemma 3.5.1 we have that $\bigcup_{i \in I} N(F_i)$ is an S -pure submodule of M . Since $\bigcup_{i \in I} N(F_i) = N(T)$ it follows that $N(T)$ is an S -pure submodule of M . \square

Definition 3.5.3 Let S be a class of finitely presented left R -modules, let N be a submodule of a left R -module M and let T_0 be the set of all indecomposable submodules of M . A subclass $T \subseteq T_0$ is said to be **S - N -independent** (in M) if $N(T) = N \oplus (\sum_{B \in T} B)$ and $N(T)$ is an S -pure submodule of M . This will be the case if and only if every finite subset of T is S - N -independent in M .

Theorem 3.5.4 Let S be any set of finitely presented left R -modules and let M be a left R -module. Suppose that every S -pure submodule M_0 of M for which M/M_0 is indecomposable is a direct summand of M . Then every S -pure submodule of M is a direct summand of M and M is a direct sum of indecomposable submodules.

PROOF: (The following proof is based on an argument in [8, Proposition 1.13, p. 53]). Let N be any S -pure submodule of M . If $N = M$ then N is a direct summand of M . Assume that $N \neq M$, thus there is $x \in M \setminus N$. Let $F = \{K \mid N \subseteq K, x \notin K \text{ and } K \text{ is an } S\text{-pure submodule of } M\}$. Since $N \in F$ it follows that F is a non-empty family. Let $\{M_i\}_{i \in I}$ be any directed subfamily of F and let $A = \bigcup_{i \in I} M_i$. It is clear that $N \subseteq A$ and $x \notin A$. By Lemma 3.5.1, A is an S -pure submodule of M and hence $A \in F$. By Zorn's lemma, F has a maximal element, say M_0 , thus M_0 is an S -pure submodule of M with $N \subseteq M_0$ and $x \notin M_0$. We will prove that M/M_0 is indecomposable.

Assume that M/M_0 is not indecomposable, thus there are two non-zero submodules $M_1/M_0, M_2/M_0$ of M/M_0 such that $M/M_0 = (M_1/M_0) \oplus (M_2/M_0)$. Therefore $M_0 \subsetneq M_1, M_0 \subsetneq M_2$ and $M_1 \cap M_2 = M_0$. Since M_1/M_0 and M_2/M_0 are direct summands of M/M_0 they are S -pure submodules of M/M_0 . Since M_0 is an S -pure submodule of M it follows from [79, 33.3(4), p. 276] that M_1 and M_2 are S -pure submodules of M . Thus, by maximality of M_0 , we have that $x \in M_1 \cap M_2$ and this is a contradiction.

Hence M/M_0 is a non-zero indecomposable left R -module. By assumption, M_0 is a direct summand of M , say $M = N_0 \oplus M_0$. Thus $N_0 \simeq M/M_0$ is a non-zero indecomposable submodule of M with $N + N_0 = N \oplus N_0$. Since N is an S -pure submodule of M and $N \subseteq M_0 \subseteq M$ it follows that N is S -pure submodule of M_0 and hence $N \oplus N_0$ is an S -pure submodule of $N_0 \oplus M_0 = M$. Thus, for any proper S -pure submodule N of M , there exists a non-zero indecomposable submodule N_0 of M such that $N \cap N_0 = 0$ and $N \oplus N_0$ is an S -pure submodule of M .

Let T be the family of all S - N -independent subsets in M . Since $\{0\} \in T$ it follows that T is non-empty. Let D be any directed subfamily of T and let U be the union of all members of D . Then $U \in T$ since every finite subset of U is S - N -independent. By Zorn's lemma, T has a maximal element, say W . Now we will prove that $N(W) = M$. Assume that $N(W) \neq M$, thus $N(W)$ is a proper S -pure submodule of M . Hence there exists a non-zero indecomposable

submodule B of M such that $N(W) \cap B = 0$ and $N(W) + B = N(W) \oplus B = N \oplus (\sum_{A \in W} \oplus A) \oplus B$ is an S -pure submodule of M , as seen above. Hence $W \cup \{B\}$ properly contains W and is S - N -independent in M . This contradicts the maximality of W in T . Therefore, $N(W) = M$. Since $N(W) = N \oplus (\sum_{A \in W} \oplus A)$ it follows that N is a direct summand of M and $M/N \simeq \sum_{A \in W} \oplus A$ is a direct sum of indecomposable submodules. If we take $N = 0$ then we see that M is a direct sum of indecomposable submodules. \square

Corollary 3.5.5 Let S be any set of finitely presented left R -modules. Then the following statements are equivalent.

- (1) Every indecomposable left R -module is S -pure-projective.
- (2) For any left R -module M , every S -pure submodule of M is a direct summand of M .
- (3) Every left R -module is S -pure-projective.
- (4) Every left R -module is S -pure-injective.
- (5) Every left R -module is a direct sum of modules in $\text{ind}(S \cup \{R\})$.

PROOF: (1) \Rightarrow (2) Let M be any left R -module and let N be any S -pure submodule of M such that M/N is indecomposable. By hypothesis, M/N is S -pure-projective hence the S -pure exact sequence $0 \rightarrow N \xrightarrow{i} M \xrightarrow{\pi} M/N \rightarrow 0$ splits and hence N is a direct summand of M . By Theorem 3.5.4, every S -pure submodule of M is a direct summand of M .

(2) \Rightarrow (3) Let M be any left R -module and let $\Sigma : 0 \rightarrow L \xrightarrow{f} N \xrightarrow{g} M \rightarrow 0$ be any S -pure exact sequence of left R -modules. By hypothesis, $\text{im}(f)$ is a direct summand of N and hence Σ is split so M is S -pure-projective.

(3) \Rightarrow (5) Assume that every left R -module is S -pure-projective, thus every left R -module is pure-projective. By [8, Proposition 4.4, p. 73], R is a left artinian ring and hence R is Krull-Schmidt, by [52, p. 164]. Let M be any left R -module. By hypothesis and Proposition 3.4.1, M is isomorphic to a direct sum of modules in $\text{ind}(S \cup \{R\})$. Thus every left R -module is a direct sum of modules in $\text{ind}(S \cup \{R\})$.

(5) \Rightarrow (1) Assume that every left R -module is a direct sum of modules in $\text{ind}(S \cup \{R\})$. Let M be an indecomposable left R -module, thus $M \in \text{ind}(S \cup \{R\})$. Since each module in $\text{ind}(S \cup \{R\})$ is S -pure-projective, M is S -pure-projective. Hence every indecomposable left R -module is S -pure-projective.

(2) \Leftrightarrow (4) This follows by using Theorem 2.2.19. \square

Remark 3.5.6 It is clear that any ring satisfying the conditions in Corollary 3.5.5 is a left pure semisimple ring but the following example shows that the converse is not true. Let $R = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ be the ring of all lower triangular 2×2 -matrices over a field k . Then R is of finite representation type, in particular is pure semisimple. Let $S = \left\{ \begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} 0 \\ k \end{pmatrix} \right\}$, thus $\text{ind}(S \cup \{R_R\}) = S$. Then $M = \begin{pmatrix} k \\ 0 \end{pmatrix}$ is a left R -module which is not a direct sum of modules in $\text{ind}(S \cup \{R_R\})$ and hence R does not satisfy the conditions in Corollary 3.5.5 for this choice of S .

Chapter 4

Definability of the classes $(\varinjlim S)$ and $(\text{Prod } S^*)$

4.1 Introduction

A ring R is said to be **right (resp. left) noetherian** if every right (resp. left) ideal of R is finitely generated. As a generalization, the concept of a left (resp. right) coherent ring was introduced. A ring R is said to be **right (resp. left) coherent** if every finitely generated right (resp. left) ideal of R is finitely presented.

It is well-known (see, e.g. [2, Proposition 18.13, p. 209]) that a ring R is right noetherian if and only if the class Inj_R of injective right R -modules is closed under direct sums. An equivalent condition (see [17, Theorem 3.19, p. 266]) is that this class is definable.

Also, Chase in [11, Theorem 2.1, p. 460] proved that a ring R is right coherent if and only if the class ${}_R\mathcal{F}\text{lat}$ is closed under direct products. Equivalently, the class ${}_R\mathcal{F}\text{lat}$ is definable (see [63, Theorem 4, p. 629]).

Recently, Holm in [26], in the process of extending results of Lenzing from [36], introduced the concepts of right S -coherent (resp. S -noetherian) rings as generalizations of right coherent (resp. noetherian) rings as follows: a ring R is said to be **right S -coherent** (resp. **S -noetherian**) if the class $\varinjlim(\text{add } S)$ (resp. $\text{Prod } S^*$) is definable, where $S \subseteq R\text{-mod}$. In fact, if we take $S = \{{}_R R\}$ then $\varinjlim(\text{add } S) = {}_R\mathcal{F}\text{lat}$ (by Lazard [35]) and $\text{Prod } S^* = \text{Prod } \{{}_R R\}^* = \text{Inj}_R$ and hence a ring R is right S -coherent (resp. S -noetherian) if and only if it is right coherent (resp. noetherian).

Holm proved in [26, Theorem 1.5, p. 545] the following result.

Theorem 4.1.1 Let S be a class of finitely presented left R -modules such that ${}_R R \in S$. Then $\text{Prod } S^*$ is definable if and only if

- (i) $\varinjlim(\text{add } S)$ is definable, and
- (ii) a right R -module M is in $\text{Prod } S^*$ if and only if $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is in $\varinjlim(\text{add } S)$.

In the literature, many characterizations of definability of $\varinjlim S$ have been given but without using the duality of modules, for example: if $S \subseteq R\text{-mod}$ and $S = \text{add } S$, then (1) $\varinjlim S$ is definable \Leftrightarrow (2) $\varinjlim S$ is closed under products in $R\text{-Mod}$ \Leftrightarrow (3) $\varinjlim S$ is preenveloping in $R\text{-Mod}$ \Leftrightarrow (4) S is preenveloping in $R\text{-mod}$. This result due to Crawley-Boevey [15, Theorem 4.2, p. 1665] and Krause [32, Proposition 3.11, p. 27].

The main purposes of this chapter are to obtain new results on definable classes by using duality of modules and applying these results to generalize many well-known characterizations of right coherent (resp. right noetherian) rings and other results in the literature to the setting of definability of the class $\varinjlim(\text{add } S)$ (resp. $\text{Prod } S^*$), where $S \subseteq R\text{-mod}$ and where $*$ denotes any duality in the sense of Proposition 4.2.2. Another aim is to prove Holm's theorem above without his assumptions (${}_R R \in S$ and $\varinjlim(\text{add } S)$ is definable) and to show that the converse of another theorem of Holm, [26, Theorem 5.6, p. 556], (see Theorem 4.4.3) is true too.

For these main purposes, we will introduce the concept of almost dual pairs.

Throughout this, and later, chapters we will use the following conventions.

If S and T are classes of finitely presented left R -modules, we will denote by \mathcal{H} and \mathcal{K} sets of matrices over R such that $L_{\mathcal{H}} = S$ and $L_{\mathcal{K}} = T$.

4.2 Almost dual pairs

4.2.1 Definition, examples and some properties

Definition 4.2.1 An **almost dual pair** over R is a pair $(\mathcal{F}, \mathcal{G})$, where \mathcal{F} (resp. \mathcal{G}) is a class of left (resp. right) R -modules such that the following conditions hold.

- (1) For any left R -module M , $M \in \mathcal{F}$ if and only if $M^* = \text{Hom}_{\mathcal{K}}(M, E) \in \mathcal{G}$.

(2) \mathcal{G} is closed under direct summands and direct products.

First, we will show that the definition of an almost dual pair is independent of the duality used. This is the following proposition.

Proposition 4.2.2 Let $*$ (resp. \natural) be dualities over $R\text{-Mod}$, induced by modules ${}_K E$ (resp. ${}_{K'} E'$). Then:

(1) $\text{Prod} M^* = \text{Prod} M^\natural$, for any $M \in R\text{-Mod}$.

(2) If $(\mathcal{F}, \mathcal{G})$ is an almost dual pair with respect to $*$, then it is an almost dual pair with respect to \natural .

PROOF: (1) This is [45, Theorem 3.2]. The proof uses techniques from the model theory of modules.

(2) Suppose that $(\mathcal{F}, \mathcal{G})$ is an almost dual pair with respect to $*$. Let $M \in \mathcal{F}$, thus $M^* \in \mathcal{G}$ and hence $\text{Prod} M^* \subseteq \mathcal{G}$. Since $\text{Prod} M^* = \text{Prod} M^\natural$ (by (1) above) it follows that $\text{Prod} M^\natural \subseteq \mathcal{G}$ and hence $M^\natural \in \mathcal{G}$.

Conversely, let $M^\natural \in \mathcal{G}$ thus by reversing the above argument we have $M \in \mathcal{F}$ and hence $(\mathcal{F}, \mathcal{G})$ is an almost dual pair with respect to \natural . \square

Theorem 4.2.3 (see [80, 1.6, p. 383]) A ring R is left coherent if and only if M^* is flat right R -module whenever M is an absolutely pure left R -module.

Remark 4.2.4 A pair $(\mathcal{F}, \mathcal{G})$ is said to be a **duality pair** if for any left R -module M , we have $M \in \mathcal{F}$ if and only if $M^* = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in \mathcal{G}$ and \mathcal{G} is closed under direct summands and finite direct sums [28, p. 625]. It is clear that every almost dual pair over a ring R is a duality pair but the converse is not true, for example: let $\mathcal{F}\text{lat}_R$ be the class of flat right modules over a non left coherent ring R and let $\mathcal{F} = \{M \mid M^* \text{ is a flat right } R\text{-module}\}$. Then $(\mathcal{F}, \mathcal{F}\text{lat}_R)$ is a duality pair but it is not an almost dual pair because some product of flat right R -modules is not flat [11, Theorem 2.1, p. 460]. Here \mathcal{F} will be a (proper) subclass of the class of absolutely pure left R -modules, by Theorem 4.2.3.

In the following, we will give some examples of almost dual pairs.

Examples 4.2.5 (1) $(R\text{-Mod}, \text{Pinj}_R)$, $(R\text{-Mod}, \text{Mod-}R)$ and $(0, 0)$ are almost dual pairs, the first by Proposition 2.2.12.

(2) Let $\mathcal{L} \subseteq \text{Mod-}R$. A right R -module M is said to be \mathcal{L} -**injective**, if $\text{Ext}_R^1(L, M) = 0$ for all $L \in \mathcal{L}$ [21]. Also, a left R -module N is said to be \mathcal{L} -**flat**, if $\text{Tor}_1^R(L, N) = 0$ for all $L \in \mathcal{L}$ [85]. Let $\mathcal{L}\text{-}_R\mathcal{F}lat$ (resp. $\mathcal{L}\text{-Inj}_R$) be the class of \mathcal{L} -flat left (resp. \mathcal{L} -injective right) R -modules. Since $\mathcal{L}\text{-Inj}_R$ is closed under direct summands and products (by e.g. [85, Proposition 2.2, p. 2943]) and for any left R -module M , we have $M \in \mathcal{L}\text{-}_R\mathcal{F}lat$ if and only if $M^* \in \mathcal{L}\text{-Inj}_R$ (by [85, Proposition 2.4, p. 2943]) it follows that $(\mathcal{L}\text{-}_R\mathcal{F}lat, \mathcal{L}\text{-Inj}_R)$ is an almost dual pair.

(3) As special cases of (2), we have the following:

(i) If $\mathcal{L} = \text{Mod-}R$, then $({}_R\mathcal{F}lat, \text{Inj}_R)$ is an almost dual pair by using Lemma 2.2.8 and Lemma 2.2.10. In fact this is Theorem 2.2.11.

(ii) If $\mathcal{L} = \text{mod-}R$, then $({}_R\mathcal{F}lat, \mathcal{A}Pure_R)$ is an almost dual pair, by using Lemma 2.2.7 and Lemma 2.2.8.

(iii) Recall (see, e.g. [42]) that a left (resp. right) R -module M is said to be **torsion-free** (resp. **divisible**) if $\text{Tor}_1^R(L, M) = 0$ (resp. $\text{Ext}_R^1(L, M) = 0$) for all $(1, 1)$ -presented right R -modules L . Let ${}_R\mathcal{T}\mathcal{F}$ (resp. Div_R) be the class of torsion-free left (resp. divisible right) R -modules. If we take $\mathcal{L} = \{L \mid L \text{ is a } (1, 1)\text{-presented right } R\text{-module}\}$, then ${}_R\mathcal{T}\mathcal{F} = \mathcal{L}\text{-}_R\mathcal{F}lat$ and $\text{Div}_R = \mathcal{L}\text{-Inj}_R$ and hence $({}_R\mathcal{T}\mathcal{F}, \text{Div}_R)$ is an almost dual pair.

(iv) Let $n, m \in \mathbb{Z}^+$. Recall (see, [84]) that a left (resp. right) R -module M is said to be (m, n) -**flat** (resp. (m, n) -**injective**) if $\text{Tor}_1^R(L, M) = 0$ (resp. $\text{Ext}_R^1(L, M) = 0$) for all (m, n) -presented right R -modules L . Let $(m, n)\text{-}_R\mathcal{F}lat$ (resp. $(m, n)\text{-Inj}_R$) be the class of (m, n) -flat left (resp. (m, n) -injective right) R -modules. If we take $\mathcal{L} = \{L \mid L \text{ is an } (m, n)\text{-presented right } R\text{-module}\}$, then $(m, n)\text{-}_R\mathcal{F}lat = \mathcal{L}\text{-}_R\mathcal{F}lat$ and $(m, n)\text{-Inj}_R = \mathcal{L}\text{-Inj}_R$ and hence $((m, n)\text{-}_R\mathcal{F}lat, (m, n)\text{-Inj}_R)$ is an almost dual pair.

(v) If $\mathcal{L} = \{R/I \mid I \text{ is a simple right ideal of } R\}$, then $(\text{Min-}_R\mathcal{F}lat, \text{Min-}\mathcal{I}nj_R)$ is an almost dual pair, where $\text{Min-}_R\mathcal{F}lat$ is the class of min-flat left R -modules and $\text{Min-}\mathcal{I}nj_R$ is the class of min-injective right R -modules in the sense of [38].

(4) Let $n \in \mathbb{Z}^+$. A right R -module L is said to be FP_n , if there is an exact sequence of

right R -modules: $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow L$ where each F_i is finitely generated and free. Let $\mathcal{X} \subseteq \text{Mod-}R$ such that the class $\mathcal{X}_n = \{L \mid L \text{ is an } FP_n \text{ module in } \mathcal{X}\}$ is nonempty. Recall (see, [9, 2.5, p. 132]) that a left (resp. right) R -module M is said to be n - \mathcal{X} -**flat** (resp. n - \mathcal{X} -**injective**) if $\text{Tor}_n^R(L, M) = 0$ (resp. $\text{Ext}_R^n(L, M) = 0$) for all $L \in \mathcal{X}_n$. Let n - \mathcal{X} - $\mathcal{F}lat$ (resp. n - \mathcal{X} - Inj_R) be the class of n - \mathcal{X} -flat left (resp. n - \mathcal{X} -injective right) R -modules. Since a left R -module M is n - \mathcal{X} -flat if and only if M^* is n - \mathcal{X} -injective (see [9, Lemma 2.8, p. 134]) and the class n - \mathcal{X} - Inj_R is closed under direct summands and products (see [9, Lemma 2.7(2), p. 133]) it follows that $(n$ - \mathcal{X} - $\mathcal{F}lat, n$ - \mathcal{X} - Inj_R) is an almost dual pair.

(5) If R is left coherent so by Theorem 4.2.3 we have $M^* \in \mathcal{F}lat_R$ whenever $M \in {}_R\mathcal{A}P\text{ure}$ and hence $({}_R\mathcal{A}P\text{ure}, \mathcal{F}lat_R)$ is an almost dual pair.

(6) A ring R is von Neumann regular if and only if $(R\text{-Mod}, \text{Inj}_R)$ is an almost dual pair. In fact, if a ring R is von Neumann regular, thus every pure-injective right R -module is injective (by [79, 37.6, p. 316]) and hence $\text{Inj}_R = \text{Pinj}_R$. Since $(R\text{-Mod}, \text{Pinj}_R)$ is an almost dual pair, $(R\text{-Mod}, \text{Inj}_R)$ is an almost dual pair. Conversely, if we assume that $(R\text{-Mod}, \text{Inj}_R)$ is an almost dual pair then the character module of every left R -module is injective and hence quasi-injective. Since a ring R is von Neumann regular if and only if the character module of every left R -module is quasi-injective [23, Theorem A, p. 402] it follows that a ring R is von Neumann regular.

Let \mathcal{G} be a class of modules. We will use \mathcal{G}^+ to denote the class $\mathcal{G}^+ = \{M \mid M \text{ is a pure submodule of a module in } \mathcal{G}\}$. Also, we will use $\text{Pinj}\mathcal{G}$ to denote the class of all pure-injective modules in \mathcal{G} .

The proof of the following lemma is clear.

Lemma 4.2.6 If $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, then $(\mathcal{F}, \mathcal{G}^+)$, $(\mathcal{F}, \text{Pinj}\mathcal{G})$ and $(\mathcal{F}, \text{Prod}\mathcal{F}^*)$ are almost dual pairs.

For examples of this lemma, see Examples 4.2.5 (1,3(i),3(ii)).

Proposition 4.2.7 Let $\mathcal{F} \subseteq R\text{-Mod}$ and let \mathcal{G} be a class of pure-injective right R -modules. If \mathcal{G} is closed under direct summands, then $(\mathcal{F}, \mathcal{G})$ is an almost dual pair if and only if $(\mathcal{F}, \mathcal{G}^+)$ is so.

PROOF: (\Rightarrow) By Lemma 4.2.6, $(\mathcal{F}, \mathcal{G}^+)$ is an almost dual pair.

(\Leftarrow) Suppose that $(\mathcal{F}, \mathcal{G}^+)$ is an almost dual pair. Let $M \in R\text{-Mod}$, if $M \in \mathcal{F}$, then $M^* \in \mathcal{G}^+$ and hence M^* is a pure submodule of N for some $N \in \mathcal{G}$. Since M^* is pure-injective it follows from Theorem 2.2.19 that M^* is a direct summand of N . Since \mathcal{G} is closed under summands (by hypothesis), $M^* \in \mathcal{G}$. Conversely, if $M^* \in \mathcal{G}$, then $M^* \in \mathcal{G}^+$ and hence $M \in \mathcal{F}$. Let $\{M_i\}_{i \in I}$ be a family of modules in \mathcal{G} , thus $M_i \in \mathcal{G}^+$ and hence $\prod_{i \in I} M_i \in \mathcal{G}^+$ and this implies that $\prod_{i \in I} M_i$ is a pure submodule of B for some $B \in \mathcal{G}$. Since $\prod_{i \in I} M_i$ is pure-injective, $\prod_{i \in I} M_i$ is a direct summand of B and hence $\prod_{i \in I} M_i \in \mathcal{G}$. Therefore, $(\mathcal{F}, \mathcal{G})$ is an almost dual pair. \square

In the following propositions, we will give some properties of \mathcal{F} and \mathcal{G} , when $(\mathcal{F}, \mathcal{G})$ is an almost dual pair.

Proposition 4.2.8 If $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, then:

- (1) \mathcal{F} is closed under pure submodules, pure homomorphic images, pure extensions, direct sums and direct limits.
- (2) \mathcal{F} is covering in $R\text{-Mod}$.
- (3) \mathcal{F} is definable if and only if it is closed under direct products if and only if every left R -module has a \mathcal{F} -preenvelope.

PROOF: (1) Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be any pure exact sequence of left R -modules. By Lemma 2.2.3, the exact sequence $0 \rightarrow N^* \rightarrow M^* \rightarrow L^* \rightarrow 0$ is split and hence by Lemma 2.1.2 we have that $M^* \simeq N^* \oplus L^*$. If $M \in \mathcal{F}$, then $M^* \in \mathcal{G}$. Since \mathcal{G} is closed under direct summands, $L^*, N^* \in \mathcal{G}$ and hence $L, N \in \mathcal{F}$. Thus \mathcal{F} is closed under pure submodules and pure homomorphic images.

Conversely, if $L, N \in \mathcal{F}$, then $L^*, N^* \in \mathcal{G}$. Since \mathcal{G} is closed under direct products, $M^* \in \mathcal{G}$ and hence $M \in \mathcal{F}$. Thus \mathcal{F} is closed under pure extensions. Let $\{M_i\}_{i \in I}$ be such that $M_i \in \mathcal{F}$ for each i , thus $M_i^* \in \mathcal{G}$ and hence $\prod_{i \in I} M_i^* \in \mathcal{G}$. Since $(\bigoplus_{i \in I} M_i)^* = \prod_{i \in I} M_i^*$ (by Lemma 2.1.5(2)) so $\bigoplus_{i \in I} M_i \in \mathcal{F}$ and hence \mathcal{F} is closed under direct sums. Also, if $\{M_i\}_{i \in I}$ is a directed system then there is a pure epimorphism $\eta : \bigoplus_{i \in I} M_i \rightarrow \varinjlim M_i$ (by Lemma 2.2.6(2)). Since \mathcal{F} is closed under pure homomorphic images, $\varinjlim M_i \in \mathcal{F}$.

- (2) By (1), \mathcal{F} is closed under direct sums and pure homomorphic images and hence by

Theorem 2.3.2 we have that \mathcal{F} is covering in $R\text{-Mod}$.

(3) This follows from (1) and Lemma 2.3.5. \square

Theorem 2.16 in [9, p. 137] is a special case of Proposition 4.2.8 (3) if we take $(\mathcal{F}, \mathcal{G})$ to be the almost dual pair $(n\text{-}\mathcal{X}\text{-}R\text{-}\mathcal{F}\text{lat}, n\text{-}\mathcal{X}\text{-}\text{Inj}_R)$ in Examples 4.2.5 (4).

Proposition 4.2.9 If $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, then:

- (1) \mathcal{G}^+ is closed under direct sums.
- (2) $\text{Pinj } \mathcal{G}$ is enveloping in $\text{Mod-}R$.
- (3) \mathcal{G}^+ is definable if and only if it is closed under direct limits.

PROOF: (1) Let $\{M_i\}_{i \in I}$ be any subclass of \mathcal{G}^+ . Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, $\prod_{i \in I} M_i \in \mathcal{G}^+$. Since \mathcal{G}^+ is closed under pure submodules and the canonical embedding $\bigoplus_{i \in I} M_i \rightarrow \prod_{i \in I} M_i$ is a pure monomorphism (by Lemma 2.2.6(1)) it follows that $\bigoplus_{i \in I} M_i \in \mathcal{G}^+$.

(2) By Lemma 2.3.4.

(3) Suppose that \mathcal{G}^+ is closed under direct limits. It is clear that \mathcal{G}^+ is closed under pure submodules. By Lemma 4.2.6, $(\mathcal{F}, \mathcal{G}^+)$ is an almost dual pair and hence \mathcal{G}^+ is closed under direct products. This implies that \mathcal{G}^+ is definable. The converse is immediate from the definition. \square

Proposition 4.2.10 If $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, then $\text{Pinj } \mathcal{F} \subseteq \text{Prod } \mathcal{G}^*$ and $\text{Prod } \mathcal{F}^* \subseteq \text{Pinj } \mathcal{G}$.

PROOF: Let $M \in \text{Pinj } \mathcal{F}$, thus $M \in \mathcal{F}$ and M is a pure-injective. Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, $M^* \in \mathcal{G}$ and hence $M^{**} \in \text{Prod } \mathcal{G}^*$. Since $\delta_M : M \rightarrow M^{**}$ is a pure monomorphism (by Lemma 2.2.4) and M is pure-injective it follows from Theorem 2.2.19 that M is a direct summand of M^{**} and hence $M \in \text{Prod } \mathcal{G}^*$. Thus $\text{Pinj } \mathcal{F} \subseteq \text{Prod } \mathcal{G}^*$.

The second part is immediate from the assumption that $(\mathcal{F}, \mathcal{G})$ is an almost dual pair. \square

4.2.2 Natural bijections induced by almost dual pairs

Let \mathcal{G} be a class of modules. We will use \mathcal{G}^\ominus to denote the class:

$$\mathcal{G}^\ominus = \{M \mid M^* \in \mathcal{G}\}.$$

Proposition 4.2.11 Let \mathcal{G} be a class of right R -modules closed under direct summands and products. Then the class $\mathcal{F} = \mathcal{G}^\ominus$ is the unique class of left R -modules such that $(\mathcal{F}, \mathcal{G})$ is an almost dual pair.

PROOF: It is clear that $(\mathcal{F}, \mathcal{G})$ is an almost dual pair. Assume that \mathcal{E} is another class of left R -modules such that $(\mathcal{E}, \mathcal{G})$ is an almost dual pair. Let $M \in R\text{-Mod}$, then $M \in \mathcal{E}$ if and only if $M^* \in \mathcal{G}$ if and only if $M \in \mathcal{F}$ and hence $\mathcal{E} = \mathcal{F}$. Thus the class $\mathcal{F} = \mathcal{G}^\ominus$ is the unique class of left R -modules such that $(\mathcal{F}, \mathcal{G})$ is an almost dual pair. \square

Proposition 4.2.12 There is a natural bijection between classes of right R -modules closed under direct summands and products and almost dual pairs.

PROOF: Let $X = \{\mathcal{G} \subseteq \text{Mod-}R \mid \mathcal{G} = \text{Prod } \mathcal{G}\}$ and let $Y = \{(\mathcal{F}, \mathcal{G}) \mid (\mathcal{F}, \mathcal{G}) \text{ is an almost dual pair}\}$. Define $\Phi : X \rightarrow Y$ by $\Phi(\mathcal{G}) = (\mathcal{G}^\ominus, \mathcal{G})$, for all $\mathcal{G} \in X$.

Let $\mathcal{G}_1, \mathcal{G}_2 \in X$ such that $\Phi(\mathcal{G}_1) = \Phi(\mathcal{G}_2)$, thus $(\mathcal{G}_1^\ominus, \mathcal{G}_1) = (\mathcal{G}_2^\ominus, \mathcal{G}_2)$ and hence $\mathcal{G}_1 = \mathcal{G}_2$ and this implies that Φ is an injection.

Let $(\mathcal{F}, \mathcal{G}) \in Y$, thus $\mathcal{G} \in X$ and $\Phi(\mathcal{G}) = (\mathcal{G}^\ominus, \mathcal{G})$. By Proposition 4.2.11, $\mathcal{G}^\ominus = \mathcal{F}$ and hence $\Phi(\mathcal{G}) = (\mathcal{F}, \mathcal{G})$ and this implies that Φ is a surjection. Therefore, Φ is a bijection. \square

Theorem 4.2.13 There are natural bijections between the elements of the following:

- (1) $X = \{\mathcal{G} \subseteq \text{Mod-}R \mid \mathcal{G} = \text{Prod } \mathcal{G} = \text{Pinj } \mathcal{G}\}$;
- (2) $Y = \{(\mathcal{F}, \mathcal{G}) \mid (\mathcal{F}, \mathcal{G}) \text{ is an almost dual pair with } \mathcal{G} = \text{Pinj } \mathcal{G}\}$;
- (3) $Z = \{(\mathcal{F}, \mathcal{G}) \mid (\mathcal{F}, \mathcal{G}) \text{ is an almost dual pair with } \mathcal{G} = \mathcal{G}^+ \text{ and } \mathcal{G} \text{ is closed under pure-injective hulls}\}$.

PROOF: Define $\Phi : X \rightarrow Y$ by $\Phi(\mathcal{G}) = (\mathcal{G}^\ominus, \mathcal{G})$, for all $\mathcal{G} \in X$. By a similar argument to the proof of Proposition 4.2.12, we can prove Φ is a bijection. Define $\Theta : Y \rightarrow Z$ by $\Theta((\mathcal{F}, \mathcal{G})) = (\mathcal{F}, \mathcal{G}^+)$, for all $(\mathcal{F}, \mathcal{G}) \in Y$.

First, we will prove that $(\mathcal{F}, \mathcal{G}^+) \in Z$ for all $(\mathcal{F}, \mathcal{G}) \in Y$. Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair it follows from Lemma 4.2.6 that $(\mathcal{F}, \mathcal{G}^+)$ is an almost dual pair. Since \mathcal{G}^+ is closed

under pure submodules, $(\mathcal{G}^+)^+ = \mathcal{G}^+$. Let $M \in \mathcal{G}^+$; thus M is a pure submodule of N for some $N \in \mathcal{G}$. By [22, Exercise 2.3, p.433], $PE(M)$ is a direct summand of $PE(N)$. Since $\mathcal{G} = \text{Pinj } \mathcal{G}$ it follows that $PE(N) = N$ and hence $PE(M) \in \mathcal{G} \subseteq \mathcal{G}^+$. Thus \mathcal{G}^+ is closed under pure injective hulls and hence $(\mathcal{F}, \mathcal{G}^+) \in Z$ for all $(\mathcal{F}, \mathcal{G}) \in Y$.

Now we will prove Θ is a bijection. Let $(\mathcal{F}_1, \mathcal{G}_1), (\mathcal{F}_2, \mathcal{G}_2) \in Y$ such that $\Theta((\mathcal{F}_1, \mathcal{G}_1)) = \Theta((\mathcal{F}_2, \mathcal{G}_2))$, thus $(\mathcal{F}_1, \mathcal{G}_1^+) = (\mathcal{F}_2, \mathcal{G}_2^+)$ and hence $\mathcal{F}_1 = \mathcal{F}_2$ and $\mathcal{G}_1^+ = \mathcal{G}_2^+$. Let $M \in \mathcal{G}_1$, thus $M \in \mathcal{G}_1^+$ and hence $M \in \mathcal{G}_2^+$ and this implies that M is a pure submodule of B for some $B \in \mathcal{G}_2$. Since $\mathcal{G}_1 = \text{Pinj } \mathcal{G}_1$ it follows that M is pure-injective and hence M is a direct summand of B and this implies that $M \in \mathcal{G}_2$ and so $\mathcal{G}_1 \subseteq \mathcal{G}_2$. Similarly, we can prove that $\mathcal{G}_2 \subseteq \mathcal{G}_1$ and hence $\mathcal{G}_1 = \mathcal{G}_2$. Thus $(\mathcal{F}_1, \mathcal{G}_1) = (\mathcal{F}_2, \mathcal{G}_2)$ and hence Θ is an injection. Let $(\mathcal{F}, \mathcal{G}) \in Z$, thus $(\mathcal{F}, \text{Pinj } \mathcal{G}) \in Y$ and $\Theta(\mathcal{F}, \text{Pinj } \mathcal{G}) = (\mathcal{F}, (\text{Pinj } \mathcal{G})^+)$. We will prove $(\text{Pinj } \mathcal{G})^+ = \mathcal{G}$. Since $\text{Pinj } \mathcal{G} \subseteq \mathcal{G}$ it follows that $(\text{Pinj } \mathcal{G})^+ \subseteq \mathcal{G}^+ = \mathcal{G}$. Let $M \in \mathcal{G}$, thus $PE(M) \in \mathcal{G}$ and hence $PE(M) \in \text{Pinj } \mathcal{G}$. Since M is a pure submodule of $PE(M)$ it follows that $M \in (\text{Pinj } \mathcal{G})^+$ and hence $\mathcal{G} \subseteq (\text{Pinj } \mathcal{G})^+$ and this implies that $(\text{Pinj } \mathcal{G})^+ = \mathcal{G}$. Thus $(\mathcal{F}, (\text{Pinj } \mathcal{G})^+) = (\mathcal{F}, \mathcal{G})$ and hence Θ is a surjection. Therefore, Θ is a bijection. \square

A non-zero right R -module M is said to be **superdecomposable** if it has no (non-zero) indecomposable direct summands.

Theorem 4.2.14 If M is a non-zero left R -module, then M^* is not superdecomposable.

PROOF: See [45, Theorem 3.3, p. 5]. \square

Recall (see, e.g. [2, p. 72]) that a non-zero submodule N of a right R -module M is said to be **essential** in M if $N \cap B \neq 0$ for any non-zero submodule B of M .

Also, we recall that a right R -module M is said to be **uniform** if every non-zero submodules of M is essential in M (see, e.g. [2, p. 294]).

Lemma 4.2.15 Let R be a domain which is not uniform as a right R -module and let $E = E(R_R)$ be its injective hull. If E is a non-zero superdecomposable module, then E^I is also a non-zero superdecomposable right R -module for any index set I .

PROOF: Let M be a non-zero submodule of E^I . Choose a be a non-zero element in M , say

$a = (a_i)_{i \in I}$. Choose $j \in I$ such that $a_j \neq 0$. Since $a_j \in E$ and R is essential in E , there is $s \in R$ such that $a_j s \neq 0$, $a_j s \in R$. Consider the R -homomorphism $f : R \rightarrow E^I$ defined by $f(r) = asr$ for all $r \in R$. Then $\ker(f) = 0$, because if $r \in \ker(f)$ then $asr = 0$ so $a_j sr = 0$. Since R is a domain, this implies $r = 0$. Therefore $R \simeq \text{im}(f)$ and $\text{im}(f)$ is a submodule of M . Since R_R is not a uniform R -module, neither is M . Therefore, E^I contains no non-zero uniform submodule. Assume that E^I has a non-zero indecomposable direct summand N . Thus N is an indecomposable injective submodule of E^I . Since a non-zero module is uniform if and only if its injective hull is indecomposable (by [2, p.294]) it follows that N is a uniform submodule of E^I and this is a contradiction. Thus E^I is a non-zero superdecomposable module. \square

The following example shows that if $(\mathcal{F}, \mathcal{G})$ is an almost dual pair then it is not necessarily the case that $\text{Prod } \mathcal{F}^* = \text{Pinj } \mathcal{G}$.

