THE STRUCTURE OF THE SECOND DERIVED IDEAL OF FREE CENTRE-BY-METABELIAN LIE RINGS

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We study the free centre-by-metabelian Lie ring, that is, the free Lie ring with the property that the second derived ideal is contained in the centre. We exhibit explicit generating sets for the homogeneous components and the fine homogeneous components of the second derived ideal. Each of these components is a direct sum of a free abelian group and a (possibly trivial) elementary abelian 2-group. Our generating sets are such that some of their elements generate the torsion subgroup while the remaining ones freely generate a free abelian group. A key ingredient of our approach is the determination of the dimensions of the corresponding homogeneous components of the free centre-by-metabelian Lie algebra over fields of characteristic other than 2. For this we exploit a 6-term exact sequence of modules over a polynomial ring that is originally defined over the integers, but turns into a sequence whose terms are projective modules after tensoring with a suitable field. Our results correct a partly erroneous theorem in the literature.

Moreover, we study the product of three homogeneous components of a free Lie algebra. Let L be a free Lie algebra of finite rank over a field and let L_n denote the degree n homogeneous component of L. Formulae for the dimension of the subspaces $[L_n, L_m]$ for all n and m were obtained by Ralph Stöhr and Michael Vaughan-Lee. Formulae for the dimension of the subspaces of the form $[L_n, L_m, L_k]$ under certain conditions on n, m and k were obtained by Nil Mansuroğlu and Ralph Stöhr. Surprisingly, in contrast to the case of a product of two homogeneous components, the dimension of such products may depend on the characteristic of the field. For example, the dimension of $[L_2, L_2, L_1]$ over fields of characteristic 2 is different from the dimension over fields of characteristic other than 2.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Dedication

To my parents Makbule, Ahmet and my siblings
Necim, Doğan, Ebru whom I love very much.

Publications

- The material in Chapter 3 is based on the paper:

  Nil Mansuroğlu and Ralph Stöhr, **"Free centre-by-metabelian Lie rings"**, *Quart.J.Math.*, Advance access published May 17, 2013, doi: 10.1093/qmath/hat 017.

- The material in Chapter 4 is based on the paper:

  Nil Mansuroğlu and Ralph Stöhr, **"On the dimension of products of homogeneous subspaces in free Lie algebras"**, *Internat. J. Algebra Comput.* 23, no.1, 205-213, (2013).

Chapter 1

Introduction

We study the free centre-by-metabelian Lie ring which is the free Lie ring satisfying the property that the second derived ideal is contained in the centre. The free centre-by-metabelian Lie rings are interesting structures with some unusual properties. One of these properties is that the 2-torsion occurring in odd degrees is drastically different from the 2-torsion in even degrees. Another peculiar feature is that the free centre-by-metabelian Lie ring is not isomorphic to the Lie ring of the free centre-by-metabelian group of the same rank.

Let X be a set consisting of the elements $x_1, x_2, \ldots, x_r$ with $r \geq 2$ to be ordered by $x_1 < x_2 < \ldots < x_r$. Let G denote the free centre-by-metabelian Lie ring of finite rank r on the free generating set X. The free centre-by-metabelian Lie ring G is a central extension of the free metabelian Lie ring G/G'', where G'' is the second derived ideal of G. The additive structure of the free metabelian Lie ring G/G'' is well-understood. This structure is clearly a free abelian group. A basis for the free metabelian Lie ring G/G'' is a $\mathbb{Z}$-basis in the form of the simple basic monomials (see, for example, Section 4.7.1 in [1]). This basis appeared already in 1951 in the work of Chen [6] and in this paper Chen derived a formula for the number of such basis elements in every degree. Therefore, when we study the free centre-by-metabelian Lie ring G, it is natural to focus on the second derived ideal G''.

We denote the degree n homogeneous component of G for $n \geq 1$ by G_n, and we write G''_n for the degree n homogeneous component of the second derived ideal G''.

For $n \geq 1$, the homogeneous component G''_n is expressed as

$$G''_n = G'' \cap G_n.$$

In the case where $n = 4$, it is easy to see that G''_4 is isomorphic to a free abelian group $G_2 \wedge G_2$.

In the case where $n \geq 5$, the situation gets interesting. Namely, for $n \geq 5$ there is a direct sum decomposition

$$G''_n = F_n \oplus T_n, \tag{1.1}$$

where F_n is a free abelian group and T_n is a (possibly trivial) elementary abelian 2-group. This result has appeared already in Theorem 4 in the section on Lie rings in Yu.V. Kuz'min's ground-breaking paper [20]. This theorem was given as the above result (1.1) together with explicit bases for F_n as a free $\mathbb{Z}$-module and for T_n as a free $\mathbb{Z}/2\mathbb{Z}$-module. This theorem shows that for even degree $n \geq 6$, the additive group of the homogeneous component for $r \geq 2$ has non-trivial 2-torsion. For odd degree $n \geq 5$, the 2-torsion occurs for $r \geq 5$. In 1991, R. Zerck, in his paper [45], pointed out that the sets asserted to be bases for the torsion subgroups T_n for even degree $n \geq 6$ in [20] are actually not sufficient to generate those groups. Unfortunately, Zerck's preprint [45] was never published properly and it is not easily accessible. Moreover, in this preprint Zerck did not give full proofs, but referred in certain instances to the methods in 1977 Kuz'min's paper [20], which is hardly satisfactory as some of the arguments in that paper are incorrect. Now it turns out that, apart from the shortcomings related to T_n, the sets claimed in his pioneering 1977 paper [20] to be bases of F_n for even degree $n \geq 6$ fail to be linearly independent over $\mathbb{Z}$. Some of the details in 1977 Kuz'min's paper [20] are in need of correction. Our main goal in this thesis is to put this right.

The free centre-by-metabelian Lie rings are closely related to the lower central quotients of the free centre-by-metabelian groups. These groups are themselves highly curious objects. This was discovered by C.K. Gupta in her papers [8] and [9]. In the paper [9], for rank 4 onwards, she has proved that the centre of these groups contains an elementary 2-subgroup of rank $\binom{r}{4}$. The existence of elements of order 2 in the centre was a very surprising result at the time.

Let $\mathfrak{F}$ be the free group of rank r. The quotient $\mathfrak{F}/[\mathfrak{F}'', \mathfrak{F}]$ is the free centre-by-metabelian group of rank r, that is a free central extension of the free metabelian group $\mathfrak{F}/\mathfrak{F}''$ (see Bourbaki's book [2] and Neumann's book [28]). Let $\mathfrak{G}$ denote the free centre-by-metabelian group of rank r on a free generating set X with $r > 1$, and let $\gamma_n(\mathfrak{G})$ denote the nth term of the lower central series of $\mathfrak{G}$ for all $n \geq 1$. For all $n = 1, 2, 3, \ldots$, the lower central quotients $\gamma_n(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G})$ are abelian groups. By taking a direct sum of lower central quotients $\gamma_n(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G})$, we can obtain an additive abelian group. Moreover, this group can be turned into a Lie ring by defining the Lie commutator of two homogeneous elements $a \in \gamma_s(\mathfrak{G})/\gamma_{s+1}(\mathfrak{G})$ and $b \in \gamma_t(\mathfrak{G})/\gamma_{t+1}(\mathfrak{G})$ as follows: Suppose that $a = x\gamma_{s+1}(\mathfrak{G})$ and $b = y\gamma_{t+1}(\mathfrak{G})$ with $x \in \gamma_s(\mathfrak{G})$ and $y \in \gamma_t(\mathfrak{G})$, then we get

$$[a, b] = (x, y)\gamma_{s+t+1}(\mathfrak{G}) \in \gamma_{s+t}(\mathfrak{G})/\gamma_{s+t+1}(\mathfrak{G}),$$

where $(x, y) = x^{-1}y^{-1}xy$ is the group commutator in $\mathfrak{G}$. This Lie ring is called the Lie ring of a group $\mathfrak{G}$ and we denote it by $L(\mathfrak{G})$. Therefore, the Lie ring $L(\mathfrak{G})$ of the group $\mathfrak{G}$ is a direct sum of the lower central quotients $\gamma_n(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G})$ for $n = 1, 2, 3, \ldots$ with the Lie bracket induced by the commutator in $\mathfrak{G}$. Further information about the lower central series of a free group can be found in ([1], Section 8.2.4).

Now, it is clear to see that the ring $L(\mathfrak{G})$ is a centre-by-metabelian Lie ring. Hence, the Lie ring $L(\mathfrak{G})$ is a homomorphic image of the free centre-by-metabelian Lie ring G, and the lower central quotients $\gamma_n(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G})$ are homomorphic images of the respective homogeneous components G_n. In [41], J.N. Ridley showed that for $r = 2$, the lower central quotients of a free centre-by-metabelian group are isomorphic to the homogeneous components of a free centre-by-metabelian Lie ring. In contrast to free nilpotent (or free soluble) groups and Lie rings, the canonical map $G \to L(\mathfrak{G})$ is not an isomorphism ([21], Theorem 2). However, there are isomorphisms $L_n(\mathfrak{G}) \otimes \mathbb{Q} \cong (\gamma_n(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G})) \otimes \mathbb{Q}$ for all $n \geq 1$, where $\mathbb{Q}$ is the field of rationals ([21], Theorem 1).

A.L. Smelkin in [36] and M.A. Ward in [43] have shown that the lower central quotients of free polynilpotent groups are torsion-free, but for $n \geq 5$, the nth lower central quotient $\gamma_n(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G})$ of $\mathfrak{G}$ of rank $r \geq 2$ may have 2-torsion. This was proved for $r = 2$ and even degree $n \geq 6$ by Ridley in [41] and then Hurley showed it for $r \geq 5$ and odd degree $n \geq 5$ (unpublished). If $n \leq 4$, the lower central quotient $\gamma_n(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G})$ is same as the corresponding quotient of a free group of rank r.

The lower central quotients of $\mathfrak{G}$ were studied by N.D. Gupta, F. Levin and T.C. Hurley in [11]. According to this paper,

$$\mathfrak{G}''_n = (\gamma_n(\mathfrak{G}) \cap \mathfrak{G}'')\gamma_{n+1}(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G}),$$

in other words, $\mathfrak{G}''_n$ is the image of G''_n under the canonical epimorphism $G \to L(\mathfrak{G})$. It was shown that

$$\mathfrak{G}''_n \cong \mathfrak{F}_n \oplus \mathfrak{T}_n,$$

where $\mathfrak{F}_n$ is a free abelian group and $\mathfrak{T}_n$ is an elementary abelian 2-group. Gupta and Levin in [10] showed that the structure of the torsion group $\mathfrak{T}_n$ differs depending on even or odd degree n. Then in [11] (Theorem 1 and Theorem 4), N.D. Gupta, F. Levin and T.C. Hurley obtained explicit generating sets for both $\mathfrak{F}_n$ and $\mathfrak{T}_n$. For odd degree n, the generating sets of $\mathfrak{F}_n$ in [11] are exactly the canonical images of the basis for F_n given in Kuz'min's paper [20]. For even degree n, however, the generating sets for $\mathfrak{F}_n$ in [11] have fewer elements which are alleged bases for F_n given in [20]. In Theorem 1 of [11] the authors use the word basis rather than generating set. It appears, though, that this is used in the sense of generating set as the question of linear independence over $\mathbb{Z}$ is not addressed in the proof. Also, in [11] the part relating to $\mathfrak{F}_n$ in the part (ii) for even degree of Theorem 4 has not been proved.

In this work our main aim is to correct the Kuz'min's Theorem 4 in [20]. Firstly, we focus on the torsion-free part of G'' and we obtain $\mathbb{Z}$-bases for the free abelian parts of the fine homogeneous components of G'', then we derive formulae for the ranks of these groups as free $\mathbb{Z}$-modules. To do this our approach is as follows: First we obtain generating sets for the fine homogeneous components of G''. Then we show that certain subsets of those generating sets span torsion subgroups, more precisely, elementary abelian 2-groups. Finally, we show that the remaining elements in those generating sets are linearly independent over $\mathbb{Z}$, that is, they actually freely generate the components modulo their torsion subgroups as free $\mathbb{Z}$-modules. This yields our main result, Theorem 3.16, which gives a detailed description of the fine homogeneous components of G''.

Our approach is based on an isomorphism, due to Kuz'min's paper [20], between G'' and a certain tensor product. More precisely, the adjoint representation of G induces on the quotient G'/G'' the structure of a module for the polynomial ring $U = \mathbb{Z}[X]$, which is, in fact, the universal envelope for the abelian Lie algebra G/G'. Namely, the elements of the quotient G'/G'' can be written as the following form

$$[y_1, y_2, y_3, \ldots, y_n] = [y_1, y_2]y_3y_4 \ldots y_n$$

for all $y_i \in X$ with $1 \leq i \leq n$ and $n \geq 2$. We denote the quotient G'/G'' by M and we define the exterior square $M \wedge M$ to be a U-module with derivation action:

$$(m_1 \wedge m_2)y = m_1y \wedge m_2 + m_1 \wedge m_2y$$

for $m_1, m_2 \in M$ and $y \in X$. Then we observe that G'' is isomorphic to $(M \wedge M) \otimes_U \mathbb{Z}$, where the exterior square $M \wedge M$ is regarded as a U-module with derivation action and $\mathbb{Z}$ is the trivial U-module. Most of the work in this research is carried out in that tensor product. The elements $[y_1, y_2] \wedge_* [y_3, y_4]y_5y_6 \ldots y_n$ for all $y_1, y_2, \ldots, y_n \in X$ form a generating set for $(M \wedge M) \otimes_U \mathbb{Z}$. The methods we employ to find spanning sets for the fine homogeneous components and to prove that part of the spanning sets which generate elementary abelian 2-groups are essentially the same in Kuz'min's paper [20] except that we take advantage of the fine homogeneous structure. This makes it possible to obtain very simple generating sets for fine homogeneous components in which at least one of the free generators occurs with multiplicity greater than one. Where our approach significantly differs from [20] is our method for proving that the generating sets which we obtain for the torsion-free part are linearly independent over $\mathbb{Z}$.

