# THE PRINCIPLE OF PREDICATE EXCHANGEABILITY IN PURE INDUCTIVE LOGIC

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# Contents

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We investigate the Principle of Predicate Exchangeability in the framework of Pure Inductive Logic.

While this principle was known to Rudolf Carnap, who started research in Inductive Logic, the principle has been somewhat neglected in the past.

After providing the framework of Pure Inductive Logic, we will show Representation Theorems for probability functions satisfying Predicate Exchangeability, filling the gap in the list of Representation Theorems for functions satisfying certain rational principles.

We then introduce a new principle, called the Principle of Strong Predicate Exchangeability, which is weaker than the well-known Principle of Atom Exchangeability, but stronger than Predicate Exchangeability and give examples of functions that satisfy this principle.

Finally, we extend the framework of Inductive Logic to Second Order languages, which allows for increasing a rational agent's expressive strength. We introduce Wilmers' Principle, a rational principle that rational agents might want to adopt in this extended framework, and give a representation theorem for this principle.

# Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

# Copyright Statement

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# Chapter 1

# Introduction

Whenever we find ourselves in a situation where we need to make a decision, this decision is influenced by a number of factors. For instance, leaving the house, how likely is it to rain while we are out? Should we therefore take an umbrella with us? A look out of the window may help us with this. If it is raining outside, the decision is easy to make. If it is not raining, however, we can still make a good guess on what to do. We may see dark clouds, and in the past it has rained most of the time when dark clouds were in the sky. Or maybe it is sunny, and no clouds can be seen, so it is very unlikely to start raining in the next few minutes.

While certainly observation (in this case observing whether there are clouds in the sky) contributes to the decision whether to take an umbrella or not, it is our *experience* that will greatly influence this decision. We may have listened to the weather report, which we know was accurate in the past, saying that it will rain today. So despite seeing no clouds, this might influence us to take an umbrella.

A rational agent, trying to give an answer to questions like the above one, faces a number of problems. She would have to decide which information is relevant to the question asked, which is irrelevant and therefore can be ignored, and how the information gained by observation should influence her predictions.

The goal of Inductive Logic, as it was introduced by Rudolf Carnap (see e.g. [3], [5], [6]), is to provide an agent with the means to answer such questions on the basis of observations in the past and to allow a rational agent to learn from experience. This is achieved by representing an agent's *belief* in a statement being true via belief functions. The agent being rational is reflected by certain rational principles holding for these functions.

We will use the following example to illustrate some rational principles that we wish our agents to adhere to.

**Example 1.1.** *Suppose we have an urn with a number of coloured balls. We do not know what this number is, nor do we know the distribution of colours in the urn. Picking balls with replacement from the urn, one after another, and writing down their colours, we may then attempt to answer the question of how likely it is that the next ball being picked will be Red.*

*Knowing that the colours that balls in the urn can have are amongst Red, Green, Yellow, Blue, say, without any other knowledge it seems to be the best guess that all four colours are evenly distributed, and that the chance of picking a Red ball is a quarter.*

*Now suppose we have picked five balls, with colours Red, Red, Green, Red, Blue, then we may feel that the proportion of Red balls in the urn is much larger than those of the other colours, and we might think it reasonable to increase our initial guess of the chance that the next ball is Red. At the very least, picking more Red balls than any other colour, would make it unreasonable for us to decrease the chance of picking a Red ball.*

In the above example, we would come to the same conclusions if the order in which the balls were picked had changed. Similarly, if instead of three Red balls, we had picked three Yellow balls, it would be reasonable to draw the same conclusion for the distribution of Yellow balls in the urn that we drew for the Red balls in the example.

This illustrates the idea that we should come to the same conclusions if there is some *symmetry* present, here represented by the order in which the balls are picked, and the symmetry between the colours.

The conclusion that we should increase our guess of the chance of picking a Red ball in the next go, on the other hand, illustrates the concept of *relevance*: the previous picks from the urn contain information that is relevant to our decision of whether we should change our guess or not.

## 1.1 Predicate Exchangeability in historical context

A number of rational principles based on symmetry and relevance have been favoured from an early stage, and are now widely accepted, most notably the principles of Constant Exchangeability and Instantial Relevance. In fact, Constant Exchangeability can be found amongst Carnap's basic requirements for probability functions, see [4].

The main focus in this thesis lies with the rational principle of Predicate Exchangeability. Carnap was very much aware of this principle, and in fact the Axiom A8 for probability functions, as stated in [4], is a version of this principle. We would argue that it comes as no surprise that Predicate Exchangeability is found in the list of axioms, since the principles of Constant and Predicate Exchangeability are motivated by the same symmetry.

However, up to this point Predicate Exchangeability has been somewhat neglected. A reason for that may be found in Carnap's favouring a principle based on relevance, Johnson's Sufficientness Postulate. The continuum of probability functions that are characterized by this principle satisfy a much stronger symmetry principle, called Atom Exchangeability. One might argue that since Carnap aimed to find the belief functions of rational agents amongst the members of this continuum, this stronger principle would be of more interest, and hence the weaker Predicate Exchangeability, while still an important requirement, might have been neglected as a principle to study in its own right.

## 1.2 Notation and conventions

The framework we will use in this thesis is that of formal languages and probability functions defined on these. Up to Chapter 4 we will work with First Order languages, with variables $x, x_1, x_2, \dots$, constant symbols $\{a_i \,|\, i \in \mathbb{N}^+\}$ and unary predicate symbols $\{P_i \,|\, i \in \mathbb{N}^+\}$. All languages considered will have a subset of $\{P_i \,|\, i \in \mathbb{N}^+\}$ as constant symbols. Thus, the languages will only differ in the predicates occurring, and we can treat a language as being defined by the predicates. If $L$ is a language such that the predicates in $L$ are $\{P_{i_1}, \dots, P_{i_n}\}$, then, in line with the above observation, we will identify $L$ with this set and write $L = \{P_{i_1}, \dots, P_{i_n}\}$. For each $q \in \mathbb{N}^+$, let $L_q = \{P_1, \dots, P_q\}$. While the main focus will be on these finite languages, there is an infinite language that will be of special significance for us. This is the language containing countably many predicates, one for each $n \in \mathbb{N}^+$, which we will denote by $L_\infty$, and is given by $L_\infty = \{P_n \,|\, n \in \mathbb{N}^+\}$. We define $L$-formulas and $L$-sentences in the usual way.

**Definition 1.2:** *Let $L$ be a First Order language with predicate symbol $\{P_{i_1}, \dots, P_{i_q}\}$ for distinct $i_1, \dots, i_n \in \mathbb{N}^+$.*

- *Let QFFL be the set of* quantifier-free $L$-formulas.
- *Let QFSL be the set of* quantifier-free $L$-sentences.
- *Let FL be the set of* $L$-formulas.
- *Let SL be the set of* $L$-sentences.
- *An* atom *of $L$ is an $L$-formula*

$$\bigwedge_{k=1}^{q} P_{i_k}^{\varepsilon_k}(x),$$

  *where $\varepsilon_k \in \{0,1\}$, $P_{i_k}^1 = P_{i_k}$, $P_{i_k}^0 = \neg P_{i_k}$ and $L = \{P_{i_1}, \dots, P_{i_q}\}$. Let* $\mathrm{At}_L$ *denote the set of $L$-atoms.*

- *A* state description for $a_{j_1}, \dots, a_{j_m}$ *of $L$ is an $L$-sentence*

$$\bigwedge_{s=1}^{m} \bigwedge_{k=1}^{q} P_{i_k}^{\varepsilon_{s,k}}(a_{j_s}).$$

  *Let* SDL *denote the set of state descriptions of $L$.*

**Remark 1.3.** Note that if a finite language $L$ has $q$ predicates, then a straightforward combinatorial argument shows that there are $2^q$ distinct atoms in the set $\mathrm{At}_L$. There is a common and natural way to enumerate these atoms, the *lexicographic ordering*. Assume that $L = \{P_{i_1}, \ldots, P_{i_q}\}$, for $i_1 < i_2 < \cdots < i_q$ distinct. Then the lexicographic ordering of $\mathrm{At}_L$ is given as follows.

Let $\alpha_1$ be the atom given by

$$\alpha_1(x) = P_{i_1}(x) \wedge P_{i_2}(x) \wedge \cdots \wedge P_{i_q}(x),$$

$\alpha_2$ the atom given by

$$\alpha_2(x) = P_{i_1}(x) \wedge P_{i_2}(x) \wedge \cdots \wedge P_{i_{q-1}}(x) \wedge \neg P_{i_q}(x),$$

$\alpha_3$ given by

$$\alpha_3(x) = P_{i_1}(x) \wedge P_{i_2}(x) \wedge \cdots \wedge \neg P_{i_{q-1}}(x) \wedge P_{i_q}(x),$$

and so on, up to $\alpha_{2^q}$ given by

$$\alpha_{2^q}(x) = \neg P_{i_1}(x) \wedge \neg P_{i_2}(x) \wedge \cdots \wedge \neg P_{i_q}(x).$$

Unless otherwise stated, we will assume that atoms are indexed by their lexicographic ordering.

We will use $\varphi, \psi, \vartheta$ to range over formulas and sentences, $\alpha, \beta, \gamma$ to range over atoms and $\Theta, \Phi, \Psi$ to range over state descriptions. Note that by definition, a state description is a conjunction of atoms. Furthermore, as any two distinct atoms are incompatible (in the sense that their conjunction is logically equivalent to a contradiction), it follows that any two distinct state descriptions are also incompatible.

By the Disjunctive Normal Form Theorem, we obtain that each quantifier-free sentence $\vartheta(a_1, \ldots, a_n)$ of a finite language $L$ is logically equivalent to a disjunction of pairwise distinct state descriptions of $L$,

$$\vartheta(a_1, \ldots, a_n) \equiv \bigvee_{\Theta(a_1,\ldots,a_n) \models \vartheta(a_1,\ldots,a_n)} \Theta(a_1, \ldots, a_n).$$

Whenever $\vartheta$ is a sentence (or formula) of $L$, we write $\vartheta(a_1, \ldots, a_m)$ ($\vartheta(x_1, \ldots, x_m)$) to say that the constants (variables) of $\vartheta$ are among $a_1, \ldots, a_m$ ($x_1, \ldots, x_m$). We will omit mentioning these whenever they are clear from the context.

We now look at the notion of $L$-structures $M$, which are defined as usual. In the context of Pure Inductive Logic, we are interested in a specific collection of $L$-structures, which we will denote by $\mathcal{T}L$.

**Definition 1.4:** *Let $\mathcal{T}L$ be the set of $L$-structures $M$ such that $M$ has the universe $\{a_i \,|\, i \in \mathbb{N}^+\}$, with each $a_i$ interpreted as itself.*

Throughout this thesis, we will be assuming belief as probability, therefore all belief functions dealt with will be probabilistic belief functions.

One way of justifying this is the Dutch Book Theorem, see Theorem 5.1 in [22], and the discussion preceding it, which we will briefly summarize here.

Suppose we identify an agent's belief as her willingness to bet. Assume that an agent is living in an ambient structure $M$, and for a statement $\vartheta$, gets the choice of either

(B1) win $s(1-p)$ if $M \models \vartheta$, lose $sp$ if $M \not\models \vartheta$,

or

(B2) win $sp$ if $M \not\models \vartheta$, lose $s(1-p)$ if $M \models \vartheta$,

for some stake $s > 0$ and $0 \leq p \leq 1$, to wager. If the agent is not happy to accept the wager (B1), because she thinks it would be to her disadvantage, then certainly (B2) would be acceptable, as this would mean swapping roles with the bookmaker. Thus, we can suppose that for any $0 \leq p \leq 1$, at least one of the above bets would be acceptable for the agent.

An agent could then be "Dutch Booked", if there was some set of simultaneous bets (possibly infinitely many), each single one acceptable for the agent, but when combined these cause a certain loss for the agent, independent of the structure $M$ the agent is living in.

The Dutch Book Theorem then tells us that if an agent's belief function is such that the agent cannot be Dutch Booked, then the belief function is a probability function. Note that the converse also holds, i.e. probability functions cannot be Dutch Booked (see Theorem 5.3 in [22]).

**Definition 1.5:** *Let $L$ be a language. A function $w : SL \to [0,1]$ is a* probabilistic belief function *on $SL$, if for $\vartheta, \varphi, \exists x\, \psi(x) \in SL$, $w$ satsifies the following three properties:*

*(P1) if* $\models \vartheta$, *then* $w(\vartheta) = 1$,

*(P2) if* $\vartheta \models \neg\varphi$, *then* $w(\vartheta \vee \varphi) = w(\vartheta) + w(\varphi)$,

*(P3)* $w(\exists x\, \psi(x)) = \lim_{n\to\infty} w\left(\bigvee_{i=1}^{n} \psi(a_i)\right)$.

We will hereafter refer to probabilistic belief functions just as probability functions.

**Definition 1.6:** *For $\vartheta$, $\varphi \in QFSL$, $w$ a probability function on $SL$ and $w(\varphi) > 0$, we define the* conditional probability of $\vartheta$, given $\varphi$ *by the Bayes formula,*

$$w(\vartheta \mid \varphi) = \frac{w(\vartheta \wedge \varphi)}{w(\varphi)}.$$

In case $w(\varphi) = 0$, it will be convenient to let the conditional probability be such that

$$w(\vartheta \wedge \varphi) = w(\vartheta \mid \varphi) \cdot w(\varphi),$$

so the expression is meaningful in those cases as well.

Immediately from the definition we obtain some technical properties of probability functions. Proofs for these can be found in e.g. [22].

**Lemma 1.7.** *Let $w$ be a probability function on $SL$. Then for $\vartheta, \varphi \in SL$, we have*

*(a)* $w(\neg\vartheta) = 1 - w(\vartheta)$.

*(b) If* $\models \neg\vartheta$, *then* $w(\vartheta) = 0$.

*(c) If* $\vartheta \models \varphi$, *then* $w(\vartheta) \leq w(\varphi)$.

*(d) If* $\vartheta \equiv \varphi$, *then* $w(\vartheta) = w(\varphi)$.

*(e)* $w(\vartheta \vee \varphi) = w(\vartheta) + w(\varphi) - w(\vartheta \wedge \varphi)$.

**Remark 1.8.** Using part (a) from the above lemma, a straightforward calculation using the property (P3) will yield that

$$w\left(\forall x\,\psi(x)\right) = \lim_{n\to\infty} w\left(\bigwedge_{i=1}^{n}\psi(a_i)\right).$$

**Example 1.9.** *We obtain a first, simple example for a probability function via the structures $M \in \mathcal{T}L$. For such a structure $M$, we can define a function $V_M : SL \to [0,1]$ by*

$$V_M(\vartheta) = \begin{cases} 1 & \text{if } M \models \vartheta, \\ 0 & \text{if } M \not\models \vartheta. \end{cases}$$

*Then it is easy to see that $V_M$ satisfies the properties (P1-3) of Definition 1.5.*

## 1.3 Rational principles

In this section we will introduce rational principles that rational agents, and thus their belief functions, may satisfy.

The first three rational principles appeal to *symmetry*: if sentences satisfy some symmetry, then a rational agent may feel that it is reasonable that those sentences should be assigned the same probability.

In the definitions that follow, let $w$ be a probability function on some finite language $L$.

**Constant Exchangeability, Ex**

*$w$ satisfies Ex, if for all $\vartheta \in SL$ and all permutations $\sigma$ of $\mathbb{N}^+$,*

$$w(\vartheta(a_1,\ldots,a_n)) = w(\vartheta(a_{\sigma(1)},\ldots,a_{\sigma(n)})).$$

**Predicate Exchangeability, Px**

*$w$ satisfies Px, if for all $\vartheta \in SL$, and all permutations $\sigma$ of the (indices of the) predicates in $L$,*

$$w\left(\vartheta(P_1,\ldots,P_m,a_1,\ldots,a_n)\right) = w\left(\vartheta(P_{\sigma(1)},\ldots,P_{\sigma(m)},a_1,\ldots,a_n)\right)$$

*whenever $\vartheta(P_{\sigma(1)},\ldots,P_{\sigma(m)},a_1,\ldots,a_n)$ is the result of replacing each occurrence of $P_j$ in $\vartheta$ with $P_{\sigma(j)}$.*

**Atom Exchangeability, Ax**

*w satisfies Ax, if for all permutations $\tau$ of the (indices of the) atoms of L and constants $a_1, \ldots, a_n$,*

$$w\left(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) = w\left(\bigwedge_{i=1}^{n} \alpha_{\tau(h_i)}(a_i)\right).$$

At this point, it will be convenient to introduce some notation. The principles based on symmetry defined above make use of permutations of either constants, atoms, or predicates of the language $L$. In fact, they are most commonly regarded as permutations of the *index set* used to enumerate these – which is a subset of $\mathbb{N}^+$ in our case. These permutations naturally induce permutations of the set of sentences $SL$:

- If $\sigma$ is a permutation of constants (i.e. a permutation of $\mathbb{N}^+$), we can extend $\sigma$ to $SL$ by letting

$$\sigma(\vartheta(a_{i_1}, \ldots, a_{i_n}) = \vartheta(a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)}).$$

- If $\sigma$ is a permutation of predicates (i.e. a permutation of $\{1, \ldots, q\}$, for $L = L_q$), we obtain $\sigma$ as a permutation of the atoms of $L$ by letting

$$\sigma\vartheta(P_1, \ldots, P_m, a_1, \ldots, a_n) = \vartheta(P_{\sigma(1)}, \ldots, P_{\sigma(m)}, a_1, \ldots, a_n).$$

- If $\sigma$ is a permutation of atoms (i.e. a permutation of the index set $\{1, \ldots, 2^q\}$ for atoms of $L = L_q$), we can write $\alpha_{\sigma(i)}$ as $\sigma(\alpha_i)$ and then extend $\sigma$ to $SL$ via the clauses

$$\sigma\left(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) = \bigwedge_{i=1}^{n} \sigma(\alpha_{h_i})(a_i),$$

$$\sigma(\neg\vartheta) = \neg\sigma(\vartheta),$$

$$\sigma(\vartheta \vee \varphi) = \sigma(\vartheta) \vee \sigma(\varphi)$$

and

$$\sigma\left(\exists x\, \psi(x)\right) = \exists x\, \sigma(\psi(x)).$$

We will say that a permutation $\sigma$ of $SL$ (or some subset of $SL$) *is induced by* Ex / Px / Ax, if it arises from some permutation of constants / predicates / atoms given by the principle Ex / Px / Ax.

**Remark 1.10.** Note that if we have $\vartheta \equiv \varphi$ for sentences $\vartheta, \varphi \in SL$, then we would like $\sigma\vartheta \equiv \sigma\varphi$ to hold. If $\sigma$ is a permutation of constants or predicates, this is immediately clear.

If $\sigma$ is a permutation of atoms, a bit more work is needed in order to see that this holds. We can show this by induction on the definition of $\sigma$ as a permutation of $SL$, as above.

If $\vartheta$, $\varphi$ are state descriptions, this is immediate. So suppose $\vartheta = \psi_1 \vee \psi_2$, $\varphi = \chi_1 \vee \chi_2$. Since $\vartheta \equiv \varphi$, assume without loss of generality that $\psi_i \equiv \chi_i$ for $i = 1, 2$. Then by inductive hypothesis we have $\sigma\psi_i \equiv \sigma\chi_i$ for $i = 1, 2$, and thus we get $\sigma\vartheta \equiv \sigma\varphi$. Similarly, this works in the cases that $\vartheta = \neg\psi$, $\varphi = \neg\chi$ and $\vartheta = \exists x\, \psi(x)$, $\varphi = \exists x\, \chi(x)$.

**Regularity, Reg**

*$w$ satisfies Reg, if for all $\vartheta \in QFSL$ such that $\vartheta$ is satisfiable, $w(\vartheta) > 0$.*

**Super Regularity, SReg**

*$w$ satisfies SReg, if for all $\vartheta \in SL$ such that $\vartheta$ is satisfiable, $w(\vartheta) > 0$.*

The rational principles Regularity and Super Regularity are motivated as follows. If a sentence is satisfiable in some structure $M$, then having no prior knowledge of which structure the agent lives in, it would be unreasonable for her to assign this sentence probability 0 and thus dismiss it right from the start.

We need to distinguish between Regularity and Super Regularity here, for suppose a probability function $w$ on some language $L$ satisfied Regularity, then we have that for any $\varphi(x) \in QFFL$, the sentences $\varphi(a_1), \varphi(a_1) \wedge \varphi(a_2), \dots, \varphi(a_1) \wedge \varphi(a_2) \wedge \cdots \wedge \varphi(a_n)$ will get assigned positive probabilities by $w$, and by Lemma 1.7, (c) it is easy to see that these values need to be decreasing. In general, we cannot guarantee the limit of this sequence to be positive, and thus in general we do not have $w\,(\forall x\, \varphi(x)) > 0$, and so $w$ need not satisfy Super Regularity, even if it satisfies Regularity.

The next three principles appeal to relevance. If two sentences $\varphi$, $\psi$ are relevant to each other, then each should have some influence on an agent's belief in the other.

**Principle of Instantial Relevance, PIR**

*w satisfies PIR, if for all $\vartheta(a_1, \ldots, a_n) \in SL$, all atoms $\alpha$ of L,*

$$w(\alpha(a_{n+2}) \,|\, \alpha(a_{n+1}) \wedge \vartheta(a_1, \ldots, a_n)) \geq w(\alpha(a_{n+1}) \,|\, \vartheta(a_1, \ldots, a_n)).$$

This principle is the main principle instantiating an agent's ability to learn from experience:

If an agent has seen something happening in the past, then this should increase her belief in this happening again, or at the very least, it should not decrease her belief. This is reflected by the inequality above, since conditioning on the sentences $\vartheta(a_1, \ldots, a_n)$ and $\alpha(a_{n+1}) \wedge \vartheta(a_1, \ldots, a_n)$ can be seen as having witnessed these as properties of the constants $a_1, \ldots, a_n, a_{n+1}$. The additional $\alpha(a_{n+1})$ occurring in the second sentence is then interpreted as additional support for the agent's belief in $\alpha$.

This principle, which is a formulation of Carnap's original suggestion of capturing an agent's ability to learn from past experience (see [5]), can be generalized to also allow an agent to increase her belief if instead of seeing the exact properties before, she sees a logical consequence of these.

**Generalized Principle of Instantial Relevance, GPIR**

*w satisfies GPIR, if for all $\vartheta(a_{n+2})$, $\varphi(a_{n+1})$, $\psi(a_1, \ldots, a_n) \in QFSL$ such that $\vartheta(x) \models \varphi(x)$,*

$$w(\vartheta(a_{n+2}) \,|\, \varphi(a_{n+1}) \wedge \psi(a_1, \ldots, a_n)) \geq w(\vartheta(a_{n+1}) \,|\, \psi(a_1, \ldots, a_n)).$$

**Constant Irrelevance, IP**

*w satisfies IP, if for any $\vartheta, \varphi \in QFSL$ that have no constants in common,*

$$w(\vartheta \wedge \varphi) = w(\vartheta) \cdot w(\varphi).$$

**Weak Irrelevance Principle, WIP**

*$w$ satisfies WIP, if for all $\vartheta, \varphi \in QFSL$ such that $\vartheta$ and $\varphi$ have no predicates or constants in common, then $w(\vartheta \wedge \varphi) = w(\vartheta) \cdot w(\varphi)$.*

The last two principles appeal to *irrelevance* rather than relevance, on the basis that if two sentences have no constants in common and, in the case of WIP, are from distinct languages and thus are syntactically disjoint, then it seems unreasonable that observing one provides us with any information relevant to the other, and therefore witnessing one should not influence an agent's belief in the other.

Constant Irrelevance is much stronger than Weak Irrelevance, and in fact, in presence of Ex, IP characterizes a class of functions, see e.g. Proposition 8.1 in [22].

In the list of principles appealing to irrelevance we also have Johnson's Sufficientness Postulate.

**Johnson's Sufficientness Postulate, JSP**

*$w$ satisfies JSP, if*

$$w\left(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right)$$

*depends only on the number of atoms $n$, and the number of times that $\alpha_j$ occurs amongst the $\alpha_{h_i}$.*

This principle is of special significance, as it characterizes Carnap's Continuum of Inductive Methods, see e.g. [2].

The following principle is not a property of a single function, but of a *family* of functions.

**Language Invariance, Li**

*Let $\mathcal{W}$ be a family of probability functions such that for each finite language $L$ there is a probability function $w = w^L$ on $SL$ with $w \in \mathcal{W}$. Then $\mathcal{W}$ is a family satisfying Language Invariance if each $w \in \mathcal{W}$ satisfies Predicate Exchangeability and whenever $w$, $w' \in \mathcal{W}$ are probability functions on $L$, $L'$, respectively such that $L \subseteq L'$, then the restriction of $w'$ to $SL$, denoted by $w' \restriction SL$, equals $w$.*

*Let $\mathcal{P}$ be a rational principle. Then we say that a family of functions $\mathcal{W}$ satisfies Language Invariancce* with $\mathcal{P}$*, if each of the $w \in \mathcal{W}$ also satisfies the principle $\mathcal{P}$.*

If $\mathcal{W}$ is a family of probability functions on purely unary languages, then Language Invariance will be called Unary Language Invariance (ULi).

We say that a function $w$ satisfies (Unary) Language Invariance if $w$ is a member of a family satisfying (Unary) Language Invariance.

**Remark 1.11.** In the statement of the principle of Language Invariance, we require that all functions in the family satisfy Predicate Exchangeability. There are two reasons for having this requirement.

We can think of a language invariant family $\mathcal{W}$ of probability functions as representing some rational agent's belief, with each member $w^L$ of this family representing the agent's belief in statements just involving the predicates occurring in $L$. Since we assume that the agent does not have any prior knowledge of the structure she is living in, it seems unreasonable to have $w^L$ not only depend on the number of predicates in $L$, but also on the particular predicates occurring in $L$, as this would impose some a priori semantics on the languages.

For instance, consider the probability function $w \in \mathcal{W}$ on the language $L_q$ and let $R$, $Q$ be distinct predicates that both do not occur amongst $P_1, P_2, \ldots, P_q$. We have $w_1, w_2 \in \mathcal{W}$ such that $w_1$ is a probability function on $L_q \cup \{R\}$ and $w_2$ is a probability function on $L_q \cup \{Q\}$, with $w_1$ and $w_2$ both extending $w$. Then having $w_1(R(a_1)) \neq w_2(Q(a_1))$ would mean that the agent does have some a priori reason for distinguishing the predicates $R$ and $Q$ before she has even examined any object in the universe.

The second reason is that it is straightforward to see that the functions $w_{\vec{c}}$, which we will introduce in the next section, would all be members of a Language Invariant family if we were to drop the condition of Px. From this it would follow that any probability function then is a member of such a family, making Language Invariance a trivial statement.

As a consequence, it will be enough to consider the languages of the form $L_q$ for language invariant families. Let $w$ be a probability function satisfying ULi on $L =$

$\{Q_1, \ldots, Q_q\}$, where each $Q_i$ is some $P_{j_i}$ with $j_i$ not necessarily the same as $i$. Then we can extend $w$ to a probability function $v$ on $L' = \{P_1, \ldots, P_q\} \cup \{Q_1, \ldots, Q_q\}$. Let $\sigma$ be a permutation of the predicates of $L'$ such that each $Q_i$ will be replaced by $P_i$ when applying $\sigma$ to sentences of $L'$. Then for each sentence $\vartheta \in SL$, we have $w(\vartheta) = v(\vartheta) = v(\sigma\vartheta)$, and $\sigma\vartheta$ is a sentence of $L_q$. Then we obtain a probability function $w'$ on $L_q$ such that $w'(\vartheta) = w(\sigma^{-1}\vartheta)$ for each $\vartheta \in SL_q$. Thus we can think of $w$ as a function on $L_q$ via the translation $w'$.

## 1.4 Some technical results for probability functions

We will quote some well-known results that will be needed later on, for proofs and detailed discussion of these see e.g. [22].

**Lemma 1.12.** *Let $\exists x_1 \ldots \exists x_k \, \psi(x_1, \ldots, x_k, \vec{a}) \in SL$ and let $b_1, b_2, b_3, \ldots$ be an infinite sequence of distinct constants including each of the $a_i$ in $\vec{a}$. Then for $w$ a probability function on SL satisfying Ex,*

$$\begin{aligned} w\left(\exists x_1 \ldots \exists x_k \;\; \psi(x_1, \ldots, x_k, \vec{a})\right) & \\ &= \lim_{n\to\infty} w\left(\bigvee_{i_1,\ldots,i_k \leq n} \psi(b_{i_1}, \ldots, b_{i_k}, \vec{a})\right) \\ w\left(\forall x_1 \ldots \forall x_k \;\; \psi(x_1, \ldots, x_k, \vec{a})\right) & \\ &= \lim_{n\to\infty} w\left(\bigwedge_{i_1,\ldots,i_k \leq n} \psi(b_{i_1}, \ldots, b_{i_k}, \vec{a})\right). \end{aligned}$$

The following theorem, which is due to Gaifman, enables us to significantly reduce the work needed to prove theorems, since it will be enough to prove statements for quantifier-free sentences.

**Theorem 1.13 (Gaifman, [10]).** *Suppose that $w : QFSL \to [0,1]$ satisfies the properties (P1) and (P2) for $\vartheta$, $\varphi \in QFSL$. Then $w$ has a unique extension to SL satisfying (P1),(P2) and (P3) for $\vartheta$, $\varphi$, $\exists x\, \psi(x) \in SL$. Furthermore if $w$ satisfies Ex, Px (respectively) on QFSL then so will its extension to SL.*

**Remark 1.14.** Since by the Disjunctive Normal Form Theorem, every $\vartheta \in QFSL$ is logically equivalent to a disjunction of state descriptions, probability functions are determined by the values of state descriptions of $L$ via

$$w(\vartheta(a_1,\ldots,a_n)) = w\left(\bigvee_{\Theta(a_1,\ldots,a_n)\models\vartheta(a_1,\ldots,a_n)} \Theta(a_1,\ldots,a_n)\right)$$
$$= \sum_{\Theta(a_1,\ldots,a_n)\models\vartheta(a_1,\ldots,a_n)} w(\Theta(a_1,\ldots,a_n)).$$

Conversely, if $w$ is defined on state descriptions $\Theta(a_1,\ldots,a_m)$ such that

(i) $w(\Theta(a_1,\ldots,a_m)) \geq 0$,

(ii) $w(\top) = 1$,

(iii) $w(\Theta(a_1,\ldots,a_m)) = \sum_{\Phi(a_1,\ldots,a_{m+1})\models\Theta(a_1,\ldots,a_m)} w(\Phi(a_1,\ldots,a_{m+1}))$,

then $w$ extends to a probability function on $QFSL$ by letting

$$w(\vartheta(a_1,\ldots,a_m)) = \sum_{\Theta(a_1,\ldots,a_m)\models\vartheta(a_1,\ldots,a_m)} w(\Theta(a_1,\ldots,a_m)).$$

It will sometimes be easier to show that (i)-(iii) holds for a function in order to show that it is (extends to) a probability function on $SL$.

In what follows we aim to provide *representation theorems* for probability functions satisfying various principles. We will start with a basic one, due to de Finetti. Before stating the theorem, we will need some more notation. Define the set $\mathbb{D}_{2^q} \subseteq \mathbb{R}^{2^q}$ by

$$\mathbb{D}_{2^q} := \left\{ \vec{c} = \langle c_1,\ldots,c_{2^q}\rangle \mid \forall i \in \{1,\ldots,2^q\}\, c_i \geq 0 \wedge \sum_{i=1}^{2^q} c_i = 1 \right\}.$$

Then for $\vec{c} \in \mathbb{D}_{2^q}$ we can define the probability function $w_{\vec{c}}$ by letting

$$w_{\vec{c}}\left(\bigwedge_{i=1}^{m} \alpha_{h_i}(a_i)\right) = \prod_{i=1}^{2^q} c_i^{m_i},$$

where $\langle m_1,\ldots,m_{2^q}\rangle$ is the *signature* of the state description $\bigwedge_{i=1}^{m} \alpha_{h_i}(a_i)$ given by $m_i = |\{k \mid h_k = i\}|$ for $i \in \{1,\ldots,2^q\}$.

**Theorem 1.15 (de Finetti's Representation Theorem).** *Let $w$ be a probability function on $L$. Then $w$ satisfies Ex if and only if there exists a $\sigma$-additive measure $\mu$ on the set $\mathbb{D}_{2^q}$ such that*

$$w = \int_{\mathbb{D}_{2^q}} w_{\vec{c}}\, d\mu(\vec{c}).$$

We will assume that all probability functions satisfy Constant Exchangeability. The next theorem will show that these satisfy the Principle of Instantial Relevance, see [11], [14].

**Theorem 1.16.** *Let $w$ be a probability function on $L$ satisfying Ex. Then $w$ satisfies PIR.*

As a final result in this section, we present a convenient characterization for ULi families.

**Proposition 1.17.** *Let $w$ be a probability function on $L$. Then $w$ satisfies Unary Language Invariance if and only if there exists a probability function $w_\infty$ on $L_\infty$ satisfying Ex + Px and $w_\infty \restriction SL = w$.*

**Proof:** Suppose that $w$ satisfies Unary Language Invariance. Then by definition there exists a family $\mathcal{W}$ of belief functions witnessing that. Without loss of generality we can assume that $L = L_q$ for some $q \in \mathbb{N}^+$. For $n \in \mathbb{N}^+$, fix $w_n$ on $L_n$ such that $w_n \in \mathcal{W}$. Then by definition of Unary Language Invariance, each $w_n$ satisfies Ex + Px and whenever $n \leq m$, we have

$$w_m \restriction SL_n = w_n.$$

Note that by this, we have $w = w_q$. As also $L_n \subseteq L_m$ whenever $n \leq m$ and $L_n \subseteq L_\infty$ for each $n \in \mathbb{N}^+$, we have

$$L_\infty = \bigcup_{n \in \mathbb{N}^+} L_n.$$

As each $w_n$ is a function with domain $SL_n$ and range $[0, 1]$ we can view $w_n$ as a set

$$w_n \subseteq SL_n \times [0, 1].$$

Since $L_n \subseteq L_m$ whenever $n \leq m$, we also have that $SL_n \subseteq SL_m$ whenever $n \leq m$, which gives

$$SL_n \times [0,1] \subseteq SL_m \times [0,1].$$

Thus, for $n \leq m$, we have

$$w_n \subseteq SL_m \times [0,1].$$

Since $w_m \restriction SL_n = w_n$ for $n \leq m$ by definition, we must have

$$w_m \cap (SL_n \times [0,1]) = w_n.$$

We can then define $w_\infty$ by

$$w_\infty := \bigcup_{n \in \mathbb{N}^+} w_n \subseteq \bigcup_{n \in \mathbb{N}^+} (SL_n \times [0,1]) = SL_\infty \times [0,1].$$

Since the $w_i$ extend each other, $w_\infty$ is a function on $SL_\infty$. By this definition, it is immediate that $w_\infty \restriction SL_q = w_q = w$.

It remains to show that $w_\infty$ satisfies Ex and Px.

For Ex, let $\sigma$ be a permutation of constants, let $\varphi \in SL_\infty$ and suppose that $\sigma\varphi \in SL_\infty$ is the sentence obtained from $\varphi$ by applying $\sigma$ to the constants occurring in $\varphi$. Now $\varphi \in SL_n$ for some $n \in \mathbb{N}^+$, and since $\sigma\varphi$ is obtained just by permuting constants, $\sigma\varphi \in SL_n$ as well. But $w_n$ satisfies Ex and therefore

$$w_n(\varphi) = w_n(\sigma\varphi).$$

Since $w_\infty \restriction SL_n = w_n$, we must have $w_\infty(\varphi) = w_\infty(\sigma\varphi)$ and thus $w_\infty$ satisfies Ex.

For Px, let $\tau$ be a permutation of predicates. Let $\varphi \in SL_\infty$, and $\tau\varphi$ be the sentence obtained from $\varphi$ by applying $\tau$ to the predicates occurring in $\varphi$. As above, we have $\varphi \in SL_n$ for some $n \in \mathbb{N}^+$ and $w_\infty(\varphi) = w_n(\varphi)$. Then the predicates occurring in $\varphi$ are a subset of

$$L_n = \{P_1, \ldots, P_n\}.$$

It is enough to consider $\tau$ on $L_n$. Let $m$ be maximal such that $P_m \in \{P_{\tau(1)}, \ldots, P_{\tau(n)}\}$. Then certainly

$$\{P_{\tau(1)}, \ldots, P_{\tau(n)}\} \subseteq \{P_1, \ldots, P_m\} = L_m.$$

As $P_1, \ldots, P_n$ are distinct predicates, we must have $m \geq n$. Then we have $\varphi, \tau\varphi \in SL_m$ and we obtain by ULi

$$w_n(\varphi) = w_m(\varphi),$$

and by Px for $w_m$,

$$w_m(\varphi) = w_m(\tau\varphi).$$

As $w_\infty \restriction SL_m = w_m$, we have $w_\infty(\varphi) = w_\infty(\tau\varphi)$, so Px holds for $w_\infty$.

For the other direction, assume there is a probability function $w_\infty$ on $L_\infty$ such that $w_\infty$ satisfies Ex + Px and $w_\infty \restriction SL = w$. We need to construct a family $\mathcal{W}$ satisfying ULi with $w \in \mathcal{W}$.

Let $L'$ be a finite language. We may assume without loss of generality that $L' \subseteq L_\infty$. Define

$$w^{L'} := w_\infty \restriction SL'$$

and let

$$\mathcal{W} := \{w^{L'} \mid L' \text{ is a finite language}\}.$$

Then each $w^{L'} \in \mathcal{W}$ inherits Ex and Px from $w_\infty$ and $w = w^L \in \mathcal{W}$. Now let $L' \subseteq L''$. Then by a straightforward calculation,

$$w^{L''} \restriction SL' = (w_\infty \restriction SL'') \restriction SL' = w_\infty \restriction SL' = w^{L'}.$$

Therefore $\mathcal{W}$ is a family witnessing that $w$ satisfies Unary Language Invariance. $\dashv$

## 1.5 Non-Standard Analysis

Some of the proofs we will carry out will make use of tools and methods from Non-Standard Analysis. We will give a brief overview of the topic here, introducing the definitions and notions we will be employing here. A more detailed introduction to Non-Standard Analysis can be found e.g. in [7],[8] and [12].

Non-Standard Analysis deals with the set of *hyperreals*, denoted ${}^*\mathbb{R}$. These contain all *standard* reals and include *infinitesimal* and *infinite* reals, and are most commonly constructed as the ultraproduct $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$ via a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

**Definition 1.18:** *Let $x \in {}^*\mathbb{R}$. Then*

*(i) $x$ is* infinitesimal *if $|x| < \varepsilon$ for all $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$,*

*(ii) $x$ is* finite *if $|x| < r$ for some $r \in \mathbb{R}$,*

*(iii) $x$ is* infinite *if $|x| > r$ for all $r \in \mathbb{R}$.*

Here, $|\cdot|$ is the extension of the modulus function on $\mathbb{R}$ to ${}^*\mathbb{R}$ in the obvious way. Since $\mathbb{R} \subseteq {}^*\mathbb{R}$, we obtain an embedding $\mathbb{R} \to {}^*\mathbb{R}$, denoted $^*$. In the other direction, we obtain the *standard part map*, denoted $^\circ : {}^*\mathbb{R} \to \mathbb{R}$. This is well-defined on all finite hyperreals by the following proposition.

**Proposition 1.19.** *Let $x \in {}^*\mathbb{R}$ be finite. Then there exists a unique $r \in \mathbb{R}$ such that $|x - r|$ is infinitesimal.*

Similarly, we obtain the set of the *hypernatural numbers*, denoted ${}^*\mathbb{N}$, as the extension of $\mathbb{N}$ to the hyperreal universe. We then obtain an analogous notion to the finite sets.

**Definition 1.20:** *Let $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$, let $N_\nu$ be the subset*

$$N_\nu = \{n \in {}^*\mathbb{N} \,|\, n < \nu\}.$$

*Let $X \subseteq {}^*\mathbb{R}$. Then $X$ is* hyperfinite *if there exists a bijection*

$$f : X \to N_\nu$$

*for some $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$.*

Note that if in the above definition we have instead $\nu \in \mathbb{N}$, then $X$ is finite.

We will be using two key notions of Non-standard Analysis. One is the *Transfer Principle*. To give a definition, we will first need to define the mathematical universe in which we will be using Non-standard Analysis. We will denote this by $\mathbb{V} = \mathbb{V}(\mathbb{R})$ and define it inductively via the clauses

$$\begin{aligned} \mathbb{V}_0(\mathbb{R}) &= \mathbb{R} \\ \mathbb{V}_{n+1}(\mathbb{R}) &= \mathbb{V}_n(\mathbb{R}) \cup \mathcal{P}(\mathbb{V}_n(\mathbb{R})), n \in \mathbb{N} \\ \mathbb{V} = \mathbb{V}(\mathbb{R}) &= \bigcup_{n\in\mathbb{N}} V_n(\mathbb{R}). \end{aligned}$$

We then define the non-standard universe as

$$^*\mathbb{V} := \{x \,|\, x \in {}^*A \text{ for some } A \in \mathbb{V}\},$$

and we obtain $^*\mathbb{V} \subseteq \mathbb{V}(^*\mathbb{R})$ and a mapping $^* : \mathbb{V}(\mathbb{R}) \to \mathbb{V}(^*\mathbb{R})$. The sets in $^*\mathbb{V}$ are called *internal sets*, the sets in $\mathbb{V}(^*\mathbb{R}) \setminus {}^*\mathbb{V}$ are called *external sets.*

**Definition 1.21 (Transfer Principle):** *Let $\mathbb{V}$ be the standard universe. Let $\varphi$ be a First Order bounded quantifier statement. Then*

$$\varphi \textit{ holds in } \mathbb{V} \iff {}^*\varphi \textit{ holds in } {}^*\mathbb{V}.$$

Here, *bounded quantifier statement* means that all quantifiers are bounded by sets. The statement $^*\varphi$ is then obtained from $\varphi$ by replacing each bound $A$ occurring in $\varphi$ by $^*A$, and each constant $c$ in $\varphi$ by $^*c$. We will also require the universe $\mathbb{V}$ to satisfy $\aleph_1$*-saturation*, by which we mean that any countable decreasing sequence of non-empty internal sets has non-empty intersection. Formally, if $\{A_i \,|\, i \in \omega_0\}$ is such that $A_i \neq \emptyset$ for each $i \in \omega_0$ and $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, then $\bigcap_{i\in\omega_0} A_i \neq \emptyset$. This ensures that all the sets we require to exist do exist in the universe. Note that if we construct the non-standard universe by the usual ultrapower construction, then it will satisfy $\aleph_1$-saturation, see Definition 1.10 and Theorem 1.11 in [8].

The other key notion is that of *Loeb measures.* These provide a way of constructing measures in the standard universe via defining measures in the non-standard universe $^*\mathbb{V}$ and then pushing them down to the standard part.

Loeb measures are constructed as follows: Let $\mathcal{A}$ be an internal algebra on an internal set $A$[1], by which we mean that $\mathcal{A}$ is a collection of subsets of $A$ containing $A$ and closed

[1]Here, *internal algebra* means that $\mathcal{A}$ is an algebra containing only internal subsets of A. Since any internal set is either uncountable or finite, this means that if $A$ is uncountable, then $\mathcal{A}$ does not contain any countably infinite subset of $A$.

under intersections and unions, let $\mu$ be a finite internal finitely additive measure on $\mathcal{A}$, i.e.

$$\begin{aligned}\mu :&\mathcal{A} \to {}^*[0,\infty),\\ \mu(B \cup C) &= \mu(B) + \mu(C) \text{ for disjoint } B, C \in \mathcal{A},\\ \mu(A) &< \infty.\end{aligned}$$

Since $\mu$ is finite, we can take the standard part of $\mu$, $^\circ\mu$ to obtain a mapping

$$^\circ\mu : \mathcal{A} \to [0,\infty)$$

by $^\circ\mu(A) = {}^\circ(\mu(A))$. Then we have that $^\circ\mu$ is finitely additive, but in general, $(A, \mathcal{A}, {}^\circ\mu)$ is not a measure space, since $\mathcal{A}$ is not a $\sigma$-algebra in general. Here, Loeb measure theory helps us to obtain a measure space, using the following theorem.

