ABSOLUTE CONTINUITY OF THE LAWS, EXISTENCE AND UNIQUENESS OF SOLUTIONS OF SOME SDES AND SPDES

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Engineering and Physical Sciences

2014

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Contents

Word count xxxxx

The University of Manchester

Wen Yue

Doctor of Philosophy

Absolute continuity of the laws, existence and uniqueness of solutions of some SDEs and SPDEs

March 30, 2014

This thesis consists of four parts. In the first part we recall some background theory that will be used throughout the thesis. In the second part, we studied the absolute continuity of the laws of the solutions of some perturbed stochastic differential equaitons(SDEs) and perturbed reflected SDEs using Malliavin calculus. Because the extra terms in the perturbed SDEs involve the maximum of the solution itself, the Malliavin differentiability of the solutions becomes very delicate. In the third part, we studied the absolute continuity of the laws of the solutions of the parabolic stochastic partial differential equations(SPDEs) with two reflecting walls using Malliavin calculus. Our study is based on Yang and Zhang [YZ1], in which the existence and uniqueness of the solutions of such SPDEs was established. In the fourth part, we gave the existence and uniqueness of the solutions of the elliptic SPDEs with two reflecting walls and general diffusion coefficients.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Acknowledgements

I would like to express my sincere gratitude to my supervisor Professor Tusheng Zhang at University of Manchester. He has provided me a lot of encouragement and help not only in my study but also in my daily life. He showed me the beauty of stochastic analysis and led me to the research field. I learned a lot from him, not only the professional knowledge but also the attitude towards research. His advice will continue to bring me a far-reaching impact on my research.

Furthermore, I especially thank my mother Xuelan Zhai and my father Shaozhou Yue for their permernant love, understand and support for me. I am so grateful to my boyfriend, Yuan Gao. He unconditionally loves and supports me during my good and bad times and gives me a lot of advice and encouragement in my life.

I thank all the staff and students in School of Mathematics and other friends I met in Manchester. They shared my enjoyments and sorrows and made my life here the most memorable.

Last but not the least, I thank the financial support from School of Mathematics here and the Chinese Government Scholarship.

Statements

The work contained in Chapter 2 has been published in *Journal of Theoretical Probability* under the title *Absolute Continuity of the Laws of Perturbed Diffusion Processes and Perturbed Reflected Diffusion Processes* with the co-author Tusheng Zhang.

The work contained in Chapter 4 has been submitted to *Infinite Dimensional Analysis, Quantum Probability and Related Topics* under the title *Elliptic Stochastic Partial Differential Equations with Two Reflecting Walls* with the co-author Tusheng Zhang.

Notations

Through this thesis, I numbered equations, lemmas, theorems, etc., separately per chapter.

Table of Symbols

a.s., a.e.	almost surely, almost everywhere
w.r.t	with respect to
$\emptyset$	the empty set
$\sum$	summation
$\prod$	product
A^c	the complement of A
$\subseteq$	contained in
$\cap$, $\cup$	intersection, union
z^+, z^-	$\max(z, 0)$, $-\min(z, 0)$
sup	supermum
inf	infimum
:=	equal to by definition
lim sup	superior limit
lim inf	inferior limit
$P(\cdot), E(\cdot)$	probability, expectation
$\mathcal{N}(\mu, \sigma^2)$	the normal distribution with mean μ, covariance σ^2
$\mathbb{R}^d$	the d-dimensional Euclidean space
$\mathbb{R}_+$ or $\mathbb{R}^+$	the set of all the non-negative real numbers
$\mathbb{R}^{+*}$	the set of all strictly positive real numbers

$L^p(U;H)$	the space of all the p-integrable mappings from U to H
$C(X;Y)$	the set of all the continuous mappings from X to Y
$C(X)$	the set of all the real continuous functions defined on X
$C_b^k(X)$	the set of all the bounded functions with continuous derivatives up to order k on X
C_p^∞	the set of all infinitely continuously differentiable functions such that the functions and their derivatives have polynomial growth
C_k^∞	the set of infinitely continuously differentiable functions with compact support
$f', f'', \nabla f$	first, second derivatives, and gradient of f
1_B	the indicator function of set B

I close all the proof with the symbol $\square$. I give all the references by number enclosed within square brackets.

Chapter 1

Introduction

1.1 Motivation and Contribution

This thesis is devoted to the study of the existence and uniqueness of the solutions to some stochastic differential equations(SDEs) and stochastic partial differential equations(SPDEs) and Malliavin calculus of the solutions.

There now exists a considerable body of literatures devoted to the study of perturbed SDEs, see e.g.[CPY],[CD],[D],[RD],[DZ],[IW],[GY1],[GY2],[PW],[W1]. The idea of perturbed SDE originated from the following equation:

$$X_t = B_t + \alpha \sup_{s \leq t} X_s. \tag{1.1.1}$$

The solution of the above equation behaves like a Brownian Motion except when it attains a new maximum: it is called an α-perturbed Brownian Motion. The first of this arose in a study of the windings of planar Brownian Motion in [GY2]. And in [CD], it was proved that for $\alpha < 1$, Equation (1.1.1) has a pathwise unique solution and is adapted to the filtration of B. Furthermore, it was proved in [DZ] that perturbed SDE and perturbed reflected SDE with general diffusion admit unique solutions under some conditions.

In Chapter 2, we will establish the absolute continuity of the laws of perturbed diffusion processes as well as perturbed reflected diffusion processes under appropriate conditions. The absolute continuity of the laws of the solutions is of fundamental importance both theoretically and numerically. The absolute continuity of the laws of the solutions to stochastic differential equations has been studied by many people in Book [N] and [S]. The tool we use is naturally Malliavin calculus. Because the extra terms in the equation involve the maximum of the solution itself, the Malliavin differentiability of the solutions becomes very delicate. For the absolute continuity of the laws of the solutions, we need a careful analysis of the time points where the solution X reaches its maximum.

Malliavin calculus extends the calculus of variations from functions to stochastic processes. The Malliavin calculus is also called the stochastic calculus of variations. In particular, it allows the computation of derivatives of random variables. Malliavin ideas led to a proof that under some conditions the probability law of the solution to stochastic differential equations and stochastic partial differential equations are absolute continuous with respect to Lebesgue measure.

For stochastic partial differential equations, Malliavin calculus associated to white noise was also used by Pardoux and Zhang in [PZ], Bally and Pardoux in [BP1] to establish the existence of the density of the solutions to parabolic SPDEs. The case of parabolic stochastic partial differential equations with one reflecting wall was studied by Donati-Martin and Pardoux in [DP]. The existence and uniqueness of the solution to parabolic stochastic partial differential equations with two reflecting walls was proved by Juan Yang and Tusheng Zhang in [YZ1]. In Chapter 3, we focus on the existence of the density for the law of the solutions to parabolic SPDEs with two reflecting walls using Malliavin calculus. But we still leave the absolute continuity of the law of the solutions in the case of hitting the reflected walls as open questions.

For the elliptic SPDEs, R. Buckdahn and E. Pardoux in [BP2] established the existence and uniqueness results for nonlinear elliptic stochastic partial differential equations when the diffusion coefficient is constant. Based on this, nonlinear elliptic SPDEs with one reflected wall has been studied by David Nualart and Samy Tindel in [NT], in which the diffusion coefficient is still constant. In Chapter 4, we give the existence and uniqueness of the solutions to nonlinear elliptic SPDEs with non-constant diffusion coefficient and two reflecting walls. This is the first time that the case of non-constant diffusion coefficients is studied. We will use the technique developed by Nuarlart and Pardoux in [NP] and Tiange Xu and Tusheng Zhang in [XZ].

1.2 Background

In this section, we recall some background material which will be used in the following chapters.

1.2.1 Malliavin calculus

Let $W = \{W(h), h \in H\}$ denote an isonormal Gaussian Process associated with the Hilbert space H. We assume that W is defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and that $\mathcal{F}$ is generated by W.
We denote by $C_p^\infty(\mathcal{R}^n)$ the set of all infinitely continuously differentiable functions $f : \mathcal{R}^n \to \mathcal{R}$ such that f and all of its partial derivatives have polynomial growth.
Let $\mathcal{S}$ denote the class of smooth random variables such that a random variable $F \in \mathcal{S}$ has the form

$$F = f(W(h_1), ..., W(h_n)), \tag{1.2.1}$$

where f belongs to $C_p^\infty(\mathcal{R}^n)$, $h_1, ..., h_n$ are in H, and $n \geq 1$.

Definition 1.2.1 *The derivative of a smooth random variable F of the form is the H-valued random variable given by*

$$DF = \Sigma_{i=1}^n \partial_i f(W(h_1), W(h_2), ..., W(h_n))h_i, \tag{1.2.2}$$

Define its norm by

$$||F||_{1,2} = [E(|F|^2) + E(||DF||_H^2)]^{\frac{1}{2}}. \tag{1.2.3}$$

Let $\mathbb{D}^{1,2}$ be the completion of S under the norm $||.||_{1,2}$.

The following result is from Nualart [N].

Theorem 1.2.1 *(Theorem 2.1.3 in [N]) Let $F \in \mathbb{D}^{1,2}$. If $||DF||_H > 0$ a.s., then the law of the random variable F is absolutely continuous with respect to Lebesgue measure.*

Theorem 1.2.2 *(Lemma 1.2.3 in [N]) Let $\{F_n, n \geq 1\}$ be a sequence of random variables in $\mathbb{D}^{1,2}$ that converges to F in $L^2(\Omega)$ and such that*

$$\sup_n E(||DF_n||_H^2) < \infty. \tag{1.2.4}$$

Then F belongs to $\mathbb{D}^{1,2}$, and the sequence of derivatives $\{DF_n, n \geq 1\}$ converges to DF in the weak topology of $L^2(\Omega; H)$.

Theorem 1.2.3 *(Proposition 2.1.10 in [N]) Let $X = \{X(t), t \in S\}$ be a continuous process parametrized by a compact metric space S. Suppose that*
(i) $E(\sup_{t \in S} X(t)^2) < \infty$,
(ii) for any $t \in S, X_t \in \mathbb{D}^{1,2}$ and the H-valued process $\{DX(t), t \in S\}$ possesses a continuous version, and $E(\sup_{t\in S} ||DX(t)||_H^2) < \infty$,
Then the random variable $M = \sup_{t \in S} X(t)$ belongs to $\mathbb{D}^{1,2}$.

1.2.2 Stochastic Integration with Respect to White Noises

The following content is from Walsh [W2].

Let $(E, \mathcal{E}, \nu)$ be a σ-finite measure space. A white noise based on ν is a random set function W on the sets $A \in \mathcal{E}$ of finite ν-measure such that
(i) $W(A)$ is a $N(0, \nu(A))$ random variable;
(ii) if $A \cap B = \emptyset$, then $W(A)$ and $W(B)$ are independent and $W(A \cup B) = W(A) + W(B)$.

We see that it is a mean-zero Gaussian process with covariance function

$$E\{W(A)W(B)\} = \nu(A \cap B). \tag{1.2.5}$$

White noise can be thought of the derivative of Wiener process. The Brownian sheet on R_+^2 is the process $W(x,t) := W((0,x] \times (0,t])$. This is a mean-zero Gaussian process.

In the classical case, one constructs the stochastic integral as a process rather than as a random variable. One can then say that the integral is a martingale. The analogue of "martingale" in our setting is "martingale measure". Accordingly, we will define our stochastic integral as a martingale measure.

Definition 1.2.2 *Let $(\mathcal{F}_t)$ be a right continuous filtration. A process $\{M_t(A), \mathcal{F}_t, t \geq 0, A \in \mathcal{E}\}$ is a martingale measure if*
(i) $M_0(A) = 0$;
(ii) if $t > 0$, M_t is a σ-finite L^2-valued measure;
(iii)$\{M_t(A), \mathcal{F}_t, t \geq 0\}$ is a martingale.

Definition 1.2.3 *A martingale measure M is worthy if there exists a random σ-finite measure $K(A \times B \times C, \omega), A \times B \times C \in \mathcal{E} \times \mathcal{E} \times \mathcal{B}, \mathcal{B} = \mathcal{B}(\mathcal{R}^+), \omega \in \Omega$ such that*
(i) K is positive definite and symmetric in A and B;
(ii) for fixed A and B, $\{K(A \times B \times (0, t]), t > 0\}$ is predictable;
(iii)$E\{K(E \times E \times [0, T])\} < \infty$;
(iv)for any rectangle D,$|C(D)| \leq K(D)$, where C is the covariance functional of M:

$$C(A \times B \times (s, t]) =< M(A), M(B) >_t - < M(A), M(B) >_s . \qquad (1.2.6)$$

We call K the dominating measure of M.

A function $f : E \times [0, \infty) \times \Omega \to \mathcal{R}$ is elementary if it is of the form $f(x, s, w) = X(\omega)I_{(a,b]}(s)I_A(x)$, where $0 \leq a \leq t$, X is bounded, $\mathcal{F}_a$-measurable and $A \in \mathcal{E}$. f is simple if it is a finite sum of elementary functions, we denote the class of simple functions by $\mathcal{S}$. Define a martingale measure $f \cdot M$ by

$$f \cdot M_t(B) = X(\omega)(M_{t\wedge b}(A \cap B) - M_{t\wedge a}(A \cap B)). \qquad (1.2.7)$$

A function is predictable if it is $\mathcal{P}$-measurable, where the predictable σ-field $\mathcal{P}$ on $\Omega \times E \times R_+$ is the σ-field generated by $\mathcal{S}$. $\mathcal{P}_M$ be the class of all predictable f for which $||f||_M := [E(\int_{E\times E\times R_+} |f(x, s)|^2 K(dxdyds))]^{\frac{1}{2}} < \infty$. $\mathcal{S}$ is dense in $\mathcal{P}_M$. From Walsh [W2], we have the following theorem:

Theorem 1.2.4 *If $f \in \mathcal{P}_M$, then $f \cdot M$ is a worthy martingale measure. It is orthogonal if M is. Its covariance and dominating measures respectively are given by*

$$Q_{f\cdot M}(dxdyds) = f(x,s)f(y,s)Q_M(dxdyds); \tag{1.2.8}$$

$$K_{f\cdot M}(dxdyds) = |f(x,s)f(y,s)|K_M(dxdyds). \tag{1.2.9}$$

Moreover, if $g \in \mathcal{P}_M$ and $A, B \in \mathcal{E}$, then

$$< f\cdot M(A), g\cdot M(B) >_t = \int_{A\times B\times[0,t]} f(x,s)g(y,s)Q_M(dxdyds); \tag{1.2.10}$$

$$E\{(f\cdot M_t(A))^2\} \le ||f||_M^2. \tag{1.2.11}$$

Definition 1.2.4 *A martingale measure M is orthogonal if, for any two disjoint sets A and B in $\mathcal{E}$, the martingale $\{M_t(A), t \ge 0\}$ and $\{M_t(B), t \ge 0\}$ are orthogonal.*

If the integrator M is orthogonal, the covariance measure Q_M sits on the diagonal and is positive, so that $Q_M = K$. Instead of having two measures on $E \times E \times R_+$, we need only concern ourselves with a single measure ν on $E \times R_+$ where $\nu(A \times [0,t]) = Q_M(A \times A \times [0,t])$.

Corollary 1.2.1 *If M is an orthogonal martingale measure and $f \in \mathcal{P}_M$, then $f \cdot M$ is an orthogonal martingale measure. Its covariance and dominating measures respectively are given by*

$$Q_{f\cdot M}(dxdyds) = f(x,s)f(y,s)\nu(dxdyds) \tag{1.2.12}$$

Moreover, if $g \in \mathcal{P}_M$ and $A, B \in \mathcal{E}$, then

$$< f\cdot M(A), g\cdot M(B) >_t = \int_{A\times B\times[0,t]} f(x,s)g(y,s)\nu(dxdyds); \tag{1.2.13}$$

$$E\{(f\cdot M_t(A))^2\} = E\int_{A\times[0,t]} f(x,s)^2\nu(dxds). \tag{1.2.14}$$

Theorem 1.2.5 *Let M be an orthogonal martingale measure, and suppose that for each $A \in \mathcal{E}, t \to M_t(A)$ is continuous. Then, M is a white noise if and only if its covariance measure C is deterministic.*

Consider the following Parabolic SPDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(x,t,u) + \sigma(x,t,u)\dot{W}(x,t) \\ u(0,t) = 0, u(1,t) = 0, \ for\ t \geq 0, \\ u(x,0) = u_0(x) \in C([0,1]), \end{cases} \tag{1.2.15}$$

where f, σ are Lipschitz continuous functions and u_0 is a continuous function on $[0,1]$, which vanishes at 0 and 1.

Several authors, including Walsh [W2], have shown that (1.2.15) has a unique continuous solution, in the sense that u is the unique continuous adapted process which satisfies:
$\forall t \in R^+$, $\psi \in C^2([0,1])$ with $\psi(0) = \psi(1) = 0$,

$$\begin{aligned}(u(t), \psi) + \int_0^t (u(s), A\psi) ds \quad & + \int_0^t (f(u(s)), \psi) ds = (u_0, \psi) \\ & + \int_0^t \int_0^1 \psi(x)\sigma(u(x,s)) W(dxds) a.s.\end{aligned}$$

or equivalently $u(x,t)$ satisfies the integral equation

$$\begin{aligned} u(x,t) \quad = \quad & \int_0^1 G_t(x,y) u_0(y) dy + \int_0^t \int_0^1 G_{t-s}(x,y) f(u(y,s)) dyds \\ & + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u(y,s)) W(dyds), \end{aligned}$$

where G is the Green's function associated to the operator $A = -\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions.
The following comparison theorem is from C. Donati-Martin and E. Pardoux [DP].

Theorem 1.2.6 (Comparison Theorem) *(Theorem 2.1 [DP]) Let the two pairs of coefficients f, σ and g, σ be Lipschitz, with $f \leq g$. We denote by $u(resp. v)$ the solution of (1.2.15), corresponding to $f(resp. g)$ with the same initial condition. Then, a.s. for all $(x,t) \in [0,1] \times R_+$, $u(x,t) \leq v(x,t)$.*

1.2.3 Useful Lemmas

In this subsection, we list some lemmas which will be used frequently in the following chapters.

Lemma 1.2.1 (Gronwall Inequality) *Let I denote an interval of the real line of the form $[a,\infty)$ or $[a,b]$ or $[a,b)$ with $a<b$. Let α, β and u be real-valued functions defined on I. Assume that β and u are continuous and that the negative part of α is integrable on every closed and bounded subinterval of I.*
(1)If β is non-negative and if u satisfies the integral inequality

$$u(t)\leq \alpha(t)+\int_a^t \beta(s)u(s)ds,\ t\in I,$$

then

$$u(t)\leq \alpha(t)+\int_a^t \alpha(s)\beta(s)e^{\int_s^t\beta(r)dr}ds,\ t\in I.$$

(2) If, in addition, the function α is non-decreasing, then

$$u(t)\leq \alpha(t)e^{\int_a^t\beta(r)dr},\ t\in I.$$

The following lemma is an elegant result of Garsia-Rodemich-Rumsey(see Corollary 1.2, P.273 in [W2]).

Lemma 1.2.2 (Kolmogorov) *Let $\mathcal{R}$ be a unit cube in $\mathbb{R}^n$ and $\{X_\alpha,\alpha\in\mathcal{R}\}$ be a real valued stochastic process. Suppose that there exist constants $k>1, K>0, \varepsilon>0$ such that*

$$E|X_\alpha-X_\beta|^k\leq K|\alpha-\beta|^{n+\varepsilon},$$

then (1) *X has a continuous version.*
(2) *there exist constants a,γ depending only on n,k and ε and a random variable Y such that a.s. for all $(\alpha,\beta)\in\mathcal{R}^2$,*

$$|X_\alpha-X_\beta|\leq Y|\alpha-\beta|^{\frac{\varepsilon}{k}}\bigg(\log\big(\frac{\gamma}{|\alpha-\beta|}\big)\bigg)^{\frac{2}{k}}$$

and $EY^k\leq aK$.

The following lemma is from P.166 in [KS].

Lemma 1.2.3 (The Burkholder-Davis-Gundy Inequalites) *Let M be a continuous local martingale. For every $m>0$ there exists universal positive constants k_m, K_m(depending only on m), such that*

$$k_mE(<M>_T^m)\leq E[(\sup_{0\leq t\leq T}|M_t|)^{2m}]\leq K_mE(<M>_T^m) \tag{1.2.16}$$

holds for every stopping time T.

Lemma 1.2.4 *(Theorem 1.6.11 in [A]) If μ is a σ-finite measure on σ-field $\mathcal{F}$ of Ω, g and h are Borel measurable, $\int_\Omega g d\mu$ and $\int_\Omega h d\mu$ exist, and $\int_A g d\mu \le \int_A h d\mu$ for all $A \in \mathcal{F}$, then $g \le h$ a.e. $[\mu]$.*

Chapter 2

Absolute Continuity of the Laws of Perturbed Diffusion Processes and Perturbed Reflected Diffusion Processes

2.1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space with filtration$\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. $\{B_t\}_{t\geq 0}$ is a one-dimensional standard $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian Motion. Suppose that $\sigma(x), b(x)$ are Lipschitz continuous functions on R. There now exists a considerable body of literature devoted to the study of perturbed stochastic differential equations(SDEs), see e.g. [CPY],[CD],[D],[RD], [DZ],[IW], [GY1],[GY2],[PW],[W1]. It was proved in [DZ] that the following perturbed SDE:

$$Y_t = y_0 + \int_0^t \sigma(Y_s)dB_s + \int_0^t b(Y_s)ds + \alpha \max_{0\leq s\leq t} Y_s. \tag{2.1.1}$$

and the perturbed reflected SDE:

$$\begin{cases} X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \alpha \max\limits_{0\leq s\leq t} X_s + L_t \\ X_t \geq 0 \\ \int_0^t \chi_{\{X_s=0\}} dL_s = L_t. \end{cases} \tag{2.1.2}$$

admit unique solutions, where L_t in (2.1.2) denotes a local time at zero of X. Perturbed Brownian motion arose in a study of the windings of planar Brownian motion, see [GY1]. Perturbed diffusion processes are also continuous versions of self-interacting random walks.

The purpose of this chapter is to establish the absolute continuity of the laws of perturbed diffusion processes as well as perturbed reflected diffusion processes under appropriate conditions. The absolute continuity of the laws of the solutions is of fundamental importance both theoretically and numerically. The absolute continuity of the laws of the solutions to stochastic differential equations has been studied by many people. We refer the reader to the books [N], [S] and references therein.

The tool we use is naturally Malliavin calculus. Because the extra terms in equation (2.1.1) and (2.1.2) involve the maximum of the solution itself, the Malliavin differentiability of the solutions becomes very delicate. For the absolute continuity of the laws of the solutions, we need a careful analysis of the time points where the solution X reaches its maximum. The local property of the Malliavin derivative and a comparison theorem for stochastic differential equations play a crucial role.

This chapter is organized as follows. In Section 2.2, we collect some results of Malliavin calculus to be used later in this chapter. In Section 2.3, we prove that the perturbed diffusion process is Malliavin differentiable and establish the absolute continuity of the laws of the perturbed diffusion processes. In Section 2.4, we study the reflected perturbed diffusion processes. The Malliavin differentiability and the absolute continuity of the solutions are obtained.

2.2 Preliminaries

Let $\Omega = C_0(R_+)$ be the space of continuous functions on R_+ which are zero at zero. Denote by $\mathcal{F}$ the Borel σ-field on Ω and P the Wiener measure. Then the canonical coordinate process $\{B_t,\ t \in R_+\}$ on Ω is a Brownian motion. Define $\mathcal{F}_t^0 = \sigma(B_s, s \le t)$. Denote by $\mathcal{F}_t$ the completion of $\mathcal{F}_t^0$ with respect to the P-null sets of $\mathcal{F}$. Let $h \in L^2(R_+)$. $W(h)$ will stand for the Wiener integral as follows:

$$W(h) = \int_0^\infty h(t)dB_t. \tag{2.2.1}$$

$\{W(h), h \in H\}$ is a Gaussian Process on $H := L^2(R_+, \mathcal{B}, \mu)$, where $(R_+, \mathcal{B})$ is a measurable space, $\mathcal{B}$ is the Borel σ-field of R_+ and μ is the Lebesgue measure on R_+.

We denote by $C_p^\infty(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth. Let S be the set of smooth random variables defined by

$$S = \{F = f(W(h_1), W(h_2), ..., W(h_n));\ h_1, ..., h_n \in L^2(R_+), n \ge 1, f \in C_p^\infty(R^n)\}. \tag{2.2.2}$$

Let $F \in S$. Define its Malliavin derivative D_tF by

$$D_tF = \Sigma_{i=1}^n \partial_i f(W(h_1), W(h_2), ..., W(h_n))h_i(t), \tag{2.2.3}$$

and its norm by

$$||F||_{1,2} = [E(|F|^2) + E(||DF||_H^2)]^{\frac{1}{2}}. \tag{2.2.4}$$

Let $\mathbb{D}^{1,2}$ be the completion of S under the norm $||.||_{1,2}$. The following result is from [N].

Theorem 2.2.1 *Let $F \in \mathbb{D}^{1,2}$. If $||DF||_H > 0$ a.s., then the law of the random variable F is absolutely continuous with respect to Lebesgue measure.*

2.3 Absolute continuity of the laws of perturbed diffusion processes

Let $\sigma(x), b(x)$ be Lipschitz continuous functions on R, i.e., there exists a constant C such that

$$|\sigma(x) - \sigma(y)| \le C|x - y|, \tag{2.3.1}$$

$$|b(x) - b(y)| \leq C|x - y|. \tag{2.3.2}$$

For $\alpha < 1$, $y_0 \in R$, consider the following stochastic differential equation:

$$Y_t = y_0 + \int_0^t \sigma(Y_s)dB_s + \int_0^t b(Y_s)ds + \alpha \max_{0\leq s\leq t} Y_s. \tag{2.3.3}$$

It was shown in [DZ] that equation (2.3.3) admits a unique, continuous, adapted solution. We have the following result.