Example 4.2.16 Let $R = k \langle X, Y \rangle$ be the **free associative algebra** (ring of non-commutative polynomials) generated by two indeterminates X, Y over a field k and let $E = E(R_R)$ be its injective hull. By [52, Example 4.4.3, p. 172], R is a domain which is not uniform as a right R -module and E is a non-zero superdecomposable pure-injective right R -module and hence from Lemma 4.2.15 we have that E^I is a non-zero superdecomposable module for any index set I . Since the class of non-zero superdecomposable right R -modules is closed under non-zero direct summands, the non-zero modules in $\text{Prod } E$ are superdecomposable. Assume that $(0, \text{Prod } E)$ is not an almost dual pair, thus there is $M^* \in \text{Prod } E$ and $M \neq 0$. Since $M^* \neq 0$ it follows that M^* is superdecomposable and this contradicts Theorem 4.2.14. Thus $(0, \text{Prod } E)$ is an almost dual pair. Since $\text{Pinj}(\text{Prod } E) = \text{Prod } E \neq 0$, thus $\text{Prod } 0^* = \{0\} \subsetneq \text{Pinj}(\text{Prod } E)$. Thus, in contrast to Proposition 4.2.11, given \mathcal{F} as in Proposition 4.2.8 (1) (and assuming it is part of an almost dual pair) there is not in general a unique almost dual pair with \mathcal{F} as first component.

In the following theorem, we will give another natural bijection involving almost dual pairs.

Theorem 4.2.17 There is a natural bijection between the elements of the following:

$$(1) X = \{\mathcal{G} \subseteq \text{Mod-}R \mid \mathcal{G} = \text{Prod } \mathcal{G} = \text{Pinj } \mathcal{G} = \text{Prod}(\mathcal{G}^\ominus)^*\};$$

$$(2) Y = \{\mathcal{F} \subseteq R\text{-Mod} \mid \text{there is } \mathcal{G} \subseteq \text{Mod-}R \text{ such that } (\mathcal{F}, \mathcal{G}) \text{ is an almost dual pair and } \text{Pinj } \mathcal{G} = \text{Prod } \mathcal{F}^*\}.$$

PROOF: Define $\Psi : X \rightarrow Y$ by $\Psi(\mathcal{G}) = \mathcal{G}^\ominus$, for all $\mathcal{G} \in X$. Let $\mathcal{G}_1, \mathcal{G}_2 \in X$ be such that $\Psi(\mathcal{G}_1) = \Psi(\mathcal{G}_2)$, thus $\mathcal{G}_1^\ominus = \mathcal{G}_2^\ominus$ and hence $\text{Prod}(\mathcal{G}_1^\ominus)^* = \text{Prod}(\mathcal{G}_2^\ominus)^*$. By hypothesis, $\mathcal{G}_1 = \text{Prod}(\mathcal{G}_1^\ominus)^*$ and $\mathcal{G}_2 = \text{Prod}(\mathcal{G}_2^\ominus)^*$ and hence $\mathcal{G}_1 = \mathcal{G}_2$ and this implies that Ψ is an injection. Let $\mathcal{F} \in Y$, thus there is $\mathcal{G} \subseteq \text{Mod-}R$ such that $(\mathcal{F}, \mathcal{G})$ is an almost dual pair with $\text{Pinj } \mathcal{G} = \text{Prod } \mathcal{F}^*$. Thus $(\mathcal{F}, \text{Pinj } \mathcal{G})$ is an almost dual pair and hence from Proposition 4.2.11 we have that $(\text{Pinj } \mathcal{G})^\ominus = \mathcal{F}$ and so $\text{Pinj } \mathcal{G} \in X$. Then $\Psi(\text{Pinj } \mathcal{G}) = (\text{Pinj } \mathcal{G})^\ominus = \mathcal{F}$ and hence Ψ is a surjection and this implies that Ψ is a bijection. \square

4.2.3 Almost dual pairs induced by classes of finitely presented modules

Let $S, T \subseteq R\text{-mod}$. In the following, we will introduce types of almost dual pair defined in terms of conditions on S and T which cover many well known examples of almost dual pairs. But first we need the following results.

Lemma 4.2.18 Let R be an algebra over a commutative ring K , let $S \subseteq R\text{-mod}$ and let M be a submodule of a module in $\text{Prod } S^*$. If $\alpha : M \rightarrow N$ is an R -homomorphism such that $\alpha \otimes_R A : M \otimes_R A \rightarrow N \otimes_R A$ is a K -monomorphism for all $A \in S$, then α is a $D_{\mathcal{H}}$ -pure monomorphism.

PROOF: Suppose that M is a submodule of a module in $\text{Prod } S^*$, thus M is a submodule of $\prod_{i \in I} M_i^*$ with $M_i \in S$. Suppose that $\alpha \otimes_R A : M \otimes_R A \rightarrow N \otimes_R A$ is a K -monomorphism for all $A \in S$. Since E is an injective K -module, the sequence $\text{Hom}_K(N \otimes_R A, E) \rightarrow \text{Hom}_K(M \otimes_R A, E) \rightarrow 0$ is exact. By Lemma 2.1.6, the sequence $\text{Hom}_R(N, \text{Hom}_K(A, E)) \rightarrow \text{Hom}_R(M, \text{Hom}_K(A, E)) \rightarrow 0$ is exact for all $A \in S$. Since $M_i \in S$, the sequence $\text{Hom}_R(N, M_i^*) \rightarrow \text{Hom}_R(M, M_i^*) \rightarrow 0$ is exact, for all $i \in I$ and hence the sequence $\prod_{i \in I} \text{Hom}_R(N, M_i^*) \rightarrow \prod_{i \in I} \text{Hom}_R(M, M_i^*) \rightarrow 0$ is exact and this implies that the sequence $\text{Hom}_R(N, \prod_{i \in I} M_i^*) \rightarrow \text{Hom}_R(M, \prod_{i \in I} M_i^*) \rightarrow 0$ is exact, by Lemma 2.1.5(1). Let $i : M \rightarrow \prod_{i \in I} M_i^*$

be the inclusion, thus there is $f \in \text{Hom}_R(N, \prod_{i \in I} M_i^*)$ such that $f\alpha = i$ and hence α is an R -monomorphism. Since $\alpha \otimes_R A$ is a K -monomorphism for all $A \in S$ it follows from Theorem 3.1.1 that α is a $D_{\mathcal{H}}$ -pure monomorphism. \square

Proposition 4.2.19 Let R be an algebra over a commutative ring K , let $S \subseteq R\text{-mod}$ (not necessarily containing ${}_R R$) and let $M \in R\text{-Mod}$ be such that M^* is a submodule of a module in $\text{Prod } S^*$. Then there is an S -pure epimorphism $\alpha : \bigoplus_{i \in I} M_i \rightarrow M$ with $M_i \in S$ for all $i \in I$.

PROOF: Let M be a left R -module such that $M^* = \text{Hom}_K(M, E)$ is a submodule of a module in $\text{Prod } S^*$. Let \mathcal{F} be the class of modules which are isomorphic to a direct sum of modules in S . By [57, Corollary 3.7(a), p.905], every left R -module has an \mathcal{F} -precover. Let $\alpha : \bigoplus_{i \in I} M_i \rightarrow M$ be an \mathcal{F} -precover of M where $M_i \in S$ and let $\Sigma = 0 \rightarrow \ker(\alpha) \xrightarrow{i} \bigoplus_{i \in I} M_i \xrightarrow{\alpha} M \rightarrow 0$. For any $A \in S$, since $\text{Hom}_R(A, -)$ is a left exact functor (by e.g. [2, Proposition 16.6, p. 183]) and α is an \mathcal{F} -precover of M it follows that the induced sequence of K -modules $0 \rightarrow \text{Hom}_R(A, \ker(\alpha)) \rightarrow \text{Hom}_R(A, \bigoplus_{i \in I} M_i) \rightarrow \text{Hom}_R(A, M) \rightarrow 0$ is exact. Since E is an injective K -module, the sequence $0 \rightarrow \text{Hom}_K(\text{Hom}_R(A, M), E) \rightarrow \text{Hom}_K(\text{Hom}_R(A, \bigoplus_{i \in I} M_i), E) \rightarrow \text{Hom}_K(\text{Hom}_R(A, \ker(\alpha)), E) \rightarrow 0$ is exact. By Lemma 2.1.9, the sequence $0 \rightarrow \text{Hom}_K(M, E) \otimes_R A \rightarrow \text{Hom}_K(\bigoplus_{i \in I} M_i, E) \otimes_R A \rightarrow \text{Hom}_K(\ker(\alpha), E) \otimes_R A \rightarrow 0$ is exact, for all $A \in S$. Since M^* is a submodule of a module in $\text{Prod } S^*$ (by hypothesis), it follows from Lemma 4.2.18 that the sequence $0 \rightarrow M^* \rightarrow (\bigoplus_{i \in I} M_i)^* \rightarrow (\ker(\alpha))^* \rightarrow 0$ is $D_{\mathcal{H}}$ -pure exact. By Lemma 2.1.10, the sequence Σ is exact and hence from Theorem 3.1.1 we have that Σ is an S -pure exact sequence. Hence there is an S -pure epimorphism $\alpha : \bigoplus_{i \in I} M_i \rightarrow M$ with $M_i \in S$ for all $i \in I$. \square

Lemma 4.2.20 Let $S, T \subseteq R\text{-mod}$. Then the following statements are equivalent for a left R -module M .

- (1) There is a T -pure exact sequence of left R -modules $0 \rightarrow A \rightarrow \bigoplus_{j \in J} M_j \xrightarrow{\alpha} M \rightarrow 0$ with $M_j \in S$ for all $j \in J$.
- (2) M^* is a $D_{\mathcal{K}}$ -pure submodule of a module in $\text{Prod } S^*$.
- (3) M^* is a submodule of a module in $\text{Prod } S^*$ and every $D_{\mathcal{H}}$ -pure exact sequence of right R -modules $0 \rightarrow M^* \xrightarrow{\alpha} A \rightarrow B \rightarrow 0$ is $D_{\mathcal{K}}$ -pure.

(4) M^* is a submodule of a module in $\text{Prod } S^*$ and every S -pure exact sequence of left R -modules $0 \rightarrow P \rightarrow Q \rightarrow M \rightarrow 0$ is T -pure.

PROOF: (1) \Rightarrow (2) Suppose that there is a T -pure exact sequence of left R -modules $0 \rightarrow A \rightarrow \bigoplus_{j \in J} M_j \xrightarrow{\alpha} M \rightarrow 0$ with $M_j \in S$ for all $j \in J$. By Theorem 3.1.1, the exact sequence $0 \rightarrow M^* \xrightarrow{\alpha^*} \prod_{j \in J} M_j^* \rightarrow A^* \rightarrow 0$ is $D_{\mathcal{K}}$ -pure and hence M^* is a $D_{\mathcal{K}}$ -pure submodule of $\prod_{j \in J} M_j^* \in \text{Prod } S^*$.

(2) \Rightarrow (3) Suppose that M^* is a $D_{\mathcal{K}}$ -pure submodule of $N \in \text{Prod } S^*$ and let $i : M^* \rightarrow N$ be the inclusion. Let $\Sigma : 0 \rightarrow M^* \xrightarrow{\alpha} A \rightarrow B \rightarrow 0$ be any $D_{\mathcal{H}}$ -pure exact sequence of right R -modules. By Theorem 3.1.4, N is $D_{\mathcal{H}}$ -pure-injective and hence there is a homomorphism $g : A \rightarrow N$ such that $g\alpha = i$. Since i is a $D_{\mathcal{K}}$ -pure monomorphism it follows from Lemma 2.2.16(2) that α is a $D_{\mathcal{K}}$ -pure monomorphism and hence the exact sequence Σ is $D_{\mathcal{K}}$ -pure.

(3) \Rightarrow (4) This follows by Theorem 3.1.1.

(4) \Rightarrow (1) Suppose that M^* is a submodule of a module in $\text{Prod } S^*$ and that every S -pure exact sequence of left R -modules $0 \rightarrow P \rightarrow Q \rightarrow M \rightarrow 0$ is T -pure. By Proposition 4.2.19, there is an S -pure exact sequence of left R -modules $\Sigma : 0 \rightarrow A \rightarrow \bigoplus_{j \in J} M_j \xrightarrow{\alpha} M \rightarrow 0$ with $M_j \in S$ for all $j \in J$ and hence by hypothesis we have Σ is T -pure. \square

Corollary 4.2.21 Let $S, T \subseteq R\text{-mod}$. Then the following statements are equivalent for a left R -module M .

(1) There is a T -pure exact sequence of left R -modules $0 \rightarrow A \rightarrow \bigoplus_{j \in J} M_j \xrightarrow{\alpha} M \rightarrow 0$ with $M_j \in S \cup \{R\}$ for all $j \in J$.

(2) M^* is a $D_{\mathcal{K}}$ -pure submodule of a module in $D_{\mathcal{H}}\text{-Pinj}$.

(3) Every $D_{\mathcal{H}}$ -pure exact sequence of right R -modules $0 \rightarrow M^* \xrightarrow{\alpha} A \rightarrow B \rightarrow 0$ is $D_{\mathcal{K}}$ -pure.

(4) Every S -pure exact sequence of left R -modules $0 \rightarrow P \rightarrow Q \rightarrow M \rightarrow 0$ is T -pure.

PROOF: Since $\text{Prod}(S \cup \{R\})^* = D_{\mathcal{H}}\text{-Pinj}$ (by Theorem 3.1.4) it follows from Corollary 3.1.3 that M^* is a submodule of a module in $D_{\mathcal{H}}\text{-Pinj}$. Thus the result follows from Lemma 4.2.20. \square

Example 4.2.22 Let $T \subseteq R\text{-mod}$ and let $S \subseteq \text{mod-}R$. Recall that a right (resp. left) R -module A (resp. M) is said to be **absolutely S -pure (resp. T -flat) in the sense of Wisbauer** [79] if every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ (resp. $0 \rightarrow L \rightarrow N \rightarrow M \rightarrow 0$) is S -pure (resp. T -pure). Let $\mathcal{A}\text{-}S\text{-Pure}_R$ (resp. $T\text{-}R\mathcal{F}\mathcal{L}$) be the class of absolutely S -pure right (resp. T -flat left) R -modules in the sense of Wisbauer. Since $\mathcal{A}\text{-}S\text{-Pure}_R$ is closed under direct summands and direct products (by [79, 35.2 (1), p. 298]) it follows from Corollary 4.2.21 that $(T\text{-}R\mathcal{F}\mathcal{L}, \mathcal{A}\text{-}D_{\mathcal{K}}\text{-Pure}_R)$ is an almost dual pair.

Lemma 4.2.23 Let S be any class of finitely presented right R -modules. For all $i \in I$, suppose that $0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0$ is an exact sequence of right R -modules. Then the exact sequence $0 \rightarrow \prod_{i \in I} A_i \rightarrow \prod_{i \in I} B_i \rightarrow \prod_{i \in I} C_i \rightarrow 0$ is S -pure if and only if the exact sequence $0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0$ is S -pure, for all $i \in I$.

PROOF: Assume that the exact sequence $0 \rightarrow \prod_{i \in I} A_i \rightarrow \prod_{i \in I} B_i \rightarrow \prod_{i \in I} C_i \rightarrow 0$ of right R -modules is S -pure. Let \mathcal{H} be a class of matrices over R such that $D_{\mathcal{H}} = S$ and let $M \in L_{\mathcal{H}}$. By Theorem 3.1.1, the sequence $0 \rightarrow (\prod_{i \in I} A_i) \otimes_R M \rightarrow (\prod_{i \in I} B_i) \otimes_R M \rightarrow (\prod_{i \in I} C_i) \otimes_R M \rightarrow 0$ is exact. Since $(\prod_{i \in I} N_i) \otimes_R M \simeq \prod_{i \in I} (N_i \otimes_R M)$, for any family $\{N_i\}_{i \in I}$ of right R -modules (by the left version of Lemma 2.1.8), the sequence $0 \rightarrow \prod_{i \in I} (A_i \otimes_R M) \rightarrow \prod_{i \in I} (B_i \otimes_R M) \rightarrow \prod_{i \in I} (C_i \otimes_R M) \rightarrow 0$ is exact. Therefore, for each $i \in I$, the sequence $0 \rightarrow A_i \otimes_R M \rightarrow B_i \otimes_R M \rightarrow C_i \otimes_R M \rightarrow 0$ is exact (by e.g. [31, p. 86]) and hence the sequence $0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0$ is S -pure, by Theorem 3.1.1. The converse follows by reversing the above argument. \square

Let $S, T \subseteq R\text{-mod}$ and let $\mathcal{L} \subseteq \text{mod-}R$. We will define the classes $T\text{-}S\text{-}R\mathcal{F}\text{lat}$ and $\mathcal{L}\text{-}S\text{-Pinj}_R$ as follows:

$T\text{-}S\text{-}R\mathcal{F}\text{lat} = \{M \mid \text{there is a } T\text{-pure exact sequence of left } R\text{-modules } 0 \rightarrow A \rightarrow \bigoplus_{j \in J} M_j \xrightarrow{\alpha} M \rightarrow 0 \text{ with } M_j \in S\}$ and

$\mathcal{L}\text{-}S\text{-Pinj}_R = \{N \mid N \text{ is an } \mathcal{L}\text{-pure submodule of a module in } \text{Prod} S^*\}$.

The following theorem is the main result in this subsection.

Theorem 4.2.24 Let $S, T \subseteq R\text{-mod}$. Then:

(1) $(T\text{-}S\text{-}R\mathcal{F}\text{lat}, D_{\mathcal{K}}\text{-}S\text{-Pinj}_R)$ is an almost dual pair;

(2) $T\text{-}S\text{-}R\mathcal{F}lat$ is covering in $R\text{-Mod}$ and it is closed under pure submodules, pure homomorphic images, pure extensions, direct sums and direct limits.

PROOF: (1) Let $M \in R\text{-Mod}$. By Lemma 4.2.20, $M \in T\text{-}S\text{-}R\mathcal{F}lat$ if and only if $M^* \in D_{\mathcal{K}}\text{-}S\text{-}Pinj_R$. Also, from Lemma 4.2.23 we have that $D_{\mathcal{K}}\text{-}S\text{-}Pinj_R$ is closed under direct products and summands. Hence $(T\text{-}S\text{-}R\mathcal{F}lat, D_{\mathcal{K}}\text{-}S\text{-}Pinj_R)$ is an almost dual pair.

(2) By Proposition 4.2.8. □

Corollary 4.2.25 [26, Theorem 1.4, p. 545] Let $S \subseteq R\text{-mod}$. Then $(\varinjlim(\text{add } S), \text{Prod } S^*)$ is an almost dual pair.

PROOF: Let $M \in R\text{-Mod}$. Thus $M \in \varinjlim(\text{add } S)$ if and only if $M \in L_{\mathcal{K}}\text{-add } S\text{-}R\mathcal{F}lat$, where \mathcal{K} is the class of matrices over R with finitely many rows and finitely many columns (by Lemma 2.2.5) if and only if M^* is a pure submodule of a module in $\text{Prod}(\text{add } S)^*$ (by Theorem 4.2.24(1)) if and only if M^* is a pure submodule of a module in $\text{Prod } S^*$ if and only if $M^* \in \text{Prod } S^*$. Hence $(\varinjlim(\text{add } S), \text{Prod } S^*)$ is an almost dual pair. □

Let $S \subseteq \text{mod-}R$. Stenström in [70] introduced the concept of an $(S\text{-pure})\text{-flat}$ left R -module as follows: a left R -module M is said to be **$(S\text{-pure})\text{-flat}$** if for every S -pure exact sequence of right R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, the sequence $0 \rightarrow A \otimes_R M \rightarrow B \otimes_R M \rightarrow C \otimes_R M \rightarrow 0$ is exact.

Lemma 4.2.26 (see, e.g. [67, Lemma 8.1, p. 118]) Let $S \subseteq \text{mod-}R$. A left R -module M is $(S\text{-pure})\text{-flat}$ if and only if M^* is S -pure-injective.

Corollary 4.2.27 Let $S \subseteq R\text{-mod}$. Then $\varinjlim(\text{add}(S \cup \{{}_R R\})) = \{M \mid M \text{ is a } (D_{\mathcal{H}}\text{-pure})\text{-flat left } R\text{-modules}\}$.

PROOF: For any left R -modules M , we have $M \in \varinjlim(\text{add}(S \cup \{{}_R R\}))$ if and only if $M^* \in \text{Prod}(S \cup \{{}_R R\})^*$ (by Corollary 4.2.25) if and only if M^* is $D_{\mathcal{H}}\text{-pure-injective}$ (by Theorem 3.1.4) if and only if M is $(D_{\mathcal{H}}\text{-pure})\text{-flat}$, by Lemma 4.2.26. □

As special cases of Corollary 4.2.25, we have the following examples:

Examples 4.2.28 (1) If $S = \{{}_R R\}$, then $\text{add } S = {}_R \text{proj}$ (the class of finitely generated projective left R -modules). Hence $\varinjlim(\text{add } S) = \varinjlim({}_R \text{proj}) = {}_R \mathcal{F}lat$ (by Lazard [35]) and $\text{Prod } S^* = \text{Prod } \{{}_R R\}^* = \text{Inj}_R$ (by Theorem 3.1.4). Also, $(\text{Prod } S^*)^+ = (\text{Inj}_R)^+ = \mathcal{A}P\text{ure}_R$ (by Lemma 2.2.7). Thus $({}_R \mathcal{F}lat, \text{Inj}_R)$ and $({}_R \mathcal{F}lat, \mathcal{A}P\text{ure}_R)$ are almost dual pairs.

(2) If $S = R\text{-mod}$, then $\text{add } S = R\text{-mod}$ and hence $\varinjlim(\text{add } S) = \varinjlim(R\text{-mod}) = R\text{-Mod}$ (by Lemma 2.1.12), and $\text{Prod } S^* = \text{Prod } (R\text{-mod})^* = \text{Pinj}_R$ (by Theorem 3.1.4). Also, $(\text{Prod } S^*)^+ = (\text{Pinj}_R)^+ = \text{Mod-}R$ (by Theorem 2.2.14). Thus $(R\text{-Mod}, \text{Pinj}_R)$ and $(R\text{-Mod}, \text{Mod-}R)$ are almost dual pairs.

(3) Recall (see [14]) that a left R -module M is said to be **RD -flat** if for every RD -pure exact sequence of right R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, the sequence $0 \rightarrow A \otimes_R M \rightarrow B \otimes_R M \rightarrow C \otimes_R M \rightarrow 0$ is exact. We will use ${}_R \mathcal{RD}\text{-Flat}$ to denote the class of RD -flat left modules. Let \mathcal{H} be the set of 1×1 matrices over R and let $S = L_{\mathcal{H}}$. Thus S (resp. $D_{\mathcal{H}}$) is the class of $(1, 1)$ -presented left (resp. right) R -modules and hence a left R -module M is $(D_{\mathcal{H}}\text{-pure})$ -flat if and only if M is RD -flat. By Corollary 4.2.27, $\varinjlim(\text{add } S) = {}_R \mathcal{RD}\text{-Flat}$. Since $\text{Prod } S^* = \mathcal{RD}\text{-Inj}_R$ it follows that $({}_R \mathcal{RD}\text{-Flat}, \mathcal{RD}\text{-Inj}_R)$ and $({}_R \mathcal{RD}\text{-Flat}, (\mathcal{RD}\text{-Inj}_R)^+)$ are almost dual pairs.

(4) Let R be a tame hereditary finite-dimensional algebra over a field k and let ${}_R \mathbf{P}$ (resp. ${}_R \mathbf{I}$, resp. ${}_R \mathbf{R}$) be the class of finitely generated indecomposable preprojective (resp. preinjective, resp. regular) left R -modules and write $\mathbf{P}_R, \mathbf{I}_R, \mathbf{R}_R$ for the corresponding classes of right modules.

(i) If $S = \{M\}$, where $M \in R\text{-mod}$, then from [33, Lemma 1.2, p. 633] we have that $\varinjlim(\text{add } S) = \text{Add}(\text{add } M) = \text{Add } M$ and $\varinjlim(\text{add } S) = \text{Prod}(\text{add } M) = \text{Prod } M$ and hence $(\text{Add } M, \text{Prod } M^*) = (\text{Prod } M, \text{Prod } M^*)$ is an almost dual pair.

(ii) If $S = {}_R \mathbf{P}$, then from [3, Corollary 11, p. 11] we have $\varinjlim(\text{add } {}_R \mathbf{P}) = {}_R \mathcal{T}\mathcal{F}\text{ree}$ (the class of torsion free left R -modules in the sense of Ringel [58], i.e., the class ${}^\perp({}_R \mathbf{R}) = \{M \mid \text{Ext}_R^1(M, L) = 0 \text{ for all } L \in {}_R \mathbf{R}\}$). Since $\text{Prod } {}_R \mathbf{P}^* = \text{Prod } \mathbf{I}_R$ it follows that $({}_R \mathcal{T}\mathcal{F}\text{ree}, \text{Prod } \mathbf{I}_R)$ is an almost dual pair. Also, since a hereditary algebra R is tame if and only if the class $\text{Prod } \mathbf{I}_R$ is definable (by [83, Theorem 3.2, p. 351]) it follows that $\text{Prod } \mathbf{I}_R$ is definable and

hence $(\text{Prod } \mathbf{I}_R)^+ = \text{Prod } \mathbf{I}_R$.

(iii) If $S = {}_R\mathbf{P} \cup {}_R\mathbf{R}$, then from [4, Example 5.2(1), p. 308] we have $\varinjlim(\text{add } S) = {}^\perp(\text{add } {}_R\mathbf{I}) = {}^\perp({}_R\mathbf{I})$. Since $\text{Prod } S^* = \text{Prod}({}_R\mathbf{P} \cup {}_R\mathbf{R})^* = \text{Prod}({}_R\mathbf{P}^* \cup {}_R\mathbf{R}^*) = \text{Prod}(\mathbf{I}_R \cup \mathbf{R}_R)$ it follows that $({}^\perp({}_R\mathbf{I}), \text{Prod}(\mathbf{I}_R \cup \mathbf{R}_R))$ and $({}^\perp({}_R\mathbf{I}), (\text{Prod}(\mathbf{I}_R \cup \mathbf{R}_R))^+)$ are almost dual pairs.

(iv) A right R -module M is said to be **separable** if every finite subset of M is contained in a finitely presented direct summand of M [48, p. 401]. Let $\mathcal{TF}S_R$ be the class of torsion-free (in the sense of Ringel) separable right R -modules. By [3, Proposition 21, p. 15], $(\text{Prod } \mathbf{P}_R)^+ = \mathcal{TF}S_R$ and hence $(\varinjlim(\text{add } {}_R\mathbf{I}), \text{Prod } \mathbf{P}_R)$ and $(\varinjlim(\text{add } {}_R\mathbf{I}), \mathcal{TF}S_R)$ are almost dual pairs.

(v) If $S = {}_R\mathbf{P} \cup {}_R\mathbf{R} \cup {}_R\mathbf{I}$, then $\text{add}({}_R\mathbf{P} \cup {}_R\mathbf{R} \cup {}_R\mathbf{I}) = R\text{-mod}$. Thus $\varinjlim(\text{add } S) = \varinjlim(R\text{-mod}) = R\text{-Mod}$, $\text{Prod } S^* = \text{Prod}(R\text{-mod})^* = \text{Pinj}_R$ and $(\text{Prod } S^*)^+ = (\text{Pinj}_R)^+ = \text{Mod-}R$. Hence $(R\text{-Mod}, \text{Pinj}_R)$ and $(R\text{-Mod}, \text{Mod-}R)$ are almost dual pairs.

Corollary 4.2.29 Let $S \subseteq R\text{-mod}$. Then:

- (1) $(\varinjlim(\text{add } S), (\text{Prod } S^*)^+)$ is an almost dual pair.
- (2) The following statements are equivalent for a left R -module M :
 - (i) $M \in \varinjlim(\text{add}(S \cup \{{}_R\mathbf{R}\}))$;
 - (ii) M^* is a $D_{\mathcal{H}}$ -pure-injective module;
 - (iii) every $D_{\mathcal{H}}$ -pure exact sequence of right R -modules $0 \rightarrow M^* \xrightarrow{\alpha} A \rightarrow B \rightarrow 0$ is pure;
 - (iv) every S -pure exact sequence of left R -modules $0 \rightarrow P \rightarrow Q \rightarrow M \rightarrow 0$ is pure.

PROOF: (1) By Corollary 4.2.25 and Lemma 4.2.6.

(2) The result follows by taking $T = L_{\mathcal{K}}$ where \mathcal{K} is the class of matrices over R with finitely many rows and finitely many columns and applying Corollary 4.2.21. \square

Lemma 4.2.30 (see [32, Theorem 3.12, p. 28]) Let \mathcal{F} be a definable subcategory of $R\text{-Mod}$. Then the following statements are equivalent:

- (1) There is a subcategory $S = \text{add } S$ of $R\text{-mod}$ such that $\mathcal{F} = \varinjlim S$.
- (2) Every finitely presented left R -module has an \mathcal{F} -preenvelope in $R\text{-mod}$.

Corollary 4.2.31 Let \mathcal{F} be a definable subcategory of $R\text{-Mod}$. Then the following statements are equivalent:

- (1) $(\mathcal{F}, \text{Prod } S^*)$ is an almost dual pair, for some subcategory $S = \text{add } S$ of $R\text{-mod}$.
- (2) Every finitely presented left R -module has an \mathcal{F} -preenvelope in $R\text{-mod}$.

PROOF: (1) \Rightarrow (2) Suppose that there is a subcategory $S = \text{add } S$ of $R\text{-mod}$ such that $(\mathcal{F}, \text{Prod } S^*)$ is an almost dual pair. By Corollary 4.2.25, $(\varinjlim S, \text{Prod } S^*)$ is an almost dual pair and hence by Proposition 4.2.11 we have that $\mathcal{F} = \varinjlim S$. Thus the result follows from Lemma 4.2.30.

(2) \Rightarrow (1) Reverse the above argument. □

Corollary 4.2.32 Let R be a tame hereditary finite-dimensional algebra over a field k , let \mathbf{R}_1 be a class of indecomposable regular right R -modules and let S be a simple regular left R -module. Then $S^*[\infty] \in \varinjlim(\text{add } \mathbf{R}_1)$ if and only if $(\mathbf{R}_1)^* \cap T_S$ is infinite, where $T_S = \{M \mid M \text{ is an indecomposable regular left } R\text{-module with } \text{Hom}_R(M, S) \neq 0\}$.

PROOF: (\Rightarrow) Suppose that $S^*[\infty] \in \varinjlim(\text{add } \mathbf{R}_1)$. Since $(\varinjlim(\text{add } \mathbf{R}_1), \text{Prod } (\mathbf{R}_1)^*)$ is an almost dual pair (by Corollary 4.2.25), $(S^*[\infty])^* \in \text{Prod } (\mathbf{R}_1)^*$. Since $(S^*[\infty])^* = \hat{S}$ (by [59, p. 109]) it follows that $\hat{S} \in \text{fsc}((\mathbf{R}_1)^*)$. By Theorem 3.4.19, $(\mathbf{R}_1)^* \cap T_S$ is infinite.

(\Leftarrow) Reverse the above argument. □

Corollary 4.2.33 Let R be a tame hereditary finite-dimensional algebra over a field k . Then the following statements are equivalent for a left R -module M :

- (1) M is torsion-free (i.e., $M \in {}_R\mathcal{TFree}$);
- (2) $M^* \in \text{Prod } \mathbf{I}_R$;
- (3) Every $D_{\mathcal{H}}$ -pure exact sequence of right R -modules $0 \rightarrow M^* \xrightarrow{\alpha} A \rightarrow B \rightarrow 0$ is pure, where \mathcal{H} is a class of matrices over R such that $L_{\mathcal{H}} = {}_R\mathbf{P}$;
- (4) Every ${}_R\mathbf{P}$ -pure exact sequence of left R -modules $0 \rightarrow C \rightarrow D \rightarrow M \rightarrow 0$ is pure.

PROOF: (1) \Leftrightarrow (2) By Examples 4.2.28(4(ii)).

(2) \Leftrightarrow (3) \Leftrightarrow (4) Let \mathcal{H} be a class of matrices over R such that $L_{\mathcal{H}} = {}_R\mathbf{P}$. Thus $D_{\mathcal{H}}\text{-Pinj} = \text{Prod}(\text{ind}((L_{\mathcal{H}} \cup \{{}_R R\})^*))$ (by Proposition 3.4.5) $= \text{Prod}(\text{ind}(({}_R\mathbf{P} \cup \{{}_R R\})^*)) = \text{Prod}(\text{ind}(\mathbf{I}_R \cup \{{}_R R\})^*) = \text{Prod}\mathbf{I}_R$. Thus the result follows by taking $S = {}_R\mathbf{P}$ and applying Corollary 4.2.29(2). \square

In the following result, we will compare two purities by using the class $\varinjlim(\text{add}S)$.

Proposition 4.2.34 Let $S, T \subseteq R\text{-mod}$ and let \mathcal{H}, \mathcal{K} be sets of matrices over R such that $L_{\mathcal{H}} = S$ and $L_{\mathcal{K}} = T$. Then the following statements are equivalent.

- (1) Every T -pure short exact sequence of left R -modules is S -pure.
- (2) $\varinjlim(\text{add}S) \subseteq \varinjlim(\text{add}(T \cup \{{}_R R\}))$.
- (3) $S \subseteq \varinjlim(\text{add}(T \cup \{{}_R R\}))$.
- (4) The corresponding assertions hold for right modules (for example, every $D_{\mathcal{K}}$ -pure short exact sequence of right R -modules is $D_{\mathcal{H}}$ -pure).

PROOF: (1) \Rightarrow (2) By Theorem 3.2.1, $S \subseteq \text{add}(T \cup \{{}_R R\})$ and hence $\varinjlim(\text{add}S) \subseteq \varinjlim(\text{add}(T \cup \{{}_R R\}))$.

(2) \Rightarrow (3) This is obvious.

(3) \Rightarrow (1) Suppose that $S \subseteq \varinjlim(\text{add}(T \cup \{{}_R R\}))$, thus $S^* \subseteq (\varinjlim(\text{add}(T \cup \{{}_R R\})))^* \subseteq \text{Prod}(T \cup \{{}_R R\})^*$ (by Corollary 4.2.25). By Theorem 3.2.1, every T -pure short exact sequence of left R -modules is S -pure.

(1) \Leftrightarrow (4) This follows by Theorem 3.2.1. \square

Theorem I.4 in [14, p. 3679] is a special case of the following corollary by taking T to be the class of $(1, 1)$ -presented left R -modules. Also, if we take $T = \{{}_R R\}$ and apply Corollary 4.2.35 then we will get many well-known equivalent characterizations of von Neumann regular rings, see for example [79, 37.6, p. 316].

Corollary 4.2.35 Let $T \subseteq R\text{-mod}$ and let \mathcal{K} be a set of matrices over R such that $L_{\mathcal{K}} = T$. Then the following statements are equivalent:

- (1) T -purity = purity for short exact sequences of left R -modules;
- (2) ${}_R\text{Pinj} = T\text{-Pinj}$;

- (3) ${}_R\text{Pproj} = T\text{-Pproj}$;
- (4) $R\text{-mod} = \text{add}(T \cup \{{}_R R\})$;
- (5) $R\text{-mod} \subseteq T\text{-Pproj}$;
- (6) $R\text{-mod} \subseteq \varinjlim(\text{add}(T \cup \{{}_R R\}))$;
- (7) $R\text{-Mod} = \varinjlim(\text{add}(T \cup \{{}_R R\}))$;
- (8) $R\text{-Mod} = (T\text{-Pinj})^+$;

(9) The corresponding assertions hold for right modules (for example, $D_{\mathcal{K}}\text{-purity} = \text{purity}$ for short exact sequences of right R -modules).

PROOF: (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) By [56, Theorem 2.5, p. 2136].

(4) \Rightarrow (5) Since $\text{add}(T \cup \{{}_R R\}) \subseteq T\text{-Pproj}$ it follows from (4) that $R\text{-mod} \subseteq T\text{-Pproj}$.

(5) \Rightarrow (6) Since $\varinjlim(\text{add}(T \cup \{{}_R R\}))$ is closed under direct sums and summands and since $T\text{-Pproj} = \text{Add}(T \cup \{{}_R R\})$ (by Proposition 3.1.2(2)) it follows that $T\text{-Pproj} \subseteq \varinjlim(\text{add}(T \cup \{{}_R R\}))$. By (5), $R\text{-mod} \subseteq \varinjlim(\text{add}(T \cup \{{}_R R\}))$.

(6) \Rightarrow (7) \Rightarrow (1) \Leftrightarrow (9) This follows by taking $S = R\text{-mod}$ and applying Proposition 4.2.34.

(2) \Rightarrow (8) Suppose that ${}_R\text{Pinj} = T\text{-Pinj}$, thus $({}_R\text{Pinj})^+ = (T\text{-Pinj})^+$. Since $({}_R\text{Pinj})^+ = R\text{-Mod}$ (by Theorem 2.2.14) it follows that $R\text{-Mod} = (T\text{-Pinj})^+$.

(8) \Rightarrow (2) Suppose that $R\text{-Mod} = (T\text{-Pinj})^+$, thus $R\text{-Mod} \cap {}_R\text{Pinj} = (T\text{-Pinj})^+ \cap {}_R\text{Pinj}$. Since $(T\text{-Pinj})^+ \cap {}_R\text{Pinj} = T\text{-Pinj}$ it follows that ${}_R\text{Pinj} = T\text{-Pinj}$. \square

Corollary 4.2.36 Let $S, T \subseteq R\text{-mod}$ and let \mathcal{H}, \mathcal{K} be sets of matrices over R such that $L_{\mathcal{H}} = S$ and $L_{\mathcal{K}} = T$. Then the following statements are equivalent.

- (1) $T\text{-purity} = S\text{-purity}$ for short exact sequences of left R -modules.
- (2) $\varinjlim(\text{add}(S \cup \{{}_R R\})) = \varinjlim(\text{add}(T \cup \{{}_R R\}))$.
- (3) $D_{\mathcal{K}}\text{-purity} = D_{\mathcal{H}}\text{-purity}$ for short exact sequences of right R -modules.
- (4) $\varinjlim(\text{add}(D_{\mathcal{H}} \cup \{{}_R R\})) = \varinjlim(\text{add}(D_{\mathcal{K}} \cup \{{}_R R\}))$.

PROOF: By Corollary 3.2.3 and Proposition 4.2.34. \square

4.3 Definable classes induced by almost dual pairs

Proposition 4.3.1 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. Then \mathcal{F} is definable if and only if $\mathcal{F}^{**} \subseteq \mathcal{F}$.

PROOF: (\Rightarrow) Suppose that \mathcal{F} is definable and let $M \in \mathcal{F}$. By Lemma 2.4.6, $M^{**} \in \mathcal{F}$ and hence $\mathcal{F}^{**} \subseteq \mathcal{F}$.