**Theorem 1.22.** *There exists a unique $\sigma$-additive extension of $^\circ\mu$ to the $\sigma$-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. The completion of this measure is the Loeb measure corresponding to $\mu$, denoted $\mu^L$. The completion of $\sigma(\mathcal{A})$ is the Loeb $\sigma$-algebra, denoted $L(\mathcal{A})$.*

**Remark 1.23.** Since we are interested in obtaining measures in the standard universe, we might be concerned whether the algebras $\sigma(\mathcal{A})$ and $L(\mathcal{A})$ are generated in the standard universe, or are obtained via pushing down algebras that have been constructed in the non-standard universe.

By the way that Loeb measureable sets are defined, Loeb algebra in the non-standard universe will contain also external subsets of $A$, but they will only differ from a subset in $\mathcal{A}$ via a $\mu$-null set.

It will therefore not matter which of these two possible ways we will take to construct the algebras.

Since we will be using Loeb measures for integration, we will also require a theorem that allows us to obtain standard descriptions for integrals that we push down to the standard universe. Since our measures will be probability measures, they will all be finite.

**Theorem 1.24.** *Let $A$ be an internal set, $\mathcal{A}$ an internal algebra on $A$ and $\mu$ a finitely additive internal probability measure on $\mathcal{A}$. Let $\mu^L$ be the corresponding Loeb measure on $L(\mathcal{A})$. Let $f$ be a finitely bounded internal measurable function, then*

$$^\circ\!\!\int f\,d\mu = \int {}^\circ\! f\,d\mu^L.$$

For a detailed discussion of Loeb measures and Loeb integration, see e.g. [7], [8].

# Chapter 2

# Representation Theorems for Predicate Exchangeability

In this chapter we will focus on results concerning Pure Inductive Logic on purely unary languages.

The main results are the Representation Theorem for probability functions satisfying Predicate Exchangeability and Unary Language Invariance and the General Representation Theorem for functions satisfying Predicate Exchangeability.

We will first start with a theorem for functions of the form $w_{\vec{c}}$, the building blocks in de Finetti's Theorem, as these will play a role in later chapters as well.

The results presented in this chapter are joint work with Jeff Paris, and appear here as they do in [16], with the occasional proof presented in more detail.

## 2.1 Unary Language Invariance and Constant Irrelevance

Apart from the characterization via de Finetti's Representation Theorem, we can characterize the functions of the form $w_{\vec{c}}$ as those that satisfy the Principle of Constant Irrelevance, see e.g. Proposition 8.1 in [22]. The Representation Theorem we will show in this chapter will thus characterize the functions satisfying Unary Language Invariance and Constant Irrelevance.

We will begin with a few observations concerning Predicate Exchangeability.

Suppose a probability function $w$ on some language $L$ satisfied Predicate Exchangeability. Then the probability that $w$ assigns any atom $\alpha$ of $L$ only depends on the number of predicates in $\alpha$ that occur negated.[1] To see this notice that if $\alpha, \alpha'$ are atoms then $\alpha'$ can be obtained from $\alpha$ by a permutation of predicates just if both atoms have the same number of negated predicates.

It is thus convenient to introduce a function assigning each atom the corresponding number of predicates:

**Definition 2.1:** *Let $L = L_q$. Define $\gamma_q : \{1, \dots, 2^q\} \to \{0, \dots, q\}$ by*

$$\gamma_q(i) = k \Leftrightarrow \alpha_i \textit{ contains } k \textit{ negated predicates.}$$

*We shall drop the index $q$ whenever it is understood from the context.*

Slightly abusing notation, we write '$\gamma_q(\alpha) = j$' to mean '$\gamma_q(i) = j$ for that $i$ with $\alpha = \alpha_i$' for atoms of $L_q$.

Now considering $\vec{c} \in \mathbb{D}_{2^q}$ it follows that $w_{\vec{c}}$ satisfies Predicate Exchangeability if and only if $c_i = c_j$ whenever $\gamma(i) = \gamma(j)$. With this in mind we shall assume that our enumeration of the atoms is such that the number of negated predicates is non-decreasing as we move right through $\alpha_1, \alpha_2, \dots, \alpha_{2^q}$. Since for each $i \in \{0, \dots, q\}$ there are $\binom{q}{i}$ atoms of $L_q$ with $i$ predicates occurring negatively we therefore have that for $w_{\vec{c}}$ satisfying Px

$$\vec{c} = \langle \mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_2, \dots, \mathcal{C}_{q-1}, \dots, \mathcal{C}_{q-1}, \mathcal{C}_q \rangle,$$

i.e. $c_i = \mathcal{C}_{\gamma(i)}$ for $i = 1, 2, \dots, 2^q$, and

$$\sum_{i=0}^{q} \binom{q}{i} \mathcal{C}_i = 1.$$

[1] This is an arbitrary choice. One could also count the number of predicates that occur positively in $\alpha$, as the argument is symmetrical.

Thus any such $\vec{c}$ gives us a unique $\vec{\mathcal{C}} = \langle \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_q \rangle$ with the properties

$$\forall i \in \{0, \dots, q\}\ \mathcal{C}_i \geq 0 \text{ and } 1 = \sum_{i=0}^{q} \binom{q}{i} \mathcal{C}_i.$$

Conversely, any $\vec{\mathcal{C}}$ with these properties provides a unique $\vec{c} \in \mathbb{D}_{2^q}$ such that $w_{\vec{c}}$ satisfies Px, giving us a 1-1 correspondence between these $\vec{c} \in \mathbb{D}_{2^q}$ and the elements of

$$\widehat{\mathbb{D}}_q := \left\{ \vec{\mathcal{C}} = \langle \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_q \rangle \,|\, \forall i \in \{0, \dots, q\}\, \mathcal{C}_i \geq 0 \text{ and } 1 = \sum_{i=0}^{q} \binom{q}{i} \mathcal{C}_i \right\}. \quad (2.1)$$

We shall refer to elements of the set above as the *alternative notation* for such a $\vec{c} \in \mathbb{D}_{2^q}$.

Given an atom $\alpha$ of $L_q$, we can view this atom as a quantifier-free sentence in the extended language $L_{q+1}$, and obtain

$$\alpha(x) \equiv \alpha^+(x) \vee \alpha^-(x) = (\alpha(x) \wedge P_{q+1}(x)) \vee (\alpha(x) \wedge \neg P_{q+1}(x)).$$

Now suppose $\vec{c} \in \mathbb{D}_{2^q}$, $\vec{d} \in \mathbb{D}_{2^{q+1}}$ are such that $w_{\vec{d}} \restriction SL_q = w_{\vec{c}}$ and both satisfy Px. Then by the logical equivalence given above, we must have

$$w_{\vec{c}}(\alpha) = w_{\vec{d}}(\alpha) = w_{\vec{d}}(\alpha^+) + w_{\vec{d}}(\alpha^-).$$

Suppose $\vec{\mathcal{C}} \in \widehat{\mathbb{D}}_q$, $\vec{\mathcal{D}} \in \widehat{\mathbb{D}}_{q+1}$ are the corresponding alternative notations for $\vec{c}$ and $\vec{d}$. Then we obtain for each $i \in \{0, \dots, q\}$,

$$\mathcal{C}_i = \mathcal{D}_i + \mathcal{D}_{i+1}.$$

The following proposition generalizes this to ULi families.

**Proposition 2.2.** *Let $w_{\vec{c}}$ be a probability function on $L_q$. Suppose $w_{\vec{c}}$ is a member of a ULi with IP family $\mathcal{W}$ and assume $w_{\vec{d}} \in \mathcal{W}$ is a probability function on $L_r$ for some $r > q$. Let $\vec{\mathcal{C}}, \vec{\mathcal{D}}$ be the corresponding alternative notations for $\vec{c}$, $\vec{d}$. Then for each $j \in \{0, \dots, q\}$, we have*

$$\mathcal{C}_j = \sum_{k=j}^{r-q+j} \binom{r-q}{k-j} \mathcal{D}_k. \quad (2.2)$$

**Proof:** We show this by induction on $r - q$. In case $r - q = 1$, we have for each $j \in \{0, \dots, q\}$,

$$\mathcal{C}_j = \mathcal{D}_j + \mathcal{D}_{j+1},$$

since for $\alpha$ an atom of $L_q$ with $\gamma_q(\alpha) = j$, we have in $L_r$ $(= L_{q+1})$

$$\alpha \equiv \alpha^+ \vee \alpha^-,$$

where $\alpha^+, \alpha^-$ are atoms of $L_r$ with $\gamma_r(\alpha^+) = j$, $\gamma_r(\alpha^-) = j+1$.

Now let $r-q = p+1$ and assume the result holds for $p$. Let $\mathcal{D}'_i$ denote the corresponding values for the atoms of $L_{q+p}$. By the inductive hypothesis we have

$$\mathcal{C}_j = \sum_{k=j}^{(q+p)-q+j} \binom{(q+p)-q}{k-j} \mathcal{D}'_k.$$

Just as in the case $r - q = 1$ we have $\mathcal{D}'_k = \mathcal{D}_k + \mathcal{D}_{k+1}$ for each $0 \leq k \leq q+p$, so we obtain

$$\mathcal{C}_j = \sum_{k=j}^{p+j} \binom{p}{k-j} (\mathcal{D}_k + \mathcal{D}_{k+1}) = \sum_{k=j}^{p+1+j} \binom{p+1}{k-j} \mathcal{D}_k = \sum_{k=j}^{r-q+j} \binom{r-q}{k-j} \mathcal{D}_k,$$

as required. ⊣

With this proposition in mind, we are ready to proceed to the first Representation Theorem.

**Theorem 2.3.** *Let $\vec{c} \in \mathbb{D}_{2^q}$ and $w_{\vec{c}}$ be a probability function satisfying Px. Then $w_{\vec{c}}$ is a member of a ULi with IP family $\mathcal{W} = \{w_{\vec{d_r}} \mid \vec{d_r} \in \mathbb{D}_{2^r}, r \in \mathbb{N}^+\}$ if and only if each entry $c_i$ of $\vec{c}$ is of the form*

$$c_i = \int_{[0,1]} x^{\gamma(i)}(1-x)^{q-\gamma(i)}\, d\rho(x) \tag{2.3}$$

*for some normalized $\sigma$-additive measure $\rho$ on $[0,1]$.*

**Proof:** We will use methods from Non-Standard Analysis working in a suitable non-standard universe ${}^*\mathbb{V}$, see [7] and section 1.5. The key idea to the proof is to marginalize some $w_{\vec{c}}$ on some infinite language to finite languages, rather than constructing extensions of some $w_{\vec{d}}$ on a finite language for each finite level.

Suppose we have such a ULi with IP family $\mathcal{W}$ of probability functions, so for each $r \in \mathbb{N}$, we have some $w^{(r)}$ on $L_r$ in this family. By the Transfer Principle this holds for each $r \in {}^*\mathbb{N}$, so we can pick some non-standard natural number $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ and consider $w^{(\nu)}$. Now $w^{(\nu)} \restriction SL_r = w^{(r)}$ for each $r < \nu$, as these are members of the same ULi family and we can retrieve our original family $\mathcal{W}$ by looking at functions of the form $w^{(\nu)} \restriction SL_r$ for $r \in \mathbb{N}$, taking standard parts – denoted as usual by $^\circ$ – where necessary.

In more detail let $^*\mathbb{V}$ be a non-standard universe that contains at least $\mathbb{D}_{2^q}$ for finite $q \in \mathbb{N}$, all probability functions $w_{\vec{b}}$ satisfying Px and everything else needed in this proof. Let $\nu \in {}^*\mathbb{N}$ be non-standard and consider $\vec{b} \in \mathbb{D}_{2^\nu}$ such that $w_{\vec{b}}$ on $L_\nu$ satisfies Px. Assume that $\vec{\mathcal{B}}$ is the alternative notation for $\vec{b}$ given by (2.1). For each $q < \nu$, we can define a probability function on $L_q$ *in* $^*\mathbb{V}$ satisfying Px by letting

$$\mathcal{C}_j = \sum_{\kappa=j}^{\nu-q+j} \binom{\nu-q}{\kappa-j} \mathcal{B}_\kappa \tag{2.4}$$

for $j = 0, \ldots, q$. In general, this gives $\vec{c} \in {}^*\mathbb{D}_{2^q}$, so we need to take the standard part of $\vec{c}$, denoted $^\circ\vec{c}$, to get a probability function $w_{^\circ\vec{c}}$ in $\mathbb{V}$.

We will first look at $\vec{\mathcal{B}}$ when all weight is concentrated on a single $\mathcal{B}_\kappa$, $0 \leq \kappa \leq \nu$. Since we need to have $\sum_{\kappa=0}^{\nu} \binom{\nu}{\kappa} \mathcal{B}_\kappa = 1$, we obtain

$$\mathcal{B}_\kappa = \binom{\nu}{\kappa}^{-1}.$$

Then we get for $0 \leq j \leq q$

$$\begin{aligned} \mathcal{C}_j = \binom{\nu-q}{\kappa-j}\mathcal{B}_\kappa &= \binom{\nu-q}{\kappa-j} \cdot \binom{\nu}{\kappa}^{-1} \\ &= \frac{(\nu-q)! \cdot \kappa! \cdot (\nu-\kappa)!}{(\kappa-j)! \cdot (\nu-q-\kappa+j)! \cdot \nu!} \\ &= \frac{\kappa \cdot (\kappa-1) \cdots (\kappa-j+1) \cdot (\nu-\kappa) \cdots (\nu-\kappa-q+j+1)}{\nu \cdot (\nu-1) \cdots (\nu-q+1)}, \end{aligned} \tag{2.5}$$

thus leading to the standard part being

$$^\circ\mathcal{C}_j = {}^\circ\left(\left(\frac{\kappa}{\nu}\right)^j \cdot \left(1 - \frac{\kappa}{\nu}\right)^{q-j}\right) = {}^\circ\left(\frac{\kappa}{\nu}\right)^j \cdot \left(1 - {}^\circ\left(\frac{\kappa}{\nu}\right)\right)^{q-j}. \tag{2.6}$$

Now consider an arbitrary $\vec{\mathcal{B}} = \langle \mathcal{B}_0, \ldots, \mathcal{B}_\nu \rangle$. Then for each $0 \leq \kappa \leq \nu$ there exists $\gamma_\kappa \in {}^*[0,1]$ such that we can write

$$\mathcal{B}_\kappa = \gamma_\kappa \cdot \binom{\nu}{\kappa}^{-1}.$$

Note that since

$$\sum_{\kappa=0}^{\nu} \binom{\nu}{\kappa} \mathcal{B}_\kappa = 1$$

we must have

$$\sum_{\kappa=0}^{\nu} \gamma_\kappa = 1.$$

Then using (2.5) we see that each summand in $\mathcal{C}_j$ will be of the form

$$\gamma_\kappa \cdot \binom{\nu - q}{\kappa - j} \binom{\nu}{\kappa}^{-1},$$

thus ${}^\circ\mathcal{C}_j$ will become

$$ {}^\circ\mathcal{C}_j = {}^\circ\left( \sum_{\kappa=j}^{\nu-q+j} \gamma_\kappa \cdot \binom{\nu - q}{\kappa - j} \binom{\nu}{\kappa}^{-1} \right). \tag{2.7}$$

Since we are only interested in the standard part, we can add the finitely many summands for $\kappa = 0, \ldots, j-1, \nu - q + j + 1, \ldots, \nu$ without changing ${}^\circ\mathcal{C}_j$ (assuming that $0 < j < q$), as we have

$$\begin{aligned}
&{}^\circ\left( \sum_{\kappa=0}^{\nu} \gamma_\kappa \cdot \binom{\nu - q}{\kappa - j} \binom{\nu}{\kappa}^{-1} \right) - {}^\circ\mathcal{C}_j \\
&= {}^\circ\left( \sum_{\kappa=0}^{j-1} \gamma_\kappa \cdot \binom{\nu - q}{\kappa - j} \binom{\nu}{\kappa}^{-1} \right) + {}^\circ\left( \sum_{\kappa=\nu-q+j+1}^{\nu} \gamma_\kappa \cdot \binom{\nu - q}{\kappa - j} \binom{\nu}{\kappa}^{-1} \right) \\
&= \sum_{\kappa=0}^{j-1} {}^\circ\left( \gamma_\kappa \cdot \binom{\nu - q}{\kappa - j} \binom{\nu}{\kappa}^{-1} \right) + \sum_{\kappa=\nu-q+j+1}^{\nu} {}^\circ\left( \gamma_\kappa \cdot \binom{\nu - q}{\kappa - j} \binom{\nu}{\kappa}^{-1} \right) \\
&= 0 + 0,
\end{aligned}$$

because for $\kappa \in \{0, \ldots, j-1, \nu-q+j+1, \ldots, \nu\}$, either ${}^\circ(\kappa/\nu) = 0$ or ${}^\circ(1-\kappa/\nu) = 0$, so the first and last sum vanish as each consists of finitely many terms. Note that in case $j = 0, q$, either the first or the second summand is empty, and therefore we can apply the same argument for $j = 0, q$ as well, giving

$$ {}^\circ\mathcal{C}_j = {}^\circ\left( \sum_{\kappa=0}^{\nu} \gamma_\kappa \cdot \binom{\nu - q}{\kappa - j} \binom{\nu}{\kappa}^{-1} \right) \tag{2.8}$$

for $j \in \{0, \ldots, q\}$.

Now let $N = \{0, \ldots, \nu\}$ and (in ${}^*\mathbb{V}$ of course) let $\mu$ be the Loeb counting measure on $N$ (see example (1), section 2 in [7]). Then we can write (2.8) as

$$^\circ\mathcal{C}_j = {}^\circ\!\!\int_N \gamma_\kappa \cdot \binom{\nu - q}{\kappa - j} \binom{\nu}{\kappa}^{-1} d\mu(\kappa). \tag{2.9}$$

Let $\mu'$ be the discrete measure on ${}^*[0,1]$ which for $\kappa \in N$ gives the point $\kappa/\nu$ measure $\gamma_\kappa$. Then we get

$$\int_N \gamma_\kappa \cdot \binom{\nu - q}{\kappa - j} \cdot \binom{\nu}{\kappa}^{-1} d\mu(\kappa) = \int_{{}^*[0,1]} \binom{\nu - q}{x \cdot \nu - j} \cdot \binom{\nu}{x \cdot \nu}^{-1} d\mu'(x). \tag{2.10}$$

Now let $\rho$ be the measure *in* $\mathbb{V}$ on $[0,1]$ which for a Borel subset $A$ of $[0,1]$ gives

$$\rho(A) = {}^\circ\mu'({}^*A). \tag{2.11}$$

By well known results from Loeb Measure Theory, see Theorem 1.24,

$$^\circ\!\!\int_{{}^*[0,1]} \binom{\nu - q}{x \cdot \nu - j} \cdot \binom{\nu}{x \cdot \nu}^{-1} d\mu'(x) = \int_{[0,1]} {}^\circ\!\left(\binom{\nu - q}{x \cdot \nu - j} \cdot \binom{\nu}{x \cdot \nu}^{-1}\right) d\rho(x). \tag{2.12}$$

Combining (2.6),(2.9),(2.10),(2.12) now gives that

$$^\circ\mathcal{C}_j = \int_{[0,1]} x^j \cdot (1 - x)^{q-j} \, d\rho(x) \tag{2.13}$$

We obtain a $\vec{c} \in \mathbb{D}_{2^q}$ by letting

$$\vec{c} = \langle {}^\circ\mathcal{C}_0, {}^\circ\mathcal{C}_1, \ldots, {}^\circ\mathcal{C}_1, \ldots, {}^\circ\mathcal{C}_{q-1}, \ldots, {}^\circ\mathcal{C}_{q-1}, {}^\circ\mathcal{C}_q \rangle.$$

As we can marginalize $\vec{b}$ in the above way to any $r \in \mathbb{N}$, we obtain that given a family of functions $\{w_{\vec{d_r}} \mid \vec{d_r} \in \mathbb{D}_{2^r}\}$ such that each $\vec{d_r}$ is obtained by marginalizing this $\vec{b} \in \mathbb{D}_{2^\nu}$ and therefore satisfies (2.3), this family satisfies Unary Language Invariance.

For the converse it is straightforward to check that any $w_{\vec{c}}$ for which all the $c_i$ in $\vec{c}$ are of the form (2.3) does satisfy ULi, the required family member on $L_r$ being obtained simply by changing $q$ to $r$ with the same measure $\rho$:

We have for any $j \in \{0, \ldots, q\}$ that

$$x^j(1-x)^{q+1-j} + x^{j+1}(1-x)^{q+1-(j+1)} = x^j(1-x)^{q-j} \cdot (x + (1-x)) = x^j(1-x)^{q-j}$$

from which – using the linearity of the integral – we get that for $\vec{d} \in \mathbb{D}_{2^{q+1}}$ defined by the same measure $\rho$ as $w_{\vec{c}}$ we obtain

$$w_{\vec{d}}(\alpha) = w_{\vec{c}}(\alpha) \tag{2.14}$$

for any atom $\alpha$ of $L_q$. Since $w_{\vec{c}}$, $w_{\vec{d}}$ satisfy Constant Irrelevance, it is enough to show (2.14) for the atoms of $L_q$ to obtain $w_{\vec{d}} \restriction SL_q = w_{\vec{c}}$. ⊣

However, as the following example will show, the probability functions of the form $w_{\vec{c}}$ satisfying ULi with IP are not the building blocks that generate all probability functions satisfying ULi:

**Example 2.4.** *Let $c_0^{L_2}$ be the probability function on $L_2$ given by*

$$c_0^{L_2} = 4^{-1}\left(w_{\langle 1,0,0,0\rangle} + w_{\langle 0,1,0,0\rangle} + w_{\langle 0,0,1,0\rangle} + w_{\langle 0,0,0,1\rangle}\right).$$

*Then $c_0^{L_2}$ satisfies ULi as it is a member of Carnap's Continuum of Inductive Methods (see e.g. [22]). However, both $\langle 0,1,0,0\rangle$ and $\langle 0,0,1,0\rangle$ are not of the form (2.3), and thus $c_0^{L_2}$ shows that we cannot have a Representation Theorem for $w$ satisfying ULi of the form*

$$w = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}\ d\mu(\vec{x})$$

*with $\mu$ giving all weight to $\vec{c}$ of the form (2.3).*

One might argue that in the example above, we have just picked the wrong representation of $c_0$, and we may yet see that in some other representation, it is indeed a mixture of functions $w_{\vec{c}}$ with $\vec{c}$ of the form (2.3). We shall show that this is not the case. The key to this is the Principle of Regularity. In fact, we obtain the following corollary to Theorem 2.3.

**Corollary 2.5.** *Let $w_{\vec{c}}$ satisfy Unary Language Invariance with $\vec{c} \in \mathbb{D}_{2^q}$ of the form (2.3) for some measure $\rho$ on $[0,1]$ and $q > 1$. Then $w_{\vec{c}}$ violates Regularity if and only if $\rho$ is a discrete measure with all weight concentrated on $\{0,1\}$.*

**Proof:** If $\rho$ is discrete with $\rho(0) = \lambda$, $\rho(1) = 1 - \lambda$, then clearly we have that $c_i = 0$ for each $i \neq 1, 2^q$ and therefore $w_{\vec{c}}$ violates Regularity.

So suppose $w_{\vec{c}}$ violates Regularity. Then there exists some quantifier-free $\vartheta(a_1, \ldots, a_n)$ such that $w_{\vec{c}}(\vartheta(a_1, \ldots, a_n)) = 0$. By the Disjunctive Normal Form Theorem, we obtain that for each state description $\Theta(a_1, \ldots, a_m)$ with $\Theta(a_1, \ldots, a_m) \models \vartheta(a_1, \ldots, a_n)$ we must have

$$w(\Theta(a_1, \ldots, a_m)) = 0.$$

Therefore, there must be some $i \in \{1, \ldots, 2^q\}$ such that $c_i = 0$.

Assume for a contradiction that there is some $c_j \neq 0$ with $j \neq 1, 2^q$. We may assume without loss of generality that $c_{i+1} \neq 0$, since in the case that all $c_j = 0$ for $j > i$, there must be some $c_j \neq 0$ for $j < i$ maximal, i.e. we can assume $c_{i-1} \neq 0$ and the argument used in this case is symmetric.

Suppose $\vec{\mathcal{C}}$ is the alternative notation of $\vec{c}$. Working with the alternative notations for the extensions $w_{\vec{d}}$ of $w_{\vec{c}}$ on languages extending $L_q$, we obtain that since

$$0 = \mathcal{C}_i = \sum_{k=i}^{r-q+i} \binom{r-q}{k-i} \mathcal{D}_k,$$

we must have $\mathcal{D}_k = 0$ for $k \in \{i, \ldots, r-q+i\}$, for any $r > q$ and $\vec{\mathcal{D}}$ the alternative notation for $\vec{d}$ with $w_{\vec{d}}$ the function extending $w_{\vec{c}}$ to $L_r$. It now follows that

$$\mathcal{C}_{i+1} = \sum_{k=i+1}^{r-q+i+1} \binom{r-q}{k-(i+1)} \mathcal{D}_k = \mathcal{D}_{r-q+i+1} > 0.$$

Since we assumed $i+1 < 2^q$, $\mathcal{C}_{i+2}$ exists and we obtain

$$\mathcal{C}_{i+2} = (r-q) \cdot \mathcal{D}_{r-q+i+1} + \mathcal{D}_{r-q+i+2} = (r-q) \cdot \mathcal{C}_{i+1} + \mathcal{D}_{r-q+i+2}.$$

Pick $r$ large so that $r - q > \frac{1}{\mathcal{C}_{i+1}}$. Then clearly $(r-q) \cdot \mathcal{C}_{i+1} > 1$, and since $w_{\vec{c}}$ is a probability function, we must have that $\mathcal{D}_{r-q+i+2} < 0$, contradicting the assumption that there exists $c_j \neq 0$ for some $j$ other than $1, 2^q$. $\dashv$

Recalling Example 2.4, we can now observe that if $c_0$ was a mixture of $w_{\vec{d}}$ satisfying ULi, a measure 1 set of these $w_{\vec{d}}$ must violate Regularity, as we have $c_0(\alpha_i \wedge \alpha_j) = 0$ for any two distinct atoms $\alpha_i$, $\alpha_j$ of $L_2$; for if there was one $w_{\vec{c}}$ in this mixture satisfying Regularity, we would have $c_0(\alpha_i \wedge \alpha_j) > 0$. However, by the corollary above this would imply that $c_0(\alpha_i) = 0$ for $i \in \{2, 3\}$, which is clearly not the case.

## 2.2 The Representation Theorem for $w$ satisfying ULi

In the previous section, we used a probability function satisfying Px + IP on the infinite language $L_\nu$ to construct a language invariant family by marginalizing to each finite level.

In this section we shall instead derive a representation theorem for just ULi by using an arbitrary state description $\Upsilon$ of $L_\nu$ to construct a probability function satisfying Px by averaging over all permutations of predicates, similarly to the definition of $c_0^{L_2}$ in Example 2.4.

Let $\Upsilon(P_1, \ldots, P_\nu, a_1, \ldots, a_\nu)$ be the state description of $L_\nu$ given by

$$\Upsilon(P_1, \ldots, P_\nu, a_1, \ldots, a_\nu) = \bigwedge_{i=1}^{\nu} \bigwedge_{j=1}^{\nu} P_i^{\varepsilon_{i,j}}(a_j).$$

Then we can represent $\Upsilon$ by the $\nu \times \nu$ - matrix

$$\begin{pmatrix} \varepsilon_{1,1} & \varepsilon_{1,2} & \cdots & \varepsilon_{1,\nu} \\ \varepsilon_{2,1} & \varepsilon_{2,2} & \cdots & \varepsilon_{2,\nu} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{\nu,1} & \varepsilon_{\nu,2} & \cdots & \varepsilon_{\nu,\nu} \end{pmatrix}. \tag{2.15}$$

Now consider the $q \times \nu$ - matrix $\Psi$ where the $j$'th row of $\Psi$ is the $i_j$'th row of $\Upsilon$, for some $i_1, \ldots, i_q \in \{1, \ldots, \nu\}$, not necessarily distinct. Then we can similarly think of $\Psi$ as a state description $\Psi(a_1, \ldots, a_\nu)$ of $L_q$. So each column of $\Psi$ represents an atom of $L_q$, and we obtain $\vec{c} \in {}^*\mathbb{D}_{2^q}$ by letting

$$c_i = \frac{|\{j \mid \Psi \models \alpha_i(a_j)\}|}{\nu}.$$

We thus obtain for each $\langle i_1, \ldots, i_q \rangle$ with $1 \leq i_1, \ldots, i_q \leq \nu$ some $w_{\vec{c}}$ for $\vec{c} \in {}^*\mathbb{D}_{2^q}$, which we shall denote by $w^{\Upsilon}_{\langle i_1, \ldots, i_q \rangle}$.

We can now define the functions that we will then use to prove the representation theorem for general ULi functions.

**Definition 2.6:** *Let $\Upsilon(P_1, \dots, P_\nu, a_1, \dots, a_\nu)$ be a state description of $L_\nu$ for $\nu$ distinct constants. Let $L = L_q$ for some finite $q$. For $i_1, \dots, i_q \in \{1, \dots, \nu\}$, not necessarily distinct, let $w^{\Upsilon}_{\langle i_1,\dots,i_q\rangle}$ be given as above.*

*Define the function $\nabla^L_\Upsilon$ on $SL$ by*

$$\nabla^L_\Upsilon = \sum_{e:\{1,\dots,q\}\to\{1,\dots,\nu\}} \frac{1}{\nu^q} w^{\Upsilon}_{\langle e(1),\dots,e(q)\rangle}.$$

Instead of just marginalizing to the first $q$ rows, as we did in the case of $w_{\vec{c}}$, $\nabla^L_\Upsilon$ now also averages over all permutations of the predicates. One can think of this as picking $q$ rows from the matrix representing $\Upsilon$ *with replacement* to obtain the predicates $P_1, \dots, P_q$ of $L_q$.

**Theorem 2.7.** *Let $\Upsilon(P_1, \dots, P_\nu, a_1, \dots, a_\nu)$ be a state description of $L_\nu$ and let $L = L_q$. Then the function ${}^\circ\nabla^L_\Upsilon$ is (can be extended to) a probability function on SL satisfying ULi + WIP.*

**Proof:** From the definition of $\nabla^L_\Upsilon$ it is obvious that ${}^\circ\nabla^L_\Upsilon$ is a probability function satisfying Ex.

For Px, let $\sigma$ be a permutation of the predicates of $L$. Then we obtain

$$\begin{aligned} {}^\circ\nabla^L_\Upsilon(\sigma\Theta) &= {}^\circ\left[\sum_{e:\{1,\dots,q\}\to\{1,\dots,\nu\}} \frac{1}{\nu^q} \cdot w^{\Upsilon}_{\langle e(1),\dots,e(q)\rangle}(\sigma\Theta)\right] \\ &= {}^\circ\left[\sum_{e:\{1,\dots,q\}\to\{1,\dots,\nu\}} \frac{1}{\nu^q} \cdot w^{\Upsilon}_{\langle e(\sigma^{-1}(1)),\dots,e(\sigma^{-1}(q))\rangle}(\Theta)\right], \end{aligned}$$

since $\sigma$ permutes the predicates of $L$,

$$\begin{aligned} &= {}^\circ\left[\sum_{e\circ\sigma^{-1}:\{1,\dots,q\}\to\{1,\dots,\nu\}} \frac{1}{\nu^q} \cdot w^{\Upsilon}_{\langle e\circ\sigma^{-1}(1),\dots,e\circ\sigma^{-1}(q)\rangle}(\Theta)\right] \\ &= {}^\circ\left[\sum_{e':\{1,\dots,q\}\to\{1,\dots,\nu\}} \frac{1}{\nu^q} \cdot w^{\Upsilon}_{\langle e'(1),\dots,e'(q)\rangle}(\Theta)\right] = {}^\circ\nabla^L_\Upsilon(\Theta). \end{aligned}$$

To show that ULi holds, notice that for $\Theta(a_1, \dots, a_n)$ the state description

$$\Theta(a_1, \dots, a_n) = \bigwedge_{j=1}^{n} \alpha_{h_j}(a_j),$$

we obtain on $L_{q+1}$,

$$\Theta(a_1,\dots,a_n) \equiv \bigvee_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}} \bigwedge_{j=1}^{n} \alpha_{h_j}^{\varepsilon_j}(a_j),$$

where

$$\alpha_{h_j}^{\varepsilon_j}(x) = \alpha_{h_j}(x) \wedge P_{q+1}^{\varepsilon_j}(x).$$

We obtain

$$\begin{aligned}
&{}^\circ\nabla_{\Upsilon}^{L_{q+1}}(\Theta)\\
&\quad= \sum_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}} {}^\circ\nabla_{\Upsilon}^{L_{q+1}}\left(\bigvee_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}} \bigwedge_{j=1}^{n} \alpha_{h_j}^{\varepsilon_j}\right)\\
&\quad= \sum_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}} {}^\circ\left[\sum_{e:\{1,\dots,q+1\}\to\{1,\dots,\nu\}} \frac{1}{\nu^{q+1}} w^{\Upsilon}_{\langle e(1),\dots,e(q+1)\rangle}\left(\bigvee_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}} \bigwedge_{j=1}^{n} \alpha_{h_j}^{\varepsilon_j}\right)\right]\\
&\quad= {}^\circ\left[\sum_{e:\{1,\dots,q+1\}\to\{1,\dots,\nu\}} \frac{1}{\nu^{q+1}} \sum_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}} w^{\Upsilon}_{\langle e(1),\dots,e(q+1)\rangle}\left(\bigvee_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}} \bigwedge_{j=1}^{n} \alpha_{h_j}^{\varepsilon_j}\right)\right]\\
&\quad= {}^\circ\left[\sum_{e':\{1,\dots,q\}\to\{1,\dots,\nu\}} \frac{1}{\nu^{q}} \cdot\right.\\
&\qquad\qquad \left.\sum_{f:\{1\}\to\{1,\dots,\nu\}} \frac{1}{\nu} \sum_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}} w^{\Upsilon}_{\langle e'(1),\dots,e'(q),f(1)\rangle}\left(\bigvee_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}} \bigwedge_{j=1}^{n} \alpha_{h_j}^{\varepsilon_j}\right)\right],
\end{aligned}$$

where

$$e(i) = \begin{cases} e'(i) & \text{if } i \in \{1,\dots,q\},\\ f(1) & \text{if } i = q+1.\end{cases}$$

It now remains to show that

$$\sum_{f:\{1\}\to\{1,\dots,\nu\}} \frac{1}{\nu} \sum_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}} w^{\Upsilon}_{\langle e'(1),\dots,e'(q),f(1))\rangle}\left(\bigvee_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}} \bigwedge_{j=1}^{n} \alpha_{h_j}^{\varepsilon_j}\right) = w^{\Upsilon}_{\langle e'(1),\dots,e'(q)\rangle}(\Theta) \tag{2.16}$$

for arbitrary $e' : \{1,\dots,q\} \to \{1,\dots,\nu\}$. There are $\vec{c} \in {}^*\mathbb{D}_{2^q}$, $\vec{d} \in {}^*\mathbb{D}_{2^{q+1}}$ such that

$$\begin{aligned} w^{\Upsilon}_{\langle e'(1),\dots,e'(q)\rangle} &= w_{\vec{c}},\\ w^{\Upsilon}_{\langle e'(1),\dots,e'(q),f(1)\rangle} &= w_{\vec{d}}. \end{aligned}$$

Given $\beta_j$ an atom of $L_{q+1}$, there is a unique atom $\alpha_i$ of $L_q$ and a unique $\varepsilon \in \{0,1\}$ such that

$$\beta_j = \alpha_i^{\varepsilon}.$$

Thus, we can unambiguously write $d_j = c_i^{\varepsilon}$ for these $i$, $\varepsilon$. We then obtain

$$\begin{aligned} w^{\Upsilon}_{\langle e'(1),\ldots,e'(q),f(1)\rangle}\left(\bigvee_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}}\bigwedge_{j=1}^{n}\alpha_{h_j}^{\varepsilon_j}\right) &= \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} w^{\Upsilon}_{\langle e'(1),\ldots,e'(q),f(1)\rangle}\left(\bigwedge_{j=1}^{n}\alpha_{h_j}^{\varepsilon_j}\right) \\ &= \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}}\prod_{j=1}^{n} c_{h_j}^{\varepsilon_j} \\ &= \prod_{j=1}^{n}(c_{h_j}^{0} + c_{h_j}^{1}). \qquad (2.17) \end{aligned}$$

Since by picking row $f(1)$ as the $q+1$'st row we partition the occurrences of the atom $\alpha_j$ of $L_q$ obtained by picking rows $e'(1),\ldots,e'(q)$ into occurrences of the atoms $\alpha_j^1$ and $\alpha_j^0$ of $L_{q+1}$, and this is the only way in which we obtain these atoms, we must have $c_i^0 + c_i^1 = c_i$ for each $i \in \{1,\ldots,2^q\}$. Thus (2.17) gives

$$\prod_{j=1}^{n}(c_{h_j}^{0} + c_{h_j}^{1}) = \prod_{j=1}^{n} c_{h_j} = w_{\langle e'(1),\ldots,e'(q)\rangle}(\Theta).$$

The equation (2.16) now follows.

It remains to show that Weak Irrelevance (i.e. WIP) holds for ${}^{\circ}\nabla^{L}_{\Upsilon}$. Let $\vartheta(a_1,\ldots,a_m)$, $\varphi(a_{m+1},\ldots,a_{m+n})$ be state descriptions of $L$ having no constants or predicates in common. We can assume that $\vartheta \in QFSL^1$, $\varphi \in QFSL^2$, where $L^1 \cap L^2 = \emptyset$ and $L^1 \cup L^2 = L$. Let $\alpha_i$ range over the atoms of $L^1$, $\beta_j$ over the atoms of $L^2$. Then we obtain in $L^1$ and $L^2$, respectively,

$$\vartheta(a_1,\ldots,a_m) = \bigwedge_{i=1}^{m}\alpha_{h_i}(a_i),$$

$$\varphi(a_{m+1},\ldots,a_{m+n}) = \bigwedge_{j=1}^{n}\beta_{g_j}(a_{m+j}).$$

Suppose that $L^1 = \{P_1,\ldots,P_p\}$, $L^2 = \{P_{p+1},\ldots,P_{p+r}\}$. Then we obtain in $L$

$$\vartheta(a_1,\ldots,a_m) \equiv \bigvee_{1\le s_1,\ldots,s_m\le 2^r}\bigwedge_{i=1}^{m}\alpha_{h_i}(a_i)\wedge\beta_{s_i}(a_i),$$

$$\varphi(a_{m+1},\ldots,a_{m+n}) \equiv \bigvee_{1\le t_1,\ldots,t_n\le 2^p}\bigwedge_{j=1}^{n}\alpha_{t_j}(a_{m+j})\wedge\beta_{g_j}(a_{m+j}),$$

and by ULi for ${}^{\circ}\nabla^{L}_{\Upsilon}$,

$$ {}^{\circ}\nabla^{L^1}_{\Upsilon}(\vartheta) = {}^{\circ}\nabla^{L}_{\Upsilon}\left(\bigvee_{1\leq s_1,\ldots,s_m\leq 2^r}\bigwedge_{i=1}^{m}\alpha_{h_i}\wedge\beta_{s_i}\right), \tag{2.18} $$

$$ {}^{\circ}\nabla^{L^2}_{\Upsilon}(\varphi) = {}^{\circ}\nabla^{L}_{\Upsilon}\left(\bigvee_{1\leq t_1,\ldots,t_n\leq 2^p}\bigwedge_{j=1}^{n}\alpha_{t_j}\wedge\beta_{g_j}\right). \tag{2.19} $$

Now for $\vartheta\wedge\varphi$, we obtain in $L$

$$ \begin{aligned} &{}^{\circ}\nabla^{L}_{\Upsilon}(\vartheta\wedge\varphi)\\ &= {}^{\circ}\nabla^{L}_{\Upsilon}\left(\bigvee_{1\leq s_1,\ldots,s_m\leq 2^r}\ \bigvee_{1\leq t_1,\ldots,t_n\leq 2^p}\left(\bigwedge_{i=1}^{n}\alpha_{h_i}\wedge\beta_{s_i}\right)\wedge\left(\bigwedge_{j=1}^{n}\alpha_{t_j}\wedge\beta_{g_j}\right)\right)\\ &= \sum_{1\leq s_1,\ldots,s_m\leq 2^r}\ \sum_{1\leq t_1,\ldots,t_n\leq 2^p}{}^{\circ}\left[\sum_{e:\{1,\ldots,q\}\to\{1,\ldots,\nu\}}\frac{1}{\nu^q}\cdot\right.\\ &\qquad\left. w^{\Upsilon}_{\langle e(1),\ldots,e(q)\rangle}\left(\left(\bigwedge_{i=1}^{n}\alpha_{h_i}\wedge\beta_{s_i}\right)\wedge\left(\bigwedge_{j=1}^{n}\alpha_{t_j}\wedge\beta_{g_j}\right)\right)\right]\\ &= \sum_{1\leq s_1,\ldots,s_m\leq 2^r}\ \sum_{1\leq t_1,\ldots,t_n\leq 2^p}{}^{\circ}\left[\sum_{e:\{1,\ldots,q\}\to\{1,\ldots,\nu\}}\frac{1}{\nu^q}\cdot w^{\Upsilon}_{\langle e(1),\ldots,e(q)\rangle}\left(\bigwedge_{i=1}^{n}\alpha_{h_i}\wedge\beta_{s_i}\right)\right.\\ &\qquad\left.\cdot w^{\Upsilon}_{\langle e(1),\ldots,e(q)\rangle}\left(\bigwedge_{j=1}^{n}\alpha_{t_j}\wedge\beta_{g_j}\right)\right], \end{aligned} $$

by IP for $w^{\Upsilon}_{\langle e(1),\ldots,e(q)\rangle}$,

$$ \begin{aligned} &= \left(\sum_{1\leq s_1,\ldots,s_m\leq 2^r}{}^{\circ}\left[\sum_{e:\{1,\ldots,q\}\to\{1,\ldots,\nu\}}\frac{1}{\nu^q}\cdot w^{\Upsilon}_{\langle e(1),\ldots,e(q)\rangle}\left(\bigwedge_{i=1}^{n}\alpha_{h_i}\wedge\beta_{s_i}\right)\right]\right)\\ &\quad\cdot\left(\sum_{1\leq t_1,\ldots,t_n\leq 2^p}{}^{\circ}\left[\sum_{e:\{1,\ldots,q\}\to\{1,\ldots,\nu\}}\frac{1}{\nu^q}\cdot w^{\Upsilon}_{\langle e(1),\ldots,e(q)\rangle}\left(\bigwedge_{j=1}^{n}\alpha_{t_j}\wedge\beta_{g_j}\right)\right]\right)\\ &= \left(\sum_{1\leq s_1,\ldots,s_m\leq 2^r}{}^{\circ}\nabla^{L}_{\Upsilon}\left(\bigwedge_{i=1}^{m}\alpha_{h_i}\wedge\beta_{s_i}\right)\right)\cdot\left(\sum_{1\leq t_1,\ldots,t_n\leq 2^p}{}^{\circ}\nabla^{L}_{\Upsilon}\left(\bigwedge_{j=1}^{n}\alpha_{t_j}\wedge\beta_{h_j}\right)\right)\\ &= {}^{\circ}\nabla^{L}_{\Upsilon}(\vartheta)\cdot{}^{\circ}\nabla^{L}_{\Upsilon}(\varphi), \end{aligned} $$

by (2.18) and (2.19). ⊣

**Theorem 2.8.** *Let $w$ be a probability function on $L = L_q$. Then $w$ satisfies ULi if and only if there exists some normalized $\sigma$-additive measure $\rho$ such that*

$$w = \int {}^\circ\nabla^L_\Upsilon \, d\rho(\Upsilon). \tag{2.20}$$

**Proof:** By Theorem 2.7, it is straightforward to see that any $w$ in the form (2.20) satisfies ULi, as it is a convex combination of ULi functions.