In this thesis we define the t-elements which will play an important role in studying the homogeneous components of $(M \wedge M) \otimes_U \mathbb{Z}$ as the elements of the form

$$\begin{aligned} w(y_1, y_2, y_3, y_4; y_5 y_6 \dots y_n) = [y_1, y_2] \wedge_* [y_3, y_4] y_5 y_6 \dots y_n + [y_2, y_3] \wedge_* [y_1, y_4] y_5 y_6 \dots y_n \\ + [y_3, y_1] \wedge_* [y_2, y_4] y_5 y_6 \dots y_n \end{aligned}$$

and moreover we use some homological methods: The exterior square $M \wedge M$ fits into a 6-term exact sequence. In the main part of this work we assume that K is a field of characteristic other than 2. By tensoring G'' with K we obtain the free centre-by-metabelian Lie algebra over a field K. The 6-term exact sequence enables us to work out the dimensions of the fine homogeneous components of $G'' \otimes K$, where K is a field of characteristic other than 2. It turns out that these dimensions coincide with the number of non-torsion elements in our generating sets for the corresponding fine homogeneous components of G'', which implies that these elements freely generate free $\mathbb{Z}$-modules. Our main result is the following theorem:

Main Theorem (Theorem 3.16 in Chapter 3): *Let G be the free centre-by-metabelian Lie ring of rank $r > 1$ on a free generating set $X = \{x_1, x_2, \dots, x_r\}$, let $q = (q_1, q_2, \dots, q_r) \models n$ be a composition of $n \geq 5$, and let G''_q denote the fine homogeneous component of multidegree q of the second derived ideal $G'' \subseteq G$.*

(i) Suppose that $q = (q_1, q_2, \dots, q_r) \models n$ is multilinear with $q_i = 1$ for $i = i_1, i_2, \dots, i_n$, where $1 \leq i_1 \leq \dots \leq i_n \leq r$. Then,

(a) if n is odd, G''_q is generated by the Kuz'min elements of multidegree q and the t-element $w(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}; x_{i_5} \dots x_{i_n})$. The former freely generate a free abelian

group of rank $\frac{1}{2}n(n-3)$ *and the latter generates a cyclic group of order at most* 2*,*

(b) if n *is even, then* G''_q *is a free abelian group of rank* $\binom{n-1}{2}$*, and the Kuz'min elements of multidegree* q *together with the element* $[x_{i_3}, x_{i_2}] \wedge_* [x_{i_4}, x_{i_1}]x_{i_5}x_{i_6} \dots x_{i_n}$ *form a free generating set for it.*

(ii) Suppose that $q = (q_1, q_2, \dots, q_r) \models n$ *is a composition of* n *with* k *non-zero parts such that* $q_i \geq 2$ *for some* i *with* $1 \leq i \leq n$*, and* m *parts of* q *are* 1*. Then*

(a) if n *is odd,* G''_q *is a free abelian group of rank* $\binom{k}{2} - m$*, and the elements* $[y_1, y_2] \wedge_* [y_3, y_4]y_5 \dots y_n$ *of multidegree* q *with* $y_2 = y_4 = x_i$*,* $y_1 \geq y_3$ *and* $y_1, y_3 \neq x_i$*, form a free generating set for it,*

(b) if n *is even, then* G''_q *is a direct sum of a free abelian group of rank* $\binom{k-1}{2}$ *that is freely generated by the elements* $[y_1, y_2] \wedge_* [y_3, y_4]y_5 \dots y_n$ *of multidegree* q *with* $y_2 = y_4 = x_i$*,* $y_1 > y_3$ *and* $y_1, y_3 \neq x_i$*, and an elementary abelian* 2*-group generated by the elements* $[y_1, y_2] \wedge_* [y_3, y_4]y_5 \dots y_n$ *of multidegree* q *such that* $y_2 = y_4 = x_i$*,* $y_1 = y_3 \neq x_i$*. If all parts of* q *are even, then all of the latter elements are zero, and the torsion subgroup of* G''_q *is trivial.*

□

This theorem which is the main result of the thesis confirms the results in Kuz'min's paper [20] on the torsion-free part of G''_n for odd degree $n \geq 5$ and it corrects the results for even degree $n \geq 6$. Our results also show that the generating sets for the torsion-free part of $\mathfrak{G}''_n$ given in [11] are linearly independent over $\mathbb{Z}$. However, this main theorem does not address the question of whether or not the torsion subgroups featuring in parts (i.a) and (ii.b) are actually non-trivial. L.G. Kovács and Ralph Stöhr in their work [19] have answered this question by working over a field K with characteristic 2. Their paper shows that the torsion subgroups in the parts (i.a) and (ii.b) of our main theorem are actually non-trivial.

In Chapter 4, we focus on the dimension of the product of three homogeneous components in a free Lie algebra. Let L be a free Lie algebra over a field and let L_n denote the degree n homogeneous component of L. Formulae for the dimensions of the subspaces $[L_n, L_m]$ for all n and m were obtained by Ralph Stöhr and Michael

Vaughan-Lee in their work [39]. Formulae for the dimensions of triple products of the form $[L_n, L_m, L_k]$ were derived by Nil Mansuroğlu and Ralph Stöhr in Theorem 3.1 of [25] and in the author's master's thesis [24]. This theorem gives formulae for the dimension of subspaces of the form $[L_n, L_m, L_k]$ under certain conditions on n, m and k. Since we have explicit formulae for the dimension of $[L_n, L_m]$ for all n and m, the terms of the formulae in the theorem can be expressed in terms of Witt's dimension function, the exception is the product of the form $\dim[L_s(L_k), L_t(L_k), L_k]$ in the formula in a part of the theorem. In this theorem the obstacle is the products $[L_s(L_k), L_t(L_k), L_k]$. In Chapter 4, we investigate the smallest possible instance of such a product. The smallest product of the form $[L_s(L_k), L_t(L_k), L_k]$ is $[L_2, L_2, L_1]$. Surprisingly, it turns out that its dimension depends on the characteristic of the ground field. By using the main results obtained in Chapter 3, we find that if $r \geq 5$, then the dimension of $[L_2, L_2, L_1]$ over a field of characteristic 2 is strictly less than the dimension of $[L_2, L_2, L_1]$ over a field of characteristic other than 2.

This thesis is organized as follows:

In Chapter 2 we give some preliminary notions and notation to be used throughout this work.

Chapter 3 is the main chapter of this thesis. In this chapter, firstly we describe our aim and then, we explain the isomorphism which our new approach is based on. We give some details about the obtaining of the isomorphism between the second derived ideal G'' of G and the tensor product $(M \wedge M) \otimes_U \mathbb{Z}$. In Section 3, we derived the generating sets for the fine homogeneous components of G''. Then we examine torsion elements in G'' by defining the t-elements in Section 4. In next section, we introduce a 6-term exact sequence that includes the exterior square $M \wedge M$, and we examine the U-modules occurring in that sequence. Some of the results require to work over a field rather than over the ring of integers in Section 6. Then, these results are exploited to work out the dimensions of the homogeneous components and the fine homogeneous components of $G'' \otimes K$ for an arbitrary field K of characteristic other than 2 in Section 7. Then in Section 8, we state and prove our main result. Moreover, we discuss how our results relate to those in Kuz'min's 1977 paper [20] and in the 1985 paper [11] by N.D. Gupta, T.C. Hurley, F. Levin. Here we give a proof for the parts relating to $\mathfrak{F}_n$ of Theorem 1 and Theorem 4 in [11] by using the main result. We show that the generating sets for $\mathfrak{F}_n$ are linearly independent over $\mathbb{Z}$. Moreover, we give a new, more direct proof for the parts 1) and 2) of Theorem 4 in [20]. In Section 9, the only open problem of this chapter is the question of whether or not the torsion subgroups featuring in parts (i.a) and (ii.b) in the main theorem are actually non-trivial. We give some details how to solve this problem by L.G. Kovács and Ralph Stöhr in [19]. In last section, we present a short direct proof for the fact G'' is a direct sum of a free abelian group and an elementary abelian 2-group. We show that the torsion subgroup of G'' is annihilated by 2, and that the quotient by it is free abelian.

In Chapter 4, we work out the dimension of the product of the form $[L_2, L_2, L_1]$ and we observe that the dimension of this product over fields of characteristic 2 is different from the dimension over fields of characteristic other than 2.

Chapter 2

Preliminaries

For this thesis we need a very good understanding of Lie algebras, a good knowledge of free Lie algebras and some homological methods. In this chapter, we establish the notation and we introduce the basic concepts to be used throughout this work. We also recall some well-known results. Most of these statements will be provided without proofs, but the references where they can be found will be given.

2.1 Preliminaries on Homology

We record some background on homology. The main sources of information in this section are [5] and [15]. For more information about homological methods we refer the reader to [42].

2.1.1 Modules

In this subsection we will establish basic results of the modules which form the building blocks of homological algebra. The concept of a module over a ring is a generalization of the notion of vector space over a field. Let K be a commutative ring with identity element.

A *right module* over K or a *right K-module* is an abelian group $(A, +)$ with a map $A \times K \to A$ defined by $(a,k) \mapsto ak \in A$ for $k \in K$ and $a \in A$ satisfying

$$(a_1 + a_2)k = a_1k + a_2k, \qquad a(k_1 + k_2) = ak_1 + ak_2,$$
$$a(k_1k_2) = (ak_1)k_2, \qquad a1 = a,$$

for $k, k_1, k_2 \in K$; $a, a_1, a_2 \in A$.

Similarly, a *left K-module* is an abelian group $(A, +)$ with an operation $ka \in A$ for $k \in K$ and $a \in A$ satisfying

$$k(a_1 + a_2) = ka_1 + ka_2, \qquad (k_1 + k_2)a = k_1a + k_2a,$$
$$(k_1k_2)a = (k_1a)k_2, \qquad 1a = a,$$

for $k, k_1, k_2 \in K$; $a, a_1, a_2 \in A$.

We have obvious examples of K-modules. In the special case $K = \mathbb{Z}$ the ring of integers, every abelian group A has a structure of $\mathbb{Z}$-module by defining the action as for any $a \in A$, $k \in \mathbb{Z}$, if $k = 0$, $ak = 0$; if $k > 0$, $ak = \underbrace{a + a + a + \ldots + a}_{k \text{ times}}$; if $k < 0$, $ak = -(-ak)$.

If K is a field, any K-module becomes a vector space over K.

Let A and B be two right K-modules. A homomorphism (or linear transformation, or mapping) $\eta : A \to B$ of K-modules is a homomorphism of abelian groups such that $(ak)\eta = (a\eta)k$ for all $a \in A$, $k \in K$. The identity map of A is clearly a homomorphism of K-modules and it is denoted by $1_A : A \to A$. A homomorphism $\eta : A \to B$ is called a monomorphism (or injective homomorphism) if $\mathrm{Ker}\eta = 0$, we use the symbol $\eta : A \rightarrowtail B$. The homomorphism η is called an epimorphism (or surjective homomorphism) if $\mathrm{Coker}\eta = 0$ or $\mathrm{Im}\eta = B$, we use the symbol $\eta : A \twoheadrightarrow B$. Moreover, if η is both surjective and injective, then η is called an isomorphism, we use the symbol $\eta : A \xrightarrow{\sim} B$. Similarly, let A and B two left K-modules. We define a homomorphism $\eta : A \to B$ of K-modules as a homomorphism of abelian groups such that $(ka)\eta = k(a\eta)$ for all $a \in A$, $k \in K$.

Suppose that A is a right (or left) K-module and B is a subgroup of A. If for any

$b \in B$ and any $k \in K$, the product bk (or kb) is in B, then B is called a *K-submodule.*

Let F be a left K-module and $\mathcal{F}$ be a subset of F. A subset $\mathcal{F}$ is said to be a *basis* for the K-module F if every element of F can be written uniquely as a finite linear combination of elements from $\mathcal{F}$. In other words, the following conditions satisfy

(i) for any $f \in F$, there exist $f_1, f_2, \ldots, f_n \in \mathcal{F}$ and $k_1, k_2, \ldots, k_n \in K$ such that $f = k_1f_1 + k_2f_2 + \ldots + k_nf_n$. This means that the set $\mathcal{F}$ spans (or generates) F,

(ii) for $f_1, f_2, \ldots, f_n \in \mathcal{F}$ and $k_1, k_2, \ldots, k_n \in K$, if $f = k_1f_1+k_2f_2+\ldots+k_nf_n = 0$, then $k_1 = k_2 = \ldots = k_n = 0$. Namely, the elements of $\mathcal{F}$ are linearly independent.

The K-module F is called a *free K-module* if it has a basis $\mathcal{F}$. The elements of $\mathcal{F}$ are called *free generators.*

Proposition 2.1 Let F be a K-module. Then F is a free K-module if and only if F is isomorphic to a direct sum of copies of K.

□

Let A be a right K-module and B be a left K-module. Let S be a free abelian group on the set $Y = \{(a, b);\ a \in A,\ b \in B\}$ and I be the subgroup of S generated by the elements

$$(a_1 + a_2, b) - (a_1, b) - (a_2, b),\ (a, b_1 + b_2) - (a, b_1) - (a, b_2),$$
$$(ak, b) - (a, kb),$$

for all $a, a_1, a_2 \in A$; $b, b_1, b_2 \in B$ and $k \in K$. The tensor product of A and B over K is the quotient S/I. We denote the tensor product of A and B over K by $A \otimes_K B$. The elements of $A \otimes_K B$ are written as linear combinations of the images of (a, b) in $A \otimes_K B$ which are denoted by $a \otimes b$ with $a \in A$ and $b \in B$. By setting module action as $(a \otimes b)k = ak \otimes b$ or $k(a \otimes b) = a \otimes kb$, the tensor product $A \otimes_K B$ turns into K-module. The elements of the form $a \otimes b$ are called *tensors.* We have the following relations between tensors:

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b, \; a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2,$$
$$ak \otimes b = a \otimes kb,$$

where $a, a_1, a_2 \in A$; $b, b_1, b_2 \in B$ and $k \in K$.

Let $\{B_i\}_{i\in I}$ be a family of left K-modules. We have that

$$A \otimes_K (\bigoplus_{i\in I} B_i) \cong \bigoplus_{i\in I}(A \otimes_K B_i).$$

Similarly, let $\{A_i\}_{i\in I}$ be a family of right K-modules. We have that

$$(\bigoplus_{i\in I} A_i) \otimes_K B \cong \bigoplus_{i\in I}(A_i \otimes_K B).$$

Let A be a free K-module with free generators $a_1, a_2, \ldots, a_s$ and B be a free K-module with free generators $b_1, b_2, \ldots, b_t$. The tensor product of these two free K-modules is free K-module with module action $(a \otimes b)k = ak \otimes b = a \otimes kb$ for $a \in A$, $b \in B$ and $k \in K$. To prove this first we may write that

$$A \cong \bigoplus_{i=1}^{s} a_i K \cong \bigoplus_{i=1}^{s} K \text{ and } B \cong \bigoplus_{j=1}^{t} b_j K \cong \bigoplus_{j=1}^{t} K.$$

Then, by tensoring A and B over K, we have

$$\begin{aligned} A \otimes_K B &\cong (\bigoplus_{i=1}^{s} K) \otimes_K (\bigoplus_{j=1}^{t} K) \\ &\cong \bigoplus_{i=1}^{s}(K \otimes_K (\bigoplus_{j=1}^{t} K)) \\ &\cong \bigoplus_{i}^{s} \bigoplus_{j}^{t}(K \otimes_K K) \\ &\cong \bigoplus_{p=1}^{st} K. \end{aligned}$$

Thus, by Proposition 2.1, the tensor product of two free K-modules is free K-module with free generators $(a_i \otimes b_j)$, where $a_i \in A$ and $b_j \in B$ with $1 \leq i \leq s$ and $1 \leq j \leq t$.