Theorem 2.3.1 *Let Y_t be the unique solution to equation (2.3.3). Then $Y_t \in \mathbb{D}^{1,2}$ for any $t \geq 0$.*

Proof. Consider Picard approximations given by

$$Y_t^0 = \frac{y_0}{1-\alpha}, \; 0 \leq t < \infty. \tag{2.3.4}$$

For $n \geq 0$, define Y_t^{n+1} to be the unique, continuous, adapted solution to the following equation:

$$Y_t^{n+1} = y_0 + \int_0^t \sigma(Y_s^n)dB_s + \int_0^t b(Y_s^n)ds + \alpha \max_{0\leq s\leq t} Y_s^{n+1}. \tag{2.3.5}$$

Such a solution exists and can be expressed explicitly as

$$\begin{aligned} Y_t^{n+1} \;\; = \;\; & \frac{y_0}{1-\alpha} + \int_0^t \sigma(Y_s^n)dB_s + \int_0^t b(Y_s^n)ds \\ & + \frac{\alpha}{1-\alpha} \max_{0\leq s\leq t}\left(\int_0^s \sigma(Y_u^n)dB_u + \int_0^s b(Y_u^n)du\right) \end{aligned} \tag{2.3.6}$$

It was shown in [DZ] that the solution Y_t is the limit of Y_t^n in $L^2(\Omega)$.

We will prove the following property by induction on n:

(P) $Y_t^n \in \mathbb{D}^{1,2}, E[\int_0^t ||DY_u^n||_H^2 du] < \infty, t \geq 0.$

Clearly, **(P)** holds for n=0. Suppose $Y_t^n \in \mathbb{D}^{1,2}$ and $E[\int_0^t ||DY_u^n||_H^2 du] < \infty$. Applying Proposition 1.2.4 in [N] to the random variable Y_s^n and to σ and b, we deduce that the random variables $\sigma(Y_s^n)$ and $b(Y_s^n)$ belong to $\mathbb{D}^{1,2}$ and that there exist adapted processes $\overline{\sigma}^n(s)$ and $\overline{b}^n(s)$, which are uniformly bounded by some constant K, such that:

$$D_r(\sigma(Y_s^n)) = \overline{\sigma}^n(s) D_r(Y_s^n) I_{\{r\leq s\}}, \tag{2.3.7}$$

and

$$D_r(b(Y_s^n)) = \overline{b}^n(s) D_r(Y_s^n) I_{\{r \le s\}}. \tag{2.3.8}$$

From (2.3.7) and (2.3.8) we get

$$|D_r(\sigma(Y_s^n))| \le K|D_r(Y_s^n)|, \tag{2.3.9}$$

and

$$|D_r(b(Y_s^n))| \le K|D_r(Y_s^n)|. \tag{2.3.10}$$

By Lemma 1.3.4 in [N], we conclude that

$$\int_0^t \sigma(Y_s^n) dB_s \in \mathbb{D}^{1,2}. \tag{2.3.11}$$

For $r \le t$, by Proposition 1.3.8 in [N],

$$D_r[\int_0^t \sigma(Y_s^n) dB_s] = \sigma(Y_r^n) + \int_r^t D_r(\sigma(Y_s^n)) dB_s \tag{2.3.12}$$

Similarly, we have

$$\int_0^t b(Y_s^n) ds \in \mathbb{D}^{1,2}, \tag{2.3.13}$$

$$D_r[\int_0^t b(Y_s^n) ds] = \int_r^t D_r(b(Y_s^n)) ds. \tag{2.3.14}$$

Let $Z_s^n = \int_0^s \sigma(Y_u^n) dB_u$, $X_s^n = \int_0^s b(Y_u^n) du$. Then

$$Z_s^n + X_s^n \in \mathbb{D}^{1,2}, \tag{2.3.15}$$

and

$$\begin{aligned} E[\sup_{0 \le s \le t} (Z_s^n + X_s^n)^2] &\le E[\sup_{0 \le s \le t} 2((Z_s^n)^2 + (X_s^n)^2)] \le 2E[\sup_{0 \le s \le t} (Z_s^n)^2] \\ &\quad + 2E[\sup_{0 \le s \le t} (X_s^n)^2] < \infty \end{aligned} \tag{2.3.16}$$

Since

$$\begin{aligned} E[\sup_{0 \le s \le t} (Z_s^n)^2] &= E[\sup_{0 \le s \le t} (\int_0^s \sigma(Y_u^n) dB_u)^2] \\ &\le K_1 E[\int_0^t \sigma(Y_u^n)^2 du] \\ &\le K_1 K_2 E[\int_0^t (Y_u^n)^2 du] \end{aligned}$$

$$< \infty,$$

and

$$\begin{aligned} E[\sup_{0\le s\le t}(X_s^n)^2] &= E[\sup_{0\le s\le t}(\int_0^s b(Y_u^n)du)^2] \\ &\le E[t\int_0^t b(Y_u^n)^2 du] \\ &\le K_3 t E[\int_0^t (Y_u^n)^2 du] \\ &< \infty \end{aligned}$$

Next we show that

$$E[\sup_{0\le s\le t} ||D(Z_s^n + X_s^n)||_H^2] < \infty. \tag{2.3.17}$$

Now

$$\begin{aligned} E[\sup_{0\le s\le t} ||D(Z_s^n + X_s^n)||_H^2] &= E[\sup_{0\le s\le t}\int_0^s |D_r(Z_s^n + X_s^n)|^2 dr] \\ &\le 3E\{\sup_{0\le s\le t}\int_0^s [\sigma(Y_r^n)^2 + |\int_r^s D_r(\sigma(Y_u^n))dB_u|^2 \\ &\quad +|\int_r^s D_r(b(Y_u^n))du|^2]dr\} \le 3E\int_0^t \sigma(Y_r^n)^2 dr \\ &\quad +3\int_0^t E[\sup_{r\le s\le t}|\int_r^s D_r(\sigma(Y_u^n))dB_u|^2]dr \\ &\quad +3E\int_0^t [\int_r^t |D_r(b(Y_u^n))|du]^2 dr \\ &\le 3\int_0^t E[\sigma(Y_r^n)^2]dr + 3C\int_0^t\int_r^t E[D_r(\sigma(Y_u^n))]^2 dudr \\ &\quad +3\int_0^t\int_r^t E[D_r(b(Y_u^n))^2]du(t-r)dr \\ &\le 3\int_0^t E[\sigma(Y_r^n)^2]dr + 3CK^2\int_0^t\int_r^t E[D_r(Y_u^n)^2]dudr \\ &\quad +3K^2\int_0^t\int_r^t E[D_r(Y_u^n)^2]du(t-r)dr \\ &\le 3\int_0^t E[\sigma(Y_r^n)^2]dr \\ &\quad +(3CK^2 + 3K^2 t)\int_0^t\int_r^t E[D_r(Y_u^n)^2]dudr < \infty. \end{aligned} \tag{2.3.18}$$

So we have proved (2.3.17).

From (2.3.15),(2.3.16) and (2.3.17), and by Proposition 2.1.10 in [N], we conclude

$$\max_{0\le s\le t}(Z_s^n + X_s^n) \in \mathbb{D}^{1,2}, \tag{2.3.19}$$

and

$$E[||D(\max_{0\le s\le t}(Z_s^n + X_s^n))||_H^2] \le E[\max_{0\le s\le t}||D(Z_s^n + X_s^n)||_H^2]. \tag{2.3.20}$$

It follows from (2.3.6) that $Y_t^{n+1} \in \mathbb{D}^{1,2}$. Moreover,

$$\begin{aligned}
& E\int_0^t ||D(Y_u^{n+1})||_H^2 du \\
\le\ & 4\int_0^t\int_0^u E[\sigma(Y_r^n)^2]drdu \\
& +4\int_0^t\int_0^u E(\int_r^u D_r(\sigma(Y_v^n))dB_v)^2 drdu \\
& +4\int_0^t\int_0^u E[\int_r^u D_r(b(Y_v^n))dv]^2 drdu \\
& +4(\frac{\alpha}{1-\alpha})^2\int_0^t\int_0^u E[D_r(\sup_{0\le v\le u}(Z_v^n + X_v^n))]^2 drdu \\
\le\ & 4t\int_0^t E[\sigma(Y_r^n)^2]dr \\
& +4K^2(t+1)\int_0^t\int_0^u\int_r^u E(D_r(Y_v^n))^2 dvdrdu \\
& +4(\frac{\alpha}{1-\alpha})^2\int_0^t\int_0^u E[D_r(\sup_{0\le v\le u}(Z_v^n + X_v^n))]^2 drdu \\
\le\ & 4t\int_0^t E[\sigma(Y_r^n)^2]dr + 4K^2(t+1)\int_0^t [E\int_0^u ||D(Y_v^n)||_H^2 dv]du \\
& +4(\frac{\alpha}{1-\alpha})^2\int_0^t E||D(\sup_{0\le v\le u}(Z_v^n + X_v^n))||_H^2 du \\
\le\ & 4t\int_0^t E[\sigma(Y_r^n)^2]dr + 4K^2(t+1)\int_0^t [E\int_0^u ||D(Y_v^n)||_H^2 dv]du \\
& +4(\frac{\alpha}{1-\alpha})^2\int_0^t E[\sup_{0\le v\le u}||D(Z_v^n + X_v^n)||_H^2]du \\
<\ & \infty,
\end{aligned} \tag{2.3.21}$$

where (2.3.17) has been used. Property (**P**) is proved.

Now we prove

$$\sup_n E(||DY_t^n||_H^2) < \infty. \tag{2.3.22}$$

Note that

$$\begin{aligned} D_r(Y_t^n) &= \sigma(Y_r^{n-1}) + \int_r^t D_r(\sigma(Y_s^{n-1}))dB_s + \int_r^t D_r(b(Y_s^{n-1}))ds \\ & +\frac{\alpha}{1-\alpha} D_r[\max_{0\le s\le t}(Z_s^{n-1}+X_s^{n-1})]. \end{aligned}$$

$$\begin{aligned} & E(||DY_t^n||_H^2) \\ = & E\int_0^t |D_rY_t^n|^2 dr \\ \le & 4\{E[\int_0^t |\sigma(Y_r^{n-1})|^2 dr] + E[\int_0^t | \int_r^t D_r(\sigma(Y_s^{n-1}))dB_s|^2 dr] \\ & +E[\int_0^t | \int_r^t D_r(b(Y_s^{n-1}))ds|^2 dr] \\ & +(\frac{\alpha}{1-\alpha})^2 E\int_0^t |D_r \max_{0\le s\le t}[\int_0^s \sigma(Y_u^{n-1})dB_u + \int_0^s b(Y_u^{n-1})du]|^2 dr\} \\ \le & 4\int_0^t E|\sigma(Y_r^{n-1})|^2 dr + 4\int_0^t E(\int_r^t |D_r(\sigma(Y_s^{n-1}))|^2 ds)dr \\ & +4t\int_0^t E[\int_r^t |D_r(b(Y_s^{n-1})|^2 ds]dr \\ & +4(\frac{\alpha}{1-\alpha})^2 E[\sup_{0\le s\le t} ||D(\int_0^s \sigma(Y_u^{n-1})dB_u + \int_0^s b(Y_u^{n-1})du)||_H^2] \\ \le & 4E[\int_0^t |\sigma(Y_r^{n-1})|^2 dr] + 4K^2\int_0^t E[\int_r^t |D_r(Y_s^{n-1})|^2 ds]dr \\ & +4K^2 t\int_0^t E[\int_r^t |D_r(Y_s^{n-1})|^2 ds]dr \\ & +4(\frac{\alpha}{1-\alpha})^2\{3\int_0^t E(\sigma(Y_r^{n-1}))^2 dr \\ & +(3CK^2+3K^2t)\int_0^t\int_r^t E(D_r(Y_u^{n-1}))^2 du dr\} \\ \le & C_1\int_0^t E|\sigma(Y_r^{n-1})|^2 dr + C_2\int_0^t E||DY_u^{n-1}||_H^2 du. \end{aligned} \tag{2.3.23}$$

Where (2.3.17) and (2.3.20) were used.

Note that $A = \sup_n \int_0^t E|\sigma(Y_r^{n-1})|^2 dr \le C\sup_n \int_0^t E(1+|Y_r^{n-1}|^2)dr < \infty$, because Y_n converges to Y uniformly w.r.t time parameter from [DZ].

Iterating (2.3.23) gives $\sup_n E||DY_t^n||_H^2 < \infty$.

Thus by Theorem 1.2.2 (Lemma 1.2.3 in [N]) we deduce that $Y_t \in \mathbb{D}^{1,2}$ and $DY_t^n \to DY_t$ weakly in $L^2(\Omega; H)$. □

Theorem 2.3.2 *Assume that $\sigma(\cdot)$ and $b(\cdot)$ are Lipschitz continuous, and $|\sigma(x)| > 0$, for all $x \in R$. Then, for $t > 0$, the law of Y_t is absolutely continuous with respect to Lebesgue measure.*

PROOF. According to Theorem 2.2.1, we just need to show that $||DY_t||_H^2 > 0$ a.s..

Note that,

$$\begin{aligned} D_r Y_t &= \sigma(Y_r) + \int_r^t D_r(\sigma(Y_s))dB_s \\ &\quad + \int_r^t D_r(b(Y_s))ds + \alpha D_r(\max_{0\le s\le t} Y_s), \ r \le t \end{aligned} \tag{2.3.24}$$

Using inequality $(a+b)^2 \ge \frac{1}{2}a^2 - b^2$, we have

$$\begin{aligned} (D_r Y_t)^2 &\ge \frac{1}{2}\sigma(Y_r)^2 - [\int_r^t D_r(\sigma(Y_s))dB_s + \int_r^t D_r(b(Y_s))ds + \alpha D_r(\max_{0\le s\le t} Y_s)]^2 \\ &\ge \frac{1}{2}\sigma(Y_r)^2 - 3\{[\int_r^t D_r(\sigma(Y_s))dB_s]^2 + [\int_r^t D_r(b(Y_s))ds]^2 \\ &\quad + \alpha^2[D_r(\max_{0\le s\le t} Y_s)]^2\}. \end{aligned}$$

Since $\sigma(Y_r)^2$ is continuous w.r.t r, it follows that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t-\epsilon}^t \sigma(Y_r)^2 dr = \sigma(Y_t)^2. \tag{2.3.25}$$

Now,

$$\begin{aligned} & E\{\int_{t-\epsilon}^t [\int_r^t D_r(b(Y_s))ds]^2 dr\} \\ \le\ & \int_{t-\epsilon}^t E[\int_r^t |D_r(b(Y_s))|^2 ds(t-r)]dr \\ \le\ & K^2 \int_{t-\epsilon}^t \int_r^t E|D_r(Y_s)|^2(t-r)dsdr \\ \le\ & K^2\epsilon \int_{t-\epsilon}^t \int_r^t E|D_r Y_s|^2 dsdr \\ =\ & K^2\epsilon \int_{t-\epsilon}^t \int_{t-\epsilon}^s E|D_r Y_s|^2 drds \\ \le\ & K^2 M\epsilon^2, \end{aligned}$$

where we used Holder inequality in the fist inequality, used (2.3.10) in the second inequality and we used $E||DY_s||_H^2 < \infty$ and its continuity in the last inequality.

$$\begin{aligned} E\int_{t-\epsilon}^{t}[\int_{r}^{t} D_r(\sigma(Y_s))dB_s]^2dr &\leq \int_{t-\epsilon}^{t} E[\int_{r}^{t} D_r(\sigma(Y_s))^2ds]dr \\ &\leq K^2\int_{t-\epsilon}^{t} E[\int_{r}^{t}(D_rY_s)^2ds]dr \\ &\leq K^2\int_{t-\epsilon}^{t}\int_{t-\epsilon}^{s} E(D_rY_s)^2drds \\ &\leq K^2\int_{t-\epsilon}^{t}\int_{s-\epsilon}^{s} E(D_rY_s)^2drds. \end{aligned}$$

Next we show that $\int_{s-\epsilon}^{s} E[(D_rY_s)]^2dr \leq C\epsilon$, where C is independent of s. Because

$$\begin{aligned} D_rY_s^n &= \sigma(Y_r^{n-1}) + \int_r^s D_r(\sigma(Y_u^{n-1}))dB_u \\ &\quad + \int_r^s D_r(b(Y_u^{n-1}))du + \frac{\alpha}{1-\alpha}D_r[\max_{0\leq u\leq s}(Z_u^{n-1}+X_u^{n-1})], \end{aligned}$$

we have,

$$\begin{aligned} & E\int_{s-\epsilon}^{s}(D_rY_s^n)^2dr \\ \leq\ & 4E\int_{s-\epsilon}^{s}\sigma(Y_r^{n-1})^2dr + 4E\int_{s-\epsilon}^{s}[\int_r^s D_r(\sigma(Y_u^{n-1}))dB_u]^2dr \\ & +4E\int_{s-\epsilon}^{s}[\int_r^s D_rb(Y_u^{n-1})du]^2dr \\ & +4(\frac{\alpha}{1-\alpha})^2E\int_{s-\epsilon}^{s}(D_r(\max_{0\leq u\leq s}(Z_u^{n-1}+X_u^{n-1}))^2dr \qquad (2.3.26) \\ \leq\ & 4\int_{s-\epsilon}^{s}E[\sigma(Y_r^{n-1})^2]dr + 4\int_{s-\epsilon}^{s}E\int_r^s D_r(\sigma(Y_u^{n-1}))^2dudr \\ & +4E\int_{s-\epsilon}^{s}[\int_r^s D_r(b(Y_u^{n-1}))^2du(s-r)]dr \\ & +4(\frac{\alpha}{1-\alpha})^2E\sup_{0\leq u\leq s}\int_{s-\epsilon}^{s}[D_r(Z_u^{n-1}+X_u^{n-1})]^2dr \qquad (2.3.27) \\ \leq\ & 4\int_{s-\epsilon}^{s}E[\sigma(Y_r^{n-1})^2]dr + 4K^2\int_{s-\epsilon}^{s}\int_r^s E(D_r(Y_u^{n-1}))^2dudr \\ & +4K^2\epsilon\int_{s-\epsilon}^{s}\int_r^s E[D_r(Y_u^{n-1})]^2dudr \\ & +4(\frac{\alpha}{1-\alpha})^2E\sup_{0\leq u\leq s}\int_{s-\epsilon}^{s}[D_r(\int_0^u\sigma(Y_v^{n-1})dB_v + \int_0^u b(Y_v^{n-1})dv)]^2dr \\ \leq\ & 4\int_{s-\epsilon}^{s}E[\sigma(Y_r^{n-1})^2]dr + (4K^2+4K^2\epsilon)\int_{s-\epsilon}^{s}\int_r^s E(D_rY_u^{n-1})^2dudr \\ & +4(\frac{\alpha}{1-\alpha})^2\{3\int_{s-\epsilon}^{s}E[\sigma(Y_r^{n-1})^2]dr \end{aligned}$$

$$
\begin{aligned}
&+(3C_1K^2+3K^2\epsilon)\int_{s-\epsilon}^{s}\int_{r}^{s}E(D_rY_v^{n-1})^2dvdr\} \\
= \;& (4+12(\frac{\alpha}{1-\alpha})^2)\int_{s-\epsilon}^{s}E[\sigma(Y_r^{n-1})^2]dr \\
&+[4K^2+4K^2\epsilon+4(\frac{\alpha}{1-\alpha})^2(3C_1K^2+3K^2\epsilon)] \\
&\cdot\int_{s-\epsilon}^{s}\int_{r}^{s}E(D_rY_v^{n-1})^2dvdr \\
= \;& C'\int_{s-\epsilon}^{s}E[\sigma(Y_r^{n-1})^2]dr+C''\int_{s-\epsilon}^{s}\int_{s-\epsilon}^{v}E(D_rY_v^{n-1})drdv \\
\leq \;& C'\int_{s-\epsilon}^{s}E[\sigma(Y_r^{n-1})^2]dr+C''\int_{s-\epsilon}^{s}\int_{v-\epsilon}^{v}E(D_r(Y_v^{n-1}))^2drdv,
\end{aligned}
$$

where we have used Theorem 1.2.2 (Proposition 2.1.10 in [N]) from (2.3.26) to (2.3.27).

Let $\phi_n(s)=E\int_{s-\epsilon}^{s}(D_rY_s^n)^2dr$, and $\phi(s)=E\int_{s-\epsilon}^{s}(D_rY_s)^2dr$, then $\phi_n(s)\leq C^*\epsilon+C''\int_{s-\epsilon}^{s}\phi_{n-1}(v)dv$. Iterating it, we get

$$
\begin{aligned}
\phi_n(s) &\leq C^*\epsilon(1+C''\epsilon+(C''\epsilon)^2+...+(C''\epsilon)^n) && (2.3.28)\\
&\leq C^*\epsilon\frac{1}{1-C''\epsilon} && (2.3.29)\\
&\leq 2C^*\epsilon, && (2.3.30)
\end{aligned}
$$

when ϵ is sufficiently small. Then by Fatou lemma:

$$
\phi(s)\leq\liminf_{n\to\infty}\phi_n(s)\leq 2C^*\epsilon. \tag{2.3.31}
$$

So

$$
E\int_{t-\epsilon}^{t}[\int_{r}^{t}D_r(\sigma(Y_s))dB_s]^2dr\leq K^2\int_{t-\epsilon}^{t}\phi(s)ds\leq 2C^*K^2\epsilon^2. \tag{2.3.32}
$$

Therefore

$$
\lim_{\epsilon\to 0}\frac{1}{\epsilon}E\{\int_{t-\epsilon}^{t}([\int_{r}^{t}D_r(\sigma(Y_s))dB_s]^2+[\int_{r}^{t}D_r(b(Y_s))ds]^2)dr\}=0.
$$

Hence, there exists $\epsilon_n\downarrow 0$, such that

$$
\lim_{\epsilon_n\to 0}\frac{1}{\epsilon_n}\int_{t-\epsilon_n}^{t}([\int_{r}^{t}D_r(\sigma(Y_s))dB_s]^2+[\int_{r}^{t}D_r(b(Y_s))ds]^2)dr=0\ a.s.. \tag{2.3.33}
$$

Set

$$A_n = \{\omega : \max_{0\le s\le t} Y_s(w) = \max_{0\le s\le t-\epsilon_n} Y_s(\omega)\},$$

and

$$A = \{\max_{0\le s\le t} Y_s = Y_t\}.$$

It is clear that $\Omega = \bigcup_{m=1}^{\infty} A_m \bigcup A$.

For $\omega \in A_m$, $\forall n > m$, we have

$$\int_{t-\epsilon_n}^{t} \alpha^2 D_r(\max_{0\le s\le t-\epsilon_m} Y_s(\omega))^2 dr = 0.$$

By the local property of the Malliavin derivative (Proposition 1.3.16 in [N]) on A_m, we have

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{\epsilon_n} \int_{t-\epsilon_n}^{t} \alpha^2 (D_r(\max_{0\le s\le t} Y_s(\omega))^2 dr \\ = & \lim_{n\to\infty} \frac{1}{\epsilon_n} \int_{t-\epsilon_n}^{t} \alpha^2 (D_r(\max_{0\le s\le t-\epsilon_m} Y_s(\omega))^2 dr \\ = & \ 0. \end{aligned} \tag{2.3.34}$$

Since m is arbitrary, by (2.3.33) and (2.3.34), we conclude that

$$\lim_{\epsilon_n\to 0} \frac{1}{\epsilon_n} \int_{t-\epsilon_n}^{t} (D_r Y_t)^2 dr \ge \frac{1}{2}\sigma(Y_t)^2 > 0 \ a.s. \text{ on } \bigcup_{m=1}^{\infty} A_m.$$

For $\omega \in A$, according to (2.3.24), we have

$$(1-\alpha) D_r Y_t = \sigma(Y_r) + \int_r^t D_r(\sigma(Y_s)) dB_s + \int_r^t D_r(b(Y_s)) ds,$$

$$(1-\alpha)^2 (D_r Y_t)^2 \ge \frac{1}{2}\sigma(Y_r)^2 - [\int_r^t D_r(\sigma(Y_s)) dB_s + \int_r^t D_r(b(Y_s)) ds]^2,$$

Since $\alpha < 1$,

$$(D_r Y_t)^2 \ge \frac{1}{2(1-\alpha)^2}\sigma(Y_r)^2 - \frac{1}{(1-\alpha)^2}[\int_r^t D_r(\sigma(Y_s)) dB_s + \int_r^t D_r(b(Y_s)) ds]^2.$$

Here on A,

$$\lim_{\epsilon_n\to 0} \frac{1}{\epsilon_n} \int_{t-\epsilon_n}^{t} (D_r Y_t)^2 dr \ \ge \ \frac{1}{2(1-\alpha)^2} \lim_{\epsilon_n\to 0} \frac{1}{\epsilon_n} \int_{t-\epsilon_n}^{t} \sigma(Y_r)^2 dr$$

$$= \frac{\sigma(Y_t)^2}{2(1-\alpha)^2} > 0. \tag{2.3.35}$$

Therefore

$$||DY_t||_H^2 = \int_0^t (D_r Y_t)^2 dr > 0 \quad a.s.. \tag{2.3.36}$$

By Theorem 2.2.1, we conclude that the law of Y_t is absolutely continuous with respect to Lebesgue measure. □

2.4 Absolute continuity of the laws of perturbed reflected diffusion processes

Consider the reflected, perturbed stochastic differential equation:

$$X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \alpha \max_{0\le s\le t} X_s + L_t. \tag{2.4.1}$$

Definition 2.4.1 *We say that $(X_t, L_t, t \ge 0)$ is a solution to the equation (2.4.1) if*

(i) $X_0 = 0, X_t \ge 0$ for $t \ge 0$, a.s.

(ii) X_t, L_t are continuous and adapted to the filtration of B,

(iii) L_t is non-decreasing with $L_0 = 0$ and $\int_0^t \chi\{X_s = 0\}dL_s = L_t$,

(iv) $(X_t, L_t, t \ge 0)$ satisfies the equation (2.4.1) almost surely for every $t > 0$.

we need the following lemma which strengthens the result of Theorem 1.2.3 (Proposition 2.1.10 in [N]).

Lemma 2.4.1 *Let $X = \{X_s, 0 \le s \le t\}$ be a continuous process. Suppose that*
(i) $E(\sup_{0\le s\le t} X_s^2) < \infty$,
(ii) for any $0 \le s \le t, X_s \in \mathbb{D}^{1,2}$ and $E(\sup_{0\le s\le t} ||DX_s||_H^2) < \infty$,
Then the random variable $M_t = \sup_{0\le s\le t} X_s$ belongs to $\mathbb{D}^{1,2}$ and moreover,
$||DM_t||_H^2 \le \sup_{0\le s\le t} ||DX_s||_H^2$ a.s..