(\Leftarrow) Suppose that $\mathcal{F}^{**} \subseteq \mathcal{F}$ and let $\{M_i\}_{i \in I}$ be any family of modules in \mathcal{F} . By Proposition 4.2.8(1), $\bigoplus_{i \in I} M_i \in \mathcal{F}$ and hence $(\prod_{i \in I} M_i^*)^* = (\bigoplus_{i \in I} M_i)^{**} \in \mathcal{F}$. By Lemma 2.2.6(1), the canonical embedding $\bigoplus_{i \in I} M_i^* \rightarrow \prod_{i \in I} M_i^*$ is a pure monomorphism and hence from Lemma 2.2.3 we have that $(\bigoplus_{i \in I} M_i^*)^*$ is isomorphic to a direct summand of $(\prod_{i \in I} M_i^*)^*$ and this implies that $\prod_{i \in I} M_i^{**} = (\bigoplus_{i \in I} M_i^*)^* \in \mathcal{F}$. Since the canonical monomorphism $\delta_{M_i} : M_i \rightarrow M_i^{**}$ is pure (by Lemma 2.2.4) it follows from Lemma 2.2.2(2) that $\prod_{i \in I} \delta_{M_i} : \prod_{i \in I} M_i \rightarrow \prod_{i \in I} M_i^{**}$ is a pure monomorphism. Since \mathcal{F} is closed under pure submodules (by Proposition 4.2.8(1)) it follows that $\prod_{i \in I} M_i \in \mathcal{F}$ and hence \mathcal{F} is closed under direct products. It follows by Proposition 4.2.8(3) that \mathcal{F} is definable. \square

Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. In general, $\mathcal{G}^* \not\subseteq \mathcal{F}$ (for example, see Example 4.4.5(5)). In the following theorem, we give equivalent characterizations of definability of \mathcal{G}^+ in terms of dual modules; these characterizations are equivalent to the condition $\mathcal{G}^* \subseteq \mathcal{F}$.

Theorem 4.3.2 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. Then the following statements are equivalent.

- (1) \mathcal{G}^+ is definable.
- (2) $(\mathcal{G}^+)^* \subseteq \mathcal{F}$.
- (3) $\mathcal{G}^* \subseteq \mathcal{F}$.
- (4) $\mathcal{G}^{**} \subseteq \mathcal{G}$.
- (5) $(\mathcal{G}^+)^{**} \subseteq \mathcal{G}$.
- (6) $(\mathcal{G}^+)^{**} \subseteq \mathcal{G}^+$.
- (7) \mathcal{G}^+ is closed under pure homomorphic images.
- (8) For every $M \in \mathcal{G}^+$, there is a pure exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ with $N^* \in \mathcal{F}$.
- (9) For every $M \in \mathcal{G}$, there is a pure exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ with $N^* \in \mathcal{F}$.

PROOF: (1) \Rightarrow (2) Suppose that \mathcal{G}^+ is definable and let $M \in \mathcal{G}^+$. By Lemma 2.4.6, $M^{**} \in \mathcal{G}^+$ and hence M^{**} is a pure submodule of N , for some $N \in \mathcal{G}$. Since M^{**} is pure-injective (by Proposition 2.2.12) it follows from Theorem 2.2.19 that M^{**} is a direct summand of N and hence $M^{**} \in \mathcal{G}$. Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, $M^* \in \mathcal{F}$ and hence $(\mathcal{G}^+)^* \subseteq \mathcal{F}$.

(2) \Rightarrow (3) Suppose that $(\mathcal{G}^+)^* \subseteq \mathcal{F}$. Since $\mathcal{G} \subseteq \mathcal{G}^+$ it follows that $\mathcal{G}^* \subseteq \mathcal{F}$.

(3) \Rightarrow (4) Suppose that $\mathcal{G}^* \subseteq \mathcal{F}$, thus $\mathcal{G}^{**} \subseteq \mathcal{F}^*$. Since $\mathcal{F}^* \subseteq \mathcal{G}$ it follows that $\mathcal{G}^{**} \subseteq \mathcal{G}$.

(4) \Rightarrow (5) Let $M \in \mathcal{G}^+$, thus M is a pure submodule of N for some $N \in \mathcal{G}$ and hence M^{**} is a direct summand of N^{**} . By (4), $N^{**} \in \mathcal{G}$ and hence $M^{**} \in \mathcal{G}$. Thus $(\mathcal{G}^+)^{**} \subseteq \mathcal{G}$.

(5) \Rightarrow (6) This is obvious.

(6) \Rightarrow (1) Let $(M_i, f_{ij})_I$ be a direct system of modules in \mathcal{G}^+ . By Proposition 4.2.9(1), $\bigoplus_{i \in I} M_i \in \mathcal{G}^+$ and from (6) we have $(\bigoplus_{i \in I} M_i)^{**} \in \mathcal{G}^+$. Since the canonical epimorphism $\eta : \bigoplus_{i \in I} M_i \rightarrow \varinjlim M_i$ is pure, $(\varinjlim M_i)^{**}$ is isomorphic to a direct summand of $(\bigoplus_{i \in I} M_i)^{**}$ and hence $(\varinjlim M_i)^{**} \in \mathcal{G}^+$. Since $\delta_{\varinjlim M_i} : \varinjlim M_i \rightarrow (\varinjlim M_i)^{**}$ is a pure monomorphism, $\varinjlim M_i \in \mathcal{G}^+$ and hence by Proposition 4.2.9(3) we have that \mathcal{G}^+ is definable.

(1) \Rightarrow (7) By Theorem 2.4.2.

(7) \Rightarrow (1) Let $(M_i, f_{ij})_I$ be any direct system of modules in \mathcal{G}^+ . Since the canonical epimorphism $\eta : \bigoplus_{i \in I} M_i \rightarrow \varinjlim M_i$ is pure (by Lemma 2.2.6(2)) and $\bigoplus_{i \in I} M_i \in \mathcal{G}^+$ (by Proposition 4.2.9(1)) it follows from (7) that $\varinjlim M_i \in \mathcal{G}^+$ and hence \mathcal{G}^+ is definable.

(3) \Rightarrow (8) Let $M \in \mathcal{G}^+$, thus M is a pure submodule of N for some $N \in \mathcal{G}$. By (3), $N^* \in \mathcal{F}$. Thus there is a pure exact sequence $0 \rightarrow M \rightarrow N \rightarrow N/M \rightarrow 0$ with $N^* \in \mathcal{F}$.

(8) \Rightarrow (9) This is obvious.

(9) \Rightarrow (3) Let $M \in \mathcal{G}$. By (9), there is a pure exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ with $N^* \in \mathcal{F}$. By Lemma 2.2.3, the exact sequence $0 \rightarrow L^* \rightarrow N^* \rightarrow M^* \rightarrow 0$ is split and hence by Lemma 2.1.2 we have that M^* is isomorphic to a direct summand of N^* . Since $N^* \in \mathcal{F}$ it follows that $M^* \in \mathcal{F}$ and hence $\mathcal{G}^* \subseteq \mathcal{F}$. \square

Remark 4.3.3 Let $n \in \mathbb{Z}^+$ and let $\mathcal{X} \subseteq \text{Mod-}R$ such that the class $\mathcal{X}_n = \{L \mid L \text{ is an } FP_n \text{ module in } \mathcal{X}\}$ is nonempty. Recall (see [9, 2.6, p. 132]) that a ring R is said to be left n - \mathcal{X} -coherent if the class n - \mathcal{X} - $R\mathcal{F}lat$ (see Examples 4.2.5(4)) is closed under direct products. If we take $(\mathcal{F}, \mathcal{G})$ to be the almost dual pair $(n$ - \mathcal{X} - $R\mathcal{F}lat, n$ - \mathcal{X} - $\text{Inj}_R)$ in Examples 4.2.5(4) and apply Proposition 4.3.1 and Theorem 4.3.2, then we get some of Bennis's characterizations

of n - \mathcal{X} -coherent rings in [9, Theorem 2.13, p. 135].

Corollary 4.3.4 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. If $\mathcal{G}^* \subseteq \mathcal{F}$, then every right R -module has a \mathcal{G}^+ -cover.

PROOF: Suppose that $\mathcal{G}^* \subseteq \mathcal{F}$. By Theorem 4.3.2, \mathcal{G}^+ is closed under pure homomorphic images. By Proposition 4.2.9(1), \mathcal{G}^+ is closed under direct sums and hence from Theorem 2.3.2 we have that every right R -module has a \mathcal{G}^+ -cover. \square

Remark 4.3.5 Several well-known results in the literature are special cases of Corollary 4.3.4, for example:

(1) when \mathcal{F} is the class of flat left R -modules and \mathcal{G} is the class of injective right R -modules, this yields [51, Theorem 2.6, p. 2193];

(2) when \mathcal{F} is the class of torsion-free left R -modules and \mathcal{G} is the class of divisible right R -modules as in [42] then we have [42, Theorem 2.10, p. 713].

In Example 4.2.16 we show that if $(\mathcal{F}, \mathcal{G})$ is an almost dual pair then it is not necessarily the case that $\text{Prod } \mathcal{F}^* = \text{Pinj } \mathcal{G}$. The following result gives conditions under which we get $\text{Prod } \mathcal{F}^* = \text{Pinj } \mathcal{G}$.

Lemma 4.3.6 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. Consider the following conditions.

- (1) If $M \in \mathcal{G}$ then M is a pure submodule of N^* for some module N such that $N^* \in \mathcal{G}$.
- (2) $\text{Prod } \mathcal{F}^* = \text{Pinj } \mathcal{G}$.
- (3) If $M \in \text{Pinj } \mathcal{G}$ then M is a direct summand of N^* for some module N such that $N^* \in \mathcal{G}$.

Then (1) \Rightarrow (2) \Rightarrow (3) and if \mathcal{G} is closed under pure-injective hulls then (3) \Rightarrow (1).

PROOF: (1) \Rightarrow (2) By Proposition 4.2.10, $\text{Prod } \mathcal{F}^* \subseteq \text{Pinj } \mathcal{G}$. Let $M \in \text{Pinj } \mathcal{G}$. By (1), M is a pure submodule of N^* for some module N such that $N^* \in \mathcal{G}$. Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, $N \in \mathcal{F}$ and hence $N^* \in \mathcal{F}^*$. Since M is pure-injective, M is a direct summand of N^* and hence $M \in \text{Prod } \mathcal{F}^*$. Thus $\text{Pinj } \mathcal{G} \subseteq \text{Prod } \mathcal{F}^*$ and hence $\text{Prod } \mathcal{F}^* = \text{Pinj } \mathcal{G}$.

(2) \Rightarrow (3) Let $M \in \text{Pinj } \mathcal{G}$. By (2), $M \in \text{Prod } \mathcal{F}^*$ and hence M is a direct summand of $\prod_{i \in I} M_i^*$ for some $M_i \in \mathcal{F}$ and this implies that M is a direct summand of $(\bigoplus_{i \in I} M_i)^*$. Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, $\bigoplus_{i \in I} M_i \in \mathcal{F}$ (by Proposition 4.2.8(1)) and hence $(\bigoplus_{i \in I} M_i)^* \in \mathcal{G}$.

(3) \Rightarrow (1) Suppose that \mathcal{G} is closed under pure-injective hulls. Let $M \in \mathcal{G}$, thus $PE(M) \in \text{Pinj } \mathcal{G}$. By (3), $PE(M)$ is a direct summand of N^* for some module N such that $N^* \in \mathcal{G}$. Since M is a pure submodule of $PE(M)$ it follows that M is a pure submodule of N^* . \square

Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. In the following proposition, we will give the relation between definability of \mathcal{F} and definability of \mathcal{G}^+ .

Proposition 4.3.7 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. Consider the following statements.

- (1) \mathcal{G}^+ is definable.
- (2) \mathcal{F} is definable and if $M \in \mathcal{G}$ then M is a pure submodule of N^* for some $N^* \in \mathcal{G}$.
- (3) \mathcal{F} is definable and $\text{Prod } \mathcal{F}^* = \text{Pinj } \mathcal{G}$.
- (4) \mathcal{F} is definable and if $M \in \text{Pinj } \mathcal{G}$ then M is a direct summand of N^* for some $N^* \in \mathcal{G}$.

Then:

- (a) (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).
- (b) If \mathcal{G} is closed under pure-injective hulls then the four statements are equivalent.

PROOF: (a) (1) \Rightarrow (2) Let $M \in \mathcal{F}$, thus $M^* \in \mathcal{G}$ and by Theorem 4.3.2 we have that $M^{**} \in \mathcal{F}$ and hence $\mathcal{F}^{**} \subseteq \mathcal{F}$. By Proposition 4.3.1, \mathcal{F} is definable. Let $M \in \mathcal{G}$, thus by Theorem 4.3.2 we have that $M^{**} \in \mathcal{G}$. Since $\delta_M : M \rightarrow M^{**}$ is a pure monomorphism, M is a pure submodule of N^* , where $N = M^*$.

(2) \Rightarrow (3) \Rightarrow (4) By Lemma 4.3.6.

(b) Suppose that \mathcal{G} is closed under pure-injective hulls, thus from Lemma 4.3.6 we have that the statements (2), (3) and (4) are equivalent.

Now we will prove that (2) implies (1). Suppose that (2) holds, thus from Proposition 4.3.1 we have $\mathcal{F}^{**} \subseteq \mathcal{F}$. Let $M \in \mathcal{G}$. By hypothesis, M is a pure submodule of N^* for some $N \in \mathcal{F}$ and hence M^* is isomorphic to a direct summand of N^{**} . Since $\mathcal{F}^{**} \subseteq \mathcal{F}$ it follows that $N^{**} \in \mathcal{F}$ and hence $M^* \in \mathcal{F}$ and this implies that $\mathcal{G}^* \subseteq \mathcal{F}$. By Theorem 4.3.2, \mathcal{G}^+ is definable. \square

Proposition 4.3.8 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. Then \mathcal{G} is definable if and only if $(\mathcal{G}, \mathcal{F})$ is an almost dual pair.

PROOF: (\Rightarrow) Suppose that \mathcal{G} is definable, thus $\mathcal{G}^+ = \mathcal{G}$ and hence \mathcal{G}^+ is definable. By Proposition 4.3.7(a), \mathcal{F} is definable and hence \mathcal{F} is closed under direct products and summands. Let M be a left R -module. Suppose that $M \in \mathcal{G}$. Since \mathcal{G}^+ is definable it follows from Theorem 4.3.2 that $M^* \in \mathcal{F}$. Now suppose that $M^* \in \mathcal{F}$. Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, $M^{**} \in \mathcal{G}$. Since the canonical monomorphism $\delta_M : M \rightarrow M^{**}$ is pure and \mathcal{G} is closed under pure submodules, $M \in \mathcal{G}$. Hence $(\mathcal{G}, \mathcal{F})$ is an almost dual pair.

(\Leftarrow) Suppose that $(\mathcal{G}, \mathcal{F})$ is an almost dual pair, thus $\mathcal{G}^* \subseteq \mathcal{F}$. Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair it follows from Theorem 4.3.2 that \mathcal{G}^+ is definable. Since $(\mathcal{G}, \mathcal{F})$ is an almost dual pair it follows from Proposition 4.2.8(1) that \mathcal{G} is closed under pure submodules and hence $\mathcal{G}^+ = \mathcal{G}$ and this implies that \mathcal{G} is definable. \square

Corollary 4.3.9 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. Then:

- (1) \mathcal{G}^+ is definable if and only if $(\mathcal{G}^+, \mathcal{F})$ is an almost dual pair.
- (2) $\text{Pinj } \mathcal{G}$ is definable if and only if $(\text{Pinj } \mathcal{G}, \mathcal{F})$ is an almost dual pair.
- (3) $\text{Prod } \mathcal{F}^*$ is definable if and only if $(\text{Prod } \mathcal{F}^*, \mathcal{F})$ is an almost dual pair.
- (4) If $\text{Pinj } \mathcal{G}$ and $\text{Prod } \mathcal{F}^*$ are definable, then $\text{Prod } \mathcal{F}^* = \text{Pinj } \mathcal{G} = \mathcal{F}^\ominus$.

PROOF: Suppose that $(\mathcal{F}, \mathcal{G})$ is an almost dual pair. By Lemma 4.2.6 we have that $(\mathcal{F}, \mathcal{G}^+)$, $(\mathcal{F}, \text{Pinj } \mathcal{G})$ and $(\mathcal{F}, \text{Prod } \mathcal{F}^*)$ are almost dual pairs. Thus (1), (2) and (3) follow from Proposition 4.3.8.

(4) Suppose that $\text{Pinj } \mathcal{G}$ and $\text{Prod } \mathcal{F}^*$ are definable. By (2) and (3) we have that $(\text{Pinj } \mathcal{G}, \mathcal{F})$ and $(\text{Prod } \mathcal{F}^*, \mathcal{F})$ are almost dual pairs and hence Proposition 4.2.11 implies that $\text{Prod } \mathcal{F}^* = \text{Pinj } \mathcal{G} = \mathcal{F}^\ominus$. \square

Theorem 4.4(2) in [61, p. 17] is a special case of the following result if we take $\mathcal{F} = {}_R\mathcal{T}\mathcal{F}$ and $\mathcal{G} = \text{Div}_R$ as in Examples 4.2.5(3(iii)).

Proposition 4.3.10 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. If \mathcal{G}^+ is definable, then:

- (1) $\mathcal{D}(\mathcal{G}^+) = \mathcal{F}$.

$$(2) \mathcal{D}(\mathcal{F}) = \mathcal{G}^+.$$

PROOF: (1) Suppose that \mathcal{G}^+ is definable. Let $M \in \mathcal{D}(\mathcal{G}^+)$, thus $M^* \in \mathcal{D}^2(\mathcal{G}^+)$ (by Lemma 2.4.5) and hence $M^* \in \mathcal{G}^+$, by Lemma 2.4.4. Since $(\mathcal{F}, \mathcal{G}^+)$ is an almost dual pair, $M \in \mathcal{F}$ and hence $\mathcal{D}(\mathcal{G}^+) \subseteq \mathcal{F}$.

Conversely, let $N \in \mathcal{F}$, thus $N^* \in \mathcal{G}^+$. By Lemma 2.4.5, $N^{**} \in \mathcal{D}(\mathcal{G}^+)$. Since $\mathcal{D}(\mathcal{G}^+)$ is definable (by Lemma 2.4.4) and N is a pure submodule of N^{**} it follows that $N \in \mathcal{D}(\mathcal{G}^+)$ and hence $\mathcal{F} \subseteq \mathcal{D}(\mathcal{G}^+)$. Therefore, $\mathcal{D}(\mathcal{G}^+) = \mathcal{F}$.

(2) Suppose that \mathcal{G}^+ is definable. By Lemma 2.4.4, $\mathcal{D}^2(\mathcal{G}^+) = \mathcal{G}^+$ and from (1) we have $\mathcal{D}(\mathcal{F}) = \mathcal{G}^+$. □

4.4 Definability of the class $(\varinjlim S)$

4.4.1 Definability of the class $(\varinjlim S)$ by using duality of modules

As an application of the above results, we have the following main result in this subsection, in which we give several equivalent characterizations of definability of the class $(\varinjlim S)$ by using duality of modules.

Theorem 4.4.1 Let $S \subseteq R\text{-mod}$. Then the following statements are equivalent.

- (1) $\varinjlim(\text{add } S)$ is definable.
- (2) $(\varinjlim(\text{add } S))^{**} \subseteq \varinjlim(\text{add } S)$.
- (3) $(\text{Prod } S^*)^+$ is definable.
- (4) $((\text{Prod } S^*)^+)^* \subseteq \varinjlim(\text{add } S)$.
- (5) $(\text{Prod } S^*)^* \subseteq \varinjlim(\text{add } S)$.
- (6) $((\text{Prod } S^*)^+)^{**} \subseteq \text{Prod } S^*$.
- (7) $((\text{Prod } S^*)^+)^{**} \subseteq (\text{Prod } S^*)^+$.
- (8) $(\text{Prod } S^*)^+$ is closed under pure homomorphic images.
- (9) For every $M \in (\text{Prod } S^*)^+$, there is a pure exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ with $N^* \in \varinjlim(\text{add } S)$.
- (10) For every $M \in \text{Prod } S^*$, there is a pure exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ with

$N^* \in \varinjlim(\text{add } S)$.

(11) $((\text{Prod } S^*)^+, \varinjlim(\text{add } S))$ is an almost dual pair.

(12) $\langle S^* \rangle = (\text{Prod } S^*)^+$, where $\langle S^* \rangle$ is the definable subcategory generated by S^* .

(13) $\mathcal{D}(\varinjlim(\text{add } S)) = (\text{Prod } S^*)^+ = \langle S^* \rangle$.

(14) $\mathcal{D}((\text{Prod } S^*)^+) = \varinjlim(\text{add } S) = \langle S \rangle$.

PROOF: By Corollary 4.2.25, $(\varinjlim(\text{add } S), \text{Prod } S^*)$ is an almost dual pair. By Proposition 4.3.1 we have that (1) and (2) are equivalent. By Theorem 4.3.2, the statements from (3) to (10) are equivalent. The direction (3) \Rightarrow (1) follows from Proposition 4.3.7(a). Now we will prove the direction (1) \Rightarrow (3). Suppose that $\varinjlim(\text{add } S)$ is definable. Let $M \in \text{Prod } S^*$, thus M is a direct summand of $\prod_{i \in I} M_i^*$ for some $M_i \in S$. Since $\prod_{i \in I} M_i^* = (\bigoplus_{i \in I} M_i)^*$ (by Lemma 2.1.5(2)) it follows from Lemma 2.2.2(1) that M is a pure submodule of $(\bigoplus_{i \in I} M_i)^* \in \text{Prod } S^*$. Since $\text{Prod } S^*$ is closed under pure-injective hulls it follows from Proposition 4.3.7(b) that $(\text{Prod } S^*)^+$ is definable.

(3) \Leftrightarrow (11) By Corollary 4.3.9(1).

(3) \Rightarrow (12) Suppose that $(\text{Prod } S^*)^+$ is definable. Since $S^* \subseteq (\text{Prod } S^*)^+$ it follows that $\langle S^* \rangle \subseteq (\text{Prod } S^*)^+$. Since $\langle S^* \rangle$ is definable, it is closed under direct products and pure submodules and hence $(\text{Prod } S^*)^+ \subseteq \langle S^* \rangle$. Thus $\langle S^* \rangle = (\text{Prod } S^*)^+$.

(12) \Rightarrow (13) Suppose that $\langle S^* \rangle = (\text{Prod } S^*)^+$, thus $(\text{Prod } S^*)^+$ is definable. Since $(\varinjlim(\text{add } S), (\text{Prod } S^*)^+)$ is an almost dual pair it follows from Proposition 4.3.10(2) that $\mathcal{D}(\varinjlim(\text{add } S)) = (\text{Prod } S^*)^+$.

(13) \Rightarrow (14) Suppose that $\mathcal{D}(\varinjlim(\text{add } S)) = (\text{Prod } S^*)^+ = \langle S^* \rangle$, thus $\mathcal{D}^2(\varinjlim(\text{add } S)) = \mathcal{D}((\text{Prod } S^*)^+)$ and hence $\varinjlim(\text{add } S) = \mathcal{D}((\text{Prod } S^*)^+)$, by Lemma 2.4.4. Since $\varinjlim(\text{add } S)$ is definable, $\varinjlim(\text{add } S) = \langle S \rangle$.

(14) \Rightarrow (1) This is obvious. □

Remark 4.4.2 As mentioned previously, there are many well-known characterizations of right coherent rings. For examples, Stenström in [71, Theorem 3.2, p. 325] characterized right coherence by the fact that direct limits of absolutely pure right R -modules are absolutely pure. In [80, 1.6, p. 383], Würfel proved that a ring R is right coherent if and only if M^* is flat for any absolutely pure right R -module M . Also, in [12, Theorem 1, p. 176],

Cheatham and Stone proved that the following statements are equivalent:

- (1) a ring R is right coherent;
- (2) a right R -module M is absolutely pure if and only if M^{**} is injective;
- (3) a left R -module M flat if and only if M^{**} is flat.

It is clear that all these examples of characterizations of right coherent rings are special cases of Theorem 4.4.1 if we take $S = \{{}_R R\}$.

The equivalence of (1) and (5) in Theorem 4.4.1 shows that the condition ${}_R R \in S$ in the following theorem of Holm is not necessary and also the converse of this theorem is true.

Theorem 4.4.3 [26, Theorem 5.6, p. 556] Assume that ${}_R R \in S$ and that $\varinjlim(\text{add } S)$ is definable, where $S \subseteq R\text{-mod}$. Then $M^* \in \varinjlim(\text{add } S)$ whenever a right R -module $M \in \text{Prod } S^*$.

The following corollary is immediate from Theorem 4.4.1 and Corollary 4.3.4.

Corollary 4.4.4 Let $S \subseteq R\text{-mod}$. If $\varinjlim(\text{add } S)$ is definable, then $(\text{Prod } S^*)^+$ is covering in $\text{Mod-}R$.

The notations in the following example are as in Example 4.2.28(4).

Example 4.4.5 Let R be a tame hereditary finite-dimensional algebra over a field k . Then:

- (1) $(\text{Prod } \mathbf{I}_R, {}_R \mathcal{TF}\text{free})$ is an almost dual pair.
- (2) $((\text{Prod}(\mathbf{I}_R \cup \mathbf{R}_R))^+, {}^\perp({}_R \mathbf{I}))$ is an almost dual pair.
- (3) $(\text{Prod } \mathbf{P}_R, \varinjlim(\text{add } {}_R \mathbf{I}))$ is not an almost dual pair.
- (4) $\mathcal{TF}\mathcal{S}_R$ is not definable and the pair $(\mathcal{TF}\mathcal{S}_R, \varinjlim(\text{add } {}_R \mathbf{I}))$ is not almost dual.
- (5) $((\text{Prod } \mathbf{P}_R)^+)^* \not\subseteq \varinjlim(\text{add } {}_R \mathbf{I})$.

PROOF: (1) By Example 4.2.28(4(ii)), $({}_R \mathcal{TF}\text{free}, \text{Prod } \mathbf{I}_R)$ is an almost dual pair. Since $\text{Prod } \mathbf{I}_R$ is definable (see Example 4.2.28(4(ii))) it follows from Theorem 4.4.1 that $(\text{Prod } \mathbf{I}_R, {}_R \mathcal{TF}\text{free})$ is an almost dual pair.

(2) By [4, Examples 3.4, p. 303], ${}^\perp({}_R \mathbf{I})$ is definable. Since $\varinjlim(\text{add}({}_R \mathbf{P} \cup {}_R \mathbf{R})) = {}^\perp({}_R \mathbf{I})$ (by Example 4.2.28(4(iii))) it follows that $\varinjlim(\text{add}({}_R \mathbf{P} \cup {}_R \mathbf{R}))$ is definable and hence by Theorem 4.4.1 that $((\text{Prod}(\mathbf{I}_R \cup \mathbf{R}_R))^+, {}^\perp({}_R \mathbf{I}))$ is an almost dual pair.

(3) Assume that $(\text{Prod } \mathbf{P}_R, \varinjlim(\text{add } \mathbf{I}))$ is an almost dual pair, thus it follows from Proposition 4.3.8 that $\text{Prod } \mathbf{P}_R$ is definable. Since a hereditary algebra is of finite representation type if and only if $\text{Prod } \mathbf{P}_R$ is definable (by [83, Theorem 4.1, p. 353]) it follows that the algebra R is of finite representation type and hence it is not tame and this is a contradiction. Thus $(\text{Prod } \mathbf{P}_R, \varinjlim(\text{add } \mathbf{I}))$ is not an almost dual pair.

(4) Assume that $\mathcal{TF}S_R$ is definable. By Example 4.2.28(4(iv)), $(\text{Prod } \mathbf{P}_R)^+ = \mathcal{TF}S_R$ and hence $(\text{Prod } \mathbf{P}_R)^+$ is definable. By Theorem 4.4.1, $\varinjlim(\text{add } \mathbf{I})$ is definable and this implies that $\text{Prod } \mathbf{I} \subseteq \varinjlim(\text{add } \mathbf{I})$. Put $\mathcal{X} = (\text{Prod } \mathbf{I}) \cap {}_R\text{pinj}$. Since $\text{Prod } \mathbf{I}$ is definable it follows from Theorem 2.4.7 that \mathcal{X} is a closed set in the Ziegler spectrum. Since $\mathbf{I} \subseteq \mathcal{X}$ it follows that \mathcal{X} contains infinite many finitely generated modules and hence from [59, Theorem, p. 106] we have that the generic module $G \in \text{Prod } \mathbf{I}$ and this implies that $G \in \varinjlim(\text{add } \mathbf{I})$. By [4, Example 5.2, p. 308], $G \notin \varinjlim(\text{add } \mathbf{I})$ and this is a contradiction. Thus $\mathcal{TF}S_R$ is not definable. By Theorem 4.4.1, the pair $(\mathcal{TF}S_R, \varinjlim(\text{add } \mathbf{I}))$ is not almost dual.

(5) Since $(\text{Prod } \mathbf{P}_R)^+ = \mathcal{TF}S_R$ it follows from (4) that $(\text{Prod } \mathbf{P}_R)^+$ is not definable. By Theorem 4.4.1, $((\text{Prod } \mathbf{P}_R)^+)^* \not\subseteq \varinjlim(\text{add } \mathbf{I})$. \square

Theorem 3 in [12, p.176] is a special case of the following result if we take $S = \{{}_R R\}$.

Corollary 4.4.6 Let $S \subseteq R\text{-mod}$. Then the following statements are equivalent.

- (1) $((\text{Prod } S^*)^+, (\text{Add } S)^+)$ is an almost dual pair.
- (2) $M \in (\text{Prod } S^*)^+$ if and only if $M^* \in \text{Add } S$.
- (3) $\varinjlim(\text{add } S)$ is definable and $\varinjlim(\text{add } S) = (\text{Add } S)^+$.

PROOF: (1) \Rightarrow (2) This is obvious.

(2) \Rightarrow (3) Since $\varinjlim(\text{add } S)$ is closed under pure submodules and direct sums (by Proposition 4.2.8(1)) it follows that $(\text{Add } S)^+ \subseteq \varinjlim(\text{add } S)$. Let $M \in \varinjlim(\text{add } S)$, thus $M^* \in (\text{Prod } S^*)^+$ and from hypothesis we have that $M^{**} \in (\text{Add } S)^+$. Since M is a pure submodule of M^{**} it follows that $M \in (\text{Add } S)^+$ and hence $\varinjlim(\text{add } S) \subseteq (\text{Add } S)^+$ and this implies that $\varinjlim(\text{add } S) = (\text{Add } S)^+$. Let $N \in (\text{Prod } S^*)^+$. By (2), $N^* \in \text{Add } S \subseteq \varinjlim(\text{add } S)$ and hence $((\text{Prod } S^*)^+)^* \subseteq \varinjlim(\text{add } S)$. By Theorem 4.4.1, $\varinjlim(\text{add } S)$ is definable.

(3) \Rightarrow (1) Suppose that $\varinjlim(\text{add } S)$ is definable and $\varinjlim(\text{add } S) = (\text{Add } S)^+$. By Theorem 4.4.1, $((\text{Prod } S^*)^+, \varinjlim(\text{add } S))$ is an almost dual pair and hence $((\text{Prod } S^*)^+, (\text{Add } S)^+)$

is an almost dual pair. □

Let $S \subseteq \text{mod-}R$. Sklyarenko in [67, p. 127] introduced the concept of a right module being pure relative to the class of all S -pure exact sequences (shortly, relatively S -pure) as follows. A right R -module M is said to be **relatively S -pure** if every S -pure exact sequence of right R -modules $0 \rightarrow M \rightarrow N \rightarrow K \rightarrow 0$ is pure.

Proposition 4.4.7 Let $S \subseteq R\text{-mod}$. Then $(\text{Prod}(S \cup \{R\})^*)^+ = \{M \mid M \text{ is a relatively } D_{\mathcal{H}}\text{-pure right } R\text{-module}\}$.

PROOF: Let $M \in (\text{Prod}(S \cup \{R\})^*)^+$, thus M is a pure submodule of N with N $D_{\mathcal{H}}$ -pure-injective. Let $\Sigma : 0 \rightarrow M \xrightarrow{f} A \rightarrow B \rightarrow 0$ be any $D_{\mathcal{H}}$ -pure exact sequence of right R -modules. Since N is an $D_{\mathcal{H}}$ -pure-injective, there exists a homomorphism $g : A \rightarrow N$ such that $gf = i$, where $i : M \rightarrow N$ is the inclusion. Since i is a pure monomorphism it follows from Lemma 2.2.16(2(ii)) that f is a pure monomorphism and hence Σ is a pure exact sequence and this implies that M is a relatively $D_{\mathcal{H}}$ -pure right R -module.

Conversely, let M be any relatively $D_{\mathcal{H}}$ -pure right R -module. By Corollary 3.1.3, there is an $D_{\mathcal{H}}$ -pure monomorphism $\alpha : M \rightarrow F$ with F a $D_{\mathcal{H}}$ -pure-injective module. Since M is relatively $D_{\mathcal{H}}$ -pure it follows that α is pure and hence $M \in (\text{Prod}(S \cup \{R\})^*)^+$. □

Corollary 4.4.8 Let $S \subseteq R\text{-mod}$. Then the following statements are equivalent.

- (1) $\varinjlim(\text{add } S)$ is definable.
- (2) For every $D_{\mathcal{H}}$ -pure exact sequence of right R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, if $A, B \in (\text{Prod } S^*)^+$ then $C \in (\text{Prod } S^*)^+$.

PROOF: (1) \Rightarrow (2) Suppose that $\varinjlim(\text{add } S)$ is definable, thus $(\text{Prod } S^*)^+$ is definable, by Theorem 4.4.1. Let $\Sigma : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be any $D_{\mathcal{H}}$ -pure exact sequence of right R -modules such that $A, B \in (\text{Prod } S^*)^+$. Since $A \in (\text{Prod } S^*)^+$ it follows from Proposition 4.4.7 that A is a relatively $D_{\mathcal{H}}$ -pure and hence Σ is pure. Since $B \in (\text{Prod } S^*)^+$ and $(\text{Prod } S^*)^+$ is definable it follows from Theorem 2.4.2 that $C \in (\text{Prod } S^*)^+$.

(2) \Rightarrow (1) Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be any pure exact sequence of right R -modules with $B \in (\text{Prod } S^*)^+$. Since $(\text{Prod } S^*)^+$ is closed under pure submodules, $A \in (\text{Prod } S^*)^+$. By (2),

$C \in (\text{Prod } S^*)^+$ and hence $(\text{Prod } S^*)^+$ is closed under pure homomorphic images and this implies that $\varinjlim(\text{add } S)$ is definable, by Theorem 4.4.1. \square

Remark 4.4.9 Couchot, in [14], introduced the concept of an RD -coflat right R -module. A right R -module M is said to be RD -coflat if every RD -pure exact sequence $0 \rightarrow M \rightarrow P \rightarrow Q \rightarrow 0$ is pure exact. We will use $\mathcal{RD}\text{-Coflat}_R$ (resp. ${}_R\mathcal{RD}\text{-Coflat}$) to denote the class of RD -coflat right (resp. left) R -modules. If S is the class of $(1, 1)$ -presented right R -modules, then a right R -module M is relatively S -pure if and only if M is RD -coflat and hence from Proposition 4.4.7, we have $\mathcal{RD}\text{-Coflat}_R = (\text{Prod}(S' \cup \{{}_R R\})^*)^+ = (\mathcal{RD}\text{-Inj}_R)^+$, where S' is the class of $(1, 1)$ -presented left R -modules, and hence from Example 4.2.28(3) we have $({}_R\mathcal{RD}\text{-Flat}, \mathcal{RD}\text{-Coflat}_R)$ is an almost dual pair.

Corollary 4.4.10 Let T be the class of $(1, 1)$ -presented left R -modules. The following statements are equivalent.

- (1) ${}_R\mathcal{RD}\text{-Flat}$ is definable.
- (2) $({}_R\mathcal{RD}\text{-Flat})^{**} \subseteq {}_R\mathcal{RD}\text{-Flat}$.
- (3) $\mathcal{RD}\text{-Coflat}_R$ is definable.
- (4) $(\mathcal{RD}\text{-Coflat}_R)^* \subseteq {}_R\mathcal{RD}\text{-Flat}$.
- (5) $(\mathcal{RD}\text{-Inj}_R)^* \subseteq {}_R\mathcal{RD}\text{-Flat}$.
- (6) $(\mathcal{RD}\text{-Coflat}_R)^{**} \subseteq \mathcal{RD}\text{-Inj}_R$.
- (7) $(\mathcal{RD}\text{-Coflat}_R)^{**} \subseteq \mathcal{RD}\text{-Coflat}_R$.
- (8) $\mathcal{RD}\text{-Coflat}_R$ is closed under pure homomorphic images.
- (9) For every $M \in \mathcal{RD}\text{-Coflat}_R$, there is a pure exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ with $N^* \in {}_R\mathcal{RD}\text{-Flat}$.
- (10) For every $M \in \mathcal{RD}\text{-Inj}_R$, there is a pure exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ with $N^* \in {}_R\mathcal{RD}\text{-Flat}$.
- (11) $(\mathcal{RD}\text{-Coflat}_R, {}_R\mathcal{RD}\text{-Flat})$ is an almost dual pair.
- (12) $\langle T^* \rangle = \mathcal{RD}\text{-Coflat}_R$, where $\langle T^* \rangle$ is the definable subcategory generated by T^* .
- (13) $\mathcal{D}({}_R\mathcal{RD}\text{-Flat}) = \mathcal{RD}\text{-Coflat}_R = \langle T^* \rangle$.

$$(14) \mathcal{D}(\mathcal{RD}\text{-Coflat}_R) = {}_R\mathcal{RD}\text{-Flat} = \langle T \rangle.$$

(15) For every $(1, 1)$ -pure exact sequence of right R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, if $A, B \in \mathcal{RD}\text{-Coflat}_R$ then $C \in \mathcal{RD}\text{-Coflat}_R$.

(16) Every left R -module has a $({}_R\mathcal{RD}\text{-Flat})$ -preenvelope.