For the other direction, suppose $w$ satisfied ULi. Then there is an extension $w^{L_\nu}$ of $w$ to $L_\nu$ and we obtain for $\Theta(a_1, \dots, a_n)$ a state description of $L$,

$$w(\Theta) = \sum_{\substack{\Phi(a_1,\dots,a_\nu) \\ \Phi \models \Theta}} w^{L_\nu}(\Phi), \tag{2.21}$$

where $\Phi$ ranges over the state descriptions of $L_\nu$. For a state description $\Upsilon(P_1, \dots, P_\nu, a_1, \dots, a_\nu)$, let

$$\overline{\Upsilon} = \{\Upsilon(P_{\sigma(1)}, \dots, P_{\sigma(\nu)}, a_{\tau(1)}, \dots, a_{\tau(\nu)}) \,|\, \sigma, \tau \text{ are permutations of } \{1, \dots, \nu\}\}.$$

Note that the sets $\overline{\Upsilon}$ partition the set of state descriptions of $L_\nu$:

Suppose $\Upsilon$, $\Upsilon'$ are two state descriptions of $L_\nu$. Suppose that there exist permutations $\sigma$ of predicates and $\tau$ of constants such that

$$\Upsilon(P_{\sigma(1)}, \dots, P_{\sigma(\nu)}, a_{\tau(1)}, \dots, a_{\tau(\nu)}) \models \Upsilon'(P_1, \dots, P_\nu, a_1, \dots, a_\nu). \tag{2.22}$$

Then $\overline{\Upsilon} = \overline{\Upsilon'}$, since for any $\sigma'$, $\tau'$,

$$\Upsilon'(P_{\sigma'(1)}, \dots, P_{\sigma'(\nu)}, a_{\tau'(1)}, \dots, a_{\tau'(\nu)}) = \Upsilon(P_{\sigma' \circ \sigma(1)}, \dots, P_{\sigma' \circ \sigma(\nu)}, a_{\tau' \circ \tau(1)}, \dots, a_{\tau' \circ \tau(\nu)}),$$

giving $\overline{\Upsilon'} \subseteq \overline{\Upsilon}$. We obtain the other inclusion using $\sigma^{-1}$, $\tau^{-1}$ in place of $\sigma$, $\tau$ and applying the above argument to the elements of $\overline{\Upsilon}$. Suppose there are no $\sigma$, $\tau$ such that (2.22) holds. Then we must have $\overline{\Upsilon} \cap \overline{\Upsilon'} = \emptyset$, since otherwise there would be some $\sigma_1, \sigma_2, \tau_1, \tau_2$ such that for some element of $\overline{\Upsilon'}$, we would have

$$\Upsilon(P_{\sigma_1(1)}, \dots, P_{\sigma_1(\nu)}, a_{\tau_1(1)}, \dots, a_{\tau_1(\nu)}) = \Upsilon'(P_{\sigma_2(1)}, \dots, P_{\sigma_2(\nu)}, a_{\tau_2(1)}, \dots, a_{\tau(\nu)}),$$

and thus we can find $\sigma$, $\tau$ such that (2.22) holds, contradicting the assumption.

Let $v$ be a probability function on $L_\nu$ satisfying Ex + Px. Then $v$ is constant on each $\overline{\Upsilon}$, as we have

$$\begin{aligned} v\left(\Upsilon(P_{\sigma(1)},\dots,P_{\sigma(\nu)},a_{\tau(1)},\dots,a_{\tau(\nu)})\right) &= v\left(\Upsilon(P_1,\dots,P_\nu,a_1,\dots,a_\nu)\right) \\ &= v\left(\Upsilon(P_{\sigma'(1)},\dots,P_{\sigma'(\nu)},a_{\tau'(1)},\dots,a_{\tau'(\nu)})\right) \end{aligned} \tag{2.23}$$

for any $\sigma,\sigma',\tau,\tau'$ permutations of $\{1,\dots,\nu\}$.

Let $\bigvee\overline{\Upsilon}$ denote the disjunction

$$\bigvee_{\Psi\in\overline{\Upsilon}}\Psi \tag{2.24}$$

of the elements of $\overline{\Upsilon}$. We can then write (2.21) as

$$\begin{aligned} w(\Theta) &= \sum_{\overline{\Upsilon}}\sum_{\substack{\Phi\in\overline{\Upsilon}\\ \Phi\models\Theta}} w^{L_\nu}(\Phi) \\ &= \sum_{\overline{\Upsilon}}\frac{|\{\Phi\in\overline{\Upsilon}\mid\Phi\models\Theta\}|}{|\overline{\Upsilon}|}w^{L_\nu}\left(\bigvee\overline{\Upsilon}\right), \end{aligned}$$

as $w^{L_\nu}$ is constant on $\overline{\Upsilon}$ since it satisfies Px (and Ex), and we obtain from (2.23) and (2.24) that

$$w^{L_\nu}\left(\bigvee\overline{\Upsilon}\right) = \sum_{\Psi\in\overline{\Upsilon}} w^{L_\nu}(\Psi) = |\overline{\Upsilon}|\cdot w^{L_\nu}(\Psi)$$

for any $\Psi\in\overline{\Upsilon}$. Now the ratio

$$\frac{|\{\Phi\in\overline{\Upsilon}\mid\Phi\models\Theta\}|}{|\overline{\Upsilon}|}$$

is equal to the probability that by randomly picking distinct predicates $P_{i_1},\dots,P_{i_q}$ and constants $a_{j_1},\dots,a_{j_n}$, we have that

$$\Upsilon\models\sigma\Theta(a_{j_1},\dots,a_{j_n}),$$

where $\sigma$ is (an initial segment of) the permutation of predicates of $L_\nu$ with $\sigma(k)=i_k$ for $k\in\{1,\dots,q\}$.

Note that with our definition of $\nabla^L_\Upsilon$, we allow the same row to be picked multiple times, so not all picks of rows represent a permutation of the predicates. Thus the difference between the probabilities given by $\nabla^L_\Upsilon$ and the above ratio is the difference between picking rows of $\Upsilon$ with and without replacement. However, since the probability of picking the same row twice is infinitesimal, it will disappear when taking standard parts.

Thus we obtain

$$^\circ\left(\frac{|\{\Phi \in \overline{\Upsilon} \mid \Phi \models \Theta\}|}{|\overline{\Upsilon}|}\right) = {}^\circ\nabla^L_\Upsilon(\Theta).$$

Now taking $\mu$ to be the measure on the $\overline{\Upsilon}$ given by $w^{L_\nu}$, we obtain

$$\sum_{\overline{\Upsilon}} \frac{|\{\Phi \in \overline{\Upsilon} \mid \Phi \models \Theta\}|}{|\overline{\Upsilon}|} w^{L_\nu}\left(\bigvee \overline{\Upsilon}\right) = \int \frac{|\{\Phi \in \overline{\Upsilon} \mid \Phi \models \Theta\}|}{|\overline{\Upsilon}|}\, d\mu(\overline{\Upsilon}).$$

Taking standard parts, we obtain

$$\begin{aligned} {}^\circ\!\int \frac{|\{\Phi \in \overline{\Upsilon} \mid \Phi \models \Theta\}|}{|\overline{\Upsilon}|}\, d\mu(\overline{\Upsilon}) &= \int {}^\circ\left(\frac{|\{\Phi \in \overline{\Upsilon} \mid \Phi \models \Theta\}|}{|\overline{\Upsilon}|}\right) d\rho(\overline{\Upsilon}) \\ &= \int {}^\circ\nabla^L_\Upsilon\, d\rho(\overline{\Upsilon}), \end{aligned}$$

where $\rho$ is the Loeb measure given by the non-standard measure $\mu$. ⊣

Since ${}^\circ\nabla^L_\Upsilon$ satisfies WIP we obtain the following theorem.

**Theorem 2.9.** *The ${}^\circ\nabla^L_\Upsilon$ are the only functions satisfying ULi with WIP.*

**Proof:** We follow essentially the proof for the analogous theorem for Atom Exchangeability, given in [21].

Let $w$ be a probability function satisfying ULi with WIP. Let $\vartheta \in QFSL$. Extend $w$ to $w'$ on some language $L'$ large enough so that we can permute the predicates and constants in $\vartheta$ to obtain $\vartheta'$ with no predicates nor constants in common with $\vartheta$. We can achieve this by picking $w'$ on $L'$ in the same ULi family as $w$, giving $w' \restriction SL = w$ and guaranteeing WIP for $w'$. By Px for $w'$ we then have $w'(\vartheta) = w'(\vartheta')$. Now we clearly obtain

$$\begin{aligned}
0 &= 2(w'(\vartheta \wedge \vartheta') - w'(\vartheta) \cdot w'(\vartheta')) \\
&= \int {}^{\circ}\nabla_{\Psi}^{L'}(\vartheta \wedge \vartheta')\, d\mu(\Psi) - 2 \int {}^{\circ}\nabla_{\Psi}^{L'}(\vartheta)\, d\mu(\Psi) \cdot \int {}^{\circ}\nabla_{\Phi}^{L'}(\vartheta')\, d\mu(\Phi) \\
&\quad + \int {}^{\circ}\nabla_{\Phi}^{L'}(\vartheta \wedge \vartheta')\, d\mu(\Phi) \\
&= \int \int \left( {}^{\circ}\nabla_{\Psi}^{L'}(\vartheta)^2 - 2 {}^{\circ}\nabla_{\Psi}^{L'}(\vartheta) \cdot {}^{\circ}\nabla_{\Phi}^{L'}(\vartheta) + {}^{\circ}\nabla_{\Phi}^{L'}(\vartheta)^2 \right) d\mu(\Psi)\, d\mu(\Phi) \\
&= \int \int \left( {}^{\circ}\nabla_{\Psi}^{L'}(\vartheta) - {}^{\circ}\nabla_{\Phi}^{L'}(\vartheta) \right)^2 d\mu(\Psi)\, d\mu(\Phi),
\end{aligned}$$

using the Representation Theorem. Certainly, since the function under the integral is non-negative, there must be a measure 1 set such that ${}^{\circ}\nabla_{\Psi}^{L'}$ is constant on this set for each $\vartheta \in QFSL$, giving $w' = {}^{\circ}\nabla_{\Psi}^{L'}$ for any $\Psi$ in this set. Since $w' \restriction SL = w$, i.e. $w = {}^{\circ}\nabla_{\Psi}^{L'} \restriction SL$, marginalizing $w'$ to $L$ yields $w = {}^{\circ}\nabla_{\Psi}^{L}$, as required. ⊣

## 2.3 A General Representation Theorem

In the case of Atom Exchangeability (Ax) (see e.g. [22, chapter 33]), we have a theorem stating that each $w$ satisfying Ax can be represented as a difference of scaled ULi functions with Ax. In this section, we will prove the analogous version for Px. For the remainder of this section we assume that $L = L_q$ for some $q \in \mathbb{N}$.

**Definition 2.10:** *Let $\vec{c} \in \mathbb{D}_{2^q}$. Let $\Sigma$ be the set of all permutations of atoms of $L$ that are induced by Px. Define the probability function $y_{\vec{c}}$ on QFSL by*

$$y_{\vec{c}}(\Theta(a_1, \ldots, a_n)) = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} w_{\sigma\vec{c}}(\Theta(a_1, \ldots, a_n))$$

*for state descriptions $\Theta(a_1, \ldots, a_n)$ of $L$.*

Note that by definition, $y_{\vec{c}}$ satisfies Px. By a straightforward argument we obtain the following variation on de Finetti's Theorem:

**Theorem 2.11.** *Let $w$ be a propability function on SL satisfying Px. Then there exists a normalized, $\sigma$-additive measure $\mu$ on the Borel sets of $\mathbb{D}_{2^q}$ such that*

$$w\left(\bigwedge_{j=1}^{n} \alpha_{h_j}(a_j)\right) = \int_{\mathbb{D}_{2^q}} y_{\vec{c}}\left(\bigwedge_{j=1}^{n} \alpha_{h_j}(a_j)\right) d\mu(\vec{c}). \tag{2.25}$$

*Conversely, given such a measure $\mu$, the function $w$ defined by (2.25) satisfies Px.*

The key to obtaining the desired General Representation Theorem will therefore involve finding a uniform representation of the building blocks $y_{\vec{c}}$ in terms of a difference of ULi functions. The $^\circ\nabla^L_\Upsilon$ functions used for this proof will have a specific characterization that deserves a slightly different notation. Since at this point, we will be working in the usual standard universe again, we will drop the standard part symbol $^\circ$ from the notation and assume that all $\nabla^L_\Upsilon$ from now on are given in their standard form.

Recalling the definition of $\nabla^L_\Upsilon$ note that for fixed $e : \{1, \dots, q\} \to \{1, \dots, \nu\}$, the function $w^\Upsilon_{\langle e(1),\dots,e(q)\rangle}$ is given by the $q \times \nu$ - matrix with the $i$'th row identical to the $e(i)$'th row of $\Upsilon$. Also, since with $w^\Upsilon_{\langle e(1),\dots,e(q)\rangle}$ we also have all the $w^\Upsilon_{\langle \sigma(e(1)),\dots,\sigma(e(q))\rangle}$ for $\sigma$ ranging over the permutations of the predicates of $L$ occurring in $\nabla^L_\Upsilon$, we see that $\nabla^L_\Upsilon$ is a convex combination of functions of the form $y_{\vec{c}}$.

We can now arrange $\nabla^L_\Upsilon$ to contain a copy of $y_{\vec{c}}$ for a given $\vec{c} \in \mathbb{D}_{2^q}$ as follows: Let $\Phi$ be the state description represented by the matrix

$$\begin{pmatrix} | & & | & | & & | & & | & & | \\ \alpha_1 & \cdots & \alpha_1 & \alpha_2 & \cdots & \alpha_2 & \cdots & \alpha_{2^q} & \cdots & \alpha_{2^q} \\ | & & | & | & & | & & | & & | \end{pmatrix},$$

where $\alpha_i$ occurs $[c_i \cdot \nu]$ times. Now let $\mathfrak{p}_1, \dots, \mathfrak{p}_q > 0$ be such that $\sum_{i=1}^q \mathfrak{p}_i = 1$ and let $\Upsilon$ be the $\nu \times \nu$ - matrix containing $[\mathfrak{p}_i \cdot \nu]$ copies of the $i$'th row of $\Phi$, for each $i$, and fill the remaining rows with arbitrary copies of rows from $\Phi$. Then $\nabla^L_\Upsilon$ certainly contains a copy of $y_{\vec{c}}$.

With this in mind, we can modify the notation of $\nabla^L_\Upsilon$ to

$$\vec{\mathfrak{p}}\nabla^L_\Upsilon$$

for $\vec{\mathfrak{p}} = \langle \mathfrak{p}_1, \dots, \mathfrak{p}_q \rangle$ to indicate that $\Upsilon$ contains only $q$ distinct rows, occurring with the frequency given by $\vec{\mathfrak{p}}$. We will write $\vec{\mathfrak{p}}\nabla^L_{\Upsilon(\vec{c})}$ to indicate that $\Upsilon$ arises from $\vec{c} \in \mathbb{D}_{2^q}$ in this manner.

We can represent $\vec{\mathfrak{p}}\nabla^L_{\Upsilon(\vec{c})}$ in terms of $y_{\vec{c}}$ as follows. Let

$$K = \{\vec{n} \in \mathbb{N}^q \mid \sum_{i=1}^q n_i = q\},$$

so $\vec{n} \in K$ represents the choices of picking rows from $\Upsilon$. Then we obtain the representation

$$\vec{\mathfrak{p}}\nabla^L_{\Upsilon(\vec{c})} = \sum_{\vec{n}\in K}\prod_{i=1}^{q}\mathfrak{p}_i^{n_i}(n_1,\ldots,n_q)!\, y_{\vec{c}_{\vec{n}}}, \tag{2.26}$$

where $\vec{c}_{\vec{n}}$ results from picking rows according to $\vec{n}$ and (as standard)

$$(n_1,\ldots,n_q)! = \frac{(n_1+n_2+\ldots+n_q)!}{n_1!\,n_2!\ldots n_q!} = \binom{q}{n_1,\ldots,n_q}.$$

Note that we need this multinomial coefficient here since $\vec{\mathfrak{p}}\nabla^L_{\Upsilon(\vec{c})}$ is in fact a sum of $w_{\vec{e}}$, and although each of the $w_{\vec{e}}$ occurring in $y_{\vec{c}}$ occurs, the normalizing constant exists only implicitly in $\vec{\mathfrak{p}}\nabla^L_{\Upsilon(\vec{c})}$. With this notation in mind, we can prove the first step needed to show the desired theorem.

**Lemma 2.12.** *Let $\vec{c} \in \mathbb{D}_{2^q}$. Then there exist $\lambda \geq 0$ and probability functions $w_1$, $w_2$ satisfying ULi such that*

$$y_{\vec{c}} = (1+\lambda)w_1 - \lambda w_2.$$

**Proof:** Fix $\vec{c} \in \mathbb{D}_{2^q}$. As demonstrated in the discussion above, we can easily find $\nabla^L_\Upsilon$ with $y_{\vec{c}}$ occurring in it, amongst other instances of $y_{\vec{e}}$. Thus, the problem reduces to finding a way to remove all of these other instances via ULi functions.

To this end, suppose that for each $\vec{m} \in K$ we have a function $\vec{\mathfrak{p}}_{\vec{m}}\nabla^L_{\Upsilon(\vec{c})}$ such that $\Upsilon$ is the state description obtained from $w_{\vec{c}}$ by the method discussed above. Then, since the representations of the form (2.26) of these functions only differ in the coefficients of the $y_{\vec{e}}$ occurring we obtain the equation

$$\begin{pmatrix} \vdots \\ \vec{\mathfrak{p}}_{\vec{m}}\nabla^L_{\Upsilon(\vec{c})} \\ \vdots \end{pmatrix} = A \cdot \begin{pmatrix} \vdots \\ (n_1,\ldots,n_q)!\, y_{\vec{c}_{\vec{n}}} \\ \vdots \end{pmatrix}, \tag{2.27}$$

where $A$ is the $K \times K$-matrix with entry $\langle \vec{m}, \vec{n}\rangle$ being $\prod_{k=1}^{q}\mathfrak{p}_{\vec{m},k}^{n_k}$. It suffices now to show that we can pick the $\vec{\mathfrak{p}}_{\vec{m}}$ such that $A$ is regular. For suppose this is the case. Then we obtain from (2.27) the equation

$$A^{-1}\begin{pmatrix} \vdots \\ \vec{\mathfrak{p}}_{\vec{m}}\nabla^L_{\Upsilon(\vec{c})} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ (n_1,\ldots,n_q)!\, y_{\vec{c}_{\vec{n}}} \\ \vdots \end{pmatrix}. \tag{2.28}$$

Suppose $A^{-1} = (b_{\vec{n},\vec{m}})_{\vec{n},\vec{m}\in K}$. Then for $\vec{n} = \langle 1, 1, \ldots, 1\rangle$ we obtain

$$y_{\vec{c}} = \frac{1}{(n_1, \ldots, n_q)!} \sum_{\vec{m}\in K} b_{\vec{n},\vec{m}} \vec{\mathfrak{p}}_{\vec{m}} \nabla^L_{\Upsilon(\vec{c})} = \frac{1}{q!} \sum_{\vec{m}\in K} b_{\vec{n},\vec{m}} \vec{\mathfrak{p}}_{\vec{m}} \nabla^L_{\Upsilon(\vec{c})},$$

and by collecting the functions with positive coefficients in the linear combination on the right-hand side, we obtain constants $\gamma, \lambda \geq 0$, *independent of* $\vec{c}$, such that[2]

$$\frac{1}{q!} \sum_{\vec{m}\in K} b_{k,\vec{m}} \vec{\mathfrak{p}}_{\vec{m}} \nabla^L_{\Upsilon(\vec{c})} = \gamma w_1 - \lambda w_2,$$

with $w_1$, $w_2$ convex combinations of ULi functions. Since this gives the probability function $y_{\vec{c}}$, we must have

$$1 = y_{\vec{c}}(\top) = \gamma w_1(\top) - \lambda w_2(\top) = \gamma - \lambda,$$

and thus $\gamma = 1 + \lambda$.

It remains to show that the $\vec{\mathfrak{p}}_{\vec{m}}$ can be chosen such that $A$ is regular. For this, we will show the following by induction on $j$:
Let $1 \leq i_1 < i_2 < \cdots < i_j \leq r$ and let $A_{\langle i_1,\ldots,i_j\rangle}$ be the $j \times j$ sub-matrix of $A$ obtained by taking the $i_1, \ldots, i_j$'th rows and columns of $A$. Then there is a choice of the $\vec{\mathfrak{p}}_{\vec{m}_k}$, $k = i_1, \ldots, i_j$ such that $A_{\langle i_1,\ldots,i_j\rangle}$ is regular.

For $j = 1$, this is trivial. Suppose $j = n + 1$ for some $n \geq 1$ and consider $A_{\langle i_1,\ldots,i_j\rangle}$. For a given $\vec{m} \in K$, the polynomial $\prod_{j=1}^q x_j^{m_j}$ takes its maximum value on $\mathbb{D}_{2^q}$ at $x_j = m_j/q$. Fix an enumeration of $K$. There exists $\vec{m}_{i_k} = \langle m_{i_k,1}, \ldots, m_{i_k,q}\rangle$ such that

$$\prod_{s=1}^q \left(\frac{m_{i_k,s}}{q}\right)^{m_{i_k,s}} > \prod_{s=1}^q \left(\frac{m_{i_k,s}}{q}\right)^{m_{i_j,s}}$$

for all $j \neq k$. For if not, then

$$\prod_{s=1}^q \left(\frac{m_{i_k,s}}{q}\right)^{m_{i_k,s}} \leq \prod_{s=1}^q \left(\frac{m_{i_k,s}}{q}\right)^{m_{i_j,s}} < \prod_{s=1}^q \left(\frac{m_{i_j,s}}{q}\right)^{m_{i_j,s}}$$

for some $j \neq k$, and continuing in this way we arrive at a contradiction.

By the inductive hypothesis, there exists a choice of the $\vec{\mathfrak{p}}_{\vec{m}_s}$, $s \in \{i_1, \ldots, i_j\} \setminus \{i_k\}$ such that the sub-matrix $A_{\langle i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_j\rangle}$ is regular. Thinking of the $\mathfrak{p}_{\vec{m}_{i_k},s}$ for the

[2] Note that we can safely assume $\lambda \neq 0$, since if $\lambda = 0$, then the $y_{\vec{c}}$ in question would already satisfy ULi, and therefore already has the desired representation by the Representation Theorem for ULi. We also trivially have $\gamma \neq 0$, since $y_{\vec{c}}$ is a probability function for any $\vec{c} \in \mathbb{D}_{2^q}$.

moment as unknowns we obtain for the determinant of $A_{\langle i_1,\dots,i_j\rangle}$ an expression of the form

$$\det(A_{\langle i_1,\dots,i_j\rangle}) =$$
$$\pm\prod_{s=1}^{q}\mathfrak{p}_{\vec{m}_{i_k},s}^{m_{i_k,s}}\cdot\det(A_{\langle i_1,\dots,i_{k-1},i_{k+1},\dots,i_j\rangle}) + \sum_{t\in\{i_1,\dots,i_j\}\setminus\{i_k\}}\prod_{s=1}^{q}\mathfrak{p}_{\vec{m}_{i_k},s}^{m_{t,s}}\cdot(\pm\det(A_t))\,, \quad (2.29)$$

(for some choices of $\pm$) where the $A_t$ are the corresponding sub-matrices of $A_{\langle i_1,\dots,i_j\rangle}$. Now picking $\mathfrak{p}_{\vec{m}_{i_k},s} = (m_{i_k,s}/q)^g$ for large enough $g > 0$, the term

$$\prod_{s=1}^{q}\mathfrak{p}_{\vec{m}_{i_k},s}^{m_{i_k,s}}\cdot\det(A_{\langle i_1,\dots,i_{k-1},i_{k+1},\dots,i_j\rangle})$$

becomes the dominant term of (2.29), giving that $\det(A_{\langle i_1,\dots,i_j\rangle}) \neq 0$, as certainly $\prod_{s=1}^{q}\mathfrak{p}_{\vec{m}_{i_k},s}^{n_{i_k,s}} > 0$ and $\det(A_{\langle i_1,\dots,i_{k-1},i_{k+1},\dots,i_j\rangle}) \neq 0$ by the inductive hypothesis.

Note that using this procedure we in general obtain $\vec{\mathfrak{p}}_{\vec{m}}$ with entries $\mathfrak{p}_{\vec{m}_i,j}$ not summing to 1. In that case, we can pick $\vec{\mathfrak{p}'}_{\vec{m}}$ such that

$$\mathfrak{p}'_{\vec{m}_i,j} = \frac{\mathfrak{p}_{\vec{m}_i,j}}{\sum_{s=1}^{q}\mathfrak{p}_{\vec{m}_i,s}}$$

for each $\vec{m} \in K$. Then the matrix $A'$ with entries $\prod_{s=1}^{q}\mathfrak{p}'^{n_{j,s}}_{\vec{m}_i,s}$ is regular just if $A$ is, and the $\vec{\mathfrak{p}'}_{\vec{m}}$ have the desired properties. ⊣

Using this lemma, we can now prove the desired theorem.

**Theorem 2.13 (General Representation Theorem for $w$ satisfying Px).** *Let $w$ be a probability function on SL satisfying Px. Then there exist $\lambda \geq 0$ and probability functions $w_1$, $w_2$ satisfying ULi such that*

$$w = (1+\lambda)w_1 - \lambda w_2.$$

**Proof:** Let $w$ be a probability function on *SL* satisfying Px. By the Representation Theorem for Px, we have that $w$ has a representation

$$w = \int_{\mathbb{D}_{2^q}} y_{\vec{c}}\; d\mu(\vec{c}) \quad (2.30)$$

for some measure $\mu$, and by Lemma 2.12, we have, for a fixed $\lambda \geq 0$, a representation

$$y_{\vec{c}} = (1+\lambda)w_{1_{\vec{c}}} - \lambda w_{2_{\vec{c}}}$$

for each $\vec{c} \in \mathbb{D}_{2^q}$. Now applying this to the representation (2.30), we obtain

$$\begin{aligned} w &= \int_{\mathbb{D}_{2^q}} (1+\lambda) w_{1_{\vec{c}}} - \lambda w_{2_{\vec{c}}} \, d\mu(\vec{c}) \\ &= \int_{\mathbb{D}_{2^q}} (1+\lambda) w_{1_{\vec{c}}} \, d\mu(\vec{c}) - \int_{\mathbb{D}_{2^q}} \lambda w_{2_{\vec{c}}} \, d\mu(\vec{c}) \\ &= (1+\lambda) w_1 - \lambda w_2, \end{aligned}$$

for

$$w_1 = \int_{\mathbb{D}_{2^q}} w_{1_{\vec{c}}} \, d\mu(\vec{c}), \qquad w_2 = \int_{\mathbb{D}_{2^q}} w_{2_{\vec{c}}} \, d\mu(\vec{c}),$$

as required. $\dashv$

# Chapter 3

# Strong Predicate Exchangeability

We will introduce a stronger version of the Principle of Predicate Exchangeability, arising from a continuum of functions. We obtain these functions by generalizing the $u^{\bar{p},L}$ functions, which form the building blocks of functions satisfying Ax + ULi, see e.g. [17], [21], [22].

**Definition 3.1 ($\boldsymbol{P}$-Spectrum):** *Let $L = L_q$ and $\Theta(a_1, \ldots, a_n) = \bigwedge_{i=1}^n \alpha_{h_i}(a_i)$ be a state description of $L$. We can view $\Theta$ as*

$$\Theta(a_1, \ldots, a_n) = \bigwedge_{i=0}^{q} \bigwedge_{j=1}^{s_i} \alpha_{h_{ij}}(a_{ij}),$$

*where each $\alpha_{h_{ij}}$ for $j \in \{1, \ldots, s_i\}$ is an atom of $L$ with $\gamma_q(h_{ij}) = i$, $n = \sum_{i=0}^q s_i$. For each $i \in \{0, \ldots, q\}$ let $E_i$ be the equivalence relation on $\{1, \ldots, s_i\}$ given by*

$$jE_ik \Longleftrightarrow h_{ij} = h_{ik}.$$

*For $i \in \{0, \ldots, q\}$ let $M_i$ be the multiset of sizes of equivalence classes of $E_i$. The* $P$-Spectrum *of $\Theta$ is the vector*

$$\langle M_0, M_1, \ldots, M_q \rangle.$$

**Strong Predicate Exchangeability, SPx**

*Let $L = L_q$ and $w$ a probability function on $L$. $w$ satisfies Strong Predicate Exchangeability if and only if $w(\Theta) = w(\Phi)$ whenever $\Theta$ and $\Phi$ have the same $P$-Spectrum.*

SPx is a strong version of Px in the sense that it implies Px, but Px does not imply SPx.

**Proposition 3.2.** *Let $w$ be a probability function on $L = L_q$ satisfying SPx. Then $w$ satisfies Px.*

**Proof:** Let $\sigma$ be a permutation of predicates, $\Theta$ a state description of $L$ and $\sigma\Theta$ the result of permuting all predicates occuring in $\Theta$ according to $\sigma$. It is enough to show that $\Theta$ and $\sigma\Theta$ have the same $P$-Spectrum.

Suppose $\Theta = \bigwedge_{i=0}^{q} \bigwedge_{j=1}^{s_i} \alpha_{h_{ij}}(a_{ij})$ and for some $j, k \in \{1, \dots, n\}$, we have that $h_{ij} = h_{ik}$, so $jE_ik$ for some $i$. Since $\alpha_{h_{ij}} = \alpha_{h_{ik}}$ implies $\sigma\alpha_{h_{ij}} = \sigma\alpha_{h_{ik}}$, $\sigma$ preserves the equivalence of $j$ and $k$. A similar argument gives that if $j, k \in \{1, \dots, s_i\}$ are not equivalent, then $\sigma\alpha_{h_{ij}} \neq \sigma\alpha_{h_{ik}}$ must hold. Thus we must have that the $P$-Spectrum of $\Theta$ is the same as the $P$-Spectrum of $\sigma\Theta$. By SPx for $w$, $w(\Theta) = w(\sigma\Theta)$. ⊣

The following counter-example will show that Px does not imply SPx.

**Example 3.3 (Px does not imply SPx).** *Suppose $\Theta(a_1, \dots, a_5)$, $\Phi(a_1, \dots, a_5)$ are state descriptions of $L_3$ given by*

$$\begin{aligned}\Theta = {} & (P_1 \wedge P_2 \wedge \neg P_3) \wedge (\neg P_1 \wedge P_2 \wedge P_3) \wedge (\neg P_1 \wedge \neg P_2 \wedge P_3) \\ & \wedge (\neg P_1 \wedge \neg P_2 \wedge P_3) \wedge (P_1 \wedge \neg P_2 \wedge \neg P_3), \\ \Phi = {} & (P_1 \wedge \neg P_2 \wedge P_3) \wedge (\neg P_1 \wedge P_2 \wedge P_3) \wedge (\neg P_1 \wedge P_2 \wedge \neg P_3) \\ & \wedge (\neg P_1 \wedge P_2 \wedge \neg P_3) \wedge (\neg P_1 \wedge \neg P_2 \wedge P_3).\end{aligned}$$

*With our usual way of enumerating the 8 atoms of $L_3$, we obtain*

$$\begin{aligned}\Theta &= \alpha_2\alpha_4\alpha_5\alpha_7^2, \\ \Phi &= \alpha_3\alpha_5\alpha_6^2\alpha_7.\end{aligned}$$

*Note that there are 6 permutations of the atoms of $L_3$ that are induced by Px, and that $\alpha_2$, $\alpha_3$, $\alpha_5$ each have one negated predicate, while $\alpha_4$, $\alpha_6$, $\alpha_7$ each have two negated predicates. Suppose $\vec{b} \in \mathbb{D}_8$ and let $\Sigma$ denote the set of all permutations of predicates. Then applying Definition 2.10 to $w_{\vec{b}}$, we have that $w$ given by*

$$w = \frac{1}{6} \sum_{\sigma \in \Sigma} w_{\sigma\vec{b}}$$

*satisfies Px. Notice that $\Phi$ and $\Theta$ have the same $P$-Spectrum. We obtain*

$$w(\Theta) = \frac{1}{6}\left(b_2b_4b_5b_7^2 + b_3b_4b_5b_6^2 + b_2b_6b_3b_7^2 + b_5b_6b_3b_4^2 + b_5b_7b_2b_6^2 + b_3b_7b_2b_6^2\right),$$
$$w(\Phi) = \frac{1}{6}\left(b_3b_5b_7b_6^2 + b_2b_5b_6b_7^2 + b_5b_3b_7b_4^2 + b_2b_3b_4b_7^2 + b_3b_2b_6b_4^2 + b_5b_2b_6b_4^2\right).$$

*Now letting*

$$\vec{b} = \left\langle \frac{1}{19}, \frac{2}{19}, \frac{4}{19}, \frac{5}{19}, \frac{2}{19}, \frac{3}{19}, \frac{1}{19}, \frac{1}{19} \right\rangle,$$

*we clearly have $\vec{b} \in \mathbb{D}_8$ and we obtain*

$$w(\Theta) = \frac{1094}{6 \cdot 19^5},$$
$$w(\Phi) = \frac{1224}{6 \cdot 19^5},$$

*which clearly gives $w(\Theta) \neq w(\Phi)$ and thus $w$ cannot satisfy SPx.*

**Remark 3.4.** The language $L_3$ is the minimal language in which Px does not imply SPx. The atoms $\alpha_1$ and $\alpha_{2^q}$ are always fixed by permutations of predicates. In $L_1$, we do not have any other atoms and in $L_2$, both $\alpha_2$ and $\alpha_3$ contain one predicate. Since there are only two permutations of predicates of $L_2$, we have that for $\sigma$ a permutation of predicates, either $\sigma(\alpha_2) = \alpha_3$ or $\sigma(\alpha_2) = \alpha_2$ must hold. Furthermore, in $L_2$ any state description has at most two equivalence classes of $E_1$, thus given $\Theta$ and $\Phi$ with the same $P$-Spectrum, there exists a permutation of predicates $\sigma$ such that $\sigma\Theta = \Phi$, guaranteeing SPx for any $w$ with Px on $L_2$.

Similarly, Ax implies SPx, but SPx does not imply Ax: clearly, if two state descriptions $\Theta$ and $\Phi$ have the same $P$-spectrum, then there exists a bijection $f$ between the atoms of $\Theta$ and those of $\Phi$ that preserves the $P$-spectrum. We can extend $f$ to a permutation of all atoms, and thus any $w$ satisfying Ax ensures that $\Theta$ and $\Phi$ get equal probability. On the other hand, a permutation of atoms that permutes the two atoms $\alpha_1$ and $\alpha_{2^q}$ (and acts arbitrarily on the remaining ones) clearly cannot preserve $P$-spectra; thus given such a permutation $\sigma$, a probability function $w$ satisfying SPx cannot guarantee $w(\alpha_1) = w(\sigma\alpha_1) = w(\alpha_{2^q})$ to hold, which is required for $w$ to satisfy Ax.

## 3.1 A collection of functions satisfying SPx

We will now turn to defining a continuum of probability functions that satisfy SPx.

The idea for the original definition of $u^{\overline{p},L}$ was to assign to a state description $\Theta$ the probability of arriving at $\Theta$ by a process of picking colours; each time a new colour (or the colour 'black') is picked, we are allowed to pick any atom with probability $1/2^q$, while repeated colours force us to pick the same atoms again. Thus, the sequence of colours contributing to $u^{\overline{p},L}(\Theta)$ need to be consistent with $\Theta$. We can generalize these functions now by changing the probability with which we pick atoms from $1/2^q$ (guaranteeing Ax) to some values $c_i$ for some $\vec{c} \in \mathbb{D}_{2^q}$ such that $w_{\vec{c}}$ satisfies Px. If in addition to that, we also require $\vec{c}$ to be of the special form (2.3), we arrive at a function that satisfies SPx + ULi. In fact, we may even allow the $\vec{c}$ to vary with different colours.

**Definition 3.5:** *Let* $L = L_q$ *and*

$$\mathbb{B} = \left\{ \langle p_0, p_1, p_2, \ldots \rangle \mid \sum_{i \in \mathbb{N}} p_i = 1 \wedge p_1 \geq p_2 \geq p_3 \geq \cdots \geq 0 \right\} \text{ and } \overline{p} \in \mathbb{B}.$$

*Let* $\overline{\rho} = \langle \rho_0, \rho_1, \rho_2, \ldots \rangle$ *be a sequence of normalized measures on* $[0,1]$ *and let* $\Theta(\vec{a}) = \Theta(a_1, \ldots, a_m)$ *be a state description of* $L$ *with*

$$\Theta(a_1, \ldots, a_m) = \bigwedge_{i=1}^{m} \alpha_{h_i}(a_i).$$

*Then we define the functions* $j_L^{\overline{p},\overline{\rho}}$, $v_L^{\overline{p},\overline{\rho}}$ *as follows: Let* $\vec{c}$ *be a sequence in* $\mathbb{N}$. *Then*

$$j_L^{\overline{p},\overline{\rho}}(\Theta(\vec{a}), \langle c_1, \ldots, c_m \rangle)$$
$$= \begin{cases} j_L^{\overline{p},\overline{\rho}}(\Theta^-, \langle c_1, \ldots, c_{m-1} \rangle) \cdot p_{c_m} \cdot b_{h_m, c_m} & \text{if } c_m = 0 \text{ or } c_l \neq c_m \text{ for all } l < m, \\ j_L^{\overline{p},\overline{\rho}}(\Theta^-, \langle c_1, \ldots, c_{m-1} \rangle) \cdot p_{c_m} & \text{if } c_l = c_m \neq 0 \text{ for some } \\ & l < n \text{ and } h_l = h_m, \\ 0 & \text{otherwise,} \end{cases}$$

*with* $\Theta^- = \Theta^-(a_1, \ldots, a_{m-1})$ *the unique state description such that*

$$\Theta(a_1, \ldots, a_m) \models \Theta^-(a_1, \ldots, a_{m-1})$$

*and*

$$b_{h_m, c_m} = \int_{[0,1]} x^{\gamma_q(h_m)}(1-x)^{q-\gamma_q(h_m)}\, d\rho_{c_m}(x).$$

*Define* $v_L^{\overline{p},\overline{\rho}}$ *on state descriptions of* $L$ *by*

$$v_L^{\overline{p},\overline{\rho}}(\Theta(\vec{a})) = \sum_{\vec{c}} j_L^{\overline{p},\overline{\rho}}(\Theta(\vec{a}), \vec{c}). \tag{3.1}$$

**Remark 3.6.** If each of the $\rho_i$ is a discrete measure giving all weight to a single $\tau_i \in [0,1]$, we will write $v_L^{\overline{p},\overline{\rho}}$ as $v_L^{\overline{p},\overline{\tau}}$. Furthermore, the $v_L^{\overline{p},\overline{\rho}}$ do not satisfy Ax in general: Let $\rho_i = \rho$ for all $i \in \mathbb{N}$, giving all weight to some $\tau \in [0,1]$ with $\tau \neq 1/2$. Then it is easy to check that we have $v_L^{\overline{p},\overline{\tau}}(\alpha_1) = (1-\tau)^q \neq \tau^q = v_L^{\overline{p},\overline{\tau}}(\alpha_{2^q})$:

The factors $b_{1,c_1}$ and $b_{2^q,c_1}$, respectively, occur in each summand of $v_L^{\overline{p},\overline{\tau}}(\alpha_1)$ and $v_L^{\overline{p},\overline{\tau}}(\alpha_{2^q})$, respectively. Since the measures $\rho_i$ are all identical, giving all weight to $\tau$, and the $p_{c_1}$ sum to 1, we obtain the equalities above.

**Theorem 3.7.** *Let* $\overline{\rho} = \langle \rho_0, \rho_1, \rho_2, \ldots \rangle$ *be a sequence of normalized measures on* $[0,1]$, $\overline{p} \in \mathbb{B}$ *and* $L = L_q$. *Then* $v_L^{\overline{p},\overline{\rho}}$ *is a member of a ULi family with Ex + SPx.*

**Proof:** We will first show that $v_L^{\overline{p},\overline{\rho}}$ is indeed a probability function satisfying Ex. We will show (i) - (iii) as in Remark 1.14.

For (i), let $\Theta(a_1, \ldots, a_m)$ be a state description of $L$. Then $v_L^{\overline{p},\overline{\rho}}(\Theta)$ is a sum of non-negative terms, as each $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c})$ is non-negative for $\vec{c}$ a sequence of natural numbers. Thus, $v_L^{\overline{p},\overline{\rho}}(\Theta) \geq 0$ and (i) holds.

For (iii), let $\Theta(a_1, \ldots, a_m)$, $\Phi(a_1, \ldots, a_{m+1})$ be state descriptions of $L$. We have

$$\begin{aligned} &\Phi(a_1, \ldots, a_{m+1}) \models \Theta(a_1, \ldots, a_m) \\ &\qquad \iff \Phi(a_1, \ldots, a_{m+1}) = \Theta(a_1, \ldots, a_m) \wedge \alpha(a_{m+1}) \text{ for some atom } \alpha \text{ of } L. \end{aligned}$$

Thus we obtain (denoting $\langle a_1, \ldots, a_m \rangle$ by $\vec{a}$, $\langle a_1, \ldots, a_{m+1} \rangle$ by $\vec{a}^+$, $\langle c_1, \ldots, c_m \rangle$ by $\vec{c}$ and $\langle c_1, \ldots, c_{m+1} \rangle$ by $\vec{c}^+$)

$$\begin{aligned}\sum_{\substack{\Phi(\vec{a}^+)\\ \Phi(\vec{a}^+)\models\Theta(\vec{a})}} v_L^{\overline{p},\overline{\rho}}(\Phi(\vec{a}^+)) &= \sum_{\substack{\Phi(\vec{a}^+)\\ \Phi(\vec{a}^+)\models\Theta(\vec{a})}} \sum_{\vec{c}^+\in\mathbb{N}^{m+1}} j_L^{\overline{p},\overline{\rho}}(\Phi(\vec{a}^+), \vec{c}^+) \\ &= \sum_{i=1}^{2^q} \sum_{\vec{c}^+\in\mathbb{N}^{m+1}} j_L^{\overline{p},\overline{\rho}}(\Theta(\vec{a}) \wedge \alpha_i(a_{m+1}), \vec{c}^+) \\ &= \sum_{\vec{c}\in\mathbb{N}^m} \sum_{c_{m+1}\in\mathbb{N}} \sum_{i=1}^{2^q} j_L^{\overline{p},\overline{\rho}}(\Theta(\vec{a}) \wedge \alpha_i(a_{m+1}), \vec{c}^+), \qquad (3.2)\end{aligned}$$

splitting up $\langle c_1, \ldots, c_m, c_{m+1} \rangle$ into the sequence $\vec{c}$ and $c_{m+1}$. We partition the possible extensions $c_{m+1}$ of $\vec{c}$ for each sequence into two disjoint subsets of $\mathbb{N}$ by letting

$$C_{\vec{c}} = \{c \in \mathbb{N} \mid c \neq 0 \wedge \exists i \leq m\, c = c_i\}.$$

Then $C_{\vec{c}}$ collects all those $c_{m+1}$ that do not result in a factor of $b_{i,c_{m+1}}$ being added. For each $c_{m+1} \in C_{\vec{c}}$ there is a unique $\Phi$ such that $j_L^{\overline{p},\overline{\rho}}(\Phi, \langle c_1, \ldots, c_{m+1} \rangle)$ has a chance of giving non-zero contribution to $v_L^{\overline{p},\overline{\rho}}(\Phi)$, namely that $\Phi$ with $\Phi \models \alpha_j(a_{m+1})$ such that $\alpha_j = \alpha_{h_k}$ for the $k$ witnessing $c_{m+1} \in C_{\vec{c}}$. In this situation we have

$$j_L^{\overline{p},\overline{\rho}}(\Theta \wedge \alpha_i, \langle c_1, \ldots, c_{m+1} \rangle) = j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) \cdot p_{c_{m+1}}$$

and note that this term is 0 if and only if $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) = 0$. Similarly, all terms for $c_{m+1} \notin C_{\vec{c}}$ are 0 if and only if $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) = 0$, since we either have $c_{m+1} = 0$ or $c_{m+1}$ has not previously been picked and thus cannot result in the term becoming 0 because of the choice of $c_{m+1}$.