PROOF. Consider a countable and dense subset $S_0 = \{t_n, n \ge 1\}$ of $[0,t]$. Define $M_n = \sup\{X_{t_1}, ..., X_{t_n}\}$. The function $\varphi_n : \mathbb{R}^n \to \mathbb{R}$ defined by

$\varphi_n(x_1,...,x_n) = max\{x_1,...,x_n\}$ is Lipschitz. Therefore, we deduce that M_n belongs to $\mathbb{D}^{1,2}$. Moreover, the sequence M_n converges in $L^2(\Omega)$ to M_t. In order to evaluate the Malliavin derivative of M_n, we introduce the following sets:

$$A_1 = \{M_n = X_{t_1}\},$$

......

$$A_k = \{M_n \neq X_{t_1},...,M_n \neq X_{t_{k-1}}, M_n = X_{t_k}\}, \quad 2 \leq k \leq n. \tag{2.4.2}$$

By the local property of the operator D, on the set A_k, the derivatives of the random variables M_n and X_{t_k} coincide. Hence, we can write

$$DM_n = \Sigma_{k=1}^n I_{A_k} DX_{t_k} \tag{2.4.3}$$

Consequently,

$$E(||DM_n||_H^2) \leq E(\sup_{0\leq s\leq t} ||DX_s||_H^2) < \infty \tag{2.4.4}$$

Then, $M_t = \sup_{0\leq s\leq t} X_s$ belongs to $\mathbb{D}^{1,2}$ and DM_n weakly converges to DM_t in $L^2(\Omega, P; H)$.

Now we want to show that

$$||DM_t||_H^2 \leq \sup_{0\leq s\leq t} ||DX_s||_H^2 \ \ a.e.. \tag{2.4.5}$$

According to Lemma 1.2.4 (Theorem 1.6.11 in [A]), it is equivalent to prove that for every non-negative bounded random variable ξ,

$$E[||DM_t||_H^2 \xi] \leq E[\sup_{0\leq s\leq t} ||DX_s||_H^2 \xi], \tag{2.4.6}$$

$$i.e. \int_\Omega ||DM_t||_H^2 \xi dP \leq \int_\Omega \sup_{0\leq s\leq t} ||DX_s||_H^2 \xi dP. \tag{2.4.7}$$

Define $\mu(A) = \int_A \xi dP, \ \ \forall A \in \mathcal{F}$, then (2.4.6) is equivalent to

$$\int_\Omega [||DM_t||_H^2] d\mu \leq \int_\Omega [\sup_{0\leq s\leq t} ||DX_s||_H^2] d\mu. \tag{2.4.8}$$

For $h \in L^2(\Omega, \mu; H)$, because ξ is bounded, $\xi h \in L^2(\Omega, P; H)$.

Consequently, by the weak convergence of DM_n,

$$\int_\Omega [(DM_n, h)_H] d\mu \quad = \quad \int_\Omega (DM_n, \xi h)_H dP$$

$$\begin{aligned} &\longrightarrow \int_\Omega (DM_t, \xi h)_H dP \\ &= \int_\Omega (DM_t, h) d\mu. \end{aligned}$$

This shows that $DM_n \to DM_t$ weakly in $L^2(\Omega, \mu; H)$.

Hence, we have

$$\int_\Omega (||DM_t||_H^2) d\mu \le \liminf_{n\to\infty} \int_\Omega (||DM_n||_H^2) d\mu \le \int_\Omega (\sup_{0\le s\le t} ||DX_s||_H^2) d\mu < \infty. \ \square$$

Theorem 2.4.1 *Assume $0 \le \alpha < \frac{1}{2}$. Let $(X_t, L_t, t \ge 0)$ be the unique solution to the equation (2.4.1). Then X_t belongs to $\mathbb{D}^{1,2}$ for any $t \ge 0$.*

PROOF. Consider the Picard iteration, $X_t^0 = 0$, $\forall t \in [0, T]$, $T \ge 0$, and let (X_t^{n+1}, L_t^{n+1}) be the unique solution to the following reflected equation:

$$X_t^{n+1} = \int_0^t \sigma(X_s^n) dB_s + \int_0^t b(X_s^n) ds + \alpha \max_{0\le s\le t} X_s^n + L_t^{n+1}. \tag{2.4.9}$$

By Skorohod problem:

$$L_t^{n+1} = -\inf_{s\le t}\{(\int_0^s \sigma(X_u^n) dB_u + \int_0^s b(X_u^n) du + \alpha \max_{0\le u\le s} X_u^n) \wedge 0\}. \tag{2.4.10}$$

It was shown in [DZ], there exists a unique solution X_t to (2.4.1). Next we are going to show that

$$\lim_{n\to\infty} E[\sup_{0\le s\le t} |X_s^n - X_s|^2] = 0. \tag{2.4.11}$$

Now Eqs(2.4.1) and (2.4.9) imply the following:

$$\begin{aligned} |X_t^{n+1} - X_t| \ &\le \ |\int_0^t (\sigma(X_s^n) - \sigma(X_s)) dB_s| + 2\alpha \max_{0\le s\le t} |X_s^n - X_s|. \\ &+ |\int_0^t (b(X_s^n) - b(X_s)) ds| + \max_{0\le s\le t} |\int_0^s (\sigma(X_u^n) - \sigma(X_u)) dB_u| \\ &+ \max_{0\le s\le t} |\int_0^s (b(X_u^n) - b(X_u)) du|, \end{aligned}$$

where we have used the fact:

$$L_t = -\inf_{0\le s\le t}\{(\int_0^s \sigma(X_u) dB_u + \int_0^s b(X_u) du + \alpha \max_{0\le u\le s} X_u) \wedge 0\}. \tag{2.4.12}$$

Consequently,

$$\begin{aligned}\max_{0\le s\le t}|X_s^{n+1}-X_s| &\le 2\max_{0\le s\le t}|\int_0^s(\sigma(X_u^n)-\sigma(X_u))dB_u|\\ &+2\max_{0\le s\le t}|\int_0^s(b(X_u^n)-b(X_u))du|+2\alpha\max_{0\le s\le t}|X_s^n-X_s|.\end{aligned}\tag{2.4.13}$$

For any $\epsilon>0$, using the elementary inequality, $(a+b)^2\le(1+C_\epsilon)a^2+(1+\epsilon)b^2$, we obtain

$$\begin{aligned}\max_{0\le s\le t}|X_s^{n+1}-X_s|^2 &\le 4(1+C_\epsilon)[\max_{0\le s\le t}|\int_0^s(\sigma(X_u^n)-\sigma(X_u))dB_u|\\ &+\max_{0\le s\le t}|\int_0^s(b(X_u^n)-b(X_u))du|]^2\\ &+(1+\epsilon)(2\alpha)^2\max_{0\le s\le t}|X_s^n-X_s|^2\\ &\le 8(1+C_\epsilon)[\max_{0\le s\le t}|\int_0^s(\sigma(X_u^n)-\sigma(X_u))dB_u|^2\\ &+\max_{0\le s\le t}|\int_0^s(b(X_u^n)-b(X_u))du|^2]\\ &+(1+\epsilon)(2\alpha)^2\max_{0\le s\le t}|X_s^n-X_s|^2.\end{aligned}$$

By Burkhölder inequality,

$$\begin{aligned}E[\max_{0\le s\le t}|X_s^{n+1}-X_s|^2] &\le 8(1+C_\epsilon)\{E[\max_{0\le s\le t}|\int_0^s(\sigma(X_u^n)-\sigma(X_u))dB_u|^2]\\ &+E[\max_{0\le s\le t}|\int_0^s(b(X_u^n)-b(X_u))du|^2]\}\\ &+(1+\epsilon)(2\alpha)^2E[\max_{0\le s\le t}|X_s^n-X_s|^2]\\ &\le 8(1+C_\epsilon)(K_1C^2+TC^2)E[\int_0^t|X_u^n-X_u|^2du]\\ &+(1+\epsilon)(2\alpha)^2E[\max_{0\le s\le t}|X_s^n-X_s|^2].\end{aligned}$$

Let $g_{n+1}(t)=E[\max_{0\le s\le t}|X_s^{n+1}-X_s|^2]$. The above inequality implies

$$g_{n+1}(t)\le 8(K_1C^2+TC^2)(1+C_\epsilon)\int_0^t g_n(s)ds+(1+\epsilon)(2\alpha)^2g_n(t).\tag{2.4.14}$$

Summing the above equations from 1 to M:

$$\sum_{n=1}^M g_{n+1}(t) \le 8(K_1C^2+TC^2)(1+C_\epsilon)\int_0^t\sum_{n=1}^M g_n(s)ds$$

$$+(1+\epsilon)(2\alpha)^2 \sum_{n=1}^{M} g_n(t). \tag{2.4.15}$$

And then,

$$\sum_{n=1}^{M} g_n(t) - g_1(t) \leq \sum_{n=1}^{M} g_{n+1}(t) \tag{2.4.16}$$

$$\leq C^* \int_0^t \sum_{n=1}^{M} g_n(s)ds + \beta \sum_{n=1}^{M} g_n(t), \tag{2.4.17}$$

where $\beta = (1+\epsilon)(2\alpha)^2$, C^* is a constant. Choose $\epsilon > 0$ sufficiently small so that $\beta = (1+\epsilon)(2\alpha)^2 < 1$.

It follows from (2.4.17) that

$$(1-\beta)\sum_{n=1}^{M} g_n(t) \leq g_1(t) + C^* \int_0^t \sum_{n=1}^{M} g_n(s)ds. \tag{2.4.18}$$

By Gronwall inequality,

$$\sum_{n=1}^{M} g_n(t) \leq \frac{g_1(T)}{1-\beta} e^{\frac{C^*}{1-\beta}T}. \tag{2.4.19}$$

Let $M \to \infty$ to get

$$\sum_{n=1}^{\infty} E[\max_{0\leq s\leq t} |X_s^n - X_s|^2] < \infty. \tag{2.4.20}$$

which yields that X_t^n converges to X_t in $L^2(\Omega)$.

Let $Y_s^n = \max_{0\leq u\leq s} X_u^n$. We will prove the following property by induction on n.

(**P**). $X_t^n \in \mathbb{D}^{1,2}$, $E(\max_{0\leq s\leq t} ||DX_s^n||_H^2) < \infty$, $E(\max_{0\leq s\leq t} ||DY_s^n||_H^2) < \infty$.

Clearly, (**P**) holds for $n = 0$.

Suppose (**P**)holds for n. We prove that it is valid for $n+1$.

Now we note that

$$\int_0^t \sigma(X_s^n)dB_s \in \mathbb{D}^{1,2}, \int_0^t b(X_s^n)ds \in \mathbb{D}^{1,2}, \tag{2.4.21}$$

and

$$D_r(\int_0^t \sigma(X_s^n)dB_s) = \sigma(X_r^n) + \int_r^t D_r(\sigma(X_s^n))dB_s,$$

$$D_r(\int_0^t b(X_s^n)ds) = \int_r^t D_r(b(X_s^n))ds.$$

Next we prove $\max_{0\le s\le t} X_s^n \in \mathbb{D}^{1,2}$.

As

$$\begin{aligned}\max_{0\le s\le t}|X_s^n| &\le 2\max_{0\le s\le t}|\int_0^s \sigma(X_u^{n-1})dB_u| + 2\alpha\max_{0\le s\le t}|X_s^{n-1}|\\ &\quad +2\max_{0\le s\le t}|\int_0^s b(X_u^{n-1})du|,\end{aligned}$$

we get

$$\begin{aligned}E(\max_{0\le s\le t}|X_s^n|^2) &\le 12E[\max_{0\le s\le t}|\int_0^s \sigma(X_u^{n-1})dB_u|^2] + 12\alpha^2 E(\max_{0\le s\le t}|X_s^{n-1}|^2)\\ &\quad +12E[\max_{0\le s\le t}|\int_0^s b(X_u^{n-1})du|^2]\\ &\le 12K_1E[\int_0^t \sigma(X_u^{n-1})^2du] + 12\alpha^2E(\max_{0\le s\le t}|X_s^{n-1}|^2)\\ &\quad +12TE[\int_0^t b(X_u^{n-1})^2du]\\ &\le 12(K_1+T)C^2[E\int_0^t(1+(X_u^{n-1})^2)du] + 12\alpha^2E(\max_{0\le s\le t}|X_s^{n-1}|^2)\\ &\le 12(K_1+T)C^2T + [12(K_1+T)C^2T + 12\alpha^2]E(\max_{0\le s\le t}|X_s^{n-1}|^2).\end{aligned}$$

By interation, we see that

$$E(\max_{0\le s\le t}|X_s^n|^2) < \infty. \tag{2.4.22}$$

By the induction hypothesis $E(\max_{0\le s\le t}||DX_s^n||_H^2) < \infty$ and Proposition 2.1.10 in [N], it follows that

$$\max_{0\le s\le t} X_s^n \in \mathbb{D}^{1,2}. \tag{2.4.23}$$

Now we want to show

$$L_t^{n+1} = \max_{0\le s\le t}\{-(\int_0^s \sigma(X_u^n)dB_u + \int_0^s b(X_u^n)du + \alpha\max_{0\le u\le s} X_u^n) \vee 0\} \in \mathbb{D}^{1,2}. \tag{2.4.24}$$

Let

$$V_s^n := -(\int_0^s \sigma(X_u^n)dB_u + \int_0^s b(X_u^n)du + \alpha\max_{0\le u\le s} X_u^n) \vee 0. \tag{2.4.25}$$

Firstly, $V_s^n \in \mathbb{D}^{1,2}$ by (2.4.21) and (2.4.23). Secondly,

$$E(\max_{0\le s\le t}(V_s^n)^2) \le 3E[\max_{0\le s\le t}(\int_0^s \sigma(X_u^n)dB_u)^2] + 3\alpha^2E[\max_{0\le s\le t}(X_s^n)^2]$$

$$
\begin{aligned}
& +3E[\max_{0\le s\le t}(\int_0^s b(X_u^n)du)^2] \\
= \; & 3K_1 E[\int_0^t \sigma(X_u^n)^2 du] + 3\alpha^2 E[\max_{0\le s\le t}(X_s^n)^2] + 3TE[\int_0^t b(X_s^n)^2 ds] \\
\le \; & 3(K_1+T)C^2 E\int_0^t (1+(X_u^n)^2)du + 3\alpha^2 E[\max_{0\le s\le t}(X_s^n)^2] \\
\le \; & 3C^2(K_1+T)(T+TE[\max_{0\le s\le t}(X_s^n)^2]) + 3\alpha^2 E[\max_{0\le s\le t}(X_s^n)^2] \\
= \; & 3C^2T(K_1+T) + [3C^2(K_1+T)T + 3\alpha^2]E[\max_{0\le s\le t}(X_s^n)^2] \\
< \; & \infty. \qquad (2.4.26)
\end{aligned}
$$

Thirdly,

$$
\begin{aligned}
E(\max_{0\le s\le t}||DV_s^n||_H^2) \; = \; & E(\max_{0\le s\le t}\int_0^s (D_r(V_s^n))^2 dr) \\
& \le 3(1+C_\epsilon)E\int_0^t \sigma(X_r^n)^2 dr \\
& +3(1+C_\epsilon)E[\max_{0\le s\le t}\int_0^s (\int_r^s D_r(\sigma(X_u^n))dB_u)^2 dr] \\
& +3(1+C_\epsilon)E[\max_{0\le s\le t}\int_0^s (\int_r^s D_r(b(X_u^n))du)^2 dr] \\
& +(1+\epsilon)\alpha^2 E[\max_{0\le s\le t}\int_0^s (D_r(Y_s^n))^2 dr] \\
\le \; & 3(1+C_\epsilon)E\int_0^t \sigma(X_r^n)^2 dr \\
& +3(1+C_\epsilon)\int_0^t [E\int_r^t (D_r(\sigma(X_u^n)))^2 du]dr \\
& +3(1+C_\epsilon)t\int_0^t E[\int_r^t (D_r b(X_u^n))^2 du]dr \\
& +(1+\epsilon)\alpha^2 E[\max_{0\le s\le t}||DY_s^n||_H^2] \\
\le \; & 3(1+C_\epsilon)E\int_0^t \sigma(X_r^n)^2 dr \\
& +3(1+C_\epsilon)(1+t)K^2\int_0^t E||DX_u^n||_H^2 du \\
& +(1+\epsilon)\alpha^2 E[\max_{0\le s\le t}||DY_s^n||_H^2] \\
< \; & \infty. \qquad (2.4.27)
\end{aligned}
$$

By Proposition 2.1.10 in [N], (2.4.26) and (2.4.27) yield that $L_t^{n+1} \in \mathbb{D}^{1,2}$. Thus, we conclude $X_t^{n+1} \in \mathbb{D}^{1,2}$.

Moreover,

$$\begin{aligned} D_r(X_s^{n+1}) &= \sigma(X_r^n) + \int_r^s D_r(\sigma(X_u^n))dB_u + \int_r^s D_r(b(X_u^n))du \\ &\quad +\alpha D_r(\max_{0\le u\le s} X_u^n) + D_r(L_s^{n+1}), \end{aligned}$$

and

$$\begin{aligned} ||D(X_s^{n+1})||_H^2 &= \int_0^s (D_r(X_s^{n+1}))^2 dr \\ &\le 5\int_0^s \sigma(X_r^n)^2 dr + 5\int_0^s [\int_r^s D_r(\sigma(X_u^n))dB_u]^2 dr \\ &\quad +5\int_0^s [\int_r^s (D_r(b(X_u^n)))du]^2 dr \\ &\quad +5\alpha^2 ||DY_s^n||_H^2 + 5||DL_s^{n+1}||_H^2. \end{aligned} \tag{2.4.28}$$

So

$$\begin{aligned} E[\max_{0\le s\le t} ||DX_s^{n+1}||_H^2] &\le 5E[\int_0^t \sigma(X_r^n)^2 dr] + 5E\int_0^t \max_{0\le s\le t}[\int_r^s D_r(\sigma(X_u^n))dB_u]^2 dr \\ &\quad +5TE\int_0^t\int_r^t (D_r(b(X_u^n)))^2 du dr + 5\alpha^2 E[\max_{0\le s\le t} ||DY_s^n||_H^2] \\ &\quad +5E[\max_{0\le s\le t} ||DL_s^{n+1}||_H^2] \\ &\le 5E\int_0^t \sigma(X_r^n)^2 dr + 5K_1\int_0^t E\int_r^t (D_r(\sigma(X_u^n)))^2 du dr \\ &\quad +5TE\int_0^t\int_r^t (D_r(b(X_u^n)))^2 du dr + 5\alpha^2 E[\max_{0\le s\le t} ||DY_s^n||_H^2] \\ &\quad +5E[\max_{0\le s\le t} ||DL_s^{n+1}||_H^2] \\ &\le 5E\int_0^t \sigma(X_r^n)^2 dr + (5K_1K^2 + 5TK^2)\int_0^t E||DX_u^n||_H^2 du \\ &\quad +5\alpha^2 E[\max_{0\le s\le t} ||DY_s^n||_H^2] + 5E[\max_{0\le s\le t} ||DL_s^{n+1}||_H^2]. \end{aligned}$$

To prove

$$E[\max_{0\le s\le t} ||DX_s^{n+1}||_H^2] < \infty, \tag{2.4.29}$$

we only need to prove

$$E[\max_{0\le s\le t} ||DL_s^{n+1}||_H^2] < \infty. \tag{2.4.30}$$

According to Lemma 4.1,

$$||DL_s^{n+1}||_H^2 \le \sup_{0\le u\le s} ||DV_u^n||_H^2. \tag{2.4.31}$$

Thus we have

$$\max_{0\le s\le t}||DL_s^{n+1}||_H^2 \le \max_{0\le s\le t}(\sup_{0\le u\le s}||DV_u^n||_H^2) = \max_{0\le s\le t}||DV_s^n||_H^2, \tag{2.4.32}$$

and by (2.4.27),

$$E[\max_{0\le s\le t}||DL_s^{n+1}||_H^2] \le E[\sup_{0\le s\le t}||DV_s^n||_H^2] < \infty. \tag{2.4.33}$$

Again by Lemma 4.1,

$$||DY_s^{n+1}||_H^2 \le \sup_{0\le u\le s}||DX_u^{n+1}||_H^2. \tag{2.4.34}$$

Hence,

$$E[\max_{0\le s\le t}||DY_s^{n+1}||_H^2] \le E[\sup_{0\le s\le t}||DX_s^{n+1}||_H^2] < \infty. \tag{2.4.35}$$

We've proved property (**P**).

Next we prove

$$\sup_n E||DX_t^{n+1}||_H^2 < \infty. \tag{2.4.36}$$

Because, for any $\epsilon > 0$,

$$\begin{aligned} |D_r X_s^{n+1}|^2 \le\; & (1+C_\epsilon)[3\sigma(X_r^n)^2 + 3(\int_r^s D_r(\sigma(X_u^n))dB_u)^2 + 3(\int_r^s D_r(b(X_u^n))du)^2] \\ & +(1+\epsilon)[2\alpha^2 D_r(\max_{0\le u\le s} X_u^n)^2 + 2D_r(L_s^{n+1})^2]. \end{aligned}$$

We have

$$\begin{aligned} ||DX_s^{n+1}||_H^2 \le\; & 3(1+C_\epsilon)\int_0^s \sigma(X_r^n)^2 dr + 3(1+C_\epsilon)\int_0^s [\int_r^s D_r(\sigma(X_u^n))dB_u]^2 dr \\ & +3(1+C_\epsilon)\int_0^s [\int_r^s D_r(b(X_u^n))du]^2 dr \\ & +2(1+\epsilon)\alpha^2 \int_0^s D_r(\max_{0\le u\le s} X_u^n)^2 dr + 2(1+\epsilon)\int_0^s D_r(L_s^{n+1})^2 dr \\ =\; & 3(1+C_\epsilon)\int_0^s \sigma(X_r^n)^2 dr + 3(1+C_\epsilon)\int_0^s [\int_r^s D_r(\sigma(X_u^n))dB_u]^2 dr \\ & +3(1+C_\epsilon)\int_0^s [\int_r^s D_r(b(X_u^n))du]^2 dr \\ & +2(1+\epsilon)\alpha^2 ||DY_s^n||_H^2 + 2(1+\epsilon)||DL_s^{n+1}||_H^2 \\ \le\; & 3(1+C_\epsilon)\int_0^s \sigma(X_r^n)^2 dr + 3(1+C_\epsilon)\int_0^s [\int_r^s D_r\sigma(X_u^n)dB_u]^2 dr \end{aligned}$$

$$+3(1+C_\epsilon)\int_0^s[\int_r^s D_r(b(X_u^n))du]^2dr$$
$$+2(1+\epsilon)\alpha^2\sup_{0\le u\le s}||DX_u^n||_H^2+2(1+\epsilon)\sup_{0\le u\le s}||DV_u^n||_H^2,$$

where Lemma 4.1 was used in the last step. Hence, using Ito's Isometry we have

$$\begin{aligned}
&E(\sup_{0\le s\le t}||DX_s^{n+1}||_H^2)\\
\le\ & 3(1+C_\epsilon)E\int_0^t\sigma(X_r^n)^2dr+3K_1K^2(1+C_\epsilon)\int_0^t E\int_r^t(D_r(X_u^n))^2dudr\\
&+3TK^2(1+C_\epsilon)\int_0^t E\int_r^t(D_r(X_u^n))^2dudr\\
&+2(1+\epsilon)\alpha^2E[\sup_{0\le u\le t}||DX_u^n||_H^2]+2(1+\epsilon)E[\sup_{0\le u\le t}||DV_u^n||_H^2]\\
=\ & 3(1+C_\epsilon)E\int_0^t\sigma(X_r^n)^2dr+3(K_1+T)K^2(1+C_\epsilon)\int_0^t E||DX_u^n||_H^2du\\
&+2(1+\epsilon)\alpha^2E\sup_{0\le s\le t}||DX_s^n||_H^2+2(1+\epsilon)\{3(1+C_\epsilon)E\int_0^t\sigma(X_r^n)^2dr\\
&+6(1+C_\epsilon)K^2\int_0^t E||DX_u^n||_H^2du+(1+\epsilon)\alpha^2E[\sup_{0\le s\le t}||DY_s^n||_H^2]\}\\
\le\ & 3(1+C_\epsilon)\int_0^t E[\sigma(X_r^n)^2]dr+3(K_1+T)K^2(1+C_\epsilon)\int_0^t E||DX_u^n||_H^2du\\
&+2(1+\epsilon)\alpha^2E[\sup_{0\le s\le t}||DX_s^n||_H^2]+6(1+\epsilon)(1+C_\epsilon)E\int_0^t\sigma(X_r^n)^2dr\\
&+12(1+\epsilon)(1+C_\epsilon)K^2\int_0^t E||DX_u^n||_H^2du\\
&+(1+\epsilon)^2\alpha^2E[\sup_{0\le s\le t}||DX_s^n||_H^2]\\
=\ & (9+6\epsilon)(1+C_\epsilon)\int_0^t E[\sigma(X_r^n)^2]dr\\
&+2(2+\epsilon)(1+\epsilon)\alpha^2E(\sup_{0\le s\le t}||DX_s^n||_H^2)\\
&+[12K^2(1+\epsilon)(1+C_\epsilon)+3K^2(K_1+T)(1+C_\epsilon)]\int_0^t E||DX_u^n||_H^2du.
\end{aligned}\tag{2.4.37}$$

Note that $\sup_n\int_0^t E[\sigma(X_r^n)^2]dr\le C\sup_n\int_0^t E(1+|X_r^n|^2)dr<\infty$.

Let

$$\psi_n(t)=E(\sup_{0\le s\le t}||DX_s^n||_H^2).$$

Then from (2.4.37), we have

$$\psi_{n+1}(t) \leq c_1 + c_2\psi_n(t) + c_3 \int_0^t \psi_n(u)du,$$

where $c_2 = 2(2+\epsilon)(1+\epsilon)\alpha^2 < 1$ when $\epsilon > 0$ is sufficiently small, according to $\alpha < \frac{1}{2}$.

Iterating this inequality, we obtain

$$\sup_n \psi_{n+1}(t) < \infty, \text{ i.e. } \sup_n E[\max_{0\leq s\leq t} ||DX_s^{n+1}||_H^2] < \infty.$$

According to Theorem 1.2.2 (Lemma 1.2.3 in [N]), $X_t \in \mathbb{D}^{1,2}$. □

To study the absolute continuity of the law, we need the following comparison theorem.

Lemma 2.4.2 *Assume $0 \leq \alpha < \frac{1}{2}$. Let X_t be the solution to the perturbed, reflected stochastic differential equation*

$$X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \alpha \max_{0\leq s\leq t} X_s + L_t.$$

Let Y_t be the solution to the reflected stochastic equation $Y_t = \int_0^t \sigma(Y_s)dB_s + \int_0^t b(Y_s)ds + \widetilde{L}_t$. Then, we have that $Y_t \leq X_t$ a.e..

PROOF.

Let $\Delta_t = Y_t - X_t = \widetilde{L}_t - L_t + \int_0^t (b(Y_s) - b(X_s))ds + \int_0^t (\sigma(Y_s) - \sigma(X_s))dB_s - \alpha \max_{0\leq s\leq t} X_s$.