Let $\{C_i\}_{i \in I}$ be a family of right K-modules. A *chain complex* is a sequence

$$\mathcal{C} : \ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots$$

of K-modules together with ∂_n such that $\partial_{n+1}\partial_n = 0$ for all $n \in \mathbb{Z}$. The *nth homology module* (or group) of the chain complex is defined to be the quotient

$$H_n(\mathcal{C}) = \text{Ker}\partial_n / \text{Im}\partial_{n+1}.$$

The chain complex $\mathcal{C}$ is *exact* at the nth place if $\text{Ker}\partial_n = \text{Im}\partial_{n+1}$. If it is exact at all places, we say the chain complex $\mathcal{C}$ is exact. Moreover, a complex is free if all C_n are free. A sequence $0 \to A \to B \to C \to 0$ of K-modules is called a short exact sequence if this sequence is exact at A, B and C. Shortly, we shall use $A \rightarrowtail B \twoheadrightarrow C$.

Every long exact sequence $\mathcal{C} : \ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots$ decomposes into short exact sequences as follows:

$$\vdots$$

$$0 \longrightarrow \text{Ker}\partial_{n-1} \longrightarrow C_{n-1} \longrightarrow \text{Im}\partial_{n-1} \longrightarrow 0$$

$$\text{Ker}\partial_{n-1} = \text{Im}\partial_n$$

$$0 \longrightarrow \text{Ker}\partial_n \longrightarrow C_n \longrightarrow \text{Im}\partial_n \longrightarrow 0$$

$$\text{Ker}\partial_n = \text{Im}\partial_{n+1}$$

$$0 \longrightarrow \text{Ker}\partial_{n+1} \longrightarrow C_{n+1} \longrightarrow \text{Im}\partial_{n+1} \longrightarrow 0$$

$$\vdots$$

Let B be a left K-module. By tensoring the complex $\mathcal{C}$ with B, we can form the following sequence

$$\mathcal{C} \otimes_K B : \ldots \longrightarrow C_{n+1} \otimes_K B \stackrel{\partial_{n+1}\otimes 1}{\longrightarrow} C_n \otimes_K B \stackrel{\partial_n \otimes 1}{\longrightarrow} C_{n-1} \otimes_K B \longrightarrow \ldots$$

It is straightforward to verify that this is still a complex.

A short exact sequence $A \stackrel{\tau_1}{\rightarrowtail} B \stackrel{\tau_2}{\twoheadrightarrow} C$ of K-modules *splits* if there is a homomorphism $\tau_2' : C \to B$ such that $\tau_2' \tau_2 = 1_C$. The map τ_2' is called a *splitting*.

Lemma 2.2 Suppose that $\tau_2' : C \to B$ is a splitting for the short exact sequence $A \stackrel{\tau_1}{\rightarrowtail} B \stackrel{\tau_2}{\twoheadrightarrow} C$. Then B is isomorphic to the direct sum $A \oplus C$.

□

Let B and C be any K-modules. A K-module P is *projective* if given any homomorphism $f : P \to C$ and any epimorphism $g : B \to C$ there is a homomorphism $h : P \to B$ with $hg = f$.

Proposition 2.3 A direct sum of K-modules is projective if and only if each summand is projective.

□

Every free module is projective, but it is not true that every projective module is free. For example, let $K = \mathbb{Z}_6$, the ring of integers modulo 6. Since $\mathbb{Z}_6 = \mathbb{Z}_3 \oplus \mathbb{Z}_2$ as a $\mathbb{Z}_6$-module, by Proposition 2.3, $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are projective $\mathbb{Z}_6$-modules. However, they are not free $\mathbb{Z}_6$-modules.

Theorem 2.4 *For a K-module P, the following statements are equivalent.*

(1) P is projective;

(2) if $\tau_2 : B \twoheadrightarrow P$ is surjective, then there exists a homomorphism $\tau_2' : P \to B$ such that $\tau_2' \tau_2 = 1_P$;

(3) P is a direct summand in every module of which it is a quotient;

(4) P is a direct summand in a free module.

□

A *projective resolution* of the K-module D is a complex

$$\mathcal{P} : \ldots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0,$$

if for $n \geq 0$, all P_n are projective K-modules and the sequence

$$\ldots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow D \longrightarrow 0$$

is exact.

Let A be a right K-module and D be a left K-module. For $n \geq 0$, the abelian group $\mathrm{Tor}_n^K(A, D)$ is defined as

$$\mathrm{Tor}_n^K(A, D) = H_n(A \otimes_K \mathcal{P}),$$

where $\mathcal{P}$ is a projective resolution of D. For $n = 0$, $\mathrm{Tor}_0^K(A, D) = A \otimes_K D$. If either of A or D is a free K-module, then $\mathrm{Tor}_n^K(A, D) = 0$ for all $n \geq 1$ (see [22]).

If the sequence $0 \to A \to B \to C \to 0$ is a short exact sequence of right K-modules, then we have a long exact sequence of the form

$$\begin{aligned} \ldots \to \mathrm{Tor}_2^K(B, D) \to &\mathrm{Tor}_2^K(C, D) \to \mathrm{Tor}_1^K(A, D) \to \mathrm{Tor}_1^K(B, D) \to \mathrm{Tor}_1^K(C, D) \\ &\to A \otimes_K D \to B \otimes_K D \to C \otimes_K D \to 0 \end{aligned} \tag{2.1}$$

(see [5]). Suppose that B is a free K-module in the short exact sequence $0 \to A \to B \to C \to 0$. Therefore, this short exact sequence induces a long exact sequence of the form

$$\begin{aligned} \ldots \to &\mathrm{Tor}_n^K(A, D) \to \underbrace{\mathrm{Tor}_n^K(B, D)}_{=0} \to \mathrm{Tor}_n^K(C, D) \\ &\to \mathrm{Tor}_{n-1}^K(A, D) \to \underbrace{\mathrm{Tor}_{n-1}^K(B, D)}_{=0} \to \mathrm{Tor}_{n-1}^K(C, D) \to \ldots. \end{aligned}$$

We get short exact sequences

$$0 \to \mathrm{Tor}_n^K(C, D) \to \mathrm{Tor}_{n-1}^K(A, D) \to 0$$

for $n \geq 1$. This implies that $\mathrm{Tor}_n^K(C, D) \cong \mathrm{Tor}_{n-1}^K(A, D)$ for $n \geq 1$.

The rank of a free abelian group is analogous to the dimension of a vector space. If the sequence $0 \to A \to B \to C \to 0$ is a short exact sequence of free abelian groups, then $\mathrm{rank}B = \mathrm{rank}A + \mathrm{rank}C$. If $K = \mathbb{Z}$, any finitely generated torsion-free abelian group is a free abelian group, i.e., a free $\mathbb{Z}$-module.

Let $\tilde{K}$ be a commutative ring containing K as a subring with the same unity as in $\tilde{K}$. If A is a K-algebra, then $A_{\tilde{K}}$ will be a $\tilde{K}$-algebra, where $A_{\tilde{K}} = A \otimes_K \tilde{K}$. If A is a free K-module with a basis $\{f_1, f_2, \ldots, f_n\}$, then $A \otimes_K \tilde{K}$ is a free $\tilde{K}$-module with a basis $\{f_1 \otimes 1, f_2 \otimes 1, \ldots, f_n \otimes 1\}$. In particular, this is true when K is a field (for further information see [17] S.A. Lang. 1965, Chapter XVI; [18] A.I. Kostrikin 1982, Chapter 8).

Free modules over a field are vector spaces. Let A, B and C be finite-dimensional vector spaces over a field K. The sequence $0 \to A \xrightarrow{\beta_1} B \xrightarrow{\beta_2} C \to 0$ is a short exact sequence and it splits. Indeed, due to being an exact sequence of vector spaces, $B/\mathrm{Im}\beta_1 \cong C$. This shows that

$$\dim B - \dim A = \dim C$$

or

$$\dim B = \dim A + \dim C.$$

Therefore, $B \cong A \oplus C$. This means that every short exact sequence of vector spaces splits.

As an immediate consequence, for an exact sequence $0 \to A \xrightarrow{\beta_1} B \xrightarrow{\beta_2} C \xrightarrow{\beta_3} D \to 0$

of vector spaces, we have

$$\begin{aligned}\dim A &= \dim B - \dim \mathrm{Im}\beta_2\\ &= \dim B - \dim \mathrm{Ker}\beta_3\\ &= \dim B - (\dim C - \dim D)\\ &= \dim B - \dim C + \dim D.\end{aligned}$$

2.1.2 Tensor, exterior and symmetric squares

Let $X = \{x_1, x_2, \ldots, x_r\}$ with $r \geq 2$ be an ordered set by $x_1 < x_2 < \ldots < x_r$ and U be the polynomial ring on X with coefficients in K. For a monomial $u = y_1y_2\ldots y_k \in U$ where $y_i \in X$ for $i \in \{1, 2, \ldots, k\}$, we let $l(u)$ denote the smallest of the elements $y_1, y_2, \ldots, y_k$ with respect to the ordering of X, and we write $\deg u$ for the degree of u. By $\mathcal{U}$, we denote the set of all monomials,

$$\mathcal{U} = \{y_1y_2\ldots y_k;\ y_i \in X,\ 1 \leq i \leq k,\ y_1 \leq y_2 \leq \ldots \leq y_k,\ k \geq 1\} \cup \{1\}$$

in U with the convention that 1 is the only monomial of degree 0. The set $\mathcal{U}$ is a basis for U. We calculate the number of the monomials of degree n by

$$\binom{n+r-1}{n}. \tag{2.2}$$

We define a map $\varepsilon : U \to K$ by

$$\sum_{u\in\mathcal{U}} \alpha_u u \mapsto \alpha_1,$$

for each $\alpha_u \in K$ and α_1 is the coefficient of the monomial 1 of degree 0, that maps every polynomial to its constant term. This map is the augmentation of U. We let Δ denote the augmentation ideal of U, that is the ideal of all polynomials with zero constant term. Hence, Δ is the kernel of the augmentation map $\varepsilon : U \to K$. The set

$\mathcal{U} \setminus \{1\}$ is a basis of Δ. The short exact sequence

$$0 \longrightarrow \Delta \longrightarrow U \xrightarrow{\varepsilon} K \longrightarrow 0 \tag{2.3}$$

is known as the augmentation sequence.

Let A be any U-module. The U-module A is called a *trivial U-module* if for all $a \in A$ and $u \in \Delta$, $au = 0$. For example, if $U = \mathbb{Z}[X]$, $\mathbb{Z}$ is a trivial U-module.

For U-modules A and B, the tensor product $A \otimes B$ over K will be regarded as a U-module with the derivation action, that is

$$(a \otimes b)y = ay \otimes b + a \otimes by,$$

for $a \in A$, $b \in B$ and $y \in X$.

Likewise, the exterior squares and symmetric squares of a U-module A are denoted by $A \wedge A$ and $A \circ A$, respectively. The *exterior and the symmetric squares of A* are defined by

$$A \wedge A = A \otimes A / \langle a \otimes a;\ a \in A \rangle$$

and

$$A \circ A = A \otimes A / \langle a_1 \otimes a_2 - a_2 \otimes a_1;\ a_1, a_2 \in A \rangle,$$

respectively, and will be regarded as U-modules with derivation actions:

$$(a \wedge b)y = ay \wedge b + a \wedge by$$

and

$$(a \circ b)y = ay \circ b + a \circ by$$

for $a, b \in A$ and $y \in X$. Moreover, in the exterior and symmetric squares of A, the following relations hold:

$$a \wedge b = -b \wedge a \text{ and } a \circ b = b \circ a.$$

There is the projection map

$$\pi : A \otimes A \to A \wedge A,\ a_1 \otimes a_2 \mapsto a_1 \wedge a_2$$

for $a_1, a_2 \in A$. Moreover, we have the embedding map

$$\nu : A \wedge A \to A \otimes A,\ a_1 \wedge a_2 \mapsto a_1 \otimes a_2 - a_2 \otimes a_1$$

for $a_1, a_2 \in A$. Under π, the image of $a_1 \otimes a_2 - a_2 \otimes a_1$ in $A \wedge A$ is equal to $a_1 \wedge a_2 - a_2 \wedge a_1$. Since $a_2 \wedge a_1 = -a_1 \wedge a_2$, it will be $2(a_1 \wedge a_2)$. Therefore, the composite $\nu\pi$,

$$A \wedge A \xrightarrow{\nu} A \otimes A \xrightarrow{\pi} A \wedge A,$$

is determined by $a_1 \wedge a_2 \mapsto 2(a_1 \wedge a_2)$.

Similarly, the canonical image of $a_1 \otimes a_2 \in A \otimes A$ in $A \circ A$ is denoted by $a_1 \circ a_2$ and there is the embedding map

$$\theta : A \circ A \to A \otimes A,\ a_1 \circ a_2 \mapsto a_1 \otimes a_2 + a_2 \otimes a_1$$

for $a_1, a_2 \in A$. By definition, it follows that there is a short exact sequence

$$0 \longrightarrow A \wedge A \xrightarrow{\nu} A \otimes A \longrightarrow A \circ A \longrightarrow 0, \tag{2.4}$$

where the maps are given by $a \wedge b \mapsto a \otimes b - b \otimes a$ and $a \otimes b \mapsto a \circ b$ for $a, b \in A$.

If the characteristic of K is equal to 2, then since the exterior square $A \wedge A$ is a homomorphic image of the symmetric square, we have a short exact sequence

$$0 \longrightarrow \mathrm{Ker} f \longrightarrow A \circ A \xrightarrow{f} A \wedge A \longrightarrow 0, \tag{2.5}$$

where the map f is the natural projection defined by $a \circ b \mapsto a \wedge b$ for all $a, b \in A$. Let $\mathcal{A}$ be a K-basis of A; therefore, the kernel of the map f will be the span of all squares $a \circ a$ for all $a \in \mathcal{A}$. Moreover, it is easy to show that $\mathrm{Ker} f$ is a trivial U-module. For any $y \in X$, by using derivation action we have

$$(a \circ a)y = ay \circ a + a \circ ay = 2(ay \circ a) = 2(a \circ ay).$$

By our assumption that $\mathrm{Char} K = 2$, we have $(a \circ a)y = 0$. This implies that $\mathrm{Ker} f$ is a trivial U-module. If K is a field of characteristic other than 2, the exact sequence (2.5) is not available.