PROOF: Let S be the class of $(1, 1)$ -presented left R -modules. By Example 4.2.28(3), $\varinjlim(\text{add} S) = {}_R\mathcal{RD}\text{-Flat}$. Thus the result follows by applying Theorem 4.4.1, Corollary 4.4.8 and Proposition 4.2.8(3). \square

4.4.2 Definability of the class $(\varinjlim S)$ in terms of $(\varinjlim S)^\ominus$

In the following theorem, we give equivalent characterizations of definability of any class of modules \mathcal{F} in terms of the class \mathcal{F}^\ominus .

Theorem 4.4.11 Let $\mathcal{F} \subseteq R\text{-Mod}$. Then the following statements are equivalent.

- (1) \mathcal{F} is definable.
- (2) $\mathcal{F}^{**} \subseteq \mathcal{F}$ and \mathcal{F} is closed under direct sums and pure submodules.
- (3) $(\mathcal{F}, \mathcal{F}^\ominus)$ is an almost dual pair.
- (4) $(\mathcal{F}^\ominus)^+$ is definable and $M \in \mathcal{F}$ if and only if $M^* \in (\mathcal{F}^\ominus)^+$.

PROOF: (1) \Rightarrow (2) This is by Lemma 2.4.6.

(2) \Rightarrow (3) Let $M \in \mathcal{F}$. By (2), $M^{**} \in \mathcal{F}$ and hence $M^* \in \mathcal{F}^\ominus$. Conversely, let M be a module such that $M^* \in \mathcal{F}^\ominus$, thus $M^{**} \in \mathcal{F}$. Since \mathcal{F} is closed under pure submodules (by hypothesis), $M \in \mathcal{F}$. Hence $M \in \mathcal{F}$ if and only if $M^* \in \mathcal{F}^\ominus$. Let $M \in \mathcal{F}^\ominus$ and let N be any direct summand of M , thus N^* is isomorphic to a direct summand of M^* . Since $M^* \in \mathcal{F}$ and \mathcal{F} is closed under direct summands it follows that $N^* \in \mathcal{F}$ and hence $N \in \mathcal{F}^\ominus$. Thus \mathcal{F}^\ominus is closed under direct summands. Let $\{M_i\}_{i \in I}$ be any family of modules in \mathcal{F}^\ominus , thus $M_i^* \in \mathcal{F}$. By hypothesis, $\bigoplus_{i \in I} M_i^* \in \mathcal{F}$ and hence $(\bigoplus_{i \in I} M_i^*)^{**} \in \mathcal{F}$ and this implies that $(\prod_{i \in I} M_i^{**})^* \in \mathcal{F}$. Since $\prod_{i \in I} M_i$ is a pure submodule of $\prod_{i \in I} M_i^{**}$ it follows that $(\prod_{i \in I} M_i)^*$ is isomorphic to a direct summand of $(\prod_{i \in I} M_i^{**})^*$ and hence $(\prod_{i \in I} M_i)^* \in \mathcal{F}$ and this implies that $\prod_{i \in I} M_i \in \mathcal{F}^\ominus$. Thus \mathcal{F}^\ominus is closed under direct products and hence $(\mathcal{F}, \mathcal{F}^\ominus)$ is an almost dual pair.

(3) \Rightarrow (4) Suppose that $(\mathcal{F}, \mathcal{F}^\ominus)$ is an almost dual pair, thus $M \in \mathcal{F}$ if and only if $M^* \in \mathcal{F}^\ominus$ if and only if $M^* \in (\mathcal{F}^\ominus)^+$. Let $M \in \mathcal{F}^\ominus$, thus $M^* \in \mathcal{F}$ and hence $(\mathcal{F}^\ominus)^* \subseteq \mathcal{F}$. By Theorem 4.3.2, $(\mathcal{F}^\ominus)^+$ is definable.

(4) \Rightarrow (1) Suppose that $(\mathcal{F}^\ominus)^+$ is definable and $M \in \mathcal{F}$ if and only if $M^* \in (\mathcal{F}^\ominus)^+$, thus $(\mathcal{F}, (\mathcal{F}^\ominus)^+)$ is an almost dual pair. Since $(\mathcal{F}^\ominus)^+$ is definable it follows from Proposition 4.3.7(a) that \mathcal{F} is definable. \square

The following corollary is immediately obtained from Theorem 4.4.11.

Corollary 4.4.12 There is a natural bijection between all definable classes of left R -modules and all almost dual pairs of the form $(\mathcal{F}, \mathcal{F}^\ominus)$, where $\mathcal{F} \subseteq R\text{-Mod}$. The bijection is given by $\mathcal{F} \mapsto (\mathcal{F}, \mathcal{F}^\ominus)$.

Proposition 4.4.13 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. Then:

- (1) $\mathcal{F}^\ominus \subseteq \mathcal{G}^+$.
- (2) $\mathcal{F}^\ominus = \mathcal{G}^+$ if and only if \mathcal{G}^+ is definable.

PROOF: (1) Let $M \in \mathcal{F}^\ominus$, thus $M^* \in \mathcal{F}$ and hence $M^{**} \in \mathcal{G}^+$. Since M is a pure submodule of M^{**} it follows that $M \in \mathcal{G}^+$ and hence $\mathcal{F}^\ominus \subseteq \mathcal{G}^+$.

(2) (\Rightarrow) Suppose that $\mathcal{F}^\ominus = \mathcal{G}^+$. Let $M \in \mathcal{G}^+$, thus $M \in \mathcal{F}^\ominus$ and hence $M^* \in \mathcal{F}$. Thus $(\mathcal{G}^+)^* \subseteq \mathcal{F}$ and from Theorem 4.3.2 we have \mathcal{G}^+ is definable.

(\Leftarrow) Suppose that \mathcal{G}^+ is definable. Let $M \in \mathcal{G}^+$. By Theorem 4.3.2, $M^* \in \mathcal{F}$ and hence $M \in \mathcal{F}^\ominus$. Thus $\mathcal{G}^+ \subseteq \mathcal{F}^\ominus$ and from (1) we have $\mathcal{F}^\ominus = \mathcal{G}^+$. \square

The following corollary gives equivalent characterizations of definability of $\varinjlim(\text{add } S)$ in terms of the class $(\varinjlim(\text{add } S))^\ominus$.

Corollary 4.4.14 Let $S \subseteq R\text{-mod}$. Then the following statements are equivalent.

- (1) $\varinjlim(\text{add } S)$ is definable.
- (2) $(\varinjlim(\text{add } S), (\varinjlim(\text{add } S))^\ominus)$ is an almost dual pair.
- (3) $((\varinjlim(\text{add } S))^\ominus)^+$ is definable and $M \in \varinjlim(\text{add } S)$ if and only if $M^* \in ((\varinjlim(\text{add } S))^\ominus)^+$.

$$(4) (\varinjlim(\text{add} S))^\ominus = (\text{Prod} S^*)^+.$$

PROOF: The proof follows from Theorem 4.4.11, Theorem 4.4.1 (1) \Leftrightarrow (3) and Proposition 4.4.13(2). \square

The following proposition gives an equivalent characterization of definability of $\varinjlim(\text{add} S)$ in terms of the class $(\varinjlim(\text{add} S))^\ominus$.

Proposition 4.4.15 Let S be a class of finitely presented left R -modules such that ${}_R R \in S$. Then $\varinjlim(\text{add} S)$ is definable if and only if every right R -module has an $D_{\mathcal{H}}$ -pure monomorphic $(\varinjlim(\text{add} S))^\ominus$ -preenvelope.

PROOF: (\Rightarrow) Suppose that $\varinjlim(\text{add} S)$ is definable, thus $(\text{Prod} S^*)^+$ is definable (by Theorem 4.4.1). By Theorem 3.1.4, $(\text{Prod} S^*)^+ = (D_{\mathcal{H}}\text{-Pinj})^+$ and hence $(D_{\mathcal{H}}\text{-Pinj})^+$ is definable. Let $M \in \text{Mod-}R$. Since $(D_{\mathcal{H}}\text{-Pinj})^+$ is closed under pure submodules and direct products it follows from Lemma 2.3.5 that $(D_{\mathcal{H}}\text{-Pinj})^+$ is preenveloping in $\text{Mod-}R$. Let $\alpha : M \rightarrow N$ be a $(D_{\mathcal{H}}\text{-Pinj})^+$ -preenvelope of M . By Corollary 3.1.3, there is a $D_{\mathcal{H}}$ -pure monomorphism $\beta : M \rightarrow F$ with $F \in D_{\mathcal{H}}\text{-Pinj} \subseteq (D_{\mathcal{H}}\text{-Pinj})^+$. Thus there is a homomorphism $f : N \rightarrow F$ such that $f\alpha = \beta$ and hence by Lemma 2.2.16(2(ii)) we have that α is a $D_{\mathcal{H}}$ -pure monomorphism. Since $(D_{\mathcal{H}}\text{-Pinj})^+$ is definable it follows from Corollary 4.4.14 that $(\varinjlim(\text{add} S))^\ominus = (D_{\mathcal{H}}\text{-Pinj})^+$. Therefore, M has a $D_{\mathcal{H}}$ -pure monomorphic $(\varinjlim(\text{add} S))^\ominus$ -preenvelope.

(\Leftarrow) Let $M \in \text{Prod} S^*$. By hypothesis, M has a $D_{\mathcal{H}}$ -pure monomorphic $(\varinjlim(\text{add} S))^\ominus$ -preenvelope $\alpha : M \rightarrow N$ and hence $N^* \in \varinjlim(\text{add} S)$. Since M is $D_{\mathcal{H}}$ -pure-injective it follows from Theorem 2.2.19 that M is a direct summand of N and hence M^* is isomorphic to a direct summand of N^* and this implies that $M^* \in \varinjlim(\text{add} S)$. Thus $(\text{Prod} S^*)^* \subseteq \varinjlim(\text{add} S)$ and hence from Theorem 4.4.1 we have that $\varinjlim(\text{add} S)$ is definable. \square

Corollary 4.4.16 A ring R is a right coherent if and only if every right R -module has a monomorphic $(({}_R\mathcal{F}lat)^\ominus)$ -preenvelope.

PROOF: By taking $S = \{{}_R R\}$ and apply Example 4.2.28(1) and Proposition 4.4.15. \square

4.4.3 Definability of the class $(\varinjlim S)$ using duality between preenvelopes and precovers

The following result can be found in [18, 3.2, p. 1137].

Lemma 4.4.17 (1) Let \mathcal{F} be a subcategory of $R\text{-Mod}$ and \mathcal{G} a subcategory of $\text{Mod-}R$ such that $\mathcal{F}^* \subseteq \mathcal{G}$ and $\mathcal{G}^* \subseteq \mathcal{F}$. If $f : A \rightarrow C$ is a \mathcal{F} -preenvelope of a module A in $R\text{-Mod}$, then $f^* : C^* \rightarrow A^*$ is a \mathcal{G} -precover of A^* in $\text{Mod-}R$.

(2) Let $f : M \rightarrow N$ be a homomorphism in $R\text{-Mod}$. If $f^* : N^* \rightarrow M^*$ is right minimal, then f is left minimal.

Proposition 4.4.18 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair such that \mathcal{G}^+ is a definable class. Let $M \in R\text{-Mod}$ and let $N \in \text{Mod-}R$.

(1) If $\varphi : M \rightarrow X$ is a \mathcal{F} -preenvelope of M then $\varphi^* : X^* \rightarrow M^*$ is a \mathcal{G}^+ -precover and \mathcal{G} -precover of M^* .

(2) If $\varphi : N \rightarrow Y$ is a \mathcal{G}^+ -preenvelope or \mathcal{G} -preenvelope of N then $\varphi^* : Y^* \rightarrow N^*$ is a \mathcal{F} -precover of N^* .

PROOF: (1) Suppose that $\varphi : M \rightarrow X$ is a \mathcal{F} -preenvelope of M . Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair (by hypothesis), $\mathcal{F}^* \subseteq \mathcal{G} \subseteq \mathcal{G}^+$. Since \mathcal{G}^+ is definable (by hypothesis) it follows from Theorem 4.3.2 that $\mathcal{G}^* \subseteq (\mathcal{G}^+)^* \subseteq \mathcal{F}$. By Lemma 4.4.17(1), $\varphi^* : X^* \rightarrow M^*$ is a \mathcal{G}^+ -precover and \mathcal{G} -precover of M^* .

(2) By a similar proof to that of (1). □

The following corollary is immediate from Corollary 4.2.25, Theorem 4.4.1 and Proposition 4.4.18.

Corollary 4.4.19 Let $S \subseteq R\text{-mod}$ be such that $S = \text{add} S$ and $\varinjlim S$ is a definable class, let $M \in R\text{-Mod}$ and let $N \in \text{Mod-}R$.

(1) If $\varphi : M \rightarrow X$ is a $(\varinjlim S)$ -preenvelope of M then $\varphi^* : X^* \rightarrow M^*$ is a $(\text{Prod} S^*)^+$ -precover and a $(\text{Prod} S^*)$ -precover of M^* .

(2) If $\varphi : N \rightarrow Y$ is a $(\text{Prod } S^*)^+$ -preenvelope or a $(\text{Prod } S^*)$ -preenvelope of N then $\varphi^* : Y^* \rightarrow N^*$ is a $(\varinjlim S)$ -precover of N^* .

Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. Since \mathcal{G}^+ is closed under pure submodules and direct products it follows from Lemma 2.3.5 that \mathcal{G}^+ is preenveloping in $\text{Mod-}R$. Also, if $\mathcal{G} \subseteq \text{Pinj}_R$ then Lemma 2.3.4 implies that \mathcal{G} is enveloping in $\text{Mod-}R$. In the following theorem, we give an equivalent characterization of definability of the class \mathcal{G}^+ in terms of the above duality between \mathcal{G}^+ -preenvelopes (or \mathcal{G} -preenvelopes) and \mathcal{F} -precovers.

Theorem 4.4.20 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair with $\mathcal{G} \subseteq \text{Pinj}_R$. Then the following statements are equivalent:

- (1) \mathcal{G}^+ is definable;
- (2) if a monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a \mathcal{G}^+ -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* in $R\text{-Mod}$;
- (3) if a pure monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a \mathcal{G}^+ -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* in $R\text{-Mod}$;
- (4) $(1_M)^* : M^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* for any $M \in \mathcal{G}^+$, where $1_M : M \rightarrow M$ is the identity homomorphism;
- (5) if a monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a \mathcal{G} -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* in $R\text{-Mod}$;
- (6) if a pure monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a \mathcal{G} -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* in $R\text{-Mod}$.

PROOF: (1) \Rightarrow (2) and (1) \Rightarrow (5) Suppose that \mathcal{G}^+ is definable and that a monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a \mathcal{G}^+ -preenvelope (or a \mathcal{G} -preenvelope) of M . By Proposition 4.4.18(2), $\varphi^* : X^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* .

(2) \Rightarrow (3) and (5) \Rightarrow (6) are obvious.

(3) \Rightarrow (4) Let $M \in \mathcal{G}^+$. It is clear that 1_M is a pure monomorphism and it is a \mathcal{G}^+ -preenvelope of M . By (3), $(1_M)^* : M^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* .

(4) \Rightarrow (1) Let $M \in \mathcal{G}^+$. By (4), $(1_M)^* : M^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* and hence $M^* \in \mathcal{F}$ and this implies that $(\mathcal{G}^+)^* \subseteq \mathcal{F}$. By Theorem 4.3.2, \mathcal{G}^+ is definable.

(6) \Rightarrow (1) Let $M \in \mathcal{G}^+$, thus M is a pure submodule of N for some $N \in \mathcal{G}$. Let $i : M \rightarrow N$ be the inclusion. Let $X \in \mathcal{G}$ and let $f : M \rightarrow X$ be any homomorphism. Since $\mathcal{G} \subseteq \text{Pinj}_R$, there is a homomorphism $g : N \rightarrow X$ such that $gi = f$. Since $N \in \mathcal{G}$, the inclusion $i : M \rightarrow N$ is therefore a \mathcal{G} -preenvelope of M . By (6), $i^* : N^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* and hence $N^* \in \mathcal{F}$. Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, $N^{**} \in \mathcal{G}$. Since $i : M \rightarrow N$ is a pure monomorphism it follows that $i^{**} : M^{**} \rightarrow N^{**}$ is split and hence M^{**} is a direct summand of N^{**} and this implies that $M^{**} \in \mathcal{G}$. Thus $(\mathcal{G}^+)^{**} \subseteq \mathcal{G}$ and hence by Theorem 4.3.2 we have that \mathcal{G}^+ is definable. \square

By applying Lemma 4.2.6 and Theorem 4.4.20 we have the following result.

Corollary 4.4.21 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. Then the following statements are equivalent:

- (1) $(\text{Pinj } \mathcal{G})^+$ is definable;
- (2) if a monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a $(\text{Pinj } \mathcal{G})^+$ -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* in $R\text{-Mod}$;
- (3) if a pure monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a $(\text{Pinj } \mathcal{G})^+$ -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* in $R\text{-Mod}$;
- (4) $(1_M)^* : M^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* for any $M \in (\text{Pinj } \mathcal{G})^+$;
- (5) if a monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a $(\text{Pinj } \mathcal{G})$ -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* in $R\text{-Mod}$;
- (6) if a pure monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a $(\text{Pinj } \mathcal{G})$ -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a \mathcal{F} -precover of M^* in $R\text{-Mod}$.

Enochs and Huang in [18, Theorem 3.5, p. 1138] proved the following result.

Theorem 4.4.22 The following statements are equivalent for a ring R :

- (1) R is right coherent;
- (2) if a monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is an absolutely pure preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a flat precover of M^* in $R\text{-Mod}$;

(3) if a monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is an injective preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a flat precover of M^* in $R\text{-Mod}$.

Theorem 4.4.22 is the special case $S = \text{add}({}_R R)$ of the following corollary of Theorem 4.4.20, in which we give an equivalent characterization of definability of the class $\varinjlim S$ in terms of the duality between $(\text{Prod } S^*)^+$ -preenvelopes (or $(\text{Prod } S^*)$ -preenvelopes) and $(\varinjlim S)$ -precovers.

Corollary 4.4.23 Let $S \subseteq R\text{-mod}$ be such that $S = \text{add } S$. Then the following statements are equivalent:

- (1) $\varinjlim S$ is definable;
- (2) if a monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a $(\text{Prod } S^*)^+$ -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a $(\varinjlim S)$ -precover of M^* in $R\text{-Mod}$;
- (3) if a pure monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a $(\text{Prod } S^*)^+$ -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a $(\varinjlim S)$ -precover of M^* in $R\text{-Mod}$;
- (4) $(1_M)^*$ is a $(\varinjlim S)$ -precover of M^* for any $M \in (\text{Prod } S^*)^+$;
- (5) if a monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a $(\text{Prod } S^*)$ -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a $(\varinjlim S)$ -precover of M^* in $R\text{-Mod}$.
- (6) if a pure monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a $(\text{Prod } S^*)$ -preenvelope of M , then $\varphi^* : X^* \rightarrow M^*$ is a $(\varinjlim S)$ -precover of M^* in $R\text{-Mod}$.

PROOF: This follows by Corollary 4.2.25, Theorem 4.4.1 and applying Theorem 4.4.20.

□

4.5 Definability of the class $\text{Prod } S^*$

4.5.1 Definability of $\text{Prod } S^*$ in terms of preenvelopes and precovers

Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair such that $\mathcal{G} \subseteq \text{Pinj}_R$. In the following theorem, we give several characterizations of definability of \mathcal{G} .

Theorem 4.5.1 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair such that $\mathcal{G} \subseteq \text{Pinj}_R$. Then the following statements are equivalent:

- (1) \mathcal{G} is definable;
- (2) \mathcal{G} is closed under direct limits;
- (3) \mathcal{G} is closed under direct sums;
- (4) every module in \mathcal{G} is Σ -pure-injective;
- (5) $\mathcal{G} = \mathcal{G}^+$;
- (6) \mathcal{G} is closed under pure submodules;
- (7) \mathcal{G} is closed under pure submodules and pure quotients;
- (8) \mathcal{G} is closed under pure submodules and direct limits;
- (9) \mathcal{G} is covering in $\text{Mod-}R$;
- (10) \mathcal{G} is precovering in $\text{Mod-}R$;
- (11) $(\mathcal{G}, \mathcal{F})$ is an almost dual pair.

PROOF: (1) \Rightarrow (2) This is obvious.

(2) \Rightarrow (3) Assume that \mathcal{G} is closed under direct limits. Let $\{M_i\}_{i \in I}$ be any family of modules in \mathcal{G} and let $M = \bigoplus_{i \in I} M_i$. By [79, Exercies 24.13(3), p. 206], $M = \varinjlim \{ \bigoplus_{i \in J} M_i \mid J \subset I \text{ and } J \text{ finite} \}$. For each $J \subset I$ and J finite, since \mathcal{G} is closed under products, $\bigoplus_{i \in J} M_i \in \mathcal{G}$. Hence M is a direct limit of a direct system of modules in \mathcal{G} . By hypothesis, $M \in \mathcal{G}$ and hence \mathcal{G} is closed under direct sums.

(3) \Rightarrow (4) Let $M \in \mathcal{G}$ and let I be any index set. By (3), $M^{(I)} \in \mathcal{G}$ and hence it is pure-injective. Thus every module in \mathcal{G} is Σ -pure-injective.

(4) \Rightarrow (5) It is clear that $\mathcal{G} \subseteq \mathcal{G}^+$. Conversely, let $N \in \mathcal{G}^+$. Thus N is a pure submodule of M with $M \in \mathcal{G}$. By (4), M is Σ -pure-injective. Since every pure submodule of a Σ -pure-injective module is a direct summand of it (by [8, Proposition 3.5, p. 71]) it follows that N is a direct summand of M . Since \mathcal{G} is closed under direct summands, $N \in \mathcal{G}$ and hence $\mathcal{G}^+ \subseteq \mathcal{G}$. Thus $\mathcal{G} = \mathcal{G}^+$.

(5) \Rightarrow (6) This is obvious.

(6) \Rightarrow (7) Assume that \mathcal{G} is closed under pure submodules. Let $\alpha : N \rightarrow M$ be any pure epimorphism with $N \in \mathcal{G}$. Consider the following pure exact sequence

$$\Sigma : 0 \rightarrow \ker(\alpha) \xrightarrow{i} N \xrightarrow{\alpha} M \rightarrow 0$$

By (6), $\ker(\alpha) \in \mathcal{G}$ and hence from the hypothesis we have that $\ker(\alpha)$ is pure-injective. By Theorem 2.2.19, the sequence Σ is split and hence Lemma 2.1.2 implies that M is isomorphic to a direct summand of N . Since \mathcal{G} is closed under isomorphisms and direct summands, $M \in \mathcal{G}$. Hence \mathcal{G} is closed under pure submodules and pure quotients.

(7) \Rightarrow (8) Let $(M_i, f_{ij})_I$ be a direct system of modules in \mathcal{G} . By hypothesis, $\prod_{i \in I} M_i \in \mathcal{G}$. By Lemma 2.2.6(1), the canonical embedding $f : \bigoplus_{i \in I} M_i \rightarrow \prod_{i \in I} M_i$ is pure. Since \mathcal{G} is closed under pure submodules, $\bigoplus_{i \in I} M_i \in \mathcal{G}$. By Lemma 2.2.6(2), the canonical epimorphism $\eta : \bigoplus_{i \in I} M_i \rightarrow \varinjlim M_i$ is pure. Since \mathcal{G} is closed under pure quotients, $\varinjlim M_i \in \mathcal{G}$. Thus \mathcal{G} is closed under direct limits.

(8) \Rightarrow (1) By hypothesis, \mathcal{G} is closed under products, direct limits and pure submodules and hence \mathcal{G} is definable.

(1) \Rightarrow (9) Suppose that \mathcal{G} is definable, thus \mathcal{G} is closed under pure quotient modules and direct sums. By Theorem 2.3.2, \mathcal{G} is covering in $\text{Mod-}R$.

(9) \Rightarrow (10) This is obvious.

(10) \Rightarrow (3) Suppose that \mathcal{G} is precovering in $\text{Mod-}R$. Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, \mathcal{G} is closed under direct summands. Since every precovering class in $\text{Mod-}R$ which is closed under direct summands is closed under direct sums (see, e.g. [27, Proposition 1.2, p. 694]) it follows that \mathcal{G} is closed under direct sums.

(1) \Leftrightarrow (11) By Proposition 4.3.8. □

Remarks 4.5.2 (a) It is well-known that a ring R is right noetherian if and only if the class Inj_R is precovering (or covering) in $\text{Mod-}R$ (see, e.g., [19, Theorem 5.4.1, p. 120]). Also, Megibben in [43, Theorem 3, p. 564] proved that a ring R is right noetherian if and only if every absolutely pure right R -module is injective. These characterizations of right noetherian rings are special cases of Theorem 4.5.1 if we take $\mathcal{G} = \text{Inj}_R$.

(b) Let $S \subseteq R\text{-mod}$. In [26, Theorem 1.3], Holm proved that the following statements are equivalent:

(1) $\text{Prod}S^*$ is closed under direct sums;

- (2) $\text{Prod } S^*$ is closed under direct limits;
- (3) $\text{Prod } S^*$ is precovering in $\text{Mod-}R$;
- (4) $\text{Prod } S^*$ is covering in $\text{Mod-}R$;
- (5) $\text{Prod } S^*$ is closed under pure submodules.

It is clear that Holm's theorem is a special case of Theorem 4.5.1 if we take $\mathcal{G} = \text{Prod } S^*$.

Cheatham and Stone in [12, Theorem 2, p. 176] proved the following.

Theorem 4.5.3 The following statements are equivalent for a ring R :

- (1) R is right noetherian;
- (2) a right R -module M is injective if and only if M^{**} is injective;
- (3) a right R -module M is injective if and only if M^* is flat.

This theorem is a special case of the next result if we take $\mathcal{G} = \text{Inj}_R$.

Theorem 4.5.4 Let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair such that $\mathcal{G} \subseteq \text{Pinj}_R$. Then the following statements are equivalent:

- (1) \mathcal{G} is definable;
- (2) $M \in \mathcal{G}$ if and only if $M^{**} \in \mathcal{G}$;
- (3) $M \in \mathcal{G}$ if and only if $M^* \in \mathcal{F}$.

Moreover, these equivalent conditions imply the following condition:

- (4) \mathcal{F} is definable.

PROOF: (1) \Rightarrow (2) Suppose that \mathcal{G} is definable. Note that Theorem 4.5.1, $\mathcal{G}^+ = \mathcal{G}$ so if $M \in \mathcal{G}$, then $M^{**} \in \mathcal{G}$ by Theorem 4.3.2. Suppose that $M^{**} \in \mathcal{G}$. Since M is a pure submodule of M^{**} and \mathcal{G} is closed under pure submodules, $M \in \mathcal{G}$.

(2) \Rightarrow (3) Suppose that $M \in \mathcal{G}$, thus from (2) we have $M^{**} \in \mathcal{G}$. Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair, $M^* \in \mathcal{F}$. Conversely, suppose that $M^* \in \mathcal{F}$, thus $M^{**} \in \mathcal{G}$ and from (2) we have $M \in \mathcal{G}$.

(3) \Rightarrow (1) Let $M \in \mathcal{G}^+$, thus M is a pure submodule of N for some $N \in \mathcal{G}$. Thus the exact sequence $0 \rightarrow M^{**} \rightarrow N^{**} \rightarrow (N/M)^{**} \rightarrow 0$ is split and hence M^{**} is a direct summand of N^{**} . Since $N \in \mathcal{G}$ it follows from (3) that $N^* \in \mathcal{F}$ and hence $N^{**} \in \mathcal{G}$. Since \mathcal{G} is closed

under direct summands, $M^{**} \in \mathcal{G}$ and hence $M^* \in \mathcal{F}$. By (3), $M \in \mathcal{G}$ and hence $\mathcal{G}^+ \subseteq \mathcal{G}$ and this implies that $\mathcal{G}^+ = \mathcal{G}$. By Theorem 4.5.1, \mathcal{G} is definable.

(1) \Rightarrow (4) Suppose that \mathcal{G} is definable. As noted already, $\mathcal{G}^+ = \mathcal{G}$ and hence \mathcal{G}^+ is definable. By Proposition 4.3.7(a), \mathcal{F} is definable. \square

The following corollary shows that we can prove Holm's theorem [26, Theorem 1.5, p. 545] (see Theorem 4.1.1) without his assumptions (${}_R R \in S$ and $\varinjlim(\text{add } S)$ is definable).

Corollary 4.5.5 Let $S \subseteq R\text{-mod}$. Then the following statements are equivalent:

- (1) $\text{Prod } S^*$ is definable;
- (2) $M \in \text{Prod } S^*$ if and only if $M^{**} \in \text{Prod } S^*$;
- (3) $M \in \text{Prod } S^*$ if and only if $M^* \in \varinjlim(\text{add } S)$.

Moreover, these equivalent conditions imply the following condition:

- (4) $\varinjlim(\text{add } S)$ is definable.

PROOF: Since $\text{Prod } S^* \subseteq \text{Pinj}_R$ and $(\varinjlim(\text{add } S), \text{Prod } S^*)$ is an almost dual pair (by Corollary 4.2.25), the result follows from Theorem 4.5.4. \square

Cheatham and Stone in [12, Theorem 4, p. 177] proved the following.

Theorem 4.5.6 The following statements are equivalent for a ring R :

- (1) R is right artinian;
- (2) a right R -module M is injective if and only if M^* is projective.

This theorem is a special case of the next result if we take $S = \{{}_R R\}$.

Corollary 4.5.7 Let $S \subseteq R\text{-mod}$. Then the following statements are equivalent.

- (1) $(\text{Prod } S^*, (\text{Add } S)^+)$ is an almost dual pair.
- (2) $M \in \text{Prod } S^*$ if and only if $M^* \in \text{Add } S$.
- (3) $\text{Prod } S^*$ is definable and $\varinjlim(\text{add } S) = (\text{Add } S)^+$.

PROOF: (1) \Rightarrow (2) This is obvious.

(2) \Rightarrow (3) In a similar way to the proof of (2) \Rightarrow (3) in Corollary 4.4.6, we can prove that $\varinjlim(\text{add } S) = (\text{Add } S)^+$. Let $M \in \text{Prod } S^*$, thus from (2) we have $M^* \in \text{Add } S \subseteq \varinjlim(\text{add } S)$. Conversely, let $M \in \text{Mod-}R$ such that $M^* \in \varinjlim(\text{add } S)$. Since $\varinjlim(\text{add } S) = (\text{Add } S)^+$ it follows that $M^* \in (\text{Add } S)^+$. Since M^* is pure-injective, $M^* \in \text{Add } S$ and hence from (2) we have $M \in \text{Prod } S^*$. By Corollary 4.5.5, $\text{Prod } S^*$ is definable.

(3) \Rightarrow (1) Suppose that $\text{Prod } S^*$ is definable and $\varinjlim(\text{add } S) = (\text{Add } S)^+$. Since $(\varinjlim(\text{add } S), \text{Prod } S^*)$ is an almost dual pair (by Corollary 4.2.25) it follows from Proposition 4.3.8 that $(\text{Prod } S^*, \varinjlim(\text{add } S))$ is an almost dual pair and hence $(\text{Prod } S^*, (\text{Add } S)^+)$ is an almost dual pair. \square

If we take S to be the class of $(1, 1)$ -presented left R -modules and apply Corollary 4.5.7, then we get the following corollary.

Corollary 4.5.8 The following statements are equivalent.

- (1) $(\mathcal{RD}\text{-Inj}_R, ({}_R\mathcal{RD}\text{-Proj})^+)$ is an almost dual pair.
- (2) $M \in \mathcal{RD}\text{-Inj}_R$ if and only if $M^* \in {}_R\mathcal{RD}\text{-Proj}$.
- (3) $\mathcal{RD}\text{-Inj}_R$ is definable and ${}_R\mathcal{RD}\text{-Flat} = ({}_R\mathcal{RD}\text{-Proj})^+$.

Enochs and Huang in [18, Theorem 3.7, p. 1139] proved the following result.

Theorem 4.5.9 The following statements are equivalent for a ring R :

- (1) R is right noetherian;
- (2) a monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is an injective preenvelope of M if and only if $\varphi^* : X^* \rightarrow M^*$ is a flat precover of M^* in $R\text{-Mod}$;
- (3) R is right coherent and a monomorphism $\varphi : M \rightarrow X$ is an injective envelope of M if $\varphi^* : X^* \rightarrow M^*$ is a flat cover of M^* .

In the following theorem, we give equivalent characterizations of definability of $\text{Prod } S^*$ in terms of preenvelopes and precovers. Theorem 4.5.9 is a special case of Theorem 4.5.10 if we take $S = \text{add}(\{{}_R R\})$.

Theorem 4.5.10 Let $S \subseteq R\text{-mod}$ be such that $S = \text{add } S$. Then the following statements are equivalent:

- (1) $\text{Prod } S^*$ is definable;

(2) a $D_{\mathcal{H}}$ -pure monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a $(\text{Prod}S^*)$ -preenvelope of M if and only if $\varphi^* : X^* \rightarrow M^*$ is a $(\underline{\lim}S)$ -precover of M^* in $R\text{-Mod}$;

(3) a pure monomorphism $\varphi : M \rightarrow X$ in $\text{Mod-}R$ is a $(\text{Prod}S^*)$ -preenvelope of M if and only if $\varphi^* : X^* \rightarrow M^*$ is a $(\underline{\lim}S)$ -precover of M^* in $R\text{-Mod}$;

(4) for any $M \in \text{Mod-}R$, the identity homomorphism $1_M : M \rightarrow M$ is a $(\text{Prod}S^*)$ -preenvelope of M if and only if $(1_M)^* : M^* \rightarrow M^*$ is a $(\underline{\lim}S)$ -precover of M^* ;

(5) $\underline{\lim}S$ is definable and a $D_{\mathcal{H}}$ -pure monomorphism $\varphi : M \rightarrow X$ is a $(\text{Prod}S^*)$ -envelope of M if $\varphi^* : X^* \rightarrow M^*$ is a $(\underline{\lim}S)$ -cover of M^* .

(6) $\underline{\lim}S$ is definable and a pure monomorphism $\varphi : M \rightarrow X$ is a $(\text{Prod}S^*)$ -envelope of M if $\varphi^* : X^* \rightarrow M^*$ is a $(\underline{\lim}S)$ -cover of M^* .

PROOF: (1) \Rightarrow (2) Suppose that $\text{Prod}S^*$ is definable, thus $\underline{\lim}S$ is definable (by Corollary 4.5.5). Let $\varphi : M \rightarrow X$ be a $D_{\mathcal{H}}$ -pure monomorphism in $\text{Mod-}R$. If φ is a $(\text{Prod}S^*)$ -preenvelope of M , then from Corollary 4.4.23 we have $\varphi^* : X^* \rightarrow M^*$ is a $(\underline{\lim}S)$ -precover of M^* in $R\text{-Mod}$. Conversely, if $\varphi^* : X^* \rightarrow M^*$ is a $(\underline{\lim}S)$ -precover of M^* in $R\text{-Mod}$, thus $X^* \in \underline{\lim}S$ and hence from Corollary 4.5.5 we have $X \in \text{Prod}S^*$. Let $f : M \rightarrow Y$ be any homomorphism with $Y \in \text{Prod}S^*$. Since φ is a $D_{\mathcal{H}}$ -pure monomorphism and Y is $D_{\mathcal{H}}$ -pure-injective, there is a homomorphism $g : X \rightarrow Y$ such that $g\varphi = f$ and hence φ is a $(\text{Prod}S^*)$ -preenvelope of M in $\text{Mod-}R$.

(2) \Rightarrow (3) This is obvious.

(3) \Rightarrow (4) Let $M \in \text{Mod-}R$. Since 1_M is a pure monomorphism it follows from (3) that 1_M is a $(\text{Prod}S^*)$ -preenvelope of M if and only if $(1_M)^* : M^* \rightarrow M^*$ is a $(\underline{\lim}S)$ -precover of M^* .

(4) \Rightarrow (1) Let $M \in \text{Prod}S^*$. It is clear that 1_M is a $(\text{Prod}S^*)$ -preenvelope of M . By (4), $(1_M)^* : M^* \rightarrow M^*$ is a $(\underline{\lim}S)$ -precover of M^* and hence $M^* \in \underline{\lim}S$. Conversely, let $M \in \text{Mod-}R$ such that $M^* \in \underline{\lim}S$. It is clear that 1_{M^*} is a $(\underline{\lim}S)$ -precover of M^* . Since $1_{M^*} = (1_M)^*$ it follows that $(1_M)^*$ is a $(\underline{\lim}S)$ -precover of M^* . By (4), 1_M is a $(\text{Prod}S^*)$ -preenvelope of M and hence $M \in \text{Prod}S^*$. Thus $M \in \text{Prod}S^*$ if and only if $M^* \in \underline{\lim}S$ and hence from Corollary 4.5.5 we have that $\text{Prod}S^*$ is definable.

(2) \Rightarrow (5) Suppose that $\text{Prod}S^*$ is definable. By Corollary 4.5.5, $\underline{\lim}S$ is definable. Let $\varphi : M \rightarrow X$ be an $D_{\mathcal{H}}$ -pure monomorphism such that $\varphi^* : X^* \rightarrow M^*$ is a $(\underline{\lim}S)$ -cover of

M^* . Thus φ^* is a right minimal homomorphism and a $(\varinjlim S)$ -precover of M^* . By (2), φ is $(\text{Prod } S^*)$ -preenvelope of M . Also, by Lemma 4.4.17(2) we have that φ is a left minimal. Thus φ is a $(\text{Prod } S^*)$ -envelope of M .

(5) \Rightarrow (6) This is obvious.

(6) \Rightarrow (1) Let $M \in \text{Prod } S^*$. Since $\varinjlim S$ is definable it follows from Theorem 4.4.1 that $M^* \in \varinjlim S$. Conversely, let $M \in \text{Mod-}R$ such that $M^* \in \varinjlim S$. Since $(1_M)^*$ is a pure monomorphism and it is a $(\varinjlim S)$ -cover of M^* it follows from (6) that 1_M is a $(\text{Prod } S^*)$ -envelope of M and hence $M \in \text{Prod } S^*$. Thus $M \in \text{Prod } S^*$ if and only if $M^* \in \varinjlim S$ and hence by Corollary 4.5.5 we have that $\text{Prod } S^*$ is definable. \square

4.5.2 Definability of S -Pinj in terms of the functors Axt_S^1 and Pext_S^1

Let $\mathcal{F} \subseteq \text{Mod-}R$ and let $M \in \text{Mod-}R$. A **right \mathcal{F} -resolution** of M is a cochain complex

$$0 \rightarrow M \rightarrow F_0 \rightarrow F_1 \rightarrow F_2 \rightarrow \dots$$

(not necessarily exact) with $F_i \in \mathcal{F}$ such that the sequence

$$\dots \rightarrow \text{Hom}_R(F_1, G) \rightarrow \text{Hom}_R(F_0, G) \rightarrow \text{Hom}_R(M, G) \rightarrow 0$$

is exact for each $G \in \mathcal{F}$ [19, p. 168].