There exists a strictly decreasing sequence $\{a_n\}_{n=0}^{\infty} \subseteq (0,1]$ with $a_0 = 1$, $\lim_{n\to\infty} a_n = 0$ and $\int_{a_n}^{a_{n-1}} \frac{1}{c^2u^2}du = n$, for every $n \geq 1$. For each $n \geq 1$, there exists a continuous function ρ_n on R with support in (a_n, a_{n-1}) so that $0 \leq \rho_n(x) \leq \frac{2}{nC^2x^2}$ holds for every $x > 0$, and $\int_{a_n}^{a_{n-1}} \rho_n(x)dx = 1$. Then the function

$$\phi_n(x) = \int_0^{|x|} \int_0^y \rho_n(u)dudy I_{(0,\infty)}(x), x \in R.$$

is twice continuously differentiable, with $0 \leq \phi_n'(x) \leq 1$ and $\lim_{n\to\infty} \phi_n(x) = x^+$ for $x \in R$.

By the Ito rule:

$$\phi_n(\Delta_t) = \int_0^t \phi_n'(\Delta_s)d\widetilde{L_s} - \int_0^t \phi_n'(\Delta_s)dL_s - \alpha \int_0^t \phi_n'(\Delta_s)d(\max_{0\leq u\leq s} X_u)$$

$$
\begin{aligned}
& +\int_0^t \phi_n^{'}(\Delta_s)(b(Y_s)-b(X_s))ds+\int_0^t \phi_n^{'}(\Delta_s)(\sigma(Y_s)-\sigma(X_s))dB_s \\
& +\frac{1}{2}\int_0^t \phi_n^{''}(\Delta_s)(\sigma(Y_s)-\sigma(X_s))^2 ds \\
\leq \; & \int_0^t \phi_n^{'}(\Delta_s)d\widetilde{L_s}+C\int_0^t \phi_n^{'}(\Delta_s)I_{\{Y_s>X_s\}}|Y_s-X_s|ds \\
& +\int_0^t \phi_n^{'}(\Delta_s)(\sigma(Y_s)-\sigma(X_s))dB_s \\
& +\frac{1}{2}\int_0^t \phi_n^{''}(\Delta_s)(\sigma(Y_s)-\sigma(X_s))^2 ds
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E[\phi_n(\Delta_t)] \;\leq\; & E\int_0^t \phi_n^{'}(\Delta_s)\chi_{\{Y_s>0\}}d\widetilde{L_s}+CE\int_0^t (Y_s-X_s)^+ ds \\
& +\frac{1}{2}E\int_0^t \phi_n^{''}(\Delta_s)(\sigma(Y_s)-\sigma(X_s))^2 ds \\
\leq\; & C\int_0^t E(Y_s-X_s)^+ ds+\frac{t}{n}
\end{aligned}
$$

Letting $n\to\infty$, we get $E\Delta_t^+ \leq C\int_0^t E\Delta_s^+ ds$. By Gronwall Inequality, $E\Delta_t^+ = 0$. Hence $Y_t \leq X_t$ a.e.. □

Theorem 2.4.2 *Assume $0\leq\alpha<\frac{1}{2}$. Let X_t be the solution to the equation (2.4.1). Suppose that $\sigma(\cdot)$ and $b(\cdot)$ are Lipschitz continuous and $|\sigma(x)|>0$ for $x\in R$. Then for $t>0$, the law of X_t is absolutely continuous with respect to Lebesgue measure.*

PROOF. It is sufficient to prove $||DX_t||_H^2>0$ a.s. according to Theorem 2.2.1. Now,

$$
X_t=\int_0^t \sigma(X_s)dB_s+\int_0^t b(X_s)ds+\alpha\max_{0\leq s\leq t}X_s+L_t,
$$

Let

$$
V_s=-(\int_0^s \sigma(X_u)dB_u+\int_0^s b(X_u)du+\alpha\max_{0\leq u\leq s}X_u)\vee 0.
$$

Then, by reflection principle,

$$
\begin{aligned}
L_t &= \max_{0\leq s\leq t}[-(\int_0^s \sigma(X_u)dB_u+\int_0^s b(X_u)du+\alpha\max_{0\leq u\leq s}X_u)\vee 0] \\
&= \max_{0\leq s\leq t}V_s,
\end{aligned}
$$

$$
\begin{aligned}
D_r X_t &= \sigma(X_r) + \int_r^t D_r(\sigma(X_s))dB_s + \int_r^t D_r(b(X_s))ds \\
&\quad + \alpha D_r(\max_{0\le s\le t} X_s) + D_r(\max_{0\le s\le t} V_s). \\
(D_r X_t)^2 &\ge \frac{1}{2}\sigma(X_r)^2 - [\int_r^t D_r(\sigma(X_s))dB_s + \int_r^t D_r(b(X_s))ds \\
&\quad + \alpha D_r(\max_{0\le s\le t} X_s) + D_r(\max_{0\le s\le t} V_s)]^2
\end{aligned}
$$

Similar as in Section 3, we have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}E\{\int_{t-\epsilon}^t ([\int_r^t D_r(\sigma(X_s))dB_s]^2 + [\int_r^t D_r(b(X_s))ds]^2)dr\} = 0.$$

Hence, there exists $\epsilon_n \downarrow 0$, such that

$$\lim_{\epsilon_n\to 0}\frac{1}{\epsilon_n}\int_{t-\epsilon_n}^t ([\int_r^t D_r(\sigma(X_s))dB_s]^2 + [\int_r^t D_r(b(X_s))ds]^2)dr = 0 \ a.s.. \quad (2.4.38)$$

Let

$$
\begin{aligned}
A_n &= \{\omega : \max_{0\le s\le t} X_s = \max_{0\le s\le t-\epsilon_n} X_s\}, \\
A &= \{\omega : \max_{0\le s\le t} X_s = X_t\}.
\end{aligned}
$$

Then,

$$\Omega = \cup_{m=1}^{\infty} A_m \cup A. \quad (2.4.39)$$

Let

$$
\begin{aligned}
B_n &= \{\omega : \max_{0\le s\le t} V_s = \max_{0\le s\le t-\epsilon_n} V_s\}, \\
B &= \{\omega : \max_{0\le s\le t} V_s = V_t\}.
\end{aligned}
$$

We have,

$$\Omega = \cup_{n=1}^{\infty} B_n \cup B. \quad (2.4.40)$$

Firstly, if $\omega \in A_m \bigcap B_n$, for $l > m, n$, we have

$$
\begin{aligned}
\int_{t-\epsilon_l}^t (D_r(\max_{0\le s\le t-\epsilon_m} X_s))^2 dr = 0, \\
\int_{t-\epsilon_l}^t (D_r(\max_{0\le s\le t-\epsilon_n} V_s))^2 dr = 0.
\end{aligned}
$$

This gives

$$\lim_{l\to\infty}\frac{1}{\epsilon_l}\int_{t-\epsilon_l}^{t}\alpha^2(D_r\max_{0\le s\le t}X_s)^2dr=0,\ \lim_{l\to\infty}\frac{1}{\epsilon_l}\int_{t-\epsilon_l}^{t}(D_r(\max_{0\le s\le t}V_s))^2dr=0,$$

a.e. on $A_m\cap B_n$.

Hence,

$$\lim_{l\to\infty}\frac{1}{\epsilon_l}\int_{t-\epsilon_l}^{t}(D_r(X_t))^2dr\ge\frac{1}{2}\sigma(X_t)^2>0, \tag{2.4.41}$$

on $A_m\cap B_n$.

Secondly, if $\omega\in A_m\bigcap B$, for fixed $m\ge 1$,

$$\begin{aligned} X_t &= \int_0^t\sigma(X_s)dB_s+\int_0^t b(X_s)ds+\alpha\max_{0\le s\le t}X_s \\ &\quad +[-(\int_0^t\sigma(X_s)dB_s+\int_0^t b(X_s)ds+\alpha\max_{0\le s\le t}X_s)\vee 0]. \end{aligned}$$

If $\int_0^t\sigma(X_s)dB_s+\int_0^t b(X_s)ds+\alpha\max_{0\le s\le t}X_s>0$, then $X_t=\int_0^t\sigma(X_s)dB_s+\int_0^t b(X_s)ds+\alpha\max_{0\le s\le t}X_s$. In this case, we can see from the proof in Section 2.3 that $||DX_t||_H^2>0$.
If $\int_0^t\sigma(X_s)dB_s+\int_0^t b(X_s)ds+\alpha\max_{0\le s\le t}X_s\le 0$, then $X_t=0$.
But $\{X_t=0\}$ is an event with probability zero. Indeed, according to Lemma 2.4.2, $0\le Y_t\le X_t$. According to Proposition 4.1 in [LNS], the law of Y_t is absolutely continuous with respect to Lesbegue measure, and then we have $P(Y_t=0)=0$.
Therefore, $P(X_t=0)\le P(Y_t=0)=0$.

Thirdly, if $\omega\in A\bigcap B_n$, for fixed $n\ge 1$,

$$D_rX_t=\sigma(X_r)+\int_r^t D_r(\sigma(X_s))dB_s+\int_r^t D_r(b(X_s))ds+\alpha D_r(X_t)+D_r(\max_{0\le s\le t-\epsilon_n}V_s).$$

Hence,

$$(1-\alpha)D_rX_t=\sigma(X_r)+\int_r^t D_r(\sigma(X_s))dB_s+\int_r^t D_r(b(X_s))ds+D_r(\max_{0\le s\le t-\epsilon_n}V_s).$$

Thus, for $l>n$,

$$\begin{aligned} &\frac{1}{\epsilon_l}\int_{t-\epsilon_l}^{t}(1-\alpha)^2(D_rX_t)^2dr \\ &\ge\frac{1}{2}\sigma(X_t)^2-\frac{3}{\epsilon_l}\int_{t-\epsilon_l}^{t}[\int_r^t D_r(\sigma(X_s))dB_s]^2dr \end{aligned}$$

$$-\frac{3}{\epsilon_l}\int_{t-\epsilon_l}^{t}[\int_r^t D_r(b(X_s))ds]^2dr - \frac{3}{\epsilon_l}\int_{t-\epsilon_l}^{t}[D_r(\max_{0\le s\le t-\epsilon_n} V_s)]^2dr.$$

This implies,

$$\lim_{l\to\infty}\frac{1}{\epsilon_l}\int_{t-\epsilon_l}^{t}(1-\alpha)^2(D_rX_t)^2dr \ge \frac{1}{2}\sigma(X_t)^2 > 0 \quad \text{on a.e. } A\cap B_n. \qquad (2.4.42)$$

Finally, let $\omega \in A\bigcap B$. Then

$$X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \alpha X_t + L_t, \qquad (2.4.43)$$

$$L_t = -(\int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \alpha X_t)\vee 0. \qquad (2.4.44)$$

If $\int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \alpha X_t \ge 0$, then $L_t = 0$, and $X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \alpha X_t$. In this case we see that $||DX_t||_H^2 > 0$ from the proof in section 3.

If $\int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \alpha X_t < 0$, then $L_t = -(\int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \alpha X_t)$, and $X_t = 0$. But $X_s \le X_t$ for $0\le s\le t$ on A. Therefore we deduce that $X_s = 0$, for $0\le s\le t$.

Note that

$$X_s = \int_0^s \sigma(X_u)dB_u + \int_0^s b(X_u)du + \alpha X_s + L_s. \qquad (2.4.45)$$

Thus we have

$$\begin{aligned} -\int_0^s \sigma(X_u)dB_u &= \max_{0\le u\le s}\{-(\int_0^u \sigma(X_v)dB_v + \int_0^u b(X_v)dv + \alpha X_u)\vee 0\} \\ &\quad + \int_0^s b(X_u)du,\ s\le t. \end{aligned} \qquad (2.4.46)$$

Notice that the right side is a process of bounded variation, so the equation (2.4.46) is not possible. Combining all the cases, we get $||DX_t||_H^2 > 0$. a.s. □

Chapter 3

Absolute Continuity of the Laws of the Solutions to Parabolic SPDEs with Two Reflecting Walls

3.1 Introduction

Parabolic SPDEs with reflection are natural extension of the widely studied deterministic parabolic obstacle problems. They also can be used to model fluctuations of an interface near a wall, see Funaki and Olla [FO]. In recent years, there is a growing interest on the study of SPDEs with reflection. Several works are devoted to the existence and uniqueness of the solutions. In the case of a constant diffusion coefficient and a single reflecting barrier $h_1 = 0$, Naulart and Pardoux [NP] proved the existence and uniqueness of the solutions. In the case of a non-constant diffusion coefficient and a single reflecting barrier $h_1 = 0$, the existence of a minimal solution was obtained by Donati-Martin and Pardoux [DP]. The existence and particularly the uniqueness of the solutions for a fully non-linear SPDE with reflecting barrier 0 have been established by Xu and Zhang [XZ]. In the case of double reflecting barriers, Otobe [O] obtained the existence and uniqueness of the solutions of a SPDE driven by an additive white noise.

The existence and uniqueness of the solution to a fully non-linear SPDE with two reflecting walls was proved by Yang and Zhang [YZ1]. We focus here on the existence of the density of the law of the solution, using Malliavin calculus. Malliavin calculus associated with white noise was also used by Pardoux and Zhang [PZ], Bally and Pardoux [BP1] to establish the existence of the density of the law of the solution to parabolic SPDE. The case of parabolic stochastic partial differential equation with one reflecting wall was studied by Donati-martin and Pardoux [DP1].

This chapter is organized as follows: Section 2 is devoted to fundamental knowledge of parabolic stochastic partial differential equations with two reflecting walls and Malliavin calculus associated with white noise. In Section 3, we recall some results obtained by Yang and Zhang [YZ1] about the existence and uniqueness of the solution to parabolic SPDEs with two reflecting walls and we prove the Malliavin differentiability of the solution. Finally, we give the existence of the density of the law of the solution.

3.2 Preliminaries

Notation: Let $Q = [0,1] \times R_+$, $Q_T = [0,1] \times [0,T]$, $V = \{u \in H^1([0,1]), u(0) = u(1) = 0\}$ where $H^1([0,1])$ denotes the usual Sobolev space of absolutely continuous funcitons defined on $[0,1]$ whose derivative belongs to $L^2([0,1])$, and $A = -\frac{\partial^2}{\partial x^2}$.

Consider the following stochastic partial differential equation with two reflecting walls:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(x,t,u) + \sigma(x,t,u)\dot{W}(x,t) + \eta - \xi, \\ u(0,t) = 0, u(1,t) = 0, \ for\ t \geq 0, \\ u(x,0) = u_0(x) \in C([0,1]), \\ h^1(x,t) \leq u(x,t) \leq h^2(x,t), for (x,t) \in Q, a.s. \end{cases} \tag{3.2.1}$$

where $\dot{W}$ denotes the space-time white noise defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\mathcal{F}_t = \sigma(W(x,s) : x \in [0,1], 0 \leq s \leq t)$, u_0 is a continuous function on $[0,1]$, which vanishes at 0 and 1.

We assume that the reflecting walls $h^1(x,t)$,$h^2(x,t)$ are continuous functions satisfying $h^1(0,t), h^1(1,t) \leq 0$, $h^2(0,t), h^2(1,t) \geq 0$, and
(H1)$h^1(x,t) < h^2(x,t)$ for $x \in (0,1)$ and $t \in R_+$;
(H2)$\frac{\partial h^i}{\partial t} + \frac{\partial^2 h^i}{\partial x^2} \in L^2([0,1] \times [0,T])$, where $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$ are interpreted in a distributional sense;
(H3)$\frac{\partial}{\partial t} h^i(0,t) = \frac{\partial}{\partial t} h^i(1,t) = 0$ for $t \geq 0$;
(H4)$\frac{\partial}{\partial t}(h^2 - h^1) \geq 0$.
We also assume that the coefficients: $f, \sigma(x,t,u(x,t)) : [0,1] \times R_+ \times R \to R$ are measurable and satisfy:
(F) : f, σ are of class of C^1 with bounded derivatives with respect to the third element and σ is bounded.

The following is the definition of the solution to a parabolic SPDE with two reflecting walls h^1, h^2.

Definition 3.2.1 *A triplet (u, η, ξ) defined on a filtered probability space $(\Omega, P, \mathcal{F}; \{\mathcal{F}_t\})$ is a solution to the SPDE(3.2.1), denoted by $(u_0; 0, 0; f, \sigma; h^1, h^2)$, if*
(i) $u = \{u(x,t); (x,t) \in Q\}$ is a continuous, adapted random field (i.e. $u(x,t)$ is $\mathcal{F}_t$-measuralbe $\forall t \geq 0$, $x \in [0,1]$) satisfying $h^1(x,t) \leq u(x,t) \leq h^2(x,t)$, $u(0,t) = 0$ and $u(1,t) = 0$, a.s.;
(ii) $\eta(dx, dt)$ and $\xi(dx, dt)$ are positive and adapted (i.e. $\eta(B)$ and $\xi(B)$ are $\mathcal{F}_t$-measurable if $B \subset (0,1) \times [0,t]$) random measures on $(0,1) \times R_+$ satisfying

$$\eta((\theta, 1-\theta) \times [0,T]) < \infty, \xi((\theta, 1-\theta) \times [0,T]) < \infty \ a.s. \tag{3.2.2}$$

for $0 < \theta < \frac{1}{2}$ and $T > 0$;
(iii) for all $t \geq 0$ and $\phi \in C_k^\infty((0,1) \times (0,\infty))$ (the set of smooth functions with compact supports) we have

$$\begin{aligned}(u(t), \phi) - \int_0^t (u(s), \phi'')ds - \int_0^t (f(y,s,u), \phi)ds - \int_0^t \int_0^1 \phi\sigma(y,s,u)W(dx,ds) \\ = (u_0, \phi) + \int_0^t \int_0^1 \phi\eta(dxds) - \int_0^t \int_0^1 \phi\xi(dxds) a.s.;\end{aligned} \tag{3.2.3}$$

(iv) $\int_Q (u(x,t) - h^1(x,t))\eta(dx,dt) = \int_Q (h^2(x,t) - u(x,t))\xi(dx,dt) = 0$ *a.s.*.

Remarks: We note that the stochastic integral in (3.2.3) is an Ito integral with respect to the Brownian sheet $\{W(x,t); (x,t) \in [0,1] \times R_+\}$ defined on the canonical space $\Omega = C_0([0,1] \times R_+)$ (the space of continuous functions on $[0,1] \times R_+$ which are zero whenever one of their arguments is zero). The Brownian sheet is equipped with its Borel σ-field $\mathcal{F}$, the filtration $\mathcal{F}_t = \{\sigma(W(x,s)), x \in [0,1], s \le t\}$ and the Wiener measure P.

Next, we recall Malliavin calculus associated with white noise:

Let S denote the set of "simple random variables" of the form

$$F = f(W(h_1), ..., W(h_n)), n \in N,$$

where $h_i \in H := L^2([0,1] \times R_+)$ and $W(h_i)$ represent the Wiener integral of h_i, $f \in C_p^\infty(R^n)$. For such a variable F, we define its derivative DF, a random variable with values in $L^2([0,1] \times R_+)$ by

$$D_{x,t}F = \Sigma_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n)) \cdot h_i(x,t).$$

We denote by $D^{1,2}$ the closure of S with respect to the norm:

$$||F||_{1,2} = (E(F^2))^{\frac{1}{2}} + [E(||DF||^2_{L^2([0,1]\times R^+)})]^{\frac{1}{2}}.$$

$D^{1,2}$ is a Hilbert space. It is the domain of the closure of derivation operator D.

We go back to consider the following parabolic SPDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(x,t,u) + \sigma(x,t,u)\dot{W}, \\ u(0,t) = 0, u(1,t) = 0, \ for \ t \ge 0, \\ u(x,0) = u_0(x) \in C([0,1]), \end{cases} \tag{3.2.4}$$

where f, σ satisfy (F).

According to [XZ], we know u also satisfies the integral equation:

$$u(x,t) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)f(u(y,s))dyds$$

$$+\int_0^t\int_0^1 G_{t-s}(x,y)\sigma(u(y,s))W(dyds)$$

And we have the following result from [PZ].

Proposition 3.2.1 *[PZ] For all* $(x,t)\in(0,1)\times R_+$, $u(x,t)$ *is the solution to (3.2.4), Then* $u(x,t)\in D^{1,2}$ *and* $D_{y,s}u(x,t)$ *is the solution of SPDE:*

$$D_{y,s}u(x,t) = G_{t-s}(x,y)\sigma(u(y,s)) + \int_s^t\int_0^1 G_{t-r}(x,z)f'(u(z,r))D_{y,s}u(z,r)dzdr \\ + \int_s^t\int_0^1 G_{t-r}(x,z)\sigma'(u(z,r))D_{y,s}(u(z,r))W(dzdr).$$

3.3 The Main Result and The Proof

We consider the penalized SPDE as follows:

$$\begin{cases} \dfrac{\partial u^{\epsilon,\delta}(x,t)}{\partial t} - \dfrac{\partial^2 u^{\epsilon,\delta}(x,t)}{\partial x^2} + f(u^{\epsilon,\delta}(x,t)) = \sigma(u^{\epsilon,\delta}(x,t))\dot{W}(x,t) \\ \qquad +\dfrac{1}{\delta}(u^{\epsilon,\delta}(x,t)-h^1(x,t))^- - \dfrac{1}{\epsilon}(u^{\epsilon,\delta}(x,t)-h^2(x,t))^+, \\ u^{\epsilon,\delta}(0,t)=u^{\epsilon,\delta}(1,t)=0, t\geq 0, \\ u^{\epsilon,\delta}(x,0)=u_0(x), \end{cases} \tag{3.3.1}$$

and we can get the following proposition.

Proposition 3.3.1 *If we have (H1),(H2), (H3), (H4) and (F). Then for any* $p\geq 1, T>0$, $\sup_{\epsilon,\delta}E(||u^{\epsilon,\delta}||_\infty^T)<\infty$ *and* $u^{\epsilon,\delta}$ *converges uniformly on* $[0,1]\times[0,T]$ *to* u *as* $\epsilon,\delta\to 0$*, where* $u, u^{\epsilon,\delta}$ *are the solutions of SPDE (3.2.1) and the penalized SPDE (3.3.1).*

PROOF. Let $u^{\epsilon,\delta}$ be the solution to the penalized SPDE (3.3.1).

Step 1: we prove that there exists $u(x,t)$ such that

$$u := \lim_{\epsilon\downarrow 0} u^\epsilon = \lim_{\epsilon\downarrow 0}\lim_{\delta\downarrow 0} u^{\epsilon,\delta} a.s. \tag{3.3.2}$$

First fix ϵ,

let $v^{\epsilon,\delta}$ be the solution of equation:

$$\begin{cases} \dfrac{\partial v^{\epsilon,\delta}(x,t)}{\partial t} - \dfrac{\partial^2 v^{\epsilon,\delta}(x,t)}{\partial x^2} + f(v^{\epsilon,\delta}(x,t)) = \sigma(u^{\epsilon,\delta}(x,t))\dot{W}(x,t) \\ \qquad\qquad - \dfrac{1}{\epsilon}(u^{\epsilon,\delta}(x,t) - h^2(x,t))^+, \\ v^{\epsilon,\delta}(x,0) = u_0(x), v^{\epsilon,\delta}(0,t) = v^{\epsilon,\delta}(1,t) = 0. \end{cases} \tag{3.3.3}$$

Then $z^{\epsilon,\delta} = v^{\epsilon,\delta} - u^{\epsilon,\delta}$ is the unique solution in $L^2((0,T)\times(0,1))$ of

$$\begin{cases} \dfrac{\partial z_t^{\epsilon,\delta}}{\partial t} + Az_t^{\epsilon,\delta} + f(v_t^{\epsilon,\delta}) - f(u_t^{\epsilon,\delta}) = -\dfrac{1}{\delta}(u^{\epsilon,\delta}(x,t) - h^1(x,t))^-, \\ z^{\epsilon,\delta}(x,0) = 0, z^{\epsilon,\delta}(0,t) = z^{\epsilon,\delta}(1,t) = 0. \end{cases} \tag{3.3.4}$$

Multiplying Eq(3.3.4) by $(z_s^{\epsilon,\delta})^+$ and integrating it to obtain:

$$\begin{aligned} & \int_0^t (\frac{\partial z^{\epsilon,\delta}(x,s)}{\partial s}, (z^{\epsilon,\delta}(x,s))^+)ds + \int_0^t (\frac{\partial z^{\epsilon,\delta}(x,s)}{\partial x}, \frac{\partial (z^{\epsilon,\delta}(x,s))^+}{\partial x})ds \\ & + \int_0^t (f(v^{\epsilon,\delta}(x,s)) - f(u^{\epsilon,\delta}(x,s)), (z^{\epsilon,\delta}(x,s))^+)ds \\ = & -\frac{1}{\delta}\int_0^t ((u^{\epsilon,\delta}(x,s) - h^1(x,s))^-, (z^{\epsilon,\delta}(x,s))^+)ds. \end{aligned} \tag{3.3.5}$$

According to Bensoussan and Lions [BL] (Lemma 6.1, P132), $(z_s^{\epsilon,\delta})^+ \in L^2(0,T;V) \cap C([0,T];H)$ a.s.

$$\int_0^t (\frac{\partial}{\partial s} z_s^{\epsilon,\delta}, (z_s^{\epsilon,\delta})^+)ds = \frac{1}{2}|(z_t^{\epsilon,\delta})^+|_H^2$$

and similarly

$$\int_0^t (\frac{\partial}{\partial x} z_s^{\epsilon,\delta}, \frac{\partial}{\partial x}(z_s^{\epsilon,\delta})^+)ds = \int_0^t |\frac{\partial}{\partial x}(z_s^{\epsilon,\delta})^+|^2 ds \geq 0,$$

and by Lipschitz continuity of f, we have

$$\int_0^t (f(v^{\epsilon,\delta}(x,s)) - f(u^{\epsilon,\delta}(x,s)), (z^{\epsilon,\delta}(x,s))^+)ds \geq -c\int_0^t |(z^{\epsilon,\delta}(x,s))^+|_H^2 ds,$$

and we deduce that

$$\begin{aligned} 0 & \geq \frac{1}{2}|(z^{\epsilon,\delta}(x,t))^+|_H^2 + \int_0^t |\frac{\partial (z^{\epsilon,\delta}(x,s))^+}{\partial x}|_H^2 ds - c\int_0^t |(z^{\epsilon,\delta}(x,s))^+|_H^2 ds \\ & \geq \frac{1}{2}|(z^{\epsilon,\delta}(x,t))^+|_H^2 - c\int_0^t |z^{\epsilon,\delta}(x,s)^+|_H^2 ds. \end{aligned}$$

Hence,

$$c\int_0^t |(z^{\epsilon,\delta}(x,s))^+|_H^2 ds \geq \frac{1}{2}|z^{\epsilon,\delta}(x,t)^+|_H^2 \tag{3.3.6}$$

From Gronwall's Lemma: $|(z^{\epsilon,\delta}(x,t))^+|_H^2 = 0$, $\forall t \geq 0.a.s.$

Then,

$$u^{\epsilon,\delta}(x,t) \geq v^{\epsilon,\delta}(x,t), \forall x \in [0,1], t \geq 0.a.s. \tag{3.3.7}$$

From Theorem 3.1 in [DP], we get that the following equation has a unique solution $\{w^{\epsilon,\delta}(x,t); x \in [0,1], t \geq 0\}$:

$$\begin{cases} \dfrac{\partial w^{\epsilon,\delta}(x,t)}{\partial t} - \dfrac{\partial^2 w^{\epsilon,\delta}(x,t)}{\partial x^2} + f(w^{\epsilon,\delta}(x,t) + \sup\limits_{s\leq t, y\in[0,1]}(w^{\epsilon,\delta}(y,s) - h^1(y,s))^-) \\ \qquad = \sigma(u^{\epsilon,\delta}(x,t))\dot{W}(x,t) - \dfrac{1}{\epsilon}(u^{\epsilon,\delta}(x,t) - h^2(x,t))^+, \\ \qquad w^{\epsilon,\delta}(\cdot,0) = u_0, w^{\epsilon,\delta}(0,t) = w^{\epsilon,\delta}(1,t) = 0. \end{cases}$$

We set

$$\overline{w}^{\epsilon,\delta}(x,t) = w^{\epsilon,\delta}(x,t) + \sup_{s\geq t, y\in[0,1]}(w^{\epsilon,\delta}(y,s) - h^1(y,s))^- = w^{\epsilon,\delta}(x,t) + \Phi_t^{\epsilon,\delta} \tag{3.3.8}$$

$\overline{w}^{\epsilon,\delta}(x,t) - h^1(x,t) \geq 0$ and $\Phi_t^{\epsilon,\delta}$ is an increasing process.