2.1.3 Trivializations of tensor, symmetric and exterior squares

Let A be an arbitrary U-module and let K be the trivial U-module. The trivialization of $A \otimes A$ is defined as

$$(A \otimes A) \otimes_U K = (A \otimes A)/(A \otimes A)\Delta,$$

where $(A \otimes A)\Delta$ is the K-submodule generated by the elements of the form

$$(a \otimes b)y = ay \otimes b + a \otimes by,$$

for $a, b \in A$, $y \in X$. The canonical homomorphism $A \otimes A \to (A \otimes A) \otimes_U K$ is determined by

$$a \otimes b \mapsto (a \otimes b) \otimes 1,$$

for $a, b \in A$. We use simply $a \otimes_* b$ instead of $(a \otimes b) \otimes 1$.

Similarly, for the exterior square and the symmetric square of A, we obtain the trivialization of $A \wedge A$ as the quotient of the exterior square $A \wedge A$ by the K-submodule generated by the elements $ay \wedge b + a \wedge by$ for $a, b \in A$, and $y \in X$. We write $a \wedge_* b$ for the image of the element $a \wedge b$ of $A \wedge A$ in the trivialization of the exterior square of A.

The trivialization of the symmetric square of A is obtained by forming the quotient $(A\circ A)/(A\circ A)\Delta$. The K-submodule $(A\circ A)\Delta$ is generated by the elements $ay\circ b+a\circ by$ for $a, b \in A$ and $y \in X$. The canonical homomorphism $A \circ A \to (A \circ A) \otimes_U K$ is determined by $a \circ b \mapsto (a \circ b) \otimes 1$ for $a, b \in A$. We use $a \circ_* b$ for the image in the trivialization of the symmetric square of A.

Moreover, in the trivializations of the tensor, exterior and symmetric squares of A, the following relations hold

$$a \otimes_* by = -(ay \otimes_* b),\ a \wedge_* by = -(ay \wedge_* b)$$

and

$$a \circ_* by = -(ay \circ_* b)$$

for all $a, b \in A$ and $y \in X$. It is easy to prove that by the derivation action, we have $(a \otimes b)y = ay \otimes b + a \otimes by \in (A \otimes A)\Delta$. It follows that $ay \otimes_* b + a \otimes_* by = 0$ in $(A \otimes A) \otimes_U K$. Hence, we obtain $a \otimes_* by = -(ay \otimes_* b)$. Similarly, using derivation action for exterior and symmetric squares of A, we have

$$(a \wedge b)y = ay \wedge b + a \wedge by \in (A \wedge A)\Delta$$

and

$$(a \circ b)y = ay \circ b + a \circ by \in (A \circ A)\Delta$$

respectively. Therefore, we obtain the relations $a \wedge_* by = -(ay \wedge_* b)$ and $a \circ_* by = -(ay \circ_* b)$ for the trivializations of exterior and symmetric squares of A.

From (2.4) and the right-exactness of $\otimes$ which follows from the long exact sequence

(2.1), these trivializations yield an exact sequence

$$(A \wedge A) \otimes_U K \to (A \otimes A) \otimes_U K \to (A \circ A) \otimes_U K \to 0$$

with the first map defined by $a \wedge_* b \mapsto a \otimes_* b - b \otimes_* a$ and the second map defined by $a \otimes_* b \mapsto a \circ_* b$.

2.2 Free Lie algebras

In this section we present classical information on Lie algebras and free Lie algebras.

Let K be a commutative ring with identity element and let A be a K-module. We say that A over K is a *K-algebra* if A has a product

$$A \times A \longrightarrow A, \ (a, b) \mapsto ab,$$

satisfying the bilinear conditions:

$$\begin{aligned}(a_1 + a_2)b =& a_1 b + a_2 b, \ a(b_1 + b_2) = ab_1 + ab_2, \\ k(ab) =& (ka)b = a(kb),\end{aligned}$$

for $a, a_1, a_2, b, b_1, b_2 \in A$ and $k \in K$.

We say that ab is the product of a and b. If this algebra has the associative property $(ab)c = a(bc)$ for $a, b, c \in A$, then A is called an *associative algebra.*

A *Lie algebra* L over K is a K-algebra with a bilinear map,

$$L \times L \longrightarrow L, \ (a, b) \mapsto [a, b],$$

satisfying the following properties:

(i) $[a, a] = 0$ for all $a \in L$,

(ii) $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ for all $a, b, c \in L$.

We say that $[a,b]$ is the Lie bracket, and also it is called the *commutator* of a and b.

If L is a Lie algebra, then by using the first (i) of these conditions, we have

$$0 = [a+b, a+b] = [a,a] + [a,b] + [b,a] + [b,b] = [a,b] + [b,a],$$

for all $a, b \in L$. This implies that $[a,b] = -[b,a]$ for all $a, b \in L$ and this is called *anti-commutativity*. Conversely, if anti-commutativity holds, then $2[a,a] = 0$, so that, if K is a field of characteristic other than 2, then $[a,a] = 0$. The second condition (ii) is known as the *Jacobi identity*.

A *Lie subalgebra* $\hat{L}$ of a Lie algebra L is a K-submodule closed under the Lie bracket of L, $[a,b] \in \hat{L}$ for all $a, b \in \hat{L}$. A K-subalgebra I of a Lie algebra L is called an *ideal* of L if $[a,b] \in I$ for all $a \in L$ and $b \in I$. The quotient of L by I is an algebra defined as a quotient module under the Lie bracket. This algebra is called the *quotient (factor) algebra* and it is denoted by L/I.

A *Lie homomorphism* between two Lie algebras over K is a linear map $f : L \to \tilde{L}$, such that $([a,b])f = [(a)f, (b)f]$ for all a and b in L. The kernel of f is an ideal in L and the image of f is a subalgebra of $\tilde{L}$.

For a Lie algebra L and any element $a \in L$, we define a map

$$\mathrm{ad}_a : L \to L, \;\; b \mapsto [b,a]$$

for all $b \in L$ which is called the *adjoint action*. Since the Lie bracket $[,]$ is bilinear and $\mathrm{ad}_a \in gl(L)$, where $gl(L)$ is the general linear algebra of L, which is the Lie algebra of linear maps on L, we have a linear map

$$\mathrm{ad} : L \to gl(L), \;\; a \mapsto \mathrm{ad}_a,$$

for $a \in L$, this map is called the *adjoint representation*. The adjoint representation is one of the famous examples of Lie homomorphisms.

We define a module of a Lie algebra L in such a way that an action of L on a K-module V is given by a map $V \times L \to V$, $(v,a) \mapsto va$ such that the following properties hold:

$$v(k_1a + k_2b) = k_1(va)+k_2(vb),\ (k_1u + k_2v)a = k_1(ua) + k_2(va),$$
$$v[a, b] = (va)b - (vb)a$$

for $k_1, k_2 \in K$; $a, b \in L$; $u, v \in V$.

We say that V is called an *L-module*. Each L-module corresponds to a representation $L \to gl(V)$, where $gl(V)$ is the general linear algebra of V. For example, the adjoint representation corresponds to viewing L as an L-module by the adjoint action.

The *center* of a Lie algebra L is defined to be

$$Z(L) = \{a \in L;\ [a, b] = 0 \text{ for all } b \in L\}.$$

The center of a Lie algebra of L is an ideal in L and also the kernel of the adjoint representation $\mathrm{ad} : L \to gl(L)$.

The Lie algebra L is called an *abelian Lie algebra* if we have $[a, b] = 0$ for all $a, b \in L$.

The *derived algebra* of a Lie algebra L is denoted by L' which is analogous to the commutator subgroup of a group. The derived algebra is the ideal $[L, L]$ generated by all $[a, b]$ for $a, b \in L$. For all non-negative integer n, the ideal $L^{(n)}$ of L is defined using induction by setting

$$L^{(0)} = L,\ L^{(1)} = [L, L],\ L^{(2)} = [L^{(1)}, L^{(1)}], \ldots,\ L^{(n+1)} = [L^{(n)}, L^{(n)}].$$

Hence, we can iterate to define the derived series of L as follows:

$$L = L^{(0)} \supseteq L^{(1)} \supseteq \ldots \supseteq L^{(n)} \supseteq \ldots .$$

The first ideal $L^{(1)}$ and the second ideal $L^{(2)}$ are often denoted by L' and L'', respectively. If $L^{(n)} = 0$ for some n, then L is called *solvable*.

The *lower central series* of L, denoted by $\{\gamma_n(L)\}_{n\geq 1}$, is defined inductively as follows:

$$\gamma_1(L) = L,\ \gamma_2(L) = [\gamma_1(L), L] = [L, L] = L', \ldots$$

If $\gamma_n(L)$ is already defined, we have $\gamma_{n+1}(L) = [\gamma_n(L), L]$, where $\gamma_n(L)$ is called the nth term of the lower central series of L. If $\gamma_n(L) = 0$ for some n, L is called *nilpotent*. Clearly, $L^{(n)} \subseteq \gamma_n(L)$ for all n. Hence, nilpotent algebras are solvable.

Let A be any associative algebra with the product ab for $a, b \in A$. Now, we define a new product on A by

$$(a, b) \mapsto [a, b] = ab - ba,$$

where ab denotes the product of a and b in A. It is easy to check that the associative algebra A with the new product satisfies the Lie algebra conditions, anticommutativity and the Jacobi identity. Hence, any associative algebra A over K becomes a Lie algebra over K with the Lie bracket $[a, b] = ab - ba$.

Let L be a Lie algebra over K and X be a subset of L. If every map from X into an arbitrary Lie algebra $\tilde{L}$ over K extends uniquely to a Lie algebra homomorphism of L to $\tilde{L}$, then we say that L is *free* on X. If L is a free Lie algebra, L is freely generated by X. The free Lie algebra on X is denoted by $L(X)$. If X is finite, the cardinality of X is called the rank of $L(X)$. Otherwise, the free Lie algebra is said to be of infinite rank.

Free Lie algebras have been a substantial topic of research since the 1930's. Many mathematicians have been studying on free Lie algebras and they proved that free Lie algebras are closely related to free groups. There are many results on free groups analogous to results on free Lie algebras. For instance, the well-known Nielsen-Schreier Theorem (Nielsen [29], Schreier [31]) on free groups states that every subgroup of a free group is free (in [23]). When K is a field, for free Lie algebras over K a corresponding result has been proved by Shirshov [34] and Witt [44] in the celebrated

Shirshov-Witt Theorem [40] which says that every subalgebra of a free Lie algebra is free.

The free monoid on X consists of the all finite sequences of the form $a_1a_2\dots a_n$ including the empty sequence, where $a_i \in X$ for $i = 1, 2, \dots, n$. These sequences are called words on X and we let denote the free monoid on X by $W(X)$. Let $A(X)$ be the free K-module with basis $W(X)$, namely, every element of $A(X)$ can be written as a K-linear combination of the words from $W(X)$. The multiplication of $W(X)$ can be extended to $A(X)$ distributively. In fact, $A(X)$ is the free associative algebra on X. We may give $A(X)$ the structure of a Lie algebra in the usual way by setting $[a, b] = ab - ba$, where ab denotes the usual product of a and b for $a, b \in A(X)$. Let L be the Lie subalgebra of $A(X)$ generated by X. The Lie subalgebra L is isomorphic to the free Lie algebra on X. This shows that the free Lie algebra $L(X)$ is embedded in $A(X)$, that is a result of Witt (see [16]). In fact, the free associative algebra $A(X)$ is the universal enveloping algebra of $L(X)$. This means that every Lie homomorphism from L into an arbitrary associative algebra $\tilde{A}$ over K extends uniquely to an algebra homomorphism from A into $\tilde{A}$ (see [16]). Lie elements in $A(X)$ are the elements of $L(X)$ considered as a subalgebra of $A(X)$. When $K = \mathbb{Z}$, $L(X)$ and $A(X)$ are called the free Lie ring on X and the free associative ring on X, respectively.

In a free Lie algebra $L(X)$, we say that the elements of X are Lie monomials, and if a and b are Lie monomials, the product of a and b, $[a, b] \in L(X)$ is a Lie monomial. The degree is defined inductively by setting $\deg a = 1$ for $a \in X$ and $\deg([a, b]) = \deg a + \deg b$ for distinct Lie monomials a and b. We introduce the left-normed Lie products inductively by $[a_1] = a_1$ and $[a_1, a_2, \dots, a_{n-1}, a_n] = [[a_1, a_2, \dots, a_{n-1}], a_n]$, where $a_i \in L(X)$ for all $1 \leq i \leq n$. Hence, we can write

$$[a_1, a_2, a_3, \dots, a_n] = [\dots[[a_1, a_2], a_3], \dots, a_n],$$

for all $a_1, a_2, \dots, a_n \in L(X)$. In this thesis, we will use of the following version of the

Jacobi identity

$$[a, b, c] = [a, c, b] + [a, [b, c]]$$

for all a, b, c belonging to L. In general, we have

$$\begin{aligned}[a_1, a_2, a_3, \ldots, [a_i, a_{i+1}], \ldots, a_n] = [a_1, a_2, a_3, \ldots, a_i, a_{i+1}, \ldots, a_n] \\ - [a_1, a_2, a_3, \ldots, a_{i+1}, a_i, \ldots, a_n],\end{aligned}$$

for all $a_1, a_2, \ldots, a_n \in L(X)$, $1 < i < n$.

If each $a_i \in X$ for $i \in \{1, 2, \ldots, n\}$, then $[a_1, a_2, \ldots, a_n]$ is a left-normed Lie monomial. Thus, a left-normed Lie monomial is a left-normed Lie product. Each element of $L(X)$ can be written as a linear combination of the left-normed monomials. In other words, the set of all left-normed Lie monomials spans the whole free Lie algebra L on X. The left-normed Lie monomials are not linearly independent. Bases of free Lie algebra were constructed by M. Hall [13] using the procedure in P. Hall's work where P. Hall introduced Hall sets in free groups [12]. These bases are called Hall bases.