Since the class \mathcal{APure}_R is preenveloping in $\text{Mod-}R$ (by Lemma 2.3.5) it follows from [19, Proposition 8.1.3, p. 168] that every right R -module has a right (\mathcal{APure}_R) -resolution.

Let $n \geq 0$ and let $M, N \in \text{Mod-}R$. Pinzon in [50] derived the functor $\text{Axt}_R^n(M, N)$ using right (\mathcal{APure}_R) -resolutions as follows: let $\Sigma : 0 \rightarrow M \xrightarrow{\alpha} A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \rightarrow \dots$ be a right (\mathcal{APure}_R) -resolution of M and let $\Sigma_0 : 0 \rightarrow A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \rightarrow \dots$ be the deleted (\mathcal{APure}_R) -resolution of M . Then $\text{Axt}_R^n(N, M)$ is the cohomology group:

$$\text{Axt}_R^n(N, M) = H^n(\text{Hom}_R(N, \Sigma_0)) = (\ker \alpha_n^{\otimes}) / (\text{im } \alpha_{n-1}^{\otimes}), n = 0, 1, 2, \dots,$$

where $\alpha_n^{\otimes} : \text{Hom}_R(N, A_n) \rightarrow \text{Hom}_R(N, A_{n+1})$ is defined by $\alpha_n^{\otimes}(f) = \alpha_n f$ and $\alpha_{-1}^{\otimes} = 0$.

Pinzon in [50, Theorem 5.4, p. 51] proved the following result:

Theorem 4.5.11 The following statements are equivalent for a ring R .

- (1) R is right noetherian.
- (2) $\text{Axt}_R^n(M, N) \simeq \text{Ext}_R^n(M, N)$, for all $M, N \in \text{Mod-}R$ and $n \geq 1$.

(3) $\text{Axt}_R^1(M, N) \simeq \text{Ext}_R^1(M, N)$, for all $M, N \in \text{Mod-}R$.

The main purpose in this subsection is to generalize Pinzon's result from definability of the class Inj_R (which is equivalent to R being right noetherian) to definability of the class $S\text{-Pinj}$, where $S \subseteq \text{mod-}R$. This is in Theorem 4.5.16.

Remarks 4.5.12 (1) Let $\mathcal{F} \subseteq \text{Mod-}R$ and let $M \in \text{Mod-}R$. If \mathcal{F} is preenveloping in $\text{Mod-}R$ then a right \mathcal{F} -resolution of M exists [19, Proposition 8.1.3, p. 168] and it is unique up to homotopy [19, p. 169].

(2) Let $S \in \text{mod-}R$. By Lemma 2.3.4, the class $S\text{-Pinj}$ is enveloping in $\text{Mod-}R$. Also, from Lemma 2.3.5 we have $(S\text{-Pinj})^+$ is preenveloping in $\text{Mod-}R$ and hence from (1) we have that every right R -module has a right $(S\text{-Pinj})$ -resolution and a right $((S\text{-Pinj})^+)$ -resolution.

(3) Let $M, N \in \text{Mod-}R$, let $S \subseteq \text{mod-}R$, let $\Sigma : 0 \rightarrow M \xrightarrow{\alpha} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \rightarrow \dots$ be a right $(S\text{-Pinj})$ -resolution of M and let Σ_0 be the deleted $(S\text{-Pinj})$ -resolution of M (i.e., $\Sigma_0 : 0 \xrightarrow{d_{-1}} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \rightarrow \dots$). We will use $\text{Pext}_S^n(N, M)$ to denote the cohomology group:

$$\text{Pext}_S^n(N, M) = H^n(\text{Hom}_R(N, \Sigma_0)) = (\ker d_n^{\otimes}) / (\text{im } d_{n-1}^{\otimes}), n = 0, 1, 2, \dots,$$

where $d_n^{\otimes} : \text{Hom}_R(N, I_n) \rightarrow \text{Hom}_R(N, I_{n+1})$ is defined by $d_n^{\otimes}(f) = d_n f$ and $d_{-1}^{\otimes} = 0$.

By (1) this is well-defined.

(4) Let $M, N \in \text{Mod-}R$ and let $S \subseteq \text{mod-}R$. By (2) above, M has a right $((S\text{-Pinj})^+)$ -resolution $\Sigma : 0 \rightarrow M \xrightarrow{\alpha} A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \rightarrow \dots$. Let $\Sigma_0 : 0 \rightarrow A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \rightarrow \dots$ be the deleted $((S\text{-Pinj})^+)$ -resolution of M . We will use $\text{Axt}_S^n(N, M)$ to denote the cohomology group:

$$\text{Axt}_S^n(N, M) = H^n(\text{Hom}_R(N, \Sigma_0)) = (\ker \alpha_n^{\otimes}) / (\text{im } \alpha_{n-1}^{\otimes}), n = 0, 1, 2, \dots,$$

where $\alpha_n^{\otimes} : \text{Hom}_R(N, A_n) \rightarrow \text{Hom}_R(N, A_{n+1})$ is defined by $\alpha_n^{\otimes}(f) = \alpha_n f$ and $\alpha_{-1}^{\otimes} = 0$.

By (1) this is well-defined.

(5) From Examples 4.2.28, we have the following.

(i) Let \mathcal{H} be a set of matrices over R such that $L_{\mathcal{H}} = \{R\mathcal{R}\}$. If $S = D_{\mathcal{H}}$, then $S\text{-Pinj} = \text{Prod}(L_{\mathcal{H}})^* = \text{Prod}(\{R\mathcal{R}\})^* = \text{Inj}_R$ and hence $(S\text{-Pinj})^+$ is the class of absolutely pure right

R -modules. Thus for every $N, M \in \text{Mod-}R$, we have that $\text{Axt}_S^n(N, M) = \text{Axt}_R^n(N, M)$ in the sense of Pinzon [50] and $\text{Pext}_S^n(N, M) = \text{Ext}_R^n(N, M)$.

(ii) Let $S = \text{mod-}R$, thus $S\text{-Pinj} = \text{Pinj}_R$. Hence for every $N, M \in \text{Mod-}R$, we have that $\text{Pext}_S^n(N, M) = \text{Pext}^n(N, M)$ as in [52, p. 55].

(iii) Let S be the class of $(1, 1)$ -presented right R -modules, thus $S\text{-Pinj} = \mathcal{RD}\text{-Inj}_R$. Hence for every $N, M \in \text{Mod-}R$, we have that $\text{Pext}_S^n(N, M) = \text{Ext}_{\mathcal{RD}}^n(N, M)$ as in [39, p. 12].

Lemma 4.5.13 Let \mathcal{F} be a class of right R -modules closed under direct sums and let $\{M_i\}_{i \in I}$ be a family of right R -modules. If $\Sigma_i : 0 \rightarrow M_i \xrightarrow{\alpha_i} A_i^0 \xrightarrow{\alpha_i^0} A_i^1 \xrightarrow{\alpha_i^1} A_i^2 \rightarrow \dots$ is a right \mathcal{F} -resolution of M_i for all $i \in I$, then the sequence $\Sigma : 0 \rightarrow \bigoplus_{i \in I} M_i \rightarrow \bigoplus_{i \in I} A_i^0 \rightarrow \bigoplus_{i \in I} A_i^1 \rightarrow \dots$ is a right \mathcal{F} -resolution of $\bigoplus_{i \in I} M_i$.

PROOF: By hypothesis, $A_i^n \in \mathcal{F}$ for all $i \in I$ and hence $\bigoplus_{i \in I} A_i^n \in \mathcal{F}$ for all $n \geq 0$. Since Σ_i is a right \mathcal{F} -resolution, the sequence $\dots \rightarrow \text{Hom}_R(A_i^1, M) \rightarrow \text{Hom}_R(A_i^0, M) \rightarrow \text{Hom}_R(M_i, M) \rightarrow 0$ is exact for all $M \in \mathcal{F}$ and $i \in I$ and hence the sequence $\dots \rightarrow \prod_{i \in I} \text{Hom}_R(A_i^1, M) \rightarrow \prod_{i \in I} \text{Hom}_R(A_i^0, M) \rightarrow \prod_{i \in I} \text{Hom}_R(M_i, M) \rightarrow 0$ is exact. By Lemma 2.1.5, it follows that the sequence $\dots \rightarrow \text{Hom}_R(\bigoplus_{i \in I} A_i^1, M) \rightarrow \text{Hom}_R(\bigoplus_{i \in I} A_i^0, M) \rightarrow \text{Hom}_R(\bigoplus_{i \in I} M_i, M) \rightarrow 0$ is exact for all $M \in \mathcal{F}$. Since Σ_i is a cochain complex it follows that Σ is a cochain complex. Hence the sequence $\Sigma : 0 \rightarrow \bigoplus_{i \in I} M_i \rightarrow \bigoplus_{i \in I} A_i^0 \rightarrow \bigoplus_{i \in I} A_i^1 \rightarrow \dots$ is a right \mathcal{F} -resolution of $\bigoplus_{i \in I} M_i$. \square

Theorem 5.5 in [50, p. 53] is a special case of the following theorem if we take $S = \{R_R\}$.

Theorem 4.5.14 Let $S \subseteq \text{mod-}R$ and let $\{M_i\}_{i \in I}$ be a family of right R -modules. If N is a finitely generated right R -module, then

$$\text{Axt}_S^n(N, \bigoplus_{i \in I} M_i) \simeq \bigoplus_{i \in I} \text{Axt}_S^n(N, M_i), \text{ for any } n \geq 0.$$

PROOF: By Remark 4.5.12(2), each M_i has a right $(S\text{-Pinj})^+$ -resolution $\Sigma_i : 0 \rightarrow M_i \rightarrow A_i^0 \rightarrow A_i^1 \rightarrow A_i^2 \rightarrow \dots$. Since $(S\text{-Pinj})^+$ is closed under direct sums (by Proposition 4.2.9(1)) it follows from Lemma 4.5.13 that the sequence $0 \rightarrow \bigoplus_{i \in I} M_i \rightarrow \bigoplus_{i \in I} A_i^0 \rightarrow \bigoplus_{i \in I} A_i^1 \rightarrow \bigoplus_{i \in I} A_i^2 \rightarrow \dots$ is a right $(S\text{-Pinj})^+$ -resolution of $\bigoplus_{i \in I} M_i$. By [62, Example 6.6(i), p. 328], the sequence $0 \rightarrow \text{Hom}_R(N, \bigoplus_{i \in I} A_i^0) \rightarrow \text{Hom}_R(N, \bigoplus_{i \in I} A_i^1) \rightarrow \text{Hom}_R(N, \bigoplus_{i \in I} A_i^2) \rightarrow \dots$ is

a cochain complex. Since N is finitely generated it follows from [2, Exercise 16.3, p. 189] that there is an R -isomorphism $\alpha_j : \text{Hom}_R(N, \bigoplus_{i \in I} A_i^j) \rightarrow \bigoplus_{i \in I} \text{Hom}_R(N, A_i^j)$ for all $j \geq 0$ and hence we have the following commutative diagram with isomorphic cochain complex rows.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}_R(N, \bigoplus_{i \in I} A_i^0) & \longrightarrow & \text{Hom}_R(N, \bigoplus_{i \in I} A_i^1) & \longrightarrow & \text{Hom}_R(N, \bigoplus_{i \in I} A_i^2) \longrightarrow \cdots \\
 & & \alpha_0 \downarrow & & \alpha_1 \downarrow & & \alpha_2 \downarrow \\
 0 & \longrightarrow & \bigoplus_{i \in I} \text{Hom}_R(N, A_i^0) & \longrightarrow & \bigoplus_{i \in I} \text{Hom}_R(N, A_i^1) & \longrightarrow & \bigoplus_{i \in I} \text{Hom}_R(N, A_i^2) \longrightarrow \cdots
 \end{array}$$

Since isomorphic cochain complexes have the same cohomology (by [62, Exercise 6.2, p. 338]) it follows that $\text{Axt}_S^n(N, \bigoplus_{i \in I} M_i) \simeq H^n(\bigoplus_{i \in I} \text{Hom}_R(N, \Sigma_i'))$, where Σ_i' is the deleted $(S\text{-Pinj})^+$ -resolution of M_i . Since direct sum commutes with cohomology (by [78, 1.2.1, p. 5]) it follows that $H^n(\bigoplus_{i \in I} \text{Hom}_R(N, \Sigma_i')) \simeq \bigoplus_{i \in I} H^n(\text{Hom}_R(N, \Sigma_i'))$. Since $\bigoplus_{i \in I} H^n(\text{Hom}_R(N, \Sigma_i')) = \bigoplus_{i \in I} \text{Axt}_S^n(N, M_i)$ it follows that $\text{Axt}_S^n(N, \bigoplus_{i \in I} M_i) \simeq \bigoplus_{i \in I} \text{Axt}_S^n(N, M_i)$, for any $n \geq 0$. \square

Lemma 4.5.15 Let $S \subseteq \text{mod-}R$. If $M \in (S\text{-Pinj})^+$ then $\text{Axt}_S^1(N, M) = 0$ for all $N \in \text{Mod-}R$.

PROOF: Let $M \in (S\text{-Pinj})^+$. It is clear that the exact sequence $\Sigma : 0 \rightarrow M \xrightarrow{1_M} M \xrightarrow{d_0} 0 \xrightarrow{d_1} 0 \rightarrow \cdots$ is a right $(S\text{-Pinj})^+$ -resolution of M . Let $N \in \text{Mod-}R$, thus $\text{Axt}_S^1(N, M)$ is the cohomological group $H^1(\text{Hom}_R(N, \Sigma_0))$, where $\Sigma_0 : 0 \rightarrow M \xrightarrow{d_0} 0 \xrightarrow{d_1} 0 \rightarrow \cdots$ and hence $\text{Axt}_S^1(N, M) = (\ker d_1^{\otimes}) / (\text{im } d_0^{\otimes}) = 0/0 = 0$. \square

The following theorem is the main result in this subsection in which we give equivalent characterizations of definability of the class $S\text{-Pinj}$ in terms of the functors $\text{Axt}_S^1(-, -)$ and $\text{Pext}_S^1(-, -)$. Pinzon's theorem (see Theorem 4.5.11) is the special case of the following theorem where we take $S = \{R_R\}$.

Theorem 4.5.16 Let $S \subseteq \text{mod-}R$. Then the following statements are equivalent.

- (1) $S\text{-Pinj}$ is a definable class.
- (2) $\text{Axt}_S^n(N, M) \simeq \text{Pext}_S^n(N, M)$, for all $N, M \in \text{Mod-}R$ and $n \geq 1$.
- (3) $\text{Axt}_S^1(N, M) \simeq \text{Pext}_S^1(N, M)$, for all $N, M \in \text{Mod-}R$.
- (4) $\text{Pext}_S^1(N, M) = 0$, for any $N \in \text{Mod-}R$ and $M \in (S\text{-Pinj})^+$.

PROOF: (1) \Rightarrow (2) Suppose that $S\text{-Pinj}$ is a definable class. By Theorem 4.5.1, $(S\text{-Pinj})^+ =$

S -Pinj and hence $\text{Axt}_S^n(N, M) \simeq \text{Pext}_S^n(N, M)$, for all $N, M \in \text{Mod-}R$ and $n \geq 1$.

(2) \Rightarrow (3) This is obvious.

(3) \Rightarrow (4) Let $N \in \text{Mod-}R$ and let $M \in (S\text{-Pinj})^+$. By Lemma 4.5.15, $\text{Axt}_S^1(N, M) = 0$ and hence from hypothesis we have that $\text{Pext}_S^1(N, M) = 0$.

(4) \Rightarrow (1) Let $A \in (S\text{-Pinj})^+$. By hypothesis, $\text{Pext}_S^1(N, A) = 0$ for all $N \in \text{Mod-}R$ and hence A is S -pure-injective, by Theorem 2.2.19. Thus $S\text{-Pinj} = (S\text{-Pinj})^+$ and hence from Theorem 4.5.1 we have that $S\text{-Pinj}$ is definable. \square

Definition 4.5.17 Let $S \subseteq \text{mod-}R$. A long exact sequence of right R -modules:

$$\cdots \rightarrow A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \xrightarrow{\alpha_{n+1}} \cdots$$

is said to be S -pure if and only if for each n the short exact sequence at A_n , which is $0 \rightarrow \text{im}(\alpha_{n-1}) \rightarrow A_n \rightarrow \text{im}(\alpha_n) \rightarrow 0$ is S -pure.

Lemma 4.5.18 Let $S \subseteq \text{mod-}R$ and let $\Sigma : 0 \rightarrow M \xrightarrow{\alpha_0} A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} \cdots$ be a sequence of right R -modules. Then Σ is a right $(S\text{-Pinj})$ -resolution of M if and only if Σ is an S -pure exact sequence with $A_i \in S\text{-Pinj}$, for all $i \geq 0$.

PROOF: (\Rightarrow) Suppose that Σ is a right $(S\text{-Pinj})$ -resolution of M . First, I will prove that the sequence Σ is exact. Let $n \geq 0$. Put $N = A_n / \text{im}(\alpha_{n-1})$ where $\alpha_{-1} = \alpha$. By Corollary 3.1.3, there is an S -pure monomorphism $g : N \rightarrow G$ with $G \in S\text{-Pinj}$. By hypothesis, the sequence

$$\cdots \rightarrow \text{Hom}_R(A_{n+1}, G) \xrightarrow{\alpha_n^\circ} \text{Hom}_R(A_n, G) \xrightarrow{\alpha_{n-1}^\circ} \text{Hom}_R(A_{n-1}, G) \rightarrow \cdots \rightarrow \text{Hom}_R(M, G) \rightarrow 0$$

is exact, where $\alpha_{n-1}^\circ : \text{Hom}_R(A_n, G) \rightarrow \text{Hom}_R(A_{n-1}, G)$ is defined by $\alpha_{n-1}^\circ(\beta) = \beta \alpha_{n-1}$ for all $\beta \in \text{Hom}_R(A_n, G)$ and $n \geq 0$ and $A_{-1} = M$. Set $f = g \pi_n : A_n \rightarrow G$ where $\pi_n : A_n \rightarrow N$ is the natural epimorphism. Thus $\alpha_{n-1}^\circ(f) = f \alpha_{n-1}$. Let $a \in A_{n-1}$, thus $(f \alpha_{n-1})(a) = (g \pi_n)(\alpha_{n-1}(a)) = g(\pi_n(\alpha_{n-1}(a))) = g(\alpha_{n-1}(a) + \text{im}(\alpha_{n-1})) = g(0) = 0$. Thus $f \alpha_{n-1} = 0$ and hence $f \in \ker(\alpha_{n-1}^\circ)$. Since $\text{im}(\alpha_{n-1}^\circ) = \ker(\alpha_{n-1}^\circ)$ it follows that $f \in \text{im}(\alpha_n^\circ)$ and hence there is $h \in \text{Hom}_R(A_{n+1}, G)$ such that $\alpha_n^\circ(h) = f$. Thus $h \alpha_n = f$ and hence $\ker(\alpha_n) \subseteq \ker(f)$. Also, since $f = g \pi_n$ and g is a monomorphism so $\ker(f) \subseteq \ker(\pi_n) = \text{im}(\alpha_{n-1})$ and hence $\ker(\alpha_n) \subseteq \text{im}(\alpha_{n-1})$. Since Σ is a right $(S\text{-Pinj})$ -resolution of M it follows that Σ is a cochain complex and hence $\alpha_n \alpha_{n-1} = 0$ and this implies that $\text{im}(\alpha_{n-1}) \subseteq \ker(\alpha_n)$. Thus $\text{im}(\alpha_{n-1}) = \ker(\alpha_n)$ for all $n \geq 0$ and hence Σ is an exact sequence.

Now we will prove that the exact sequence Σ is S -pure. Let $\rho_{n-1} : A_{n-1} \rightarrow \text{im}(\alpha_{n-1})$ be the natural epimorphism and let $i : \text{im}(\alpha_{n-1}) \rightarrow A_n$ be the inclusion. So $\alpha_{n-1} = i\rho_{n-1}$. We show that i is an S -pure embedding, using Proposition 3.1.6. Let $L \in S\text{-Pinj}$ and let $f : \text{im}(\alpha_{n-1}) \rightarrow L$ be any homomorphism. Since $\alpha_{n-1}\alpha_{n-2} = 0$ it follows that $i\rho_{n-1}\alpha_{n-2} = 0$. Since i is a monomorphism, $\rho_{n-1}\alpha_{n-2} = 0$. Therefore $f\rho_{n-1}\alpha_{n-2} = 0$ and hence $f\rho_{n-1} \in \ker(\alpha_{n-2}^\odot) = \text{im}(\alpha_{n-1}^\odot)$. Thus there is $g : A_n \rightarrow L$ such that $f\rho_{n-1} = \alpha_{n-1}^\odot(g)$ and hence $f\rho_{n-1} = g\alpha_{n-1} = gi\rho_{n-1}$. Since ρ_{n-1} is an epimorphism, $f = gi$. It follows by Proposition 3.1.6 that the exact sequence $0 \rightarrow \text{im}(\alpha_{n-1}) \rightarrow A_n \rightarrow \text{im}(\alpha_n) \rightarrow 0$ is S -pure for all $n \geq 0$ and hence Σ is an S -pure exact sequence.

(\Leftarrow) Suppose that Σ is S -pure exact and that each A_i is S -pure-injective. Let $G \in S\text{-Pinj}$. We show that the sequence

$$\cdots \rightarrow \text{Hom}_R(A_{n+1}, G) \xrightarrow{\alpha_n^\odot} \text{Hom}_R(A_n, G) \xrightarrow{\alpha_{n-1}^\odot} \text{Hom}_R(A_{n-1}, G) \rightarrow \cdots$$

is exact at $\text{Hom}_R(A_n, G)$. Let $f \in \ker(\alpha_n^\odot)$, thus $f \in \text{Hom}_R(A_n, G)$ and $f\alpha_{n-1} = 0$. Set $\alpha_{n-1} = i\rho_{n-1}$ as above. So $fi\rho_{n-1} = 0$ and, since ρ_{n-1} is an epimorphism, $fi = 0$. By assumption the sequence $0 \rightarrow \text{im}(\alpha_{n-1}) \xrightarrow{i} A_n \xrightarrow{\rho_n} \text{im}(\alpha_n) \rightarrow 0$ is exact so, since $fi = 0$, there is $g : \text{im}(\alpha_n) \rightarrow G$ such that $g\rho_n = f$. Also, $i' : \text{im}(\alpha_n) \rightarrow A_{n+1}$ is, by assumption, an S -pure embedding so, since G is S -pure-injective, there is $g' : A_{n+1} \rightarrow G$ such that $g'i' = g$. Since $\alpha_n = i'\rho_n$ it follows that $g'\alpha_n = g'i'\rho_n = g\rho_n = f$ and hence $\alpha_n^\odot(g') = f$. Thus $f \in \text{im}(\alpha_n^\odot)$ and hence $\ker(\alpha_{n-1}^\odot) \subseteq \text{im}(\alpha_n^\odot)$. Since Σ is cochain complex it follows from [62, 6.6(ii), p. 328] that the sequence

$$\cdots \rightarrow \text{Hom}_R(A_{n+1}, G) \xrightarrow{\alpha_n^\odot} \text{Hom}_R(A_n, G) \xrightarrow{\alpha_{n-1}^\odot} \text{Hom}_R(A_{n-1}, G) \rightarrow \cdots$$

is chain complex and hence $\text{im}(\alpha_n^\odot) \subseteq \ker(\alpha_{n-1}^\odot)$. Thus $\text{im}(\alpha_n^\odot) = \ker(\alpha_{n-1}^\odot)$ and hence the sequence

$$\cdots \rightarrow \text{Hom}_R(A_{n+1}, G) \xrightarrow{\alpha_n^\odot} \text{Hom}_R(A_n, G) \xrightarrow{\alpha_{n-1}^\odot} \text{Hom}_R(A_{n-1}, G) \rightarrow \cdots$$

is exact at $\text{Hom}_R(A_n, G)$ for each $n \geq 0$. Therefore Σ is a right $(S\text{-Pinj})$ -resolution of M . \square

Lemma 4.5.19 Let $S \subseteq \text{mod-}R$. Then $\text{Pext}_S^0(N, M) \simeq \text{Hom}_R(N, M)$, for any $N, M \in \text{Mod-}R$.

PROOF: Let $\Sigma : 0 \rightarrow M \xrightarrow{\alpha} A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \rightarrow \cdots$ be a right $(S\text{-Pinj})$ -resolution of M . Thus the cohomology group $H^0(\text{Hom}_R(N, \Sigma_0)) = \text{Pext}_S^0(N, M)$ where Σ_0 is the deleted $(S\text{-Pinj})$ -resolution of M and hence $\text{Pext}_S^0(N, M) \simeq \ker(d_0^*)$ where $d_0^* : \text{Hom}_R(N, A_0) \rightarrow \text{Hom}_R(N, A_1)$

defined by $d_0^{\otimes}(f) = d_0f$, for all $f \in \text{Hom}_R(N, A_0)$. Since Σ is a right (S -Pinj)-resolution of M it follows from Lemma 4.5.18 that $0 \rightarrow M \xrightarrow{\alpha} A_0 \xrightarrow{d_0} A_1$ is an exact sequence. Since the functor $\text{Hom}_R(N, -)$ is a left exact, the sequence $0 \rightarrow \text{Hom}_R(N, M) \xrightarrow{\alpha^{\otimes}} \text{Hom}_R(N, A_0) \xrightarrow{d_0^{\otimes}} \text{Hom}_R(N, A_1)$ is exact and hence $\text{Hom}_R(N, M) \simeq \text{im}(\alpha^{\otimes}) = \ker(d_0^{\otimes}) \simeq \text{Pext}_S^0(N, M)$. \square

Mao in [39, Theorem 3.1, p. 13] proved the following result.

Theorem 4.5.20 Let $\{M_i\}_{i \in I}$ be a family of right R -modules. If the class of RD -injective right R -modules is closed under direct sums, then $\bigoplus_{i \in I} \text{Ext}_{RD}^n(N, M_i) \simeq \text{Ext}_{RD}^n(N, \bigoplus_{i \in I} M_i)$, for any $n \geq 0$ and for any finitely generated right R -module N .

Theorem 4.5.20 is the special case $S = \{M \mid M \text{ is a } (1, 1)\text{-presented right } R\text{-module}\}$ of the following corollary of Theorem 4.5.16.

Corollary 4.5.21 Let $S \subseteq \text{mod-}R$ and let $\{M_i\}_{i \in I}$ be a family of right R -modules. If S -Pinj is definable, then $\bigoplus_{i \in I} \text{Pext}_S^n(N, M_i) \simeq \text{Pext}_S^n(N, \bigoplus_{i \in I} M_i)$, for any $n \geq 0$ and for any finitely generated right R -module N .

PROOF: Suppose that S -Pinj is definable. Let N be a finitely generated right R -module and let $\{M_i\}_{i \in I}$ be a family of right R -modules. Thus for any $n \geq 1$, we have:

$$\begin{aligned} \bigoplus_{i \in I} \text{Pext}_S^n(N, M_i) &\simeq \bigoplus_{i \in I} \text{Axt}_S^n(N, M_i) && \text{(by Theorem 4.5.16)} \\ &\simeq \text{Axt}_S^n(N, \bigoplus_{i \in I} M_i) && \text{(by Theorem 4.5.14)} \\ &\simeq \text{Pext}_S^n(N, \bigoplus_{i \in I} M_i) && \text{(by Theorem 4.5.16)}. \end{aligned}$$

By Lemma 4.5.19, $\text{Pext}_S^0(N, M) \simeq \text{Hom}_R(N, M)$ for any $M \in \text{Mod-}R$. Since $\bigoplus_{i \in I} \text{Hom}_R(N, M_i) \simeq \text{Hom}_R(N, \bigoplus_{i \in I} M_i)$ (by [2, Exercise 16.3, p. 189]) it follows that $\bigoplus_{i \in I} \text{Pext}_S^0(N, M_i) \simeq \text{Pext}_S^0(N, \bigoplus_{i \in I} M_i)$ and hence $\bigoplus_{i \in I} \text{Pext}_S^n(N, M_i) \simeq \text{Pext}_S^n(N, \bigoplus_{i \in I} M_i)$, for any $n \geq 0$. \square

Chapter 5

\mathcal{S} -Cotorsion pairs

In his study of abelian groups, Salce introduced in [64] the notion of a cotorsion pair. The concept readily generalized to the category $R\text{-Mod}$. A cotorsion pair in $R\text{-Mod}$ is a pair of classes $(\mathcal{F}, \mathcal{G})$ of $R\text{-Mod}$ which are orthogonal with respect to $\text{Ext}_R^1(-, -)$. The cotorsion pairs are analogues of torsion pairs where $\text{Hom}_R(-, -)$ is substituted by $\text{Ext}_R^1(-, -)$.

In [29] Hovey introduced the notion of **cotorsion pairs relative to a proper class** \mathcal{P} . They are defined as complete orthogonal classes with respect to the functor $\text{Ext}_{\mathcal{P}}^1(-, -)$ instead of $\text{Ext}_R^1(-, -)$.

Let $S \subseteq R\text{-mod}$ and let \mathcal{P}_S be the class of S -pure short exact sequences of left R -modules. In this chapter, we will study the notion of cotorsion pairs relative to the proper class \mathcal{P}_S (in short, S -cotorsion pairs). In the first section we recall the definition and some basic properties of cotorsion pairs, covers and envelopes relative to the proper class \mathcal{P}_S which will be used throughout this chapter. In the second section, we will induce S -cotorsion pairs by using S' -Pinj, where $S, S' \subseteq R\text{-mod}$. The main purpose of the third section is to generalize the results of Holm and Jørgensen [27, Theorem 3.4, p. 697] (see Theorem 5.3.1) and [28, Theorem 3.1(c), p. 629] (see Theorem 5.3.2) to the case of S -cotorsion pairs. In the final section we provide some examples and applications of some results in this chapter.

5.1 Cotorsion pairs, covers and envelopes relative to the proper class \mathcal{P}_S

Let $S \subseteq R\text{-mod}$. In this section we recall the definition and some basic properties of cotorsion pairs, covers and envelopes relative to the proper class \mathcal{P}_S which will be used throughout this chapter. The material in this section is covered in [46] and [37].

Before introducing S -cotorsion pairs, we define Ext-orthogonal classes and cotorsion pairs.

Definition 5.1.1 Let $\mathcal{F} \subseteq R\text{-Mod}$. We define the classes

$${}^{\perp}\mathcal{F} = \{M \in R\text{-Mod} \mid \text{Ext}_R^1(M, \mathcal{F}) = 0\},$$

$$\mathcal{F}^{\perp} = \{M \in R\text{-Mod} \mid \text{Ext}_R^1(\mathcal{F}, M) = 0\}.$$

Definition 5.1.2 Let \mathcal{F} and \mathcal{G} be classes of left R -modules. The pair $(\mathcal{F}, \mathcal{G})$ is said to be a **cotorsion pair** if ${}^{\perp}\mathcal{G} = \mathcal{F}$ and $\mathcal{F}^{\perp} = \mathcal{G}$. The cotorsion pair is called **perfect** if \mathcal{F} is covering and \mathcal{G} is enveloping.

Remark 5.1.3 Let $S \subseteq R\text{-mod}$ and let $M, N \in R\text{-Mod}$. Denote by $\text{Ext}_{\mathcal{P}_S}^1(N, M)$ the set of equivalence classes of S -pure short exact sequences $0 \rightarrow M \rightarrow L \rightarrow N \rightarrow 0$. By [37, Theorem 4.3, p. 369], $\text{Ext}_{\mathcal{P}_S}^1(N, M)$ is an abelian group where the addition on $\text{Ext}_{\mathcal{P}_S}^1(N, M)$ is the Baer sum, (see for example [37, p. 69] or [46, p. 78]). By [37, p. 371] or [67, p. 98] we have that $\text{Pext}_S^1(N, M) \simeq \text{Ext}_{\mathcal{P}_S}^1(N, M)$.

Definition 5.1.4 Let $S \subseteq R\text{-mod}$ and let $\mathcal{F} \subseteq R\text{-Mod}$. We define the classes

$${}^{\perp_S}\mathcal{F} = \{M \mid \text{Ext}_{\mathcal{P}_S}^1(M, \mathcal{F}) = 0\} = \{M \mid \text{Pext}_S^1(M, \mathcal{F}) = 0\},$$

$$\mathcal{F}^{\perp_S} = \{M \mid \text{Ext}_{\mathcal{P}_S}^1(\mathcal{F}, M) = 0\} = \{M \mid \text{Pext}_S^1(\mathcal{F}, M) = 0\}.$$

We now recall the definition of a cotorsion pair relative to the proper class \mathcal{P}_S . This notion was considered in [29] for any proper class of short exact sequences.

Definition 5.1.5 Let $S \subseteq R\text{-mod}$ and let \mathcal{F} and \mathcal{G} be classes of left R -modules. The pair $(\mathcal{F}, \mathcal{G})$ is said to be a cotorsion pair relative to the proper class \mathcal{P}_S (for short, an S -cotorsion pair) if ${}^{\perp_S}\mathcal{G} = \mathcal{F}$ and $\mathcal{F}^{\perp_S} = \mathcal{G}$.

Proposition 5.1.6 Let $S \subseteq R\text{-mod}$ and let $\mathcal{F} \subseteq R\text{-Mod}$. Then $\mathcal{F} \subseteq ({}^{\perp_S}\mathcal{F})^{\perp_S}$ and $\mathcal{F} \subseteq {}^{\perp_S}(\mathcal{F}^{\perp_S})$.

PROOF: Let $M \in \mathcal{F}$, thus $\text{Pext}_S^1({}^{\perp_S}\mathcal{F}, M) = 0$ and $\text{Pext}_S^1(M, \mathcal{F}^{\perp_S}) = 0$ and hence $M \in ({}^{\perp_S}\mathcal{F})^{\perp_S}$ and $M \in {}^{\perp_S}(\mathcal{F}^{\perp_S})$. Thus $\mathcal{F} \subseteq ({}^{\perp_S}\mathcal{F})^{\perp_S}$ and $\mathcal{F} \subseteq {}^{\perp_S}(\mathcal{F}^{\perp_S})$. \square

Examples 5.1.7 (as [46, p. 60]) Let $S \subseteq R\text{-mod}$ and let $\mathcal{F} \subseteq R\text{-Mod}$. Then:

- (1) $(S\text{-Pproj}, R\text{-Mod})$ and $(R\text{-Mod}, S\text{-Pinj})$ are S -cotorsion pairs;
- (2) $({}^{\perp_S}\mathcal{F}, ({}^{\perp_S}\mathcal{F})^{\perp_S})$ and $({}^{\perp_S}(\mathcal{F}^{\perp_S}), \mathcal{F}^{\perp_S})$ are S -cotorsion pairs.

Let $S \subseteq R\text{-mod}$. Recall that a class of left R -modules \mathcal{F} is said to be **closed under S -pure extensions** if, whenever $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an S -pure short exact sequence in which L and N are in \mathcal{F} , then $M \in \mathcal{F}$.

Proposition 5.1.8 Let $S \subseteq R\text{-mod}$. If $(\mathcal{F}, \mathcal{G})$ is an S -cotorsion pair, then:

- (1) \mathcal{F} and \mathcal{G} are closed under S -pure extensions and direct summands.
- (2) $S\text{-Pproj} \subseteq \mathcal{F}$ and $S\text{-Pinj} \subseteq \mathcal{G}$.

PROOF: See [46, Proposition 15.3, p. 60]. \square

We now recall the definition of (pre)covers, (pre)envelopes, special precovers and special preenvelopes relative to the proper class \mathcal{P}_S . These notions were considered in [46] for any proper class of short exact sequences.

Definition 5.1.9 Let $S \subseteq R\text{-mod}$ and let $\mathcal{F} \subseteq R\text{-Mod}$. An \mathcal{F} -precover relative to \mathcal{P}_S of a left R -module M is an S -pure short exact sequence $\Sigma : 0 \rightarrow L \rightarrow N \xrightarrow{g} M \rightarrow 0$ with g an \mathcal{F} -precover of M . An \mathcal{F} -precover Σ relative to \mathcal{P}_S of a left R -module M is said to be an \mathcal{F} -cover relative to \mathcal{P}_S of M if g is an \mathcal{F} -cover of M . We say that a class \mathcal{F} is **precovering**

(resp. **covering**) **relative to** \mathcal{P}_S if every module in $R\text{-Mod}$ has an \mathcal{F} -precover (resp. \mathcal{F} -cover) relative to \mathcal{P}_S . An \mathcal{F} -precover Σ of M relative to \mathcal{P}_S is said to be **special** if $\ker(g) \in \mathcal{F}^{\perp_S}$.

Definition 5.1.10 Let $S \subseteq R\text{-mod}$ and let $\mathcal{F} \subseteq R\text{-Mod}$. An \mathcal{F} -**preenvelope relative to** \mathcal{P}_S of a left R -module M is an S -pure short exact sequence $\Sigma : 0 \rightarrow M \xrightarrow{f} N \rightarrow L \rightarrow 0$ with f an \mathcal{F} -preenvelope of M . An \mathcal{F} -preenvelope Σ relative to \mathcal{P}_S of a left R -module M is said to be an \mathcal{F} -**envelope relative to** \mathcal{P}_S of M if f is an \mathcal{F} -envelope of M . We say that a class \mathcal{F} is **preenveloping** (resp. **enveloping**) **relative to** \mathcal{P}_S if every module in $R\text{-Mod}$ has an \mathcal{F} -preenvelope (resp. \mathcal{F} -envelope) relative to \mathcal{P}_S . An \mathcal{F} -preenvelope Σ of M relative to \mathcal{P}_S is said to be **special** if $\text{coker}(f) \in {}^{\perp_S}\mathcal{F}$.