We can then rephrase (3.2) as

$$\begin{aligned}&\sum_{\langle c_1,\ldots,c_m\rangle\in\mathbb{N}^m} \sum_{c_{m+1}\in\mathbb{N}} \sum_{i=1}^{2^q} j_L^{\overline{p},\overline{\rho}}(\Theta(a_1, \ldots, a_m) \wedge \alpha_i(a_{m+1}), \langle c_1, \ldots, c_{m+1} \rangle) \\ &\quad= \sum_{\vec{c}\in\mathbb{N}^m} \left( \sum_{c_{m+1}\in C_{\vec{c}}} j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) \cdot p_{c_{m+1}} + \sum_{c_{m+1}\notin C_{\vec{c}}} \sum_{i=1}^{2^q} j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) \cdot p_{c_{m+1}} \cdot b_{i,c_{m+1}} \right) \\ &\quad= \sum_{\vec{c}\in\mathbb{N}^m} \left( j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) \cdot \sum_{c_{m+1}\in C_{\vec{c}}} p_{c_{m+1}} + j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) \cdot \sum_{c_{m+1}\notin C_{\vec{c}}} \sum_{i=1}^{2^q} p_{c_{m+1}} \cdot b_{i,c_{m+1}} \right) \\ &\quad= \sum_{\vec{c}\in\mathbb{N}^m} \left( j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) \cdot \sum_{c_{m+1}\in C_{\vec{c}}} p_{c_{m+1}} + j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) \cdot \sum_{c_{m+1}\notin C_{\vec{c}}} p_{c_{m+1}} \right),\end{aligned}$$

since $\sum_{i=1}^{2^q} b_{i,c_{m+1}} = 1$ for fixed $c_{m+1}$,

$$= \sum_{\vec{c}\in\mathbb{N}^m} j_L^{\overline{p},\overline{\rho}}(\Theta,\vec{c}) \cdot \sum_{c_{m+1}\in\mathbb{N}} p_{c_{m+1}}$$
$$= \sum_{\vec{c}\in\mathbb{N}^m} j_L^{\overline{p},\overline{\rho}}(\Theta,\vec{c}) = v_L^{\overline{p},\overline{\rho}}(\Theta),$$

giving (iii). We now obtain (ii) from (iii) by

$$v_L^{\overline{p},\overline{\rho}}(\top) = \sum_{i=1}^{2^q} v_L^{\overline{p},\overline{\rho}}(\alpha_i) = \sum_{i=1}^{2^q}\sum_{c\in\mathbb{N}} p_c \cdot b_{i,c} = 1,$$

using the same argument for the right-most equation as above.

To show that $v_L^{\overline{p},\overline{\rho}}$ satisfies Ex, let $\Theta(a_1,\dots,a_m)$ be a state description of $L$, $\sigma$ a permutation of the constants $a_1,\dots,a_m$, and $\Theta'$ the result of permuting the constants of $\Theta$ according to $\sigma$. The aim is to produce a bijection $f$ of the sequences $\vec{c}\in\mathbb{N}^m$ such that

$$j_L^{\overline{p},\overline{\rho}}(\Theta,\vec{c}) = j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c}))$$

for each $\vec{c}\in\mathbb{N}^m$, thus showing the required equality for $v_L^{\overline{p},\overline{\rho}}$. For this, we define

$$f(\vec{c}) = \langle c_1^f,\dots,c_m^f\rangle$$

by

$$c_i^f = c_{\sigma^{-1}(i)}, i\in\{1,\dots,m\}.$$

Note that this does define a bijection. Viewing $\Theta$ and $\Theta'$ as

$$\Theta(a_1,\dots,a_m) = \bigwedge_{i=1}^{m} \alpha_{h_i}(a_i) \text{ and } \Theta'(a_1,\dots,a_m) = \bigwedge_{i=1}^{m} \alpha_{g_i}(a_i), \tag{3.3}$$

it is easy to check that $h_i = j$ if and only if $g_{\sigma(i)} = j$. We then get that if $j_L^{\overline{p},\overline{\rho}}(\Theta,\vec{c}) = 0$, since $c_i = c_j$, but $h_i \neq h_j$ for some $i$, $j$, then obviously $g_{\sigma(i)} \neq g_{\sigma(j)}$ and thus since

$$c_{\sigma(i)}^f = c_{\sigma^{-1}(\sigma(i))} = c_{\sigma^{-1}(\sigma(j))} = c_{\sigma(j)}^f,$$

we must have $j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c})) = 0$ for the same reason. Similarly, if $j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c})) = 0$, then this must hold for $j_L^{\overline{p},\overline{\rho}}(\Theta,\vec{c})$ as well.

Now suppose that $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) \neq 0$. Then for no pair $i$, $j$ of indices with $c_i = c_j \neq 0$ can we have that $h_i \neq h_j$. This must hold for $f(\vec{c})$ as well, as we have seen above. Therefore, the product $\prod_{i=1}^{m} p_{c_i}$ occurs in both $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c})$ and $j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c}))$. It remains to check that the other factors in $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c})$ occur in $j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c}))$ as well.

Suppose for some $i$ we have $c_i = 0$. Then we obtain the factor $b_{h_i,0}$ occuring in $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c})$ paired with $p_{c_i}$. Now in $j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c}))$, we have $c_j^f = c_{\sigma^{-1}(j)} = 0$ for that $j$ with $\sigma(i) = j$. Thus, $b_{g_j,0}$ occurs in $j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c}))$ paired with $p_{c_j^f}$. But then since $h_i = g_{\sigma(i)} = g_j$, we have $b_{h_i,0} = b_{g_j,0}$. Now let $c_i \neq 0$, say $c_i = k$. Without loss of generality, we may assume that $i$ is minimal with $c_i = k$, since $k$ occuring in $\vec{c}$ gives us at most one additional factor. We obtain that $b_{h_i,k}$ occurs in $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c})$ paired with $p_{c_i}$. We have that $c_j^f = k$ for $j$ such that $\sigma(i) = j$. If $j$ is the least index such that $c_j^f = k$, then $b_{g_j,k}$ occurs in $j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c}))$ paired with $p_{c_j^f}$. As before, we have $b_{h_i,k} = b_{g_j,k}$, and this factor appears in both $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c})$ and $j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c}))$ only once. Now suppose $j$ is not minimal, i.e. there exists $j' < j$ such that $c_{j'}^f = k$. Then the factor $b_{g_{j'},k}$ occurs in $j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c}))$ paired with $p_{c_{j'}}$. By the assumption that $j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c})) \neq 0$, we must have $g_{j'} = g_j$ and thus $b_{g_{j'},k} = b_{h_i,k}$. Since multiplication is commutative, we may treat this factor as occuring in $j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c}))$ paired with $p_{c_j^f}$. Thus we have that $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) = j_L^{\overline{p},\overline{\rho}}(\Theta', f(\vec{c}))$. Since $f$ was a bijection from the set $\mathbb{N}^m$ onto itself, Ex for $v_L^{\overline{p},\overline{\rho}}$ follows.

To show SPx, let $\Theta(a_1, \dots, a_m)$, $\Phi(a_1, \dots, a_m)$ be state descriptions of $L$ given by

$$\Theta(a_1, \dots, a_m) = \bigwedge_{i=1}^{m} \alpha_{h_i}(a_i) \text{ and } \Phi(a_1, \dots, a_m) = \bigwedge_{i=1}^{m} \alpha_{g_i}(a_i)$$

with the same $P$-spectrum, say

$$\langle M_0, M_1, \dots, M_q \rangle$$

with $M_i$ the multiset of sizes of equivalence classes of $E_i$. For $k \in \{1, \dots, m\}$ let $[k]_\Theta$ denote the equivalence class of $k$ according to the $P$-spectrum of $\Theta$. Since Constant Exchangeability holds for $v_L^{\overline{p},\overline{\rho}}$, we can assume that for each $k \in \{1, \dots, m\}$

$$[k]_\Theta = [k]_\Phi \text{ and } [k]_\Theta \in M_i \iff [k]_\Phi \in M_i. \tag{3.4}$$

We will now show that for each $\vec{c} \in \mathbb{N}^m$, we have

$$j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) = j_L^{\overline{p},\overline{\rho}}(\Phi, \vec{c}). \tag{3.5}$$

Since we have (3.4), it is immediate that if for $j$, $k$ we have $c_j = c_k$, but $h_j \neq h_k$, then we also must have $g_j \neq g_k$, and thus

$$j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) = 0 \Longleftrightarrow j_L^{\overline{p},\overline{\rho}}(\Phi, \vec{c}) = 0.$$

It remains to show the equality (3.5) in case that $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) \neq 0$.

Let $k \in \{1, \dots, m\}$. Suppose $c_k = 0$. Then in $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c})$, we have the factor $p_0 \cdot b_{j,0}$ occuring, where $j$ is such that $\Theta \models \alpha_j(a_k)$. Similarly, we have $p_0 \cdot b_{s,0}$ occuring in $j_L^{\overline{p},\overline{\rho}}(\Phi, \vec{c})$, with $s$ given by $\Phi \models \alpha_s(a_k)$. Now by the second statement in (3.4), we must have that $\gamma_q(s) = \gamma_q(j)$, and thus obtain $b_{s,0} = b_{j,0}$.

Suppose $c_k = t$ for some $t \neq 0$. Assume that there exists $j < k$ such that $[j]_\Theta = [k]_\Theta$, i.e. $t$ appears at an earlier point in $\vec{c}$. Then $c_k$ only contributes the factor $p_t$ to $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c})$, and since by the first statement of (3.4), $[j]_\Phi = [k]_\Phi$ must hold, the same holds for $j_L^{\overline{p},\overline{\rho}}(\Phi, \vec{c})$. So assume that $k$ is the least index such that $c_k = t$. Then in $j_L^{\overline{p},\overline{\rho}}(\Theta, \vec{c})$ the factor $p_t \cdot b_{j,t}$ occurs, with $j$ such that $\Theta \models \alpha_j(a_k)$, while in $j_L^{\overline{p},\overline{\rho}}(\Phi, \vec{c})$ we have $p_t \cdot b_{s,t}$ occuring with $\Phi \models \alpha_s(a_k)$. Again, by the second part of (3.4) we must have $b_{j,t} = b_{s,t}$ and thus (3.5) holds.

To show Unary Language Invariance, we will use the $b_{i,c_j}$ in their integral form

$$\int_{[0,1]} x^k (1-x)^{q-k} \, d\rho_{c_j}(x).$$

So let $\Theta(a_1, \dots, a_m)$ be a state desctription of $L = L_q$. Consider $\Theta$ as a quantifier-free sentence of $L_{q+1}$ and expand $\Theta$ to its representation in terms of state descriptions of $L_{q+1}$,

$$\Theta \equiv \bigvee_{\varepsilon_1, \dots, \varepsilon_m \in \{0,1\}} \bigwedge_{i=1}^{m} \alpha_{h_i}^{\varepsilon_i}(a_i).$$

We obtain, for arbitrary $\vec{c}$,

$$\sum_{\varepsilon_1, \dots, \varepsilon_m \in \{0,1\}} j^{\overline{p},\overline{\rho}} \left( \bigwedge_{i=1}^{m} \alpha_{h_i}^{\varepsilon_i}(a_i), \vec{c} \right)$$
$$= \sum_{\varepsilon_1, \dots, \varepsilon_{m-1} \in \{0,1\}} \left( j^{\overline{p},\overline{\rho}} \left( \bigwedge_{i=1}^{m-1} \alpha_{h_i}^{\varepsilon_i}(a_i), \langle c_1, \dots, c_{m-1} \rangle \right) \cdot p_{c_m} \cdot (x_m^{+,\vec{\varepsilon}} + x_m^{-,\vec{\varepsilon}}) \right),$$

where $x_m^{+,\vec{\varepsilon}}$ is the $b_{j,s}$ corresponding to $\alpha_{h_m}^{+}$ and $x_m^{-,\vec{\varepsilon}}$ that $b_{j,s}$ corresponding to $\alpha_{h_m}^{-}$. In case $c_m = c_l$ for some $l < m$, $x_m^{+,\vec{\varepsilon}} + x_m^{-,\vec{\varepsilon}}$ is either 1 (if $h_l = h_m$, in which case either $\alpha_{h_l}^{+}$

or $\alpha^-_{h_l}$ must have occured), or 0 in case $h_l \neq h_m$. It remains to see that in the other case

$$x_m^{+,\vec{\varepsilon}} + x_m^{-,\vec{\varepsilon}} = \int_{[0,1]} x^{\gamma_q(h_m)}(1-x)^{q-\gamma_q(h_m)}\, d\rho_{c_m}(x).$$

Note that for both $x_m^{+,\vec{\varepsilon}}$ and $x_m^{-,\vec{\varepsilon}}$ the correct measure $\rho_{c_m}$ is chosen, and the equation now follows appealing to the ULi part of Theorem 2.3.

Thus we have showed that the required Language Invariance property holds for each summand $j_L^{\overline{p},\overline{\rho}}$, and therefore we have $v^{\overline{p},\overline{\rho}}_{L_{q+1}}(\Theta) = v^{\overline{p},\overline{\rho}}_{L_q}(\Theta)$, giving ULi for $v_L^{\overline{p},\overline{\rho}}$. ⊣

In case each $\rho_i$ gives all weight to a single point $\tau_i \in [0,1]$, i.e. $v_L^{\overline{p},\overline{\rho}} = v_L^{\overline{p},\overline{\tau}}$ (see Remark 3.6), then $v_L^{\overline{p},\overline{\tau}}$ also satisfies Weak Irrelevance. In fact, these are the only $v_L^{\overline{p},\overline{\rho}}$ that have this property. In order to prove this, we will introduce an alternative definition for the $v_L^{\overline{p},\overline{\rho}}$, similar to the alternative definition of $u^{\overline{p},L}$ (see e.g. [21]).

**Definition 3.8:** *Let $\overline{p} \in \mathbb{B}$, $\overline{\rho} = \langle \rho_0, \rho_1, \rho_2, \ldots \rangle$ and $n \in \mathbb{N}$. Let $q \in \mathbb{N}$, $L = L_q$. Define $f^q : \{1, \ldots, 2^q\} \times \mathbb{N} \to [0,1]$ by*

$$f^q(j,r) = \int_{[0,1]} x^{\gamma_q(j)}(1-x)^{q-\gamma_q(j)}\, d\rho_r(x).$$

*Let $Z_n^q = \{e : \{1, \ldots, n\} \to \{1, \ldots, 2^q\}\}$. Then define the function $v_{n,L}^{\overline{p},\overline{\rho}}$ on state descriptions on $L$ by*

$$v_{n,L}^{\overline{p},\overline{\rho}}(\Theta) = \sum_{e \in Z_n^q} \prod_{r=1}^{n} f^q(e(r),r) \prod_{s=1}^{2^q} \left( f^q(s,0) R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{m_s}, \tag{3.6}$$

*where as usual $\langle m_1, \ldots, m_{2^q} \rangle$ is the signature of $\Theta$ and $R_{\overline{p},n} = 1 - \sum_{i=1}^n p_i$.*

We will show that for all $n \in \mathbb{N}$, $v_{n,L}^{\overline{p},\overline{\rho}}$ satisfies Ex + SPx + ULi and that $\lim_{n\to\infty} v_{n,L}^{\overline{p},\overline{\rho}} = v^{\overline{p},\overline{\rho}}$. For WIP, we will need the additional property that the $\rho_i$ concentrate the measure on a single support point $\tau_i$.

**Lemma 3.9.** *Let $n \in \mathbb{N}$, $\overline{p} \in \mathbb{B}$ and $\overline{\rho} = \langle \rho_0, \rho_1, \ldots, \rangle$ with $\rho_i$ a measure on $[0,1]$ for all $i \in \mathbb{N}$. Then $v_{n,L}^{\overline{p},\overline{\rho}}$ is a probability function satisftying Ex, SPx and ULi.*

**Proof:** Constant Exchangeability follows immediately from the definition; since $m_s = |\{i \mid h_i = s\}|$ is independent of the particular indices $i$, and only depends on their total number, the signature $\langle m_1, \ldots, m_{2^q} \rangle$ of $\Theta$, and thus $v_{n,L}^{\overline{p},\overline{\rho}}(\Theta)$ are invariant under permutations of constants.

We will again show properties (i) - (iii) for $v_{n,L}^{\overline{p},\overline{\rho}}$. For (i), notice that for $\Theta$ a state description of $L$, $v_{n,L}^{\overline{p},\overline{\rho}}(\Theta)$ is a sum of non-negative terms and therefore $v_{n,L}^{\overline{p},\overline{\rho}}(\Theta) \geq 0$.

To show (iii), let $\Theta(a_1, \ldots, a_m)$ be a state description of $L$ and consider

$$\sum_{\substack{\Phi(a_1,\ldots,a_{m+1})\\ \Phi \models \Theta}} v_{n,L}^{\overline{p},\overline{\rho}}(\Phi).$$

As above we have $\Phi(a_1, \ldots, a_{m+1}) = \Theta(a_1, \ldots, a_m) \wedge \alpha_j(a_{m+1})$ for some $j \in \{1, \ldots, 2^q\}$. Then

$$\begin{aligned}
\sum_{\substack{\Phi(a_1,\ldots,a_{m+1})\\ \Phi \models \Theta}} v_{n,L}^{\overline{p},\overline{\rho}}(\Phi) &= \sum_{j=1}^{2^q} v_{n,L}^{\overline{p},\overline{\rho}}(\Theta \wedge \alpha_j) \\
&= \sum_{j=1}^{2^q} \sum_{e \in Z_n^q} \prod_{r=1}^{n} f^q(e(r), r) \prod_{s=1}^{2^q} \left( f^q(s,0) \cdot R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{m_{s,j}},
\end{aligned}$$

where $m_{s,j} = \begin{cases} m_s + 1 & \text{if } s = j, \\ m_s & \text{otherwise.} \end{cases}$

$$\begin{aligned}
&= \sum_{e \in Z_n^q} \prod_{r=1}^{n} f^q(e(r), r) \cdot \sum_{j=1}^{2^q} \prod_{s=1}^{2^q} \left( f^q(s,0) \cdot R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{m_{s,j}} \\
&= \sum_{e \in Z_n^q} \prod_{r=1}^{n} f^q(e(r), r) \cdot \prod_{s=1}^{2^q} \left( f^q(s,0) \cdot R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{m_s} \\
&\quad \cdot \sum_{j=1}^{2^q} \left( f^q(j,0) \cdot R_{\overline{p},n} + \sum_{e(i)=j} p_i \right)
\end{aligned}$$

$$= \sum_{e \in Z_n^q} \prod_{r=1}^{n} f^q(e(r), r) \cdot \prod_{s=1}^{2^q} \left( f^q(s,0) \cdot R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{m_s}$$
$$\cdot \left( R_{\overline{p},n} \cdot \sum_{j=1}^{2^q} f^q(j,0) + \sum_{i=1}^{n} p_i \right)$$
$$= \sum_{e \in Z_n^q} \prod_{r=1}^{n} f^q(e(r), r) \cdot \prod_{s=1}^{2^q} \left( f^q(s,0) \cdot R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{m_s}$$
$$\cdot (R_{\overline{p},n} + \sum_{i=1}^{n} p_i)$$
$$= \sum_{e \in Z_n^q} \prod_{r=1}^{n} f^q(e(r), r) \cdot \prod_{s=1}^{2^q} \left( f^q(s,0) \cdot R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{m_s}$$
$$= v_{n,L}^{\overline{p},\overline{\rho}}(\Theta),$$

since $\sum_{j=1}^{2^q} f^q(j,0) = 1$, as $\rho_0$ defines a probability function $w_{\vec{b}}$ with $b_i = f^q(i,0)$.

Using (iii), we then obtain (ii) since

$$v_{n,L}^{\overline{p},\overline{\rho}}(\top) = \sum_{j=1}^{2^q} v_{n,L}^{\overline{p},\overline{\rho}}(\alpha_j)$$
$$= \sum_{j=1}^{2^q} \sum_{e \in Z_n^q} \prod_{r=1}^{n} f^q(e(r), r) \cdot \prod_{s=1}^{2^q} \left( f^q(s,0) \cdot R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)$$
$$= \sum_{e \in Z_n^q} \prod_{r=1}^{n} f^q(e(r), r) \cdot \sum_{j=1}^{2^q} \prod_{s=1}^{2^q} \left( f^q(s,0) \cdot R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)$$
$$= \sum_{e \in Z_n^q} \prod_{r=1}^{n} f^q(e(r), r)$$
$$= \sum_{e' \in Z_{n-1}} \prod_{r=1}^{n-1} f^q(e'(r), r) \cdot \sum_{j=1}^{2^q} f^q(j, n)$$
$$= \sum_{e' \in Z_{n-1}} \prod_{r=1}^{n-1} f^q(e'(r), r) = \cdots = \sum_{e' \in Z_1} f^q(e'(1), 1) = 1.$$

We will now turn to SPx. Let $\Theta(a_1, \ldots, a_m)$, $\Phi(a_1, \ldots, a_m)$ be state descriptions of $L$ with the same $P$-spectrum. There exists a permutation $\pi$ of atoms such that $\pi(\alpha_s) = \alpha_k$ if and only if $\alpha_s$ and $\alpha_k$ are instantiated by constants belonging to the same equivalence class of some $E_i$ in the spectra of $\Theta$, $\Phi$, respectively. Denote $\alpha_k$ such that $\alpha_k = \pi(\alpha_s)$ by $\alpha_{\pi(s)}$. Let $\langle m_1, \ldots, m_{2^q}\rangle$ and $\langle l_1, \ldots, l_{2^q}\rangle$ denote the signatures of $\Theta$ and $\Phi$, respectively. Then the permutation $\pi$ of atoms induces a permutation of the signatures such that $l_k = \pi(m_s)$ if and only if $\alpha_k = \alpha_{\pi(s)}$. The notation $\alpha_{\pi(s)}$ for $\alpha_k$ suggest to do the same for the $l_i$, and thus we identify $l_k$ with $l_{\pi(s)}$. This gives us

$$\begin{aligned} v^{\overline{p},\overline{\rho}}_{n,L}(\Theta) &= \sum_{e\in Z^q_n} \prod_{r=1}^{n} f^q(e(r), r) \cdot \prod_{s=1}^{2^q} \left( f^q(s,0)R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{m_s} \\ &= \sum_{e\in Z^q_n} \prod_{r=1}^{n} f^q(e(r), r) \cdot \prod_{s=1}^{2^q} \left( f^q(s,0)R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{\pi(m_s)} \\ &= \sum_{e\in Z^q_n} \prod_{r=1}^{n} f^q(e(r), r) \cdot \prod_{s=1}^{2^q} \left( f^q(s,0)R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{l_{\pi(s)}} . \end{aligned} \tag{3.7}$$

Observe that $m_s = \pi(m_s) = l_{\pi(s)}$ since $\Theta$ and $\Phi$ have identical $P$-spectra. The permutation $\pi$ induces a bijection $\Pi$ from $Z^q_n$ onto $Z^q_n$ such that

$$\Pi(e)(i) = \pi(e(i)).$$

Thus we can write the right-hand side of (3.7) as

$$\begin{aligned} &\sum_{\Pi(e)\in Z^q_n} \prod_{r=1}^{n} f^q(\Pi(e)(r), r) \cdot \prod_{s=1}^{2^q} \left( f^q(s,0)R_{\overline{p},n} + \sum_{\Pi(e)(i)=\pi(s)} p_i \right)^{n_{\pi(s)}} \\ &\quad = \sum_{e\in Z^q_n} \prod_{r=1}^{n} f^q(e(r), r) \cdot \prod_{s=1}^{2^q} \left( f^q(s,0)R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{n_s} = v^{\overline{p},\overline{\rho}}_{n,L}(\Phi), \end{aligned}$$

as $\Pi$ is a bijection and $f^q(\Pi(e)(r), r) = f^q(e(r), r)$ since by definition of $\Pi$, $e(r)$ and $\pi(e(r))$ are atoms with $\gamma_q(e(r)) = \gamma_q(\pi(e(r)))$. Thus $v^{\overline{p},\overline{\rho}}_{n,L}$ satisfies SPx.

For ULi, let $\Theta(a_1, \ldots, a_m)$ be a state description of $L = L_q$ and let

$$\Theta(a_1, \ldots, a_m) = \bigvee_{\varepsilon_1,\ldots,\varepsilon_m\in\{0,1\}} \bigwedge_{i=1}^{m} \alpha^{\varepsilon_i}_{h_i}(a_i).$$

It will be more convenient to use a slightly different notation for $v^{\overline{p},\overline{\rho}}_{n,L}$. Notice that for any state description $\Phi(a_1, \ldots, a_m)$ and any $q \in \mathbb{N}$,

$$\prod_{s=1}^{2^q} \left( f^q(s,0) \cdot R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{m_s} = \prod_{j=1}^{m} \left( f^q(h_j,0) \cdot R_{\overline{p},n} + \sum_{e(i)=s} p_i \right),$$

where $\langle m_1, \ldots, m_{2^q} \rangle$ is the signature of $\Phi$. Then we obtain

$$\begin{aligned}
v^{\overline{p},\overline{\rho}}_{n,L_{q+1}}(\Theta(a_1,\ldots,a_m)) &= v^{\overline{p},\overline{\rho}}_{n,L_{q+1}}\left(\bigvee_{\varepsilon_1,\ldots,\varepsilon_m\in\{0,1\}}\bigwedge_{i=1}^{m}\alpha^{\varepsilon_i}_{h_i}(a_i)\right)\\
&= \sum_{\varepsilon_1,\ldots,\varepsilon_m\in\{0,1\}} v^{\overline{p},\overline{\rho}}_{n,L_{q+1}}\left(\bigwedge_{i=1}^{m}\alpha^{\varepsilon_i}_{h_i}(a_i)\right)\\
&= \sum_{\varepsilon_1,\ldots,\varepsilon_m\in\{0,1\}}\ \sum_{e\in Z_n^{q+1}}\prod_{r=1}^{n} f^{q+1}(e(r),r)\\
&\qquad \cdot\prod_{j=1}^{m}\left(f^{q+1}(h_j^{\varepsilon_j},0)R_{\overline{p},n}+\sum_{e(i)=h_j^{\varepsilon_j}} p_i\right)\\
&= \sum_{e\in Z_n^{q+1}}\prod_{r=1}^{n} f^{q+1}(e(r),r)\\
&\qquad \cdot\sum_{\varepsilon_1,\ldots,\varepsilon_m\in\{0,1\}}\prod_{j=1}^{m}\left(f^{q+1}(h_j^{\varepsilon_j},0)R_{\overline{p},n}+\sum_{e(i)=h_j^{\varepsilon_j}} p_i\right),
\end{aligned}$$

where $h_j^{\varepsilon_j} = k$ for $\alpha^{\varepsilon_j}_{h_j} = \alpha_k$ as an atom of $L_{q+1}$. Now fix $e \in Z_n^{q+1}$. We obtain

$$\begin{aligned}
&\prod_{r=1}^{n} f^{q+1}(e(r),r)\cdot\sum_{\varepsilon_1,\ldots,\varepsilon_m\in\{0,1\}}\prod_{j=1}^{m}\left(f^{q+1}(h_j^{\varepsilon_j},0)\cdot R_{\overline{p},n}+\sum_{e(i)=h_j^{\varepsilon_j}} p_i\right)\\
&\quad=\prod_{r=1}^{n} f^{q+1}(e(r),r)\cdot\sum_{\varepsilon_1,\ldots,\varepsilon_{m-1}\in\{0,1\}}\prod_{j=1}^{m-1}\left(f^{q+1}(h_j^{\varepsilon_j},0)\cdot R_{\overline{p},n}+\sum_{e(i)=h_j^{\varepsilon_j}} p_i\right)\\
&\qquad\cdot\left[\left(f^{q+1}(h_m^0,0)R_{\overline{p},n}+\sum_{e(i)=h_m^0} p_i\right)+\left(f^{q+1}(h_m^1,0)R_{\overline{p},n}+\sum_{e(i)=h_m^1} p_i\right)\right]\\
&\quad=\prod_{r=1}^{n} f^{q+1}(e(r),r)\cdot\sum_{\varepsilon_1,\ldots,\varepsilon_{m-1}\in\{0,1\}}\prod_{j=1}^{m-1}\left(f^{q+1}(h_j^{\varepsilon_j},0)\cdot R_{\overline{p},n}+\sum_{e(i)=h_j^{\varepsilon_j}} p_i\right)\\
&\qquad\cdot\left[\left(f^{q+1}(h_m^0,0)+f^{q+1}(h_m^1,0)\right)R_{\overline{p},n}+\sum_{e(i)\in\{h_m^0,h_m^1\}} p_i\right]
\end{aligned}$$

$$= \prod_{r=1}^{n} f^{q+1}(e(r), r) \cdot \sum_{\varepsilon_1,\ldots,\varepsilon_{m-1}\in\{0,1\}} \prod_{j=1}^{m-1} \left( f^{q+1}(h_j^{\varepsilon_j}, 0) \cdot R_{\overline{p},n} + \sum_{e(i)=h_j^{\varepsilon_j}} p_i \right)$$
$$\cdot \left( f^q(h_m, 0) R_{\overline{p},n} + \sum_{e(i)\in\{h_m^0, h_m^1\}} p_i \right)$$
$$= \cdots = \prod_{r=1}^{n} f^{q+1}(e(r), r) \cdot \prod_{j=1}^{m} \left( f^q(h_j, 0) \cdot R_{\overline{p},n} + \sum_{e(i)=\in\{h_j^0, h_j^1\}} p_i \right).$$

Now consider $\hat{e} \in Z_n^q$. Define the subset $Z_n^{q+1}(\hat{e}) \subseteq Z_n^{q+1}$ by

$$Z_n^{q+1}(\hat{e}) := \{e \in Z_n^{q+1} \mid \forall i \in \{1, \ldots, n\} \ \big(e(i) = (\hat{e}(i))^0\big) \vee \big(e(i) = (\hat{e}(i))^1\big)\}.$$

Thus, some $e \in Z_n^{q+1}$ is in $Z_n^{q+1}(\hat{e})$ if and only if

$$\left( f^q(h_j, 0) R_{\overline{p},n} + \sum_{e(i)\in\{h_j^0, h_j^1\}} p_i \right) = \left( f^q(h_j, 0) R_{\overline{p},n} + \sum_{\hat{e}(i)=h_j} p_i \right)$$

holds for each $j \in \{1, \ldots, m\}$. This then implies that

$$\prod_{j=1}^{m} \left( f^q(h_j, 0) R_{\overline{p},n} + \sum_{e(i)\in\{h_j^0, h_j^1\}} p_i \right) = \prod_{j=1}^{m} \left( f^q(h_j, 0) R_{\overline{p},n} + \sum_{\hat{e}(i)=h_j} p_i \right)$$

holds, which in turn provides us with a partition

$$Z_n^{q+1} = \bigcup_{\hat{e}\in Z_n^q} Z_n^{q+1}(\hat{e})$$

of $Z_n^{q+1}$ into disjoint sets. We obtain

$$v_{n,L_{q+1}}^{\overline{p},\overline{\rho}}(\Theta(a_1, \ldots, a_m))$$
$$= \sum_{\hat{e}\in Z_n^q} \sum_{e\in Z_n^{q+1}(\hat{e})} \prod_{r=1}^{n} f^{q+1}(e(r), r) \cdot \prod_{j=1}^{m} \left( f^q(h_j, 0) \cdot R_{\overline{p},n} + \sum_{e(i)=\in\{h_j^0, h_j^1\}} p_i \right)$$

and it now suffices to show that for any $\hat{e} \in Z_n^q$ we have

$$\sum_{e\in Z_n^{q+1}(\hat{e})} \prod_{r=1}^{n} f^{q+1}(e(r), r) \cdot \prod_{j=1}^{m} \left( f^q(h_j, 0) \cdot R_{\overline{p},n} + \sum_{e(i)=\in\{h_j^0, h_j^1\}} p_i \right)$$
$$= \prod_{r=1}^{n} f^q(\hat{e}(r), r) \cdot \prod_{j=1}^{m} \left( f^q(h_j, 0) \cdot R_{\overline{p},n} + \sum_{\hat{e}(i)=h_j} p_i \right). \tag{3.8}$$

So fix $\hat{e}$. Let $k = |\{i \mid \hat{e}(i) = \hat{e}(1)\}|$. Notice that $k \neq 0$, so we can let $i$ be maximal such that $\hat{e}(i) = \hat{e}(1)$. Pick $e \in Z_n^{q+1}(\hat{e})$ such that $e(i) = (\hat{e}(1))^1$, i.e. that $j \in \{1, \ldots, 2^{q+1}\}$ with $\beta_j = \alpha^1_{\hat{e}(1)}$, with $\alpha$ ranging over the atoms of $L_q$ and $\beta$ ranging over those of $L_{q+1}$. Note that such an $e$ exists. There is a unique $e' \in Z_n^{q+1}(\hat{e})$ such that

$$e'(i) = (\hat{e}(1))^0 \wedge \forall j \neq i \, e'(j) = e(j).$$

We obtain

$$\begin{aligned}
&\prod_{r=1}^{n} f^{q+1}(e(r), r) \cdot \prod_{j=1}^{m} \left( f^q(h_j, 0) R_{\overline{p},n} + \sum_{e(i) \in \{h_j^0, h_j^1\}} p_i \right) \\
&\quad + \prod_{r=1}^{n} f^{q+1}(e'(r), r) \cdot \prod_{j=1}^{m} \left( f^q(h_j, 0) R_{\overline{p},n} + \sum_{e'(i) \in \{h_j^0, h_j^1\}} p_i \right) \\
&= \left( f^{q+1}(e(i), i) + f^{q+1}(e'(i), i) \right) \cdot \prod_{\substack{r=1 \\ r \neq i}}^{n} f^{q+1}(e(r), r) \\
&\quad \cdot \prod_{j=1}^{m} \left( f^q(h_j, 0) R_{\overline{p},n} + \sum_{e(i) \in \{h_j^0, h_j^1\}} p_i \right)
\end{aligned}$$

and note that in the last summation of the $p_i$ we can safely neglect mentioning $e'$, as for fixed $h_j$ both $\sum_{e \in \{h_j^0, h_j^1\}} p_i$ and $\sum_{e' \in \{h_j^0, h_j^1\}} p_i$ are equal,

$$\begin{aligned}
&= \left( f^{q+1}((\hat{e}(1))^1, i) + f^{q+1}((\hat{e}(1))^0, i) \right) \cdot \prod_{\substack{r=1 \\ r \neq i}}^{n} f^{q+1}(e(r), r) \cdot \\
&\quad \prod_{j=1}^{m} \left( f^q(h_j, 0) R_{\overline{p},n} + \sum_{e(i) \in \{h_j^0, h_j^1\}} p_i \right) \\
&= f^q(1, i) \cdot \prod_{\substack{r=1 \\ r \neq i}}^{n} f^{q+1}(e(r), r) \cdot \prod_{j=1}^{m} \left( f^q(h_j, 0) R_{\overline{p},n} + \sum_{e(i) \in \{h_j^0, h_j^1\}} p_i \right)
\end{aligned}$$

$$= f^q(\hat{e}(i), i) \cdot \prod_{\substack{r=1 \\ r \neq i}}^{n} f^{q+1}(e(r), r) \cdot \prod_{j=1}^{m} \left( f^q(h_j, 0) R_{\overline{p},n} + \sum_{e(i) \in \{h_j^0, h_j^1\}} p_i \right).$$

We can now continue to pair off functions $e$, $e'$ refining $\hat{e}$ in this way until we exhaust all the $i$ with $\hat{e}(i) = \hat{e}(1)$, and then repeat this process with the least $j$ such that $\hat{e}(j) \neq \hat{e}(1)$, and so on, until we have moved through the domain of $\hat{e}$.

Thus we obtain for this fixed $\hat{e}$ that (3.8) holds, and therefore have

$$v^{\overline{p},\overline{\rho}}_{n,L_{q+1}}(\Theta(a_1,\dots,a_m)) = v^{\overline{p},\overline{\rho}}_{n,L}(\Theta(a_1,\dots,a_m)),$$

giving ULi for $v^{\overline{p},\overline{\rho}}_{n,L}$. ⊣

Our aim is to show that the $v^{\overline{p},\overline{\rho}}_{n,L}$ functions satisfy WIP under certain conditions. To this end, we will make use of the operation $\oplus_\lambda$, which preservers WIP and enables us to use induction over $n$ to prove the desired result.

**Definition 3.10:** *Let $w_1$, $w_2$ be discrete probability functions on $L = L_q$ whose de Finetti priors put measure $e_x$ on point $\vec{c}_x \in \mathbb{D}_{2^q}$ for $x = 1,\dots,X$ and measure $f_y$ on point $\vec{d}_y \in \mathbb{D}_{2^q}$ for $y = 1,\dots,Y$ respectively. Let $\lambda \in [0,1]$. Define the discrete probability function $w = w_1 \oplus_\lambda w_2$ to put measure*

$$\sum\{e_x f_y \,|\, \vec{k} = \lambda\vec{c}_x + (1-\lambda)\vec{d}_y, 1 \le x \le X, 1 \le y \le Y\}$$

*on point $\vec{k} \in \mathbb{D}_{2^q}$.*

The main result for $\oplus_\lambda$ is the following theorem, appearing in [26].

**Theorem 3.11.** *If $w_1$, $w_2$ are discrete probability functions on $L$ satisfying Ex, Ax, (Unary) Language Invariance and WIP and $\lambda \in [0,1]$ then $w = w_1 \oplus_\lambda w_2$ also satisfies these principles.*

The result, however, deals with Atom Exchangeability rather than Predicate Exchangeability. We therefore need to check that Px is preserved as well, which is straightforward using the same arguments as in the proof of Theorem 3.11, as the following corollary shows.

**Corollary 3.12.** *Let $w_1$, $w_2$ be discrete probability functions on $L$ satisfying Px. Let $\lambda \in [0,1]$ and let $w = w_1 \oplus_\lambda w_2$. Then $w$ satisfies Px.*

**Proof:** Let $\Theta(a_1,\dots,a_m)$ be the state description

$$\Theta(a_1,\dots,a_m) = \bigwedge_{j=1}^{m}\bigwedge_{i=1}^{q} P_i^{\varepsilon_{i,j}}(a_j)$$

with signature $\langle m_1, \dots, m_{2^q} \rangle$. Let $\sigma$ be a permutation of predicates and let $\sigma\Theta$ the state description

$$\sigma\Theta(a_1, \dots, a_m) = \bigwedge_{j=1}^{m} \bigwedge_{i=1}^{q} P_{\sigma(i)}^{\varepsilon_{i,j}}(a_j)$$

with signature $\langle n_1, \dots, n_{2^q} \rangle$. Note that the permutation $\sigma$ induces a permutation $\sigma'$ of atoms giving $m_r = n_{\sigma'(r)}$ for $r = 1, \dots, 2^q$.[1] Then by Px for $w_1$, $w_2$ we have

$$w_1(\Theta) = w_1(\sigma\Theta),$$
$$w_2(\Theta) = w_2(\sigma\Theta).$$

We need to show that

$$w(\Theta) = w(\sigma\Theta).$$

For $\vec{c}_x$, $\vec{d}_y \in \mathbb{D}_{2^q}$, let $c_{xr}$, $d_{yr}$ be the coordinates corresponding to the atom $\alpha_r$ of $L$. We obtain

$$w(\Theta) = \sum_{x=1}^{X} \sum_{y=1}^{Y} e_x f_y \cdot \prod_{r=1}^{2^q} (\lambda c_{xr} + (1-\lambda) d_{yr})^{m_r}.$$

Now let $Z_{\vec{m}}$ be the set

$$Z_{\vec{m}} = \{\langle g_1, \dots, g_{2^q} \rangle \mid g_t \le m_t \text{ for } t = 1, \dots, 2^q\}.$$

Then we obtain

$$w(\sigma\Theta) = \sum_{x=1}^{X} \sum_{y=1}^{Y} e_x f_y \cdot \prod_{r=1}^{2^q} \left(\lambda c_{x\sigma'(r)} + (1-\lambda) d_{y\sigma'(r)}\right)^{m_r},$$

[1] Since $\sigma$ preserves the number of negations in each atom, we have that $\sigma'(1) = 1$ and $\sigma'(2^q) = 2^q$, giving $n_1 = m_1$ and $n_{2^q} = m_{2^q}$.

since atom $\alpha_{\sigma'(r)}$ occurs $m_r$ times in $\sigma\Theta$ according to its signature,

$$
\begin{aligned}
&= \sum_{x=1}^{X}\sum_{y=1}^{Y} e_x f_y \cdot \prod_{r=1}^{2^q}\left(\sum_{g_r=0}^{m_r}\binom{m_r}{g_r}\lambda^{g_r}(1-\lambda)^{m_r-g_r}c_{x\sigma'(r)}^{g_r}d_{y\sigma'(r)}^{m_r-g_r}\right)\\
&= \sum_{x=1}^{X}\sum_{y=1}^{Y} e_x f_y \cdot \left(\sum_{\langle g_1,\dots,g_{2^q}\rangle\in Z_{\vec{m}}}\prod_{r=1}^{2^q}\binom{m_r}{g_r}\lambda^{g_r}(1-\lambda)^{m_r-g_r}c_{x\sigma'(r)}^{g_r}d_{y\sigma'(r)}^{m_r-g_r}\right)\\
&= \sum_{\langle g_1,\dots,g_r\rangle\in Z_{\vec{m}}}\left(\sum_{x=1}^{X} e_x\prod_{r=1}^{2^q}c_{x\sigma'(r)}^{g_r}\right)\left(\sum_{y=1}^{Y} f_y\prod_{r=1}^{2^q}d_{y\sigma'(r)}^{m_r-g_r}\right)\\
&\qquad\left(\prod_{r=1}^{2^q}\binom{m_r}{g_r}\lambda^{g_r}(1-\lambda)^{m_r-g_r}\right)\\
&= \sum_{\langle g_1,\dots,g_r\rangle\in Z_{\vec{m}}}\left(\sum_{x=1}^{X} e_x\prod_{r=1}^{2^q}c_{xr}^{g_r}\right)\left(\sum_{y=1}^{Y} f_y\prod_{r=1}^{2^q}d_{yr}^{m_r-g_r}\right)\\
&\qquad\left(\prod_{r=1}^{2^q}\binom{m_r}{g_r}\lambda^{g_r}(1-\lambda)^{m_r-g_r}\right),
\end{aligned}
$$

this last equation holding since both $w_1$ and $w_2$ satisfy Px and $\sigma'$ is a permutation induced by Px. Now reversing this argument with $c_{xr}$ and $d_{yr}$ in place of $c_{x\sigma'(r)}$ and $d_{y\sigma'(r)}$, respectively, we obtain that this is indeed $w(\Theta)$, giving the required equality $w(\sigma\Theta) = w(\Theta)$. ⊣

**Lemma 3.13.** *Let $\overline{p} \in \mathbb{B}$, $\overline{\rho} = \langle \rho_0, \rho_1, \rho_2, \dots \rangle$ such that $\rho_1$ gives all weight to a single $\tau_1 \in [0,1]$. Let $L = L_q$. Then $v_{1,L}^{\overline{p},\overline{\rho}}$ satisfies WIP.*

**Proof:** Let $\vartheta$, $\varphi$ be quantifier-free sentences of $L_q$ with no constants and predicates in common. Then there are disjoint sublanguages $L'$, $L'' \subseteq L_q$ such that $\vartheta \in QFSL'$, $\varphi \in QFSL''$ and $L_q = L' \cup L''$. Let $\alpha_i$ range over the atoms of $L'$, $\beta_j$ over those of $L''$. We may assume $L' = L_r = \{P_1, \dots, P_r\}$ and $L'' = \{P_{r+1}, \dots, P_{r+p}\}$ with $r + p = q$. Then we obtain

$$
\begin{aligned}
\vartheta(a_1,\dots,a_n) &= \bigwedge_{i=1}^{n}\alpha_{h_i}(a_i),\\
\varphi(a_{n+1},\dots,a_{n+m}) &= \bigwedge_{j=1}^{m}\beta_{g_j}(a_{n+j}),
\end{aligned}
$$

viewed in $L'$ and $L''$, respectively and assuming that $\vartheta$ and $\varphi$ are state descriptions in these sublanguages of $L$. In $L_q$, we get

$$\vartheta(a_1,\dots,a_n) \equiv \bigvee_{1\le k_1,\dots,k_n\le 2^p} \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\wedge\beta_{k_i}(a_i),$$

$$\varphi(a_{n+1},\dots,a_{n+m}) \equiv \bigvee_{1\le l_1,\dots,l_m\le 2^r} \bigwedge_{j=1}^{m} \alpha_{l_j}(a_{n+j})\wedge\beta_{g_j}(a_{n+j}).$$

Let $\langle n_1,\dots,n_{2^r}\rangle$ be the signature of $\vartheta$ in $L'$, and $\langle m_1,\dots,m_{2^p}\rangle$ the signature of $\varphi$ in $L''$. Note that taking $\varphi^*$ to be the result of replacing each occurrence of $P_{r+i}$ in $\varphi$ – as sentence of $L''$ – by $P_i$, we obtain

$$v^{\overline{p},\overline{\rho}}_{1,L}(\varphi) = v^{\overline{p},\overline{\rho}}_{1,L''}(\varphi) = v^{\overline{p},\overline{\rho}}_{1,L_p}(\varphi^*).$$

Therefore, using Unary Language Invariance, we obtain

$$v^{\overline{p},\overline{\rho}}_{1,L}(\vartheta) = v^{\overline{p},\overline{\rho}}_{1,L_r}(\vartheta) = \sum_{x=1}^{2^r} f^r(x,1)\left(p_0\cdot f^r(x,0)+p_1\right)^{n_x}\cdot\prod_{\substack{k=1\\k\neq x}}^{2^r}\left(p_0 f^r(k,0)\right)^{n_k}, \tag{3.9}$$

$$v^{\overline{p},\overline{\rho}}_{1,L}(\varphi) = v^{\overline{p},\overline{\rho}}_{1,L_p}(\varphi^*) = \sum_{y=1}^{2^p} f^p(y,1)\left(p_0\cdot f^p(y,0)+p_1\right)^{m_y}\cdot\prod_{\substack{l=1\\l\neq y}}^{2^p}\left(p_0 f^p(l,0)\right)^{n_l}. \tag{3.10}$$

For $s\in\{1,\dots,2^r\}$, $t\in\{1,\dots,2^p\}$, let $\langle s,t\rangle$ denote the index of the atom $\alpha_s\wedge\beta_t$ in $L$. Then, as the family of functions $f^q$ was defined to take care of the Language Invariance aspect of $v^{\overline{p},\overline{\rho}}_{1,L}$, we obtain

$$f^r(k,i) = \sum_{t=1}^{2^p} f^q(\langle k,t\rangle,i),$$

$$f^p(l,i) = \sum_{s=1}^{2^r} f^q(\langle s,l\rangle,i)$$

for all $i\in\mathbb{N}$. Thus we can expand (3.9) and (3.10) to

$$v^{\overline{p},\overline{\rho}}_{1,L}(\vartheta) = \sum_{x=1}^{2^r}\sum_{y=1}^{2^p} f^q(\langle x,y\rangle,1)\left(p_0\sum_{t=1}^{2^p} f^q(\langle x,t\rangle,0)+p_1\right)^{n_x} \cdot\prod_{\substack{k=1\\k\neq x}}^{2^r}\left(p_0\sum_{t=1}^{2^p} f^q(\langle k,t\rangle,0)\right)^{n_k},$$

$$v^{\overline{p},\overline{\rho}}_{1,L}(\varphi) = \sum_{x=1}^{2^r}\sum_{y=1}^{2^p} f^q(\langle x,y\rangle,1)\left(p_0\sum_{s=1}^{2^r} f^q(\langle s,y\rangle,0)+p_1\right)^{m_y} \cdot\prod_{\substack{l=1\\l\neq y}}^{2^p}\left(p_0\sum_{s=1}^{2^r} f^q(\langle s,y\rangle,0)\right)^{m_l},$$

respectively.