For any $T > 0$, $\overline{z}^{\epsilon,\delta} = u^{\epsilon,\delta} - \overline{w}^{\epsilon,\delta}$ is the unique solution in $L^2((0,T); H^1(0,1))$ of

$$\begin{cases} \dfrac{\partial \overline{z}^{\epsilon,\delta}(x,t)}{\partial t} + A\overline{z}^{\epsilon,\delta}(x,t) + f(u^{\epsilon,\delta}(x,t)) - f(\overline{w}^{\epsilon,\delta}(x,t)) + \dfrac{d\Phi_t^{\epsilon,\delta}}{dt} \\ \qquad = \dfrac{1}{\delta}(u^{\epsilon,\delta}(x,t) - h^1(x,t))^-, \\ \qquad \overline{z}^{\epsilon,\delta}(\cdot,0) = 0, \\ \qquad \overline{z}^{\epsilon,\delta}(0,t) = \overline{z}^{\epsilon,\delta}(1,t) = -\Phi_t^{\epsilon,\delta}. \end{cases}$$

Multiplying this equation by $(\overline{z}^{\epsilon,\delta}(x,s))^+$, we obtain by the same arguments as above:

$$\begin{aligned} &\int_0^t (\frac{\partial \overline{z}^{\epsilon,\delta}(x,s)}{\partial s}, (\overline{z}^{\epsilon,\delta}(x,s))^+)ds + \int_0^t (\frac{\partial \overline{z}^{\epsilon,\delta}(x,s)}{\partial x}, \frac{\partial(\overline{z}^{\epsilon,\delta}(x,s))^+}{\partial x})ds \\ &+ \int_0^t (f(u^{\epsilon,\delta}(x,s)) - f(\overline{w}^{\epsilon,\delta}(x,s)), (\overline{z}^{\epsilon,\delta}(x,s))^+)ds \\ &+ \int_0^t \int_0^1 (\overline{z}^{\epsilon,\delta}(x,s))^+ dx d\Phi_s^{\epsilon,\delta} \end{aligned}$$

$$= \frac{1}{\delta} \int_0^t ((u^{\epsilon,\delta}(x,s) - h^1(x,s))^-, (\overline{z}^{\epsilon,\delta}(x,s))^+) ds \tag{3.3.9}$$

The right-hand side of the above equality is zero because $(\overline{z}^{\epsilon,\delta}(x,s))^+ > 0$ implies $u^{\epsilon,\delta}(x,s) - h^1(x,s) > \overline{w}^{\epsilon,\delta}(x,s) - h^1(x,s) \geq 0$.

Hence we again deduce from Gronwall's Lemma:

$$u^{\epsilon,\delta}(x,t) \leq \overline{w}^{\epsilon,\delta}(x,t) \tag{3.3.10}$$

By (3.3.7),(3.3.10),

$$\begin{aligned} |u^{\epsilon,\delta}(x,t)| &\leq |v^{\epsilon,\delta}(x,t)| + |w^{\epsilon,\delta}(x,t)| + \sup_{s\leq t, y\in[0,1]} (w^{\epsilon,\delta}(y,s) - h^1(y,s))^- \\ &\leq |v^{\epsilon,\delta}(x,t)| + 2 \sup_{s\leq t, y\in[0,t]} [|w^{\epsilon,\delta}(y,s)| + |h^1(y,s)|]. \end{aligned} \tag{3.3.11}$$

From Lemma 6.1 in [DP], for arbitrarily large p and any $T > 0$, consider that $f'(v^{\epsilon,\delta}(x,t)) = f(v^{\epsilon,\delta}(x,t)) + \frac{1}{\epsilon}(u^{\epsilon,\delta}(x,t) - h^2(x,t))^+$ is Lipschitz continuous with respect to $v^{\epsilon,\delta}$ and $f'(w^{\epsilon,\delta}(x,t) + \sup_{s\geq t, y\in[0,1]}(w^{\epsilon,\delta}(y,s) - h^1(y,s))^-) =$
$f(w^{\epsilon,\delta}(x,t) + \sup_{s\geq t, y\in[0,1]}(w^{\epsilon,\delta}(y,s) - h^1(y,s))^-) + \frac{1}{\epsilon}(u^{\epsilon,\delta}(x,t) - h^2(x,t))^+$ is Lipschitz continuous with respect to $w^{\epsilon,\delta}(x,t) + \sup_{s\geq t, y\in[0,1]}(w^{\epsilon,\delta}(y,s) - h^1(y,s))^-$, we have that $\sup_\delta E[\sup_{(x,t)\in\overline{Q}_T} |v^{\epsilon,\delta}(x,t)|^p] < \infty$ and $\sup_\delta E[\sup_{(x,t)\in\overline{Q}_T} |w^{\epsilon,\delta}(x,t)|^p] < \infty$,

which imply

$$\sup_\delta E[\sup_{(x,t)\in\overline{Q}_T} |u^{\epsilon,\delta}(x,t)|^p] < \infty. \tag{3.3.12}$$

So $u^\epsilon = \sup_\delta u^{\epsilon,\delta}$ is a.s. bounded on $\overline{Q}_T$.

Let

$$\eta^\epsilon = \lim_{\delta\to 0} \frac{(u^{\epsilon,\delta}(x,t) - h^1(x,t))^-}{\delta} \tag{3.3.13}$$

Similar as the proof of Th4.1 in [DP], u^ϵ is continuous and u^ϵ is the solution to:

$$\frac{\partial u^\epsilon}{\partial t} + Au^\epsilon + f(u^\epsilon) = \sigma(u^\epsilon)\dot{W}(x,t) + \eta^\epsilon(x,t) - \frac{1}{\epsilon}(u^\epsilon(x,t) - h^2(x,t))^+ \tag{3.3.14}$$

In addition, by the definition of u^ϵ, $u^\epsilon \geq h^1$and using Theorem 1.2.6 (Comparison Theorem), u^ϵ decreases when $\epsilon \to 0$.

Hence, there exists $u(x,t)$ such that

$$u := \lim_{\epsilon\downarrow 0} u^\epsilon = \lim_{\epsilon\downarrow 0} \lim_{\delta\downarrow 0} u^{\epsilon,\delta} a.s. \tag{3.3.15}$$

Step 2: Next we prove $u(x,t)$ is continuous.

Let $\tilde{v}^{\epsilon,\delta}$ be the solution of

$$\frac{\partial \tilde{v}^{\epsilon,\delta}}{\partial t} + A\tilde{v}^{\epsilon,\delta} = \sigma(u^{\epsilon,\delta})\dot{W}, \tag{3.3.16}$$

and let $\hat{v}$ be the solution of

$$\frac{\partial \hat{v}}{\partial t} + A\hat{v} = \sigma(u)\dot{W}. \tag{3.3.17}$$

Remember

$$\frac{\partial u^{\epsilon,\delta}(x,t)}{\partial t} - \frac{\partial^2 u^{\epsilon,\delta}(x,t)}{\partial x^2} + f(u^{\epsilon,\delta}(x,t)) = \sigma(u^{\epsilon,\delta}(x,t))\dot{W}(x,t)$$
$$+\frac{1}{\delta}(u^{\epsilon,\delta}(x,t) - h^1(x,t))^- - \frac{1}{\epsilon}(u^{\epsilon,\delta}(x,t) - h^2(x,t))^+,$$

Let $\tilde{z}^{\epsilon,\delta} = u^{\epsilon,\delta} - \tilde{v}^{\epsilon,\delta}$, then $\tilde{z}^{\epsilon,\delta}$ is the solution of

$$\begin{aligned} &\frac{\partial \tilde{z}^{\epsilon,\delta}}{\partial t} + A\tilde{z}^{\epsilon,\delta} + f(\tilde{z}^{\epsilon,\delta} + \tilde{v}^{\epsilon,\delta}) \\ = \ &\frac{1}{\delta}(\tilde{z}^{\epsilon,\delta} + \tilde{v}^{\epsilon,\delta} - h^1)^- - \frac{1}{\epsilon}(\tilde{z}^{\epsilon,\delta} + \tilde{v}^{\epsilon,\delta} - h^2)^+. \end{aligned} \tag{3.3.18}$$

Let $\hat{z}^{\epsilon,\delta}$ be the solution of

$$\frac{\partial \hat{z}^{\epsilon,\delta}}{\partial t} + A\hat{z}^{\epsilon,\delta} + f(\hat{z}^{\epsilon,\delta} + \hat{v}) = \frac{1}{\delta}(\hat{z}^{\epsilon,\delta} + \hat{v} - h^1)^- - \frac{1}{\epsilon}(\hat{z}^{\epsilon,\delta} + \hat{v} - h^2)^+. \tag{3.3.19}$$

We have

$$||\tilde{z}^{\epsilon,\delta} - \hat{z}^{\epsilon,\delta}||_{T,\infty} \le ||\tilde{v}^{\epsilon,\delta} - \hat{v}||_{T,\infty}. \tag{3.3.20}$$

$\hat{z}^{\epsilon,\delta}$ is continuous. According to proof of Theorem 2.1 in [O] , $\hat{z}^{\epsilon,\delta} \to \hat{z}$ (*continuous*). It means

$$\hat{z} = \lim_{\epsilon\to 0} \hat{z}^{\epsilon} = \lim_{\epsilon\to 0}\lim_{\delta\to 0} \hat{z}^{\epsilon,\delta}.$$

Fix ϵ, $\hat{z}^{\epsilon,\delta} \uparrow \hat{z}^{\epsilon}(continuous)$, and from Dini theorem, $\hat{z}^{\epsilon,\delta}$ uniformly converges to $\hat{z}^{\epsilon}$. i.e.$||\hat{z}^{\epsilon,\delta} - \hat{z}^{\epsilon}||_{T,\infty} \to 0, \quad \delta \to 0$.

Since $\hat{z}^{\epsilon} \downarrow \hat{z}$, and from Dini theorem, $\hat{z}^{\epsilon}$ uniformly converges to $\hat{z}$. i.e.$||\hat{z}^{\epsilon} - \hat{z}||_{T,\infty} \to 0$.

Then we get

$$||\hat{z}^{\epsilon,\delta} - \hat{z}||_{T,\infty} = ||\hat{z}^{\epsilon,\delta} - \hat{z}^{\epsilon} + \hat{z}^{\epsilon} - \hat{z}||_{T,\infty} \le ||\hat{z}^{\epsilon,\delta} - \hat{z}^{\epsilon}||_{T,\infty} + ||\hat{z}^{\epsilon} - \hat{z}||_{T,\infty} \to 0$$

$$(\delta \to 0, \epsilon \to 0). \tag{3.3.21}$$

i.e. $\hat{z}^{\epsilon,\delta} \to \hat{z}$ uniformly.

Next we prove $\tilde{v}^{\epsilon,\delta} \to \hat{v}$ uniformly with respect to s, t as $\epsilon \to 0, \delta \to 0$:

Let $I(x,t) = \tilde{v}^{\epsilon,\delta}(x,t) - \hat{v}(x,t) = \int_0^t \int_0^1 G_{t-s}(x,y)(\sigma(u^{\epsilon,\delta}) - \sigma(u))W(dyds)$, from the proof of Corollary 3.4 in [W2],

$$E|I(x,t) - I(y,s)|)^p \le C_T E \int_0^{t \bigvee s} \int_0^1 (|\sigma(u^{\epsilon,\delta}) - \sigma(u)|)^p dzdr|(x,t) - (y,s)|^{\frac{p}{4}-3},$$

and following the same calculation as in the proof of Theorem 2.1 in Xu and Zhang [XZ], we deduce

$$E(\sup_{x\in[0,1],t\in[0,T]} |I(x,t)|)^p \le C_T E \int_0^T \int_0^1 (|\sigma(u^{\epsilon,\delta}) - \sigma(u)|)^p dxdt.$$

Again according to $u := \lim_{\epsilon\to 0} \lim_{\delta \to 0} u^{\epsilon,\delta}$ and $\sigma(x,t,u(x,t))$ is Lipschitz continuous and bounded, we can have

$$\begin{aligned} E(\sup_{x\in[0,1],t\in[0,T]} |I(x,t)|)^p & \le C_T E \int_0^T \int_0^1 (|\sigma(u^{\epsilon,\delta}) - \sigma(u)|)^p dtdx \\ & \to 0 \end{aligned}$$

Then we have that $\tilde{v}^{\epsilon,\delta} \to \hat{v}$ uniformly a.s. and again from (3.3.20) and (3.3.21) we deduce that $\tilde{z}^{\epsilon,\delta} \to \hat{z}$ uniformly a.s..

So

$$\lim_{\epsilon\to 0} \lim_{\delta\to 0} u^{\epsilon,\delta} = u = \hat{z} + \hat{v}$$

is continuous.

Step 3: Next we prove $u(x,t)$ is the solution of

$$\frac{\partial u}{\partial t} + Au + f(u) = \sigma(u)\dot{W}(x,t) + \eta(x,t) - \xi(x,t). \tag{3.3.22}$$

For $\psi \in C_0^\infty((0,1) \times [0,\infty))$,

$$\begin{aligned} & - \int_0^t (u^\epsilon(x,s), \psi_s(s))ds - \int_0^t (u^\epsilon(x,s), A\psi)ds + \int_0^t (f(u^\epsilon), \psi)ds \\ = & \int_0^t \int_0^1 (\sigma(u^\epsilon), \psi)W(dx,ds) + \int_0^t \int_0^1 \psi(x,t)(\eta^\epsilon(dx,dt) - \xi^\epsilon(dx,dt)) \end{aligned} \tag{3.3.23}$$

$$\eta^\epsilon = \lim_{\delta\to 0}\frac{(u^{\epsilon,\delta}-h^1)^-}{\delta}, \xi^\epsilon = \frac{(u^\epsilon - h^2)^+}{\epsilon}.$$

Let $\epsilon \to 0$,

$$\begin{aligned} & -\int_0^t (u(x,s),\psi_s)ds - \int_0^t (u(x,s),A\psi)ds + \int_0^t (f(u),\psi)ds \\ = & \int_0^t\int_0^1 (\sigma(u),\psi)W(dx,ds) + \lim_{\epsilon\to 0}\int_0^t\int_0^1 \psi(x,t)(\eta^\epsilon(dx,dt)-\xi^\epsilon(dx,dt)). \end{aligned}$$

Then it is clear that, under the limit $\epsilon \to 0$, $lim_{\epsilon\to 0}(\eta^\epsilon - \xi^\epsilon)$ exists in the sense of Schwartz distribution a.s..

Because u^ϵ uniformly converges to u, similarly as Theorem 3.1 in [YZ1] we get $\eta^\epsilon \to \eta$ and $\xi^\epsilon \to \xi$. Let $\epsilon \to 0$ to see that (u,η,ξ) satisfies condition (iii) of Def 3.2.1.

Multiplying both sides of Eq(3.3.23) by ϵ and letting $\epsilon \to 0$,

$$\lim_{\epsilon\to 0}\int_0^t\int_0^1 \psi(x,t)(\epsilon\lim_{\delta\to 0}\frac{(u^{\epsilon,\delta}-h^1)^-}{\delta} - (u^\epsilon-h^2)^+)(dx,dt) = 0 \qquad (3.3.24)$$

then $\int_0^t\int_0^1 \psi(x,t)(u-h^2)^+(dx,dt) = 0$, and we can get $u \le h^2$. And since $u^\epsilon \ge h^1$, then $u \ge h^1$. Combining these two inequalities, we have $h^1 \le u \le h^2$.

Finally, we can show that $\int_{Q_T}(u-h^1)d\eta = \int_{Q_T}(h^2-u)d\xi = 0$.

For $\epsilon \le \epsilon'$, $u^\epsilon \ge u^{\epsilon'}$, therefore $supp(\eta^\epsilon) \subset supp(\eta^{\epsilon'})$, we get $supp(\eta) \subset supp(\eta^\epsilon)$. we know $u^\epsilon - h^1 \le 0$ on $supp\eta^\epsilon$. So $\int_{Q_T}(u^\epsilon-h^1)d\eta \le 0$. Then$\int_{Q_T}(u-h^1)d\eta = 0$. Because $\xi^\epsilon = \frac{1}{\epsilon}(u^\epsilon-h^2)^+$, then $0 \ge \int_{Q_T}(u^\epsilon-h^2)d\xi^\epsilon \ge 0$. And since $\xi^\epsilon \to \xi$, then $\int_{Q_T}(u-h^2)d\xi = 0$.

By taking $\psi \in C_0^\infty((0,1)\times(0,\infty))$ such that $\psi = 1$ on $(supp\eta)\cap((\delta,1-\delta)\times[0,T])$ and $\psi = 0$ on $supp\xi$. Hence, in view of (3.2.3),

$$\eta([\delta,1-\delta]\times[0,T]) = \int_0^T\int_0^1 \psi(x,t)\eta(dx,dt) - \int_0^T \phi(x,t)\xi(dx,dt) < \infty$$

for all $0<\delta<\frac{1}{2}$ and $T>0$. Similarly we can get $\xi([\delta,1-\delta]\times[0,T]) < \infty$ for all $0<\delta<\frac{1}{2}$ and $T>0$. $\square$

Set $k_1(u^{\epsilon,\delta} - h^1(x,t)) = arctan[(u^{\epsilon,\delta} - h^1(x,t)) \wedge 0]^2$ and $k_2(u^{\epsilon,\delta} - h^2(x,t)) = arctan[(h^2(x,t) - u^{\epsilon,\delta}) \wedge 0]^2$. Consider the following penalized SPDE:

$$\begin{cases} \frac{\partial u^{\epsilon,\delta}(x,t)}{\partial t} - \frac{\partial^2 u^{\epsilon,\delta}(x,t)}{\partial x^2} + f(u^{\epsilon,\delta}(x,t)) = \sigma(u^{\epsilon,\delta}(x,t))\dot{W}(x,t) \\ \qquad +\frac{1}{\delta}k_1(u^{\epsilon,\delta} - h^1(x,t)) - \frac{1}{\epsilon}k_2(u^{\epsilon,\delta} - h^2(x,t)), \\ \qquad u^{\epsilon,\delta}(x,0) = u_0(x). \end{cases} \tag{3.3.25}$$

Notice that the corresponding penalized elements in Proposition 3.3.1 are $(u^{\epsilon,\delta} - h^1(x,t))^-$ and$(u^{\epsilon,\delta} - h^2(x,t))^+$. It was shown in [DMZ](also in [DP1]) that the choice of k_1, k_2 does not change the limit of $u^{\epsilon,\delta}$, but makes k_1, k_2 differentiable with respect to $u^{\epsilon,\delta}$.

Proposition 3.3.2 *For all $(x,t) \in [0,1] \times R^+$, $u(x,t) \in D_{1,p}$ and there exists a subsequence of $Du^{\epsilon,\delta}(x,t)$ that converges to $Du(x,t)$ in the weak topology of $L^p(\Omega; H)$ and $H = L^2([0,1] \times R^+)$.*

PROOF. Let $u^{\epsilon,\delta}$ be the solution to the following SPDE:

$$\begin{cases} \frac{\partial u^{\epsilon,\delta}(x,t)}{\partial t} - \frac{\partial^2 u^{\epsilon,\delta}(x,t)}{\partial x^2} + f(u^{\epsilon,\delta}(x,t)) = \sigma(u^{\epsilon,\delta}(x,t))\dot{W}(x,t) \\ \qquad +\frac{1}{\delta}k_1(u^{\epsilon,\delta} - h^1(x,t)) - \frac{1}{\epsilon}k_2(u^{\epsilon,\delta} - h^2(x,t)), \\ \qquad u^{\epsilon,\delta}(x,0) = u_0(x). \end{cases} \tag{3.3.26}$$

Then it can be expressed as,

$$\begin{aligned} u^{\epsilon,\delta}(x,t) &= \int_0^t G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u^{\epsilon,\delta}(x,t))W(dyds) \\ &+ \int_0^t \int_0^1 G_{t-s}(x,y)[-f(u^{\epsilon,\delta}(x,t)) + \frac{1}{\delta}k_1 - \frac{1}{\epsilon}k_2]dyds, \end{aligned}$$

where $G_t(x,y)$ is the heat kernel.

And we also know from Section 3.2 that:

$$\begin{aligned} D_{y,s}u^{\epsilon,\delta}(x,t) &= G_{t-s}(x,y)\sigma(u^{\epsilon,\delta}(y,s)) \\ &+ \int_s^t \int_0^1 G_{t-r}(x,z)\sigma'(u^{\epsilon,\delta}(z,r))D_{y,s}(u^{\epsilon,\delta}(z,r))W(dzdr) \\ &+ \int_s^t \int_0^1 G_{t-r}(x,z)[-f' + \frac{1}{\delta}k_1' - \frac{1}{\epsilon}k_2']D_{y,s}(u^{\epsilon,\delta}(z,r))dzdr \end{aligned}$$

Let

$$D_{y,s}u^{\epsilon,\delta}(x,t)=\sigma(u^{\epsilon,\delta}(y,s))S^{\epsilon,\delta}_{y,s}(x,t) \tag{3.3.27}$$

and then $S^{\epsilon,\delta}_{y,s}(x,t)$ is the solution of

$$\begin{aligned} S^{\epsilon,\delta}_{y,s}(x,t) &= G_{t-s}(x,y)+\int_s^t\int_0^1 G_{t-r}(x,z)\sigma'(u^{\epsilon,\delta}(z,r))S^{\epsilon,\delta}_{y,s}(z,r)W(dzdr) \\ &\quad +\int_s^t\int_0^1 G_{t-r}(x,z)[-f'(u^{\epsilon,\delta}(z,r))+\frac{1}{\delta}k_1'-\frac{1}{\epsilon}k_2']S^{\epsilon,\delta}_{y,s}(z,r)dyds. \end{aligned}$$

According to Theorem 1.2.6 (the comparison theorem of SPDE), we have the following properties:

(i)$S^{\epsilon,\delta}_{y,s}\geq 0$,

(ii)$0\leq S^{\epsilon,\delta}_{y,s}(x,t)\leq \widehat{S}^{\epsilon,\delta}_{y,s}(x,t)$ and $\widehat{S}^{\epsilon,\delta}_{y,s}(x,t)$ is the solution of SPDE:

$$\begin{aligned} \widehat{S}^{\epsilon,\delta}_{y,s} &= G_{t-s}(x,y)+\int_s^t\int_0^1 G_{t-r}(x,z)\sigma'(u^{\epsilon,\delta}(z,r))\widehat{S}^{\epsilon,\delta}_{y,s}(z,r)W(dzdr) \\ &\quad +\int_s^t\int_0^1 G_{t-r}(x,z)[-f'(u^{\epsilon,\delta}(z,r))]\widehat{S}^{\epsilon,\delta}_{y,s}(z,r)dzdr. \end{aligned} \tag{3.3.28}$$

Consequently,

$$|D_{y,s}u^{\epsilon,\delta}(x,t)|=|\sigma(u^{\epsilon,\delta}(y,s))|S^{\epsilon,\delta}_{y,s}(x,t)\leq|\sigma(u^{\epsilon,\delta}(y,s))|\widehat{S}^{\epsilon,\delta}_{y,s}(x,t). \tag{3.3.29}$$

According to Proposition 2.1 in [YZ2], we already have the following:

$$\sup_{\epsilon,\delta}E[\sup_{(y,s)\in[0,1]\times[0,T]}|u^{\epsilon,\delta}(y,s)|^p]<\infty. \tag{3.3.30}$$

We just need to prove

$$\sup_{\epsilon,\delta}E(\int_0^t\int_0^1|\widehat{S}^{\epsilon,\delta}_{y,s}|^2dyds)^p<\infty,\forall p\geq 1, \tag{3.3.31}$$

according to Theorem 1.2.2 (Lemma 1.2.3 in [N]).