2.2.1 Preliminary results on free Lie algebras

We let $L(X)$ denote the free Lie algebra of rank $r \geq 2$ on free generating set $X = \{x_1, x_2, \ldots, x_r\}$ over K, and $A(X)$ denotes the universal enveloping algebra of $L(X)$. The universal enveloping algebra $A(X)$ is a graded algebra, we write

$$A(X) = \bigoplus_{n=0}^{\infty} A_n(X), \tag{2.6}$$

where $A_0(X) = K$ and $A_n(X)$ is the *degree* $n > 0$ *homogeneous component* of $A(X)$, that is the K-submodule of $A(X)$ spanned by all monomials of length n in the form $a_1a_2 \ldots a_n$, where $a_i \in X$ for $i = 1, 2, \ldots, n$. Due to the grading of $A(X)$, the free Lie algebra $L(X)$ is a graded algebra,

$$L(X) = \bigoplus_{n=1}^{\infty} L_n(X), \tag{2.7}$$

where the K-submodule $L_n(X)$ of $L(X)$ is called the *nth homogeneous component* of $L(X)$ spanned by Lie products of degree n in the free generators of $L(X)$. Any element in $L_n(X)$ is called a *homogeneous element* of degree n. By our assumption that X is a finite set, each of the homogeneous components of $L(X)$ is a free K-module of finite rank and the rank of $L_n(X)$ for all $n \geq 1$ is given by Witt's formula

$$\text{rank} L_n(X) = f(n,r) = \frac{1}{n}\sum_{d|n} \mu(d) r^{\frac{n}{d}}, \tag{2.8}$$

where μ is the Möbius function ([23], Theorem 5.11).
The function $\mu : \mathbb{N} \mapsto \{-1, 0, 1\}$ is defined as follows: If d has a prime factorization,

$$d = p_1^{n_1} p_2^{n_2} \dots p_m^{n_m},\ n_i > 0 \text{ for } i \in \{1, 2, \dots, m\},$$

where m is the number of distinct prime divisors of d, then

$$\mu(d) = \begin{cases} 1, & \text{if } d = 1, \\ 0, & \text{if } p_i^2 | d, \text{ for some prime } p_i, \\ (-1)^m, & \text{otherwise.} \end{cases}$$

Let $L_n(X)$ and $L_m(X)$ be nth and mth homogeneous components of $L(X)$ respectively. The product $[L_n, L_m]$ of these homogeneous components is a K-submodule of L_{n+m} spanned by the elements $[a, b]$, where $a \in L_n$ and $b \in L_m$. Ranks of such submodules were obtained by R. Stöhr and M. Vaughan-Lee in [39]. The authors proved that if $n > m$ and $m \nmid n$, then

$$\text{rank}([L_n, L_m]) = \text{rank} L_n \text{rank} L_m.$$

If $n = sm$ with $s \geq 1$, then

$$\text{rank}([L_n, L_m]) = (\text{rank}L_n - f(s, \text{rank}L_m))\text{rank}L_m + f(s+1, \text{rank}L_m).$$

We define the *fine homogeneous components* of $L(X)$ as the submodules of $L(X)$ spanned by all left-normed monomials of the same multidegree in the free generators. A *composition* q of n in r parts ($q \models n$) is a sequence $q = (q_1, q_2, \ldots, q_r)$ of non-negative integers such that $\sum_{j=1}^{r} q_j = n$. The fine homogeneous component $L_q(X)$ of $L(X)$ is the submodule of $L(X)$ generated by all left-normed Lie products of partial degree q_i with respect to x_i for $1 \leq i \leq r$. It is clear to see that each of the homogeneous components $L_n(X)$ can be written as a direct sum of fine homogeneous components,

$$L_n(X) = \bigoplus_{q \models n} L_q(X).$$

If all non-zero parts of q are equal to 1, a fine homogeneous component $L_q(X)$ of multidegree q is called *multilinear*.

Similarly, we let $A_q(X)$ be the submodule of $A(X)$ spanned by all elements from $W(X)$ of partial degree q_i with respect to x_i for $1 \leq i \leq r$. Each of the homogeneous components $A_n(X)$ can be written as a direct sum of fine homogeneous components,

$$A_n(X) = \bigoplus_{q \models n} A_q(X).$$

The derived ideal L' of the free Lie algebra $L(X)$ is a free Lie algebra and it can be expressed as

$$\begin{aligned} L' =& [L, L] \\ =& \langle [a, b];\ a, b \in L \rangle \\ =& L_2 \oplus L_3 \oplus L_4 \oplus \ldots \end{aligned} \tag{2.9}$$

The Lie monomials $[y_1, y_2, \ldots, y_n]$, where $y_1 > y_2 \leq y_3 \leq y_4 \leq \ldots \leq y_n$ for all $y_i \in X$ with $1 \leq i \leq n$ and $n \geq 2$ form a free generating set for L' (for example, see Section 4.2.2 in [1]). We denote this generating set by X'. The lexicographic order is defined on X'.

Similarly, the second derived ideal L'' of the free Lie algebra $L(X)$ is free and by the definition of L'', we can show it as

$$\begin{aligned} L'' =& [L', L'] \\ =& [L_2 \oplus L_3 \oplus L_4 \oplus \ldots, L_2 \oplus L_3 \oplus L_4 \oplus \ldots] \\ =& [L_2, L_2] \oplus [L_3, L_2] \oplus ([L_4, L_2] + [L_3, L_3]) \oplus \ldots \end{aligned} \tag{2.10}$$

The second derived ideal is generated by $[y_1, y_2, \ldots, y_n]$, where $y_1 > y_2 \leq y_3 \leq y_4 \leq \ldots \leq y_n$ for all $y_i \in X'$ with $1 \leq i \leq n$ and $n \geq 2$.

We can find further information on Lie algebras in the book of N. Bourbaki [2] and in the book of N. Jacobson [16]. Moreover, further reference on free Lie algebras can be found in [[2], Chapter IV] and [[16], Chapter V, Section 4]. We recommend the book of Yu.A. Bakhturin [[1], Chapter 2] and the book of C. Reutenauer [40] for more information about free Lie algebras.

2.3 Free centre-by-metabelian Lie algebras

As we defined before, X is a set consisting of the elements $x_1, x_2, \ldots, x_r$ with $r \geq 2$ to be ordered by $x_1 < x_2 < \ldots < x_r$. Let K be a commutative ring with identity element and L be the free Lie algebra on free generating set X with the finite rank r over K. We define the *free metabelian Lie algebra* D as the quotient L/L'', where L'' is the second derived ideal of L. The grading of L induces a grading of D and thus,

$$D = \bigoplus_{n=1}^{\infty} D_n,$$

where D_n denotes the degree n homogeneous component of D, which is defined by $D_n = L_n/(L_n \cap L'')$. It is well-known that D is a free K-module (see [1]). The left-normed basic commutators $[y_1, y_2, \ldots, y_n]$ for all $y_i \in X$ with $1 \leq i \leq n$ and $y_1 > y_2 \leq y_3 \leq \ldots \leq y_n$ form a K-basis of the homogeneous component D_n. For a finite rank r and $n \geq 2$, the rank of D_n is given by

$$\text{rank}D_n = (n-1)\begin{pmatrix} n+r-2 \\ r-2 \end{pmatrix} \tag{2.11}$$

(page 349, (2.2) in [3]). Moreover, this formula already appeared in Chen's article [6] of 1951. In the nth homogeneous component D_n some properties hold as follows: for all $y_i \in X$ with $i \in \{1, 2, \ldots, n\}$,

(i) $[y_1, y_2, \ldots, y_n] + [y_2, y_1, \ldots, y_n] = 0$ (*anti-commutativity*)

(ii) $[y_1, y_2, y_3, \ldots, y_n] + [y_3, y_1, y_2, \ldots, y_n] + [y_2, y_3, y_1 \ldots, y_n] = 0$ (*Jacobi identity*)

and

(iii) $[y_1, \ldots, y_j, y_{j+1}, \ldots, y_n] - [y_1, \ldots, y_{j+1}, y_j, \ldots, y_n] = 0$ for $j \geq 3$.

It is easy to prove the case (iii). Using the Jacobi identity we obtain

$$[y_1, \ldots, y_j, y_{j+1}, \ldots, y_n] = [y_1, \ldots, y_{j+1}, y_j, \ldots, y_n] + [y_1, \ldots, [y_j, y_{j+1}], \ldots, y_n].$$

Since the element $[y_1, \ldots, [y_j, y_{j+1}], \ldots, y_n]$ belongs to L'', we have

$$[y_1, \ldots, y_j, y_{j+1}, \ldots, y_n] = [y_1, \ldots, y_{j+1}, y_j, \ldots, y_n].$$

This implies that $[y_1, \ldots, y_j, y_{j+1}, \ldots, y_n] - [y_1, \ldots, y_{j+1}, y_j, \ldots, y_n] = 0$.

The *free centre-by-metabelian Lie algebra* is the free Lie algebra with the property that the second derived ideal is contained in the centre. It is defined to be the quotient $L/[L'', L]$ and it is denoted by G. We let G_n denote the nth homogeneous component of the free centre-by-metabelian Lie algebra G.

In a Lie algebra (or a group), if each variable in a commutator occurs exactly once, we say this commutator is the polylinear commutator. Let c be a polylinear commutator. If $c = 1$, then this commutator is called a polylinear commutator identity in a group and similarly, if $c = 0$, we say that c is called a polylinear commutator identity in a Lie algebra.

Let $\mathcal{H}$ be the free group in the variety of all groups satisfying a set of polylinear commutator identities. The Lie ring $L(\mathcal{H})$ on the group $\mathcal{H}$ is the graded Lie ring whose homogeneous components are the quotients of the lower central series of $\mathcal{H}$,

$$L(\mathcal{H}) = \bigoplus_{n=1}^{\infty} \gamma_n(\mathcal{H})/\gamma_{n+1}(\mathcal{H}),$$

where the quotients of the lower central series are written additively and for $a = x\gamma_{s+1}(\mathcal{H}) \in \gamma_s(\mathcal{H})/\gamma_{s+1}(\mathcal{H})$ and $b = y\gamma_{t+1}(\mathcal{H}) \in \gamma_t(\mathcal{H})/\gamma_{t+1}(\mathcal{H})$ with $x \in \gamma_s(\mathcal{H})$ and $y \in \gamma_t(\mathcal{H})$, the Lie product is defined by

$$[a, b] = (x, y)\gamma_{s+t+1}(\mathcal{H}) \in \gamma_{s+t}(\mathcal{H})/\gamma_{s+t+1}(\mathcal{H}),$$

where $(x, y) = x^{-1}y^{-1}xy$ is the group commutator in $\mathcal{H}$ ([32], Chapter 4). Further information about the lower central series of a free group can be found in ([1], Section 8.2.4).

Let H be the free Lie ring in the variety of all Lie rings satisfying the corresponding set of polylinear commutator identities. We have a canonical homomorphism $H \to L(\mathcal{H})$ defined by $x_i \mapsto y_i\mathcal{H}'$, where $\mathcal{H}'$ is the commutator subgroup of $\mathcal{H}$ and $x_i \in H$, $y_i \in L(\mathcal{H})$ for $1 \leq i \leq r$. According to Theorem 1 in [21], if the additive group of H is torsion free, then $H \cong L(\mathcal{H})$. And moreover, we always have $L(\mathcal{H}) \otimes \mathbb{Q} \cong H \otimes \mathbb{Q}$, where $\mathbb{Q}$ is the field of rationals.

In general, H and $L(\mathcal{H})$ are not isomorphic.

Theorem 2.5 *(Yu.V. Kuz'min and Shapiro, Theorem 2 in [21]) Let $\mathcal{H}$ (respectively H) be a free centre-by-metabelian group (respectively free centre-by-metabelian Lie ring) of rank 3 defined by the commutator $[[x, y], [u, v], w]$. Then $L(\mathcal{H})$ and H are not* *isomorphic. More precisely, if* x_1, x_2, x_3 *(respectively* y_1, y_2, y_3*) are free generators of the ring* H *(respectively the group* $\mathcal{H}$*),* $\varphi : H \to L(\mathcal{H})$ *is a homomorphism that maps* x_i *into* $y_i\mathcal{H}'$ $(i = 1, 2, 3)$ *and*

$$e = [[x_2, x_1],[x_2, x_1, x_1, x_2, x_3, x_3]] - [[x_2, x_1], [x_2, x_1, x_1, x_2, x_2, x_3]] \\ - [[x_2, x_1], [x_2, x_1, x_1, x_1, x_2, x_3]],$$

then $e \neq 0$ *but* $(e)\varphi = 0$*.*

□

Chapter 3

Free centre-by-metabelian Lie rings

This chapter is our main chapter. In this chapter, first we will describe our problem then we give an approach to solve this problem.

3.1 Description of the problem

Let K be a commutative ring with identity element. Let $L = L(X)$ be the free Lie algebra of rank $r \geq 2$ over K on a set $X = \{x_1, x_2, \ldots, x_r\}$. We assume that the set X is ordered by $x_1 < x_2 < \ldots < x_r$. The free centre-by-metabelian Lie algebra $G = G(X)$ of rank r is defined as the quotient

$$G = L/[L'', L],$$

where L'' is the second derived ideal of L. The second derived ideal G'' of G is defined to be quotient

$$G'' = L''/[L'', L].$$

Hence, G is a central extension of the free metabelian Lie algebra G/G'' with the kernel G''.

In 1977 Kuz'min studied on the free centre-by-metabelian Lie ring, that is $K = \mathbb{Z}$

and he constructed a basis for the second derived ideal G'' of the free centre-by-metabelian Lie ring G. In his paper [20], Kuz'min defined the Lie monomials of the form

$$[y_1, y_2] \wedge_* [y_3, y_4](y_5 \dots y_n) = [[y_1, y_2], [y_3, y_4, y_5, \dots, y_n]] \tag{3.1}$$

for all $y_i \in X$ with $i \in \{1, 2, \dots, n\}$ such that

$$y_1 > y_2,\ y_3 > y_4,\ y_1 \geq y_3,\ y_4 \leq y_2 \leq y_5 \leq \dots \leq y_n.$$

Throughout this thesis we call such elements Kuz'min elements. Moreover, Kuz'min defined the Jacobian elements as the elements of the form

$$[[y_1, y_2], [y_3, y_4, y_5, \dots, y_n]] + [[y_1, y_3], [y_4, y_2, y_5, \dots, y_n]] + [[y_1, y_4], [y_2, y_3, y_5, \dots, y_n]]$$

for all $y_i \in X$ with $i \in \{1, 2, \dots, n\}$.

Kuz'min discovered that the second derived ideal of G contains elements of order 2 and the 2-torsion in even degrees is very different from the 2-torsion in odd degrees. The main result (Theorem 4) in Kuz'min's ground-breaking paper [20] shows that for $n \geq 5$, there is a direct sum decomposition

$$G''_n = F_n \oplus T_n,$$

where F_n is a free abelian group and T_n is a (possibly trivial) elementary abelian 2-group. Moreover, this result (Theorem 4 in [20]) for bases of G'' implies that for $r \geq 2$, the additive group of the even degree $n \geq 6$ homogeneous component has non-trivial 2-torsion. For odd degree, 2-torsion occurs for $r \geq 5$.