The following results are proved in [46] for any proper class of short exact sequences.

Proposition 5.1.11 Let $S \subseteq R\text{-mod}$ and let \mathcal{F} be a class of left R -modules closed under isomorphisms. Then the following statements are equivalent.

- (1) \mathcal{F} is enveloping relative to \mathcal{P}_S .
- (2) \mathcal{F} is enveloping and $S\text{-Pinj} \subseteq \mathcal{F}$.

PROOF: See [46, Proposition 16.3, p. 73]. □

Proposition 5.1.12 Let $S \subseteq R\text{-mod}$ and let \mathcal{F} be a class of left R -modules closed under isomorphisms. Then the following statements are equivalent.

- (1) \mathcal{F} is covering relative to \mathcal{P}_S .
- (2) \mathcal{F} is covering and $S\text{-Pproj} \subseteq \mathcal{F}$.

PROOF: See [46, Proposition 16.4, p. 74]. □

Lemma 5.1.13 Let $S \subseteq R\text{-mod}$ and let $(\mathcal{F}, \mathcal{G})$ be an S -cotorsion pair in $R\text{-Mod}$. Then the following statements are equivalent.

- (1) Every left R -module has a special \mathcal{F} -precover relative to \mathcal{P}_S .
- (2) Every left R -module has a special \mathcal{G} -preenvelope relative to \mathcal{P}_S .

PROOF: See [46, Lemma 16.5, p. 74]. \square

Lemma 5.1.14 (Relative Wakamatsu's lemma) Let $S \subseteq R\text{-mod}$ and let \mathcal{F} be a class of left R -modules closed under S -pure extensions. Then:

(1) if $\Sigma_A : 0 \rightarrow A \rightarrow X_A \rightarrow Y_A \rightarrow 0$ is an \mathcal{F} -envelope of A relative to \mathcal{P}_S , then Σ_A is special;

(2) if $\Sigma_C : 0 \rightarrow Y_C \rightarrow X_C \rightarrow C \rightarrow 0$ is an \mathcal{F} -cover of C relative to \mathcal{P}_S , then Σ_C is special.

PROOF: See [46, Lemma 16.6, p. 75]. \square

Definition 5.1.15 Let $S \subseteq R\text{-mod}$. An S -cotorsion pair $(\mathcal{F}, \mathcal{G})$ in $R\text{-Mod}$ is said to be **perfect**, if \mathcal{F} is covering relative to \mathcal{P}_S and \mathcal{G} is enveloping relative to \mathcal{P}_S .

5.2 S -Cotorsion pairs induced by S' -Pinj

Let $S \subseteq R\text{-mod}$ and let $\mathcal{F} \subseteq R\text{-Mod}$. The following proposition gives conditions under which $({}^{\perp_S} \mathcal{F}, \mathcal{F})$ is an S -cotorsion pair.

Proposition 5.2.1 Let $S \subseteq R\text{-mod}$ and let $\mathcal{F} \subseteq R\text{-Mod}$ be such that every left R -module has an \mathcal{F} -envelope. Then the following statements are equivalent:

- (1) $S\text{-Pinj} \subseteq \mathcal{F}$ and \mathcal{F} is closed under S -pure extensions and direct summands;
- (2) $({}^{\perp_S} \mathcal{F})^{\perp_S} \subseteq \mathcal{F}$;
- (3) $({}^{\perp_S} \mathcal{F}, \mathcal{F})$ is an S -cotorsion pair.

PROOF: (1) \Rightarrow (2) Let $M \in ({}^{\perp_S} \mathcal{F})^{\perp_S}$. By hypothesis, M has an \mathcal{F} -envelope. Let $\alpha : M \rightarrow N$ be an \mathcal{F} -envelope of M . By Corollary 3.1.3, there is an S -pure monomorphism $f : M \rightarrow F$ with $F \in S\text{-Pinj}$. Since $S\text{-Pinj} \subseteq \mathcal{F}$ (by hypothesis) it follows that there is a homomorphism $g : N \rightarrow F$ such that $g\alpha = f$ and hence α is an S -pure monomorphism, by Lemma 2.2.16(2). Thus the sequence $\Sigma : 0 \rightarrow M \xrightarrow{\alpha} N \rightarrow \text{coker}(\alpha) \rightarrow 0$ is an \mathcal{F} -envelope of M relative to \mathcal{P}_S . Since \mathcal{F} is closed under S -pure extensions (by hypothesis) it follows from Lemma 5.1.14(1) that $\text{coker}(\alpha) \in {}^{\perp_S} \mathcal{F}$. Since $\text{Pext}_S^1({}^{\perp_S} \mathcal{F}, M) = 0$ it follows that $\text{Pext}_S^1(\text{coker}(\alpha), M) = 0$ and

hence Σ is split. Thus M is a direct summand of N . Since \mathcal{F} is closed under direct summands (by hypothesis) it follows that $M \in \mathcal{F}$ and hence $({}^{\perp S}\mathcal{F})^{\perp S} \subseteq \mathcal{F}$.

(2) \Rightarrow (3) Since $\mathcal{F} \subseteq ({}^{\perp S}\mathcal{F})^{\perp S}$ (by Proposition 5.1.6) it follows from (2) that $\mathcal{F} = ({}^{\perp S}\mathcal{F})^{\perp S}$ and hence $({}^{\perp S}\mathcal{F}, \mathcal{F})$ is an S -cotorsion pair.

(3) \Rightarrow (1) By Proposition 5.1.8. □

Corollary 5.2.2 Let $S, T \subseteq R\text{-mod}$. Then the following statements are equivalent.

(1) Every T -pure short exact sequence of left R -modules is S -pure and $T\text{-Pinj}$ is closed under S -pure extensions.

(2) $({}^{\perp S}(T\text{-Pinj}))^{\perp S} = T\text{-Pinj}$.

(3) $({}^{\perp S}(T\text{-Pinj}), T\text{-Pinj})$ is an S -cotorsion pair.

PROOF: By Lemma 2.3.4, the class $T\text{-Pinj}$ is enveloping in $R\text{-Mod}$. By Theorem 3.2.1 we have that every T -pure short exact sequence of left R -modules is S -pure if and only if $S\text{-Pinj} \subseteq T\text{-Pinj}$. Thus the result follows by taking $\mathcal{F} = T\text{-Pinj}$ and applying Proposition 5.2.1. □

Let $T \subseteq R\text{-mod}$. In the next corollary, we will consider when the class of T -pure-injective left R -modules is closed under extensions. This generalizes Theorem 2.18 in [39, p. 11], if we take T to be the class of $(1, 1)$ -presented left R -modules.

Corollary 5.2.3 Let $T \subseteq R\text{-mod}$. Then the following statements are equivalent.

(1) $T\text{-Pinj}$ is closed under extensions.

(2) $({}^{\perp}(T\text{-Pinj}))^{\perp} = T\text{-Pinj}$.

(3) $({}^{\perp}(T\text{-Pinj}), T\text{-Pinj})$ is a cotorsion pair.

(4) $({}^{\perp}(T\text{-Pinj}), T\text{-Pinj})$ is a perfect cotorsion pair.

PROOF: (1) \Leftrightarrow (2) \Leftrightarrow (3) Take $S = \{{}_R R\}$ and apply Corollary 5.2.2.

(3) \Rightarrow (4) By Lemma 2.3.4, the class $T\text{-Pinj}$ is enveloping in $R\text{-Mod}$. Since the class ${}^{\perp}\mathcal{F}$ is covering for any $\mathcal{F} \subseteq {}_R\text{Pinj}$ (by [27, Theorem 4.3(i), p. 699]) it follows that ${}^{\perp}(T\text{-Pinj})$ is covering in $R\text{-Mod}$ and hence $({}^{\perp}(T\text{-Pinj}), T\text{-Pinj})$ is a perfect cotorsion pair.

(4) \Rightarrow (3) This is clear. □

Let $S \subseteq R\text{-mod}$ and let $\mathcal{F} \subseteq R\text{-Mod}$. The following proposition gives conditions under which $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is an S -cotorsion pair.

Proposition 5.2.4 Let $S \subseteq R\text{-mod}$ and let $\mathcal{F} \subseteq R\text{-Mod}$ such that every left R -module has an \mathcal{F} -cover. Then the following statements are equivalent.

- (1) $S\text{-Pproj} \subseteq \mathcal{F}$ and \mathcal{F} is closed under S -pure extensions and direct summands.
- (2) ${}^{\perp_S}(\mathcal{F}^{\perp_S}) \subseteq \mathcal{F}$.
- (3) $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is an S -cotorsion pair.

PROOF: (1) \Rightarrow (2) Let $M \in {}^{\perp_S}(\mathcal{F}^{\perp_S})$. By hypothesis, M has an \mathcal{F} -cover. Let $\alpha : N \rightarrow M$ be an \mathcal{F} -cover of M . By Proposition 3.1.2(1), there is an S -pure epimorphism $f : F \rightarrow M$ with $F \in S\text{-Pproj}$. Since $S\text{-Pproj} \subseteq \mathcal{F}$ (by hypothesis) it follows that there is a homomorphism $g : F \rightarrow N$ such that $\alpha g = f$ and hence α is an S -pure epimorphism, by Lemma 2.2.16(1). Thus the sequence $\Sigma : 0 \rightarrow \ker(\alpha) \rightarrow N \xrightarrow{\alpha} M \rightarrow 0$ is an \mathcal{F} -cover of M relative to \mathcal{P}_S . Since \mathcal{F} is closed under S -pure extensions (by hypothesis) it follows from Lemma 5.1.14 that $\ker(\alpha) \in \mathcal{F}^{\perp_S}$. Since $\text{Pext}_S^1(M, \mathcal{F}^{\perp_S}) = 0$ it follows that $\text{Pext}_S^1(M, \ker(\alpha)) = 0$ and hence Σ is split. Thus M is isomorphic to a direct summand of N . Since \mathcal{F} is closed under direct summands (by hypothesis) it follows that $M \in \mathcal{F}$ and hence ${}^{\perp_S}(\mathcal{F}^{\perp_S}) \subseteq \mathcal{F}$.

(2) \Rightarrow (3) Since $\mathcal{F} \subseteq {}^{\perp_S}(\mathcal{F}^{\perp_S})$ (by Proposition 5.1.6) it follows from (2) that $\mathcal{F} = {}^{\perp_S}(\mathcal{F}^{\perp_S})$ and hence $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is an S -cotorsion pair.

(3) \Rightarrow (1) By Proposition 5.1.8. □

Corollary 5.2.5 Let $S \subseteq R\text{-mod}$ and let $(\mathcal{F}, \mathcal{G})$ be an almost dual pair. Then the following statements are equivalent.

- (1) $S\text{-Pproj} \subseteq \mathcal{F}$ and \mathcal{F} is closed under S -pure extensions.
- (2) ${}^{\perp_S}(\mathcal{F}^{\perp_S}) \subseteq \mathcal{F}$.
- (3) $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is an S -cotorsion pair.

PROOF: By Proposition 4.2.8(2), \mathcal{F} is covering in $R\text{-Mod}$. Thus the result follows from Proposition 5.2.4. □

Example 5.2.6 Let T be a class of finitely presented left R -modules containing ${}_R R$. Then $\varinjlim(\text{add } T)$ is closed under extensions if and only if $(\varinjlim(\text{add } T), (\varinjlim(\text{add } T))^{\perp})$ is a cotorsion pair.

PROOF: By Corollary 4.2.25, $(\varinjlim(\text{add } T), \text{Prod } T^*)$ is an almost dual pair and hence from Proposition 4.2.8(1) we have that $\varinjlim(\text{add } T)$ is closed under direct summands and direct sums. Since $T\text{-Pproj} = \text{Add } T$ (by Proposition 3.1.2(2)) it follows that $T\text{-Pproj} \subseteq \varinjlim(\text{add } T)$. Thus the result follows by taking $(\mathcal{F}, \mathcal{G}) = (\varinjlim(\text{add } T), \text{Prod } T^*)$, $S = \{{}_R R\}$ and applying Corollary 5.2.5. \square

In the next result we consider S -cotorsion pairs induced by $S\text{-Pinj}$ and we give equivalent characterizations of a ring over which every module in $(S\text{-Pinj})^+$ is S -pure-projective.

Theorem 5.2.7 Let $S \subseteq R\text{-mod}$. Then the following statements are equivalent:

- (1) $((S\text{-Pinj})^+)^{\perp_S} = R\text{-Mod}$;
- (2) $(S\text{-Pproj}, ((S\text{-Pinj})^+)^{\perp_S})$ is an S -cotorsion pair;
- (3) every module in $(S\text{-Pinj})^+$ is S -pure-projective;
- (4) if $0 \rightarrow A \rightarrow B \xrightarrow{g} C \rightarrow 0$ is an S -pure exact sequence of left R -modules with $B \in (S\text{-Pinj})^+$, then $g : B \rightarrow C$ is a $(S\text{-Pinj})^+$ -precover of C ;
- (5) every left R -module is the kernel of a $(S\text{-Pinj})^+$ -precover $g : B \rightarrow C$ with g an S -pure epimorphism and B S -pure-injective;
- (6) every left R -module is injective with respect to every S -pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $C \in (S\text{-Pinj})^+$.

PROOF: (1) \Rightarrow (2) Suppose that $((S\text{-Pinj})^+)^{\perp_S} = R\text{-Mod}$. Since $(S\text{-Pproj}, R\text{-Mod})$ is an S -cotorsion pair (by Example 5.1.7(1)) it follows that $(S\text{-Pproj}, ((S\text{-Pinj})^+)^{\perp_S})$ is an S -cotorsion pair.

(2) \Rightarrow (3) Suppose that $(S\text{-Pproj}, ((S\text{-Pinj})^+)^{\perp_S})$ is an S -cotorsion pair, thus ${}^{\perp_S}((S\text{-Pinj})^+)^{\perp_S} = S\text{-Pproj}$. Since $(S\text{-Pinj})^+ \subseteq {}^{\perp_S}((S\text{-Pinj})^+)^{\perp_S}$ (by Proposition 5.1.6) it follows that $(S\text{-Pinj})^+ \subseteq S\text{-Pproj}$.

(3) \Rightarrow (4) Let $\Sigma : 0 \rightarrow A \rightarrow B \xrightarrow{g} C \rightarrow 0$ be an *S*-pure exact sequence of left *R*-modules with $B \in (S\text{-Pinj})^+$. Let $f : M \rightarrow C$ be any *R*-homomorphism with $M \in (S\text{-Pinj})^+$. By hypothesis, *M* is *S*-pure-projective and hence there is an *R*-homomorphism $h : M \rightarrow B$ such that $gh = f$. Thus $g : B \rightarrow C$ is a $(S\text{-Pinj})^+$ -precover of *C*.

(4) \Rightarrow (5) Let *M* be any left *R*-module. By Corollary 3.1.3, there exists an *S*-pure exact sequence $0 \rightarrow M \xrightarrow{f} N \xrightarrow{g} L \rightarrow 0$ with $N \in S\text{-Pinj}$. Since $S\text{-Pinj} \subseteq (S\text{-Pinj})^+$ it follows that $N \in (S\text{-Pinj})^+$. By (4), $g : N \rightarrow L$ is a $(S\text{-Pinj})^+$ -precover of *L*.

(5) \Rightarrow (1) Let $M \in R\text{-Mod}$ and let $N \in (S\text{-Pinj})^+$. By (5), *M* is the kernel of a $(S\text{-Pinj})^+$ -precover $f : A \rightarrow B$ with *f* an *S*-pure epimorphism and *A* an *S*-pure-injective module. Thus we have an *S*-pure exact sequence $0 \rightarrow M \rightarrow A \xrightarrow{\pi} A/M \rightarrow 0$. By [37, Theorem 5.1, p. 372], the sequence $\text{Pext}_S^0(N, A) \rightarrow \text{Pext}_S^0(N, A/M) \rightarrow \text{Pext}_S^1(N, M) \rightarrow \text{Pext}_S^1(N, A)$ is exact. Since $A \in S\text{-Pinj}$ it follows from Theorem 2.2.19 that $\text{Pext}_S^1(N, A) = 0$. By Lemma 4.5.19, the sequence $\text{Hom}_R(N, A) \xrightarrow{\pi^\otimes} \text{Hom}_R(N, A/M) \rightarrow \text{Pext}_S^1(N, M) \rightarrow 0$ is exact. Since $\pi : A \rightarrow A/M$ is a $(S\text{-Pinj})^+$ -precover of *A*/*M*, the sequence $\text{Hom}_R(N, A) \xrightarrow{\pi^\otimes} \text{Hom}_R(N, A/M) \rightarrow 0$ is exact. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc} \text{Hom}_R(N, A) & \longrightarrow & \text{Hom}_R(N, A/M) & \longrightarrow & \text{Pext}_S^1(N, M) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \\ \text{Hom}_R(N, A) & \longrightarrow & \text{Hom}_R(N, A/M) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

By Lemma 2.1.3, $\text{Pext}_S^1(N, M) = 0$ and hence $M \in ((S\text{-Pinj})^+)^{\perp_s}$. Therefore, $((S\text{-Pinj})^+)^{\perp_s} = R\text{-Mod}$.

(3) \Rightarrow (6) Let $\Sigma : 0 \rightarrow A \xrightarrow{\alpha} B \rightarrow C \rightarrow 0$ be any *S*-pure exact sequence of left *R*-modules with $C \in (S\text{-Pinj})^+$ and let $M \in R\text{-Mod}$. Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow f & & \swarrow g & & \\ & & M & & & & \end{array}$$

By (3), *C* is *S*-pure-projective and hence from Theorem 2.2.18 we have that α is split and so there is an *R*-homomorphism $\varphi : B \rightarrow A$ such that $\varphi\alpha = 1_A$. Put $g = f\varphi$, thus $g\alpha = f\varphi\alpha = f$ and hence *M* is injective with respect to Σ .

(6) \Rightarrow (3) Let $N \in (S\text{-Pinj})^+$ and let $M \in R\text{-Mod}$. By Proposition 3.1.2, there exists an *S*-pure exact sequence $0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0$ with *F* an *S*-pure-projective module. By [37, Theorem 5.1, p. 372], the sequence $\text{Pext}_S^0(F, M) \rightarrow \text{Pext}_S^0(K, M) \rightarrow \text{Pext}_S^1(N, M) \rightarrow \text{Pext}_S^1(F, M)$

is exact. Since F is S -pure-projective it follows from Theorem 2.2.18 that $\text{Pext}_S^1(F, M) = 0$. By Lemma 4.5.19, the sequence $\text{Hom}_R(F, M) \rightarrow \text{Hom}_R(K, M) \rightarrow \text{Pext}_S^1(N, M) \rightarrow 0$ is exact. By (6), the sequence $\text{Hom}_R(F, M) \rightarrow \text{Hom}_R(K, M) \rightarrow 0$ is exact and hence from Lemma 2.1.3 we get that $\text{Pext}_S^1(N, M) = 0$. Thus N is an S -pure-projective module, by Theorem 2.2.18. \square

Recall that R is said to be a **quasi-Frobenius ring** if R is a left noetherian and ${}_R R$ is injective.

Theorem 5.2.7 together with the following example show that there is a ring R and $S \subseteq R\text{-mod}$ such that $(S\text{-Pinj})^+ \not\subseteq S\text{-Pproj}$.

Example 5.2.8 Let F be a field and let K be a proper subfield. Let $A = \prod_{i=1}^{\infty} F_i$ with $F_i = F$ and let $R = \{(x_i) \in A \mid \text{all but finitely many } x_i \in K\}$. It is clear that R is commutative. By [30, Example 1.11(vi)], R is a von Neumann regular ring but it is not self-injective and hence R is not a quasi-Frobenius ring. Since a ring R is quasi-Frobenius if and only if $({}_R \mathcal{APure})^\perp = R\text{-Mod}$ (by [41, Proposition 2.8, p. 372]) it follows that there is an R -module M such that $\text{Ext}_R^1(G, M) \neq 0$ for some absolutely pure R -module G . Since a ring R is von Neumann regular if and only if $\text{Ext}_R^1(A, B) \simeq \text{Pext}_S^1(A, B)$ for all R -modules A and B , where S is the class of $(1, 1)$ -presented R -modules (by [39, Proposition 3.3, p. 14]) it follows that $\text{Pext}_S^1(G, M) \neq 0$. Since $G \in (S\text{-Pinj})^+$ it follows that $R\text{-Mod} \not\subseteq ((S\text{-Pinj})^+)^{\perp S}$ and from Theorem 5.2.7 we have that $(S\text{-Pinj})^+ \not\subseteq S\text{-Pproj}$.

The following corollary is immediate from Theorem 5.2.7 and Remark 4.4.9.

Corollary 5.2.9 Let S be the class of $(1, 1)$ -presented left R -modules. Then the following statements are equivalent:

- (1) $({}_R \mathcal{RD}\text{-Coflat})^{\perp S} = R\text{-Mod}$;
- (2) $({}_R \mathcal{RD}\text{-Proj}, ({}_R \mathcal{RD}\text{-Coflat})^{\perp S})$ is an S -cotorsion pair;
- (3) every RD -coflat left R -module is RD -projective;
- (4) if $0 \rightarrow A \rightarrow B \xrightarrow{g} C \rightarrow 0$ is a RD -pure exact sequence of left R -modules with B RD -coflat, then $g : B \rightarrow C$ is a $({}_R \mathcal{RD}\text{-Coflat})$ -precover of C ;

(5) every left R -module is the kernel of a $({}_R\mathcal{RD}\text{-Coflat})$ -precover $g : B \rightarrow C$ with g a RD -pure epimorphism and B being RD -injective;

(6) every left R -module is injective with respect to every RD -pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with C RD -coflat.

In the next result we give equivalent characterizations of definability of the class $S\text{-Pinj}$ in terms of S -cotorsion pairs and $(S\text{-Pinj})^+$ -preenvelopes.

Theorem 5.2.10 Let $S \subseteq R\text{-mod}$. Then the following statements are equivalent.

- (1) $S\text{-Pinj}$ is definable.
- (2) $(R\text{-Mod}, (S\text{-Pinj})^+)$ is an S -cotorsion pair.
- (3) $({}^{\perp_S}((S\text{-Pinj})^+), S\text{-Pinj})$ is an S -cotorsion pair.
- (4) ${}^{\perp_S}((S\text{-Pinj})^+) = R\text{-Mod}$.
- (5) Every left R -module is projective with respect to every S -pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $A \in (S\text{-Pinj})^+$.
- (6) If $0 \rightarrow A \xrightarrow{f} B \rightarrow C \rightarrow 0$ is an S -pure exact sequence of left R -modules with $B \in (S\text{-Pinj})^+$, then $f : A \rightarrow B$ is a $(S\text{-Pinj})^+$ -preenvelope of A .
- (7) If $0 \rightarrow A \xrightarrow{f} B \rightarrow C \rightarrow 0$ is an S -pure exact sequence of left R -modules with $B \in S\text{-Pinj}$, then $f : A \rightarrow B$ is a $(S\text{-Pinj})^+$ -preenvelope of A .
- (8) Every $(S\text{-Pinj})$ -preenvelope of a left R -module M is a $(S\text{-Pinj})^+$ -preenvelope of M .

PROOF: (1) \Rightarrow (2) Suppose that $S\text{-Pinj}$ is definable. By Theorem 4.5.1, $(S\text{-Pinj})^+ = S\text{-Pinj}$. Since $(R\text{-Mod}, S\text{-Pinj})$ is an S -cotorsion pair (by Example 5.1.7(1)) it follows that $(R\text{-Mod}, (S\text{-Pinj})^+)$ is an S -cotorsion pair.

(2) \Rightarrow (3) Suppose that $(R\text{-Mod}, (S\text{-Pinj})^+)$ is an S -cotorsion pair, thus ${}^{\perp_S}((S\text{-Pinj})^+) = R\text{-Mod}$. Since $(R\text{-Mod}, S\text{-Pinj})$ is an S -cotorsion pair (by Example 5.1.7(1)) it follows that $({}^{\perp_S}((S\text{-Pinj})^+), S\text{-Pinj})$ is an S -cotorsion pair.

(3) \Rightarrow (4) Suppose that $({}^{\perp_S}((S\text{-Pinj})^+), S\text{-Pinj})$ is an S -cotorsion pair, thus ${}^{\perp_S}((S\text{-Pinj})^+) = {}^{\perp_S}(S\text{-Pinj}) = R\text{-Mod}$, by Theorem 2.2.19.

(4) \Rightarrow (5) Let $\Sigma : 0 \rightarrow A \rightarrow B \xrightarrow{\alpha} C \rightarrow 0$ be any S -pure exact sequence of left R -modules with $A \in (S\text{-Pinj})^+$. Let $M \in R\text{-Mod}$ and let $f : M \rightarrow C$ be any homomorphism. By (4), $\text{Pext}_S^1(C, A) = 0$ and hence the exact sequence Σ is split. Thus there is a homomorphism

$\alpha' : C \rightarrow B$ such that $\alpha\alpha' = 1_C$. Put $g = \alpha'f$, thus $\alpha g = \alpha\alpha'f = 1_C f = f$. Hence M is projective with respect to Σ .

(5) \Rightarrow (1) Let $N \in (S\text{-Pinj})^+$ and let $M \in R\text{-Mod}$. By Corollary 3.1.3, there is an S -pure exact sequence $0 \rightarrow N \rightarrow F \rightarrow K \rightarrow 0$ with $F \in S\text{-Pinj}$. By [37, Theorem 5.1, p. 372], the sequence $\text{Pext}_S^0(M, F) \rightarrow \text{Pext}_S^0(M, K) \rightarrow \text{Pext}_S^1(M, N) \rightarrow \text{Pext}_S^1(M, F)$ is exact. Since $F \in S\text{-Pinj}$ it follows that $\text{Pext}_S^1(M, F) = 0$. By Lemma 4.5.19, the sequence $\text{Hom}_R(M, F) \rightarrow \text{Hom}_R(M, K) \rightarrow \text{Pext}_S^1(M, N) \rightarrow 0$ is exact. Also, by (5) we have that the sequence $\text{Hom}_R(M, F) \rightarrow \text{Hom}_R(M, K) \rightarrow 0$ is exact and hence we have that $\text{Pext}_S^1(M, N) = 0$ and this implies that $N \in S\text{-Pinj}$, by Theorem 2.2.19. Thus the class $S\text{-Pinj}$ is definable, by Theorem 4.5.1.

(1) \Rightarrow (6) Let $0 \rightarrow A \xrightarrow{f} B \rightarrow C \rightarrow 0$ be an S -pure exact sequence of left R -modules with $B \in (S\text{-Pinj})^+$. Let $g : A \rightarrow M$ be any homomorphism with $M \in (S\text{-Pinj})^+$. By hypothesis, $(S\text{-Pinj})^+ \subseteq S\text{-Pinj}$ and hence M is S -pure-injective and this implies that there is a homomorphism $h : B \rightarrow M$ such that $hf = g$. Therefore $f : A \rightarrow B$ is a $(S\text{-Pinj})^+$ -preenvelope of A .

(6) \Rightarrow (7) This is obvious.

(7) \Rightarrow (8) Let $M \in R\text{-Mod}$ and let $f : M \rightarrow N$ be an $(S\text{-Pinj})$ -preenvelope of a module M . By Corollary 3.1.3, there is an S -pure monomorphism $g : M \rightarrow F$ with $F \in S\text{-Pinj}$. Thus there is a homomorphism $h : N \rightarrow F$ such that $hf = g$ and hence f is an S -pure monomorphism. Then the sequence $0 \rightarrow M \xrightarrow{f} N \rightarrow \text{coker}(f) \rightarrow 0$ is S -pure exact with $N \in S\text{-Pinj}$. By hypothesis, $f : M \rightarrow N$ is a $(S\text{-Pinj})^+$ -preenvelope of M .

(8) \Rightarrow (1) Let $M \in (S\text{-Pinj})^+$ and let $0 \rightarrow A \xrightarrow{\alpha} B \rightarrow C \rightarrow 0$ be any S -pure exact sequence of left R -modules. Let $f : A \rightarrow M$ be any homomorphism. By Corollary 3.1.3, there is an S -pure monomorphism $g : B \rightarrow D$ with $D \in S\text{-Pinj}$. Since $g\alpha : A \rightarrow D$ is an S -pure monomorphism it follows that $g\alpha$ is an $(S\text{-Pinj})$ -preenvelope of a module A . By hypothesis, $g\alpha$ is an $(S\text{-Pinj})^+$ -preenvelope of A and hence there is a homomorphism $h : D \rightarrow M$ such that $hg\alpha = f$. Put $\beta = hg$, thus $\beta\alpha = hg\alpha = f$ and hence M is S -pure-injective. Thus $(S\text{-Pinj})^+ = S\text{-Pinj}$ and hence from Theorem 4.5.1 we have that the class $S\text{-Pinj}$ is definable. \square

The following proposition provides characterizations of the equivalence of two purities by using relative cotorsion pairs.

Proposition 5.2.11 Let $S, S' \subseteq R\text{-mod}$. Then the following statements are equivalent:

- (1) S -purity = S' -purity for short exact sequences of left R -modules;
- (2) a pair $(\mathcal{F}, \mathcal{G})$ is an S -cotorsion pair if and only if it is an S' -cotorsion pair;
- (3) $(R\text{-Mod}, S\text{-Pinj})$ is an S' -cotorsion pair;
- (4) $(R\text{-Mod}, S'\text{-Pinj})$ is an S -cotorsion pair;
- (5) $(S\text{-Pproj}, R\text{-Mod})$ is an S' -cotorsion pair;
- (6) $(S'\text{-Pproj}, R\text{-Mod})$ is an S -cotorsion pair.

PROOF: (1) \Rightarrow (2) Suppose that S -purity = S' -purity for short exact sequences of left R -modules. Then, by Corollary 3.2.3, $S\text{-Pinj} = S'\text{-Pinj}$ and hence $\text{Pext}_S^1(M, N) \simeq \text{Pext}_{S'}^1(M, N)$ for all $M, N \in R\text{-Mod}$. It follows that $\mathcal{F}^{\perp_S} = \mathcal{F}^{\perp_{S'}}$ and ${}^{\perp_S}\mathcal{G} = {}^{\perp_{S'}}\mathcal{G}$ for any $\mathcal{F}, \mathcal{G} \subseteq R\text{-Mod}$. Hence a pair $(\mathcal{F}, \mathcal{G})$ is S -cotorsion pair if and only if it is an S' -cotorsion pair.

(2) \Rightarrow (3) By Example 5.1.7(1), the pair $(R\text{-Mod}, S\text{-Pinj})$ is an S -cotorsion pair and hence by hypothesis we have that $(R\text{-Mod}, S\text{-Pinj})$ is an S' -cotorsion pair.

(3) \Rightarrow (4) Suppose that $(R\text{-Mod}, S\text{-Pinj})$ is an S' -cotorsion pair, thus $(R\text{-Mod})^{\perp_{S'}} = S\text{-Pinj}$. Since $(R\text{-Mod}, S'\text{-Pinj})$ is an S' -cotorsion pair (by Example 5.1.7(1)) it follows that $(R\text{-Mod})^{\perp_{S'}} = S'\text{-Pinj}$ and hence $S\text{-Pinj} = S'\text{-Pinj}$. Since $(R\text{-Mod}, S\text{-Pinj})$ is an S -cotorsion pair it follows that $(R\text{-Mod}, S'\text{-Pinj})$ is an S -cotorsion pair.

(4) \Rightarrow (5) Suppose that $(R\text{-Mod}, S'\text{-Pinj})$ is an S -cotorsion pair. By a similar proof as in (3) \Rightarrow (4), we can show that $S\text{-Pinj} = S'\text{-Pinj}$. By Corollary 3.2.3, $S\text{-Pproj} = S'\text{-Pproj}$. Since $(S'\text{-Pproj}, R\text{-Mod})$ is an S' -cotorsion pair (by Example 5.1.7(1)) it follows that $(S\text{-Pproj}, R\text{-Mod})$ is an S' -cotorsion pair.

(5) \Rightarrow (6) Suppose that $(S\text{-Pproj}, R\text{-Mod})$ is an S' -cotorsion pair, thus ${}^{\perp_{S'}}(R\text{-Mod}) = S\text{-Pproj}$. Since $(S'\text{-Pproj}, R\text{-Mod})$ is an S' -cotorsion pair (by Example 5.1.7(1)) it follows that ${}^{\perp_{S'}}(R\text{-Mod}) = S'\text{-Pproj}$ and hence $S\text{-Pproj} = S'\text{-Pproj}$. Since $(S\text{-Pproj}, R\text{-Mod})$ is an S -cotorsion pair it follows that $(S'\text{-Pproj}, R\text{-Mod})$ is an S -cotorsion pair.

(6) \Rightarrow (1) Suppose that $(S'\text{-Pproj}, R\text{-Mod})$ is an S -cotorsion pair. By a similar proof as in (5) \Rightarrow (6), we can show that $S\text{-Pproj} = S'\text{-Pproj}$. By Corollary 3.2.3, S -purity = S' -purity for short exact sequences of left R -modules. \square

The following corollary is immediately obtained from Proposition 5.2.11.

Corollary 5.2.12 Let S be the class of $(1, 1)$ -presented left R -modules. Then the following statements are equivalent for a ring R :

- (1) R is an RD -ring;
- (2) a pair $(\mathcal{F}, \mathcal{G})$ is an $(R\text{-mod})$ -cotorsion pair if and only if it is an S -cotorsion pair;
- (3) $(R\text{-Mod}, {}_R\text{Pinj})$ is an S -cotorsion pair;
- (4) $(R\text{-Mod}, {}_R\mathcal{RD}\text{-Inj})$ is an $(R\text{-mod})$ -cotorsion pair;
- (5) $({}_R\text{Pproj}, R\text{-Mod})$ is an S -cotorsion pair;
- (6) $({}_R\mathcal{RD}\text{-Proj}, R\text{-Mod})$ is an $(R\text{-mod})$ -cotorsion pair.

Let S be the class of $(1, 1)$ -presented left R -modules. We deduce from the following corollary that S -cotorsion pairs and $(R\text{-mod})$ -cotorsion pairs do not coincide with cotorsion pairs over any ring R which is not von Neumann regular (recall, from Examples 3.1.9(1) the definition of von Neumann regular ring).

Corollary 5.2.13 Let T be the class of $(1, 1)$ -presented left R -modules. Then the following statements are equivalent for a ring R :

- (1) R is a von Neumann regular ring;
- (2) a pair $(\mathcal{F}, \mathcal{G})$ is cotorsion if and only if it is T -cotorsion;
- (3) $(R\text{-Mod}, {}_R\mathcal{RD}\text{-Inj})$ is a cotorsion pair;
- (4) $(R\text{-Mod}, {}_R\text{Inj})$ is an T -cotorsion pair;
- (5) $({}_R\mathcal{RD}\text{-Proj}, R\text{-Mod})$ is a cotorsion pair;
- (6) $({}_R\text{Proj}, R\text{-Mod})$ is an T -cotorsion pair;
- (7) a pair $(\mathcal{F}, \mathcal{G})$ is cotorsion if and only if it is $(R\text{-mod})$ -cotorsion;
- (8) $(R\text{-Mod}, {}_R\text{Pinj})$ is a cotorsion pair;
- (9) $(R\text{-Mod}, {}_R\text{Inj})$ is an $(R\text{-mod})$ -cotorsion pair;
- (10) $({}_R\text{Pproj}, R\text{-Mod})$ is a cotorsion pair;
- (11) $({}_R\text{Proj}, R\text{-Mod})$ is an $(R\text{-mod})$ -cotorsion pair.

PROOF: By taking $S = T$ and $S' = \{{}_R R\}$ and applying Proposition 5.2.11 we have that $(2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6)$. Also, by taking $S = R\text{-mod}$ and $S' = \{{}_R R\}$ and applying

Proposition 5.2.11 we have that (7) \Leftrightarrow (8) \Leftrightarrow (9) \Leftrightarrow (10) \Leftrightarrow (11).

By [39, Corollary 2.14, p. 10], a ring R is von Neumann regular if and only if T -purity = S' -purity for short exact sequences of left R -modules. Also, from [79, 37.6, p. 316] we deduce that a ring R is von Neumann regular if and only if $(R\text{-mod})$ -purity = S' -purity for short exact sequences of left R -modules. Hence (1), (2) and (7) are equivalent by Proposition 5.2.11. \square

5.3 When is $(\mathcal{F}, \mathcal{F}^{\perp_S})$ a perfect S -cotorsion pair?

Recall from [28] that a duality pair $(\mathcal{F}, \mathcal{G})$ is said to be a **perfect duality pair** if \mathcal{F} is closed under direct sums and extensions and ${}_R R \in \mathcal{F}$. Holm and Jørgensen in [27, Theorem 3.4, p. 697] proved the following result.

Theorem 5.3.1 If a class \mathcal{F} contains the ground ring R and is closed under extensions, direct sums, pure submodules, and pure quotient modules, then $(\mathcal{F}, \mathcal{F}^{\perp})$ is a perfect cotorsion pair.

Also, they proved in [28, Theorem 3.1(c), p. 629] the following result.

Theorem 5.3.2 If $(\mathcal{F}, \mathcal{G})$ is a perfect duality pair, then $(\mathcal{F}, \mathcal{F}^{\perp})$ is a perfect cotorsion pair.

The main purpose of this section is to generalize these results of Holm and Jørgensen to the case of S -cotorsion pairs.

Let $\mathcal{F} \subseteq R\text{-Mod}$ and let M be a left R -module. Recall (see [81, Definition 2.2.1, p. 30]) that an exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ with $L \in \mathcal{F}$ is said to be a generator for $\mathcal{E}xt(\mathcal{F}, M)$ if for each exact sequence $0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0$ with $L' \in \mathcal{F}$ there exist $f \in \text{Hom}_R(N', N)$ and $g \in \text{Hom}_R(L', L)$ such that the diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0 \\
 & & \parallel & & \downarrow f & & \downarrow g & & \\
 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0
 \end{array}$$

is commutative.

Furthermore, such a generator is said to be **minimal** if for any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

we have that g is an isomorphism (then so too is f).

For the main purpose of this section, we need the following definition, which is a generalization of [81, Definition 2.2.1, p. 30], and lemmas.