We know from (3.3.28):

$$\begin{aligned} & |\widehat{S}^{\epsilon,\delta}_{y,s}(x,t)|^2 \\ \leq\ & c\{|G_{t-s}(x,y)|^2+|\int_s^t\int_0^1 G_{t-r}(z,r)\sigma'(u^{\epsilon,\delta}(z,r))\widehat{S}^{\epsilon,\delta}_{y,s}(z,r)W(dzdr)|^2 \\ & +|\int_s^t\int_0^1 G_{t-r}(x,z)[-f'(u^{\epsilon,\delta}(z,r))]\widehat{S}^{\epsilon,\delta}_{y,s}(z,r)dzdr|^2\}. \end{aligned}$$

Then,

$$\begin{aligned}
&|\int_0^t\int_0^1 |\widehat{S}^{\epsilon,\delta}_{y,s}(x,t)|^2 dyds|^p \\
\leq\ & c_p\{(\int_0^t\int_0^1 |G_{t-s}(x,y)|^2dyds)^p \\
&+(\int_0^t\int_0^1 |\int_s^t\int_0^1 G_{t-r}(x,z)\sigma'(u^{\epsilon,\delta}(z,r))\widehat{S}^{\epsilon,\delta}_{y,s}(z,r)W(dzdr)|^2dyds)^p \\
&+(\int_0^t\int_0^1 |\int_s^t\int_0^1 G_{t-r}(x,z)[-f'(u^{\epsilon,\delta}(z,r))]\widehat{S}^{\epsilon,\delta}_{y,s}(z,r)dzdr|^2dyds)^p\}.
\end{aligned}$$

We shall use Burkholder's inequality for Hilbert space (see [BP1] Inequality(4.18) P41) to get the following:

$$\begin{aligned}
&E|\int_0^t\int_0^1 |\widehat{S}^{\epsilon,\delta}_{y,s}(x,t)|^2 dyds|^p \\
\leq\ & c_p\{M \\
&+E(\int_0^t\int_0^1 |\int_s^t\int_0^1 G_{t-r}(x,z)\sigma'(u^{\epsilon,\delta}(z,r))\widehat{S}^{\epsilon,\delta}_{y,s}(z,r)W(dzdr)|^2dyds)^p \\
&+E(\int_0^t\int_0^1 |\int_s^t\int_0^1 G_{t-r}(x,z)[-f'(u^{\epsilon,\delta}(z,r))]\widehat{S}^{\epsilon,\delta}_{y,s}(z,r)dzdr|^2dyds)^p\} \\
\leq\ & c_p\{M \\
&+KE(\int_0^t\int_0^1 (\int_0^r\int_0^1 G^2_{t-r}(x,z)(\sigma'(u^{\epsilon,\delta}(z,r)))^2(\widehat{S}^{\epsilon,\delta}_{y,s}(z,r))^2dyds)dzdr)^p \\
&+E(\int_0^t\int_0^1 |\int_s^t\int_0^1 G^2_{t-r}(x,z)[-f'(u^{\epsilon,\delta}(z,r))]^2(\widehat{S}^{\epsilon,\delta}_{y,s}(z,r))^2dzdr|dyds)^p\} \\
\leq\ & c_p\{M+KE|\int_0^t\int_0^1 (\int_0^r\int_0^1 G^2_{t-r}(x,z)(\widehat{S}^{\epsilon,\delta}_{y,s}(z,r))^2dyds)dzdr|^p\} \\
=\ & c_p\{M+KE(\int_0^t\int_0^1 G^2_{t-r}(x,z)[\int_0^r\int_0^1 (\widehat{S}^{\epsilon,\delta}_{y,s}(z,r))^2dyds]dzdr)^p\} \\
\leq\ & c_pM+c_pKE\{(\int_0^t\int_0^1 G^{2\epsilon q}_{t-r}dzdr)^{\frac{p}{q}}\cdot\int_0^t\int_0^1 G^{2(1-\epsilon)p}_{t-r}[\int_0^t\int_0^1 (\widehat{S}^{\epsilon,\delta}_{y,s}(z,r))^2dyds]^pdzdr\},
\end{aligned}$$

where $\epsilon\in(1-\frac{3}{2p},\frac{3}{2}-\frac{3}{2p}), q=\frac{p}{p-1}$.

Then,

$$\begin{aligned}
&E|\int_0^t\int_0^1 |\widehat{S}^{\epsilon,\delta}_{y,s}(x,t)|^2 dyds|^p \\
\leq\ & c_pM+c_pKM\int_0^t\int_0^1 G^{2(1-\epsilon)p}_{t-r}E[\int_0^r\int_0^1 (\widehat{S}^{\epsilon,\delta}_{y,s}(z,r))^2dyds]^pdzdr \\
\leq\ & c_pM+c_pKM\int_0^t \sup_z E(\int_0^r\int_0^1 (\widehat{S}^{\epsilon,\delta}_{y,s}(z,r))^2dyds)^p(\int_0^1 G^{2(1-\epsilon)p}_{t-r}dz)dr
\end{aligned}$$

$$\leq \ c_pM + c_pKM\int_0^t \sup_z E[\int_0^r\int_0^1 (\widehat{S}^{\epsilon,\delta}_{y,s}(z,r))^2 dyds]^p (t-r)^a dr$$

where $a = \frac{1}{2} - (1-\epsilon)p$.

It's equivalent to

$$\begin{aligned} & \sup_x E[\int_0^t\int_0^1 (\widehat{S}^{\epsilon,\delta}_{y,s}(x,t))^2 dyds]^p \\ \leq & \ c_pM + c_pKM\int_0^t \sup_z E[\int_0^r\int_0^1 (\widehat{S}^{\epsilon,\delta}_{y,s}(z,r))^2 dyds]^p (t-r)^a dr \end{aligned} \quad (3.3.32)$$

Let

$$f(t) = \sup_x E[\int_0^t\int_0^1 (\widehat{S}^{\epsilon,\delta}_{y,s}(x,t))^2 dyds]^p. \quad (3.3.33)$$

Then,

$$f(t) \leq c_pM + c_pKM\int_0^t (t-r)^a f(r) dr \quad (3.3.34)$$

According to Gronwall's Inequality, we have,

$$\begin{aligned} f(t) & \leq \ c_pM + \int_0^t c_pMc_pKM(t-r)^a exp(\int_r^t (t-s)^a ds)dr \\ & = \ C + \int_0^t C(t-r)^a e^{-\frac{1}{a+1}(t-r)^{a+1}} dr \\ & = \ C + C'(e^{\frac{1}{a+1}t^{a+1}} - 1) \\ & < \ \infty. \end{aligned}$$

It shows that

$$\sup_x E[\int_0^t\int_0^1 (\widehat{S}^{\epsilon,\delta}_{y,s}(x,t))^2 dyds]^p \leq C + C'(e^{\frac{1}{a+1}t^{a+1}} - 1). \quad (3.3.35)$$

We can deduce from (3.3.35) that:

$$\sup_{\epsilon,\delta} E[\int_0^t\int_0^1 (\widehat{S}^{\epsilon,\delta}_{y,s}(x,t))^2 dyds]^p < \infty, \forall p \geq 1$$

□

Theorem 3.3.1 *If u is the solution of SPDE with two walls $(u_0;0,0;f,\sigma;h^1,h^2)$ and $\sigma > 0$ on $[h^1,h^2]$. Then, for all $(x_0,t_0) \in (0,1)\times R^{+*}$, the restriction on $(h^1(x_0,t_0),h^2(x_0,t_0))$ of the law of $u(x_0,t_0)$ is absolutely continuous.*

we will show that, for all $a > 0$, the restriction on $[h^1(x_0,t_0)+a, h^2(x_0,t_0)-b]$, the law of $u(x_0,t_0)$ is absolute continuous. From Proposition 2.2 in [BH] and Proposition 3.3 in [DP1], it remains to prove if $\sigma > 0$, then, $||Du(x_0,t_0)||_{L^2([0,1]\times R^+)} > 0$ on

$$\Omega_{a,b} = \{u(x_0,t_0) - h^1(x_0,t_0) \geq a, h^2(x_0,t_0) - u(x_0,t_0) \geq b\}.$$

And,

$$||Du(x_0,t_0)||_{L^2(R^+\times[0,1])} > 0 \Leftrightarrow \int_0^{t_0}\int_0^1 |D_{y,s}(u(x_0,t_0))|dyds > 0 \ a.s. \qquad (3.3.36)$$

if $\sigma > 0$, then $D_{y,s}u^{\epsilon,\delta}(x_0,t_0) \geq 0$ by Eq(3.3.27). By weak limit, $D_{y,s}u(x_0,t_0) \geq 0$, for $(y,s) \in [0,1]\times[0,t_0]$. Inequality (3.3.36) is equivalent to

$$\int_0^{t_0}\int_0^1 D_{y,s}u(x_0,t_0)dyds > 0 \ on \ \Omega_{a,b} \qquad (3.3.37)$$

To demonstrate (3.3.37), we will give a lower bound of $D_{y,s}u(x_0,t_0)$.
$(x_0,t_0) \in (0,1)\times R^{+*}$, for $y < x_0$ and $s < t_0$, we note $\{w(y,s;x,t); x \in [y, \widetilde{y} = (2x_0 - y)\wedge 1], t > s\}$ is the solution of SPDE:

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} - \dfrac{\partial^2 w(x,t)}{\partial x^2} = \sigma'(u(x,t))w(x,t)\dot{W}(x,t) + f'(u(x,t))w(x,t), \\ w(x,s) = \sigma(u(x,s)), y < x < \widetilde{y}, \\ w(y,t) = w(\widetilde{y},t) = 0, t > s. \end{cases} \qquad (3.3.38)$$

(We have omitted the dependence of w of y,s for abbreviation.)

Proposition 3.3.3 *Suppose $a > 0$ and $(x_0,t_0) \in (0,1)\times R^{+*}$. For $y < x_0$ and $s < t_0$, we define*

$$B_{y,s} = \{w \in \Omega, \inf_{z\in[y,\widetilde{y}]}(u(z,s) - h^1(z,s)) > \frac{a}{2} \ and \ \inf_{z\in[y,\widetilde{y}]}(h^2(z,s) - u(z,s)) > \frac{b}{2}\},$$

$B_{y,s}$ is $\mathcal{F}_s$-measurable. If $\tau_{y,s}$ is stopping time defined by

$$\tau_{y,s} = \inf\{t \geq s, inf_{z\in[y,\widetilde{y}]}(u(z,t)-h^1(z,t)) = \frac{a}{2} \ or \ inf_{z\in[y,\widetilde{y}]}(h^2(z,t)-u(z,t)) = \frac{b}{2}\}. \qquad (3.3.39)$$

Then,

$$\int_y^{\widetilde{y}} D_{z,s}u(x_0,t_0)dz \geq w(y,s;x_0,t_0)I_{\{\tau_{y,s}>t_0\}}a.s. \qquad (3.3.40)$$

$w(y,s;x,t)$ is the solution of (3.3.38) and $w(y,s;x_0,t_0) > 0$ a.s.

Lemma 3.3.1 $v^{\epsilon,\delta}(y,s;x,t) \geq w^{\epsilon,\delta}(y,s;x,t), \forall t > s, x \in [y, \widetilde{y}]$. *a.s.*

Lemma 3.3.2 *There exists a subsequence of* $w^{\epsilon,\delta}$ *(we still note it* $w^{\epsilon,\delta}$*) such that*

$$w^{\epsilon,\delta}(y,s;x_0,t_0 \wedge \tau_{y,s})I_{B_{y,s}} \longrightarrow w(y,s;x_0,t_0 \wedge \tau_{y,s})I_{B_{y,s}},$$

and $w(y,s;x,t)$ *is solution of SPDE(3.3.38) which can be written as integral:*

$$\begin{aligned} w(y,s;x,t) &= \int_y^{\widetilde{y}} \widetilde{G_{t-s}}(x,z)\sigma(u(z,s))dz \\ &+ \int_s^t \int_y^{\widetilde{y}} \widetilde{G_{t-r}}(x,z)\sigma^{'}(u(z,r))w(y,s;z,r)W(dzdr) \\ &+ \int_s^t \int_y^{\widetilde{y}} \widetilde{G_{t-r}}(x,z)f^{'}(u(z,r))w(y,s;z,r)dzdr, t > s, y < x < \widetilde{y}. \end{aligned}$$

We leave the proofs of Lemma 3.3.1 and 3.3.2 to the end of this section.

Demonstration of Proposition 3.3.1: Observe first that $B_{y,s} = \{\tau_{y,s} > s\}$ by continuity of u and

$$\begin{aligned} \{\tau_{y,s} > t_0\} &= \{w, \inf_{z\in[y,\widetilde{y}], r\in[s,t_0]} (u(z,r) - h^1(z,r)) > \frac{a}{2} \text{ and} \\ &\qquad \inf_{z\in[y,\widetilde{y}], r\in[s,t_0]} (h^2(z,r) - u(z,r)) > \frac{b}{2}\}, \end{aligned}$$

fix $(y,s) \in [0,x_0) \times [0,t_0)$. According to Proposition 3.3.2, $\int_y^{\widetilde{y}} D_{z,s}u(x_0,t_0)dz$ is the weak limit in $L^p(\Omega)$ of the subsequence of $\int_y^{\widetilde{y}} D_{z,s}u^{\epsilon,\delta}(x_0,t_0)dz$.
Note $v(y,s;x,t) := \int_y^{\widetilde{y}} D_{z,s}u(x,t)dz$, and $v^{\epsilon,\delta}(y,s;x,t) := \int_y^{\widetilde{y}} D_{z,s}u^{\epsilon,\delta}(x,t)dz$, for $s < t$.
$v^{\epsilon,\delta}$ is the solution of linear SPDE:

$$\begin{aligned} & v^{\epsilon,\delta}(y,s;x,t) \\ = & \int_y^{\widetilde{y}} G_{t-s}(x,z)\sigma(u^{\epsilon,\delta}(z,s))dz \\ & + \int_s^t \int_0^1 G_{t-r}(x,z)\sigma^{'}(u^{\epsilon,\delta}(z,r))v^{\epsilon,\delta}(y,s;z,r)W(dzdr) \\ & + \int_s^t \int_0^1 G_{t-r}(x,z)f^{'}_{\epsilon,\delta}(u^{\epsilon,\delta}(z,r))v^{\epsilon,\delta}(y,s;z,r)drdz, t > s; \\ & f^{'}_{\epsilon,\delta}(u^{\epsilon,\delta}(z,r)) \\ = & [f(u^{\epsilon,\delta}(z,r)) + \frac{1}{\delta}k_1 - \frac{1}{\epsilon}k_2]^{'}. \end{aligned}$$

Introduce $w^{\epsilon,\delta}(y,s;x,t)$ to be the solution of the same SPDE as $v^{\epsilon,\delta}(y,s;x,t)$ restricted in the interval $[y,\widetilde{y}]$ with Dirichlet conditions at $y,\widetilde{y}$.

$$\begin{cases} \dfrac{\partial w^{\epsilon,\delta}(x,t)}{\partial t} - \dfrac{\partial^2 w^{\epsilon,\delta}(x,t)}{\partial x^2} = \sigma'(u^{\epsilon,\delta}(x,t))w^{\epsilon,\delta}(x,t)\dot{W}(x,t) \\ \qquad\qquad + f'_{\epsilon,\delta}(u^{\epsilon,\delta}(x,t))w^{\epsilon,\delta}(x,t); \\ w^{\epsilon,\delta}(x,s) = \sigma(u^{\epsilon,\delta}(x,s)), y < x < \widetilde{y}; \\ \quad w^{\epsilon,\delta}(y,t) = w^{\epsilon,\delta}(\widetilde{y},t) = 0, t > s. \end{cases} \tag{3.3.41}$$

(We have omitted the dependence of $w^{\epsilon,\delta}$ of y,s for abbreviation.)

We have the integral form:

$$\begin{aligned} w^{\epsilon,\delta}(y,s;x,t) &= \int_y^{\widetilde{y}} \widetilde{G_{t-s}}(x,z)\sigma(u^{\epsilon,\delta}(z,s))dz \\ &+ \int_s^t \int_y^{\widetilde{y}} \widetilde{G_{t-r}}(x,z)\sigma'(u^{\epsilon,\delta}(z,r))w^{\epsilon,\delta}(y,s;z,r)W(dzdr) \\ &+ \int_s^t \int_y^{\widetilde{y}} \widetilde{G_{t-r}}(x,z)f'_{\epsilon,\delta}(u^{\epsilon,\delta}(z,r))w^{\epsilon,\delta}(y,s;z,r)dzdr, \\ & t > s, y < x < \widetilde{y}, \end{aligned}$$

where $f'_{\epsilon,\delta}(u^{\epsilon,\delta}(z,r)) = [f(u^{\epsilon,\delta}(z,r)) + \frac{1}{\delta}k_1 - \frac{1}{\epsilon}k_2]'$.

$\widetilde{G}$ denotes the fundamental solution of the heat equation with Dirichlet conditions on y and $\widetilde{y}$($\widetilde{G}$ depends on y).

Next we will use Lemma 3.3.1 and Lemma 3.3.2 to get our result:

Note: $v(y,s;x_0,t_0) = \int_y^{\widetilde{y}} D_{z,s}u(x_0,t_0)dz \geq 0$,

$$\begin{aligned} v(y,s;x_0,t_0) &\geq v(y,s;x_0,t_0)I_{\{\tau_{y,s}>t_0\}} \\ &= \lim_{\epsilon,\delta\to 0} v^{\epsilon,\delta}(y,s;x_0,t_0)I_{\{\tau_{y,s}>t_0\}} \\ v^{\epsilon,\delta}(y,s;x_0,t_0)I_{\{\tau_{y,s}>t_0\}} &\geq w^{\epsilon,\delta}(y,s;x_0,t_0)I_{\{\tau_{y,s}>t_0\}} \end{aligned}$$

and

$$w^{\epsilon,\delta}(y,s;x_0,t_0)I_{\{\tau_{y,s}>t_0\}} \longrightarrow w(y,s;x_0,t_0)I_{\{\tau_{y,s}>t_0\}} a.s. \tag{3.3.42}$$

$w(y,s;x_0,t_0) > 0$ is a consequence of the result in Pardoux and Zhang [PZ](Proposition 3.1). □

Demonstration of Theorem 3.3.1: By Proposition 3.3.3, for all $s < t_0$ and $y < x_0$, there exists a measurable set $\Omega_{y,s}$ of probability 1 such that $\forall \omega \in \Omega_{y,s}$, we have:

$$v(y,s;x_0,t_0)(\omega) \geq w(y,s;x_0,t_0)I_{\tau_{y,s}>t_0}(\omega) \tag{3.3.43}$$

$$\textit{and } w(y,s;x_0,t_0) > 0. \tag{3.3.44}$$

We define $\widetilde{\Omega_s} = \cap_{y\in[0,x_0)\cap Q}\Omega_{y,s}$ and then $P(\widetilde{\Omega_s}) = 1$. In order to prove (3.3.37), we need the following estimate.

By continuity of u, there exist two random variables S_0 and Y_0 such that $Y_0 < x_0$, and $S_0 < t_0$ on $\Omega_{a,b}$ and

$$u(z,s) - h^1(z,s) > \frac{a}{2},\ h^2(z,s) - u(z,s) > \frac{b}{2}\ \forall r \in [S_0,t_0], z \in [Y_0, \widetilde{Y_0}] a.s.on \Omega_{a,b} \tag{3.3.45}$$

A sufficient condition to prove (3.3.37) is

$$\int_{S_0}^{t_0} ds \int_0^1 D_{z,s}u(x_0,t_0)dz > 0 \textit{ on } \Omega_{a,b} \tag{3.3.46}$$

Note $k(s) = \int_0^1 D_{z,s}u(x_0,t_0)dz$, (3.3.46) can be verified if we show $k(s) > 0$ a.s. on $\Omega_{a,b}$, $\forall S_0 \leq s \leq t_0$.

On $\Omega_{a,b} \cap \widetilde{\Omega_s}$,

$$\begin{aligned} k(s) &\geq v(y,s;x_0,t_0)\ \ \forall y \in Q & (3.3.47)\\ &\geq w(y,s;x_0,t_0)I_{\{\tau_{y,s}>t_0\}}. & (3.3.48)\end{aligned}$$

Take $y \in [Y_0, x_0) \cap Q$, then

$$I_{\{\tau_{y,s}>t_0\}} = 1$$

and

$$k(s) \geq w(y,s;x_0,t_0) > 0$$

according to (3.3.44).□

Demonstration of Lemma 3.3.1:

The proof of Lemma 3.3.1 is the same as Proposition 5.1 and Corollary 5.1 in Appendix of [DP1].

Demonstration of Lemma 3.3.2:

Step 1: we introduce the intermediate solution $\bar{w}^{\epsilon,\delta}$ of SPDE which is similar as $w^{\epsilon,\delta}$:

$$\begin{aligned}\bar{w}^{\epsilon,\delta}(y,s;x,t) &= \int_y^{\widetilde{y}} \widetilde{G_{t-s}}(x,z)\sigma(u^{\epsilon,\delta}(z,s))dz \\ &+ \int_s^t \int_y^{\widetilde{y}} \widetilde{G_{t-r}}(x,z)\sigma'(u^{\epsilon,\delta}(z,r))w^{\epsilon,\delta}(y,s;z,r)W(dzdr) \\ &+ \int_s^t \int_y^{\widetilde{y}} \widetilde{G_{t-r}}(x,z)f'(u^{\epsilon,\delta}(z,r))\bar{w}^{\epsilon,\delta}(y,s;z,r)dzdr, t>s, y<x<\widetilde{y}\end{aligned}$$

so that $w^{\epsilon,\delta}(y,s;x,t)-\bar{w}^{\epsilon,\delta}(y,s;x,t)$ satisfies the following PDE with random coefficients:

$$\begin{aligned}&w^{\epsilon,\delta}(y,s;x,t)-\bar{w}^{\epsilon,\delta}(y,s;x,t) \\ =& \int_s^t \int_y^{\widetilde{y}} \widetilde{G_{t-r}}(x,z)[f'_{\epsilon,\delta}(u^{\epsilon,\delta}(z,r))w^{\epsilon,\delta}(y,s;z,r)-f'(u^{\epsilon,\delta}(z,r))\bar{w}^{\epsilon,\delta}(y,s;z,r)]dzdr\end{aligned} \tag{3.3.49}$$

Next we will show that for $t>s, x\in(y,\widetilde{y})$,

$$[w^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s})-\bar{w}^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s})]I_{B_{y,s}} \longrightarrow 0. \tag{3.3.50}$$

Fix a trajectory $w\in B_{y,s}$ and consider the previous equation (3.3.49) at $t\wedge\tau_{y,s}(w)$, $\forall(z,r)\in[y,\widetilde{y}]\times[s,t\wedge\tau_{y,s}(w)]$, we have $u(z,r)-h^1(z,r)>\frac{a}{2}, h^2(z,r)-u(z,r)>\frac{b}{2}$. Since $u^{\epsilon,\delta}$ uniformly converges to u on $[0,T]\times[0,1]$, then there exists $\epsilon_0(w)>0$ such that $\epsilon<\epsilon_0$, $u^{\epsilon,\delta}(z,r)-h^1(z,r)>\frac{a}{4}$; and there exists $\delta_0(w)>0$ such that $\delta<\delta_0$, $h^2(z,r)-u^{\epsilon,\delta}(z,r)>\frac{b}{4}$.

Then for $(z,r)\in[y,\widetilde{y}]\times[s,t\wedge\tau_{y,s}]$, we have $f'_{\epsilon,\delta}(u^{\epsilon,\delta}(z,r))=f'(u^{\epsilon,\delta}(z,r))$, for $\epsilon<\epsilon_0, \delta<\delta_0$.

For $t>s, x\in[y,\widetilde{y}]$,

$$\begin{aligned}&w^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s}(w))-\bar{w}^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s}(w))(w) \\ =& \int_s^{t\wedge\tau_{y,s}} \int_y^{\widetilde{y}} \widetilde{G_{t-r}}(x,z)[f'(u^{\epsilon,\delta}(z,r))(w^{\epsilon,\delta}(y,s;z,r)-\bar{w}^{\epsilon,\delta}(y,s;z,r))]dzdr.\end{aligned}$$

Then,

$$|w^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s}(w))-\bar{w}^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s}(w))(w)|^2$$

$$= \ |\int_s^{t\wedge\tau_{y,s}}\int_y^{\widetilde{y}}\widetilde{G_{t-r}}(x,z)[f'(u^{\epsilon,\delta}(z,r))(w^{\epsilon,\delta}(y,s;z,r)-\bar{w}^{\epsilon,\delta}(y,s;z,r))]dzdr|^2$$
$$\leq \ K\int_s^{t\wedge\tau_{y,s}}\int_y^{\widetilde{y}}\widetilde{G_{t-r}}^2(x,z)dzdr\int_s^{t\wedge\tau_{y,s}}\int_y^{\widetilde{y}}|w^{\epsilon,\delta}(y,s;z,r)-\bar{w}^{\epsilon,\delta}(y,s;z,r)|^2dzdr.$$

We deduce that

$$\sup_x|w^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s}(w))-\bar{w}^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s}(w))(w)|^2$$
$$\leq \ KM_t\int_s^{t\wedge\tau_{y,s}}\sup_z|w^{\epsilon,\delta}(y,s;z,r)-\bar{w}^{\epsilon,\delta}(y,s;z,r)|^2(\widetilde{y}-y)dr.$$

According to Gronwall's Lemma:

$$\sup_x|w^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s}(w))-\bar{w}^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s}(w))(w)|^2=0.a.s. \qquad (3.3.51)$$

Then,

$$|w^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s}(\omega))(\omega)-\bar{w}^{\epsilon,\delta}(y,s;x,t\wedge\tau_{y,s}(\omega))(\omega)|=0 \ for \ \epsilon<\epsilon_0,\delta<\delta_0. \qquad (3.3.52)$$

We have proved (3.3.50).