Let

$$H(x) = \left( p_0 \sum_{t=1}^{2^p} f^q(\langle x, t\rangle, 0) + p_1 \right)^{n_x} \prod_{\substack{k=1 \\ k \neq x}}^{2^r} \left( p_0 \sum_{t=1}^{2^p} f(\langle k, t\rangle, 0) \right)^{n_k},$$

$$G(y) = \left( p_0 \sum_{s=1}^{2^r} f^q(\langle s, y\rangle, 0) + p_1 \right)^{m_y} \prod_{\substack{l=1 \\ l \neq y}}^{2^p} \left( p_0 \sum_{s=1}^{2^r} f(\langle s, y\rangle, 0) \right)^{m_l}.$$

We have

$$v_{1,L}^{\bar{p},\bar{\rho}}(\vartheta \wedge \varphi) = \sum_{1 \leq k_1, \dots, k_n \leq 2^r} \sum_{1 \leq l_1, \dots, l_m \leq 2^p} \sum_{z=1}^{2^q} f^q(z, 1) \cdot \prod_{\substack{s=1 \\ s \neq z}}^{2^q} (p_0 \cdot f^q(s, 0))^{N(s,\vec{k}) + M(s,\vec{l})} \cdot (p_0 \cdot f^q(z, 0) + p_1)^{N(z,\vec{k}) + M(z,\vec{l})},$$

where $N(s, \vec{k})$ is the number of times the $L_q$ - atom $\alpha_s$ occurs in $\bigwedge_{i=1}^n \alpha_{h_i}(a_i) \wedge \beta_{k_i}(a_i)$, and $M(s, \vec{l})$ is the number of times $\alpha_s$ occurs in $\bigwedge_{j=1}^m \alpha_{l_j}(a_{n+j}) \wedge \beta_{g_j}(a_{n+j})$,

$$\begin{aligned} &= \sum_{z=1}^{2^q} f^q(z, 1) \cdot \left( \sum_{1 \leq k_1, \dots, k_n \leq 2^r} \prod_{\substack{s=1 \\ s \neq z}}^{2^q} (p_0 f^q(s, 0))^{N(s,\vec{k})} \cdot (p_0 f^q(z, 0) + p_1)^{N(z,\vec{k})} \right) \\ &\quad \cdot \left( \sum_{1 \leq l_1, \dots, l_m \leq 2^p} \prod_{\substack{s=1 \\ s \neq z}}^{2^q} (p_0 f^q(s, 0))^{M(s,\vec{l})} \cdot (p_0 f^q(z, 0) + p_1)^{M(z,\vec{l})} \right) \\ &= \sum_{x=1}^{2^r} \sum_{y=1}^{2^p} f^q(\langle x, y\rangle, 1) H(x) G(y), \end{aligned} \tag{3.11}$$

since for fixed $z = \langle x, y\rangle$, we have

$$\begin{aligned} &\sum_{1 \leq k_1, \dots, k_n \leq 2^r} \prod_{\substack{s=1 \\ s \neq z}}^{2^q} (p_0 f^q(s, 0))^{N(s,\vec{k})} \cdot (p_0 f^q(z, 0) + p_1)^{N(z,\vec{k})} \\ &\qquad = \sum_{1 \leq k_1, \dots, k_n \leq 2^r} \prod_{\substack{s=1 \\ s \neq \langle x,y\rangle}}^{2^q} (p_0 f^q(s, 0))^{N(s,\vec{k})} \cdot (p_0 f^q(\langle x, y\rangle, 0) + p_1)^{N(\langle x,y\rangle,\vec{k})} \\ &\qquad = H(x), \end{aligned}$$

and similarly,

$$\sum_{1 \leq l_1, \dots, l_m \leq 2^p} \prod_{\substack{s=1 \\ s \neq z}}^{2^q} (p_0 f^q(s, 0))^{M(s,\vec{l})} \cdot (p_0 f^q(z, 0) + p_1)^{M(z,\vec{l})} = G(y).$$

Furthermore, we obtain

$$v_{1,L}^{\overline{p},\overline{\rho}}(\vartheta)\cdot v_{1,L}^{\overline{p},\overline{\rho}}(\varphi)=\sum_{x_1=1}^{2^r}\sum_{x_2=1}^{2^r}\sum_{y_1=1}^{2^p}\sum_{y_2=1}^{2^p} f^q(\langle x_1,y_1\rangle,1)\cdot f^q(\langle x_2,y_2\rangle,1)\cdot H(x_1)\cdot G(y_2). \tag{3.12}$$

Simplifying (3.12), we obtain

$$v_{1,L}^{\overline{p},\overline{\rho}}(\vartheta)\cdot v_{1,L}^{\overline{p},\overline{\rho}}(\varphi)=\sum_{x_1=1}^{2^r}\sum_{y_2=1}^{2^p} f^r(x_1,1)\cdot f^p(y_2,1)\cdot H(x_1)\cdot G(y_2),$$

since for fixed $x_1$, $y_2$ we have

$$\begin{aligned}\sum_{x_2=1}^{2^r}\sum_{y_1=1}^{2^p} f^q(\langle x_1,y_1\rangle,1)\cdot f^q(\langle x_2,y_2\rangle,1) &= \sum_{x_2=1}^{2^r} f^q(\langle x_2,y_2\rangle,1)\cdot\sum_{y_1=1}^{2^p} f^q(\langle x_1,y_1\rangle,1)\\ &= \sum_{x_2=1}^{2^r} f^q(\langle x_2,y_2\rangle,1)\cdot f^r(x_1,1)\\ &= f^r(x_1,1)\cdot f^p(y_2,1),\end{aligned}$$

using the definition of the $f^i$. Combining this with (3.11), it remains to show that

$$f^q(\langle x_1,y_2\rangle,1)=f^r(x_1,1)\cdot f^p(y_2,1).$$

Since $\rho_1$ gives all measure to a single $\tau_1\in[0,1]$, this last equation is

$$\tau_1^{\gamma_q(\langle x_1,y_2\rangle)}(1-\tau_1)^{q-\gamma_q(\langle x_1,y_2\rangle}=\tau_1^{\gamma_r(x_1)}(1-\tau_1)^{r-\gamma_r(x_1)}\cdot\tau_1^{\gamma_p(y_2)}(1-\tau_1)^{p-\gamma_p(y_2)}, \tag{3.13}$$

which is certainly true. If $\vartheta$, $\varphi$ are not state descriptions of $L'$, $L''$, respectively, then we obtain

$$\vartheta(a_1,\ldots,a_n)\equiv\bigvee_{\Theta\models\vartheta}\Theta(a_1,\ldots,a_n)$$

$$\varphi(a_{n+1},\ldots,a_{n+m})\equiv\bigvee_{\Phi\models\varphi}\Phi(a_{n+1},\ldots,a_{n+m})$$

where $\Theta$ ranges over state descriptions of $L'$ and $\Phi$ over those of $L''$. We obtain

$$\begin{aligned}v_{1,L}^{\overline{p},\overline{\rho}}(\vartheta\wedge\varphi) &= v_{1,L}^{\overline{p},\overline{\rho}}\left(\left(\bigvee_{\Theta\models\vartheta}\Theta(a_1,\ldots,a_n)\right)\wedge\left(\bigvee_{\Phi\models\varphi}\Phi(a_{n+1},\ldots,a_{n+m})\right)\right)\\ &= \sum_{\Theta\models\vartheta,\Phi\models\varphi} v_{1,L}^{\overline{p},\overline{\rho}}(\Theta\wedge\Phi)\\ &= \sum_{\Theta\models\vartheta,\Phi\models\varphi} v_{1,L}^{\overline{p},\overline{\rho}}(\Theta)\cdot v_{1,L}^{\overline{p},\overline{\rho}}(\Phi)\\ &= v_{1,L}^{\overline{p},\overline{\rho}}(\vartheta)\cdot v_{1,L}^{\overline{p},\overline{\rho}}(\varphi),\end{aligned}$$

so the claim holds in this case as well. ⊣

**Remark 3.14.** Note that the restriction to single point measures does not apply to $\rho_0$. This is due to the fact that $\rho_0$ is used to build the inner probability functions of the form $w_{f(\overline{p})}$, which satisfy Constant Irrelevance, and therefore WIP and Language Invariance trivially. Since $v_{1,L}^{\overline{p},\overline{\rho}}$ is a convex combination of those functions, the condition on $\rho_1$ (and as we shall show, on all other measures in the sequence $\overline{\rho}$ as well) being a single point measure is required to show that this specific convex combination is well behaved to preserve WIP.

By induction, we can extend this result to $n > 1$, using the operation $\oplus_\lambda$, which preserves WIP. See also [21] for a result along these lines in a similar context.

**Lemma 3.15.** *Let $\overline{p} \in \mathbb{B}$, $\overline{\rho} = \langle \rho_0, \rho_1, \rho_2, \ldots \rangle$ such that $\rho_i$ gives all weight to a single $\tau_i \in [0,1]$ for $1 \leq i \leq n$. Then $v_{n,L}^{\overline{p},\overline{\rho}}$ satisfies WIP.*

**Proof:** Proof by induction on $n$. The case $n = 1$ has already been shown. We will proceed as in the proof for $u_n^{\overline{p}}$ (see e.g. [21]): Let

$$\overline{r} = \langle 0, 1, 0, 0, \ldots \rangle,$$
$$\overline{q} = \left\langle 1 - \frac{\sum_{i=1}^n p_i}{1 - p_{n+1}}, \frac{p_1}{1 - p_{n+1}}, \frac{p_2}{1 - p_{n+1}}, \ldots, \frac{p_n}{1 - p_{n+1}}, 0, 0, \ldots \right\rangle.$$

furthermore, let

$$\overline{\eta} = \langle \rho_{n+1}, \rho_{n+1}, \rho_{n+1}, \ldots \rangle,$$
$$\overline{\rho} = \langle \rho_0, \rho_1, \ldots, \rho_n, \rho_{n+1}, \rho_{n+2}, \ldots \rangle$$

and consider $w = v_{1,L}^{\overline{r},\overline{\eta}} \oplus_{p_{n+1}} v_{n,L}^{\overline{q},\overline{\rho}}$. Note that for $v_{1,L}^{\overline{r},\overline{\eta}}$ to satisfy WIP, we require $\rho_{n+1}$ to give all weight to a single $\tau_{n+1} \in [0,1]$.

We shall show that $w = v_{n+1,L}^{\overline{p},\overline{\rho}}$. We get

$$v_{1,L}^{\overline{r},\overline{\eta}} \oplus_{p_{n+1}} v_{n,L}^{\overline{q},\overline{\rho}} = \sum_{e \in Z_1} f(e(1),1) \cdot w_{h(\overline{r},e)} \oplus_{p_{n+1}} \sum_{k \in Z_n^q} \prod_{r=1}^n f(k(r),r) \cdot w_{h(\overline{q},k)}, \tag{3.14}$$

where

$$h(\overline{r},e) = \langle f^q(\gamma_q(1), n+1), \ldots, f^q(\gamma_q(2^q), n+1) \rangle,$$
$$h(\overline{q},k) = \left\langle f^q(\gamma_q(1),0) + \sum_{k(i)=1} p_i, \ldots, f^q(\gamma_q(2^q),0) + \sum_{k(i)=2^q} p_i \right\rangle,$$

with $\gamma_q$ the function given by Definition 2.1. Notice that the sums do not occur in $h(\overline{r}, e)$, since $r_i = 0$ for $i \geq 1$. The right-hand side of (3.14) then becomes

$$\sum_{e \in Z_1} \sum_{k \in Z_n^q} f^q(e(1), 1) \prod_{r=1}^{n} f^q(k(r), r) \cdot w_{p_{n+1} \cdot h(\overline{r}, e) + (1-p_{n+1}) \cdot h(\overline{q}, k)}$$

and defining $g_{k,e} : \{1, \ldots, n+1\} \to \{1, \ldots, 2^q\}$ by

$$g_{k,e}(x) = \begin{cases} k(x) & \text{if } x \leq n, \\ e(1) & \text{if } x = n+1, \end{cases}$$

we obtain

$$\sum_{e \in Z_1} \sum_{k \in Z_n^q} f^q(e(1), 1) \prod_{r=1}^{n} f^q(k(r), r) \cdot w_{p_{n+1} \cdot h(\overline{r}, e) + (1-p_{n+1}) \cdot h(\overline{q}, k)}$$
$$= \sum_{g_{k,e}} \prod_{r=1}^{n+1} f^q(g_{k,e}(r), r) \cdot w_{p_{n+1} \cdot h(\overline{r}, e) + (1-p_{n+1}) \cdot h(\overline{q}, k)}.$$

It is easy to see now that the $g_{k,e}$ are all the functions in $Z_{n+1}$ and that

$$p_{n+1} \cdot h(\overline{r}, e) + (1 - p_{n+1}) \cdot h(\overline{q}, k) = h(\overline{p}, g_{k,e}),$$

as required, so that we get

$$v_{1,L}^{\overline{r}, \overline{\eta}} \oplus_{p_{n+1}} v_{n,L}^{\overline{q}, \overline{\rho}} = v_{n+1,L}^{\overline{p}, \overline{\rho}}.$$

Since $v_{1,L}^{\overline{r}, \overline{\eta}}$ and $v_{n,L}^{\overline{q}, \overline{\rho}}$ both satisfy WIP (the former by Lemma 3.13, the latter by induction hypothesis) and the operation $\oplus_\lambda$ preservers WIP, we obtain the required result. ⊣

We will now show that in the limit, the $v_{n,L}^{\overline{p}, \overline{\rho}}$ functions are the same as the $v_L^{\overline{p}, \overline{\rho}}$ functions. We will do this in two steps. First we show that $v_{n,L}^{\overline{p}, \overline{\rho}} = v_L^{\overline{r}, \overline{\rho}}$ holds for a specific $\overline{r}$. The second step will show that this alternative description of $v_{n,L}^{\overline{p}, \overline{\rho}}$ tends to $v_L^{\overline{p}, \overline{\rho}}$ as $n$ tends to $\infty$.

**Lemma 3.16.** *Let $\overline{p} \in \mathbb{B}$, $\overline{\rho} = \langle \rho_0, \rho_1, \rho_2, \ldots \rangle$. Let $L = L_q$ and define $\overline{t(n)} = \langle t_0, t_1, t_2, \ldots \rangle$ by*

$$t_i = \begin{cases} p_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i > n, \\ 1 - \sum_{i=1}^{n} p_i & \text{if } i = 0. \end{cases}$$

*Then $v^{\overline{t(n)}, \overline{\rho}} = v_{n,L}^{\overline{p}, \overline{\rho}}$.*

Note that by the definition of $\overline{t(n)}$, we have $t_0 = R_{\overline{p},n}$.

**Proof:** Let $\Theta(a_1,\dots,a_m) = \bigwedge_{i=1}^m \alpha_{h_i}(a_i)$ be a state description of $L_q$. Recall that

$$v_{n,L}^{\overline{p},\overline{\rho}}(\Theta) = \sum_{e\in Z_n^q} \prod_{r=1}^{n} f^q(e(r),r) \prod_{s=1}^{2^q} \left( f^q(s,0)R_{\overline{p},n} + \sum_{e(i)=s} p_i \right)^{m_s}$$

for $\Theta$ a state description of $L_q$.

Since $t_i = 0$ for $i > n$, we are only concerned with sequences $\vec{c}$ of the length $n$ when looking at $v^{\overline{t(n)},\overline{\rho}}(\Theta)$. We will give an alternative definition for $v^{\overline{t(n)},\overline{\rho}}$:
Let $Q_n(\Theta) = \{\eta : \{1,\dots,m\} \to \{0,\dots,n\} \,|\, \eta(i) = \eta(j) \neq 0 \to h_i = h_j\}$. Then $Q_n(\Theta)$ lists all the sequences of length $n$ that will contribute to $v^{\overline{t(n)},\overline{\rho}}(\Theta)$. Define the function $H : Q_n(\Theta) \times \{1,\dots,2^q\} \times \{0,\dots,n\} \to \mathbb{N}$ as follows:

$$H(\eta,s,j) = \begin{cases} 1 & \text{if } j \neq 0 \wedge \exists i(\eta(i) = j \wedge h_i = s), \\ |\{i \,|\, \eta(i) = 0 \wedge h_i = s\}| & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The idea here is that we collect the factors $f^q(s,j)$ in a convenient way; the function $H$ adds precisely one factor $f^q(s,j)$ in case that $j = c_i \neq 0$, and $h_i = s$, which in the original definition would result in a factor of $f^q(h_i,c_i)$, or a number of factors $f^q(s,0)$ equal to the total number of $c_i = 0$ associated with the atom $\alpha_s$.

Then, recalling the definition (3.1), we can write $v^{\overline{t(n)},\overline{\rho}}(\Theta)$ as

$$v^{\overline{t(n)},\overline{\rho}}(\Theta) = \sum_{\eta\in Q_n(\Theta)} \prod_{i=1}^{m} t_{\eta(i)} \prod_{s=1}^{2^q} \prod_{j=0}^{n} f^q(s,j)^{H(\eta,s,j)}. \tag{3.15}$$

Fix $e \in Z_n^q$ and expand the corresponding summand in $v_{n,L}^{\overline{p},\overline{\rho}}$ (replacing $R_{\overline{p},n}$ with $t_0$ and all other instances of $p_i$ with $t_i$) by multiplying out:

$$\begin{aligned}
&\prod_{r=1}^{n} f^q(e(r),r) \prod_{s=1}^{2^q} \left( f^q(s,0)t_0 + \sum_{e(i)=s} t_i \right)^{m_s} \\
&\quad = \prod_{r=1}^{n} f^q(e(r),r) \left( \sum_{\vec{x}_1} \binom{m_1}{\vec{x}_1(e)} \cdot f^q(1,0)^{x_{0,1}(e)} \cdot t_0^{x_{0,1}(e)} \prod_{j=1}^{M(e,1)} t_{i_{j,1}}^{x_{j,1}(e)} \right) \\
&\qquad \cdot \prod_{s=2}^{2^q} \left( f^q(s,0)t_0 + \sum_{e(i)=s} t_i \right)^{m_s} \\
&\quad = \cdots = \sum_{\vec{x}_1(e)} \cdots \sum_{\vec{x}_{2^q}(e)} \prod_{r=1}^{n} f^q(e(r),r) \cdot \prod_{s=1}^{2^q} \binom{m_s}{\vec{x}_s} f^q(s,0)^{x_{0,s}(e)} t_0^{x_{0,s}(e)} \prod_{j=1}^{M(e,s)} t_{i_{j,s}}^{x_{j,s}(e)},
\end{aligned} \tag{3.16}$$

where $i_{j,s}$ enumerates those $i$ with $e(i) = s$, $M(e,s) = |\{i \,|\, e(i) = s\}|$ and $\vec{x}_s(e) = \langle x_{0,s}(e), x_{1,s}(e), \ldots, x_{M(e,s),s}(e)\rangle$ such that $\sum_{i=0}^{M(e,s)} x_{j,s}(e) = m_s$.

The aim is now to match the summands in (3.16) with those in (3.15), showing that both terms are identical.

For this, fix $e$ and partitions $\vec{x}_s(e)$ for $s \in \{1, \ldots, 2^q\}$. Suppose first that each $x_{j,s} \neq 0$. Then this implies that each $t_i$ occurs in the term

$$\prod_{r=1}^{n} f^q(e(r),r) \cdot \prod_{s=1}^{2^q} \binom{m_s}{\vec{x}_s} f^q(s,0)^{x_{0,s}(e)} t_0^{x_{0,s}(e)} \prod_{j=1}^{M(e,s)} t_{i_{j,s}}^{x_{j,s}(e)}. \tag{3.17}$$

We now obtain $\eta \in Q_n(\Theta)$ such that

$$\prod_{i=1}^{m} t_{\eta(i)} = \prod_{s=1}^{2^q} \prod_{j=0}^{M(e,s)} t_{i_{j,s}}^{x_{j,s}(e)}.$$

This $\eta$ is unique up to a permutation of the colours $t_j$ associated to each atom $\alpha_s$ occurring in $\Theta$. Note that for each $s \in \{1, \ldots, 2^q\}$, there are $\binom{m_s}{\vec{x}_s(e)}$ choices to distribute the $t_j$ on the atom $\alpha_s$. Therefore, the number of functions $\xi \in Q_n(\Theta)$ with the same product of the $t_j$ occurring as for $\eta$, is

$$\prod_{s=1}^{2^q} \binom{m_s}{\vec{x}_s(e)}.$$

Note that for any two distinct functions $\eta_1, \eta_2$ in this subset of $Q_n(\Theta)$, we have $H(\eta_1, s, j) = H(\eta_2, s, j)$ for each $s, j$, by the definition of the function $H$, and so the factor of the $f^q(s,j)$ are the same in each of the summands. Since for each $i$ such that $\eta(i) = 0$ and $\alpha_{h_i} = s$, a factor $f^q(s,0)$ is added, and there are $x_{0,s}(e)$ of these factors, the factor $\prod_{s=1}^{2^q} f^q(s,0)^{x_{0,s}(e)}$ in (3.17) occurs in each summand arising from the functions $\eta$ we are considering at the moment. Similarly, each $f^q(e(r),r)$ occurs in each of these summands: since we have $e(r) = s$, this means that we must have $\eta(i) = r$ for some $i$ with $h_i = s$, and thus a factor $f^q(s,r)$ occurs.

Thus, collecting all the summands for the $\eta \in Q_n(\Theta)$, we obtain

$$\prod_{i=1}^{n} t_{\eta(i)} \prod_{s=1}^{2^q} \binom{m_s}{\vec{x}_s(e)} \prod_{j=0}^{n} f^q(s,j)^{H(\eta,s,j)}$$

as a summand of $v^{\overline{t(n)},\bar{\rho}}$, which is by the discussion above exactly the same as the summand (3.17).

Now suppose that for some $j, s$, $x_{j,s}(e) = 0$. Then not all the $t_i$ occur in (3.17). In contrast to the first case, there are functions $e'$ and partitions $\vec{x}_s(e')$, $s \in \{1, \ldots, 2^q\}$ such that the summand arising from $e'$ and $\vec{x}_s(e')$ contains the same factors $t_i$ with the same power as (3.17). For each of these $e'$ we have that if $x_{j,s}(e) \neq 0$, then for that $r$ such that $t_r = t_{i_{j,s}}$ and $e(r) = s$, we have $e'(r) = s$ as well. So $e$ and $e'$ differ only on those $r$ such that $t_r$ does not occur in (3.17). We will collect each of these summands for $e'$, $\vec{x}_s(e')$.

These summands differ only in the factors $f^q(e(r),r)$, as the factors $f^q(s,0)$ for each $s$ are determined by the values of $x_{0,s}(e')$, which are identical for each of the $e'$. If $t_r$ occurs in (3.17), then each $e'$ contains a factor $f^q(e'(r),r)$ and $e'(r) = e(r)$ for all these $e'$. Thus, these summands differ only in the factors $f^q(e'(r),r)$ with $t_r$ not occurring in (3.17). As the $e'(r)$ vary for each of the $r$, the values $f^q(e'(r),r)$ sum to 1 for fixed $r$, by definition of the $f^q(s,j)$. Thus, letting $R \subseteq \{1,\ldots,n\}$ the subset such that $r \in R$ just if $t_r$ occurs in (3.17), then simplifying the collection of summands, we obtain as the sum over all such $e'$

$$\prod_{r\in R} f^q(e(r),r) \cdot \prod_{s=1}^{2^q} \binom{m_s}{\vec{x}_s(e)} f^q(s,0)^{x_{0,s}(e)} t_0^{x_{0,s}(e)} \prod_{j=1}^{M(e,s)} t_{i_{j,s}}^{x_{j,s}(e)} \tag{3.18}$$

as summand in $v_{n,L}^{\bar{p},\bar{\rho}}$. Now as above, we obtain a subset of $Q_n(\Theta)$ such that for each $\eta$ in this subset, the same $t_j$ occur with the same power as in (3.18). Again, the number of these $\eta$ is $\prod_{s=1}^{2^q} \binom{m_s}{\vec{x}_s(e)}$. Grouping those, we obtain a summand

$$\prod_{i=1}^{n} t_{\eta(i)} \prod_{s=1}^{2^q} \binom{m_s}{\vec{x}_s(e)} \prod_{j=0}^{n} f^q(s,j)^{H(\eta,s,j)}$$

with $H(\eta, s, 0) = x_{0,s}(e)$ for each $s \in \{1, \ldots, 2^q\}$, and whenever $H(\eta, s, j) = 1$, then for some $i \in \{1, \ldots, m\}$, $\eta(i) = j$ and $h_i = s$, so since this subset of $Q_n(\Theta)$ is obtained from the $t_r$ occurring in (3.18), we have $f^q(s, j) = f^q(e(j), j)$. Thus, in this case, we have found a collection of summands in $v_{n,L}^{\overline{p},\overline{\rho}}$ that matches exactly a collection of summands in $v^{\overline{t(n)},\overline{\rho}}$.

Moving through the functions $e \in Z_n^q$ and partitions $\vec{x}_s(e)$, we obtain corresponding summands in $v^{\overline{t(n)},\overline{\rho}}$ that give the same value as $v_{n,L}^{\overline{p},\overline{\rho}}(\Theta)$. It remains to show that each $\eta \in Q_n(\Theta)$ occurs in one of these collections of summands.

Note that if $\eta$ occurs in one collection, then it cannot occur in any other collection of summands. Since these are determined by the $t_r$ occurring and the powers with which they occur, the range of $\eta$ and the size of the preimages of each element in the range of $\eta$ are uniquely determined. Furthermore, each $\eta$ gives rise to at least one possible $e$ and a partition $\vec{x}_s(e)$: for each $j$ in the range of $\eta$, we must have that for $r \neq 0$, $\{h_i \mid \eta(i) = r\}$ is a singleton $\{s\}$, and we can let $e(r) = s$. If $\{1, \ldots, n\} \not\subseteq \text{ran}(\eta)$, then for each $r \in \{1, \ldots, n\}$ with $r \notin \text{ran}(\eta)$, let $e(r)$ be arbitrary.

Having fixed $e$ this way, we obtain a partition $\vec{x}_s(e)$ by letting $x_{0,s}(e) = |\{i \mid h_i = s \wedge \eta(i) = 0\}|$ and, fixing an enumeration $i_{j,s}$ of the $r$ such that $e(r) = s$, let $x_{j,s}(e) = |\{i \mid h_i = s \wedge \eta(i) = i_{j,s}\}|$. Note that if $i_{j,s} \notin \text{ran}(\eta)$, then we obtain $x_{j,s}(e) = 0$, and in the corresponding summand (3.17), $t_{i_{j,s}}$ does not occur, as required. Note that the partition $\vec{x}_s(e)$ is unique, up to the enumeration of the $r$ with $e(r) = s$. If $\{1, \ldots, n\} \not\subseteq \text{ran}(\eta)$, we do not have that $e$ is unique, but the restriction of $e$ on $\{1, \ldots, n\} \cap \text{ran}(\eta)$ is. This is actually enough to determine the non-zero elements $x_{j,s}(e)$ of the partition, letting the other $x_{j,s}(e) = 0$. Considering the way we collected $e'$ into a summand in the case where not all $x_{j,s}(e)$ were non-zero, each possible extension of this restricted function will be in the same summand, so the choices made to obtain the unrestricted version of $e$ determine the same collection of summands independent of the actual choice made.

Therefore, carefully re-arranging the summands of $v^{\overline{t(n)},\overline{\rho}}(\Theta)$, $v^{\overline{p},\overline{\rho}}_{n,L}(\Theta)$ in this way, we see that $v^{\overline{t(n)},\overline{\rho}}(\Theta) = v^{\overline{p},\overline{\rho}}_{n,L}(\Theta)$, as the summands occurring on both side are identical. Since $\Theta$ was an arbitrary state description, the lemma follows. $\dashv$

**Lemma 3.17.** *Let* $\overline{p}, \overline{t(n)} \in \mathbb{B}$ *with* $t_i = p_i$ *for* $i \in \{1, \dots, n\}$ *and* $t_0 = 1 - \sum_{i=1}^{n} p_i$, $L = L_q$. *Then for* $\vartheta \in QFSL$, $\lim_{n\to\infty} |v^{\overline{p},\overline{\rho}}(\vartheta) - v^{\overline{t(n)},\overline{\rho}}(\vartheta)| = 0$.

**Proof:** It is enough to show this for state descriptions, so let $\overline{t(n)}$, $\overline{p}$, $n$ be as in the statement of the lemma, $\Theta = \bigwedge_{i=1}^{m} \alpha_{h_i}(a_i)$ a state description of $L$. We may assume that for all $n \geq 1$ we have $p_n > 0$, since otherwise $\overline{p} = \overline{t(n_0)}$ for some $n_0$, and the claim trivially holds for $n \geq n_0$, as shown by Lemma 3.16.

Since $\Theta$ mentions $m$ constants, we only have sequences of $\mathbb{N}^m$ occuring. For $\vec{c} \in \{1, \dots, n\}^m$, we must have

$$j^{\overline{p},\overline{\rho}}(\Theta, \vec{c}) = j^{\overline{t(n)},\overline{\rho}}(\Theta, \vec{c}) \tag{3.19}$$

by the definition of $\overline{t(n)}$. Let

$$J_{n,m}(\Theta) = \{f : \{1, \dots, m\} \to \{0, \dots, n\} \mid f(i) = f(j) \neq 0 \to h_i = h_j\},$$
$$I_{n,m}(\Theta) = \{f : \{1, \dots, m\} \to \{1, \dots, n\} \mid f(i) = f(j) \to h_i = h_j\}$$

and

$$N_m(\Theta) = \{f : \{1, \dots, m\} \to \mathbb{N} \mid f(i) = f(j) \neq 0 \to h_i = h_j\}.$$

We view the functions $f$ here as those picking the sequences of natural numbers that contribute to $v^{\overline{p},\overline{\rho}}$, $v^{\overline{t(n)},\overline{\rho}}$. Note that $I_{n,m}(\Theta)$ is exactly the set whose members give the equality (3.19). We obtain

$$\begin{aligned} v^{\overline{p},\overline{\rho}}(\Theta) &= \sum_{f\in N_m(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta, \langle f(1), \dots, f(m)\rangle), \\ v^{\overline{t(n)},\overline{\rho}}(\Theta) &= \sum_{f\in N_m(\Theta)} j^{\overline{t(n)},\overline{\rho}}(\Theta, \langle f(1), \dots, f(m)\rangle) \\ &= \sum_{f\in J_{n,m}(\Theta)} j^{\overline{t(n)},\overline{\rho}}(\Theta, \langle f(1), \dots, f(m)\rangle), \end{aligned}$$

the last equation holding by definition of $v^{\overline{t(n)},\overline{\rho}}$. For simplicity, write $\langle f(1),\dots,f(m)\rangle$ as $\overline{f}$. We obtain

$$\begin{aligned}
&|v^{\overline{p},\overline{\rho}}(\Theta) - v^{\overline{t(n)},\overline{\rho}}(\Theta)| \\
&= \left| \sum_{f\in N_m(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta,\overline{f}) - \sum_{f\in J_{n.m}(\Theta)} j^{\overline{t(n)},\overline{\rho}}(\Theta,\overline{f}) \right| \\
&= \left| \sum_{f\in I_{n,m}(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta,\overline{f}) + \sum_{f\in J_{n,m}(\Theta)\setminus I_{n,m}(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta,\overline{f}) + \sum_{f\in N_m(\Theta)\setminus J_{n,m}(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta,\overline{f}) \right. \\
&\qquad \left. - \sum_{f\in I_{n,m}(\Theta)} j^{\overline{t(n)},\overline{\rho}}(\Theta,\overline{f}) - \sum_{f\in J_{n,m}(\Theta)\setminus I_{n,m}(\Theta)} j^{\overline{t(n)},\overline{\rho}}(\Theta,\overline{f}) \right| \\
&= \left| \sum_{f\in J_{n,m}(\Theta)\setminus I_{n,m}(\Theta)} \left( j^{\overline{p},\overline{\rho}}(\Theta,\overline{f}) - j^{\overline{t(n)},\overline{\rho}}(\Theta,\overline{f}) \right) + \sum_{f\in N_m(\Theta)\setminus J_{n,m}(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta,\overline{f}) \right|
\end{aligned}$$

using (3.19) and the fact that $I_{n,m}(\Theta) \subseteq J_{n,m}(\Theta) \subseteq N_m(\Theta)$,

$$\leq \left| \sum_{f\in J_{n,m}(\Theta)\setminus I_{n,m}(\Theta)} \left( j^{\overline{p},\overline{\rho}}(\Theta,\overline{f}) - j^{\overline{t(n)},\overline{\rho}}(\Theta,\overline{f}) \right) \right| + \left| \sum_{f\in N_m(\Theta)\setminus J_{n,m}(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta,\overline{f}) \right|, \tag{3.20}$$

this last inequality holding by the triangle inequality.

It is therefore enoguh to show that the expression in (3.20) tends to 0. So fix $\varepsilon > 0$. For each $f \in J_{n,m}(\Theta) \setminus I_{n,m}(\Theta)$ we must have $0 \in \mathrm{ran}(f)$, and there are $\binom{m}{i} \cdot \binom{m-i}{m_1,\dots,m_n}$ functions with

$$|\{j \mid f(j) = 0\}| = i, |\{j \mid f(j) = k\}| = m_k, k = 1,\dots,n.$$

Thus we get

$$\left| \sum_{f\in J_{n,m}(\Theta)\setminus I_{n,m}(\Theta)} \left( j^{\overline{p},\overline{\rho}}(\Theta,\overline{f}) - j^{\overline{t(n)},\overline{\rho}}(\Theta,\overline{f}) \right) \right| \leq \left| \sum_{i=1}^{m} \binom{m}{i} \cdot (p_0^i - t_0^i) \cdot \left( \sum_{j=1}^{n} p_j \right)^{m-i} \right|,$$

as $t_i = p_i$ for $i \in \{1,\dots,n\}$,

$$\leq \sum_{i=1}^{m} \binom{m}{i} \cdot (t_0^i - p_0^i) \cdot \left( \sum_{j=1}^{n} p_j \right)^{m-i},$$

by repeated application of the triangle inequality, the fact that binomials and the $p_i$ are non-negative and $t_0 \geq p_0$ by definition,

$$\leq (1+t_0)^m - (1+p_0)^m, \tag{3.21}$$

since $\sum_{j=1}^{n} p_j < 1$. By definition of $\overline{t(n)}$, $t_0$ tends to $p_0$ as $n$ tends to $\infty$, so there exists $n_0$ such that for all $n \geq n_0$, the right-hand side of (3.21) is strictly smaller than $\varepsilon/2$, so we obtain that

$$\left| \sum_{f \in J_{n,m}(\Theta) \setminus I_{n,m}(\Theta)} \left( j^{\overline{p},\overline{\rho}}(\Theta, \overline{f}) - j^{\overline{t(n)},\overline{\rho}}(\Theta, \overline{f}) \right) \right| < \frac{\varepsilon}{2} \tag{3.22}$$

holds for all $n \geq n_0$.

Notice that

$$\sum_{f \in N_m(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta, \overline{f}) = \sum_{n \in \mathbb{N}} \left( \sum_{f \in J_{n,m}(\Theta) \setminus J_{n-1,m}(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta, \overline{f}) \right),$$

and that for $f \in J_{n,m}(\Theta) \setminus J_{n-1,m}(\Theta)$, $j^{\overline{p},\overline{\rho}}(\Theta, \overline{f}) \leq p_n$, and we obtain

$$\sum_{f \in N_m(\Theta) \setminus J_{n_1,m}(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta, \overline{f}) = \sum_{n > n_1} \left( \sum_{f \in J_{n,m}(\Theta) \setminus J_{n-1,m}(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta, \overline{f}) \right) < \frac{\varepsilon}{2} \tag{3.23}$$

for sufficiently large $n_1$, using a similar argument as above. Now let $N$ be the maximum of $n_0$, $n_1$, and (3.22) together with (3.23) shows that for all $n \geq N$,

$$\begin{aligned} &|v^{\overline{p},\overline{\rho}}(\Theta) - v^{\overline{t(n)},\overline{\rho}}(\Theta)| \\ &\quad \leq \left| \sum_{f \in J_{n,m}(\Theta) \setminus I_{n,m}(\Theta)} \left( j^{\overline{p},\overline{\rho}}(\Theta, \overline{f}) - j^{\overline{t(n)},\overline{\rho}}(\Theta, \overline{f}) \right) \right| + \left| \sum_{f \in N_m(\Theta) \setminus J_{n,m}(\Theta)} j^{\overline{p},\overline{\rho}}(\Theta, \overline{f}) \right| \\ &\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

as required. ⊣

## 3.2 The $v^{\overline{p},\overline{\rho}}$ functions in comparison

The $v^{\overline{p},\overline{\rho}}$ functions were defined as generalized versions of the $u^{\overline{p}}$ functions (see e.g. [17], [21], [22] for discussion of the properties of $u^{\overline{p}}$), in a first attempt to show the

Theorem 2.8. Thus, there is an obvious way of getting the $u^{\overline{p}}$ functions back – we certainly obtain $v^{\overline{p},\overline{\rho}} = u^{\overline{p}}$ if we let each $\rho_i$ be the measure concentrated on the single point $\frac{1}{2}$. It remains to be seen whether this is the only way in which we can obtain a function $u^{\overline{p}}$ from $v^{\overline{p},\overline{\rho}}$.

A second question is concerned with the Generalized Principle of Instantial Relevance. There are certain $u^{\overline{p}}$ functions that satisfy GPIR, namely the Nix-Paris functions $w^{\delta}$. As the $v^{\overline{p},\overline{\rho}}$ are related closely to the $u^{\overline{p}}$, this raises the question whether there is a generalized version of the $w^{\delta}$ satisfying SPx and GPIR.

We intend to provide answers to those questions in this section.

**Proposition 3.18.** *Let $\overline{p} \in \mathbb{B}$. Then $v^{\overline{p},\overline{\rho}} = u^{\overline{p}}$ if and only if each $\rho_i$ is the measure concentrated on the singleton $\{\frac{1}{2}\}$.*

**Proof:** First notice that each of the $u^{\overline{p}}$ satisfy Ax + WIP, and by Theorem 5 in [21] these are the only functions with this property. Thus, in order to have $v^{\overline{p},\overline{\rho}} = u^{\overline{p}}$, $v^{\overline{p},\overline{\rho}}$ must satisfy both Ax and WIP. By the converse of Lemma 3.15, we therefore have that each of the $\rho_i$ must be a measure concentrate on some singleton $\tau_i \in [0, 1]$.