Definition 5.3.3 Let $\mathcal{F} \subseteq R\text{-Mod}$, let $S \subseteq R\text{-mod}$ and let M be a left R -module. An S -pure exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ with $L \in \mathcal{F}$ is said to be a **generator for $\mathcal{P}ext_S(\mathcal{F}, M)$** if for each S -pure exact sequence $0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0$ with $L' \in \mathcal{F}$ there exist $f \in \text{Hom}_R(N', N)$ and $g \in \text{Hom}_R(L', L)$ such that the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \parallel & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

is commutative.

Furthermore, such a generator is said to be **minimal** if for any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

we have that g is an isomorphism (then so too is f , by the Five Lemma [62, Proposition 2.72, p. 90]).

Example 5.3.4 Let $\Sigma : 0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} L \rightarrow 0$ be any RD -pure exact sequence of left R -modules. If N is an RD -injective left R -module, then Σ is a generator for $\mathcal{P}ext_S(R\text{-Mod}, M)$, where S is the class of $(1, 1)$ -presented left R -modules. Moreover, if $\alpha : M \rightarrow N$ is a $({}_R\mathcal{RD}\text{-Inj})$ -envelope of M then this generator is minimal.

PROOF: Let $0 \rightarrow M \xrightarrow{\alpha'} N' \xrightarrow{\beta'} L' \rightarrow 0$ be any RD -pure exact sequence and consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\alpha'} & N' & \xrightarrow{\beta'} & L' & \longrightarrow & 0 \\ & & \parallel & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & M & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & L & \longrightarrow & 0 \end{array}$$

Since N is RD -injective (by hypothesis), there is a homomorphism $f : N' \rightarrow N$ such that $f\alpha' = \alpha$. By Lemma 2.1.3, there is a $g \in \text{Hom}_R(L', L)$ such that $g\beta' = \beta f$ and hence Σ is a generator for $\mathcal{P}ext_S(R\text{-Mod}, M)$. Now, suppose that $\alpha : M \rightarrow N$ is a $({}_R\mathcal{RD}\text{-Inj})$ -envelope of M and consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\alpha} & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \downarrow \lambda & & \downarrow \varphi & & \\ 0 & \longrightarrow & M & \xrightarrow{\alpha} & N & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

Since $\alpha : M \rightarrow N$ is an $({}_R\mathcal{RD}\text{-Inj})$ -envelope of M it follows that λ is an automorphism. By Lemma 2.1.3, φ is an automorphism and hence Σ is a minimal generator for $\mathcal{P}ext_S(R\text{-Mod}, M)$. □

Lemma 5.3.5 Let \mathcal{F} be a class of left R -modules closed under direct limits, let $S \subseteq R\text{-mod}$, let M be a left R -module and let $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ be a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$. Then there exists a generator $0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$ and a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0 \end{array}$$

such that $\ker(f) = \ker(f'f)$ in any commutative diagram whose rows are generators for $\mathcal{P}ext_S(\mathcal{F}, M)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \parallel & & \downarrow f' & & \downarrow g' & & \\ 0 & \longrightarrow & M & \longrightarrow & N'' & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

PROOF: Assume that the statement is not true. By induction, we will construct a direct system of generators for $\mathcal{P}ext_S(\mathcal{F}, M)$ indexed by ordinals as follows:

First let $(\Sigma_0 : 0 \rightarrow M \rightarrow N_0 \rightarrow L_0 \rightarrow 0) = (0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0)$. By assumption, there is a generator of $\mathcal{P}ext_S(\mathcal{F}, M)$, $(0 \rightarrow M \rightarrow N_1 \rightarrow L_1 \rightarrow 0)$, and a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N_0 & \longrightarrow & L_0 & \longrightarrow & 0 \\ & & \parallel & & \downarrow f_{1,0} & & \downarrow g_{1,0} & & \\ 0 & \longrightarrow & M & \longrightarrow & N_1 & \longrightarrow & L_1 & \longrightarrow & 0 \end{array}$$

such that $\ker(f_{1,0}) = \ker(f_{1,0} 1_N) \not\subseteq \ker(1_N) = 0$.

Let α be any non-zero ordinal number. Suppose that we have already constructed the direct system $(\Sigma_{\beta'}, f_{\beta', \beta}, g_{\beta', \beta})_{\beta \leq \beta' < \alpha}$ of generators for $\mathcal{P}ext_S(\mathcal{F}, M)$, where $\Sigma_\beta = 0 \rightarrow M \rightarrow N_\beta \rightarrow L_\beta \rightarrow 0$, $f_{\beta', \beta} \in \text{Hom}_R(N_\beta, N_{\beta'})$, $g_{\beta', \beta} \in \text{Hom}_R(L_\beta, L_{\beta'})$ and $f_{\beta, \beta}, g_{\beta, \beta}$ are identity maps.

If α is not a limit ordinal, then the generator $0 \rightarrow M \rightarrow N_{\alpha-1} \rightarrow L_{\alpha-1} \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$ is defined together with $f_{\alpha-1, \beta} \in \text{Hom}_R(N_\beta, N_{\alpha-1})$ and $g_{\alpha-1, \beta} \in \text{Hom}_R(L_\beta, L_{\alpha-1})$, for all $\beta \leq \alpha - 1$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \downarrow f_{\alpha-1,0} & & \downarrow g_{\alpha-1,0} & & \\ 0 & \longrightarrow & M & \longrightarrow & N_{\alpha-1} & \longrightarrow & L_{\alpha-1} & \longrightarrow & 0 \end{array}$$

By assumption, there is a generator $0 \rightarrow M \rightarrow N_\alpha \rightarrow L_\alpha \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$ and a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N_{\alpha-1} & \longrightarrow & L_{\alpha-1} & \longrightarrow & 0 \\ & & \parallel & & \downarrow f_{\alpha, \alpha-1} & & \downarrow g_{\alpha, \alpha-1} & & \\ 0 & \longrightarrow & M & \longrightarrow & N_\alpha & \longrightarrow & L_\alpha & \longrightarrow & 0 \end{array}$$

such that $\ker(f_{\alpha, \alpha-1} f_{\alpha-1, 0}) \not\subseteq \ker(f_{\alpha-1, 0})$, where $f_{\alpha, \beta} = f_{\alpha, \alpha-1} f_{\alpha-1, \beta}$ and $g_{\alpha, \beta} = g_{\alpha, \alpha-1} g_{\alpha-1, \beta}$, for all $\beta \leq \alpha - 1$ and hence $\ker(f_{\alpha, 0}) \not\subseteq \ker(f_{\alpha-1, 0})$.

If α is a limit ordinal, then put $N_\alpha = \lim_{\rightarrow \beta < \alpha} N_\beta$ and $L_\alpha = \lim_{\rightarrow \beta < \alpha} L_\beta$ and let $\Sigma_\alpha : 0 \rightarrow M \rightarrow N_\alpha \rightarrow L_\alpha \rightarrow 0$ be the direct limit of the generators $\Sigma_\beta : 0 \rightarrow M \rightarrow N_\beta \rightarrow L_\beta \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$, for all $\beta < \alpha$. Since Σ_β are *S*-pure exact sequences of left *R*-modules it follows from [79, 33.8, p. 279] that Σ_α is an *S*-pure exact sequence. Since $L_\beta \in \mathcal{F}$ for all $\beta < \alpha$ and \mathcal{F} is closed under direct limits (by hypothesis), $L_\alpha \in \mathcal{F}$. Let $0 \rightarrow M \rightarrow \bar{N} \rightarrow \bar{L} \rightarrow 0$ be any *S*-pure exact sequence with $\bar{L} \in \mathcal{F}$. Choose any $\beta < \alpha$, thus $\Sigma_\beta : 0 \rightarrow M \rightarrow N_\beta \rightarrow L_\beta \rightarrow 0$ is a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$ and hence there exist $\bar{f} \in \text{Hom}_R(\bar{N}, N_\beta)$ and $\bar{g} \in \text{Hom}_R(\bar{L}, L_\beta)$

such that the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \bar{N} & \longrightarrow & \bar{L} & \longrightarrow & 0 \\ & & \parallel & & \bar{f} \downarrow & & \bar{g} \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & N_\beta & \longrightarrow & L_\beta & \longrightarrow & 0 \end{array}$$

is commutative. Put $f = f_\beta \bar{f}$ and $g = g_\beta \bar{g}$, where $(f_\lambda, N_\alpha)_{\lambda < \alpha}$ and $(g_\lambda, L_\alpha)_{\lambda < \alpha}$ are the direct limits of the direct systems $(N_\lambda, f_{\lambda, \lambda'})_{\lambda' \leq \lambda < \alpha}$ and $(L_\lambda, g_{\lambda, \lambda'})_{\lambda' \leq \lambda < \alpha}$ respectively. By [79, 24.4, p. 199], the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N_\beta & \longrightarrow & L_\beta & \longrightarrow & 0 \\ & & \parallel & & f_\beta \downarrow & & g_\beta \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & N_\alpha & \longrightarrow & L_\alpha & \longrightarrow & 0 \end{array}$$

is commutative and this implies that the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \bar{N} & \longrightarrow & \bar{L} & \longrightarrow & 0 \\ & & \parallel & & f \downarrow & & g \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & N_\alpha & \longrightarrow & L_\alpha & \longrightarrow & 0 \end{array}$$

is commutative. Hence $\Sigma_\alpha : 0 \rightarrow M \rightarrow N_\alpha \rightarrow L_\alpha \rightarrow 0$ is a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$. Put $f_{\alpha, \beta} = \lim_{\rightarrow \beta \leq \beta' < \alpha} f_{\beta', \beta}$ and $g_{\alpha, \beta} = \lim_{\rightarrow \beta \leq \beta' < \alpha} g_{\beta', \beta}$, for all $\beta < \alpha$ (i.e., $f_{\alpha, \beta} = f_\beta$ and $g_{\alpha, \beta} = g_\beta$). By the definition of direct limit, $f_{\alpha, \beta} = f_{\alpha, \beta'} f_{\beta', \beta}$ and $g_{\alpha, \beta} = g_{\alpha, \beta'} g_{\beta', \beta}$ for all $\beta \leq \beta' < \alpha$. Thus $(\Sigma_{\beta'}, f_{\beta', \beta}, g_{\beta', \beta})_{\beta \leq \beta' < \alpha}$ is a direct system of generators for $\mathcal{P}ext_S(\mathcal{F}, M)$. Since $f_{\alpha, 0} = f_{\alpha, \beta} f_{\beta, 0}$ for all $\beta < \alpha$ it follows that $\ker(f_{\alpha, 0}) \supseteq \ker(f_{\beta, 0})$, and hence $\ker(f_{\alpha, 0}) \supsetneq \ker(f_{\beta, 0})$ for all $\beta < \alpha$. By induction, for each ordinal α we obtain a strictly increasing chain $(\ker(f_{\beta, 0}) \mid \beta < \alpha)$ of submodules of N , a contradiction. \square

Lemma 5.3.6 Let \mathcal{F} be a class of left R -modules closed under direct limits, let $S \subseteq R\text{-mod}$, let M be a left R -module and let $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ be a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$. Then there exists a generator $0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$ and a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & f \downarrow & & g \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0 \end{array}$$

such that $\ker(f') = 0$ in any commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \parallel & & f' \downarrow & & g' \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & N'' & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

whose rows are generators for $\mathcal{P}ext_S(\mathcal{F}, M)$.

PROOF: By Lemma 5.3.5, there exists a generator $0 \rightarrow M \rightarrow N_1 \rightarrow L_1 \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$ such that in any commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & M & \longrightarrow & N_1 & \longrightarrow & L_1 & \longrightarrow & 0 \\ & & \parallel & & \downarrow \bar{f} & & \downarrow \bar{g} & & \\ 0 & \longrightarrow & M & \longrightarrow & \bar{N} & \longrightarrow & \bar{L} & \longrightarrow & 0 \end{array}$$

whose rows are generators for $\mathcal{P}ext_S(\mathcal{F}, M)$, we have $\ker(\bar{f}f) = \ker(f)$. Let ω be the first infinite ordinal number. So, by induction on $n < \omega$, we can conclude from Lemma 5.3.5 that there is a countable direct system \mathcal{D} of generators $0 \rightarrow M \rightarrow N_n \rightarrow L_n \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$ with homomorphisms $f_{n+1,n} \in \text{Hom}_R(N_n, N_{n+1})$ and $g_{n+1,n} \in \text{Hom}_R(L_n, L_{n+1})$ such that the 0-th term of \mathcal{D} is the given generator $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$, the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N_n & \longrightarrow & L_n & \longrightarrow & 0 \\ & & \parallel & & \downarrow f_{n+1,n} & & \downarrow g_{n+1,n} & & \\ 0 & \longrightarrow & M & \longrightarrow & N_{n+1} & \longrightarrow & L_{n+1} & \longrightarrow & 0 \end{array}$$

is commutative, and for each commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N_n & \longrightarrow & L_n & \longrightarrow & 0 \\ & & \parallel & & \downarrow f_{n+1,n} & & \downarrow g_{n+1,n} & & \\ 0 & \longrightarrow & M & \longrightarrow & N_{n+1} & \longrightarrow & L_{n+1} & \longrightarrow & 0 \\ & & \parallel & & \downarrow \bar{f} & & \downarrow \bar{g} & & \\ 0 & \longrightarrow & M & \longrightarrow & \bar{N} & \longrightarrow & \bar{L} & \longrightarrow & 0 \end{array}$$

whose rows are generators for $\mathcal{P}ext_S(\mathcal{F}, M)$, we have that $\ker(f_{n+1,n}) = \ker(\bar{f}f_{n+1,n})$. Let $\Sigma : 0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0$ be the direct limit of \mathcal{D} . As in the proof of Lemma 5.3.5, we can prove that Σ is a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$. We claim that Σ has the desired property. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \parallel & & \downarrow f' & & \downarrow g' & & \\ 0 & \longrightarrow & M & \longrightarrow & N'' & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

whose rows are generators for $\mathcal{P}ext_S(\mathcal{F}, M)$. We will prove that f' is injective. Assume that f' is not injective, thus there is a non-zero element $x \in N'$ with $f'(x) = 0$. By [79, 24.3(2), p. 198], there is $m < \omega$ and an element $x_m \in N_m$ such that $f_m(x_m) = x$, where $(f_i, N^i)_{i < \omega}$ is the direct limit of a direct system $(N_i, f_{ij})_{j \leq i < \omega}$ where f_{ii} is the identity map. Thus $f'(x) = f' f_m(x_m) = 0$. Note that $f_m = f_{m+1} f_{m+1, m}$ (from the definition of direct limit). Since the following diagram:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & M & \longrightarrow & N_m & \longrightarrow & L_m & \longrightarrow & 0 \\
 & & \parallel & & \downarrow f_{m+1, m} & & \downarrow & & \\
 0 & \longrightarrow & M & \longrightarrow & N_{m+1} & \longrightarrow & L_{m+1} & \longrightarrow & 0 \\
 & & \parallel & & \downarrow f' f_{m+1} & & \downarrow & & \\
 0 & \longrightarrow & M & \longrightarrow & N'' & \longrightarrow & L'' & \longrightarrow & 0
 \end{array}$$

is commutative it follows that $\ker(f_{m+1, m}) = \ker(f' f_{m+1} f_{m+1, m})$. Since $x_m \in \ker(f' f_m) = \ker(f' f_{m+1} f_{m+1, m})$ it follows that $x_m \in \ker(f_{m+1, m})$. Thus $x = f_m(x_m) = (f_{m+1} f_{m+1, m})(x_m) = f_{m+1}(0) = 0$ and this is a contradiction. Hence f' is injective. \square

Lemma 5.3.7 Let \mathcal{F} be a class of left R -modules closed under direct limits, let $S \subseteq R\text{-mod}$ and let M be a left R -module. If $0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0$ is a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$ constructed as in Lemma 5.3.6, then it is a minimal generator.

PROOF: Assume that the assertion does not hold. Thus there is a commutative diagram:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0 \\
 & & \parallel & & \downarrow f' & & \downarrow g' & & \\
 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0
 \end{array}$$

with f' is injective, but not surjective. By induction, we will construct for any ordinal number α , a direct system $(\Sigma_\beta, f_{\beta, \beta'}, g_{\beta, \beta'})_{\beta' \leq \beta \leq \alpha}$ of generators $\Sigma_\beta : 0 \rightarrow M \rightarrow N_\beta \rightarrow L_\beta \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$ with injective, but not surjective, homomorphism $f_{\alpha, \beta} \in \text{Hom}_R(N_\beta, N_\alpha)$ ($\beta < \alpha$) and $f_{\beta, \beta'}, g_{\beta, \beta'}$ are identity maps as follows. First let $(0 \rightarrow M \rightarrow N_0 \rightarrow L_0 \rightarrow 0) = (0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0)$. Let α be any nonzero ordinal. Define $f_{\alpha+1, \alpha} = f'$ and $g_{\alpha+1, \alpha} = g'$.

If α is a non-limit ordinal, then define $(\Sigma_\alpha : 0 \rightarrow M \rightarrow N_\alpha \rightarrow L_\alpha \rightarrow 0) = (0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0)$, $f_{\alpha, \alpha-1} = f'$, $g_{\alpha, \alpha-1} = g'$, $f_{\alpha, \beta} = f_{\alpha, \beta+1} f_{\beta+1, \beta}$ and $g_{\alpha, \beta} = g_{\alpha, \beta+1} g_{\beta+1, \beta}$ for all $\beta < \alpha$. Since $f_{\alpha, \beta}$ is a composition of injective (not surjective) homomorphisms, $f_{\alpha, \beta}$ is

an injective, but not surjective, homomorphism. Therefore, using the induction hypothesis, (that each $f_{\beta,\beta'}$ is non surjective for $\beta' < \beta < \alpha$) we get that $\text{im}(f_{\alpha,0}) \subsetneq \text{im}(f_{\alpha,1}) \subsetneq \cdots \subsetneq \text{im}(f_{\alpha,\alpha-1}) \subsetneq N'$.

If α is a limit ordinal, suppose we have already constructed the direct system $(\Sigma_\beta, f_{\beta,\beta'}, g_{\beta,\beta'})_{\beta' \leq \beta < \alpha}$ of generators $\Sigma_\beta : 0 \rightarrow M \rightarrow N_\beta \rightarrow L_\beta \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$ with injective, but not surjective, homomorphisms $f_{\beta,\beta'} \in \text{Hom}_R(N_{\beta'}, N_\beta)$ ($\beta' < \beta < \alpha$). First take the direct limit $(\Sigma : 0 \rightarrow M \rightarrow N'_\alpha \rightarrow L'_\alpha \rightarrow 0) = (0 \rightarrow M \rightarrow \varinjlim N_\beta \rightarrow \varinjlim L_\beta \rightarrow 0)$, writing $\{f_\beta, N'_\alpha\}_{\beta < \alpha}$ for the direct limit of the direct system $\{N_\beta, f_{\beta,\beta'}\}_{\beta' \leq \beta < \alpha}$. Since $(0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0)$ is a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$ and Σ is an *S*-pure exact sequence, there are $f \in \text{Hom}_R(N'_\alpha, N')$ and $g \in \text{Hom}_R(L'_\alpha, L')$ such that the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N'_\alpha & \longrightarrow & L'_\alpha & \longrightarrow & 0 \\ & & \parallel & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0 \end{array}$$

is commutative. Define $(\Sigma_\alpha : 0 \rightarrow M \rightarrow N_\alpha \rightarrow L_\alpha \rightarrow 0) = (0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0)$ and let $f_{\alpha,\beta} = f' f f_\beta$ and $g_{\alpha,\beta} = g' g g_\beta$ for all $\beta < \alpha$. Since the ordinal $\beta + 1$ is a non-limit, $(\Sigma_{\beta+1} : 0 \rightarrow M \rightarrow N_{\beta+1} \rightarrow L_{\beta+1} \rightarrow 0) = (0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0)$. Since Σ_α is a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$ and the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N_{\beta+1} & \longrightarrow & L_{\beta+1} & \longrightarrow & 0 \\ & & \parallel & & \downarrow f_{\alpha,\beta+1} & & \downarrow g_{\alpha,\beta+1} & & \\ 0 & \longrightarrow & M & \longrightarrow & N_\alpha & \longrightarrow & L_\alpha & \longrightarrow & 0 \end{array}$$

is commutative it follows from Lemma 5.3.6 that $f_{\alpha,\beta+1}$ is injective. Since $f_{\alpha,\beta} = f_{\alpha,\beta+1} f_{\beta+1,\beta}$ and $f_{\beta+1,\beta}$ is injective it follows that $f_{\alpha,\beta}$ is injective, for all $\beta < \alpha$. Since $f_{\alpha,\beta} = f' f f_\beta$ and f' is not surjective (by assumption) it follows that $f_{\alpha,\beta}$ is not surjective, for all $\beta < \alpha$. Hence if α is a limit ordinal, then we constructed a direct system $(\Sigma_\beta, f_{\beta,\beta'}, g_{\beta,\beta'})_{\beta' \leq \beta \leq \alpha}$ of generators $\Sigma_\beta : 0 \rightarrow M \rightarrow N_\beta \rightarrow L_\beta \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$ with injective, but not surjective, homomorphisms $f_{\beta,\beta'} \in \text{Hom}_R(N_{\beta'}, N_\beta)$ ($\beta' < \beta \leq \alpha$) and $f_{\beta,\beta'} = f_{\beta,\beta''} f_{\beta'',\beta'}$ for any $\beta' < \beta'' < \beta \leq \alpha$. Consequently for any sequence of ordinal numbers: $\cdots < \mu < \lambda < \beta < \cdots < \alpha$ we have a chain of submodules of N' : $\cdots \subsetneq \text{im}(f_{\alpha,\mu}) \subsetneq \text{im}(f_{\alpha,\lambda}) \subsetneq \text{im}(f_{\alpha,\beta}) \subsetneq \cdots \subsetneq N'$ which, for some α , has more elements than N' and this is a contradiction. So f' is surjective and hence $0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0$ is a minimal generator for $\mathcal{P}ext_S(\mathcal{F}, M)$. \square

The following theorem generalizes Theorem 2.2.2 in [81, p. 31], if we take $S = \{R\}$.

Theorem 5.3.8 Let \mathcal{F} be a class of left R -modules closed under direct limits and let $S \subseteq R\text{-mod}$. Then for a left R -module M , if there is a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$, then there must be a minimal generator for $\mathcal{P}ext_S(\mathcal{F}, M)$.

PROOF: Suppose that there is a generator $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$. By Lemma 5.3.6, there exists a generator $\Sigma : 0 \rightarrow M \rightarrow N' \rightarrow L' \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0 \end{array}$$

such that $\ker(f') = 0$ in any commutative diagram whose rows are generator for $\mathcal{P}ext_S(\mathcal{F}, M)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \parallel & & \downarrow f' & & \downarrow g' & & \\ 0 & \longrightarrow & M & \longrightarrow & N'' & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

By Lemma 5.3.7, Σ is a minimal generator for $\mathcal{P}ext_S(\mathcal{F}, M)$. □

Proposition 5.3.9 Let $S \subseteq R\text{-mod}$, let \mathcal{F} be a class of left R -modules closed under S -pure extensions and let M be a left R -module. If $0 \rightarrow M \xrightarrow{f} K \rightarrow F \rightarrow 0$ is a minimal generator for $\mathcal{P}ext_S(\mathcal{F}, M)$, then $K \in \mathcal{F}^{\perp_S}$.

PROOF: Let $F' \in \mathcal{F}$ and let $\Sigma : 0 \rightarrow K \xrightarrow{h} N \rightarrow F' \rightarrow 0$ be any S -pure exact sequence of left R -modules. Using a pushout diagram, we have the following commutative diagram with

exact rows and columns.

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & M & \xrightarrow{f} & K & \longrightarrow & F \longrightarrow 0 \\
 & & \parallel & & \downarrow h & & \downarrow l \\
 0 & \longrightarrow & M & \xrightarrow{g} & N & \longrightarrow & P \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & F' & = & F' \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Since Σ is an S -pure exact sequence it follows from Lemma 2.2.17 that $0 \rightarrow F \xrightarrow{l} P \rightarrow F' \rightarrow 0$ is S -pure. Since $F, F' \in \mathcal{F}$ and \mathcal{F} is closed under S -pure extensions, $P \in \mathcal{F}$. Since f and h are S -pure monomorphisms, so is hf . Since $g = hf$, the exact sequence $0 \rightarrow M \xrightarrow{g} N \rightarrow P \rightarrow 0$ is S -pure. Since $0 \rightarrow M \xrightarrow{f} K \rightarrow F \rightarrow 0$ is a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$, there are homomorphisms h', l' such that the diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P \longrightarrow 0 \\
 & & \parallel & & \downarrow h' & & \downarrow l' \\
 0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & F \longrightarrow 0
 \end{array}$$

is commutative. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & F \longrightarrow 0 \\
 & & \parallel & & \downarrow h'h & & \downarrow l'l \\
 0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & F \longrightarrow 0
 \end{array}$$

Since $0 \rightarrow M \xrightarrow{f} K \rightarrow F \rightarrow 0$ is a minimal generator for $\mathcal{P}ext_S(\mathcal{F}, M)$ it follows that $h'h$ is an automorphism and hence $0 \rightarrow K \xrightarrow{h} N \rightarrow F' \rightarrow 0$ is split. Thus $\text{Pext}_S^1(\mathcal{F}, K) = 0$ and hence $K \in \mathcal{F}^{\perp_S}$. □

Theorem 5.3.10 Let $S \subseteq R\text{-mod}$ and let \mathcal{F} be a class of left R -modules closed under S -pure extensions and direct limits. For a given left R -module M , if there is a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$, then M has an \mathcal{F}^{\perp_S} -envelope.

PROOF: Suppose that there is a generator for $\mathcal{P}ext_S(\mathcal{F}, M)$. By Theorem 5.3.8, there is a minimal generator $\Sigma : 0 \rightarrow M \xrightarrow{f} N \rightarrow L \rightarrow 0$ for $\mathcal{P}ext_S(\mathcal{F}, M)$. By Proposition 5.3.9,

$N \in \mathcal{F}^{\perp_S}$. Let $K \in \mathcal{F}^{\perp_S}$ and let $g \in \text{Hom}_R(M, K)$. Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & N & \longrightarrow & L \longrightarrow 0 \\ & & \downarrow g & & & & \\ & & K & & & & \end{array}$$

By a pushout of the above diagram, we get the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & N & \longrightarrow & L \longrightarrow 0 \\ & & \downarrow g & & \downarrow g' & & \parallel \\ 0 & \longrightarrow & K & \xrightarrow{f'} & P & \longrightarrow & L \longrightarrow 0 \end{array}$$

Since the exact sequence $0 \rightarrow M \xrightarrow{f} N \rightarrow L \rightarrow 0$ is S -pure it follows from Lemma 2.2.17 that $0 \rightarrow K \xrightarrow{f'} P \rightarrow L \rightarrow 0$ is S -pure. Since $K \in \mathcal{F}^{\perp_S}$, the exact sequence $0 \rightarrow K \xrightarrow{f'} P \rightarrow L \rightarrow 0$ is split and hence there is a homomorphism $f'' : P \rightarrow K$ such that $f''f' = 1_K$. Put $h = f''g'$, thus $hf = f''g'f = f''f'g = g$. Thus $f : M \rightarrow N$ is an \mathcal{F}^{\perp_S} -preenvelope of M . Let $\alpha \in \text{End}_R(N)$ be such that $\alpha f = f$. By Lemma 2.1.3, there is $\beta \in \text{End}_R(L)$ such that the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & N & \longrightarrow & L \longrightarrow 0 \\ & & \parallel & & \downarrow \alpha & & \downarrow \beta \\ 0 & \longrightarrow & M & \xrightarrow{f} & N & \longrightarrow & L \longrightarrow 0 \end{array}$$

is commutative. By minimality of Σ , we have α is an automorphism and hence $f : M \rightarrow N$ is an \mathcal{F}^{\perp_S} -envelope of M . □

In the next theorem, we obtain a generalization of the result [27, Theorem 3.4, p. 697] due to Holm and Jørgensen (see Theorem 5.3.1) to the case of S -cotorsion pairs, which accomplishes our main purpose of this section.

Theorem 5.3.11 Let $S \subseteq R\text{-mod}$ and let \mathcal{F} be a class of left R -modules which contains $S \cup \{R\}$ and is closed under S -pure extensions, direct sums, pure submodules, and pure quotient modules. Then $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is a perfect S -cotorsion pair.

PROOF: Let $M \in S\text{-Pproj}$, thus $M \in \text{Add}(S \cup \{R\})$ (by Proposition 3.1.2(2)). Since $S \cup \{R\} \subseteq \mathcal{F}$ and \mathcal{F} is closed under direct sums and summands, $M \in \mathcal{F}$ and hence $S\text{-Pproj} \subseteq \mathcal{F}$. Since \mathcal{F} is closed under pure quotient modules and direct sums

(by hypothesis) it follows from Theorem 2.3.2 that \mathcal{F} is covering in $R\text{-Mod}$. Since \mathcal{F} is closed under S -pure extensions and direct summands (by hypothesis) it follows from Proposition 5.2.4 that $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is an S -cotorsion pair. By Proposition 5.1.12, \mathcal{F} is covering relative to \mathcal{P}_S . Since \mathcal{F} is closed under S -pure extensions, \mathcal{F} is special covering relative to \mathcal{P}_S in $R\text{-Mod}$ (by Lemma 5.1.14) and hence from Lemma 5.1.13 we have that every left R -module has a special \mathcal{F}^{\perp_S} -preenvelope relative to \mathcal{P}_S . We will prove that every left R -module has a \mathcal{F}^{\perp_S} -envelope relative to \mathcal{P}_S .

Let $M \in R\text{-Mod}$, thus M has a special \mathcal{F}^{\perp_S} -preenvelope relative to \mathcal{P}_S . Let $\Sigma : 0 \rightarrow M \xrightarrow{f} F \rightarrow C \rightarrow 0$ be a special \mathcal{F}^{\perp_S} -preenvelope of M relative to \mathcal{P}_S , thus Σ is an S -pure exact sequence with $F \in \mathcal{F}^{\perp_S}$ and $\text{coker}(f) = C \in {}^{\perp_S}(\mathcal{F}^{\perp_S})$. Since $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is an S -cotorsion pair, ${}^{\perp_S}(\mathcal{F}^{\perp_S}) = \mathcal{F}$ and hence $C \in \mathcal{F}$. Let $\Sigma' : 0 \rightarrow M \xrightarrow{\alpha} F' \rightarrow C' \rightarrow 0$ be any S -pure exact sequence with $C' \in \mathcal{F}$. By [37, Theorem 5.1, p. 372], the sequence $\text{Hom}_R(C', F) \rightarrow \text{Hom}_R(F', F) \xrightarrow{\alpha^{\circledast}} \text{Hom}_R(M, F) \rightarrow \text{Pext}_S^1(C', F)$ is exact, where α^{\circledast} is defined by $\alpha^{\circledast}(h) = h\alpha$ for all $h \in \text{Hom}_R(F', F)$. Since $C' \in \mathcal{F}$ and $F \in \mathcal{F}^{\perp_S}$ it follows that $\text{Pext}_S^1(C', F) = 0$ and hence the sequence $\text{Hom}_R(C', F) \rightarrow \text{Hom}_R(F', F) \xrightarrow{\alpha^{\circledast}} \text{Hom}_R(M, F) \rightarrow 0$ is exact. Since $f \in \text{Hom}_R(M, F)$, there is $g \in \text{Hom}_R(F', F)$ such that $g\alpha = f$. Thus, we get the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\alpha} & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \parallel & & \downarrow g & & \downarrow g' & & \\ 0 & \longrightarrow & M & \xrightarrow{f} & F & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

By Lemma 2.1.3, there is $g' \in \text{Hom}_R(C', C)$ such that the above diagram is commutative. Hence $\Sigma : 0 \rightarrow M \xrightarrow{f} F \rightarrow C \rightarrow 0$ is a generator for $\mathcal{P}\text{ext}_S(\mathcal{F}, M)$. Since \mathcal{F} is closed under direct sums and pure quotient modules, \mathcal{F} is closed under direct limits and hence Theorem 5.3.10 implies that M has an \mathcal{F}^{\perp_S} -envelope. Since $S\text{-Pinj} \subseteq \mathcal{F}^{\perp_S}$ and \mathcal{F}^{\perp_S} is closed under isomorphisms it follows from Proposition 5.1.11 that \mathcal{F}^{\perp_S} is enveloping relative to \mathcal{P}_S in $R\text{-Mod}$ and hence $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is a perfect S -cotorsion pair. \square

A duality pair $(\mathcal{F}, \mathcal{G})$ is said to be a **coproduct duality pair** if \mathcal{F} is closed under direct sums ([28]).

The following corollary is a generalization of [28, Theorem 3.1(c), p. 629] (see Theorem 5.3.2) to the case of S -cotorsion pairs.

Corollary 5.3.12 Let $S \subseteq R\text{-mod}$ and let $(\mathcal{F}, \mathcal{G})$ be a coproduct duality pair (or almost dual pair) such that \mathcal{F} contains $S \cup \{{}_R R\}$. If \mathcal{F} is closed under S -pure extensions, then $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is a perfect S -cotorsion pair.

PROOF: By [28, Theorem 3.1(a), p. 629], \mathcal{F} is closed under pure submodules and pure quotient modules. By Theorem 5.3.11, $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is a perfect S -cotorsion pair. \square

Remark 5.3.13 If we take $S = \{{}_R R\}$ in Theorem 5.3.11 (resp. Corollary 5.3.12) then we get [27, Theorem 3.4, p. 697] (resp. [28, Theorem 3.1(c), p. 629]).

Corollary 5.3.14 Let $S \subseteq R\text{-mod}$ and let $(\mathcal{F}, \mathcal{G})$ be a coproduct duality pair (or almost dual pair). If $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is an S -cotorsion pair, then it is perfect.

PROOF: Suppose that $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is an S -cotorsion pair. By Proposition 5.2.4, \mathcal{F} is closed under S -pure extensions and $S\text{-Pproj} \subseteq \mathcal{F}$. Since $S \cup \{{}_R R\} \subseteq S\text{-Pproj}$ it follows from Corollary 5.3.12 that $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is perfect. \square

5.4 Some examples and applications

In this section, we will present some examples and applications of some results in this chapter.

Let us begin by applying Corollary 5.3.12 to the following examples.

Example 5.4.1 Let $S \subseteq R\text{-mod}$, then $(R\text{-Mod}, S\text{-Pinj})$ is a perfect S -cotorsion pair. In particular, $S\text{-Pinj}$ is enveloping in $R\text{-Mod}$.

PROOF: It is clear that $S \cup \{{}_R R\} \subseteq R\text{-Mod}$, $R\text{-Mod}$ is closed under S -pure extensions and $(R\text{-Mod}, \text{Pinj}_R)$ is an almost dual pair. Thus, by Corollary 5.3.12, $(R\text{-Mod}, (R\text{-Mod})^{\perp_S})$ is a perfect S -cotorsion pair and hence $(R\text{-Mod}, S\text{-Pinj})$ is a perfect S -cotorsion pair. Thus every left R -module M has an $(S\text{-Pinj})$ -envelope, $\alpha : M \rightarrow M'$, with α is an S -pure monomorphism. If we take $S = \{{}_R R\}$, then we obtain the fact that $(R\text{-Mod}, {}_R\text{Inj})$ is a perfect cotorsion pair. \square

Example 5.4.2 Let $S \subseteq R\text{-mod}$ be such that containing $\{{}_R R\}$ and $S = \text{add} S$. Then $(\varinjlim S, (\varinjlim S)^{\perp_S})$ is a perfect S -cotorsion pair. In particular, $(\varinjlim S)^{\perp_S}$ is enveloping in $R\text{-Mod}$.

PROOF: By [26, Corollary 5.4(b), p. 556], $\varinjlim S$ is closed under S -pure extensions. Since $(\varinjlim S, \text{Prod} S^*)$ is an almost dual pair (by Corollary 4.2.25) and $S \cup \{{}_R R\} \subseteq \varinjlim S$ it follows from Corollary 5.3.12 that $(\varinjlim S, (\varinjlim S)^{\perp_S})$ is a perfect S -cotorsion pair. \square

Example 5.4.3 As special cases of Example 5.4.2, we have the following.

(1) If $S = \text{add}\{{}_R R\}$, then $\varinjlim S = {}_R \mathcal{F}\text{lat}$ and $(\varinjlim S)^{\perp_S} = ({}_R \mathcal{F}\text{lat})^{\perp} = {}_R \mathcal{C}\mathcal{T}$ (the class of cotorsion left R -modules as in [81, p. 52]). Hence $({}_R \mathcal{F}\text{lat}, {}_R \mathcal{C}\mathcal{T})$ is a perfect cotorsion pair.

(2) Let T be the class of $(1, 1)$ -presented left R -modules. If $S = \text{add} T$, then as in Example 4.2.28(3) we have $\varinjlim S = {}_R \mathcal{R}\mathcal{D}\text{-Flat}$. Thus $({}_R \mathcal{R}\mathcal{D}\text{-Flat}, ({}_R \mathcal{R}\mathcal{D}\text{-Flat})^{\perp_S})$ is a perfect S -cotorsion pair.

(3) Let R be a tame hereditary finite-dimensional algebra over a field k . Then from Example 4.2.28 and Example 5.4.2, we have:

(i) If $S = \text{add} M$, where $M \in R\text{-mod}$, then $(\text{Prod} S, (\text{Prod} S)^{\perp_S})$ is a perfect S -cotorsion pair.

(ii) If $S = \text{add}({}_R \mathbf{P})$, then $({}_R \mathcal{T}\mathcal{F}\text{ree}, ({}_R \mathcal{T}\mathcal{F}\text{ree})^{\perp_S})$ is a perfect S -cotorsion pair.

(iii) If $S = \text{add}({}_R \mathbf{P} \cup {}_R \mathbf{R})$, then $(\perp(\text{add} {}_R \mathbf{I}), (\perp(\text{add} {}_R \mathbf{I}))^{\perp_S})$ is a perfect S -cotorsion pair.

In the following corollary, we give a relation between cotorsion pairs and S -cotorsion pairs.

Corollary 5.4.4 Let $(\mathcal{F}, \mathcal{G})$ be a coproduct duality pair (or an almost dual pair). Then the following statements are equivalent.