Step 2: $\bar{w}^{\epsilon,\delta}\longrightarrow w$

Note that the sequence of $w^{\epsilon,\delta}$ and $\bar{w}^{\epsilon,\delta}$ are bounded in $L^p(\Omega;L^p([y,\widetilde{y}]\times[s,t]))$ *i.e.*

$$\sup_{\epsilon,\delta}E[\int_s^t\int_y^{\widetilde{y}}(w^{\epsilon,\delta}(y,s;z,r))^pdrdz]<\infty, \qquad (3.3.53)$$

$$\sup_{\epsilon,\delta}E[\int_s^t\int_y^{\widetilde{y}}(\bar{w}^{\epsilon,\delta}(y,s;z,r))^pdzdr]<\infty, \qquad (3.3.54)$$

The convergence *a.s.* obtained in (3.3.50) together with Inequalities (3.3.53) and (3.3.54) obtained for p, ensuring the convergence of

$$[w^{\epsilon,\delta}(y,s;\cdot,\cdot\wedge\tau_{y,s})-\bar{w}^{\epsilon,\delta}(y,s;\cdot,\cdot\wedge\tau_{y,s})]I_{B_{y,s}} \ to \ 0$$

in $L^p(\Omega;L^p([y,\widetilde{y}]\times[s,T]))$, that is to say

$$E[\int_s^{T\wedge\tau_{y,s}}\int_y^{\widetilde{y}}(w^{\epsilon,\delta}(y,s;z,r)-\bar{w}^{\epsilon,\delta}(y,s;z,r))^pdzdr]\longrightarrow 0, \quad \epsilon,\delta\to 0$$

$$w(x,t)-\bar{w}^{\epsilon,\delta}(x,t)$$

$$
\begin{aligned}
= & \int_y^{\widetilde{y}} \widetilde{G_{t-s}}(x,z)[\sigma(u(z,s)) - \sigma(u^{\epsilon,\delta}(z,s))]dz \\
& + \int_s^t \int_y^{\widetilde{y}} \widetilde{G_{t-r}}(x,z)[\sigma^{'}(u(z,r))w(z,r) - \sigma^{'}(u^{\epsilon,\delta}(z,r))w^{\epsilon,\delta}(y,s;z,r)]W(dzdr) \\
& + \int_s^t \int_y^{\widetilde{y}} \widetilde{G_{t-r}}(x,z)[f^{'}(u(z,r))w(z,r) - f^{'}(u^{\epsilon,\delta}(z,r))\bar{w}^{\epsilon,\delta}(y,s;z,r)]dzdr, \\
& for \quad t > s, y < x < \widetilde{y}
\end{aligned}
$$

Let

$$
F^{\epsilon,\delta}(t) = \sup_{x\in[y,\widetilde{y}]} E[|w(x,t\wedge\tau_{y,s}) - \bar{w}^{\epsilon,\delta}(x,t\wedge\tau_{y,s})|^p I_{B_{y,s}}], t > s \tag{3.3.55}
$$

Following the similar steps as P.417 in [DP1], we can show

$$
F^{\epsilon,\delta}(t) \le K_p(C^{\epsilon,\delta} + \int_s^t F^{\epsilon,\delta}(r)dr) \ and \ C^{\epsilon,\delta} \longrightarrow 0 \tag{3.3.56}
$$

From Gronwall Lemma: $F^{\epsilon,\delta}(t) \longrightarrow 0, \ \epsilon,\delta \to 0$

So we have a subsequence of $\bar{w}^{\epsilon,\delta}$ (still denote it $\bar{w}^{\epsilon,\delta}$) such that

$$
|w(x,t\wedge\tau_{y,s}) - \bar{w}^{\epsilon,\delta}(x,t\wedge\tau_{y,s})|^p I_{B_{y,s}} \longrightarrow 0 \ (\epsilon,\delta \to 0). \tag{3.3.57}
$$

□

Chapter 4

Existence and Uniqueness of the Solutions of Elliptic Stochastic Partial Differential Equations with Two Reflecting Walls

4.1 Introduction

In this chapter we will consider the following elliptic stochastic partial differential equations (SPDEs) with Dirichlet boundary condition on a bounded domain D of $\mathbb{R}^k$, $k = 1, 2, 3$.

$$-\Delta u(x) + f(x; u(x)) = \eta(x) - \xi(x) + \sigma(x; u(x))\dot{W}(x), \quad x \in D, \tag{4.1.1}$$

where $\{\dot{W}(x),\ x \in D\}$ is a white noise in D. We are looking for a continuous random field $u(x),\ x \in D$ which is the solution of equation (4.1.1) satisfying $h^1(x) \leq u(x) \leq h^2(x)$, where h^1 and h^2 are given two walls. When $u(x)$ hits $h^1(x)$ or $h^2(x)$, the additional forces are added to prevent u from leaving $[h^1, h^2]$. These forces are expressed by random measures ξ and η in equation(4.1.1) which play a similar role as the local time in the usual Skorokhod equation constructing Brownian motions with reflecting barriers. SPDEs with two reflecting walls can be used to model the evolution of random interfaces near two hard walls, see T. Funaki and S. Olla [FO]. For nonlinear elliptic PDEs with measures as right side or boundary condition, we refer to Boccardo, Gallouet [BG] and Rockner, Zegarlinski [RZ].

For elliptic SPDEs without reflection, R. Buckdahn and E. Pardoux in [BP2] established the existence and uniqueness of the solutions of nonlinear elliptic stochastic partial differential equations driven by additive noise. Based on this, elliptic SPDEs with reflection at zero driven by additive noise, have been studied by David Nualart and Samy Tindel in [NT].

In our present chapter, we will study the elliptic SPDEs with two reflecting walls driven by multiplicative noise. This is the first time to consider the case of multiplicative noise. We will establish the existence and uniqueness of the solutions. A similar problem for reflected stochastic heat equations has been studied by Nualart and Pardoux in [NP], Donati-Martin and Pardoux in [DP], Yang and Zhang in [YZ1] and by Xu and Zhang in [XZ]. Our approaches were inspired by the ones in [NP], [NT], [O] and [XZ].

The rest of the chapter is organized as follows. In Section 2, we lay down the framework of the chapter. In Section 3, we study deterministic reflected elliptic PDEs and obtain some a priori estimates. The main result is established in Section 4.

4.2 Framework

Let D be an open bounded subset of $\mathbb{R}^k$, with $k \in \{1, 2, 3\}$. Consider a Gaussian family of random variables $\{W = W(B), B \in \mathcal{B}(D)\}$, where $\mathcal{B}(D)$ is the Borel σ-field on D, defined in a complete probability space $(\Omega, \mathcal{F}, P)$, such that $E(W(B)) = 0$ and

$$E(W(A)W(B)) = |A \cap B|, \tag{4.2.1}$$

where $|A \cap B|$ denotes the Lebesgue measure of the set $A \cap B$. We want to study a reflected nonlinear stochastic elliptic equation with Dirichlet condition driven by multiplicative noise:

$$-\Delta u(x) + f(x, u(x)) = \sigma(x; u(x))\dot{W}(x), \qquad (4.2.2)$$

where $x \in D$ while $h^1(x) \le u(x) \le h^2(x)$, $\dot{W}(x)$ is the formal derivative of W with respect to the Lebesgue measure and the symbol Δ denotes the Laplace operator on $L^2(D)$. If $u(x)$ hits $h^1(x)$ or $h^2(x)$, additional forces are added in order to prevent u from leaving $[h^1, h^2]$. Such an effect will be expressed by adding extra(unknown) terms ξ and η in (4.2.2) which play a similar role as the local time in the usual Skorokhod equation constructing Brownian motions with reflecting boundaries.

$\mathcal{C}_0^\infty(D)$ denotes the set of infinitely differentiable functions on D with compact supports. We will denote by $(\cdot, \cdot)$ the scalar product in $L^2(D)$, and by $||\cdot||_\infty$ the supremum norm on D. Let $f, \sigma : D \times \mathbb{R} \to \mathbb{R}$ be measurable functions. We will also denote by $f(u)$ the function $f(u)(x) = f(x, u(x))$, $\sigma(u)$ the function $\sigma(u)(x) = \sigma(x, u(x))$. We introduce the following hypotheses on f and σ:

(F1) The function f is locally bounded, continuous and nondecreasing as a function of the second variable.

(Σ 1) The function σ is Lipschitz continuous:

$$|\sigma(x, z_1) - \sigma(x, z_2)| \le C_\sigma |z_1 - z_2|.$$

(H1) The walls $h^i(x), i = 1, 2$, are continuous functions satisfying $h^1(x) < h^2(x)$ for $x \in D$ and $h^1(x) \le 0 \le h^2(x)$ for $x \in \partial D$.

The solution to Eq(4.1.1) will be a triplet (u, η, ξ) such that $h^1(x) \le u(x) \le h^2(x)$ on D which satisfies Eq(4.1.1) in the sense of distributions, and $\eta(dx)$, $\xi(dx)$ are random measures on D which force the process u to be in the interval $[h^1, h^2]$. More precisely, a rigorous definition of the solution to Eq(4.1.1) is given as follows:

Definition 4.2.1 *A triplet* (u, η, ξ) *defined on a complete probability space* $(\Omega, \mathcal{F}, P)$ *is a solution to the SPDE (4.1.1), denoted by* $(0; f; \sigma; h^1, h^2)$, *if* *(i)* $\{u(x), x \in D\}$ *is a continuous random field on* D *satisfying* $h^1(x) \leq u(x) \leq h^2(x)$ *and* $u_{|\partial D} = 0$ *a.s.*

(ii) $\eta(dx)$ *and* $\xi(dx)$ *are random measures on* D *such that* $\eta(K) < \infty$ *and* $\xi(K) < \infty$ *for all compact subset* $K \subset D$.

(iii) For all $\phi \in \mathcal{C}_0^\infty(D)$, *we have*

$$-(u, \Delta\phi)+(f(u), \phi) = \int_D \phi(x)\sigma(u)W(dx)+\int_D \phi(x)\eta(dx)-\int_D \phi(x)\xi(dx). \quad P-a.s. \tag{4.2.3}$$

(iv) $\int_D (u(x) - h^1(x))\eta(dx) = \int_D (h^2(x) - u(x))\xi(dx) = 0.$

4.3 Deterministic obstacle problem

Let h^1, h^2 be as in Section 2, and f satisfies $(F1)$. Let $v(x) \in C(D)$ with $v|_{\partial D} = 0$. Consider a deterministic elliptic PDE with two reflecting walls:

$$\begin{cases} -\Delta z + f(z+v) = \eta - \xi \\ h^1 \leq z + v \leq h^2 \\ z|_{\partial D} = 0. \end{cases} \tag{4.3.1}$$

Here is a precise definition of the solution of equation (4.3.1).

Definition 4.3.1 *A triplet* (z, η, ξ) *is called a solution to the PDE (4.3.1) if*

(i) $z = z(x); x \in D$ *is a continuous function satisfying* $h^1(x) \leq z(x) + v(x) \leq h^2(x)$, $z|_{\partial D} = 0$.

(ii) $\eta(dx)$ *and* $\xi(dx)$ *are measures on* D *such that* $\eta(K) < \infty$ *and* $\xi(K) < \infty$ *for all compact subset* $K \subset D$.

(iii) For all $\phi \in \mathcal{C}_0^\infty(D)$ *we have*

$$-(z, \Delta\phi) + (f(z+v), \phi) = \int_D \phi(x)\eta(dx) - \int_D \phi(x)\xi(dx). \tag{4.3.2}$$

(iv)$\int_D (z(x) + v(x) - h^1(x))\eta(dx) = \int_D (h^2(x) - z(x) - v(x))\xi(dx) = 0.$

The following result is the existence and uniqueness of the solutions of the PDE with two reflecting walls (4.3.1).

Theorem 4.3.1 *Equation (4.3.1) admits a unique solution (z, η, ξ).*

We first consider the problem of a single reflecting barrier, denoted by $(0; f; h^1)$:

$$\begin{cases} -\Delta z + f(z+v) = \eta(x) \\ z + v \geq h^1 \\ z|_{\partial D} = 0 \\ \int_D (z + v - h^1)\eta(dx) = 0, \end{cases} \tag{4.3.3}$$

where the coefficient f satisfies $(F1)$ and h^1 satisfies $(H1)$ in Section 2.

In the next lemma, we give the existence and uniqueness of the solution of $(0; f; h^1)$, and it follows from Theorem 2.2 in David Nualart and Samy Tindle [NT] using similar methods.

Lemma 4.3.1 *Let v be a continuous function on $\bar{D}$ such that $v|_{\partial D} = 0$. There exists a unique pair (z, η) such that:*
(i) z is a continuous function on $\bar{D}$ such that $z|_{\partial D} = 0$ and $z + v \geq h^1$.
(ii) η is a measure on on D such that $\eta(K) < \infty$ for any compact set $K \subset D$.
(iii) For every $\phi \in \mathcal{C}_k^\infty(\mathcal{D})$, we have

$$-(z, \Delta\phi) + (f(z+v), \phi) = \int_D \phi(x)\eta(dx).$$

(iv)$\int_D (z(x) + v(x) - h^1(x))\eta(dx) = 0.$

Theorem 2.2 from David Nualart and Samy Tindel:
Let v be a continuous function on $\bar{D}$ such that $v|_{\partial D=0}$. There exist a unique pair (z, η) such that:
(i) z is a continuous function on $\bar{D}$ such that $z|_{\partial D} = 0$ and $z \geq -v$.
(ii) η is a measure on D such that $\eta(K) < \infty$ for any compact set $K \subset D$.
(iii) For every $\phi \in \mathcal{C}_k^\infty(\mathcal{D})$, we have

$$-(z, \Delta\phi) + (f(z+v), \phi) = \int_D \phi(x)\eta(dx).$$

(iv)$\int_D (z(x)+v(x))\eta(dx)=0.$

Next lemma is a comparison theorem for the PDE with reflection.

Lemma 4.3.2 *(comparison)*
Let (z_1,η_1) and (z_2,η_2) be solutions to single reflection problems $(0;f_1,h_1)$ and $(0;f_2,h_2)$ respectively as in (4.3.3). If $f_1 \le f_2$, and $h_1 \ge h_2$, for every $x\in D$, then we have $z_1(x)\ge z_2(x)$.

Proof. Let z_1^ϵ and z_2^ϵ be the solutions of the following PDEs:

$$\begin{cases} -\Delta z_1^\epsilon(x) + f_1(z_1^\epsilon+v)(x) = \dfrac{1}{\epsilon}(z_1^\epsilon+v-h_1)^-(x) \\ z_1^\epsilon|_{\partial D}=0. \end{cases} \tag{4.3.4}$$

$$\begin{cases} -\Delta z_2^\epsilon(x) + f_2(z_2^\epsilon+v)(x) = \dfrac{1}{\epsilon}(z_2^\epsilon+v-h_2)^-(x) \\ z_2^\epsilon|_{\partial D}=0. \end{cases} \tag{4.3.5}$$

According to [NT], $z_1^\epsilon\to z_1$ and $z_2^\epsilon\to z_2$ uniformly on $\bar{D}$ as $\epsilon\to 0$.
Let $\psi = z_2^\epsilon - z_1^\epsilon$, then

$$\begin{cases} -\Delta\psi + f_2(z_2^\epsilon+v) - f_1(z_1^\epsilon+v) = \dfrac{1}{\epsilon}[(z_2^\epsilon+v-h_2)^- - (z_1^\epsilon+v-h_1)^-] \\ \psi|_{\partial D}=0. \end{cases} \tag{4.3.6}$$

Multiplying (4.3.6) by ψ^+, we obtain

$$(-\Delta\psi,\psi^+) + (f_2(z_2^\epsilon+v) - f_1(z_1^\epsilon+v),\psi^+) = \frac{1}{\epsilon}([(z_2^\epsilon+v-h_2)^- - (z_1^\epsilon+v-h_1)^-],\psi^+) \tag{4.3.7}$$

Note that,

$$-(\Delta\psi,\psi^+) = (\nabla\psi,\nabla\psi^+) = (\nabla\psi^+,\nabla\psi^+) = ||\nabla\psi^+||^2_{L^2(D)} \ge 0. \tag{4.3.8}$$

If $\psi^+(x)\ne 0$, we have $z_2^\epsilon(x) > z_1^\epsilon(x)$. Because f_2 is increasing and $f_1\le f_2$, we also have

$$(f_2(z_2^\epsilon+v) - f_1(z_1^\epsilon+v),\psi^+) \ge 0. \tag{4.3.9}$$

Since $h_1 \geq h_2$, we have $z_2^\epsilon(x) + v(x) - h_2(x) \geq z_1^\epsilon(x) + v(x) - h^1(x)$ and then

$$\frac{1}{\epsilon}([(z_2^\epsilon(x) + v(x) - h_2(x))^- - (z_1^\epsilon(x) + v(x) - h_1(x))^-], \psi^+(x)) \leq 0. \quad (4.3.10)$$

Thus it follows from (4.3.7),(4.3.8),(4.3.9) and (4.3.10) that:

$$||\nabla \psi^+||^2_{L^2(D)} = 0.$$

Hence, by the boundary condition $\psi^+|_{\partial D} = 0$, we get $\psi^+ = 0$ and then $z_2^\epsilon \leq z_1^\epsilon$, for every $\epsilon > 0$. Hence, the lemma follows immediately by taking $\epsilon \to 0$. □

Lemma 4.3.3 *Let v and $\hat{v}$ be given continuous functions and let $z^{\epsilon,\delta}$ be a unique solution to the following deterministic PDE:*

$$\begin{cases} -\Delta z^{\epsilon,\delta}(x) + f(z^{\epsilon,\delta} + v)(x) = \frac{1}{\delta}(z^{\epsilon,\delta}(x) + v(x) - h^1(x))^- - \frac{1}{\epsilon}(z^{\epsilon,\delta}(x) + v(x) - h^2(x))^+, \\ z^{\epsilon,\delta}|_{\partial D} = 0. \end{cases} \quad (4.3.11)$$

We also denote by $\hat{z}^{\epsilon,\delta}$ the solution to the above PDE replacing v by $\hat{v}$. Then we have, $||z^{\epsilon,\delta} - \hat{z}^{\epsilon,\delta}||_\infty \leq ||v - \hat{v}||_\infty$, where $||w||_\infty = \sup_{x \in D} |w(x)|$.

PROOF. Define $w(x) = z^{\epsilon,\delta}(x) - \hat{z}^{\epsilon,\delta}(x) - l$, where $l = ||v - \hat{v}||_\infty$.
Then, w satisfies the following PDE:

$$\begin{aligned} -\Delta w + f(z^{\epsilon,\delta} + v) - f(\hat{z}^{\epsilon,\delta} + \hat{v}) &= \frac{1}{\delta}[(z^{\epsilon,\delta} + v - h^1)^- - (\hat{z}^{\epsilon,\delta} + \hat{v} - h^1)^-] \\ &\quad -\frac{1}{\epsilon}[(z^{\epsilon,\delta} + v - h^2)^+ - (\hat{z}^{\epsilon,\delta} + \hat{v} - h^2)^+] \end{aligned} \quad (4.3.12)$$

Set

$$F_{\epsilon,\delta}(u) = f(u) - \frac{1}{\delta}(u - h^1)^- + \frac{1}{\epsilon}(u - h^2)^+$$

Now we note that, if $w^+(x) > 0$, we have $z^{\epsilon,\delta}(x) + v(x) > \hat{z}^{\epsilon,\delta}(x) + \hat{v}(x)$ and hence

$$\begin{cases} f(z^{\epsilon,\delta} + v)(x) \geq f(\hat{z}^{\epsilon,\delta} + \hat{v})(x) \\ \frac{1}{\delta}[(z^{\epsilon,\delta} + v - h^1)^-(x) - (\hat{z}^{\epsilon,\delta} + \hat{v} - h^1)^-(x)] \leq 0 \\ \frac{1}{\epsilon}[(z^{\epsilon,\delta} + v - h^2)^+(x) - (\hat{z}^{\epsilon,\delta} + \hat{v} - h^2)^+(x)] \geq 0, \end{cases} \quad (4.3.13)$$

Consequently, on the set $\{x \in D; w^+(x) > 0\}$, we have

$$F_{\epsilon,\delta}(z^{\epsilon,\delta} + v)(x) - F_{\epsilon,\delta}(\hat{z}^{\epsilon,\delta} + \hat{v})(x) \geq 0 \tag{4.3.14}$$

On the other hand, multiplying (4.3.12) by w^+, we obtain:

$$-(\Delta w, w^+) + (F_{\epsilon,\delta}(z^{\epsilon,\delta} + v) - F_{\epsilon,\delta}(\hat{z}^{\epsilon,\delta} + \hat{v}), w^+) \quad = \quad 0$$

Because

$$-(\Delta w, w^+) = ||\nabla w^+||^2_{L^2(D)} \geq 0,$$

it follows from (4.3.15) that

$$||\nabla w^+||^2_{L^2(D)} = 0$$

and

$$(F_{\epsilon,\delta}(z^{\epsilon,\delta} + v) - F_{\epsilon,\delta}(\hat{z}^{\epsilon,\delta} + \hat{v}), w^+) = 0$$

Taking into account the fact $w^+ = 0$ on ∂D, we deduce $w^+ = 0$. Hence $z^{\epsilon,\delta} - \hat{z}^{\epsilon,\delta} \leq l$. Interchanging the role of $z^{\epsilon,\delta}$ and $\hat{z}^{\epsilon,\delta}$, we prove the lemma. □

The next lemma is a straight consequence of the above lemma.

Lemma 4.3.4 *Let v and $\hat{v}$ be given continuous functions and let $(z^\epsilon, \eta^\epsilon)$ and $(\hat{z}^\epsilon, \hat{\eta}^\epsilon)$ be the unique solutions to single reflection problems $(0; f + \frac{(\cdot + v - h^2)^+}{\epsilon}; h^1)$ and $(0; f + \frac{(\cdot + \hat{v} - h^2)^+}{\epsilon}; h^1)$, respectively. Then we have $||z^\epsilon - \hat{z}^\epsilon||_\infty \leq ||v - \hat{v}||_\infty$.*

Sketch of Proof: Let $\delta \to 0$ in Lemma 4.3.3, we will get the result.

Proof of Theorem 3.1:

Denote by z^ϵ the solution of the following single barrier problem:

$$\begin{cases} -\Delta z^\epsilon + f(z^\epsilon + v) + \dfrac{1}{\epsilon}(z^\epsilon + v - h^2)^+ = \eta^\epsilon \\ z^\epsilon + v \geq h^1 \\ \displaystyle\int_D (z^\epsilon + v - h^1)\eta^\epsilon(dx) = 0, \end{cases} \tag{4.3.15}$$

By the construction in [NT], it is known that $\eta^\epsilon(dx) = \lim_{\delta\to 0}\frac{(z^{\epsilon,\delta}+v-h^1)^-}{\delta}(dx)$ and it means that the measure $\frac{(z^{\epsilon,\delta}+v-h^1)^-}{\delta}(dx)$ converges to $\eta^\epsilon(dx)$ in the sense of distribution on D. According to lemma 3.2(comparison): $z^\epsilon(x)$ is decreasing as $\epsilon \downarrow 0$. Since $z^\epsilon(x) \geq h^1(x) - v(x)$, $z^\epsilon(x)$ converge to some function $z(x)$ as $\epsilon \to 0$. Using similar arguments as in the proof of Lemma 3.2 in [NT], we can show that the function $z(x)$ is also continuous.

Next we prove that $z(x)$ is a solution of the reflected PDE with two reflected walls

$$\begin{cases} -\Delta z + f(z+v) = \eta - \xi \\ h^1 \leq z + v \leq h^2 \\ \int_D (z+v-h^1)\eta(dx) = \int_D (h^2 - z - v)\xi(dx) = 0. \end{cases} \tag{4.3.16}$$

Step1:

Now for $\psi \in C_0^\infty(D)$, z^ϵ satisfies the following integral equation:

$$-(\Delta z^\epsilon, \psi) + (f(z^\epsilon + v), \psi) + (\frac{1}{\epsilon}(z^\epsilon + v - h^2)^+, \psi) = \int \psi(x)\eta^\epsilon(dx) \tag{4.3.17}$$

i.e.

$$-(z^\epsilon, \Delta\psi) + (f(z^\epsilon + v), \psi) = \int \psi(x)(\eta^\epsilon - \xi^\epsilon)(dx), \tag{4.3.18}$$

where $\xi^\epsilon = \frac{(z^\epsilon+v-h^2)^+}{\epsilon}$. The limit of the left hand side of (4.3.18) exists as $\epsilon \to 0$. Therefore $\lim_{\epsilon\to 0}(\eta^\epsilon - \xi^\epsilon)$ exists in the space of distributions, i.e.

$$-(z, \Delta\psi) + (f(z+v), \psi) = \lim_{\epsilon\to 0}(\eta^\epsilon - \xi^\epsilon, \psi) \tag{4.3.19}$$

Next we want to show that both $\lim_{\epsilon\to 0}\eta^\epsilon$ and $\lim_{\epsilon\to 0}\xi^\epsilon$ exist. By Dini theorem, we know that $z^\epsilon(x) \to z(x)$ uniformly on compact subsets of D. For $\phi(x) \in C_0^\infty(D)$, denote by $K = supp(\phi)$, the compact support of ϕ. As $h^1(x) < h^2(x)$ in D, there exists $\theta_K > 0$ such that $h^2(x) - h^1(x) \geq \theta_K$ on K .On the other hand, there exists $\epsilon_0 > 0$, such that for $\epsilon < \epsilon_0$, $|z^\epsilon(x) - z(x)| < \frac{\theta_K}{4}$ on K. Let θ_K be chosen as above. Since

$$supp\eta^\epsilon \subseteq \{x : z^\epsilon(x) + v(x) = h^1(x)\},$$

and

$$supp\xi^\epsilon = \{x : z^\epsilon(x) + v(x) \geq h^2(x)\},$$

we have for $\epsilon \leq \epsilon_0$,

$$supp\eta^\epsilon \cap K \subseteq \{x : z(x) - \frac{\theta_K}{4} + v(x) \leq h^1(x)\} \cap K := A_K,$$

and

$$supp\xi^\epsilon \cap K \subseteq \{x : z(x) + \frac{\theta_K}{4} + v(x) \geq h^2(x)\} \cap K := B_K,$$

for $\epsilon < \epsilon_0$.

By the choice of θ_K, we see that $A_K \cap B_K = \varnothing$. Thus, we can find $\tilde{\phi}(x) \in C_0^\infty(D)$ such that $\tilde{\phi} = \phi$ on A_K, $supp\tilde{\phi} \cap B_K = \varnothing$ and $supp\tilde{\phi} \cap supp\xi^\epsilon = \varnothing$ for $\epsilon < \epsilon_0$. Hence, $\lim_{\epsilon\to 0}(\eta^\epsilon, \phi) = \lim_{\epsilon\to 0}(\eta^\epsilon, \tilde{\phi}) = \lim_{\epsilon\to 0}(\eta^\epsilon - \xi^\epsilon, \tilde{\phi})$ exists. Therefore $\eta^\epsilon \to \eta$ in the space of distributions. Similarly, $\xi^\epsilon \to \xi$. Let $\epsilon \to 0$ in equation (4.3.19) to see that (z, η, ξ) satisfies the following equation:

$$-(\Delta z, \psi) + (f(z + v), \psi) = \int_D \psi(x)(\eta - \xi)(dx). \tag{4.3.20}$$

Step 2:

Multiplying (4.3.17) by ϵ and letting $\epsilon \to 0$, we get

$$0 = ((z + v - h^2)^+, \psi).$$

This implies $z + v - h^2 \leq 0$, i.e. $z + v \leq h^2$. Since $h^1 \leq z^\epsilon + v$, we see that $h^1 \leq z + v$. So $h^1 \leq z + v \leq h^2$.