According to Kuz'min's theorem, if n is even, the elementary 2-group T_n is generated by the linearly independent elements of the form (3.1), where $y_1 = y_3$, $y_2 = y_4$ and the monomial $y_5 y_6 \dots y_n$ is not a square. Rainer Zerck in his paper [45] pointed out that the generating sets for the torsion subgroups T_n for even degree in Kuz'min's paper [20] are not sufficient to generate those groups. For instance, suppose that $n = 6$ and $r = 3$. According to Kuz'min, the elements of a basis for T_6 are

$$[[x_3, x_2], [x_3, x_2, x_2, x_3]],\ [[x_3, x_1], [x_3, x_1, x_1, x_2]],\ [[x_3, x_1], [x_3, x_1, x_1, x_3]],$$
$$[[x_3, x_1], [x_3, x_1, x_2, x_3]],\ [[x_2, x_1], [x_2, x_1, x_1, x_2]],\ [[x_2, x_1], [x_2, x_1, x_1, x_3]],$$
$$[[x_2, x_1], [x_2, x_1, x_2, x_3]].$$

According to Zerck, the elements of a basis are

$$[[x_3, x_2], [x_3, x_2, x_2, x_3]],\ [[x_3, x_2], [x_3, x_2, x_1, x_2]],\ [[x_3, x_2], [x_3, x_2, x_1, x_3]],$$
$$[[x_3, x_1], [x_3, x_1, x_1, x_2]],\ [[x_3, x_1], [x_3, x_1, x_1, x_3]],\ [[x_3, x_1], [x_3, x_1, x_2, x_3]],$$
$$[[x_2, x_1], [x_2, x_1, x_1, x_2]],\ [[x_2, x_1], [x_2, x_1, x_1, x_3]],\ [[x_2, x_1], [x_2, x_1, x_2, x_3]].$$

The second and third of Zerck's elements are not contained in Kuz'min's elements. On account of this, in Kuz'min's theorem, the sets claimed for even degree to be bases for free abelian groups are not linearly independent over $\mathbb{Z}$.

According to the above example, the element $[[x_3, x_2], [x_3, x_2, x_1, x_3]]$ is an element of the elementary 2-group T_6. By the Jacobi identity, we have

$$[[x_3, x_2], [x_3, x_2, x_1, x_3]] = [[x_3, x_2], [x_3, x_1, x_2, x_3]] - [[x_3, x_2], [x_2, x_1, x_3, x_3]].$$

Kuz'min's result claims that $[[x_3, x_2], [x_3, x_1, x_2, x_3]]$ and $[[x_3, x_2], [x_2, x_1, x_3, x_3]]$ are linearly independent elements in F_6, but since $[[x_3, x_2], [x_3, x_2, x_1, x_3]]$ is of order 2,

$$0 = 2[[x_3, x_2], [x_3, x_1, x_2, x_3]] - 2[[x_3, x_2], [x_2, x_1, x_3, x_3]].$$

Therefore, $[[x_3, x_2], [x_3, x_1, x_2, x_3]]$ and $[[x_3, x_2], [x_2, x_1, x_3, x_3]]$ are linearly dependent.

Unfortunately, Zerck did not give exact proofs in his preprint [45] which was never published.

In this thesis we focus on how to correct this problem. We refer in certain instances to some methods in 1977 Kuz'min's paper [20]. Our approach which is different from

Kuz'min's approach is to obtain generating sets for the fine homogeneous components of the second derived ideal G'' of the free centre-by-metabelian Lie ring G.

3.2 Approach to the problem

We recall that the derived ideal L' of the free Lie algebra $L(X)$ over K has a free generating set consisting of the elements of the form

$$[y_1, y_2, \ldots, y_n], \tag{3.2}$$

where $y_1 > y_2 \leq y_3 \leq \ldots \leq y_n$ with $y_i \in X$ for $1 \leq i \leq n$ and $n \geq 2$ (see, for example, [1], Section 2.4.2). It follows that the elements (3.2) with their cosets modulo L'' form a basis of the quotient $M = L'/L''$. The adjoint representation of L is effective in inducing on M the structure of an L/L'-module. Indeed, let a be any element in M and y be any free generator in L/L'. We define the (right) module action of L/L' on M by $a \bullet y = [a, y]$. As a result, we have the following general form

$$a \bullet (y_1 y_2 \ldots y_n) = [a, y_1, y_2, \ldots, y_n],$$

where $a \in M$ and for all $1 \leq i \leq n$ each y_i is a free generator in L/L'. Hence, M will be of a module for the universal envelope of the abelian Lie algebra L/L'. Since L is the free Lie algebra over K, actually, this universal envelope is the ring of polynomials on X with coefficients from K, namely $U = K[X]$. As we have seen in Chapter 2, the set of all monomials, $\mathcal{U} = \{y_1 y_2 \ldots y_k;\ y_i \in X,\ 1 \leq i \leq k,\ y_1 \leq y_2 \leq \ldots \leq y_k,\ k \geq 1\} \cup \{1\}$, is a basis for U and in the polynomial ring $U = K[X]$, a basis for each of fine homogeneous components has only one element. For example, suppose that $X = \{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$, for a composition $q = (2, 1, 2)$ of degree 5 we have only one monomial $x_1 x_1 x_2 x_3 x_3$. For a composition $q = (3, 0, 3)$ of degree 6, the only had monomial is $x_1 x_1 x_1 x_3 x_3 x_3$. Therefore, the fine homogeneous component with multidegree q of U is spanned by the unique monomial of multidegree q in U.

Moreover, we have the augmentation sequence

$$0 \longrightarrow \Delta \longrightarrow U \xrightarrow{\varepsilon} K \longrightarrow 0. \tag{3.3}$$

The ring K will always be regarded as a trivial U-module, and so the augmentation sequence is a sequence of U-modules.

The quotient M is generated as a U-module by the Lie products $[y_i, y_j]$ with $y_i, y_j \in X$ for $1 \leq i, j \leq r$. Using the module notation, the elements (3.2) in M can be written as

$$[y_1, y_2] \bullet (y_3 y_4 \dots y_n) \text{ with } y_1 > y_2 \leq y_3 \leq \dots \leq y_n. \tag{3.4}$$

Since the elements (3.2) with their cosets modulo L'' form a basis of the quotient $M = L'/L''$, these elements (3.4) form a K-basis of M.

Now, we consider the lower central quotients of the derived ideal L'. The adjoint representation of L induces on the lower central quotients of L' the structure of a U-module. Indeed, for any arbitrary $y \in X$ and positive integers n,

$$\mathrm{ad}_y : \gamma_n(L')/\gamma_{n+1}(L') \to \gamma_n(L')/\gamma_{n+1}(L')$$

is defined by

$$(a)\mathrm{ad}_y \mapsto [a, y] = a \bullet y$$

for all $a \in \gamma_n(L')/\gamma_{n+1}(L')$. This shows that the lower central quotients of L' have the structure of a U-module. By the definition of the lower central quotient, we have

$$\gamma_1(L') = L' \text{ and } \gamma_2(L') = [\gamma_1(L'), L'] = [L', L'] = L''.$$

Therefore, the first lower central quotient $\gamma_1(L')/\gamma_2(L')$ can be expressed as

$$\gamma_1(L')/\gamma_2(L') = L'/L''.$$

This quotient is exactly the U-module M, namely, $\gamma_1(L')/\gamma_2(L') = L'/L'' = M$.

In the second lower central quotient $\gamma_2(L')/\gamma_3(L')$, we know that $\gamma_2(L') = L''$, and the module action for L'' is given by

$$[m_1, m_2] \bullet y = [m_1, m_2, y]$$

for $m_1, m_2 \in L'$ and $y \in X$. By the Jacobi identity, we obtain $[m_1, m_2, y] = [[m_1, y], m_2] + [m_1, [m_2, y]]$. Therefore, the module action for the second lower central quotient $\gamma_2(L')/\gamma_3(L')$ is given by

$$\begin{aligned}([m_1, m_2] + \gamma_3(L')) \bullet y =& [m_1, m_2] \bullet y + \gamma_3(L') \\ =& [m_1, m_2, y] + \gamma_3(L') \\ =& [[m_1, y], m_2] + [m_1, [m_2, y]] + \gamma_3(L') \qquad (3.5)\end{aligned}$$

for $m_1, m_2 \in L'$ and $y \in X$.

The second lower central quotient $\gamma_2(L')/\gamma_3(L')$ is, as an abelian group, isomorphic to the exterior square of $M = L'/L'' = \gamma_1(L')/\gamma_2(L')$, namely

$$\gamma_2(L')/\gamma_3(L') \cong M \wedge M, \qquad (3.6)$$

via the map

$$([m_1, m_2] + \gamma_3(L')) \mapsto (m_1 + \gamma_2(L')) \wedge (m_2 + \gamma_2(L'))$$

for $m_1, m_2 \in L'$. In view of (3.5) this is, in fact, an isomorphism of U-modules where $M \wedge M$ is regarded as a U-module with derivation action. By tensoring (3.6) with

K, we get

$$(\gamma_2(L')/\gamma_3(L')) \otimes_U K \cong (M \wedge M) \otimes_U K.$$

The trivialization of $\gamma_2(L')/\gamma_3(L')$ is

$$(\gamma_2(L')/\gamma_3(L')) \otimes_U K = (\gamma_2(L')/\gamma_3(L'))/(\gamma_2(L')/\gamma_3(L'))\Delta.$$

Therefore, since $(\gamma_2(L')/\gamma_3(L'))\Delta = [\gamma_2(L'), L]/\gamma_3(L')$, we have

$$\begin{aligned}(\gamma_2(L')/\gamma_3(L')) \otimes_U K =& (\gamma_2(L')/\gamma_3(L'))/(\gamma_2(L')/\gamma_3(L'))\Delta \\ =& (\gamma_2(L')/\gamma_3(L'))/([\gamma_2(L'), L]/\gamma_3(L')) \\ =& \gamma_2(L')/[\gamma_2(L'), L] \\ =& L''/[L'', L] \\ =& G''.\end{aligned}$$

Therefore, trivializing of the U-action on both sides of the isomorphism (3.6) yields an isomorphism

$$G'' \cong (M \wedge M) \otimes_U K. \tag{3.7}$$

We focus on the case where $K = \mathbb{Z}$, $G'' \cong (M \wedge M) \otimes_U \mathbb{Z}$, that was proved by Yu. V. Kuz'min ([20], Lemma 2). This isomorphism will be one of the main tools in our approach for finding the generating sets of G''.

The free centre-by-metabelian Lie algebra G has a natural grading by degree. We let G_n denote the degree n homogeneous component of G, and write G''_n for the degree n homogeneous component of the second derived ideal:

$$G''_n = G'' \cap G_n. \tag{3.8}$$

We are also interested in the fine homogeneous components of G. These are the K-submodules of G spanned by all Lie monomials of the same multidegree in the free generators. As defined in Section 2.2.1, for a fixed composition $q = (q_1, q_2, \ldots, q_r)$ of n, let G_q be the K-submodule of G generated by all left-normed Lie products of partial degree q_j with respect to x_j for $1 \leq j \leq r$. Then

$$G_n = \bigoplus_{q \models n} G_q.$$

We write G''_q for $G'' \cap G_q$.

The smallest non-zero homogeneous component of G'' is G''_4 and when $K = \mathbb{Z}$, it is easy to see that G''_4 is isomorphic to a free abelian group $G_2 \wedge G_2$. By (3.8), we have $G''_4 = G'' \cap G_4$. It is known that the free centre-by-metabelian Lie algebra G is a graded algebra, namely, $G = G_1 \oplus G_2 \oplus G_3 \oplus \ldots$. By (2.9), the derived algebra of G can be written as $G' = G_2 \oplus G_3 \oplus \ldots$. By using (2.10), we can express the second derived ideal of G as

$$G'' = [G_2, G_2] \oplus [G_3, G_2] \oplus ([G_4, G_2] + [G_3, G_3]) \oplus \ldots$$

Thus, for degree 4,

$$\begin{aligned} G''_4 &= G'' \cap G_4 \\ &\cong [G_2, G_2] \\ &= G_2 \wedge G_2. \end{aligned}$$

Hence, for the rest of this thesis we focus on the degree n homogeneous components and fine homogeneous components of G'' for $n \geq 5$. The homogeneous components and fine homogeneous components of the polynomial ring $U = K[X]$ are defined in a natural way. The polynomial ring $U = K[X]$ is a graded ring, namely $U = \bigoplus_{j=0}^{\infty} U_j$, where U_j denote the degree jth homogeneous component of U, for $j = 0$, $U_0 = K$; for $j \neq 0$, U_j is generated by the elements $y_1 y_2 \ldots y_j$, where $y_1, y_2, \ldots, y_j \in X$. For example, suppose that $X = \{x_1\}$, $K[x_1]$ can be given by

$$\underbrace{K}_{\deg 0} \oplus \underbrace{Kx_1}_{\deg 1} \oplus \underbrace{Kx_1^2}_{\deg 2} \oplus \dots .$$

For $X = \{x_1, x_2\}$, $U = K[x_1, x_2]$ can be given by

$$\underbrace{K}_{\deg 0} \oplus \underbrace{(Kx_1 + Kx_2)}_{\deg 1} \oplus \underbrace{(Kx_1^2 + Kx_1x_2 + Kx_2^2)}_{\deg 2} \oplus \dots .$$

Since each homogeneous component can be written as a direct sum of the fine homogeneous components, namely $U_j = \bigoplus_{q \models j} U_q$, where $q = (q_1, q_2)$ is a composition of j, we have

$$\underbrace{K}_{(0,0)} \oplus \underbrace{Kx_1}_{(1,0)} \oplus \underbrace{Kx_2}_{(0,1)} \oplus \underbrace{Kx_1^2}_{(2,0)} \oplus \underbrace{Kx_1x_2}_{(1,1)} \oplus \underbrace{Kx_2^2}_{(0,2)} \oplus \dots .$$

Let P denote the free U-module of rank r with free generators $e_1, e_2, \dots, e_r$. The map $[x_i, x_j] \mapsto e_ix_j - e_jx_i$ extends to an embedding $\mu : M \to P$. If $\sigma : P \to \Delta$ is the map determined by $e_i \mapsto x_i$, then the following sequence

$$0 \longrightarrow M \xrightarrow{\mu} P \xrightarrow{\sigma} \Delta \longrightarrow 0 \tag{3.9}$$

is an exact sequence of U-modules. A proof of this can be found in [36].

For all $i \in \{1, 2, \dots, r\}$ each free generator e_i of P is assigned the same partial degree as the matching $x_i \in X$ and since U is a graded U-module, we can consider P as a graded U-module. For instance, the element $e_2x_1x_2x_4 \in P$ has degree 4 and multidegree $(1, 2, 0, 1, 0, \dots, 0)$ and the element $e_1x_1 \otimes e_2 \in P \otimes P$ has degree 3 and multidegree $(2, 1, 0, \dots, 0)$. The U-module M is graded by degree, and so is the exterior square $M \wedge M$. Moreover, the tensor product $(M \wedge M) \otimes_U K$ is graded by degree. We write $((M \wedge M) \otimes_U K)_n$ for nth degree homomorphism components, and $((M \wedge M) \otimes_U K)_q$ for the fine homogeneous components. Hence, the isomorphism $G'' \cong (M \wedge M) \otimes_U K$ is an isomorphism of graded U-modules.