- (1) \mathcal{F} is closed under extensions and ${}_R R \in \mathcal{F}$.
- (2) $(\mathcal{F}, \mathcal{F}^{\perp})$ is a perfect cotorsion pair.
- (3) $(\mathcal{F}, \mathcal{F}^{\perp})$ is a cotorsion pair.
- (4) $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is a perfect S -cotorsion pair, for all $S \subseteq \mathcal{F} \cap R\text{-mod}$.
- (5) $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is an S -cotorsion pair, for all $S \subseteq \mathcal{F} \cap R\text{-mod}$.

PROOF: (1) \Rightarrow (2) By hypothesis, $(\mathcal{F}, \mathcal{G})$ is a perfect duality pair and hence by Theorem 5.3.2 we have that $(\mathcal{F}, \mathcal{F}^\perp)$ is a perfect cotorsion pair.

(2) \Rightarrow (3) This is obvious.

(3) \Rightarrow (4) Suppose that $(\mathcal{F}, \mathcal{F}^\perp)$ is a cotorsion pair, thus \mathcal{F} is closed under extensions and ${}_R R \in \mathcal{F}$. Let $S \subseteq \mathcal{F} \cap R\text{-mod}$, thus \mathcal{F} is closed under S -pure extensions and $S \cup \{{}_R R\} \subseteq \mathcal{F}$. By Corollary 5.3.12, $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is a perfect S -cotorsion pair.

(4) \Rightarrow (5) This is obvious.

(5) \Rightarrow (1) Suppose that $(\mathcal{F}, \mathcal{F}^{\perp_S})$ is an S -cotorsion pair, for all $S \subseteq \mathcal{F} \cap R\text{-mod}$. By Proposition 5.2.4, $S\text{-Pproj} \subseteq \mathcal{F}$ for all $S \subseteq \mathcal{F} \cap R\text{-mod}$ and hence ${}_R R \in \mathcal{F}$. Let $S = \{{}_R R\}$, thus $(\mathcal{F}, \mathcal{F}^\perp)$ is a cotorsion pair (by (5)) and hence \mathcal{F} is closed under extensions. \square

Corollary 5.4.5 Let $T \subseteq R\text{-mod}$ be such that $T = \text{add} T$. Then $(\varinjlim T, (\varinjlim T)^\perp)$ is a cotorsion pair if and only if $(\varinjlim T, (\varinjlim T)^{\perp_S})$ is a perfect S -cotorsion pair, for all $S \subseteq T$.

PROOF: (\Rightarrow) Suppose that $(\varinjlim T, (\varinjlim T)^\perp)$ is a cotorsion pair. Since T is an additive subcategory of $R\text{-mod}$ it follows from [5, Lemma 1.2, p. 29] that $(\varinjlim T) \cap R\text{-mod} = T$. By Corollary 5.4.4, $(\varinjlim T, (\varinjlim T)^{\perp_S})$ is a perfect S -cotorsion pair, for all $S \subseteq (\varinjlim T) \cap R\text{-mod} = T$.

(\Leftarrow) Suppose that $(\varinjlim T, (\varinjlim T)^{\perp_S})$ is a perfect S -cotorsion pair, for all $S \subseteq T$. Since T is an additive subcategory of $R\text{-mod}$ it follows from [5, Lemma 1.2, p. 29] that $(\varinjlim T) \cap R\text{-mod} = T$. By hypothesis, $(\varinjlim T, (\varinjlim T)^{\perp_S})$ is a perfect S -cotorsion pair, for all $S \subseteq (\varinjlim T) \cap R\text{-mod}$. By Corollary 5.4.4, $(\varinjlim T, (\varinjlim T)^\perp)$ is a cotorsion pair. \square

Example 5.4.6 Let R be a tame hereditary artin algebra over a field k and let $T = \text{add}({}_R \mathbf{P} \cup {}_R \mathbf{R})$. By [4, Example 5.2(1), p. 308], $(\varinjlim T, (\varinjlim T)^\perp)$ is a perfect cotorsion pair. By Corollary 5.4.5, $(\varinjlim T, (\varinjlim T)^{\perp_S})$ is a perfect S -cotorsion pair, for all $S \subseteq T$.

Let \mathcal{P} and \mathcal{Q} be any proper classes of short exact sequences. Montaña in [46] introduced the concept of \mathcal{P} - \mathcal{Q} -flat modules as follows. A module M is said to be \mathcal{P} - \mathcal{Q} -flat if every exact sequence in \mathcal{P} ending at M belongs to \mathcal{Q} . We will use $\mathcal{P}\text{-Flat-}\mathcal{Q}$ to denote the class of all \mathcal{P} - \mathcal{Q} -flat modules.

Proposition 5.4.7 Let $S, T \subseteq R\text{-mod}$. Then $(\mathcal{P}_S\text{-Flat-}\mathcal{P}_T, (\mathcal{P}_S\text{-Flat-}\mathcal{P}_T)^{\perp_S})$ is a perfect S -cotorsion pair.

PROOF: By [46, Corollary 15.9, p. 65], $(\mathcal{P}_S\text{-Flat-}\mathcal{P}_T, (\mathcal{P}_S\text{-Flat-}\mathcal{P}_T)^{\perp_S})$ is an S -cotorsion pair. By Corollary 4.2.21, $\mathcal{P}_S\text{-Flat-}\mathcal{P}_T = T\text{-}(S \cup \{L_{0, 1 \times 1}\})\text{-Flat}$. By Theorem 4.2.24 we have $(\mathcal{P}_S\text{-Flat-}\mathcal{P}_T, D_{\mathcal{K}}\text{-}(S \cup \{L_{0, 1 \times 1}\})\text{-Pinj})$ is almost dual pair. By Corollary 5.3.14, $(\mathcal{P}_S\text{-Flat-}\mathcal{P}_T, (\mathcal{P}_S\text{-Flat-}\mathcal{P}_T)^{\perp_S})$ is a perfect S -cotorsion pair. \square

In the following examples we give special cases of Proposition 5.4.7.

Examples 5.4.8 (1) If $S = T$, then $(R\text{-Mod}, S\text{-Pinj})$ is a perfect S -cotorsion pair, since if $S = T$, then $\mathcal{P}_S\text{-Flat-}\mathcal{P}_T = R\text{-Mod}$ and $(R\text{-Mod})^{\perp_S} = S\text{-Pinj}$.

(2) If $S = \{R\}$ and $T = R\text{-mod}$, then $({}_R\mathcal{F}\text{lat}, {}_R\mathcal{C}\mathcal{T})$ is a perfect cotorsion pair.

(3) If $S = \{R\}$, then $(T\text{-}{}_R\mathcal{F}\mathcal{L}, (T\text{-}{}_R\mathcal{F}\mathcal{L})^{\perp})$ is a perfect cotorsion pair, where $T\text{-}{}_R\mathcal{F}\mathcal{L}$ is the class of T -flat left R -modules as in Example 4.2.22.

(4) If $S = \text{add}(S \cup \{L_{0, 1 \times 1}\})$ and $T = R\text{-mod}$, then $(\varinjlim S, (\varinjlim S)^{\perp_S})$ is a perfect S -cotorsion pair, since if $S = \text{add}(S \cup \{L_{0, 1 \times 1}\})$, then $\mathcal{P}_S\text{-Flat-}\mathcal{P}_T = \varinjlim S$; this follows from Lemma 2.2.5 and Corollary 4.2.21.

Let $\mathcal{F} \subseteq \text{Mod-}R$ and let $M \in \text{Mod-}R$. A **left \mathcal{F} -resolution** of M is a chain complex $\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ (not necessarily exact) with $F_i \in \mathcal{F}$ such that the sequence $\cdots \rightarrow \text{Hom}_R(G, F_1) \rightarrow \text{Hom}_R(G, F_0) \rightarrow \text{Hom}_R(G, M) \rightarrow 0$ is exact for each $G \in \mathcal{F}$ [19, p. 168].

Remarks 5.4.9 (1) Let $\mathcal{F} \subseteq \text{Mod-}R$ and let $M \in \text{Mod-}R$. If \mathcal{F} is precovering in $\text{Mod-}R$ then a left \mathcal{F} -resolution of M exists [19, Proposition 8.1.3, p. 168] and it is unique up to homotopy [19, p. 169].

(2) Let $S \in \text{mod-}R$. It is clear from Proposition 3.1.2 that every right R -module has an $(S\text{-Pproj})$ -precover and hence from (1) above we have that every right R -module has a left $(S\text{-Pproj})$ -resolution and it is unique up to homotopy.

Definition 5.4.10 Let $N \in \text{Mod-}R$, let $M \in R\text{-Mod}$ and let $S \subseteq \text{mod-}R$. Let $\Sigma : \cdots \rightarrow F_2 \xrightarrow{d_2}$

$F_1 \xrightarrow{d_1} F_0 \xrightarrow{\alpha} N \rightarrow 0$ be a left $(S\text{-Pproj})$ -resolution of N and let Σ_0 be the deleted $(S\text{-Pproj})$ -resolution of N (i.e., $\Sigma_0 : \cdots \rightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0$). Define $\text{Tor}_n^S(N, M)$ to be the homology group:

$$\text{Tor}_n^S(N, M) = H_n(\Sigma_0 \otimes_R M) = \ker(d_n \otimes_R 1_M) / \text{im}(d_{n+1} \otimes_R 1_M), \quad n = 1, 2, \dots$$

By Remark 5.4.9(2) this is well-defined.

Let $S \subseteq \text{mod-}R$ and let $\mathcal{L} \subseteq \text{Mod-}R$. We will define the classes $\mathcal{L}\text{-}S\text{-}\overline{\text{Flat}}$ and $\mathcal{L}\text{-}S\text{-}\overline{\text{Inj}}$ as follows:

$$\mathcal{L}\text{-}S\text{-}\overline{\text{Flat}} = \{M \in R\text{-Mod} \mid \text{Tor}_1^S(L, M) = 0 \text{ for all } L \in \mathcal{L}\} \text{ and}$$

$$\mathcal{L}\text{-}S\text{-}\overline{\text{Inj}} = \{N \in \text{Mod-}R \mid \text{Pext}_S^1(L, N) = 0 \text{ for all } L \in \mathcal{L}\}.$$

Lemma 5.4.11 Let $S \subseteq \text{mod-}R$ and let $\{M_i\}_{i \in I}$ be a family of right R -modules. If $\Sigma_i : 0 \rightarrow M_i \xrightarrow{\alpha_i} A_i^0 \xrightarrow{\alpha_i^0} A_i^1 \xrightarrow{\alpha_i^1} A_i^2 \rightarrow \cdots$ is a right $(S\text{-Pinj})$ -resolution of M_i for all $i \in I$, then the sequence $\Sigma : 0 \rightarrow \prod_{i \in I} M_i \xrightarrow{\prod \alpha_i} \prod_{i \in I} A_i^0 \xrightarrow{\prod \alpha_i^0} \prod_{i \in I} A_i^1 \rightarrow \cdots$ is a right $(S\text{-Pinj})$ -resolution of $\prod_{i \in I} M_i$.

PROOF: Suppose that for each $i \in I$, the sequence Σ_i is a right $(S\text{-Pinj})$ -resolution of M_i . By Lemma 4.5.18, Σ_i is an S -pure exact sequence with $A_i^n \in S\text{-Pinj}$ for all $n \geq 0$ and hence the sequence $\Sigma : 0 \rightarrow \prod_{i \in I} M_i \xrightarrow{\prod \alpha_i} \prod_{i \in I} A_i^0 \xrightarrow{\prod \alpha_i^0} \prod_{i \in I} A_i^1 \rightarrow \cdots$ is an exact sequence with $\prod_{i \in I} A_i^n \in S\text{-Pinj}$ for all $n \geq 0$. Since Σ_i is an S -pure exact sequence, the exact sequence $0 \rightarrow \text{im}(\alpha_i^{n-1}) \rightarrow A_i^n \rightarrow \text{im}(\alpha_i^n) \rightarrow 0$ is S -pure for all $n \geq 0$. By Lemma 4.2.23, the exact sequence $0 \rightarrow \prod_{i \in I} \text{im}(\alpha_i^{n-1}) \rightarrow \prod_{i \in I} A_i^n \rightarrow \prod_{i \in I} \text{im}(\alpha_i^n) \rightarrow 0$ is S -pure and it follows from [31, Lemma 4.3.2, p. 86] that the sequence $0 \rightarrow \text{im}(\prod_{i \in I} \alpha_i^{n-1}) \rightarrow \prod_{i \in I} A_i^n \rightarrow \text{im}(\prod_{i \in I} \alpha_i^n) \rightarrow 0$ is S -pure exact. Thus the sequence Σ is S -pure exact with $\prod_{i \in I} A_i^n \in S\text{-Pinj}$ for all $n \geq 0$. By Lemma 4.5.18, Σ is a right $(S\text{-Pinj})$ -resolution of $\prod_{i \in I} M_i$. \square

Lemma 5.4.12 Let $S \subseteq \text{mod-}R$, let $\{M_i\}_{i \in I}$ be a family of right R -modules and let $M \in \text{Mod-}R$. Then

$$\text{Pext}_S^n(M, \prod_{i \in I} M_i) \simeq \prod_{i \in I} \text{Pext}_S^n(M, M_i), \text{ for any } n \geq 0.$$

PROOF: By Remark 4.5.12(2), each M_i has a right $(S\text{-Pinj})$ -resolution $\Sigma_i : 0 \rightarrow M_i \rightarrow$

$A_i^0 \rightarrow A_i^1 \rightarrow A_i^2 \rightarrow \dots$. By Lemma 5.4.11, the sequence $0 \rightarrow \prod_{i \in I} M_i \rightarrow \prod_{i \in I} A_i^0 \rightarrow \prod_{i \in I} A_i^1 \rightarrow \prod_{i \in I} A_i^2 \rightarrow \dots$ is a right $(S\text{-Pinj})$ -resolution of $\prod_{i \in I} M_i$. By [62, Example 6.6(i), p. 328], the sequence $0 \rightarrow \text{Hom}_R(N, \prod_{i \in I} A_i^0) \rightarrow \text{Hom}_R(N, \prod_{i \in I} A_i^1) \rightarrow \text{Hom}_R(N, \prod_{i \in I} A_i^2) \rightarrow \dots$ is a cochain complex. Since $\text{Hom}_R(N, \prod_{i \in I} A_i^j) \simeq \prod_{i \in I} \text{Hom}_R(N, A_i^j)$ for all $j \geq 0$ (by Lemma 2.1.5(1)) it follows that we have the following commutative diagram with isomorphic cochain complex rows where α_j are isomorphisms $j \geq 0$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(N, \prod_{i \in I} A_i^0) & \longrightarrow & \text{Hom}_R(N, \prod_{i \in I} A_i^1) & \longrightarrow & \text{Hom}_R(N, \prod_{i \in I} A_i^2) \longrightarrow \dots \\ & & \alpha_0 \downarrow & & \alpha_1 \downarrow & & \alpha_2 \downarrow \\ 0 & \longrightarrow & \prod_{i \in I} \text{Hom}_R(N, A_i^0) & \longrightarrow & \prod_{i \in I} \text{Hom}_R(N, A_i^1) & \longrightarrow & \prod_{i \in I} \text{Hom}_R(N, A_i^2) \longrightarrow \dots \end{array}$$

Since isomorphic cochain complexes have the same cohomology (by [62, Exercise 6.2, p. 338]) it follows that $\text{Pext}_S^n(N, \prod_{i \in I} M_i) \simeq H^n(\prod_{i \in I} \text{Hom}_R(N, \Sigma'_i))$, where Σ'_i is the deleted $(S\text{-Pinj})$ -resolution of M_i . Since direct product commutes with cohomology (by [78, 1.2.1, p. 5]) it follows that $H^n(\prod_{i \in I} \text{Hom}_R(N, \Sigma'_i)) \simeq \prod_{i \in I} H^n(\text{Hom}_R(N, \Sigma'_i))$. Since $\prod_{i \in I} H^n(\text{Hom}_R(N, \Sigma'_i)) = \prod_{i \in I} \text{Pext}_S^n(N, M_i)$ it follows that $\text{Pext}_S^n(N, \prod_{i \in I} M_i) \simeq \prod_{i \in I} \text{Pext}_S^n(N, M_i)$, for any $n \geq 0$. \square

In the following proposition, we will induce almost dual pairs by using the functors $\text{Tor}_1^S(-, -)$ and $\text{Pext}_S^1(-, -)$. This covers many well-known examples of almost dual pairs (see for example, Examples 4.2.5(2,3)).

Proposition 5.4.13 Let $S \subseteq \text{mod-}R$ and let $\mathcal{L} \subseteq \text{Mod-}R$. Then:

- (1) $(\mathcal{L}\text{-}S\text{-}\overline{\text{Flat}}, \mathcal{L}\text{-}S\text{-}\overline{\text{Inj}})$ is an almost dual pair.
- (2) $\mathcal{L}\text{-}S\text{-}\overline{\text{Inj}}$ is closed under S -pure extensions.

PROOF: (1) Let $M \in R\text{-Mod}$. For each $L \in \mathcal{L}$, we have $M \in \mathcal{L}\text{-}S\text{-}\overline{\text{Flat}}$ if and only if $\text{Tor}_1^S(L, M) = 0$ if and only if $(\text{Tor}_1^S(L, M))^* = 0$ if and only if $\text{Pext}_S^1(L, M^*) = 0$ (by [67, Proposition 2.7, p. 106]) if and only if $M^* \in \mathcal{L}\text{-}S\text{-}\overline{\text{Inj}}$. Now we will prove that $\mathcal{L}\text{-}S\text{-}\overline{\text{Inj}}$ is closed under direct summands and products. Let $\{M_i\}_{i \in I}$ be any family of right R -modules. Then for each $i \in I$ and $L \in \mathcal{L}$, we have $M_i \in \mathcal{L}\text{-}S\text{-}\overline{\text{Inj}}$ if and only if $\text{Pext}_S^1(L, M_i) = 0$ if and only if $\prod_{i \in I} \text{Pext}_S^1(L, M_i) = 0$ if and only if $\text{Pext}_S^1(L, \prod_{i \in I} M_i) = 0$ (by Lemma 5.4.12) if and only if $\prod_{i \in I} M_i \in \mathcal{L}\text{-}S\text{-}\overline{\text{Inj}}$. Thus $\mathcal{L}\text{-}S\text{-}\overline{\text{Inj}}$ is closed under direct summands and products and hence $(\mathcal{L}\text{-}S\text{-}\overline{\text{Flat}}, \mathcal{L}\text{-}S\text{-}\overline{\text{Inj}})$ is an almost dual pair.

(2) Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an S -pure exact sequence of right R -modules with $A, C \in \mathcal{L}\text{-}S\text{-}\overline{\text{Inj}}$. Thus $\text{Pext}_S^1(L, A) = \text{Pext}_S^1(L, C) = 0$, for all $L \in \mathcal{L}$. Let $L \in \mathcal{L}$, thus by [37, Theorem 5.1, p. 372], there is an exact sequence $\text{Pext}_S^1(L, A) \rightarrow \text{Pext}_S^1(L, B) \rightarrow \text{Pext}_S^1(L, C)$ and hence the sequence $0 \rightarrow \text{Pext}_S^1(L, B) \rightarrow 0$ is exact and this implies that $\text{Pext}_S^1(L, B) = 0$. Thus $B \in \mathcal{L}\text{-}S\text{-}\overline{\text{Inj}}$ and hence $\mathcal{L}\text{-}S\text{-}\overline{\text{Inj}}$ is closed under S -pure extensions. \square

The following corollary is immediate from Proposition 4.2.8.

Corollary 5.4.14 Let $S \subseteq \text{mod-}R$ and let $\mathcal{L} \subseteq \text{Mod-}R$. Then $\mathcal{L}\text{-}S\text{-}\overline{\text{Flat}}$ is covering in $R\text{-Mod}$ and it is closed under pure submodules, pure homomorphic images, pure extensions, direct sums and direct limits.

Corollary 5.4.15 Let $S \subseteq R\text{-mod}$. Then $\varinjlim(\text{add}(S \cup \{R\mathcal{R}\})) = (\text{Mod-}R)\text{-}D_{\mathcal{H}}\text{-}\overline{\text{Flat}}$.

PROOF: By Corollary 4.2.25, $(\varinjlim(\text{add}(S \cup \{R\mathcal{R}\})), D_{\mathcal{H}}\text{-Pinj})$ is an almost dual pair. Also, from Proposition 5.4.13(1) we have $((\text{Mod-}R)\text{-}D_{\mathcal{H}}\text{-}\overline{\text{Flat}}, D_{\mathcal{H}}\text{-Pinj})$ is an almost dual pair. Thus $\varinjlim(\text{add}(S \cup \{R\mathcal{R}\})) = (\text{Mod-}R)\text{-}D_{\mathcal{H}}\text{-}\overline{\text{Flat}}$, by Proposition 4.2.11. \square

Proposition 5.4.16 Let $S \subseteq R\text{-mod}$ and let $\mathcal{L} \subseteq \text{Mod-}R$. Then $(\mathcal{L}\text{-}D_{\mathcal{H}}\text{-}\overline{\text{Flat}}, (\mathcal{L}\text{-}D_{\mathcal{H}}\text{-}\overline{\text{Flat}})^{\perp S})$ is a perfect S -cotorsion pair.

PROOF: Put $\mathcal{F} = \mathcal{L}\text{-}D_{\mathcal{H}}\text{-}\overline{\text{Flat}}$ and $\mathcal{G} = \mathcal{L}\text{-}D_{\mathcal{H}}\text{-}\overline{\text{Inj}}$. First we will prove that $S \cup \{R\mathcal{R}\} \subseteq \mathcal{F}$. Let $M \in S \cup \{R\mathcal{R}\}$, thus $M^* \in (S \cup \{R\mathcal{R}\})^*$ and hence M^* is $D_{\mathcal{H}}$ -pure-injective. Thus $\text{Pext}_{D_{\mathcal{H}}}^1(L, M^*) = 0$, for all $L \in \mathcal{L}$ (by Theorem 2.2.19) and hence $M^* \in \mathcal{G}$. By Proposition 5.4.13(1), $M \in \mathcal{F}$ and hence $S \cup \{R\mathcal{R}\} \subseteq \mathcal{F}$. Now we will prove that \mathcal{F} is closed under S -pure extensions.

Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an S -pure exact sequence of left R -modules with $A, C \in \mathcal{F}$. By Theorem 3.1.1, the exact sequence $0 \rightarrow C^* \rightarrow B^* \rightarrow A^* \rightarrow 0$ is $D_{\mathcal{H}}$ -pure. By Proposition 5.4.13(1), $A^*, C^* \in \mathcal{G}$ and hence from Proposition 5.4.13(2) we have $B^* \in \mathcal{G}$ and this implies that $B \in \mathcal{F}$. Thus \mathcal{F} is closed under S -pure extensions. Since $(\mathcal{F}, \mathcal{G})$ is an almost dual pair (by Proposition 5.4.13(1)) it follows from Corollary 5.3.12 that $(\mathcal{F}, \mathcal{F}^{\perp S})$ is a perfect S -cotorsion pair. \square

Corollary 5.4.17 Let $\mathcal{L} \subseteq \text{Mod-}R$. Then $(\mathcal{L}\text{-}_R\mathcal{F}\text{lat}, (\mathcal{L}\text{-}_R\mathcal{F}\text{lat})^\perp)$ is a perfect cotorsion pair, where $\mathcal{L}\text{-}_R\mathcal{F}\text{lat} = \{M \in R\text{-Mod} \mid \text{Tor}_1^R(L, M) = 0 \text{ for all } L \in \mathcal{L}\}$.

PROOF: Take $\mathcal{H} = \left\{ \begin{smallmatrix} 0 \\ 1 \times 1 \end{smallmatrix} \right\}$ and let $S = L_{\mathcal{H}} = \{R_R\}$. Thus $D_{\mathcal{H}} = \{R_R\}$ and hence $\text{Tor}_1^{D_{\mathcal{H}}}(-, -) = \text{Tor}_1^R(-, -)$ and $\text{Pext}_S^1(-, -) = \text{Ext}_R^1(-, -)$. Thus $\mathcal{L}\text{-}D_{\mathcal{H}}\overline{\text{Flat}} = \mathcal{L}\text{-}_R\mathcal{F}\text{lat}$ and $(\mathcal{L}\text{-}D_{\mathcal{H}}\overline{\text{Flat}})^{\perp_s} = (\mathcal{L}\text{-}_R\mathcal{F}\text{lat})^\perp$. By Proposition 5.4.16, $(\mathcal{L}\text{-}_R\mathcal{F}\text{lat}, (\mathcal{L}\text{-}_R\mathcal{F}\text{lat})^\perp)$ is a perfect cotorsion pair. \square

In the following examples we give some corollaries of Proposition 5.4.16 and Corollary 5.4.17.

Examples 5.4.18 (1) Let $S \subseteq R\text{-mod}$. If $\mathcal{L} = \text{Mod-}R$, then from Corollary 5.4.15 and Proposition 5.4.16 we have that $(\varinjlim(\text{add}(S \cup \{R_R\})), (\varinjlim(\text{add}(S \cup \{R_R\})))^{\perp_s})$ is a perfect S -cotorsion pair.

(2) Let $m, n \in \mathbb{Z}^+$. If we take $\mathcal{L} = \{M \mid M \text{ is an } (m, n)\text{-presented right } R\text{-module}\}$ and apply Corollary 5.4.17, then we get that $((m, n)\text{-}_R\mathcal{F}\text{lat}, ((m, n)\text{-}_R\mathcal{F}\text{lat})^\perp)$ is a perfect cotorsion pair, and this is the left version of [40, Theorem 2.3(2), p. 2533].

(3) Let $n \in \mathbb{Z}^+$. If we take $\mathcal{L} = \{M \mid M \text{ is a finitely presented right } R\text{-module with } \text{pd}_R(M) \leq n\}$, where $\text{pd}_R(M)$ is the projective dimension of M , and apply Corollary 5.4.17, then we get the left version of [82, Theorem 2.1(1), p. 361].

(4) If we take $\mathcal{L} = \{M \mid M \simeq R/I \text{ with } I \text{ a simple right ideal of } R\}$ and apply Corollary 5.4.17, then we get that $(\text{Min-}_R\mathcal{F}\text{lat}, (\text{Min-}_R\mathcal{F}\text{lat})^\perp)$ is a perfect cotorsion pair, and this is the left version of [38, Theorem 3.4(2), p. 638].

Bibliography

- [1] W. Al-Kawarit and F. Couchot, Comparison of some purities, flatnesses and injectivities, *Comm. Algebra*, 39(10) (2011), 3879-3896.
- [2] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Grad. Texts in Math., Vol. 13, Springer, 1974.
- [3] L. Angeleri Hügel, D. Herbera and J. Trlifaj, Baer and Mittag-Leffler modules over tame hereditary algebras, *Math. Z.*, 265 (2010), 1-19.
- [4] L. Angeleri Hügel, J. Šaroch and J. Trlifaj, On the telescope conjecture for module categories, *J. Pure Appl. Algebra*, 212(2) (2008), 297-310.
- [5] L. Angeleri Hügel and J. Trlifaj, Direct limits of modules of finite projective dimension, pp. 27-44 in A. Facchini et al. (Eds), *Rings, Modules, Algebras and Abelian Groups*, Dekker, 2004.
- [6] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras. 1: Techniques of Representation Theory*, London Math. Soc. Student Texts, Vol. 65, Cambridge University Press, 2006.
- [7] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics, Vol. 36, Cambridge University Press, 1995.
- [8] G. Azumaya, Countable generatedness version of rings of pure global dimension zero, pp. 43-79 in: H. Tachikawa, S. Brenner (Eds.), *Representations of Algebras and Related Topics*, Kyoto, 1990, London Mathematical Society, Lecture Note Series, vol. 168, Cambridge University Press, Cambridge, 1992.
- [9] D. Bennis, n - \mathcal{X} -Coherent rings, *Int. Elec. J. Algebra*, 7 (2010), 128-139.

- [10] K. Burke, Some model-theoretic properties of functor categories for modules, Doctoral Thesis, University of Manchester, 1994.
- [11] S. U. Chase, Direct products of modules, *Trans. Amer. Math. Soc.*, 97(3) (1960), 457-473.
- [12] T. J. Cheatham and D. Stone, Flat and projective character modules, *Proc. Amer. Math. Soc.* 81 (1981), 175-177.
- [13] P. M. Cohn, On the free product of associative rings, *Math. Z.*, 71(1) (1959), 380-398.
- [14] F. Couchot, *RD*-flatness and *RD*-injectivity, *Comm. Algebra*, 34(10) (2006), 3675-3689.
- [15] W. Crawley-Boevey, Locally finitely presented additive categories. *Comm. Algebra* 22(5) (1994), 1641-1674.
- [16] W. Crawley-Boevey, Infinite-dimensional modules in the representation theory of finite-dimensional algebras, pp. 29-54 in I. Reiten, S. Smalø and Ø. Solberg (Eds.), *Algebras and Modules I*, Canadian Math. Soc. Conf. Proc., Vol. 23, Amer. Math. Soc, 1998.
- [17] P. Eklof and G. Sabbagh, Model-completions and modules, *Ann. Math. Logic*, 2(3) (1971), 251-295.
- [18] E. E. Enochs and Z. Y. Huang, Injective envelopes and (Gorenstein) flat covers, *Algebr. Represent. Theor.*, 15 (2012), 1131-1145.
- [19] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, de Gruyter Expositions in Mathematics 30, Walter de Gruyter and Co. Berlin, 2000.
- [20] A. Facchini, *Module Theory: Endomorphism Rings and Direct Sum Decompositions in some Classes of Modules*, Progress in Math., Vol. 167, Birkhäuser, Basel, 1998.
- [21] T. H. Fay and S. V. Joubert, Relative injectivity, *Chinese J. Math.*, 22(1) (1994), 65-94.
- [22] L. Fuchs and L. Salce, *Modules over Non-Noetherian Domains*, Math. Surveys and Monographs, vol. 84, Amer. Math. Soc., 2001.

- [23] J. S. Golan, Characterization of rings using quasiprojective modules. III, *Proc. Amer. Math. Soc.*, 31 (1972), 401-408.
- [24] I. Herzog, Elementary duality of modules, *Trans. Amer. Math. Soc.*, 340 (1) (1993), 37-69.
- [25] V. A. Hiremath, Copure-injective modules, *Indian J. pure appl. Math.*, 20(3) (1989), 250-259.
- [26] H. Holm, Modules with cosupport and injective functors, *Algebr Represent Theor*, 13 (2010), 543-560.
- [27] H. Holm and P. Jørgensen, Covers, precovers, and purity, *Illinois J. Math.*, 52(2) (2008), 691-703.
- [28] H. Holm and P. Jørgensen, Cotorsion pairs induced by duality pairs, *J. Commut. Algebra*, 1(4) (2009), 621-633.
- [29] M. Hovey, Cotorsion pairs, model category structures, and representation theory, *Math. Z.*, 241 (2002), 553-592.
- [30] S. K. Jain and S. R. López-Permouth, A survey on the theory of weakly injective modules, pp. 205-232 in: *Computational Algebra, Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, Inc., New York, 1994.
- [31] F. Kasch, *Modules and Rings*, Academic Press, London, New York, 1982.
- [32] H. Krause, *The Spectrum of a Module Category*, Habilitationsschrift, Universität Bielefeld, 1997, published as *Mem. Amer. Math. Soc.*, No. 707, 2001.
- [33] H. Krause and O. Solberg, Applications of cotorsion pairs, *J. London Math. Soc.*, 68(3) (2003), 631-650.
- [34] J. Lambek, A module is flat if and only if its character module is injective, *Canad. Math. Bull.*, 7 (1964), 237-243.
- [35] D. Lazard, Sur les modules plats, *C. R. Acad. Sci. Paris*, 258 (1964) 6313-6316.

- [36] H. Lenzing, Homological transfer from finitely presented to infinite modules, pp. 734-761 in Abelian Group Theory, Lecture Notes in Math., Vol. 1006, Springer-Verlag, 1983.
- [37] S. Mac Lane, Homology, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
- [38] L. Mao, Min-flat modules and min-coherent rings, *Comm. Algebra*, 35(2) (2007), 635-650.
- [39] L. Mao, Properties of RD -projective and RD -injective modules, *Turk. J. Math.*, 34 (2010), 1-19.
- [40] L. Mao and N. Ding, On relative injective modules and relative coherent rings, *Comm. Algebra*, 34(7) (2006), 2531-2545.
- [41] L. Mao and N. Ding, FI -injective and FI -flat modules, *J. Algebra*, 309 (2007), 367-385.
- [42] L. Mao and N. Ding, On divisible and torsionfree modules, *Comm. Algebra*, 36(2) (2008), 708-731.
- [43] C. Megibben, Absolutely pure modules, *Proc. Amer. Math. Soc.*, 26 (1970), 561-566.
- [44] A. R. Mehdi, Purity relative to classes of finitely presented modules, *J. Algebra Apps.*, to appear.
- [45] A. R. Mehdi and M. Prest, Almost dual pairs and definable classes of modules, <http://arxiv.org/abs/1304.4481v1>, 2013.
- [46] C. F. Montaña, Proper classes of short exact sequences and structure theory of modules, Doctoral Thesis, Düsseldorf, 2010.
- [47] W. K. Nicholson and M. F. Yousif, Quasi-Frobenius Rings, Cambridge Tracts in Mathematics, 158, Cambridge University Press, Cambridge, 2003.
- [48] F. Okoh, Separable modules over finite-dimensional algebras, *J. Algebra*, 116 (1988), 400-414.
- [49] B. L. Osofsky, Rings all of whose finitely generated modules are injective, *Pacific J. Math.*, 14 (1964), 645-650.

- [50] K. Pinzon, Absolutely Pure Modules, Doctoral Thesis, University of Kentucky, 2005.
- [51] K. Pinzon, Absolutely pure covers, *Comm. Algebra*, 36(6) (2008), 2186-2194.
- [52] M. Prest, *Purity, Spectra and Localisation*, Cambridge University Press, 2009.
- [53] M. Prest and V. Puninskaya, Modules with few types over some finite dimensional algebras, *J. Symbolic Logic*, 67(2) (2002), 841-858.
- [54] M. Prest, Ph. Rothmaler and M. Ziegler, Absolutely pure and flat modules and "indiscrete" rings, *J. Algebra*, 174(2) (1995), 349-372.
- [55] G. Puninski, Some model theory over an exceptional uniserial ring and decompositions of serial modules, *J. London Math. Soc. (2)*, 64(2) (2001), 311-326.
- [56] G. Puninski, M. Prest and Ph. Rothmaler, Rings described by various purities, *Comm. Algebra*, 27(5) (1999), 2127-2162.
- [57] J. Rada and M. Saorín, Rings characterized by (pre)envelopes and (pre)covers of their modules, *Comm. Algebra* 26 (1998), 899-912.
- [58] C. M. Ringel, Infinite-dimensional representations of finite-dimensional hereditary algebras, *Symp. Math. inst. Naz. Alta. Mat.*, 23 (1979), 321-412.
- [59] C. M. Ringel, The Ziegler spectrum of a tame hereditary algebra, *Colloq. Math.*, 76 (1998), 105-115.
- [60] C. M. Ringel, Infinite length modules: some examples as introduction, pp. 1-73 in: H. Krause and C. M. Ringel (Eds.), *Infinite Length Modules*, Birkhäuser, 2000.
- [61] Ph. Rothmaler, Torsion-free, divisible, and Mittag-Leffler modules, <http://arxiv.org/abs/1301.0961v1>, 2013.
- [62] J. J. Rotman, *An Introduction to Homological Algebra*, Springer, 2009.
- [63] G. Sabbagh and P. Eklof, Definability problems for rings and modules, *J. Symbolic Logic*, 36(4) (1971), 623-649.
- [64] L. Salce, Cotorsion theories for abelian groups, *Symposia Math.*, 23 (1979), Academic Press, 11-32.

- [65] R. T. Shannon, The rank of a flat module, *Proc. Amer. Math. Soc.*, 24(3) (1970), 452-456.
- [66] G. E. Simons, Varieties of rings with definable principal congruences, *Proc. Amer. Math. Soc.*, 87(3) (1983), 397-402.
- [67] E. G. Sklyarenko, Relative homological algebra in the category of modules, *Russ. Math. Surv.*, 33(3)(1978), 97-137.
- [68] X. Song and J. Chen, Notes on pure projective modules, *J. South. Univ.*, 21(4) (2005), 506-508.
- [69] J. T. Stafford, Stable structure of noncommutative Noetherian rings, *J. Algebra*, 47(2) (1977), 244-267.
- [70] B. Stenström, Pure submodules, *Arkiv Mat.*, 7(2) (1967), 159-171.
- [71] B. Stenström, Coherent rings and FP-injective modules, *J. London Math. Soc.*, 2 (1970), 323-329.
- [72] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, 1975.
- [73] R. Ware, Endomorphism rings of projective modules, *Tran. Amer. Math. Soc.*, 155(1) (1971), 233-256.
- [74] R. B. Warfield Jr., Purity and algebraic compactness for modules, *Pacific J. Math.*, 28(3) (1969), 699-719.
- [75] R. B. Warfield Jr., Relatively injective modules, unpublished manuscript, 1969.
- [76] R. B. Warfield Jr., Decomposability of finitely presented modules, *Proc. Amer. Math. Soc.*, 25(1) (1970), 167-172.
- [77] R. B. Warfield Jr., Serial rings and finitely presented modules, *J. Algebra*, 37(2) (1975), 187-222.
- [78] Ch. A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994.

- [79] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia, 1991.
- [80] T. Würfel, Über absolut reine Ringe, *J. Reine Angew. Math.*, 262-263 (1973), 381-391.
- [81] J. Xu, *Flat Covers of Modules*, Lecture Notes in Mathematics, Vol. 1634, Springer, 1996.
- [82] X. Yang and Z. Liu, n -flat and n -FP-injective modules, *Czechoslovak Math. J.*, 61(136) (2011), 359-369.
- [83] M. Zayed, On \underline{C} -pure-injective modules, *Comm. Algebra*, 25(2) (1997), 347-355.
- [84] X. Zhang, J. Chen and J. Zhang, On (m, n) -injective modules and (m, n) -coherent rings, *Algebra Colloq.*, 12(1) (2005), 149-160.
- [85] D. Zhou, Rings characterized by a class of modules, *Comm. Algebra*, 33(9) (2005), 2941-2955.
- [86] M. Ziegler, Model theory of modules, *Ann. Pure Appl. Logic*, 26(2) (1984), 149-213.