Step 3:

Now let us show that

$$\int_D (z + v - h^1)\eta(dx) = 0$$

and

$$\int_D (z + v - h^2)\xi(dx) = 0.$$

By the definition of $\xi^\epsilon = \frac{(z^\epsilon + v - h^2)^+}{\epsilon}$, $\int_D (z^\epsilon + v - h^2)\xi^\epsilon(dx) \geq 0$, and the uniform convergence of z^ϵ on compact subsets, letting $\epsilon \to 0$, we have $\int_D (z + v - h^2)\xi(dx) \geq 0$. Hence we must have $\int_D (z + v - h^2)\xi(dx) = 0$. From the single reflecting barrier problem $(0; -\frac{(\cdot + v - h^2)^+}{\epsilon}; h^1)$, we know $\int_D (z^\epsilon + v - h^1)\eta^\epsilon(dx) = 0$. Then letting $\epsilon \downarrow 0$, we get $\int_D (z + v - h^1)\eta(dx) = 0$.

Step 4:

For any compact set $K \subset D$, since

$$-(\Delta z, \psi) + (f(z+v), \psi) = \int_D \psi(x)\eta(dx) - \int_D \psi(x)\xi(dx).$$

Choose a non-negative function $\psi \in C_0^\infty(D)$ such that $\psi(x) = 1$ on $supp(\eta) \cap K$ and $\psi(x) = 0$ on $supp(\xi) \cap K$,

$$-(\Delta z, \psi) + (f(z+v), \psi) = \int_K \eta(dx) - 0,$$

So we get $\eta(K) < \infty$. Similarly, $\xi(K) < \infty$.

Uniqueness: Let (z, η, ξ) and $(\bar{z}, \bar{\eta}, \bar{\xi})$ be solutions to a double reflection problem $(0; f; h^1, h^2)$. We set $\Psi = z - \bar{z}$. For any $\psi \in C_k^\infty(D)$, we have

$$\begin{aligned} & -\int_D \Psi(x)\Delta\psi(x)dx + \int_D [f(z+v) - f(\bar{z}+v)]\psi(x)dx \\ = & \int_D \psi(x)\eta(dx) - \int_D \psi(x)\xi(dx) - \int_D \psi(x)\bar{\eta}(dx) + \int_D \psi(x)\bar{\xi}(dx) \end{aligned} \quad (4.3.21)$$

From here, the following is similar arguments as that in the proof of Theorem 2.2 in [NT], and finally we can show that $z = \bar{z}$.

We are going to approximate Ψ by functions of $C_k^\infty(D)$ and substitute this approximation in (4.3.21).

Step 1: Let ϵ be an infinitely differentiable function with support included in $[-1,1]^k$, (which is the kernel of a nonnegative integral operator),

and $\int_{[-1,1]^k} \epsilon(x)dx = 1$. Consider the approximation of the identity $\epsilon_n(x) = n^k \epsilon(nx)$. Fix $\delta > 0$, let K_δ as in lemma 3.4 in [NT], and for $x \in K_\delta$ and $\frac{1}{n} < \delta$. Set $D_n(x) = x + [-\frac{1}{n}, \frac{1}{n}]^k$.

Let ϕ be an infinitely differentiable function whose support is contained in K_δ and set $\psi_n = (\Psi\phi * \epsilon_n)\phi$ that is, for $x \in K_\delta$, $\psi_n(x) = (\int_{D_n} \Psi(y)\phi(y)\epsilon_n(x-y)dy)\phi(x)$ and $\psi_n(x) = 0$ otherwise. As $\psi_n \in C_k^\infty(D)$, we can replace ψ by ψ_n in (4.3.21) and study the asymptotic behavior of the different terms when n tends to infinity.

Step 2:

$$\lim_{n\to\infty} \int_D \psi_n(x)\eta(dx) - \int_D \psi_n(x)\bar{\eta}(dx)$$

$$\begin{aligned}
&= \int_D \Psi(x)\phi^2(x)\eta(dx) - \int_D \Psi(x)\phi^2(x)\bar{\eta}(dx) \\
&= \int_D (z(x) + v(x) - h^1(x) - \bar{z}(x) - v(x) + h^1(x))\phi^2(x)\eta(dx) \\
&\quad - \int_D (z(x) + v(x) - h^1(x) - \bar{z}(x) - v(x) + h^1(x))\phi^2(x)\bar{\eta}(dx) \\
&= \int_D (z + v - h^1)\phi^2(x)\eta(dx) - \int_D (\bar{z} + v - h^1)\phi^2(x)\eta(dx) \\
&\quad - \int_D (z + v - h^1)\phi^2(x)\bar{\eta}(dx) + \int_D (\bar{z} + v - h^1)\phi^2(x)\bar{\eta}(dx) \\
&= -\int_D (\bar{z} + v - h^1)\phi^2(x)\eta(dx) - \int_D (z + v - h^1)\phi^2(x)\bar{\eta}(dx) \le 0
\end{aligned}$$

by properties (i) and (iv) of Definition 4.3.1.

Similarly,

$$\lim_{n\to\infty} \int_D \psi_n(x)\bar{\xi}(dx) - \int_D \psi_n(x)\xi(dx) \le 0 \tag{4.3.22}$$

Step 3:

$$\begin{aligned}
&\lim_{n\to\infty} \int_D [f(z+v) - f(\bar{z}+v)]\psi_n(x)dx \\
&= \int_D [f(z+v) - f(\bar{z}+v)](z(x) - \bar{z}(x))\phi^2(x)dx \ge 0,
\end{aligned}$$

because f is non-decreasing and, hence $[f(z+v) - f(\bar{z}+v)]$ and $(z - \bar{z})$ have the same sign.

Step 4: Suppose that Ψ is a function in $C_k^\infty(D)$. We can write the following equalities:

$$\begin{aligned}
-(\Delta\psi_n, \Psi) &= (\nabla[(\Psi\phi) * \epsilon_n\phi], \nabla\Psi)_{L^2(D,R^k)} \\
&= ([\nabla(\Psi\phi) * \epsilon_n]\phi, \nabla\Psi)_{L^2(D,R^k)} + ([(\Psi\phi) * \epsilon_n]\nabla\phi, \nabla\Psi)_{L^2(D;R^k)} \\
&= I_1 + I_2 + I_3
\end{aligned}$$

where

$$I_1 = ([\phi\nabla\Psi] * \epsilon_n, \phi\nabla\Psi)_{L^2(D;R^k)}$$

$$I_2 = ([\Psi\nabla\phi] * \epsilon_n, \phi\nabla\Psi)_{L^2(D;R^k)}$$

$$I_3 = ([(\Psi\phi) * \epsilon_n]\nabla\phi, \nabla\Psi)_{L^2(D;R^k)}$$

Let us study the sign of I_1. We set, for $i \in \{1, ..., k\}$, $f_i = (\frac{\partial\Psi}{\partial x_i})\phi$. Then

$$I_1 = \Sigma_{i=1}^k \int_{D^2} f_i(x) f_i(y) \epsilon_n(x-y) dx dy \geq 0$$

because ϵ is nonnegative definite. It follows that, if ξ is a smooth function,

$$(-\Delta\psi_n, \Psi) \geq I_2 + I_3.$$

With the classical relation,

$$div(a\vec{v}) = < \nabla a, \vec{v} > + a \; div\vec{v}$$

and the fact that $(f * \epsilon_n, g) = (f, g * \epsilon_n)$ because ϵ_n is symmetric, we obtain

$$\begin{aligned} I_3 &= -(div\{[(\Psi\phi) * \epsilon_n]\nabla\phi\}, \Psi) \\ &= -([\Psi\phi] * \epsilon_n, \Psi\Delta\phi) - ([\phi\nabla\Psi] * \epsilon_n, \Psi\nabla\phi)_{L^2(D;R^k)} - ([\Psi\nabla\phi] * \epsilon_n, \Psi\nabla\phi)_{L^2(D;R^k)} \\ &= -([\Psi\phi] * \epsilon_n, \Psi\Delta\phi) - I_2 - ([\Psi\nabla\phi] * \epsilon_n, \Psi\nabla\phi)_{L^2(D;R^k)} \end{aligned}$$

and consequently,

$$-(\Delta\psi_n, \Psi) \geq -([\Psi\phi] * \epsilon_n, \Psi\Delta\phi) - ([\Psi\nabla\phi] * \epsilon_n, \Psi\nabla\phi)_{L^2(D;R^k)}.$$

This formula still holds for any continuous function Ψ, by approximation. Thus,

$$\liminf_{n\to\infty} -(\Delta\psi_n, \Psi) \geq -(\Psi^2, \phi\Delta\phi + < \nabla\phi, \nabla\phi >) = -\frac{1}{2}(\Psi^2, \Delta\phi^2). \qquad (4.3.23)$$

As a conclusion, if we let n tend to infinity in (4.3.21) with ψ replaced by ψ_n we get

$$-(\Psi^2, \Delta\phi^2) \leq 0$$

for any $\phi \in C_k^\infty(D)$. Setting $h = -\Psi^2$, and applying Lemma 3.4 in [NT], we obtain that $z = \bar{z}$.

Recall that

$$supp\eta, supp\bar{\eta} \subset \{x \in D : z + v = h^1\} =: A,$$

$$supp\xi, supp\bar{\xi} \subset \{x \in D : z + v = h^2\} =: B.$$

Because $A \cap B = \varnothing$, for any $\psi \in C_0^\infty(D)$ with $supp\psi \subset supp\eta \cup supp\bar{\eta}$, it holds that $supp\psi \cap supp\xi = \varnothing$ and $supp\psi \cap supp\bar{\xi} = \varnothing$. Applying equation (4.3.21) to such a function ψ, we deduce that $\eta = \bar{\eta}$. Similarly $\xi = \bar{\xi}$. Then the uniqueness is proved. □

4.4 Reflected SPDEs

Recall

$$\begin{cases} -\Delta u(x) + f(x, u(x)) = \sigma(x, u(x))\dot{W}(x) + \eta(x) - \xi(x) \\ u|_{\partial D} = 0 \\ h^1(x) \le u(x) \le h^2(x) \\ \int_D (u(x) - h^1(x))\eta(dx) = \int_D (h^2(x) - u(x))\xi(dx) = 0. \end{cases} \tag{4.4.1}$$

Let $G_D(x, y)$ be the Green function on D associated to the Laplacian operator with Dirichlet boundary conditions. We recall from [BP2] (or [ST]) that if $k = 2$ or 3,

$$G_D(x, y) = G(x, y) - E_x(G(B_\tau, y)), x, y \in D$$

with

$$G(x, y) = \frac{1}{2\pi} log|x - y|, \quad \text{if} \quad k = 2;$$
$$G(x, y) = -\frac{1}{4\pi}|x - y|^{-1}, \quad \text{if} \quad k = 3;$$

and B_τ is the random variable obtained by stopping a k-dimensional Brownian motion starting at x at its first exit time of D. For $k = 1$,if $D = (0, 1)$, then $G_D(x, y) = (x \wedge y) - xy$.

The main result of this chapter is the following theorem.

Theorem 4.4.1 *Assume that (F1), (H1) and (Σ1) with C_σ satisfying $\exists p > 1$,*

$$[2^{2p-1}ac_pBr_D^{\lambda p-k} + 2^{2p-1}c_p(C_D)^{\frac{p}{2}}]C_\sigma^p < 1, \tag{4.4.2}$$

where c_p and a are universal constants appeared in the Burkholder's inequality, Komogorov's inequality, r_D is the diameter of the domain D (see(4.4.20), (4.4.21)), $C_D = \sup_x \int_D |G_D(x,y)|^2 dy$. And B is the constant appeared in the estimate of the Green function G_D in (4.4.13). And λ is any number in $(0,1]$ when the dimension $k=1$; λ is any number in $(0,1)$ when the dimension $k=2$; λ is any number in $(0,\frac{1}{2})$ when the dimension $k=3$.
Then there exists a unique solution (u,η,ξ) to the reflected SPDE Eq(4.1.1). Moreover, $E(||u||_\infty)^p < \infty$.

PROOF.
Existence:
We will use successive iteration:
Let

$$v_1(x) = \int_D G_D(x,y)\sigma(y;0)W(dy). \tag{4.4.3}$$

As in [BP2], it is seen that $v_1(x)$ is the solution of the following SPDE:

$$\begin{cases} -\Delta v_1(x) = \sigma(x;0)\dot{W}(x) \\ v_1|_{\partial D} = 0 \end{cases} \tag{4.4.4}$$

and $v_1(x) \in C(\bar{D})$.
Denote by (z_1,η_1,ξ_1) be the unique random solution of the following reflected PDE:

$$\begin{cases} -\Delta z_1(x) + f(z_1+v_1) = \eta_1(x) - \xi_1(x) \\ z_1|_{\partial D} = 0 \\ h^1(x) \le z_1(x) + v_1(x) \le h^2(x) \\ \int_D (z_1(x)+v_1(x)-h^1(x))\eta_1(dx) = \int_D (h^2(x)-z_1(x)-v_1(x))\xi_1(dx) = 0. \end{cases} \tag{4.4.5}$$

Set $u_1 = z_1 + v_1$. Then we can easily verify that (u_1,η_1,ξ_1) is the unique solution of the following reflected SPDE:

$$\begin{cases} -\Delta u_1(x) + f(x; u_1) = \sigma(x; 0)\dot{W}(x) + \eta_1(x) - \xi_1(x) \\ u_1|_{\partial D} = 0 \\ h^1(x) \leq u_1(x) \leq h^2(x) \\ \int_D (u_1(x) - h^1(x))\eta_1(dx) = \int_D (h^2(x) - u_1(x))\xi_1(dx) = 0. \end{cases} \tag{4.4.6}$$

Iterating this procedure, suppose u_{n-1} has been defined. Let

$$v_n(x) = \int_D G_D(x, y)\sigma(y; u_{n-1})W(dy), \tag{4.4.7}$$

and (z_n, η_n, ξ_n) be the unique random solution of the following reflected PDE:

$$\begin{cases} -\Delta z_n(x) + f(z_n + v_n) = \eta_n(x) - \xi_n(x) \\ z_n|_{\partial D} = 0 \\ h^1(x) \leq z_n(x) + v_n(x) \leq h^2(x) \\ \int_D (z_n(x) + v_n(x) - h^1(x))\eta_n(dx) = \int_D (h^2(x) - z_n(x) - v_n(x))\xi_n(dx) = 0. \end{cases} \tag{4.4.8}$$

Set $u_n = z_n + v_n$. Then (u_n, η_n, ξ_n) is the unique solution of the following reflected SPDE:

$$\begin{cases} -\Delta u_n(x) + f(x; u_n(x)) = \sigma(x; u_{n-1}(x))\dot{W}(x) + \eta_n - \xi_n \\ u_n|_{\partial D} = 0 \\ h^1(x) \leq u_n(x) \leq h^2(x) \\ \int_D (u_n(x) - h^1(x))\eta_n(dx) = \int_D (h^2(x) - u_n(x))\xi_n(dx) = 0. \end{cases} \tag{4.4.9}$$

From the proof of Lemma 4.3.4 (also Lemma 3.1 in [NT]), we have

$$||z_n - z_{n-1}||_\infty \leq ||v_n - v_{n-1}||_\infty, \tag{4.4.10}$$

hence

$$||u_n - u_{n-1}||_\infty \leq 2||v_n - v_{n-1}||_\infty. \tag{4.4.11}$$

Namely,

$$(||u_n - u_{n-1}||_\infty)$$

$$\leq 2\sup_{x\in D}|\int_D G_D(x,y)(\sigma(y;u_{n-1})-\sigma(y;u_{n-2}))W(dy)|. \qquad (4.4.12)$$

Set

$$I(x)=\int_D G_D(x,y)(\sigma(y;u_{n-1})-\sigma(y;u_{n-2}))W(dy).$$

Then $\forall p\geq 1$,

$$\begin{aligned}
&E[|I(x)-I(y)|^p]\\
&=E|\int_D (G_D(x,z)-G_D(y,z))(\sigma(z;u_{n-1})-\sigma(z;u_{n-2}))W(dz)|^p\\
&\leq c_pE[\int_D |G_D(x,z)-G_D(y,z)|^2\cdot|\sigma(z;u_{n-1})-\sigma(z;u_{n-2})|^2dz]^{\frac{p}{2}}\\
&\leq c_pE||\sigma(z;u_{n-1})-\sigma(z;u_{n-2})||_\infty^p[\int_D |G_D(x,z)-G_D(y,z)|^2dz]^{\frac{p}{2}},
\end{aligned}$$

where c_p is a Burkholder constant only related to p.

Similarly as the proof of Theorem 3.3 in [ST], we have

$$||G_D(x,z)-G_D(y,z)||^2_{L^2(D)}\leq B|x-y|^{2\lambda}, \qquad (4.4.13)$$

where $\lambda=1$ when $k=1$, λ is arbitrarily close to 1 when $k=2$, and λ is arbitrarily close to $\frac{1}{2}$ when $k=3$. Then,

$$E|I(x)-I(y)|^p\leq E||\sigma(z;u_{n-1})-\sigma(z;u_{n-2})||^p_\infty c_pB|x-y|^{\lambda p}. \qquad (4.4.14)$$

We next show that u_n converges uniformly on D. Let $K\subset D$ be any compact subset of D. $\forall x,y\in K$, from Kolmogrov lemma (Lemma 3.1 in [DP]), we deduce that for $\forall p>\frac{k}{\lambda}$,

$$|I(x)-I(y)|^p\leq (N(w))^p|x-y|^{\lambda p-k}(log(\frac{\gamma}{|x-y|}))^2, \qquad (4.4.15)$$

$$E(N^p)\leq ac_pBE||\sigma(z;u_{n-1})-\sigma(z;u_{n-2})||^p_\infty, \qquad (4.4.16)$$

where a is a universal constant independent of K. Choosing $y=x_0\in K$, we see that

$$\begin{aligned}
E[\sup_{x\in K}|I(x)|^p] &\leq 2^{p-1}E[\sup_{x\in K}|I(x)-I(x_0)|^p]+2^{p-1}E|I(x_0)|^p\\
&\leq 2^{p-1}E(N^p)r_D^{\lambda p-k}+2^{p-1}E|I(x_0)|^p
\end{aligned}$$

$$\leq 2^{p-1}ac_pBr_D^{\lambda p-k}E||\sigma(z;u_{n-1})-\sigma(z;u_{n-2})||_\infty^p$$
$$+2^{p-1}E|I(x_0)|^p, \qquad (4.4.17)$$

where $r_D = sup_{x,y\in D}|x-y|$ is the diameter of D. Furthermore,

$$\begin{aligned} E|I(x_0)|^p &= E|\int_D G_D(x_0,y)(\sigma(y;u_{n-1})-\sigma(y;u_{n-2}))W(dy)|^p \\ &\leq c_pE[\int_D |G_D(x_0,y)|^2\cdot|\sigma(y;u_{n-1})-\sigma(y;u_{n-2})|^2dy]^{\frac{p}{2}} \\ &\leq c_p(C_D)^{\frac{p}{2}}E||\sigma(z;u_{n-1})-\sigma(z;u_{n-2})||_\infty^p, \qquad (4.4.18)\end{aligned}$$

where $C_D = \sup_x \int_D |G_D(x,y)|^2dy < \infty$. So we have

$$\begin{aligned} & E[\sup_{x\in K}|I(x)|^p] \\ \leq\ & 2^{p-1}ac_pBr_D^{\lambda p-k}E||\sigma(z;u_{n-1})-\sigma(z;u_{n-2})||_\infty^p \\ & +2^{p-1}c_p(C_D)^{\frac{p}{2}}E||\sigma(z;u_{n-1})-\sigma(z;u_{n-2})||_\infty^p. \qquad (4.4.19)\end{aligned}$$

Since the constants on the right side of (4.4.19) are independent of the compact subset K, by Fatou's Lemma we deduce that

$$\begin{aligned} & E[\sup_{x\in D}|I(x)|^p] \\ \leq\ & 2^{p-1}ac_pBr_D^{\lambda p-k}E||\sigma(z;u_{n-1})-\sigma(z;u_{n-2})||_\infty^p \\ & +2^{p-1}c_p(C_D)^{\frac{p}{2}}E||\sigma(z;u_{n-1})-\sigma(z;u_{n-2})||_\infty^p. \qquad (4.4.20)\end{aligned}$$

Now it follows from (4.4.12) and (4.4.20) that

$$\begin{aligned} & E(||u_n-u_{n-1}||_\infty)^p \\ \leq\ & [2^{2p-1}ac_pBr_D^{\lambda p-k}+2^{2p-1}c_p(C_D)^{\frac{p}{2}}] \\ & \quad \times E||\sigma(z;u_{n-1})-\sigma(z;u_{n-2})||_\infty^p \\ \leq\ & [2^{2p-1}ac_pBr_D^{\lambda p-k}+2^{2p-1}c_p(C_D)^{\frac{p}{2}}]C_\sigma^p \\ & \quad \times E||u_{n-1}-u_{n-2}||_\infty^p \\ \leq\ & ... \\ \leq\ & \left([2^{2p-1}ac_pBr_D^{\lambda p-k}+2^{2p-1}c_p(C_D)^{\frac{p}{2}}]C_\sigma^p\right)^{n-1}E||u_1-u_0||_\infty^p \qquad (4.4.21)\end{aligned}$$

Since

$$[2^{2p-1}ac_pBr_D^{\lambda p-k}+2^{2p-1}c_p(C_D)^{\frac{p}{2}}]C_\sigma^p < 1,$$

we obtain from (4.4.21) that for any $m \geq n \geq 1$,

$$E(||u_m - u_n||_\infty)^p \to 0,$$

as $n, m \to \infty$.

Hence, there exists a continuous random field $u(\cdot) \in C(D)$, such that

$$E((||u||_\infty)^p) < \infty, \tag{4.4.22}$$

and

$$\lim_{n\to\infty} E(||u_n - u||_\infty)^p = 0. \tag{4.4.23}$$

Next we will show that u is a solution of Eq(4.4.1).

Set

$$v(x) = \int_D G_D(x,y)\sigma(y;u)W(dy). \tag{4.4.24}$$

As the proof of (4.4.20), we have

$$\lim_{n\to\infty} E||v_n - v||_\infty^p = \lim_{n\to\infty} E||u_{n-1} - u||_\infty^p = 0. \tag{4.4.25}$$

From the inequality (4.4.10), there exists a continuous random field $z(x)$ on D such that

$\lim_{n\to\infty} E||z_n - z||_\infty^p = 0$. So z_n converges to z uniformly on D. Similar to the proof of Theorem 4.3.1, we can show that $\eta(dx) = \lim_{n\to\infty} \eta_n(dx)$, $\xi(dx) = \lim_{n\to\infty} \xi_n(dx)$ exist almost surely and (z, η, ξ) is the solution of equation (4.3.1) with the above given v. Put $u(x) = z(x) + v(x)$. It is easy to verify (u, η, ξ) is a solution to the SPDE(4.4.1) with two reflecting walls.

Uniqueness:

Let (u_1, η_1, ξ_1) and (u_2, η_2, ξ_2) be two solutions of Eq(4.4.1). Set

$$v_1(x) = \int_D G_D(x,y)\sigma(y;u_1)W(dy), \tag{4.4.26}$$

$$v_2(x) = \int_D G_D(x,y)\sigma(y;u_2)W(dy), \tag{4.4.27}$$

and $z_1 = u_1 - v_1$ and $z_2 = u_2 - v_2$. Then z_1, z_2 are solutions of the following reflected random PDEs:

$$\begin{cases} -\Delta z_1(x) + f(z_1 + v_1) = \eta_1(x) - \xi_1(x) \\ z_1|_{\partial D} = 0 \\ h^1(x) \le z_1(x) + v_1(x) \le h^2(x) \\ \int_D (z_1(x) + v_1(x) - h^1(x))\eta_1(dx) = \int_D (h^2(x) - z_1(x) - v_1(x))\xi_1(dx) = 0, \end{cases} \tag{4.4.28}$$

$$\begin{cases} -\Delta z_2(x) + f(z_2 + v_2) = \eta_2(x) - \xi_2(x) \\ z_2|_{\partial D} = 0 \\ h^1(x) \le z_2(x) + v_2(x) \le h^2 \\ \int_D (z_2(x) + v_2(x) - h^1(x))\eta_2(dx) = \int_D (h^2(x) - z_2(x) - v_2(x))\xi_2(dx) = 0, \end{cases} \tag{4.4.29}$$

Similar to the inequality (4.4.10), we have

$$||z_1 - z_2||_\infty \le ||v_1 - v_2||_\infty. \tag{4.4.30}$$

Hence,

$$\begin{aligned} ||u_1 - u_2||_\infty^p &\le 2^p ||v_1 - v_2||_\infty^p \\ &\le 2^p (\sup_{x \in D} |\int_D G_D(x,y)(\sigma(y;u_1) - \sigma(y;u_2))W(dy)|^p) \end{aligned} \tag{4.4.31}$$

As the proof of (4.4.21), we deduce from (4.4.31) that

$$\begin{aligned} E||u_1 - u_2||_\infty^p &\le [2^{2p-1} a c_p C(p) r_D^{\lambda p - k} + 2^{2p-1} c_p (C_D)^{\frac{p}{2}}] C_\sigma^p \\ &\quad \times E||u_1 - u_2||_\infty^p. \end{aligned} \tag{4.4.32}$$

As

$$[2^{2p-1} a c_p C(p) r_D^{\lambda p - k} + 2^{2p-1} c_p (C_D)^{\frac{p}{2}}] C_\sigma^p < 1,$$

it follows that

$$u_1 = u_2 \quad a.s. \tag{4.4.33}$$

On the other hand, for $\phi \in C_0^\infty(D)$,

$$-(u_1(x) - u_2(x), \Delta\phi(x)) + (f(x, u_1(x)) - f(x; u_2(x)), \phi(x))$$

$$= \int_D [\sigma(x; u_1(x)) - \sigma(x; u_2(x))]\phi(x)W(dx)$$
$$+ \int_D \phi(x)(\eta_1(dx) - \eta_2(dx)) - \int_D \phi(x)(\xi_1(dx) - \xi_2(dx)). \qquad (4.4.34)$$

Therefore we have

$$\int_D \phi(x)(\eta_1(dx) - \eta_2(dx)) - \int_D \phi(x)(\xi_1(dx) - \xi_2(dx)) = 0. \qquad (4.4.35)$$

Recall that

$$supp\eta_1, supp\eta_2 \subset \{x \in D : u_1(x) = h^1(x)\} =: A$$
$$supp\xi_1, supp\xi_2 \subset \{x \in D : u_1(x) = h^2(x)\} =: B.$$

Because $A \bigcap B = \emptyset$, for any $\phi \in C_0^\infty(D)$ with $supp\phi \subset (supp\eta_1 \bigcup supp\eta_2)$, it holds that $supp\phi \bigcap supp\xi_1 = \emptyset$ and $supp\phi \bigcap supp\xi_2 = \emptyset$. Applying equation(4.4.35) to such a function ϕ, we deduce that $\eta_1 = \eta_2$. Similarly, $\xi_1 = \xi_2$. Then the uniqueness is proved. □

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