Now we record some easy facts about the fine homogeneous structure of G. For any map $f : X \to X$ of the free generating set X to itself, we let $\pi_f : G \to G$ denote the unique endomorphism of G that extends f. It is clear that the image of a fine homogeneous component G_q with $q = (q_1, q_2, \dots, q_r) \models n$ under an endomorphism of the form π_f is itself a fine homogeneous component. In fact,

$$G_q \pi_f = G_{q'}, \text{ where } q' = (q'_1, q'_2, \dots, q'_r) \text{ with } q'_i = \sum_{j:(j)f=i} q_j.$$

It is plain that $G_q \pi_f \cong G_q$ if f is a bijection. Consequently, each fine homogeneous component is isomorphic to a fine homogeneous component of the form G_q, where $q = (q_1, q_2, \dots, q_r)$ is a partition of n, that is, a composition with the additional condition that $q_1 \geq q_2 \geq \dots \geq q_r$. Finally, note that every fine homogeneous component G_q, $q \models n \leq r$ is a homomorphic image of the multilinear fine homogeneous component $q = (\underbrace{1, \dots, 1}_{n}, 0, \dots, 0)$ under some endomorphism of the form π_f. Moreover, it is easily seen that f can be chosen in such a way that it preserves the order of the free generators, i.e. if $x_i \leq x_j$, then $x_i f \leq x_j f$.

Similarly, by replacing G_q by G''_q we have the same facts recorded in the last paragraph. Namely, the image of a fine homogeneous component G''_q under an endomorphism of the form π_f is itself a fine homogeneous component. Hence,

$$G''_q \pi_f = G''_{q'}, \text{ where } q' = (q'_1, q'_2, \dots, q'_r) \text{ with } q'_i = \sum_{j:(j)f=i} q_j.$$

When f is a bijection, it is clear to see that $G''_q \pi_f \cong G''_q$. As a result, each fine homogeneous component is isomorphic to a fine homogeneous component of the form G''_q, where $q = (q_1, q_2, \dots, q_r)$ is a partition of n, that is, a composition with the additional condition that $q_1 \geq q_2 \geq \dots \geq q_r$. In conclusion, every fine homogeneous component G''_q, $q \models n \leq r$ is a homomorphic image of the multilinear fine homogeneous component $q = (\underbrace{1, \dots, 1}_{n}, 0, \dots, 0)$ under some endomorphism of the form π_f.

3.3 Generating sets

In this section, our aim is to find generating sets for the second derived ideal of the free centre-by-metabelian Lie algebra G over $\mathbb{Z}$. By the isomorphism $G'' \cong (M \wedge M) \otimes_U \mathbb{Z}$, we derive generating sets for the fine homogeneous components of $(M \wedge M) \otimes_U \mathbb{Z}$. Firstly, we know that M is generated by the elements (3.2) of the form $[y_1, y_2, \ldots, y_n]$, where $y_1 > y_2 \leq y_3 \leq y_4 \leq \ldots \leq y_n$ for all $y_i \in X$ with $1 \leq i \leq n$ and $n \geq 2$. Since M is a U-module, these elements can be written as

$$[y_1, y_2] \bullet (y_3 y_4 \ldots y_n), \text{ where } y_3 y_4 \ldots y_n \in \mathcal{U} \text{ for all } y_i \in X, \; n \geq 2.$$

Namely, using the module notation we have

$$[y_1, y_2, y_3, y_4, \ldots, y_n] = [y_1, y_2] \bullet (y_3 y_4 \ldots y_n).$$

In the light of this, it is obvious that the exterior square $M \wedge M$ is generated by all elements

$$([y_1, y_2] \bullet u_1) \wedge ([y_3, y_4] \bullet u_2), \tag{3.10}$$

where $y_i \in X$ for all $i \in \{1, 2, \ldots, n\}$ and $u_1, u_2 \in \mathcal{U}$.

For the exterior square of M, the trivialization of $M \wedge M$ is

$$(M \wedge M) \otimes_U \mathbb{Z} = (M \wedge M)/(M \wedge M)\Delta,$$

where $(M \wedge M)\Delta$ is the $\mathbb{Z}$-submodule generated by the elements of the form

$$(m_1 \wedge m_2) \bullet y = m_1 \bullet y \wedge m_2 + m_1 \wedge m_2 \bullet y$$

for $m_1, m_2 \in M$ and $y \in X$. The canonical homomorphism from $(M \wedge M)$ to

$(M \wedge M) \otimes_U \mathbb{Z}$ is defined by

$$m_1 \wedge m_2 \mapsto (m_1 \wedge m_2) \otimes 1$$

for all $m_1, m_2 \in M$.

Throughout this work, in order to save space and to simplify notation, we use the convention of writing $m_1 \wedge_* m_2$ for the image of the element $m_1 \wedge m_2$ of $M \wedge M$ in the trivialization of the exterior square of M instead of $(m_1 \wedge m_2) \otimes 1$. Also M is a U-module, for module action we use simply mu instead of $m \bullet u$ for $m \in M$ and $u \in U$.

Now we give some relations that are satisfied in $(M \wedge M) \otimes_U \mathbb{Z}$. First of all, there are the relations coming from anti-commutativity and the Jacobi identity in M:

$$[y_i, y_i] = 0, \; [y_i, y_j] = -[y_j, y_i] \tag{3.11}$$

and

$$\begin{aligned} [y_i, y_j]y_k &= [y_i, y_j, y_k] \\ &= -[y_j, y_k, y_i] + [y_i, y_k, y_j] \\ &= -[y_j, y_k]y_i + [y_i, y_k]y_j \end{aligned} \tag{3.12}$$

for all $y_i, y_j, y_k \in X$. Then there is anti-commutativity coming from the exterior square $M \wedge M$:

$$m \wedge_* m = 0, \; m_1 \wedge_* m_2 = -(m_2 \wedge_* m_1) \tag{3.13}$$

for all $m, m_1, m_2 \in M$. Finally, there are relations coming from the trivialization of the U-action:

$$m_1 \wedge_* m_2 y = -(m_1 y \wedge_* m_2) \tag{3.14}$$

for all $m_1, m_2 \in M$ and $y \in X$. Indeed, since

$$(m_1 \wedge m_2)y = m_1 y \wedge m_2 + m_1 \wedge m_2 y \in (M \wedge M)\Delta,$$

it follows that

$$m_1 y \wedge_* m_2 + m_1 \wedge_* m_2 y = 0$$

in $(M \wedge M) \otimes_U \mathbb{Z}$. Hence, we get

$$m_1 \wedge_* m_2 y = -(m_1 y \wedge_* m_2).$$

Now we generalize (3.14) for the monomials of degree n. Let $u = y_1 y_2 \dots y_n \in \mathcal{U}$, where $y_1, y_2, \dots, y_n \in X$. For $m_1, m_2 \in M$, we apply the relation (3.14) n times to $m_1 \wedge_* m_2 u$. First of all, by applying for y_1 we get

$$m_1 \wedge_* m_2 y_1 y_2 \dots y_n = -\, m_1 y_1 \wedge_* m_2 y_2 \dots y_n,$$

then applying for y_2 we have

$$-m_1 y_1 \wedge_* m_2 y_2 \dots y_n = m_1 y_1 y_2 \wedge_* m_2 y_3 \dots y_n.$$

By continuing this process up to y_n we obtain

$$(-1)^{n-1} m_1 y_1 y_2 \dots y_{n-1} \wedge_* m_2 y_n = (-1)^n m_1 y_1 y_2 \dots y_n \wedge_* m_2,$$

and at the last application, we have

$$m_1 \wedge_* m_2 u = (-1)^n m_1 u \wedge_* m_2 = (-1)^{\deg u} m_1 u \wedge_* m_2.$$

Then by (3.13) we obtain the following result

$$\begin{aligned} m_1 \wedge_* m_2 u =&(-1)^{\deg u} m_1 u \wedge_* m_2 \\ =& -(-1)^{\deg u}(m_2 \wedge_* m_1 u) \\ =&(-1)^{\deg u+1}(m_2 \wedge_* m_1 u). \end{aligned} \tag{3.15}$$

It is clear to see that the relation (3.14) is a special case where $u = y \in X$ in (3.15). This relation (3.15) will be useful in the rest of this chapter.

Now we return to the canonical homomorphism $M \wedge M \to (M \wedge M) \otimes_U \mathbb{Z}$. Under this homomorphism, the images of the elements (3.10) generate this tensor product as a $\mathbb{Z}$-module. In view of (3.14),

$$[y_1, y_2]u_1 \wedge_* [y_3, y_4]u_2 = (-1)^{\deg u_1}[y_1, y_2] \wedge_* [y_3, y_4]u_1 u_2.$$

In other words, each of those images is (up to sign) equal to an element of the form

$$[y_1, y_2] \wedge_* [y_3, y_4](y_5 y_6 \dots y_n) \tag{3.16}$$

for all $y_i \in X$ with $1 \le i \le n$. Hence, the elements (3.16) form a generating set for $(M \wedge M) \otimes_U \mathbb{Z}$. Now our task is to further reduce this generating set.

The following result will be important in the proofs throughout this section.

Lemma 3.1 Let $n \geq 4$ and $a, b \in X$. Any element (3.16) with $y_i = a$, $y_j = b$ for some i, j with $1 \le i, j \le n$ and $i \ne j$, belongs to the span of the elements

$$[z_1, z_2] \wedge_* [b, a](z_3 z_4 \dots z_{n-2}) \text{ and } [z_1, b] \wedge_* [z_2, a](z_3 z_4 \dots z_{n-2}) \tag{3.17}$$

of the same multidegree with $z_1, z_2, \dots, z_{n-2} \in X$.

Proof. We first show that under our assumptions the element (3.16)

$$[y_1, y_2] \wedge_* [y_3, y_4](y_5 y_6 \dots y_n)$$

for all $y_i \in X$ is in the span of the elements

$$[z_1, z_2] \wedge_* [z_3, a](z_4 z_5 \dots z_{n-1}), \tag{3.18}$$

where $z_1, z_2, \dots, z_{n-1} \in X$.

We focus on the position of a in the element (3.16). There are two cases:

Case(i). If a is one of y_1, y_2, y_3, y_4, it is clear that the element (3.16) is (up to sign) certainly equal to one of the elements (3.18).

Firstly, we assume that $y_1 = a$, namely, the element (3.16) is the form

$$[a, y_2] \wedge_* [y_3, y_4](y_5 y_6 \dots y_n).$$

By using anti-commutativity, we get

$$-[y_2, a] \wedge_* [y_3, y_4](y_5 y_6 \dots y_n).$$

By (3.13), we have

$$[y_3, y_4](y_5 y_6 \dots y_n) \wedge_* [y_2, a],$$

and then by (3.15), we have

$$(-1)^{n-4}[y_3, y_4] \wedge_* [y_2, a](y_5 y_6 \dots y_n).$$

This implies that in the case where $y_1 = a$, the element (3.16) is of the form of the elements (3.18). The proof for cases where $y_2 = a$, $y_3 = a$ and $y_4 = a$ is similar.

If $y_2 = a$, the element (3.16) is of the form

$$[y_1, a] \wedge_* [y_3, y_4](y_5 y_6 \dots y_n).$$

By (3.13), we have

$$-[y_3, y_4](y_5y_6 \dots y_n) \wedge_* [y_1, a],$$

then applying (3.15) we obtain

$$-(-1)^{n-4}[y_3, y_4] \wedge_* [y_1, a](y_5y_6 \dots y_n) = (-1)^{n-3}[y_3, y_4] \wedge_* [y_1, a](y_5y_6 \dots y_n).$$

This is of the required form.

If $y_3 = a$, the element (3.16) is of the form

$$[y_1, y_2] \wedge_* [a, y_4](y_5y_6 \dots y_n).$$

By anti-commutativity we get

$$-[y_1, y_2] \wedge_* [y_4, a](y_5y_6 \dots y_n),$$

as required.

If $y_4 = a$ in (3.16), then we have the element $[y_1, y_2] \wedge_* [y_3, a](y_5y_6 \dots y_n)$. This element is of required form.

Case(ii). If a is one of $y_5, y_6, \dots, y_n$, we may assume that $y_5 = a$, and then by using (3.12) we get

$$\begin{aligned}
&[y_1, y_2] \wedge_* [y_3, y_4](ay_6 \dots y_n) \\
&\quad =[y_1, y_2] \wedge_* [y_3, y_4, a](y_6 \dots y_n) \\
&\quad =[y_1, y_2] \wedge_* [y_3, a, y_4](y_6 \dots y_n) - [y_1, y_2] \wedge_* [y_4, a, y_3](y_6 \dots y_n) \\
&\quad =[y_1, y_2] \wedge_* [y_3, a](y_4y_6 \dots y_n) - [y_1, y_2] \wedge_* [y_4, a](y_3y_6 \dots y_n).
\end{aligned}$$

The two elements at the bottom of the right hand side are as required.

It therefore remains to show that any element of the form (3.18) such that b is equal to one of the elements $z_1, z_2, \dots, z_{n-1}$ is in span of the elements (3.17). This is clear if b is one of z_1, z_2, z_3. If $b = z_1$, the element (3.18) is of the form

$$[b, z_2] \wedge_* [z_3, a]z_4z_5 \ldots z_{n-1}.$$

By anti-commutativity, we obtain the required form

$$-[z_2, b] \wedge_* [z_3, a]z_4z_5 \ldots z_{n-1}.$$

Similarly, if $b = z_2$, the element (3.18) is of the form

$$[z_1, b] \wedge_* [z_3, a]z_4z_5 \ldots z_{n-1}.$$

If $b = z_3$, the element (3.18) is of the form

$$[z_1, z_2] \wedge_* [b, a]z_4z_5 \ldots z_{n-1}.$$

These elements are in the required form.

Now, we suppose that b is one of $z_4, z_5, \ldots, z_{n-1}$, and we may assume that $z_4 = b$. Then by using the relations (3.14) and (3.12) respectively, we find

$$\begin{aligned}
&[z_1, z_2] \wedge_* [z_3, a](bz_5 \ldots z_{n-1}) \\
=& -[z_1, z_2]b \wedge_* [z_3, a](z_5 \ldots z_{n-1}) \\
=&(-[z_1, b]z_2 + [z_2, b]z_1) \wedge_* [z_3, a](z_5 \ldots z_{n-1}) \\
=& -[z_1, b]z_2 \wedge_* [z_3, a](z_5 \ldots z_{n-1}) + [z_2, b]z_1 \wedge_* [z_3, a](z_5 \ldots z_{n-1}) \\
=&[z_1, b] \wedge_* [z_3, a](z_2z_5 \ldots z_{n-1}) - [z_2, b] \wedge_* [z_3, a](z_1z_5 \ldots z_{n-1}).
\end{aligned}$$

The two elements at the bottom are of the form of the elements in (3.18). Therefore, this completes the proof of the lemma

□

Armed with Lemma 3.1, it is very easy to obtain efficient generating sets for the fine homogeneous components $((M \wedge M) \otimes_U \mathbb{Z})_q$ for partitions $q \models n$ with at least one part greater than 1. Moreover, we will see later that these generating sets are actually minimal if n is odd, and also they can easily be reduced to minimal generating sets if n is even.