With Ax holding for $v^{\overline{p},\overline{\rho}}$, we obtain that for all $j \in \{1, \ldots, 2^q\}$, we have $v^{\overline{p},\overline{\tau}}(\alpha_j) = 1/2^q$. Since we only look at single atoms, we get

$$\sum_{i \in \mathbb{N}} p_i \tau_i^k (1 - \tau_i)^{q-k} = \frac{1}{2^q}$$

for all $k \in \{0, \ldots, q\}$. For $q = 1$, $k = 1$ we get

$$\sum_{i \in \mathbb{N}} p_i \tau_i = \frac{1}{2}, \tag{3.24}$$

and for $q = 2$, $k = 2$ we have

$$\sum_{i \in \mathbb{N}} p_i \tau_i^2 = \frac{1}{4}. \tag{3.25}$$

We can view (3.24) as the expected value for a random variable $\tau$, and (3.25) as the expected value of the random variable $\tau^2$, giving for the variance

$$\mathrm{Var}(\tau) = \frac{1}{4} - \left(\frac{1}{2}\right)^2 = 0.$$

But a random variable with expected value $\frac{1}{2}$ and a variance of 0 must necessarily be constant. Thus $\tau_i = 1/2$ for all $i$ and we get $v^{\overline{p},\overline{\tau}} = u^{\overline{p}}$. ⊣

We now turn to the question of which of the $v^{\overline{p},\overline{\rho}}$ satisfy GPIR. Note that the Nix-Paris functions, usually denoted by $w^\delta$, are characterized by the principles Reg, Ex, Ax and GPIR, see e.g. Theorem 24 in [18] and chapters 18 and 19 in [22]. In addition to that, one can obtain these functions from the $u^{\overline{p}}$ functions, and in fact we have $w^\delta = u^{\overline{p}}$ for $\overline{p} = \langle 1-\delta, \delta, 0, 0, \ldots\rangle$. By definition of the $u_n^{\overline{p}}$ functions, this is equivalent to $w^\delta = u_1^{\overline{p}}$ for any $\overline{p}$ with $p_1 = \delta$. This prompts the question if something similar holds for $v_1^{\overline{p},\overline{\rho}}$. The following lemma will show that this is indeed the case.

**Lemma 3.19.** *Let $\overline{p} \in \mathbb{B}$, $\overline{\rho}$ a sequence of normalized measures on $[0,1]$. Then $v_{1,L}^{\overline{p},\overline{\rho}}$ satisfies GPIR.*

**Proof:** Fix $k \in \mathbb{N}$ and consider the measure $\rho_k$. Recall that for each atom $\alpha_j$ of $L$, we have

$$f^q(j,k) = \int_{[0,1]} x^{\gamma_q(j)}(1-x)^{q-\gamma_q(j)}\, d\rho_k(x),$$

where $\gamma_q(j)$ is the number of negated predicates occuring in $\alpha_j$. We obtain for a state description $\Theta(a_1,\ldots,a_m) = \bigwedge_{i=1}^n \alpha_{h_i}(a_i)$,

$$v_{1,L}^{\overline{p},\overline{\rho}}(\Theta) = \sum_{j=1}^{2^q} f^q(j,1)\cdot\left(R_{\overline{p},1} f^q(j,0)+p_1\right)^{m_j}\cdot \prod_{\substack{s=1\\ s\neq j}}^{2^q}\left(R_{\overline{p},1} f^q(s,0)\right)^{m_s},$$

where as usual $\langle m_1,\ldots,m_{2^q}\rangle$ is the signature of $\Theta$.

Define $s_{\varphi,\psi}(\vartheta) = v_{1,L}^{\overline{p},\overline{\rho}}(\vartheta|\varphi\wedge\psi) - v_{1,L}^{\overline{p},\overline{\rho}}(\vartheta|\psi)$. Spelling out the definition of conditional probabilities, we obtain

$$s_{\varphi,\psi}(\vartheta) = \frac{v_{1,L}^{\overline{p},\overline{\rho}}(\vartheta\wedge\varphi\wedge\psi)\cdot v_{1,L}^{\overline{p},\overline{\rho}}(\psi) - v_{1,L}^{\overline{p},\overline{\rho}}(\vartheta\wedge\psi)\cdot v_{1,L}^{\overline{p},\overline{\rho}}(\varphi\wedge\psi)}{v_{1,L}^{\overline{p},\overline{\rho}}(\psi)\cdot v_{1,L}^{\overline{p},\overline{\rho}}(\varphi\wedge\psi)}.$$

Clearly, $v_{1,L}^{\overline{p},\overline{\rho}}$ satisfies GPIR if $s_{\varphi,\psi}(\vartheta)\geq 0$ whenever $\vartheta\models\varphi$. We will be only interested in the numerator, since the denominator surely is positive and does not depend on $\vartheta$. For the same reason we obtain

$$s_{\varphi,\psi}(\vartheta) = \sum_{\alpha_i\models\vartheta} s_{\varphi,\psi}(\alpha_i),$$

as we have

$$v_{1,L}^{\overline{p},\overline{\rho}}(\vartheta \wedge \eta) = \sum_{\alpha \models \vartheta} v_{1,L}^{\overline{p},\overline{\rho}}(\alpha \wedge \eta)$$

for any $\eta \in SL$ by (P2), and it is therefore enough to show that $s_{\varphi,\psi}(\alpha_i) \geq 0$ for any atom $\alpha_i$ with $\alpha_i \models \vartheta$.

Let $\psi(\vec{a}) = \bigvee_{h=1}^{r} \bigwedge_{j=1}^{m} \alpha_{hj}(a_j)$, and for $h \in \{1, \ldots, r\}$, let $k_{hj} = |\{i|\alpha_{hi} = \alpha_j\}|$. Furthermore, for $i, i_1, \ldots, i_n \in \{1, \ldots, 2^q\}$, let

$$k_{hz}^{i} = \begin{cases} k_{hz} + 1 & \text{if } i = z, \\ k_{hz} & \text{otherwise,} \end{cases}$$

$$k_{hz}^{i_1,\ldots,i_n} = \begin{cases} k_{hz}^{i_2,\ldots,i_n} + 1 & \text{if } i_1 = z, \\ k_{hz}^{i_2,\ldots,i_n} & \text{otherwise.} \end{cases}$$

Now let an atom $\alpha_i$ of $L$ be given. Then letting

$$S_1 = \left[ \sum_{\alpha_j \models \varphi} \sum_{h=1}^{r} v_{1,L}^{\overline{p},\overline{\rho}} \left( \alpha_i \wedge \alpha_j \wedge \bigwedge_{s=1}^{m} \alpha_{hs} \right) \right] \cdot \left[ \sum_{g=1}^{r} v_{1,L}^{\overline{p},\overline{\rho}} \left( \bigwedge_{s=1}^{m} \alpha_{gs} \right) \right],$$

$$S_2 = \left[ \sum_{\alpha_j \models \varphi} \sum_{h=1}^{r} v_{1,L}^{\overline{p},\overline{\rho}} \left( \alpha_j \wedge \bigwedge_{s=1}^{m} \alpha_{hs} \right) \right] \cdot \left[ \sum_{g=1}^{r} v_{1,L}^{\overline{p},\overline{\rho}} \left( \alpha_i \wedge \bigwedge_{s=1}^{m} \alpha_{gs} \right) \right]$$

and

$$S = \left[ \sum_{\alpha_j \models \varphi} \sum_{h=1}^{r} v_{1,L}^{\overline{p},\overline{\rho}} \left( \alpha_j \wedge \bigwedge_{s=1}^{m} \alpha_{hs} \right) \right] \cdot \left[ \sum_{g=1}^{r} v_{1,L}^{\overline{p},\overline{\rho}} \left( \bigwedge_{s=1}^{m} \alpha_{gs} \right) \right],$$

we obtain

$$s_{\varphi,\psi}(\alpha_i) = \frac{S_1 - S_2}{S}.$$

It remains to show that the difference $S_1 - S_2$, and therefore $s_{\varphi,\psi}(\alpha_i)$, is non-negative for $\alpha_i \models \varphi$.

Multiplying out, we obtain for $S_1$ and $S_2$

$$S_1 = \sum_{\alpha_j \models \varphi} \sum_{h=1}^{r} \sum_{g=1}^{r} \sum_{x=1}^{2^q} \sum_{y=1}^{2^q} f^q(x,1) f^q(y,1) \cdot \prod_{\substack{s=1 \\ s \neq x}}^{2^q} (R_{\overline{p},1} f^q(s,0))^{k_{hs}^{i,j}} \cdot \prod_{\substack{t=1 \\ t \neq y}}^{2^q} (R_{\overline{p},1} f^q(t,0))^{k_{gt}}$$
$$\cdot (R_{\overline{p},1} f^q(x,0) + p_1)^{k_{hx}^{i,j}} \cdot (R_{\overline{p},1} f^q(y,0) + p_1)^{k_{gy}}$$
$$S_2 = \sum_{\alpha_j \models \varphi} \sum_{h=1}^{r} \sum_{g=1}^{r} \sum_{x=1}^{2^q} \sum_{y=1}^{2^q} f^q(x,1) f^q(y,1) \cdot \prod_{\substack{s=1 \\ s \neq x}}^{2^q} (R_{\overline{p},1} f^q(s,0))^{k_{hs}^{j}} \cdot \prod_{\substack{t=1 \\ t \neq y}}^{2^q} (R_{\overline{p},1} f^q(t,0))^{k_{gt}^{i}}$$
$$\cdot (R_{\overline{p},1} f^q(x,0) + p_1)^{k_{hx}^{j}} \cdot (R_{\overline{p},1} f^q(y,0) + p_1)^{k_{gy}^{i}}.$$

To simplify this equation, let

$$E_{h,g}(x,y) = f^q(x,1) \cdot f^q(y,1) \cdot (R_{\overline{p},1} f^q(x,0) + p_1)^{k_{hx}} \cdot (R_{\overline{p},1} f^q(y,0) + p_1)^{k_{gy}}$$
$$\cdot \prod_{\substack{s=1 \\ s \neq x}}^{2^q} (R_{\overline{p},1} f^q(s,0))^{k_{hs}} \cdot \prod_{\substack{t=1 \\ t \neq y}}^{2^q} (R_{\overline{p},1} f^q(t,0))^{k_{gt}}$$

and let

$$A_1^{h,g}(x,y,i,j) = \begin{cases} (R_{\overline{p},1})^2 f^q(i,0) f^q(j,0) & \text{if } x \neq i,j, \\ (R_{\overline{p},1} f^q(i,0) + p_1)^2 & \text{if } x = i = j, \\ (R_{\overline{p},1} f^q(i,0) + p_1)(R_{\overline{p},1} f^q(j,0)) & \text{if } i \neq j \text{ and } x = i, \\ (R_{\overline{p},1} f^q(i,0))(R_{\overline{p},1} f^q(j,0) + p_1) & \text{if } i \neq j \text{ and } x = j, \end{cases} \tag{3.26}$$

and

$$A_2^{h,g}(x,y,i,j) = \begin{cases} (R_{\overline{p},1})^2 f^q(i,0) f^q(j,0) & \text{if } x \neq i \text{ and } y \neq j, \\ (R_{\overline{p},1} f^q(i,0) + p_1)(R_{\overline{p},1} f^q(j,0) + p_1) & \text{if } x = i \text{ and } y = j, \\ (R_{\overline{p},1} f^q(i,0))(R_{\overline{p},1} f^q(j,0) + p_1) & \text{if } x \neq i \text{ and } y = j, \\ (R_{\overline{p},1} f^q(i,0) + p_1)(R_{\overline{p},1} f^q(j,0)) & \text{if } x = i \text{ and } y \neq j. \end{cases} \tag{3.27}$$

Note that $A_1^{h,g}(x,y,i,j)$ does not depend on $y$. We keep $y$ in the list of parameters for notational convenience, especially as the difference of both terms does depend on $y$.

We obtain for the difference

$$S_1 - S_2 = \sum_{\alpha_j \models \varphi} \sum_{h=1}^{r} \sum_{g=1}^{r} \sum_{x=1}^{2^q} \sum_{y=1}^{2^q} E_{h,g}(x,y) \cdot (A_1^{h,g}(x,y,i,j) - A_2^{h,g}(x,y,i,j)). \tag{3.28}$$

Key to the proof are now the factors $A_1^{h,g}(x,y,i,j)$ and $A_2^{h,g}(x,y,i,j)$ and their difference.

We will consider the two main cases

(1) $i = j$ and

(2) $i \neq j$

and calculate the difference

$$A_1^{h,g}(x,y,i,j) - A_2^{h,g}(x,y,i,j) \tag{3.29}$$

as $x$ and $y$ vary. We shall show that the difference only depends on whether $x$ and $y$ take the value of $i$ and $j$ or not.

First, consider case (1) and assume $i = j$. Then we have four choices for $x$ and $y$:

(1.i) $x = y = i$,

(1.ii) $x = i$, $y \neq i$,

(1.iii) $x \neq i$, $y = i$,

(1.iv) $x \neq i$, $y \neq i$.

From (3.26) and (3.27) we obtain immediately that in cases (1.i) and (1.iv) both factors are equal and thus their difference (3.29) vanishes.

In case (1.ii), we obtain

$$\begin{aligned} A_1^{h,g}(x,y,i,j) - A_2^{h,g}(x,y,i,j) &= (R_{\overline{p},1} f^q(i,0) + p_1)^2 - (R_{\overline{p},1} f^q(i,0) + p_1) R_{\overline{p},1} f^q(i,0) \\ &= (R_{\overline{p},1})^2 f^q(i,0)^2 + 2p_1 R_{\overline{p},1} f^q(i,0) + p_1^2 \\ &\quad - (R_{\overline{p},1})^2 f^q(i,0)^2 - p_1 R_{\overline{p},1} f^q(i,0) \\ &= p_1 R_{\overline{p},1} f^q(i,0) + p_1^2, \end{aligned}$$

and for case (1.iii), the difference is

$$\begin{aligned}
A_1^{h,g}(x,y,i,j) - A_2^{h,g}(x,y,i,j) &= (R_{\overline{p},1})^2 f^q(i,0)^2 - R_{\overline{p},1} f^q(i,0)(R_{\overline{p},1} f^q(i,0) + p_1) \\
&= R_{\overline{p},1}^2 f^q(i,0)^2 - R_{\overline{p},1}^2 f^q(i,0)^2 - p_1 R_{\overline{p},1} f^q(i,0) \\
&= -p_1 R_{\overline{p},1} f^q(i,0).
\end{aligned}$$

Now suppose $i \neq j$. Then we obtain the following cases for $x$ and $y$.

(2.i) $x = y = j$,

(2.ii) $x = i$, $y = j$,

(2.iii) $x \notin \{i, j\}$, $y = j$,

(2.iv) $x = i$, $y \neq j$,

(2.v) $x = j$, $y \neq j$,

(2.vi) $x \notin \{i, j\}$, $y \neq j$.

Just as in case (1), we obtain from the definitions (3.26) and (3.27) that $A_1^{h,g}(x,y,i,j) = A_2^{h,g}(x,y,i,j)$ in the cases (2.i), (2.iv) and (2.vi), and thus the difference (3.29) vanishes in these cases.

For case (2.ii), we obtain

$$\begin{aligned}
A_1^{h,g}(x,y,i,j) - A_2^{h,g}(x,y,i,j) &= (R_{\overline{p},1} f^q(i,0) + p_1) R_{\overline{p},1} f^q(j,0) \\
&\quad - (R_{\overline{p},1} f^q(i,0) + p_1)(R_{\overline{p},1} f^q(j,0) + p_1) \\
&= (R_{\overline{p},1} f^q(i,0) + p_1)(R_{\overline{p},1} f^q(j,0) - R_{\overline{p},1} f^q(j,0) - p_1) \\
&= -p_1 R_{\overline{p},1} f^q(i,0) - p_1^2,
\end{aligned}$$

in case (2.iii), we have

$$\begin{aligned}
A_1^{h,g}(x,y,i,j) - A_2^{h,g}(x,y,i,j) &= (R_{\overline{p},1})^2 f^q(i,0) f^q(j,0) - R_{\overline{p},1} f^q(i,0)(R_{\overline{p},1} f^q(j,0) + p_1) \\
&= R_{\overline{p},1} f^q(i,0)(R_{\overline{p},1} f^q(j,0) - R_{\overline{p},1} f^q(j,0) - p_1) \\
&= -p_1 R_{\overline{p},1} f^q(i,0),
\end{aligned}$$

and for case (2.v) we have

$$\begin{aligned} A_1^{h,g}(x,y,i,j) - A_2^{h,g}(x,y,i,j) &= R_{\overline{p},1} f^q(i,0)(R_{\overline{p},1} f^q(j,0) + p_1) - (R_{\overline{p},1})^2 f^q(i,0) f^q(j,0) \\ &= R_{\overline{p},1} f^q(i,0)(R_{\overline{p},1} f^q(j,0) + p_1 - R_{\overline{p},1} f^q(j,0)) \\ &= p_1 R_{\overline{p},1} f^q(i,0). \end{aligned}$$

Thus we obtain for (3.28), separating the sum over $\alpha_j \models \varphi$ into the cases $i \neq j$ and $i = j$,

$$\begin{aligned} S_1 - S_2 = &\left[ \sum_{\substack{\alpha_j \models \varphi \\ j \neq i}} \sum_{h=1}^{r} \sum_{g=1}^{r} \left( -p_1 R_{\overline{p},1} f^q(i,0) \sum_{\substack{x=1 \\ x \neq i,j}} E_{h,g}(x,j) \right.\right. \\ &\left.\left. - (p_1^2 + p_1 R_{\overline{p},1} f^q(i,0)) E_{h,g}(i,j) + p_1 R_{\overline{p},1} f^q(i,0) \sum_{\substack{y=1 \\ y \neq j}}^{2^q} E_{h,g}(j,y) \right) \right] \\ &+ \left( (p_1 R_{\overline{p},1} f^q(i,0) + p_1^2) \sum_{\substack{y=1 \\ y \neq i}}^{2^q} E_{h,g}(i,y) - p_1 R_{\overline{p},1} f^q(i,0) \sum_{\substack{x=1 \\ x \neq i}}^{2^q} E_{h,g}(x,i) \right), \end{aligned} \tag{3.30}$$

with the expression

$$(p_1 R_{\overline{p},1} f^q(i,0) + p_1^2) \sum_{\substack{y=1 \\ y \neq i}}^{2^q} E_{h,g}(i,y) - p_1 R_{\overline{p},1} f^q(i,0) \sum_{\substack{x=1 \\ x \neq i}}^{2^q} E_{h,g}(x,i) \tag{3.31}$$

only occurring in (3.30) if $\alpha_i \models \varphi$.

Notice that for fixed $h$, $g$, $x$, $y$ we have

$$E_{h,g}(x,y) = E_{g,h}(y,x),$$

and thus the terms

$$-p_1 R_{\overline{p},1} f^q(i,0) \sum_{\substack{x=1 \\ x \neq i,j}}^{2^q} E_{h,g}(x,j)$$

and

$$p_1 R_{\overline{p},1} f^q(i,0) \sum_{\substack{y=1 \\ y \neq i,j}}^{2^q} E_{g,h}(j,y)$$

cancel by this symmetry. Simplifying (3.30) in this way, we obtain in case that $\alpha_i \models \varphi$,

$$S_1 - S_2 = \sum_{\substack{\alpha_j \models \varphi \\ j \neq i}} -p_1^2 \sum_{h=1}^{r} \sum_{g=1}^{r} E_{h,g}(i,j) + p_1^2 \sum_{h=1}^{r} \sum_{g=1}^{r} \sum_{\substack{y=1 \\ y \neq i}}^{2^q} E_{h,g}(i,y)$$

$$= p_1^2 \sum_{h=1}^{r} \sum_{g=1}^{r} \sum_{\alpha_j \not\models \varphi} E_{h,g}(i,j), \tag{3.32}$$

as the only $y$ to range over that do not get cancelled are those with $\alpha_y \not\models \varphi$. Note that the term in (3.32) is non-negative.

Since we assumed $\vartheta \models \varphi$ we have for each $\alpha_i \models \vartheta$ that

$$s_{\varphi,\psi}(\alpha_i) = p_1^2 \sum_{h=1}^{r} \sum_{g=1}^{r} \sum_{\alpha_j \not\models \varphi} \frac{E_{h,g}(i,j)}{S},$$

which is clearly non-negative as it is a sum of non-negative terms. Thus GPIR holds for $v_{1,L}^{\bar{p},\bar{\rho}}$. ⊣

## 3.3 SPx and Johnson's Sufficientness Postulate

While the Principle of Strong Predicate Exchangeability arose from a generalized version of the $u^{\bar{p}}$ functions, and as such seems a rather artificial one, there is still some justification for it in presence of a weaker version of Johnson's Sufficientness Postulate that we defined in section 1.3. We will show in this section that any function satisfying this weak version will also satisfy SPx.

**Johnson's Predicate Sufficientness Postulate, JPSP**

$$w\left(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right)$$

*depends only on* $n$, $n_j = |\{h_i \mid h_i = j\}|$ *and* $\gamma_q(j)$.

To relate this to Johnson's Sufficientness Postulate (JSP), note that JPSP differs from JSP by the additional dependence on $\gamma_q(j)$. For a more detailed discussion of JSP see e.g. [22].

**Proposition 3.20.** *Let $w$ be a probability function on $L$ satisfying JPSP + Ex. Then $w$ satisfies SPx.*

**Proof:** Let $\sigma$ be a permutation of atoms induced by SPx. Then clearly, for any atom $\alpha_i$ we have $\gamma_q(i) = \gamma_q(\sigma(i))$. Thus, as we obtain from JPSP that

$$w\left(\sigma(\alpha_j)(a_{n+1}) | \bigwedge_{i=1}^{n} \sigma(\alpha_{h_i})(a_i)\right) = w\left(\alpha_{\sigma(j)}(a_{n+1}) | \bigwedge_{i=1}^{n} \alpha_{\sigma(h_i)}(a_i)\right) \tag{3.33}$$

depends only on $n$, $n_{\sigma(j)} = |\{\sigma(h_i) \,|\, \sigma(h_i) = \sigma(j)\}|$, and $\gamma_q(\sigma(j))$, we get that

$$w\left(\alpha_{\sigma(j)}(a_{n+1}) | \bigwedge_{i=1}^{n} \alpha_{\sigma(h_i)}(a_i)\right) = w\left(\alpha_j(a_{n+1}) | \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right). \tag{3.34}$$

Let $\Theta(a_1, \dots, a_n)$, $\Phi(a_1, \dots, a_n)$ be state descriptions of $L$, say

$$\Theta(a_1, \dots, a_n) = \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)$$

and

$$\Phi(a_1, \dots, a_n) = \bigwedge_{i=1}^{n} \alpha_{g_i}(a_i),$$

with the same $P$-spectra. Then there exists a permutation of atoms induced by SPx such that $\sigma\Theta = \Phi$. Note that in general, we will not have $\sigma(h_i) = g_i$ for each $i \in \{1, \dots, n\}$, but since we have Constant Exchangeability as a standing assumption, we can assume that this is the case.

We now obtain

$$\begin{aligned} w(\Theta(a_1, \dots, a_n)) &= \frac{w(\Theta(a_1, \dots, a_n))}{w\left(\bigwedge_{i=1}^{n-1} \alpha_{h_i}(a_i)\right)} \cdot \frac{w\left(\bigwedge_{i=1}^{n-1} \alpha_{h_i}(a_i)\right)}{w((\bigwedge_{i=1}^{n-2} \alpha_{h_i}(a_i))} \cdot \dots \\ &\qquad \cdot \frac{w(\alpha_{h_1}(a_1) \wedge \alpha_{h_2}(a_2))}{w(\alpha_{h_1}(a_1))} \cdot \frac{w(\alpha_{h_1}(a_1))}{w(\top)} \\ &= w\left(\alpha_{h_n}(a_n) \,|\, \bigwedge_{i=1}^{n-1} \alpha_{h_i}(a_i)\right) \cdot w\left(\alpha_{h_{n-1}}(a_{n-1}) \,|\, \bigwedge_{i=1}^{n-2} \alpha_{h_i}(a_i)\right) \cdot \dots \\ &\qquad \cdot w(\alpha_{h_1}(a_1) \,|\, \top), \end{aligned}$$

and applying (3.34) to this situation, we obtain

$$
\begin{aligned}
&= w\left(\alpha_{\sigma(h_n)}(a_n) \mid \bigwedge_{i=1}^{n-1} \alpha_{\sigma(h_i)}(a_i)\right) \\
&\quad \cdot w\left(\alpha_{\sigma(h_{n-1})}(a_{n-1}) \mid \bigwedge_{i=1}^{n-2} \alpha_{\sigma(h_i)}(a_i)\right) \cdot \dots \cdot w(\alpha_{\sigma(h_1)}(a_1) \mid \top) \\
&= w\left(\bigwedge_{i=1}^{n} \alpha_{\sigma(h_i)}(a_i)\right) = w(\Phi(a_1, \dots, a_n)),
\end{aligned}
$$

and thus SPx holds for $w$. $\dashv$

We will now proceed to classify the functions satisfying JPSP. Since JPSP is closely related to JSP, it is perhaps not surprising that these functions will be similar to Carnap's $c_\lambda$ functions.

**Definition 3.21:** *Let $L = L_q$ and let $\vec{\mathcal{C}} = \langle \mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_q \rangle$ be such that*

$$\sum_{i=0}^{q} \binom{q}{i} \mathcal{C}_i = 1.$$

*For $0 < \lambda < \infty$, define the function $c^L_{\lambda,\vec{\mathcal{C}}}$ by*

$$c^L_{\lambda,\vec{\mathcal{C}}}\left(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) = \frac{n_j + \mathcal{C}_{\gamma_q(j)}\lambda}{n+\lambda},$$

*where $n_j = |\{h_i \mid h_i = j\}|$. Define the function $c^L_{0,\vec{\mathcal{C}}}$ by*

$$c^L_{0,\vec{\mathcal{C}}}\left(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) = \begin{cases} \mathcal{C}_{\gamma_q(h_1)} & \text{if } h_1 = h_2 = \dots = h_n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that by definition, $c^L_{\lambda,\vec{\mathcal{C}}}$ satisfies JPSP. We will now show that if $w$ satisfies JPSP + Reg and $L$ contains at least two predicates, then $w = c^L_{\lambda,\vec{\mathcal{C}}}$ for some $0 < \lambda < \infty$. The proof we are going to provide is essentially a modified version of the proof for the analogous result for JSP, see e.g. Theorem 17.2 in [22].

**Theorem 3.22.** *Let $q \geq 2$ and $w$ be a probability function on $L = L_q$ satisfying JPSP + Ex + Reg. Then there exist $\vec{\mathcal{C}}$ with $\sum_{i=0}^{q} \binom{q}{i} \mathcal{C}_i = 1$ and $0 < \lambda < \infty$ such that $w = c^L_{\lambda,\vec{\mathcal{C}}}$.*

**Proof:** Since by JPSP, $w\left(\alpha_j(a_{n+1}) \,|\, \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right)$ depends only on $n$, $n_j$ and $\gamma_q(j)$, we can set

$$w\left(\alpha_j(a_{n+1}) \,|\, \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) = g(\gamma_q(j), n, n_j).$$

As for any $\alpha_j$, $w(\bigvee_{i=1}^{2^q} \alpha_i(a_2) \,|\, \alpha_j(a_1)) = 1$, we obtain for each $\gamma_q(j) \in \{0, \ldots, q\}$

$$\sum_{i=1}^{2^q} w(\alpha_i(a_2) \,|\, \alpha_j(a_1)) = g(\gamma_q(j), 1, 1) + \sum_{\substack{i=1 \\ i \neq j}}^{2^q} g(\gamma_q(i), 0, 1) = 1, \tag{3.35}$$

which in turn gives

$$\sum_{i=1}^{2^q} g(\gamma_q(i), 1, 1) + (2^q - 1) \sum_{i=1}^{2^q} g(\gamma_q(i), 0, 1) = 2^q. \tag{3.36}$$

Since Ex holds for $w$, so does PIR, and together with Reg we obtain that

$$1 > g(\gamma_q(j), 1, 1) \geq g(\gamma_q(j), 0, 1) > 0$$

for each $j \in \{1, \ldots, 2^q\}$. Combined with (3.36), we can let

$$2^{-q} \sum_{i=1}^{2^q} g(\gamma_q(i), 1, 1) = \frac{1 + 2^{-q}\lambda}{1 + \lambda}$$

and

$$2^{-q} \sum_{i=1}^{2^q} g(\gamma_q(i), 0, 1) = \frac{2^{-q}\lambda}{1 + \lambda}, \tag{3.37}$$

for some $\lambda$ such that $0 < \lambda < \infty$. For $j \in \{1, \ldots, 2^q\}$, let $C_j$ be such that

$$C_j \sum_{i=1}^{2^q} g(\gamma_q(i), 0, 1) = g(\gamma_q(j), 0, 1).$$

Note that by Regularity, we must have $C_j > 0$. We obtain that $\sum_{i=1}^{2^q} \mathcal{C}_i = 1$, and together with (3.37) we get

$$g(\gamma_q(j), 0, 1) = \frac{C_j \lambda}{1 + \lambda}, \tag{3.38}$$

which then substituted in (3.35) gives

$$g(\gamma_q(j), 1, 1) = \frac{1 + C_j \lambda}{1 + \lambda}. \tag{3.39}$$

Also note that $C_j = C_i$ whenever $\gamma_q(i) = \gamma_q(j)$ by JPSP for $w$, so from now on we shall denote $C_j$ by $\mathcal{C}_{\gamma_q(j)}$.

We will now show for all $n \geq 0$, $s \leq n$ that

$$g(\gamma_q(j), s, n) = \frac{s + \mathcal{C}_{\gamma_q(j)}\lambda}{n + \lambda}. \tag{3.40}$$

We shall do this by induction on $n$. For this, notice first that for $r + s = n + 1$, we obtain

$$\begin{aligned}
1 &= w\left(\bigvee_{i=1}^{2^q} \alpha_i(a_{n+2}) \mid \bigwedge_{i=1}^{r} \alpha_m(a_i) \wedge \bigwedge_{i=r+1}^{n+1} \alpha_j(a_i)\right) \\
&= \sum_{t=1}^{2^q} w\left(\alpha_t(a_{n+2}) \mid \bigwedge_{i=1}^{r} \alpha_m(a_i) \wedge \bigwedge_{i=r+1}^{n+1} \alpha_j(a_i)\right) \\
&= g(\gamma_q(m), r, n+1) + g(\gamma_q(j), s, n+1) + \sum_{i \neq m,j} g(\gamma_q(i), 0, n+1) = 1.
\end{aligned} \tag{3.41}$$

Furthermore, let $r + s + t = n$ and $1 \leq j, k, m \leq 2^q$ distinct, we obtain

$$\begin{aligned}
& w\left(\alpha_j(a_{n+2}) \mid \bigwedge_{i=1}^{r} \alpha_j(a_i) \wedge \bigwedge_{i=r+1}^{r+s+1} \alpha_k(a_i) \wedge \bigwedge_{i=r+s+2}^{n+1} \alpha_m(a_i)\right) \\
&\qquad \cdot w\left(\alpha_k(a_{n+1}) \mid \bigwedge_{i=1}^{r} \alpha_j(a_i) \wedge \bigwedge_{i=r+1}^{r+s} \alpha_k(a_i) \wedge \bigwedge_{i=r+s+1}^{n} \alpha_m(a_i)\right) \\
&= w\left(\alpha_k(a_{n+2}) \mid \bigwedge_{i=1}^{r+1} \alpha_j(a_i) \wedge \bigwedge_{i=r+2}^{r+s+1} \alpha_k(a_i) \wedge \bigwedge_{i=r+s+2}^{n+1} \alpha_m(a_i)\right) \\
&\qquad \cdot w\left(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^{r} \alpha_j(a_i) \wedge \bigwedge_{i=r+1}^{r+s} \alpha_k(a_i) \wedge \bigwedge_{i=r+s+1}^{n} \alpha_m(a_i)\right),
\end{aligned}$$

from which we obtain for $j \neq k$

$$g(\gamma_q(j), r, n+1) \cdot g(\gamma_q(k), s, n) = g(\gamma_q(k), s, n+1) \cdot g(\gamma_q(j), r, n). \tag{3.42}$$

Now letting $r = s = n = 0$ and substituting (3.38) in (3.42), we obtain

$$\frac{\mathcal{C}_{\gamma_q(j)}\lambda}{1+\lambda} \cdot g(\gamma_q(k), 0, 0) = \frac{\mathcal{C}_{\gamma_q(k)}\lambda}{1+\lambda} \cdot g(\gamma_q(j), 0, 0),$$

and as the $\mathcal{C}_{\gamma(j)}$ and $g(\gamma_q(j), 0, 0)$ both sum to 1, this in turn gives

$$\begin{aligned}
\frac{\mathcal{C}_{\gamma_q(k)}\lambda}{1+\lambda} &= \sum_{j=1}^{2^q} \frac{\mathcal{C}_{\gamma_q(k)}\lambda \cdot g(\gamma_q(j), 0, 0)}{1+\lambda} \\
&= \sum_{j=1}^{2^q} \frac{\mathcal{C}_{\gamma_q(j)}\lambda \cdot g(\gamma_q(k), 0, 0)}{1+\lambda} \\
&= \frac{g(\gamma_q(k), 0, 0)\lambda}{1+\lambda},
\end{aligned}$$

and thus $\mathcal{C}_{\gamma_q(i)} = g(\gamma_q(i), 0, 0)$ for each $i \in \{1, \dots, 2^q\}$.

So assume that (3.40) holds for $n$, $s \leq n$. Taking $s = 0$ and using the inductive hypothesis (3.42) gives us for $k \neq j$ that

$$g(\gamma_q(k), r, n+1) = \frac{r + \mathcal{C}_{\gamma_q(k)}\lambda}{\mathcal{C}_{\gamma_q(j)}\lambda} \cdot g(\gamma_q(j), 0, n+1). \tag{3.43}$$

Note that this holds trivially for $r = s = 0$ in case $k = j$. Now in (3.41), let $r = 1$, $s = n$ and combining with (3.43), we obtain

$$\left( \frac{1 + \mathcal{C}_{\gamma_q(k)}\lambda}{\mathcal{C}_{\gamma_q(j)}\lambda} + \frac{n + \mathcal{C}_{\gamma_q(m)}\lambda}{\mathcal{C}_{\gamma_q(j)}\lambda} + \sum_{i \neq k,m} \frac{\mathcal{C}_{\gamma_q(i)}\lambda}{\mathcal{C}_{\gamma_q(j)}\lambda} \right) \cdot g(\gamma_q(j), 0, n+1) = 1,$$

which gives

$$g(\gamma_q(j), 0, n+1) = \frac{\mathcal{C}_{\gamma_q(j)}\lambda}{n + 1 + \lambda}.$$

Substituting in (3.43) for $n+1$ and $r = 1, \dots, n$, we obtain (3.40) holding for these as well. Finally, taking $r = n+1$, $s = 0$ in (3.41), we obtain

$$\begin{aligned} g(\gamma_q(j), n+1, n+1) &= 1 - \sum_{i \neq j} g(\gamma_q(i), 0, n+1) \\ &= \frac{n + 1 + \lambda - \sum_{i \neq j} \mathcal{C}_{\gamma_q(i)}\lambda}{n + 1 + \lambda} = \frac{n + 1 + \mathcal{C}_{\gamma_q(j)}\lambda}{n + 1 + \lambda}, \end{aligned}$$

as required. ⊣

To conclude the section on JPSP, we will show that the $c_{\lambda,\vec{\mathcal{C}}}$ satisfy ULi just if each $\mathcal{C}_i$ is of the form $\int_{[0,1]} x^i (1-x)^{q-i} \, d\rho(x)$ for some normalized measure $\rho$ on $[0, 1]$.

This is straightforward to check for $\lambda = 0$: Let $\alpha_j$ be an atom of $L_q$ with $\gamma_q(j) = k$. Suppose that $c^{L_q}_{0,\vec{\mathcal{C}^q}}$ satisfies ULi, with the corresponding function on $L_r$ being given by the parameters 0 and $\vec{\mathcal{C}^r} = \langle \mathcal{C}^r_0, \dots, \mathcal{C}^r_r \rangle$. Then on $L_{q+1}$, we have

$$\begin{aligned} c^{L_{q+1}}_{0,\vec{\mathcal{C}^{q+1}}} \left( \bigwedge_{i=1}^n \alpha_j(a_i) \right) &= \sum_{\varepsilon_1,\dots,\varepsilon_n \in \{0,1\}} c^{L_{q+1}}_{0,\vec{\mathcal{C}^{q+1}}} \left( \bigwedge_{i=1}^n \alpha_j^{\varepsilon_i}(a_i) \right) \\ &= c^{L_{q+1}}_{0,\vec{\mathcal{C}^{q+1}}} \left( \bigwedge_{i=1}^n \alpha_j^0(a_i) \right) + c^{L_{q+1}}_{0,\vec{\mathcal{C}^{q+1}}} \left( \bigwedge_{i=1}^n \alpha_j^1(a_i) \right), \end{aligned}$$

as these state descriptions of $L_{q+1}$ are the only ones contributing any weight,

$$= \mathcal{C}^{q+1}_k + \mathcal{C}^{q+1}_{k+1}.$$

We obtain that $\mathcal{C}_k^q = \mathcal{C}_k^{q+1} + \mathcal{C}_{k+1}^{q+1}$ must hold. For any state description $\Theta(a_1, \ldots, a_n) = \bigwedge_{i=1}^n \alpha_{h_i}(a_i)$ with not all the $h_i$ the same, we clearly have that $c_{0,\vec{\mathcal{C}}^q}^{L_q}(\Theta) = 0$ and the same holds for $c_{0,\vec{\mathcal{C}}^{q+1}}^{L_{q+1}}(\Theta)$. Thus it is enough to have $\mathcal{C}_k^r = \mathcal{C}_k^{r+1} + \mathcal{C}_{k+1}^{r+1}$ for each $r \in \mathbb{N}^+$, $k \in \{0, \ldots, r\}$. By Theorem 2.3, this holds just if $\mathcal{C}_k^r = \int_{[0,1]} x^k (1-x)^{r-k}\, d\rho(x)$.

To show the result for $\lambda > 0$, we first need to prove a technical result which is analogous to Proposition 16.1 in [22].

**Proposition 3.23.** *Let* $L = L_q$ *and* $\vec{\mathcal{C}} = \langle \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_q \rangle$ *be such that* $\sum_{i=0}^q \mathcal{C}_i = 1$. *Let* $\vartheta_1(x), \ldots, \vartheta_k(x)$ *be disjoint quantifier-free formulas of* $L$. *Then for* $0 < \lambda < \infty$,

$$c_{\lambda,\vec{\mathcal{C}}}^L\left(\vartheta_j(a_{n+1}) \mid \bigwedge_{i=1}^n \vartheta_{h_i}(a_i)\right) = \frac{n_j + \lambda \cdot \sum_{\alpha_r \in \Gamma_j} \mathcal{C}_{\gamma_q(r)}}{n+\lambda},$$

*and for* $n > 0$,

$$c_{0,\vec{\mathcal{C}}}^L\left(\bigwedge_{i=1}^n \vartheta_{h_i}(a_i)\right) = \begin{cases} \sum_{\alpha+r \in \Gamma_j} \mathcal{C}_{\gamma_q(j)} & \text{if } j = h_1 = h_2 = \cdots = h_n, \\ 0 & \text{otherwise,} \end{cases}$$

*where* $n_j = |\{i \mid h_i = j\}|$ *and* $\Gamma_j = \{\alpha \mid \vartheta_j(x) \models \alpha(x)\}$ *for* $j = 1, \ldots, n$.

**Proof:** For $\lambda = 0$, we obtain the result as above by a straightforward calculation. Assume $\lambda > 0$. For notational convenience, let $\psi(a_1, \ldots, a_n)$ vary over all state descriptions of the form

$$\alpha_{g_1}(a_1) \wedge \alpha_{g_2}(a_2) \wedge \cdots \wedge \alpha_{g_n}(a_n)$$

where $\alpha_{g_i}(x) \in \Gamma_{h_i}$ for $i = 1, \ldots, n$. We then obtain

$$\begin{aligned} c_{\lambda,\vec{\mathcal{C}}}^L\left(\vartheta_j(a_{n+1}) \mid \bigwedge_{i=1}^n \vartheta_{h_i}(a_i)\right) &= \sum_{\alpha_r \in \Gamma_j} c_{\lambda,\vec{\mathcal{C}}}^L\left(\alpha_r(a_{n+1}) \mid \bigwedge_{i=1}^n \bigvee_{\alpha_{g_i} \in \Gamma_{h_i}} \alpha_{g_i}(a_i)\right) \\ &= \sum_{\alpha_r \in \Gamma_j} c_{\lambda,\vec{\mathcal{C}}}^L\left(\alpha_r(a_{n+1}) \mid \bigvee_{\psi} \psi(a_1, \ldots, a_n)\right) \\ &= \sum_{\alpha_r \in \Gamma_j} \sum_{\psi} \frac{c_{\lambda,\vec{\mathcal{C}}}^L(\alpha_r(a_{n+1}) \mid \psi) \cdot c_{\lambda,\vec{\mathcal{C}}}^L(\psi)}{c_{\lambda,\vec{\mathcal{C}}}^L(\bigvee_\psi \psi)}. \end{aligned} \tag{3.44}$$

For any such state description $\psi$ we have

$$\begin{aligned} \sum_{\alpha_r \in \Gamma_j} c_{\lambda,\vec{\mathcal{C}}}^L(\alpha_r(a_{n+1}) \mid \psi) &= \sum_{\alpha_r \in \Gamma_j} \frac{s_r + \mathcal{C}_{\gamma_q(r)}\lambda}{n+\lambda} \\ &= \frac{n_j + \lambda \cdot \sum_{\alpha_r \in \Gamma_j} \mathcal{C}_{\gamma_q(r)}}{n+\lambda}, \end{aligned}$$

where $s_r$ is the number of times $\alpha_r$ occurs in $\psi$ and by definition of $\psi$ and $n_j$, the $s_r$ sum up to $n_j$. Now substituting this in (3.44), we obtain

$$\begin{aligned} c^L_{\lambda,\vec{\mathcal{C}}}\left(\vartheta_j(a_{n+1}) \mid \bigwedge_{i=1}^{n} \vartheta_{h_i}(a_i)\right) &= \sum_{\psi} \frac{n_j + \lambda \sum_{\alpha_r \in \Gamma_j} \mathcal{C}_{\gamma_q(r)}}{n+\lambda} \cdot \frac{c^L_{\lambda,\vec{\mathcal{C}}}(\psi)}{c^L_{\lambda,\vec{\mathcal{C}}}(\bigvee_\psi \psi)} \\ &= \frac{n_j + \lambda \sum_{\alpha_r \in \Gamma_j} \mathcal{C}_{\gamma_q(r)}}{n+\lambda}, \end{aligned}$$

as required. ⊣

We are now able to show the desired lemma.

**Lemma 3.24.** *Let $\lambda > 0$ and for each $q \in \mathbb{N}^+$ let $\vec{\mathcal{C}^q} = \langle \mathcal{C}^q_0, \mathcal{C}^q_1, \dots, \mathcal{C}^q_q \rangle$ be given such that $\sum_{i=0}^{q} \mathcal{C}^q_i = 1$. Then the family of probability functions $\{c^{L_q}_{\lambda,\vec{\mathcal{C}^q}} \mid q \in \mathbb{N}^+\}$ satisfies ULi just if there is a normalized measure $\rho$ on $[0,1]$ such that for each $q \in \mathbb{N}^+$, $j \in \{0, \dots, q\}$,*

$$\mathcal{C}^q_j = \int_{[0,1]} x^j (1-x)^{q-j}\, d\rho(x).$$

**Proof:** Let $\lambda > 0$ and consider $c^{L_{q+1}}_{\lambda,\vec{\mathcal{C}^{q+1}}}$. Let $\alpha_1(x), \alpha_2(x), \dots, \alpha_{2^q}(x)$ be the atoms of $L_q$, taken as quantifier free sentences of $L_{q+1}$. Then, using Proposition 3.23, we obtain

$$c^{L_{q+1}}_{\lambda,\vec{\mathcal{C}^{q+1}}}\left(\alpha_i(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) = \frac{n_j + \lambda \cdot (\mathcal{C}^{q+1}_k + \mathcal{C}^{q+1}_{k+1})}{n+\lambda},$$

where $\gamma_q(j) = k$, since then we have $\Gamma_j = \{\alpha^+_j, \alpha^-_j\}$ for $\Gamma_j$ as in Proposition 3.23, and $\gamma_{q+1}(s_1) = k, \gamma_{q+1}(s_2) = k+1$ for $\beta_{s_1}, \beta_{s_2}$ the atoms of $L_{q+1}$ such that $\beta_{s_1} = \alpha^+_j$, $\beta_{s_2} = \alpha^-_j$.

Then by ULi we need to have $\mathcal{C}^q_k = \mathcal{C}^{q+1}_k + \mathcal{C}^{q+1}_{k+1}$, and since $q$ was arbitrarily chosen, the lemma follows. ⊣

# Chapter 4

# Second Order Logic and Wilmers' Principle

In this chapter, we discuss extensions of probability functions to Second Order Logic. We will start with some motivation.

Just as in the initial examples, regard a probability function $w$ as the belief of some rational agent, making observations in the world and trying to infer which kind of world she is inhabiting. Statements about the world are commonly expressed using First Order Logic, as we have seen in the previous chapters. First Order Logic allows us not only to express simple statements such as *'this (specific) object has that (specific) property'*, but also statements of the form *'there exists some object with this property'*, omitting the name of a specific object, be it for the reason that we do not care which specific one it is that satisfies the property, or that we simply cannot tell.

In terms of the urn example Example 1.1, suppose again that our agent has just picked a red, a blue and two yellow balls from the urn, in that order. The agent would then be able to make statements such as *'the second ball is blue'* and *'there exists a red ball'*.

The agent might also want to express that a couple of balls in the urn have a colour in common, say that the first four balls are all the same colour. We would be able to express this using Second Order quantifiers:

$$\exists X \bigwedge_{i=1}^{4} X(a_i).$$

Of course, in a finite language we can easily find a First Order sentence that will express the same, given that the predicates in the language are all that will be observable. In our example of picking balls from an urn, this would clearly be the case if we knew beforehand that there is a fixed, finite number of colours that were used to colour the balls in the urn.

However, if we allowed for at least one binary relation in the language, there will be Second Order statements that cannot be expressed in First Order, a typical example being the Geach-Kaplan sentence

*"Some critics admire only one another."*

This can be formalized by the Second Order sentence

$$\exists X \; (\exists x \, \exists y \, (X(x) \wedge X(y) \wedge R(x,y)) \wedge \exists x \, \neg X(x) \wedge \forall x \, \forall y \, (X(x) \wedge R(x,y) \rightarrow X(y)))\,,$$

which can be shown not to be equivalent to any First Order sentence, see e.g. [1]. Using the same proof idea as in [1], interpreting $R$ as the set $\{(a_i, a_j) \,|\, |j-i| = 1\}$, we obtain that $X$ is a subset of the universe closed under predecessors and successors and has no minimal element. Such a set clearly cannot exist in standard structures, but there are such sets in non-standard structures. As we have used such structures in chapter 2, this example gives us justification to extend probability functions to Second Order Languages, providing rational agents with more expressive power.

## 4.1 A Second Order Framework

At this point we should discuss problems that arise when going from First Order to Second Order Logic. The main issue here is the inherent incompleteness of Second Order Logic that will cause problems trying to show the existence of probability functions on Second Order sentences.

If we adopt Full Second Order Logic, we will no longer have the Compactness Theorem and the Löwenheim-Skolem Theorem, so we would have to establish probability functions rather carefully in a suitable model of set theory. On the other hand, both the Compactness and Löwenheim-Skolem Theorems hold in Henkin models of Second Order Logic, see e.g. [24] [1].

The main justification for our approach – apart from the mentioned technical reasons – lies within the subject of Inductive Logic and how the non-logical symbols in the language are treated. The domain of discourse in the First Order case was always *countable*: the language contained countably many constant symbols, and so did the universe of each structure taken into consideration.

The interpretation for this, that each (First Order) object in the observable universe is represented by some constant (or name) in our language, can be applied in case of Second Order objects as well:

Each property (or subset of the universe) that exists in the observable universe should be represented by some name in our language.

Since the languages under consideration contain only countably many predicate symbols, the Second Order structures we want to consider need to have at most countably many subsets in the Second Order part of the universe, and we obtain a 1-1 correspondence between the Second Order part of the universe and the predicate symbols in the language. This is formalized in the following definitions.

**Definition 4.1:** *Define Second Order formulas and sentences by the same clauses as First Order formulas and sentences, with the additional clause*

- *If $\varphi$ is a Second Order formula and $X$ a Second Order variable, then $\exists X\, \varphi(X)$ is a Second Order formula.*

*For a purely unary language $L$, let $F_1L$ and $F_2L$ denote the sets of First Order and Second Order formulas of $L$, respectively. Let $S_1L$ and $S_2L$ denote the sets of First Order and Second Order sentences of $L$. Recall that $L_\infty$ is the language containing a unary predicate $P_i$ for each $i \in \mathbb{N}^+$.*

[1] What we refer to as "Full Second Order", Shapiro calls "Second Order with standard semantics".

Note that by this definition, we have $F_1 L \subseteq F_2 L$ and $S_1 L \subseteq S_2 L$ for any language $L$. Furthermore, as each $\varphi \in F_2 L_\infty$ is a string of finite length, we have $\varphi \in F_2 L_q$ for some $q \in \mathbb{N}$. Since our interpretation of the predicates in the structures is analogous to the interpretation of constants, we will work on structures for the language $L_\infty$.

Recalling Definition 1.4, let $\mathcal{T}_1 L_\infty = \mathcal{T} L_\infty$. We will extend this notion to Second Order structures.

**Definition 4.2 (Second Order structures and $\mathcal{T}_2 L_\infty$):** *A Second Order structure $\mathfrak{M}$ for $L_\infty$ is a structure*

$$\mathfrak{M} = \langle M, S_M, (P_i)_{i \in \mathbb{N}^+}, (a_i)_{i \in \mathbb{N}^+} \rangle$$

*such that $M = \{\mathfrak{a}_i \,|\, i \in \mathbb{N}^+\}$, $S_M = \{\mathfrak{P}_i \,|\, i \in \mathbb{N}^+\}$ and for all $i, j \in \mathbb{N}^+$,*

$$a_i^{\mathfrak{M}} = \mathfrak{a}_i,$$
$$P_j^{\mathfrak{M}} = \mathfrak{P}_j.$$

*Let $\mathcal{T}_2 L_\infty$ be the set of all Second Order structures for $L_\infty$.*

One should point out that the Second Order part of the universe may very well be finite, even if the First Order part is not. Take $\mathcal{M} \in \mathcal{T}_1 L_\infty$ to be a structure with countably infinite universe $M$ such that $\mathcal{M} \models \forall x \pm P_j(x)$ for each $j \in \mathbb{N}^+$, where $\pm P_j(x)$ is either $P_j(x)$ or $\neg P_j(x)$. Then we obtain a structure $\mathfrak{M} \in \mathcal{T}_2 L_\infty$ from $\mathcal{M}$ by letting $S_M = \{M, \emptyset\}$.

For simplicity, we henceforth identify $a_i$ with $\mathfrak{a}_i$, and $P_j$ with $\mathfrak{P}_j$. Given some $\mathfrak{M} \in \mathcal{T}_2 L_\infty$, we obtain a unique First Order structure $\mathcal{M}$ such that for all First Order $L_\infty$-sentences $\vartheta$

$$\mathcal{M} \models \vartheta \iff \mathfrak{M} \models \vartheta$$

by letting $\mathcal{M} = \langle M, (P_i)_{i \in \mathbb{N}^+}, (a_i)_{i \in \mathbb{N}^+} \rangle$, i.e. $\mathcal{M}$ is the restriction of $\mathfrak{M}$ to the First Order part of the structure. The following proposition shows that in fact we have a bijection between the set of Second Order structures $\mathcal{T}_2 L_\infty$ and the set of First Order structures $\mathcal{T}_1 L_\infty$.

As in the example above, there is an obvious way of obtaining Second Order structures from a First Order structure. The following proposition will show that with our definition of Second Order structures, these are in fact all the structures we obtain, and the Second Order universe of a structure is determined by its First Order theory.

**Proposition 4.3.** *For each First Order structure $\mathcal{M} \in \mathcal{T}_1 L_\infty$, there exists a unique $\mathfrak{M} \in \mathcal{T}_2 L_\infty$, the extension of $\mathcal{M}$ to the Second Order, such that for all $\vartheta \in S_1 L_\infty$,*

$$\mathcal{M} \models \vartheta \Longleftrightarrow \mathfrak{M} \models \vartheta.$$

**Proof:** To obtain $\mathfrak{M}$, it suffices to define the elements of the set $S_M$. We do this by defining the set $\mathfrak{P}_i \in S_M$ to be

$$\mathfrak{P}_i = \{x \in M \mid \mathcal{M} \models P_i(x)\}."$$

To show that this is unique, suppose $\mathfrak{M}$, $\mathfrak{M}'$ are two distinct extensions of $\mathcal{M}$. Then there must be $i \in \mathbb{N}^+$ such that

$$P_i^{\mathfrak{M}} \neq P_i^{\mathfrak{M}'},$$

i.e. there is $x \in M$ such that $\mathfrak{M} \models P_i(x)$ and $\mathfrak{M}' \models \neg P_i(x)$ (or vice-versa). Thus, the restrictions of $\mathfrak{M}$ and $\mathfrak{M}'$ to the First Order part cannot be the same First Order structure, contradicting the assumption. $\dashv$

**Remark 4.4 (Consistency for Second Order sentences).** As in the First Order case, we call a Second Order sentence $\varphi$ *consistent*, if there is a Second Order structure $\mathfrak{M}$ such that $\mathfrak{M} \models \varphi$. However, it is not immediately clear that such an $\mathfrak{M}$ will be in $\mathcal{T}_2 L_\infty$, as such an $\mathfrak{M}$ may not be countable.

In this case, we can certainly construct a Henkin structure in which $\varphi$ holds, and using the downward Löwenheim-Skolem Theorem, we may assume this is countable. We can then easily find such a structure in $\mathcal{T}_2 L_\infty$.

In the First Order case, we have for each (First Order) sentence $\vartheta$ the set

$$[\vartheta] = \{\mathcal{M} \in \mathcal{T}_1 L_\infty \mid \mathcal{M} \models \vartheta\}.$$

We will adjust this definition, extending it to the Second Order sentences.

**Definition 4.5:** *Let $\vartheta \in S_1L_\infty$, $\varphi \in S_2L_\infty$. We define $[\vartheta]_1$, $[\varphi]_2$ by*

$$[\vartheta]_1 = \{\mathcal{M} \in \mathcal{T}_1 L_\infty \,|\, \mathcal{M} \models \vartheta\},$$
$$[\varphi]_2 = \{\mathfrak{M} \in \mathcal{T}_2 L_\infty \,|\, \mathfrak{M} \models \varphi\}.$$

Note that $[\vartheta]_1 = [\vartheta]$ for $\vartheta \in S_1L_\infty$, and abusing notation by identifying $\mathcal{M} \in \mathcal{T}_1L_\infty$ with its unique extension $\mathfrak{M} \in \mathcal{T}_2L_\infty$, we have $[\vartheta]_1 = [\vartheta]_2$ for $\vartheta \in S_1L_\infty$. Now let $w$ be a probability function on $SL_\infty = S_1L_\infty$ satisfying (P1-3). Then there exists a measure $\mu_w$ on $\mathcal{T}_1L_\infty$ such that for all $\vartheta \in S_1L_\infty$,

$$w(\vartheta) = \mu_w([\vartheta]_1).$$

This framework essentially just extends the already established framework of First Order structures to allow handling of Second Order sentences and formulas, where 'extends' refers to an enrichment of the individual structures by a Second Order universe, rather than an extension of the set $\mathcal{T}_1L_\infty$. This approach allows us to easily transfer results we have established for First Order structures to Second Order structures in this context.

## 4.2 Extending First Order Probability Functions

The approach we will be taking here is to treat Second Order quantifiers in probability functions in a similar way as First Order quantifiers are treated, just as the definition of $\mathcal{T}_2L_\infty$ in the previous section suggests.

Recall that in our initial setup, we regarded the list of constants $a_i$ as exhaustive. This motivated the clause (P3) in the definition Definition 1.5 of probability functions: if there is an object satisfying the formula, then it must be one of the $a_i$'s.

As discussed in the previous section, the approach we take for Second Order quantifiers is analogous to this: Given a Second Order statement $\varphi = \exists X\, \psi(X)$, we regard this as stating '$\varphi$ holds just if there is a $P_i$ such that $\psi(P_i)$ holds'.

As a consequence of this treatment, Second Order probability functions will have to be defined on the language containing the predicate symbol $P_i$ for each $i \in \mathbb{N}$, i.e. the language $L_\infty$.

We can then, just as was the case for constants, treat the $P_i$ as exhausting the Second Order universe, and require of our probability functions that in addition to (P1-3), they also satisfy the condition

$$\text{(P4) } w(\exists X\, \psi(X)) = \lim_{n\to\infty} w(\bigvee_{i=1}^{n} \psi(P_i)).$$

We therefore arrive at the following definitions. Note that we will be concerned with purely unary languages throughout the chapter, and thus state definitions in terms of these, while they would work just as well for languages containing higher arities.

Recall that the Definition 1.5 made use of the logical consequence relation, $\models$. Since we want to give an enhanced definition for probability functions on Second Order sentences, we will need to enhance this relation as well.

For the remainder of the chapter, let $\models$ denote the logical consequence relation extended to Second Order sentences, that is for all $\varphi, \psi \in F_2L$, let

$$\varphi \models \psi$$

be shorthand for

for all Second Order structures $\mathfrak{M} \in \mathcal{T}_2 L_\infty$, if $\mathfrak{M} \models \varphi$, then $\mathfrak{M} \models \psi$.

**Definition 4.6 (Second Order Probability Functions):** *Let $L = L_\infty$. $w$ is a* probability function on $S_2L$*, if for any $\vartheta, \varphi \in S_2L$, $\psi(x), \eta(X) \in F_2L$,*

*(P1) If $\models \vartheta$, then $w(\vartheta) = 1$.*

*(P2) If $\vartheta \models \neg\varphi$, then $w(\vartheta \vee \varphi) = w(\vartheta) + w(\varphi)$.*

*(P3) $w(\exists x\, \psi(x)) = \lim_{n\to\infty} w\left(\bigvee_{i=1}^{n} \psi(a_i)\right)$.*

*(P4) $w(\exists X\, \eta(X)) = \lim_{n\to\infty} w\left(\bigvee_{i=1}^{n} \eta(P_i)\right)$.*

It is immediately clear from Proposition 1.17 that the only probability functions having a chance of possessing an extension to the Second Order are those satisfying Unary Language Invariance.

Before we will show that this is indeed the only condition needed to be able to extend First Order probability functions to Second Order probability functions, we shall provide some useful results.

**Lemma 4.7.** *Let $w$ be a probability function on $S_2L_\infty$. Then for $\vartheta, \varphi \in S_2L_\infty$, we have*

*(a)* $w(\neg\vartheta) = 1 - w(\vartheta)$.

*(b) If* $\models \neg\vartheta$, *then* $w(\vartheta) = 0$.

*(c) If* $\vartheta \models \varphi$, *then* $w(\vartheta) \leq w(\varphi)$.

*(d) If* $\vartheta \equiv \varphi$, *then* $w(\vartheta) = w(\varphi)$.

*(e)* $w(\vartheta \vee \varphi) = w(\vartheta) + w(\varphi) - w(\vartheta \wedge \varphi)$.

It is straightforward to see that the proof for the First Order version of this lemma, Lemma 1.7, can easily be seen to work for Second Order sentences, if the First Order versions of (P1), (P2) are replaced by their Second Order versions given in Definition 4.6.

**Lemma 4.8.** *Let $w$ be a probability function on $S_2L_\infty$. Let $\vartheta(x_1, \ldots, x_k, \vec{P}, \vec{a}), \psi(X_1, \ldots, X_k, \vec{P}, \vec{a}) \in F_2L_\infty$ be formulas with $x_1, \ldots, x_k$ and $X_1, \ldots, X_k$ the only free variables in $\vartheta$ and $\psi$, respectively. Then*

$$w(\exists x_1 \ldots \exists x_k \, \vartheta(x_1, \ldots, x_k, \vec{P}, \vec{a})) = \lim_{n\to\infty} w\left(\bigvee_{i_1,\ldots,i_k \leq n} \vartheta(a_{i_1}, \ldots, a_{i_k}, \vec{P}, \vec{a})\right) \tag{4.1}$$

$$w(\forall x_1 \ldots \forall x_k \, \vartheta(x_1, \ldots, x_k, \vec{P}, \vec{a})) = \lim_{n\to\infty} w\left(\bigwedge_{i_1,\ldots,i_k \leq n} \vartheta(a_{i_1}, \ldots, a_{i_k}, \vec{P}, \vec{a})\right) \tag{4.2}$$

$$w(\exists X_1 \ldots \exists X_k \, \psi(X_1, \ldots, X_k, \vec{P}, \vec{a})) = \lim_{n\to\infty} w\left( \bigvee_{i_1,\ldots,i_k \le n} \psi(P_{i_1}, \ldots, P_{i_k}, \vec{P}, \vec{a}) \right) \tag{4.3}$$

$$w(\forall X_1 \ldots \forall X_k \, \psi(X_1, \ldots, X_k, \vec{P}, \vec{a})) = \lim_{n\to\infty} w\left( \bigwedge_{i_1,\ldots,i_k \le n} \psi(P_{i_1}, \ldots, P_{i_k}, \vec{P}, \vec{a}) \right) \tag{4.4}$$

This lemma is essentially the Second Order version of Lemma 1.12. Since the proof of the latter one uses just the First Order version of (P3) and Lemma 1.7, we obtain a proof for the former by replacing the First Order notions by their Second Order counterparts, which will immediately give (4.1) and (4.2). For (4.3) and (4.4), we can just repeat the proof with (P4) in place of (P3).

**Theorem 4.9.** *Let $w$ be a probability function on $S_1L$ for some finite language $L$ satisfying Ex + Px + ULi. Then there exists a unique extension $\mathfrak{w}$ on $S_2L_\infty$ satisfying (P1-4) + Ex + Px such that $\mathfrak{w} \restriction SL = w$.*

**Proof:** Let $w$ be as in the statement of the theorem and let $w'$ be the probability function on $S_1L_\infty$ extending $w$ that exists by Proposition 1.17. By discussion above, there exists a measure $\mu$ on $\mathcal{T}_1L_\infty$ such that $w'(\vartheta) = \mu([\vartheta]_1)$ for each $\vartheta \in S_1L_\infty$. By Language Invariance, we have therefore that $w(\vartheta) = \mu([\vartheta]_1)$ for each $\vartheta \in S_1L$.

The measure $\mu$ lifts to a function $\mu_2$ on $\mathcal{T}_2L_\infty$: For $\vartheta \in S_1L_\infty$, let

$$\mu_2([\vartheta]_2) := \mu([\vartheta]_1).$$

Then $\mu_2$ satisfies (P1-3) for all $\vartheta \in S_1L_\infty$. Since by Proposition 1.17 the measure $\mu$ is unique, so is $\mu_2$.

Given a set $A \subseteq \mathcal{T}_2L_\infty$, let $A \restriction \mathcal{T}_1L_\infty$ denote the set

$$\{\mathcal{M} \in \mathcal{T}_1L_\infty \,|\, \mathcal{M} \text{ is the First Order restriction of } \mathfrak{M} \text{ for some } \mathfrak{M} \in A\}.$$

Let $\mathcal{B}$ be the $\sigma$-algebra on $\mathcal{T}_1L_\infty$ on which $\mu$ is defined on. Then $\mathcal{B}$ lifts to a $\sigma$-algebra $\mathcal{B}_2$ on $\mathcal{T}_2L_\infty$, given by

$$A \in \mathcal{B}_2 \iff A \restriction \mathcal{T}_1L_\infty \in \mathcal{B},$$

and $\mu_2$ is a normalized measure on this $\sigma$-algebra, defined by $\mu_2(A) = \mu_1(A \upharpoonright \mathcal{T}_2 L_\infty)$. It remains to check that $\mu_2$ satisfies (P1-4) for all $\varphi \in S_2 L_\infty$.

We will show by induction on the definition of "$\varphi$ is a Second Order sentence" that $[\varphi]_2 \in \mathcal{B}_2$ for each $\varphi \in S_2 L_\infty$:

Let $\varphi \in S_2 L_\infty$. If $\varphi$ is quantifier-free, then $\varphi$ is logically equivalent to a quantifier-free First Order sentence $\vartheta$. We then obtain $[\varphi]_2 = [\vartheta]_2$, and since $[\vartheta]_1 \in \mathcal{B}$, we obtain $[\varphi]_2 \in \mathcal{B}_2$.

Suppose $\varphi = \psi \vee \chi$, with $\psi$, $\chi \in S_2 L_\infty$ not necessarily quantifier-free. By the inductive hypothesis, $[\psi]_2$, $[\chi]_2 \in \mathcal{B}_2$, and since $[\varphi]_2 = [\psi]_2 \cup [\chi]_2$ and $\mathcal{B}_2$ is closed under unions, $[\varphi]_2 \in \mathcal{B}_2$.

Suppose $\varphi = \neg\psi$, then we have $[\varphi]_2 = \mathcal{T}_2 L_\infty \setminus [\psi]_2$. Since $[\psi]_2 \in \mathcal{B}_2$ by inductive hypothesis, and $\mathcal{B}_2$ is closed under complements, $[\varphi]_2 \in \mathcal{B}_2$.

Suppose that $\varphi = \exists X\, \psi(X)$ for $\psi(X) \in F_2 L_\infty$. Then we have

$$[\varphi]_2 = \bigcup_{n \in \mathbb{N}^+} [\psi(P_n)]_2.$$

By the inductive hypothesis, $\psi(P_n) \in S_2 L_\infty$ and $[\psi(P_n)]_2 \in \mathcal{B}_2$ for each $n \in \mathbb{N}^+$. As $\mathcal{B}_2$ is closed under countable unions, we have $[\varphi]_2 \in \mathcal{B}_2$. Similarly we obtain $[\varphi]_2 \in \mathcal{B}_2$ for $\varphi = \exists x\, \psi(x)$.

Since $\mu_2$ is a measure on $\mathcal{B}_2$, we can now define $\mathfrak{w}$ on $S_2 L_\infty$ by

$$\mathfrak{w}(\varphi) = \mu_2([\varphi]_2)$$

for all $\varphi \in S_2 L_\infty$. Then $\mathfrak{w}$ satisfies (P1-4) and $\mathfrak{w}(\vartheta) = \mu_2([\vartheta]_2) = \mu([\vartheta]_1) = w'(\vartheta)$ for all $\vartheta \in S_1 L_\infty$, so $\mathfrak{w} \upharpoonright S_1 L_\infty = w'$ and therefore $\mathfrak{w} \upharpoonright S_1 L = w$.

It remains to show that $\mathfrak{w}$ is unique. So suppose that $\mathfrak{v}$ is a second probability function on $S_2 L_\infty$ extending $w'$. By the Prenex Normal Form Theorem and Lemma 4.8, it is enough to show that $\mathfrak{w}$ and $\mathfrak{v}$ agree on quantifier-free sentences in $S_2 L_\infty$. But as these are logically equivalent to quantifier-free sentences in $S_1 L_\infty$ and both $\mathfrak{w}$ and $\mathfrak{v}$ extend $w'$, this holds. $\dashv$

**Corollary 4.10.** *The Principles of Constant and Predicate Exchangeability are consistent with probability functions on Second Order languages.*

**Proof:** Let $w$ be a (First Order) probability function satisfying Ex + Px + ULi. Let $\mathfrak{w}$ be its extension to Second Order languages. Then $\mathfrak{w}$ satisfies Ex + Px for $\varphi \in S_1 L_\infty$ as it extends $w$. So suppose $\varphi(P_1, \ldots, P_n, a_1, \ldots, a_m) \in S_2 L_\infty$ and let $\sigma, \tau : \mathbb{N}^+ \to \mathbb{N}^+$ be permutations of the predicates and constants in $L_\infty$, respectively. We need to show that

$$\mathfrak{w}(\varphi(P_{\sigma(1)}, \ldots, P_{\sigma(n)}, a_{\tau(1)}, \ldots, a_{\tau(m)})) = \mathfrak{w}(\varphi(P_1, \ldots, P_n, a_1, \ldots, a_m)).$$

We do this again by induction on the definition of Second Order sentences. Since by Theorem 4.9, we can view $\mathfrak{w}$ as defined via a measure $\mu_2$, it is enough to show that $\mu_2$ is invariant under permutations of predicates and constants. For readability, write $\varphi_{\sigma,\tau}$ for $\varphi(P_{\sigma(1)}, \ldots, P_{\sigma(n)}, a_{\tau(1)}, \ldots, a_{\tau(m)})$.

Since for $\varphi$ a quantifier-free sentence, $\varphi \equiv \vartheta$ for a quantifier-free First Order sentence $\vartheta$, we have $\varphi_{\sigma,\tau} \equiv \vartheta_{\sigma,\tau}$ and

$$\mu_2([\varphi_{\sigma,\tau}]_2) = \mu_2([\vartheta_{\sigma,\tau}]_2) = \mu_2([\vartheta]_2) = \mu_2([\varphi]_2),$$

as $\mu_2$ extends a First Order probability function satisfying Ex + Px.

Suppose that $\varphi = \vartheta \vee \psi$. Then $[\varphi]_2 = [\vartheta]_2 \cup [\psi]_2$. By the inductive hypothesis, we have $\mu_2([\vartheta]_2) = \mu_2([\vartheta_{\sigma,\tau}]_2)$ and $\mu_2([\psi]_2) = \mu_2([\psi_{\sigma,\tau}]_2)$. Furthermore, $\varphi_{\sigma,\tau} \equiv \vartheta_{\sigma,\tau} \vee \psi_{\sigma,\tau}$. This gives

$$\mu_2([\varphi_{\sigma,\tau}]_2) = \mu_2([\vartheta_{\sigma,\tau}]_2 \cup [\psi_{\sigma,\tau}]_2) = \mu_2([\vartheta]_2 \cup [\psi]_2) = \mu_2([\varphi]_2).$$

Suppose that $\varphi = \neg\psi$. Then $\varphi_{\sigma,\tau} \equiv \neg\psi_{\sigma,\tau}$, and by inductive hypothesis, $\mu_2([\psi_{\sigma,\tau}]_2) = \mu_2([\psi]_2)$. We obtain

$$\mu_2([\varphi_{\sigma,\tau}]_2) = \mu_2(\mathcal{T}_2 L_\infty \setminus [\psi_{\sigma,\tau}]_2) = \mu_2(\mathcal{T}_2 L_\infty \setminus [\psi]_2) = \mu_2([\varphi]_2).$$

Suppose that $\varphi = \exists X\, \psi(X)$. Then we have

$$[\varphi]_2 = \bigcup_{k \in \mathbb{N}^+} [\psi(P_k)]_2.$$

Now by the inductive hypothesis, we have that for each $k \in \mathbb{N}^+$, $\mu_2([\psi(P_k)]_2) = \mu_2([\psi(P_{\sigma(k)})_{\sigma,\tau}]_2)$, where the subscript $\sigma, \tau$ should be seen as being applied to $\psi(X)$ before the free variable $X$ is replaced by the constant $P_{\sigma(k)}$. As $\sigma$ is a permutation of $\mathbb{N}^+$, we have that there is a bijection between

$$\{[\psi(P_k)_{\sigma,\tau}]_2 \,|\, k \in \mathbb{N}^+\}$$

and

$$\{[\psi(P_{\sigma(k)})_{\sigma,\tau}]_2 \,|\, k \in \mathbb{N}^+\}.$$

Since taking the union of sets is a commutative operation, we therefore have

$$\mu_2\left(\bigcup_{k\in\mathbb{N}^+} [\psi(P_k)_{\sigma,\tau}]_2\right) = \mu_2\left(\bigcup_{k\in\mathbb{N}^+} [\psi(P_{\sigma(k)})_{\sigma,\tau}]_2\right).$$

We obtain

$$\begin{aligned} \mu_2([\varphi_{\sigma,\tau}]_2) &= \mu_2([\exists X\, \psi(X)_{\sigma,\tau}]_2) \\ &= \mu_2\left(\bigcup_{k\in\mathbb{N}^+} [\psi(P_k)_{\sigma,\tau}]_2\right) \\ &= \mu_2\left(\bigcup_{k\in\mathbb{N}^+} [\psi(P_{\sigma(k)})_{\sigma,\tau}]_2\right) \\ &= \mu_2\left(\bigcup_{k\in\mathbb{N}^+} [\psi(P_k)]_2\right), \end{aligned}$$

by the inductive hypothesis,

$$= \mu_2([\exists X\, \psi(X)]_2) = \mu_2([\varphi]_2).$$

We obtain $\mu_2([\exists x\, \psi(x)]_2) = \mu_2([\exists x\, \psi(x)_{\sigma,\tau}]_2)$ in a similar way, with $a_k$, $a_{\tau(k)}$ in place of $P_k$, $P_{\sigma(k)}$, respectively. $\dashv$

## 4.3 Consistency Results

The previous corollary showed that some of the principles can easily seen to be consistent with Second Order probability functions. For certain principles such as Regularity and Super Regularity, this proves to be not quite as simple, and in fact as the initial remarks in Chapter 26 of [22] shows, these even can cause some trouble in the First Order case.

In this section we will provide methods that can be used to show that the principles of Regularity and Super Regularity are consistent in the Second Order case as well. In fact, we shall use the same method as presented in [22] and apply it to the Second Order case.

**Proposition 4.11.** *There exists a probability function on $S_2L_\infty$ satisfying SReg.*

**Proof:** Fix an enumeration $(\vartheta_i)_{i\in\mathbb{N}^+}$ of the consistent First Order sentences of $L_\infty$. For each $i \in \mathbb{N}^+$, pick a Second Order structure $\mathfrak{M}_i \in \mathcal{T}_2L_\infty$ such that $\mathfrak{M}_i \models \vartheta_i$.

Then define the probability function $w$ by

$$w = \sum_{i\in\mathbb{N}^+} 2^{-i} V_{\mathfrak{M}_i},$$

where $V_{\mathfrak{M}_i}$ is the function with

$$V_{\mathfrak{M}_i}(\varphi) = \begin{cases} 1 & \text{if } \mathfrak{M}_i \models \varphi, \\ 0 & \text{if } \mathfrak{M}_i \not\models \varphi. \end{cases}$$

Then since $\sum_{i\in\mathbb{N}^+} 2^{-i} = 1$, one easily checks that $w$ is indeed a probability function satisfying (P1-4).

For each consistent First Order sentence $\vartheta$, there exists $i \in \mathbb{N}^+$ such that $\vartheta = \vartheta_i$, and thus we have

$$w(\vartheta) \geq 2^{-i} V_{\mathfrak{M}_i}(\vartheta) = 2^{-i} > 0,$$

and thus $w$ satisfies SReg. ⊣

**Lemma 4.12.** *There exists a probability function on $S_2L_\infty$ satisfying Ex + Px + SReg.*

**Proof:** We proceed just as in the analogous proof in the polyadic case, see Chapter 26 of [22]. We define a candidate $w$ as in the proof of the previous proposition, i.e.

$$w = \sum_{i\in\mathbb{N}^+} 2^{-i} V_{\mathfrak{M}_i}.$$

This function satisfies Super Regularity, but may fail to satisfy Ex or Px. The discussion in Chapter 26, [22] provided us with a method of obtaining some $v$ from $w$ that did satisfy Ex, the problem there being that there are countably many constants that needed to be taken care of.

The same problem applies in this case not only to Ex, but to Px as well. We will thus apply the method mentioned in [22] twice – once on constants to obtain Ex, and then on predicates to obtain Px.

Let $H$ be the set

$$H = \{f : \mathbb{N}^+ \to \mathbb{N}^+\},$$

and for distinct $i_1, \ldots, i_n \in \mathbb{N}^+$, $\sigma : \{i_1, \ldots, i_n\} \to \mathbb{N}^+$, let

$$[\sigma] = \{f \in H \mid f \restriction \{i_1, \ldots, i_n\} = \sigma\}.$$

Let $S$ be the collection of all such $[\sigma]$. Let $\mu_0$ be the measure on $\mathbb{N}^+$ given by $\mu_0\{n\} = 2^{-n}$. Define the measure $\mu$ on $S$ by

$$\mu([\sigma]) = \prod_{s=1}^{n} 2^{-\sigma(i_s)}$$

for $\sigma : \{i_1, \ldots, i_n\} \to \mathbb{N}^+$, i.e. the usual product measure extension of $\mu_0$ to $H$. Note that this extends to a countably additive normalized measure on the Borel sets generated by the $[\sigma]$.

We can now define the function $w'$ on $S_2L_\infty$ by

$$w'(\vartheta(a_{i_1}, \ldots, a_{i_n})) = \sum_{\sigma:\{i_1,\ldots,i_n\}\to\mathbb{N}^+} \mu([\sigma]) \cdot w(\vartheta(a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)})).$$

Then $w'$ satisfies (P1-4) + Ex. The proof for this, except for (P4), has been given in [22]; for the sake of providing a complete proof, we will essentially copy the given arguments, replacing $SL_\infty$ by $S_2L_\infty$ to adjust to our situation.

Note that if a sentence $\vartheta(a_{i_1}, \ldots, a_{i_n}) \in S_2L_\infty$ is a tautology, then so is $\vartheta(a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)})$. Thus we have that (P1) holds for $w'$.

It is now straightforward to obtain (P2) for $w'$. Suppose that

$$\vartheta(a_{i_1}, \ldots, a_{i_n}) \models \neg\varphi(a_{i_1}, \ldots, a_{i_n}).$$

Then we have

$$\vartheta(a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)}) \models \varphi(a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)})$$

for any $\sigma : \{i_1, \ldots, i_n\} \to \mathbb{N}^+$ and we obtain

$$\begin{aligned} &w'(\vartheta(a_{i_1}, \ldots, a_{i_n}) \vee \varphi(a_{i_1}, \ldots, a_{i_n})) \\ &\quad = \sum_{\sigma:\{i_1,\ldots,i_n\}\to\mathbb{N}^+} \mu([\sigma]) \cdot w(\vartheta(a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)}) \vee \varphi(a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)})) \\ &\quad = \sum_{\sigma:\{i_1,\ldots,i_n\}\to\mathbb{N}^+} \mu([\sigma]) \cdot \big(w(\vartheta(a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)})) + w(\varphi(a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)}))\big), \end{aligned}$$

since this holds for $w$,

$$\begin{aligned} &\quad = \sum_{\sigma:\{i_1,\ldots,i_n\}\to\mathbb{N}^+} \mu([\sigma]) \cdot w(\vartheta(a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)})) \\ &\qquad + \sum_{\sigma:\{i_1,\ldots,i_n\}\to\mathbb{N}^+} \mu([\sigma]) \cdot w(\varphi(a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)})) \\ &\quad = w'(\vartheta(a_{i_1}, \ldots, a_{i_n})) + w'(\varphi(a_{i_1}, \ldots, a_{i_n})). \end{aligned}$$

For (P3), let $\exists x\, \psi(x, a_{i_1}, \ldots, a_{i_n}) \in S_2 L_\infty$ and let $\varepsilon > 0$. Pick $m$ such that

$$\sum_{\sigma:\{i_1,\ldots,i_n\}\to\{1,\ldots,m\}} \mu([\sigma]) > 1 - \varepsilon.$$

Then

$$\sum_{\sigma:\{i_1,\ldots,i_n\}\to\{1,\ldots,m\}} \mu([\sigma]) \cdot w(\exists x\, \psi(x, a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)}))$$

is within $\varepsilon$ of $w'(\exists x\, \psi(x, a_{i_1}, \ldots, a_{i_n}))$. Pick $q$ such that for each of the $m^n$ maps $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$, we have

$$w\left(\bigvee_{j=1}^{q} \psi(a_j, a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)})\right) + \varepsilon m^{-n} \geq w(\exists x\, \psi(x, a_{\sigma(i_1)}, \ldots, a_{\sigma(i_n)}))$$

and then pick $r \geq q$, $i_1, \ldots, i_n$ such that for each $\sigma$,

$$\sum_{\tau \in A_\sigma} \mu([\tau]) \geq \mu([\sigma]) - \varepsilon m^{-n},$$

where $A_\sigma$ is the set

$$\{\tau : \{1, \ldots, r\} \to \mathbb{N}^+ \mid \tau \restriction \{i_1, \ldots, i_n\} = \sigma \text{ and } \{1, \ldots, q\} \subseteq \{\tau(1), \ldots, \tau(r)\}\}.$$

Then we obtain

$$
\begin{aligned}
&w'\left(\bigvee_{j=1}^{r}\psi(a_j,a_{i_1},\ldots,a_{i_n})\right)\\
&\quad=\sum_{\tau:\{1,\ldots,r\}\to\mathbb{N}^+}\mu([\tau])w\left(\bigvee_{j=1}^{r}\psi(a_{\tau(j)},a_{\tau(i_1)},\ldots,a_{\tau(i_n)})\right)\\
&\quad\geq\sum_{\sigma:\{i_1,\ldots,i_n\}\to\{1,\ldots,m\}}\sum_{\tau\in A_\sigma}\mu([\tau])w\left(\bigvee_{j=1}^{r}\psi(a_{\tau(j)},a_{\tau(i_1)},\ldots,a_{\tau(i_n)})\right)\\
&\quad\geq\sum_{\sigma:\{i_1,\ldots,i_n\}\to\{1,\ldots,m\}}\sum_{\tau\in A_\sigma}\mu([\tau])\left(w(\exists x\,\psi(x,a_{\tau(i_1)},\ldots,a_{\tau(i_n)}))-\varepsilon m^{-n}\right)\\
&\quad\geq\sum_{\sigma:\{i_1,\ldots,i_n\}\to\{1,\ldots,m\}}\left(\mu([\sigma])-\varepsilon m^{-n}\right)\left(w(\exists x\,\psi(x,a_{\sigma(i_1)},\ldots,a_{\sigma(i_n)}))-\varepsilon m^{-n}\right)\\
&\quad\geq\sum_{\sigma:\{i_1,\ldots,i_n\}\to\{1,\ldots,m\}}\mu([\sigma])w(\exists x\,\psi(x,a_{\sigma(i_1)},\ldots,a_{\sigma(i_n)}))-2\varepsilon\\
&\quad\geq\sum_{\sigma:\{i_1,\ldots,i_n\}\to\mathbb{N}^+}\mu([\sigma])w(\exists x\,\psi(x,a_{\sigma(i_1)},\ldots,a_{\sigma(i_n)}))-3\varepsilon\\
&\quad\geq w'(\exists x\,\psi(x,a_{i_1},\ldots,a_{i_n}))-3\varepsilon.
\end{aligned}
$$

Note that since we already have

$$w'(\exists x\,\psi(x,a_{i_1},\ldots,a_{i_n}))\geq w'\left(\bigvee_{j=1}^{r}\psi(a_j,a_{i_1},\ldots,a_{i_n})\right),$$

by Lemma 4.7, (c) and

$$\bigvee_{j=1}^{r}\psi(a_j,a_{i_1},\ldots,a_{i_n})\models\exists x\,\psi(x,a_{i_1},\ldots,a_{i_n},$$

(P3) holds for $w'$.

Using this same argument, we can show by a similar calculation that (P4) holds for $w'$ as well.

Finally, we show that Ex holds for $w'$: Let $f:\{1,\ldots,n\}\to\mathbb{N}^+$ be injective. Then we have

$$\begin{aligned} & w'(\vartheta(a_1,\dots,a_n)) \\ & \quad = \sum_{\sigma:\{1,\dots,n\}\to\mathbb{N}^+} \mu([\sigma]) w(\vartheta(a_{\sigma(1)},\dots,a_{\sigma(n)})) \\ & \quad = \sum_{\sigma f^{-1} f:\{1,\dots,n\}\to\mathbb{N}^+} \mu([\sigma f^{-1} f]) w(\vartheta(a_{\sigma f^{-1} f(1)},\dots,a_{\sigma f^{-1} f(n)})) \\ & \quad = \sum_{\sigma f^{-1}:\{f(1),\dots,f(n)\}\to\mathbb{N}^+} \mu([\sigma f^{-1}]) w(\vartheta(a_{\sigma f^{-1}(f(1))},\dots,a_{\sigma f^{-1}(f(n))})), \end{aligned}$$

since $\mu([\sigma f^{-1} f]) = \mu([\sigma f^{-1}])$,

$$= w'(\vartheta(a_{f(1)},\dots,a_{f(n)})).$$

We will now repeat this process to obtain a probability function satisfying Px. Since $\sigma$ is a function on the index set $\mathbb{N}^+$ that we use for both predicates and constants, we can use the same measure $\mu$ again, defining

$$v(\vartheta(P_{j_1},\dots,P_{j_m})) = \sum_{\tau:\{j_1,\dots,j_m\}\to\mathbb{N}^+} \mu([\tau]) \cdot w'(\vartheta(P_{\tau(j_1)},\dots,P_{\tau(j_m)})).$$

We need to show that $v$ inherits the properties (P1-4) + Ex from $w'$ and in turn now satisfies Px. Clearly, since we are defining $v$ in the same way as $w'$ was defined, with the functions $\tau$ now being applied to predicate indices instead of constant indices, we can use just the same arguments for (P1-4) as above, with the roles of predicates and constants reversed.

To see that Ex is preserved, notice that the functions $\tau$ used to obtain $v$ from $w'$ do not affect constants at all. Since we have showed above that $w'$ satisfies Ex, $v$ inherits this property from $w'$.

We now obtain that $v$ satisfies Px by the same method used to obtain Ex for $w'$.

The original $w$ was constructed to satisfy Super Regularity. Taking $\sigma$, and in turn $\tau$, to be the identity, we see that this is preserved by this construction, and thus both $w'$ and $v$ satisfy SReg. The function $v$ then is the desired probability function satisfying (P1-4) + Ex + Px + SReg. $\dashv$

**Remark 4.13.** One can easily extend this proof to show that there is a probability function $w$ satisfying Ex + Px such that

$$w(\psi) > 0$$

for all consistent Second Order sentences $\psi$, with the notion of consistency as in Remark 4.4, suggesting a Second Order version of Super Regularity as a possible rational principle. We shall see in the next section that such a principle would conflict with Wilmers' Principle as the latter forces certain Second Order sentences to be given probability 0.

## 4.4 Wilmers' Principle

So far we have established some initial technical results, allowing us to work with second order expressions in the framework of Pure Inductive Logic. In this section we suggest and discuss a rational principle for Second Order expressions that rational agents may want to accept as defining their beliefs.

The motivation for the principle is the following idea: Suppose we have a First Order formula $\vartheta(x)$ with just one free variable. Then $\vartheta(x)$ *defines a subset* of the universe, namely

$$\{a_i \,|\, \vartheta(a_i)\}.$$

A rational agent then might feel that there should be a *name* for this set in the language, and thus $\vartheta(x)$ defines not only a subset of the universe, but also a predicate of the language, i.e.

$$\{a_i \,|\, \vartheta(a_i)\} = \{a_i \,|\, P(a_i)\}$$

for some (unary) $P$ in the agent's language.

The formal definition of this principle in terms of probability functions is given by:

**Wilmers' Principle, WP**

*Let $w$ be a probability function on $S_2L_\infty$. Then $w$ satisfies* Wilmers' Principle, *if*

$$w(\exists X\, \forall x\, (\vartheta(x) \longleftrightarrow X(x))) = 1$$

*whenever $\vartheta(x) \in F_1 L_\infty$.*

**Remark 4.14.** This principle was suggested by George Wilmers, who originally intended it to be phrased in the context of Full Second Order Logic. For reasons given in the beginning of this chapter, we have adjusted it for our approach to Second Order probability functions.

We first show that Wilmers' Principle is consistent.

**Lemma 4.15.** *There exists a probability function $w$ on $S_2L_\infty$ satisfying Ex + Px + Wilmers' Principle.*

**Proof:** If we can find $\mathfrak{M} \in \mathcal{T}_2 L_\infty$ such that for each $\vartheta(x) \in F_1 L_\infty$, there is a $P_k$ with

$$\{a_i \,|\, \mathfrak{M} \models P_k(a_i)\} = \{a_i \,|\, \mathfrak{M} \models \vartheta(a_i)\},$$

then $V_\mathfrak{M}$ will satisfy Wilmers' Principle.

We could obtain such a structure if we added a new predicate $P_\vartheta$ with

$$P_\vartheta^\mathfrak{M} = \{a_i \,|\, \mathfrak{M} \models \vartheta(a_i)\}$$

for each $\vartheta \in F_1 L_\infty$. However, we face the problem that in $\mathfrak{M}$ there already exists an interpretation for each $P_i \in L_\infty$.

In order to avoid this problem, consider $M \in F_1 L_q$ for some $q \in \mathbb{N}^+$. Fix an enumeration $\vartheta_i(x), i \in \mathbb{N}^+$ of the First Order formulas in $L_q$ with one free variable. Now extend $M$ to a structure $M' \in F_1 L_\infty$ by letting

$$M' \models P_i(a_j) \Longleftrightarrow \begin{cases} M \models P_i(a_j) & \text{if } i \leq q, \\ M \models \vartheta_k(a_j) & \text{if } i = q + k \text{ for some } k \geq 1 \end{cases}$$

for each $j \in \mathbb{N}^+$. We can easily extend $M'$ to $M^* \in \mathcal{T}_2 L_\infty$. Let $\psi(x) \in F_1 L_\infty$ and consider the set

$$\{a_j \mid M^* \models \psi(a_j)\}. \tag{4.5}$$

Since we already have an interpretation for each $P_i \in L_\infty$, we need to ensure that our construction of $M^*$ already has a $P_k$ that interprets the set (4.5). Let $\psi'(x)$ be the result of replacing each occurrence of $P_k$ for $k \geq q$ in $\psi(x)$ with $\vartheta$, where $\vartheta(x)$ is the first order formula of $L_q$ such that

$$M' \models P_k(a_j) \Longleftrightarrow M \models \vartheta(a_j)$$

for each $j \in \mathbb{N}^+$. Then $\psi'(x) \in F_1 L_q$, and we have that $\psi'(x) = \vartheta_s(x)$ for some $s \in \mathbb{N}^+$. Then

$$\{a_j \mid M^* \models \psi(a_j)\} = \{a_j \mid M \models \vartheta_s(a_j)\} = P_s^{M^*}$$

and the function $V_{M^*}$ will satisfy Wilmers' Principle, as required. $\dashv$

**Proposition 4.16.** *Wilmers' Principle is consistent with Super Regularity.*

**Proof:** Let $(\vartheta_i)_{i \in \mathbb{N}^+}$ enumerate the consistent First Order sentences of $L_\infty$. We have in fact $\vartheta_i \in S_1 L_{q_i}$ for some $q_i \in \mathbb{N}^+$. By consistency of $\vartheta_i$, we can find a structure $M_i \in \mathcal{T}_1 L_{q_i}$. As in the proof of Lemma 4.15, extend $M_i$ to $M_i^* \in \mathcal{T}_2 L_\infty$. We obtain a collection of functions $V_{M_i^*}$ satisfying Wilmers' Principle.

Then $w := \sum_{i \in \mathbb{N}^+} 2^{-i} \cdot V_{M_i^*}$ is the desired probability function satisfying Wilmers' Principle and SReg, as for each consistent sentence $\vartheta_i$, we have $V_{M_i^*}(\vartheta_i) = 1$, giving $w(\vartheta_i) > 0$. $\dashv$

**Remark 4.17.** Note that since Wilmers' Principle forces certain Second Order sentences to get value 1, their negations must get value 0, and thus a Principle of Super Regularity for Second Order sentences would certainly not be satisfied in the presence of Wilmers' Principle.

## 4.5 Regularity, Super Regularity and a Ladder Theorem

In the previous section we have seen that Wilmers' Principle is consistent with Super Regularity and hence Regularity. However, as we have remarked earlier, it does not imply Regularity. In fact, with Carnap's $c_0$, we have an 'old friend' as an example for this.

**Proposition 4.18.** *The extension of $c_0$ to $S_2L_\infty$ satisfies Wilmers' Principle, and hence there exists a probability function satisfying WP + ¬ Reg.*

**Proof:** Slightly abusing notation, let $c_0$ denote the extension of the First Order function $c_0^{L_\infty}$ to the Second Order. Note that since $c_0$ satisfies Unary Language Invariance, both $c_0^{L_\infty}$ and its extension exist. We shall show that for any First Order formula $\vartheta(x)$, we have that

$$c_0(\exists X\, \forall x\, (\vartheta(x) \longleftrightarrow X(x))) = 1$$

holds.

By definition of $c_0$, we have that each constant is considered to look the same by this function, and once we have witnessed one constant and its properties, we know these for all other constants as well. In particular, for any $\vartheta(x) \in F_1L_\infty$, we have that

$$c_0(\forall x\, \vartheta(x) \vee \forall x\, \neg\vartheta(x)) = 1.$$

Notice that WP fails for $c_0$ just in case there is a constant $a_k$ such that

$$c_0\left(\forall X\, (X(a_k) \wedge \neg\vartheta(a_k)) \vee \forall X\, (\neg X(a_k) \wedge \vartheta(a_k))\right) > 0.$$

By Lemma 4.7, (e), we have

$$\begin{aligned} c_0\,(\forall X\ \ &(X(a_k) \wedge \neg\vartheta(a_k)) \vee \forall X\, (\neg X(a_k) \wedge \vartheta(a_k))) \\ &= c_0\,(\forall X\, (X(a_k) \wedge \neg\vartheta(a_k))) + c_0\,(\forall X\, (\neg X(a_k) \wedge \vartheta(a_k))) \\ &\qquad - c_0\,(\forall X\, (X(a_k) \wedge \neg\vartheta(a_k)) \wedge \forall X\, (\neg X(a_k) \wedge \vartheta(a_k))) \\ &\leq c_0\,(\forall X\, X(a_k) \wedge \neg\vartheta(a_k)) + c_0\,(\forall X\, \neg X(a_k) \wedge \vartheta(a_k)) \\ &\leq c_0\,(\forall X\, (X(a_k))) + c_0\,(\forall X\, (\neg X(a_k))\,, \end{aligned}$$

as $\forall X\, X(a_k) \wedge \neg\vartheta(a_k) \models \forall X\, X(a_k)$ and $\forall X\, \neg X(a_k) \wedge \vartheta(a_k) \models \forall X\, \neg X(a_k)$,

$$= \lim_{q\to\infty} c_0\left(\bigwedge_{i=1}^{q} P_i(a_k)\right) + \lim_{q\to\infty} c_0\left(\bigwedge_{i=1}^{q} \neg P_i(a_k)\right)$$
$$= \lim_{q\to\infty} \left(c_0(\alpha_1(a_k)) + c_0(\alpha_{2^q}(a_q))\right),$$

with $\alpha_1$, $\alpha_{2^q}$ the atoms of $L_q$, indexed by the usual lexicographic ordering,

$$= \lim_{q\to\infty} \left(\frac{1}{2^q} + \frac{1}{2^q}\right) = 0.$$

As $\vartheta(x)$ was arbitrary, $c_0$ satisfies Wilmers' Principle. $\dashv$

We will now give a characterization of those probability functions that satisfy Wilmers' Principle, but fail to satisfy Regularity. The example of $c_0$ above can be used to demonstrate the general idea behind this:

The function $c_0$ will give probability 0 to any sentence instantiating at least two distinct constants. We can therefore view $c_0$ as describing a structure with precisely one element: all constants in the language are interpreted in this structure by the same object in the universe. The probabilistic nature of $c_0$ amounts here to the uncertainty of which properties this object has exactly.

One can extend this to structures with $n$ elements, for arbitrary $n \in \mathbb{N}^+$: Suppose $M$ is a structure with universe $\{e_0, \dots, e_{n-1}\}$. Then each constant $a_i$ in our language will be interpreted by one of the $e_j$. Then in this structure, any sentence instantiating $n+1$ distinct constants will be false. Just using structures with $n$ elements and the corresponding functions $V_M$, we may construct a probability function that gives such sentences probability 0. By the methods used earlier in our proof that Wilmers' Principle is consistent with our notion of probability functions, we can construct such finite structures $M$ such that $V_M$ will satisfy Wilmers' Principle.

The aim is now to construct measures for each finite size of the universe, so that we obtain a Representation of all functions satisfying Wilmers' Principle via a convex combination of these measures.

**Definition 4.19:** *For each $n \in \mathbb{N}^+$, define $(2^n)^\infty \times n^\infty$ and structures $M_{f,g}$ as follows:*

- *Let $(2^n)^\infty$ denote the set of functions $f : \mathbb{N}^+ \to \{0, 1, \ldots, 2^n - 1\}$ and let $n^\infty$ denote the set of function $g : \mathbb{N}^+ \to \{0, 1, \ldots, n-1\}$. Let $(2^n)^\infty \times n^\infty$ denote the set of all pairs $\langle f, g \rangle$ with $f \in (2^n)^\infty$ and $g \in n^\infty$.*
- *For each pair $\langle f, g \rangle \in (2^n)^\infty \times n^\infty$, define the structure $M_{f,g}$ with universe $\{e_0, e_1, \ldots, e_{n-1}\}$ by $a_i^{M_{f,g}} = e_{g(i)}$ for each $i \in \mathbb{N}^+$ and*

$$M_{f,g} \models P_i(a_j) \iff f(i)(g(j)) = 1.$$

**Remark 4.20.** Here, we regard the elements of $2^n$ as functions $h : \{0, \ldots, n-1\} \to \{0, 1\}$. Thus the function $f$ assigns to each index for a predicate in $L_\infty$ an *evaluation function*, evaluating the predicate at the elements $e_0, \ldots, e_{n-1}$ of the universe of $M_{f,g}$, with $g$ evaluating the constants $a_i$, $i \in \mathbb{N}^+$ in $M_{f,g}$.

Analogous to the canonical ordering of the atoms of (finite) language $L_n$ we can enumerate those functions as $h_1, \ldots, h_{2^n}$, with $h_1$ the constant function with $h_1(i) = 1$ for $i \in \{0, \ldots, n-1\}$, $h_{2^n}$ the constant 0-valued function, and $h_i$ the function with $h_i(j) = 1$ iff $\alpha_i(x) \models P_{j-1}(x)$, where $\alpha_i$ is an atom of $L_n$.

**Definition 4.21:** *Let $n \in \mathbb{N}^+$ and let $\mu_n$ be a normalized, $\sigma$-additive measure on $(2^n)^\infty \times n^\infty$. We say that*

- *$\mu_n$ is* invariant under Ex *if for any $\vartheta(a_{i_1}, \ldots, a_{i_n}) \in S_2L_\infty$ and any $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$,*

$$\mu_n(\{\langle f, g\rangle \mid M_{f,g} \models \vartheta(a_{i_1}, \ldots, a_{i_n})\}) = \mu_n(\{\langle f, g \circ \sigma\rangle \mid M_{f,g} \models \vartheta(a_{i_1}, \ldots, a_{i_n})\}),$$

- *$\mu_n$ is* invariant under Px *if for any $\vartheta \in S_2L_\infty$ and any $\tau : \mathbb{N}^+ \to \mathbb{N}^+$,*

$$\mu_n(\{\langle f, g\rangle \mid M_{f,g} \models \vartheta(P_{j_1}, \ldots, P_{j_m})\}) = \mu_n(\{\langle f \circ \tau, g\rangle \mid M_{f,g} \models \vartheta(P_{j_1}, \ldots, P_{j_m})\}).$$

From the above definition, it is straightforward to see that a probability function on $S_2L_\infty$ defined via a measure $\mu_n$ invariant under Ex and Px will satisfy Constant and Predicate Exchangeability. With the methods used in Lemma 4.12, we can close any probability function on $S_2L_\infty$ under Ex and Px, and the resulting measure on $(2^n)^\infty \times n^\infty$ would then satisfy the invariance properties in the definition, so we might as well require this property to hold for the measures $\mu_n$.

**Lemma 4.22.** *Let $\mu_n$ be a normalized, $\sigma$-additive measure on $(2^n)^\infty \times n^\infty$ invariant under Ex and Px and suppose that*

$$\mu_n(\{\langle f,g\rangle \mid \operatorname{ran}(f) \neq 2^n \text{ or } \operatorname{ran}(g) \neq n\}) = 0.$$

*Let $w_{\mu_n}$ be the function on $S_2L_\infty$ defined by $w_{\mu_n}(\vartheta) = \mu_n(\{\langle f,g\rangle \mid M_{f,g} \models \vartheta\})$ for $\vartheta \in S_2L_\infty$. Then*

*(i) $w_{\mu_n}$ is a probability function on $S_2L_\infty$ satisfying Px + Ex.*

*(ii) $w_{\mu_n}$ satisfies Wilmers' Principle and*

$$w_{\mu_n}\left(\exists x_1 \ldots \exists x_n \left[\bigwedge_{1\le i<j\le n} \exists X\, \neg(X(x_i) \longleftrightarrow X(x_j)) \wedge \forall z \bigvee_{i=1}^{n} \forall X\, (X(x_i) \longleftrightarrow X(z))\right]\right) = 1.$$

*(iii) Let $v$ be a probability function on $S_2L_\infty$ satisfying Px + Ex + Wilmers' Principle and suppose that*

$$v\left(\exists x_1 \ldots \exists x_n \left[\bigwedge_{1\le i<j\le n} \exists X\, \neg(X(x_i) \longleftrightarrow X(x_j)) \wedge \forall z \bigvee_{i=1}^{n} \forall X\, (X(x_i) \longleftrightarrow X(z))\right]\right) = 1.$$

*Then there exists a normalized, $\sigma$-additive measure $\mu_n$ on $(2^n)^\infty \times n^\infty$ invariant under Ex + Px with $\mu_n(\{\langle f,g\rangle \mid \operatorname{ran}(f) \neq 2^n$ or $\operatorname{ran}(g) \neq n\}) = 0$ such that $v = w_{\mu_n}$.*

**Proof:** For notational convenience, define the *support set* of a sentence $\vartheta$ by

$$S(\vartheta) = \{\langle f,g\rangle \mid M_{f,g} \models \vartheta\} \subseteq (2^n)^\infty \times n^\infty.$$

For (i), let $\mu_n$ and $w_{\mu_n}$ be as defined in the statement of the lemma.

(P1) clearly holds for $w_{\mu_n}$: For any tautology $\vartheta$, $\{\langle f,g\rangle \mid M_{f,g} \models \vartheta\} = (2^n)^\infty \times n^\infty$. Since $\mu_n$ is a normalized measure, it follows that $w_{\mu_n}(\vartheta) = 1$.

For (P2), let $\vartheta, \varphi \in S_2L_\infty$ such that $\vartheta \models \neg\varphi$. We clearly have $S(\vartheta \vee \varphi) = S(\vartheta) \cup S(\varphi)$. We want to show that this is a disjoint union. For this, suppose $\langle f,g\rangle \in S(\vartheta)$. Then

$M_{f,g} \models \vartheta$ and since $\vartheta \models \neg\varphi$, we have $M_{f,g} \models \neg\varphi$. But then $\langle f, g\rangle \notin S(\varphi)$, since otherwise we would have $M_{f,g} \models \varphi \wedge \neg\varphi$. Similarly, if $\langle f, g\rangle \in S(\varphi)$, we must have $\langle f, g\rangle \notin S(\vartheta)$. We obtain

$$\begin{aligned} w_{\mu_n}(\vartheta \vee \varphi) &= \mu_n(S(\vartheta \vee \varphi)) = \mu_n(S(\vartheta) \cup S(\varphi)) \\ &= \mu_n(S(\vartheta)) + \mu_n(S(\varphi)) \text{ by } (\sigma\text{-})\text{additivity of } \mu_n, \\ &= w_{\mu_n}(\vartheta) + w_{\mu_n}(\varphi), \end{aligned}$$

as required.

Let $\vartheta(x) \in F_2L_\infty$ with $x$ the only free variable. Then

$$\begin{aligned} w_{\mu_n}(\exists x\, \vartheta(x)) &= \mu_n(S(\exists x\, \vartheta(x))) \\ &= \mu_n\left(\bigcup_{i=1}^{\infty} S\left(\bigvee_{k=1}^{i} \vartheta(a_k)\right)\right) \\ &= \lim_{i\to\infty} \mu_n\left(S\left(\bigvee_{k=1}^{i} \vartheta(a_k)\right)\right) \text{ by } \sigma\text{-additivity of } \mu_n, \\ &= \lim_{i\to\infty} w_{\mu_n}\left(\bigvee_{k=1}^{i} \vartheta(a_k)\right), \end{aligned}$$

and thus (P3) holds for $w_{\mu_n}$. (P4) now follows using an analogous argument for a Second Order sentence $\exists X\, \psi(X)$.

Since $\mu_n$ is invariant under Ex and Px, it follows that $w_{\mu_n}$ satisfies Ex + Px.

For (ii), we will first show that $w_{\mu_n}$ satisfies Wilmers' Principle. Let $\vartheta(x)$ be a consistent First Order formula of $L_\infty$ with $x$ the only free variable. We obtain a partition of $(2^n)^\infty \times n^\infty$ along $\vartheta$ by:

$$(2^n)^\infty \times n^\infty = \bigcup_{\varepsilon_0,\dots,\varepsilon_{n-1}\in\{0,1\}} \bigcap_{i=0}^{n-1} \{\langle f, g\rangle \mid M_{f,g} \models \vartheta^{\varepsilon_i}(e_i)\},$$

where as usual we let

$$\vartheta^{\varepsilon_i}(e_i) = \begin{cases} \vartheta(e_i) & \text{if } \varepsilon_i = 1, \\ \neg\vartheta(e_i) & \text{if } \varepsilon_i = 0. \end{cases}$$

Here, consider the occurrence of $e_i$ as abbreviating '$a_j$ with $j$ such that $g(j) = i$'.

For $\vec{\varepsilon} = \langle \varepsilon_0, \ldots, \varepsilon_{n-1} \rangle \in \{0,1\}^n$, let

$$S(\vartheta, \vec{\varepsilon}) = \bigcap_{i=0}^{n-1} \{\langle f, g\rangle \mid M_{f,g} \models \vartheta^{\varepsilon_i}(e_i)\}$$

and let

$$A(\vec{\varepsilon}) = \bigcap_{j \in \mathbb{N}^+} \left\{ \langle f, g\rangle \mid M_{f,g} \models \bigvee_{i=0}^{n-1} P_j^{1-\varepsilon_i}(e_i) \right\}.$$

Then $w_{\mu_n}$ fails to satisfy Wilmers' Principle, if for some $\vec{\varepsilon}$,

$$\mu_n(S(\vartheta, \vec{\varepsilon}) \cap A(\vec{\varepsilon})) \neq 0,$$

since then $w_{\mu_n}$ would give positive weight to structures in which there is no predicate $P_j$ that holds for each $e_i$ just if $\vartheta$ holds for $e_i$.

Now let $f$ be such that $\langle f, g\rangle \in A(\vec{\varepsilon})$. There exists an atom $\alpha_k$ of $L_n$ such that $\alpha_k(x) \models P_i^{\varepsilon_i}(x)$ for each $i \in \{0, \ldots, n-1\}$. Then by $\langle f, g\rangle \in A(\vec{\varepsilon})$, we cannot have $k \in \mathrm{ran}(f)$. But then $A(\vec{\varepsilon})$ is a subset of a $\mu_n$-null set, and thus $w_{\mu_n}(S(\vartheta, \vec{\varepsilon}) \cap A(\vec{\varepsilon})) = 0$. Since this holds for arbitrary $\vec{\varepsilon}$, we must have that $w_{\mu_n}(\exists X\, \forall x\, (\vartheta(x) \longleftrightarrow X(x))) = 1$, and thus Wilmers' Principle holds for $w_{\mu_n}$.

Let $\vartheta_n$ be the sentence

$$\vartheta_n = \exists x_1 \ldots \exists x_n \left[ \bigwedge_{1 \leq i < j \leq n} \exists X \neg (X(x_i) \longleftrightarrow X(x_j)) \right.$$
$$\left. \wedge \forall z \bigvee_{i=1}^{n} \forall X\, (X(x_i) \longleftrightarrow X(z)) \right].$$

We need to show that $w_{\mu_n}(\vartheta_n) = 1$, i.e. we need to show that $\mu_n(S(\vartheta_n)) = 1$. The latter holds if and only if $S(\neg\vartheta_n)$ is a $\mu_n$-null set. But we have

$$S(\neg\vartheta_n) = \{\langle f, g\rangle \mid M_{f,g} \models \forall X(X(a_i) \longleftrightarrow X(a_j)) \text{ for some } i \neq j \text{ with } g(i) \neq g(j)\}$$
$$\cup \{\langle f, g\rangle \mid \mathrm{ran}(g) \neq n\}.$$

Note that this is in general not a disjoint union. However, if $\langle f, g\rangle \in S(\neg\vartheta_n)$ because it is in the first set of the right-hand side, then we must have $\mathrm{ran}(f) \neq 2^n$, since $f$ cannot distinguish $e_{g(i)}$ and $e_{g(j)}$, and so for any $k \in 2^n$ such that $k(g(i)) \neq k(g(j))$ we must have $k \notin \mathrm{ran}(f)$. Thus the first set is a subset of a $\mu_n$-null set. The second set in this union can be directly seen to be a $\mu_n$-null set. This means that $S(\neg\vartheta)$ must be a $\mu_n$-null set as well, and we obtain $w_{\mu_n}(\vartheta_n) = 1$, as required.

Let $v$ be a probability function on $S_2L_\infty$ with the properties as in the statement of (iii). Define $\mu$ on $(2^n)^\infty \times n^\infty$ by

$$\mu\left(\bigcap_{i=1}^{k}\bigcap_{j=1}^{m}\{\langle f,g\rangle \mid M_{f,g} \models P_i^{\varepsilon_{i,j}}(a_j)\}\right) = v\left(\bigwedge_{i=1}^{k}\bigwedge_{j=1}^{m} P_i^{\varepsilon_{i,j}}(a_j)\right).$$

Note that since $v(\vartheta_n) = 1$, we have that $n$ is minimal so that $\mu$ is a well-defined measure on $(2^n)^\infty \times n^\infty$, as for $k < n$, we have $\{\langle f,g\rangle \in (2^k)^\infty \times k^\infty \mid M_{f,g} \models \vartheta_n\} = \emptyset$.

The function $\mu$ is a pre-measure on $(2^n)^\infty \times n^\infty$. Suppose that $\vartheta, \varphi_i \in QFS_2L_\infty$ for $i \in \mathbb{N}$, with $\{\langle f,g\rangle \mid M_{f,g} \models \varphi_i\}$ disjoint and

$$\bigcup_{i\in\mathbb{N}}\{\langle f,g\rangle \mid M_{f,g} \models \varphi_i\} = \{\langle f,g\rangle \mid M_{f,g} \models \vartheta\}. \tag{4.6}$$

Suppose that there is no $s \in \mathbb{N}$ such that

$$\bigcup_{i\leq s}\{\langle f,g\rangle \mid M_{f,g} \models \varphi_i\} = \{\langle f,g\rangle \mid M_{f,g} \models \vartheta\}. \tag{4.7}$$

Then we must have that $\{\neg\varphi_i \mid i \in \mathbb{N}\} \cup \{\vartheta\}$ is finitely satisfiable. Note that since all sentences involved are quantifier-free, $QFS_2L_\infty = QFS_1L_\infty$ and we are working with countable structures, we can apply the Compactness Theorem to obtain a structure $\mathcal{M}$ in which the set is satisfiable.

While $\mathcal{M}$ might not be in $\mathcal{T}_1L_\infty$, we can obtain a structure in $\mathcal{T}_1L_\infty$, and in turn an extension $\mathfrak{M} \in \mathcal{T}_2L_\infty$ satisfying the same quantifier-free sentences as $\mathcal{M}$ and thus $\{\neg\varphi_i \mid i \in \mathbb{N}\} \cup \{\vartheta\}$ is satisfiable in $\mathfrak{M}$. To achieve the necessary contradiction to the assumption (4.6), we require this $\mathfrak{M}$ to be of the form $M_{f,g}$ for some $\langle f,g\rangle \in (2^n)^\infty \times n^\infty$. To see that this is indeed possible, we temporarily extend[2] the language $L_\infty$ by the symbol $=$, interpreted as equality. Then, for $1 \leq k \leq n$, let $\chi_k$ be the sentence

$$\chi_k = \exists x_1, \ldots, \exists x_k \left[\left(\bigvee_{1\leq i<j\leq k} (x_i \neq x_j)\right) \wedge \forall x \left(\bigvee_{i=1}^{k} x = x_i\right)\right],$$

so $\chi_k$ expresses that there are $k$ distinct elements in the structure. Let $\chi = \bigvee_{1\leq k\leq n} \chi_k$. Then adding $\chi$ to the set $\{\neg\varphi_i \mid i \in \mathbb{N}\} \cup \{\vartheta\}$ certainly does not change finite satisfiability, as the sets in (4.6) and (4.7) only contain structures $M_{f,g}$ for some $\langle f,g\rangle$, and

[2]We will have to be a bit careful with this extension. Throughout the thesis, we only use unary languages, and even in the subject of Inductive Logic, equality is not usually present, see e.g. Chapter 37 in [22] for a discussion. Indeed, we needed to use Second Order sentences in order to express what $\chi, \chi_k$ can do within First Order Logic. However, we do this here as a means to show the existence of a certain structure. At no point do we actually extend the probability functions in question to languages including equality.

therefore all satisfy $\chi$. Then the structure $\mathfrak{M}$ will satisfy $\chi$, and "forgetting" $=$, we can find a structure $M_{f,g} \prec_{L_\infty} \mathfrak{M}$, by restricting $\mathfrak{M}$ to the language $L_\infty$. We have $\langle f, g\rangle \in (2^n)^\infty \times n^\infty$, as $\mathfrak{M} \models \chi$.

Therefore by Carathéodory's Extension Theorem, we obtain a unique $\widetilde{\mu}$ on $(2^n)^\infty \times n^\infty$ of $\mu$ which agrees with $v$ on $S_1 L_\infty$, and using the same methods as in the proof of Theorem 4.9, we see that $\widetilde{\mu}$ agrees with $v$ on $S_2 L_\infty$.

It remains to show that

$$\widetilde{\mu}(\{\langle f, g\rangle \mid \operatorname{ran}(f) \neq 2^n \text{ or } \operatorname{ran}(g) \neq n\}) = 0.$$

Suppose for a contradiction that the above set is not a $\widetilde{\mu}$-null set. Since $\widetilde{\mu}(\{\langle f, g\rangle \mid M_{f,g} \models \vartheta_n\}) = 1$, we already have that if $A \subseteq \{\langle f, g\rangle \mid \operatorname{ran}(g) \neq n\}$, then $\widetilde{\mu}(A) = 0$, since for each $\langle f, g\rangle \in A$ we must have $M_{f,g} \models \neg\vartheta_n$.

So assume that there is $A \subseteq \{\langle f, g\rangle \mid \operatorname{ran}(f) \neq 2^n\}$ such that $\widetilde{\mu}(A) > 0$, i.e. there is such an $A$ with

$$\langle f, g\rangle \in A \iff M_{f,g} \models \varphi$$

for some $\varphi$ with $v(\varphi) > 0$. If no such $A$, $\varphi$ exist, then we are done.

In the other case, note that we must have $M_{f,g} \models \vartheta_n$ for each $\langle f, g\rangle \in A$, since we would otherwise reach a contradiction as $v(\vartheta_n) = 1$ and $v$ is a probability function.

Define $\mathcal{F}$ on the subsets of $A$ by

$$B \in \mathcal{F} \iff B \subseteq A \wedge \widetilde{\mu}(A \setminus B) = 0.$$

Then $\mathcal{F}$ can easily be seen to be a filter on the subsets of $A$. Extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$ on the subsets of $A$. As $\mathcal{U}$ extends $\mathcal{F}$, we cannot have $B \in \mathcal{U}$ for $\widetilde{\mu}(B) = 0$. Now there exist $i_0, \ldots, i_{n-1}$ and $B \subseteq A$ such that for each $\langle f, g\rangle \in B$, $g(i_s) = s$, $s \in \{0, \ldots, n-1\}$, and $\widetilde{\mu}(B) > 0$. For each choice of $i_0, \ldots, i_{n-1}$ either the set $\{\langle f, g\rangle \in A \mid g(i_s) = s, s \in \{0, \ldots, n-1\}\}$ or its complement must be in $\mathcal{U}$, and some such set must be non-empty, as $\operatorname{ran}(g) = n$. If no such set is in $\mathcal{U}$, we get a subset of $A$ with positive measure under $\widetilde{\mu}$ and $\operatorname{ran}(g) < n$ for $\langle f, g\rangle$ in this set.

So fix $i_0, \dots, i_{n-1}$, $B \subseteq A$ with this property. Similarly, we can obtain $j_1, \dots, j_R$ for $R \geq n \cdot (n+1)/2$ and a subset $C \subseteq B$ such that $\widetilde{\mu}(C) > 0$ and for each $\langle f, g \rangle \in C$,

$$M_{f,g} \models \bigwedge_{1 \leq s < t \leq n} \left( \bigvee_{p=1}^{R} \neg (P_{j_p}(a_{i_s}) \longleftrightarrow P_{j_p}(a_{i_t})) \right),$$

as for each $\langle f, g \rangle \in B$ we must have $M_{f,g} \models \vartheta_n$, so some such $P_{j_p}$ witnessing this must exist.

Now for each $k \in 2^n$, define the First Order formula $\varphi_k(x)$ by

$$\varphi_k(x) = \bigwedge_{s=1}^{n} \left( \left( \bigwedge_{t=1}^{R} (P_{j_t}(a_{i_s}) \longleftrightarrow P_{j_t}(x)) \right) \to \eta_s \right),$$

where

$$\eta_s = \begin{cases} P_1(x) \vee \neg P_1(x) & \text{if } k(s) = 1, \\ P_1(x) \wedge \neg P_1(x) & \text{if } k(s) = 0. \end{cases}$$

Then we obtain for each $\langle f, g \rangle \in C$, each $k \in 2^n$ and each $i \in \mathbb{N}^+$

$$M_{f,g} \models \varphi_k(a_i) \Longleftrightarrow k(g(i)) = 1.$$

Furthermore, by Wilmers' Principle,

$$\widetilde{\mu}\left(\{\langle f, g \rangle \mid M_{f,g} \models \exists X \, \forall x \, (\varphi_k(x) \longleftrightarrow X(x))\}\right) = 1$$

for each $k \in 2^n$. Since $\widetilde{\mu}(C) > 0$, we must have that for each $\langle f, g \rangle \in C$, there exists $P_{j_{f,k}}$ such that $M_{f,g} \models P_{j_{f,k}}(a_i)$ if and only if $f(j_{f,k}) = k$, or rather the subset of $C$ on which this fails must be $\widetilde{\mu}$-null, since otherwise we contradict Wilmers' Principle holding for $v$ and thus for $\widetilde{\mu}$.

But by construction of $C$, we have $C \subseteq \{\langle f, g \rangle \mid \operatorname{ran}(f) \neq 2^n\}$, so we must fail to find $P_{j_{f,k}}$ for some $k$ on a subset of $C$ with positive measure, so we must have $\widetilde{\mu}(C) = 0$, contradicting the assumption that $A$ with $\widetilde{\mu}(A) > 0$ and $A \subseteq \{\langle f, g \rangle \mid \operatorname{ran}(f) \neq 2^n\}$ existed. $\dashv$

**Theorem 4.23 (Ladder Theorem).** *Let $w$ be a probability function on $S_2L_\infty$ satisfying (P1-4) + Ex + Px + Wilmers' Principle. Then $w$ can be represented as*

$$w = \lambda_0 \cdot w_0 + \sum_{n \in \mathbb{N}^+} \lambda_n w_{\mu_n}$$

*with $w_0$ satisfyings (P1-4) + Ex + Px + Wilmers' Principle.*

**Proof:** Let $w$ be a function on $S_2 L_\infty$ satisfying (P1-4) + Ex + Px + George's Principle. Suppose that for some $n \in \mathbb{N}^+$, $w(\vartheta_n) = \lambda_n > 0$. Then we can define $w_n = w(\cdot \mid \vartheta_n)$ and obtain a well-defined probability function.

$w_n$ inherits Ex, Px and Wilmers' Principle from $w$: Notice that since $\vartheta_n$ does not contain any constant or predicate symbol, it is invariant under any permutation induced by Ex or Px. Then we obtain for any sentence $\varphi \in S_2 L_\infty$ and $\sigma$ a permutation induced by Ex or Px, respectively,

$$\begin{aligned} w_n(\sigma\varphi) &= \frac{w(\sigma\varphi \wedge \vartheta_n)}{w(\vartheta_n)} \\ &= \frac{w(\sigma(\varphi \wedge \vartheta_n))}{w(\vartheta_n)} \\ &= \frac{w(\varphi \wedge \vartheta_n)}{w(\vartheta_n)}, \end{aligned}$$

by Ex or Px for $w$, respectively,

$$= w_n(\varphi).$$

For Wilmers' Principle, consider $\vartheta(x) \in F_1 L_\infty$ and let

$$\psi = \exists X \, \forall x \, (\vartheta(x) \longleftrightarrow X(x)).$$

Then $w(\psi) = 1$. We obtain

$$\begin{aligned} w_n(\psi) &= \frac{w(\psi \wedge \vartheta_n)}{w(\vartheta_n)} \\ &= \frac{\lambda_n}{\lambda_n} = 1. \end{aligned}$$

By Lemma 4.22, there is a measure $\mu_n$ on $(2^n)^\infty \times n^\infty$ such that $w_n = w_{\mu_n}$.

Now consider $\hat{w} := w - \sum_{n \in \mathbb{N}^+} \lambda_n \cdot w_{\mu_n}$, where for each $n \in \mathbb{N}^+$, $\lambda_n = w(\vartheta_n)$, and $w_{\mu_n}$ an arbitrary function if $\lambda_n = 0$. Let $\lambda_0 := \hat{w}(\top)$. If $\lambda_0 = 0$, we are done. So assume $\lambda_0 > 0$.

Then $w_0 := \lambda_0^{-1} \hat{w}$ is a probability function satisfying Ex + Px + Wilmers' Principle.

(P1) for $w_0$ follows straight from the definition. For (P2), let $\varphi, \vartheta \in S_2 L_\infty$ and suppose that $\varphi \models \neg\vartheta$. Then

$$\begin{aligned} w_0(\varphi \vee \vartheta) &= \lambda_0^{-1}\left(w(\varphi \vee \vartheta) - \sum_{n\in\mathbb{N}^+} \lambda_n \cdot w_{\mu_n}(\varphi \vee \vartheta)\right) \\ &= \lambda_0^{-1}\left(w(\varphi) + w(\vartheta) - \sum_{n\in\mathbb{N}^+} \lambda_n \cdot w_{\mu_n}(\varphi) - \sum_{n\in\mathbb{N}^+} \lambda_n \cdot w_{\mu_n}(\vartheta)\right) \\ &= \lambda_0^{-1}\left(w(\varphi) - \sum_{n\in\mathbb{N}^+} \lambda_n \cdot w_{\mu_n}(\varphi)\right) + \lambda_0^{-1}\left(w(\vartheta) - \sum_{n\in\mathbb{N}^+} \lambda_n \cdot w_{\mu_n}(\vartheta)\right) \\ &= w_0(\varphi) + w_0(\vartheta) \end{aligned}$$

For (P3), assume that $\vartheta(x) \in F_2 L_\infty$. We obtain

$$\begin{aligned} w_0(\exists x\, \vartheta(x)) &= \lambda_0^{-1}\left(w(\exists x\, \vartheta(x)) - \sum_{n\in\mathbb{N}^+} \lambda_n \cdot w_{\mu_n}(\exists x\, \vartheta(x))\right) \\ &= \lambda_0^{-1}\left(\lim_{k\to\infty} w\left(\bigvee_{i=1}^{k} \vartheta(a_i)\right) - \sum_{n\in\mathbb{N}^+} \lambda_n \cdot \lim_{k\to\infty} w_{\mu_n}\left(\bigvee_{i=1}^{k} \vartheta(a_i)\right)\right) \\ &= \lim_{k\to\infty}\left[\lambda_0^{-1}\left(w\left(\bigvee_{i=1}^{k} \vartheta(a_i)\right) - \sum_{n\in\mathbb{N}^+} \lambda_n \cdot w_{\mu_n}\left(\bigvee_{i=1}^{k} \vartheta(a_i)\right)\right)\right] \\ &= \lim_{k\to\infty} w_0\left(\bigvee_{i=1}^{k} \vartheta(a_i)\right), \end{aligned}$$

as required. (P4) now follows with an analogous argument.

As both $w$ and each of the $w_{\mu_n}$ are invariant under permutations induced by Ex and Px, so is $w_0$. We obtain that $w_0$ satisfies Wilmers' Principle since each of the $w_{\mu_n}$ and $w$ do. So if $\vartheta(x) \in F_1 L_\infty$ and $\psi = \exists X\, \forall x\, (\vartheta(x) \longleftrightarrow X(x))$, then

$$\begin{aligned} w_0(\psi) &= \lambda_0^{-1}\left(w(\psi) - \sum_{n\in\mathbb{N}^+} \lambda_n \cdot w_{\mu_n}(\psi)\right) \\ &= \lambda_0^{-1}\left(1 - \sum_{n\in\mathbb{N}^+} \lambda_n\right) \\ &= \lambda_0^{-1} \cdot \lambda_0 = 1, \end{aligned}$$

as required.

We now get $w = \lambda_0 \cdot w_0 + \sum_{n\in\mathbb{N}^+} \lambda_n \cdot w_{\mu_n}$ by a straightforward calculation, replacing $w_0$ by its definition. $\dashv$

**Proposition 4.24.** *The function $w_0$ defined above satisfies Regularity.*

**Proof:** By the definition of $w_0$, we clearly have $w_0(\vartheta_n) = 0$ for any $n \in \mathbb{N}^+$. As a consequence, viewing $w_0$ as a mixture of functions $V_M$, we have that each $M$ with $V_M$ contributing to $w_0$ with positive weight must have infinitely many distinct constants in the universe.

To show that $w_0$ satisfies Regularity, it is enough to show that each state description of each finite sublanguage of $L_\infty$ gets positive weight under $w_0$.

So suppose $\Theta(a_1, \ldots, a_n)$ is a state description of some finite language $L_q$. Now let $P_{j_1}, \ldots, P_{j_q}$ be distinct predicates and fix an enumeration of $\vec{\varepsilon_i} = \langle \varepsilon_{i,1}, \ldots, \varepsilon_{i,q} \rangle \in \{0,1\}^q$. Define the sentence $\Psi(a_{i_1}, \ldots, a_{i_{2^q}})$ by

$$\Psi(a_{i_1}, \ldots, a_{i_{2^q}}) = \bigwedge_{k=1}^{2^q} \bigwedge_{s=1}^{q} P_{j_s}^{\varepsilon_{k,s}}(a_{i_k}).$$

As $\Psi$ instantiates $2^q$ distinct constants, we must have $w_0(\Psi) > 0$, since otherwise there must be some $m < 2^q$ such that each structure $M$ contributing to $w_0$ with positive weight must have at most $m$ distinct elements in the universe.

Applying suitable permutations of predicates and constants, we obtain the state description

$$\Psi_q(a_1, \ldots, a_{2^q}) = \bigwedge_{i=1}^{n} \alpha_i(a_i)$$

and since $w_0$ satisfies Ex + Px, we have $w_0(\Psi_q) > 0$. Let $\langle n_1, \ldots, n_{2^q} \rangle$ be the signature of the state description $\Theta$, let $S = \{i \,|\, n_i \neq 0\}$ and let $\varphi$ be the sentence

$$\varphi = \bigwedge_{i \in S} \alpha_i(a_i).$$

Then $\Psi_q \models \varphi$ and therefore $w_0(\varphi) > 0$. Now consider the function $w' = w_0 \restriction S_1 L_q$. This is a probability function on $S_1 L_\infty$ satisfying Px + Ex and $w'(\varphi) > 0$. Therefore in the de Finetti representation of $w'$, we must have some $w_{\vec{c}}$ occurring with positive weight such that $c_i > 0$ for each $i \in S$. Therefore we can extend $\varphi$ to a state description $\Phi(a_1, \ldots, a_n)$ with the same signature as $\Theta$ and obtain $w_{\vec{c}}(\Phi) > 0$, giving $w'(\Phi) > 0$ and subsequently $w_0(\Phi) > 0$. As $\Theta$ and $\Phi$ have the same signature, we must have $w_0(\Theta) = w_0(\Phi)$ holding by Ex for $w_0$. ⊣

**Corollary 4.25.** *Let* $w = \lambda_0 w_0 + \sum_{n \in \mathbb{N}^+} \lambda_n \cdot w_{\mu_n}$ *for* $\lambda_i \geq 0$, $\sum_{i \in \mathbb{N}} \lambda_i = 1$ *be a probability function on* $S_2 L_\infty$ *satisfying (P1-4) + Ex + Px + Wilmers' Principle.*

*Then* $w$ *satisfies Regularity if and only if* $\lambda_0 > 0$, *or for arbitrarily large* $n \in \mathbb{N}^+$, $\lambda_n > 0$.

**Proof:** Since $w_0$ satisfies Regularity by Proposition 4.24, $w$ satisfies Regularity if $\lambda_0 > 0$. So suppose that $w$ satisfies Regularity but $\lambda_0 = 0$. We show that then for arbitrarily large $n \in \mathbb{N}^+$, we must have $\lambda_n > 0$.

Suppose this is not the case. Then there exists a maximal $m \in \mathbb{N}^+$ such that $\lambda_m > 0$. Now suppose that $q$ is such that $2^q > m$ and let $n$ be such that $m < n \leq 2^q$. Then the state description

$$\Theta(a_1, \ldots, a_n) = \bigwedge_{i=1}^{n} \alpha_i(a_i),$$

where $\alpha_i$ range over the atoms of $L_q$ instantiates $n$ distinct constants. Since $m$ was maximal with $\lambda_m > 0$, $w$ can give positive probability only to sentences that instantiate at most $m$ distinct constants. Thus, we must have $w(\Theta(a_1, \ldots, a_n)) = 0$, contradicting Regularity. ⊣

# Chapter 5

# Conclusion

We have provided some more insight in the principle of Predicate Exchangeability. In this final chapter, we will recall the work done in this thesis and present suggestions for further research related to each chapter.

In Chapter 2, we have shown a Representation Theorem for probability functions satisfying Unary Language Invariance. Using this, we could then show that any probability function satisfying Predicate Exchangeability is the difference of functions satisfying Language Invariance – an analogous result to one that holds for functions satisfying Atom Exchangeability. However, the results have been shown for purely unary languages only. An obvious next step would be to extend these Representation Theorems to languages involving $n$-ary relations. We believe that the approach via Non-Standard Analysis used in this chapter will be of some use to prove a Representation Theorem for those languages.

In Chapter 3, we have investigated a symmetry principle that sits in between the principles of Predicate and Atom Exchangeability. This involved describing a collection of functions that has similar properties as the building blocks of functions satisfying Atom Exchangeability. In contrast to Predicate Exchangeability and Atom Exchangeability, it is still an open question what the building blocks of functions satisfying Strong Predicate Exchangeability look like. While the principle itself appeared rather artificially, the justification via a weak Johnson's Sufficientness Postulate that we provided in Chapter 3 should give us some reason for further investigation into a Representation

Theorem for this principle.

We then extended probability functions to Second Order languages in Chapter 4, introducing a new rational principle, Wilmers' Principle, and providing a Representation Theorem for functions satisfying it. While we have limited ourselves to a rather weak version of Second Order Logic, we have provided rational agents with a lot more expressive strength. Research in this area is still in the early stages. We hope that further work will provide us with interesting rational principles involving Second Order statements besides Wilmers' Principle.

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