3.3.1 Generating sets for non-multilinear fine homogeneous components

Let $n \geq 5$ and assume that $q = (q_1, q_2, \ldots, q_r)$ is a composition of n in r parts such that some parts are greater than 1. Here, our task is to find a generating set for the fine homogeneous component $((M \wedge M) \otimes_U \mathbb{Z})_q$.

Lemma 3.2 Let $n \geq 5$ and let $q = (q_1, q_2, \ldots, q_r)$ be a composition of n in r parts such that $q_i \geq 2$ for some i. Then the elements

$$[z_1, x_i] \wedge_* [z_2, x_i](z_3 z_4 \ldots z_{n-2}),$$

where $z_1, z_2, \ldots, z_{n-2} \in X$, of multidegree q with $z_1 \geq z_2$ and $z_1, z_2 \neq x_i$ form a generating set for the homogeneous component $((M \wedge M) \otimes_U \mathbb{Z})_q$.

Proof. We assume that $a = b = x_i$ in Lemma 3.1. Hence, each element (3.16) of multidegree q in $(M \wedge M) \otimes_U \mathbb{Z}$ with $y_k = x_i$, $y_j = x_i$ for some k, j with $1 \leq k, j \leq n$, $k \neq j$ is a linear combination of elements of the form $[z_1, x_i] \wedge_* [z_2, x_i](z_3 \ldots z_{n-2})$.

If $z_1 \geq z_2$, this element is of the required form. Otherwise, if $z_2 > z_1$, by the relations (3.13) and (3.15) we have

$$\begin{aligned}
[z_1, x_i] \wedge_* [z_2, x_i](z_3 \ldots z_{n-2}) &= -[z_2, x_i](z_3 \ldots z_{n-2}) \wedge_* [z_1, x_i] \\
&= -(-1)^{n-4}[z_2, x_i] \wedge_* [z_1, x_i](z_3 \ldots z_{n-2}) \\
&= (-1)^{n-3}[z_2, x_i] \wedge_* [z_1, x_i](z_3 \ldots z_{n-2}).
\end{aligned}$$

This result implies that the elements of the form $[z_1, x_i] \wedge_* [z_2, x_i](z_3 \ldots z_{n-2})$ with $z_2 > z_1$ are (up to sign) actually equal to the elements of the required form. This completes the proof of the lemma.

□

3.3.2 Generating sets for multilinear fine homogeneous components

When investigating the generating sets for multilinear fine homogeneous components, we need to define the Kuz'min elements. Recall that if the elements of the form (3.16)

$$[y_1, y_2] \wedge_* [y_3, y_4](y_5 y_6 \dots y_n)$$

for all $y_i \in X$ with $1 \leq i \leq n$ satisfy

$$y_1 > y_2,\ y_3 > y_4,\ y_1 \geq y_3,\ y_2 \geq y_4 \text{ and } y_2 \leq y_5 \leq y_6 \leq \dots \leq y_n, \tag{3.19}$$

we call such elements *Kuz'min elements*.

In this section we deal with multilinear homogeneous components. We show that the multilinear Kuz'min elements of degree $n \geq 5$ together with one additional element form a generating set for the degree n multilinear fine homogeneous components of $(M \wedge M) \otimes_U \mathbb{Z}$.

Lemma 3.3 Suppose $|X| = n \geq 5$. Then every element (3.16) of multidegree $q = (1, 1, \dots, 1) \models n$ is a linear combination of Kuz'min elements of multidegree $(1, 1, \dots, 1)$ and the element $h = [x_3, x_2] \wedge_* [x_4, x_1](x_5 \dots x_n)$.

Proof. Let g be an arbitrary element of the form (3.16) of multidegree $(1, 1, \dots, 1)$, namely, each of the elements $x_1, x_2, \dots, x_n$ appears exactly once in the element g. By Lemma 3.1 with $a = x_1$ and $b = x_2$, we may assume that g is either

$$[z_1, z_2] \wedge_* [x_2, x_1](z_3 \dots z_{n-2}) \tag{3.20}$$

or

$$[z_1, x_2] \wedge_* [z_2, x_1](z_3 \dots z_{n-2}), \tag{3.21}$$

where $z_1, z_2, \ldots, z_{n-2} \in X$.

In the first case (3.20), the element is a Kuz'min element if $z_2 = x_3$. Also, if $z_1 = x_3$, we swap z_1 and z_2 at the expense of a sign change to obtain a Kuz'min element. If neither z_1 nor z_2 are equal to x_3, then x_3 must be one of $z_3, \ldots, z_{n-2}$, and we may assume that $z_3 = x_3$. Let $v = z_4 \ldots z_{n-2}$. Then by using the relations (3.14) and (3.12) respectively, we have

$$\begin{aligned}
[z_1, z_2] \wedge_* [x_2, x_1]x_3v &= -[z_1, z_2]x_3 \wedge_* [x_2, x_1]v \\
&= (-[z_1, x_3]z_2 + [z_2, x_3]z_1) \wedge_* [x_2, x_1]v \\
&= [-z_1, x_3]z_2 \wedge_* [x_2, x_1]v + [z_2, x_3]z_1 \wedge_* [x_2, x_1]v \\
&= [z_1, x_3] \wedge_* [x_2, x_1]z_2v - [z_2, x_3] \wedge_* [x_2, x_1]z_1v. \qquad (3.22)
\end{aligned}$$

The two elements in the bottom line of the right hand side of (3.22) are Kuz'min, as required.

Now we examine the second case (3.21). There are three possibilities for x_3's choice:

Case(i). If $z_2 = x_3$, we have $[z_1, x_2] \wedge_* [x_3, x_1](z_3 \ldots z_{n-2})$. Such elements are Kuz'min.

Case(ii). If $z_1 = x_3$, such elements are of the form $[x_3, x_2] \wedge_* [z_2, x_1]u$. Now we consider the place of x_4.

When $z_2 = x_4$, we get the element h. If this is not the case, x_4 must be one of $z_3, \ldots, z_{n-2}$, and we may choose that $z_3 = x_4$. Then the element in question is of the form

$$[x_3, x_2] \wedge_* [z_2, x_1]x_4v,$$

where $v = z_4 \ldots z_{n-2}$. Using the relations (3.12) and (3.11), we get

$$[x_3, x_2] \wedge_* [z_2, x_1]x_4v = [x_3, x_2] \wedge_* [x_4, x_1]z_2v + [x_3, x_2] \wedge_* [z_2, x_4]x_1v.$$

The first element on the right hand side is the element h. It remains to demonstrate that the second element on the right hand side is of the required form. By using the relations (3.11)-(3.14) we get

$$\begin{aligned}
&[x_3, x_2] \wedge_* [z_2, x_4]x_1 v \\
=&-[x_3, x_2]x_1 \wedge_* [z_2, x_4]v \\
=&(-[x_3, x_1]x_2 + [x_2, x_1]x_3) \wedge_* [z_2, x_4]v \\
=&[x_2, x_1]x_3 \wedge_* [z_2, x_4]v - [x_3, x_1]x_2 \wedge_* [z_2, x_4]v \\
=&-[z_2, x_4]v \wedge_* [x_2, x_1]x_3 + [z_2, x_4]v \wedge_* [x_3, x_1]x_2 \\
=&(-1)^{\deg v+1}[z_2, x_4] \wedge_* [x_2, x_1]x_3 v + (-1)^{\deg v}[z_2, x_4] \wedge_* [x_3, x_1]x_2 v.
\end{aligned}$$

We have already seen that the first of the two elements at the bottom can be written as a linear combination of Kuz'min elements (see (3.22)). For the second of these elements we have, again by using the relations (3.11), (3.12) and (3.14),

$$\begin{aligned}
&(-1)^{\deg v}[z_2, x_4] \wedge_* [x_3, x_1]x_2 v \\
=&(-1)^{\deg v+1}[z_2, x_4]x_2 \wedge_* [x_3, x_1]v \\
=&(-1)^{\deg v+1}([z_2, x_2]x_4 - [x_4, x_2]z_2) \wedge_* [x_3, x_1]v \\
=&(-1)^{\deg v+1}[z_2, x_2]x_4 \wedge_* [x_3, x_1]v + (-1)^{\deg v}[x_4, x_2]z_2 \wedge_* [x_3, x_1]v \\
=&(-1)^{\deg v}[z_2, x_2] \wedge_* [x_3, x_1]x_4 v + (-1)^{\deg v+1}[x_4, x_2] \wedge_* [x_3, x_1]z_2 v.
\end{aligned}$$

The two elements at the bottom are Kuz'min elements.

Case(iii). If x_3 is one of $z_3, \ldots, z_{n-2}$, the elements are of the form $[z_1, x_2] \wedge_* [z_2, x_1]x_3 v$. When $z_1 > z_2$, it is clear to observe that such elements are Kuz'min. Otherwise, if $z_1 < z_2$, by (3.12)

$$[z_1, x_2] \wedge_* [z_2, x_1]x_3 v = [z_1, x_2] \wedge_* [z_2, x_3]x_1 v + [z_1, x_2] \wedge_* [x_3, x_1]z_2 v.$$

The second element on the right hand side is Kuz'min. It now remains to show that the first element on the right hand side is of the required form. By using the relations

(3.13), (3.15) and (3.12) respectively, we get

$$\begin{aligned}
&[z_1, x_2] \wedge_* [z_2, x_3]x_1v \\
=& -[z_2, x_3]x_1v \wedge_* [z_1, x_2] \\
=&(-1)^{\deg v}[z_2, x_3] \wedge_* [z_1, x_2]x_1v \\
=&(-1)^{\deg v}[z_2, x_3] \wedge_* [z_1, x_1]x_2v + (-1)^{\deg v+1}[z_2, x_3] \wedge_* [x_2, x_1]z_1v.
\end{aligned}$$

The second element of two elements at the bottom is Kuz'min. For the first element, by using the relations (3.14) and (3.12) respectively, we obtain

$$\begin{aligned}
&(-1)^{\deg v}[z_2, x_3] \wedge_* [z_1, x_1]x_2v \\
=&(-1)^{\deg v+1}[z_2, x_3]x_2 \wedge_* [z_1, x_1]v \\
=&(-1)^{\deg v+1}([z_2, x_2]x_3 - [x_3, x_2]z_2) \wedge_* [z_1, x_1]v \\
=&(-1)^{\deg v+1}[z_2, x_2]x_3 \wedge_* [z_1, x_1]v + (-1)^{\deg v+1}[x_3, x_2]z_2 \wedge_* [z_1, x_1]v \\
=&(-1)^{\deg v}[z_2, x_2] \wedge_* [z_1, x_1]x_3v + (-1)^{\deg v}[x_3, x_2] \wedge_* [z_1, x_1]z_2v.
\end{aligned}$$

The first element on the right hand side is Kuz'min, and by choosing $z_1 = x_4$ we observe that the second element is the element h. If x_4 is one of the elements $z_2, z_4, z_5, \ldots, z_{n-2}$, by (3.12), we get

$$\begin{aligned}
&(-1)^{\deg v}[x_3, x_2] \wedge_* [z_1, x_1]x_4v \\
=&(-1)^{\deg v}[x_3, x_2] \wedge_* [z_1, x_4]x_1v + (-1)^{\deg v}[x_3, x_2] \wedge_* [x_4, x_1]z_1v.
\end{aligned}$$

The second element at the bottom is the element h and also we have already seen in Case(ii) that the first element at the bottom can be expressed as a linear combination of Kuz'min elements as the following :

By using the relations (3.12), (3.13) and (3.15) respectively, we get

$$
\begin{aligned}
&(-1)^{\deg v}[x_3, x_2] \wedge_* [z_1, x_4]x_1v \\
=&-(-1)^{\deg v}[z_1, x_4]x_1v \wedge_* [x_3, x_2] \\
=&[z_1, x_4] \wedge_* [x_3, x_2]x_1v \\
=&[z_1, x_4] \wedge_* [x_3, x_1]x_2v - [z_1, x_4] \wedge_* [x_2, x_1]x_3v \\
=&[z_1, x_4]x_2 \wedge_* [x_3, x_1]v - [z_1, x_4]x_3 \wedge_* [x_2, x_1]v \\
=&[z_1, x_2]x_4 \wedge_* [x_3, x_1]v - [x_4, x_2]z_1 \wedge_* [x_3, x_1]v \\
&- [z_1, x_3]x_4 \wedge_* [x_2, x_1]v + [x_4, x_3]z_1 \wedge_* [x_2, x_1]v \\
=&[z_1, x_2] \wedge_* [x_3, x_1]x_4v - [x_4, x_2] \wedge_* [x_3, x_1]z_1v \\
&- [z_1, x_3] \wedge_* [x_2, x_1]x_4v + [x_4, x_3] \wedge_* [x_2, x_1]z_1v.
\end{aligned}
$$

Having successfully dealt with all the cases, this completes the proof of the lemma.

□

3.4 t-elements

In this section, apart from the Kuz'min elements we introduce a second kind of elements that will play a crucial role in studying the homogeneous components of $(M \wedge M) \otimes_U \mathbb{Z}$ in odd degree $n \geq 5$. We call elements of the form

$$w(y_1, y_2, y_3, y_4; u) = [y_1, y_2] \wedge_* [y_3, y_4]u + [y_2, y_3] \wedge_* [y_1, y_4]u + [y_3, y_1] \wedge_* [y_2, y_4]u,$$

where $y_1, y_2, y_3, y_4 \in X$ and $u \in \mathcal{U}$, *t-elements.*

The concept of these elements was first introduced in [10]. Furthermore, they have been used by Yu.V. Kuz'min in his paper [20], where the elements of the form

$$[y_1, y_2] \wedge_* [y_3, y_4]u + [y_1, y_3] \wedge_* [y_4, y_2]u + [y_1, y_4] \wedge_* [y_2, y_3]u$$

are called Jacobian elements.

For odd degree, the Jacobian elements are the same as t-elements. Indeed, by (3.11), (3.12) and (3.15), the Jacobian element can be written as

$$\begin{aligned}&[y_1, y_2] \wedge_* [y_3, y_4]u + [y_1, y_3] \wedge_* [y_4, y_2]u + [y_1, y_4] \wedge_* [y_2, y_3]u \\ =&[y_1, y_2] \wedge_* [y_3, y_4]u - [y_1, y_3] \wedge_* [y_2, y_4]u + (-1)^{\deg u+1}[y_2, y_3] \wedge_* [y_1, y_4]u.\end{aligned}$$

Since $\deg u$ is odd, we obtain

$$\begin{aligned}[y_1, y_2] \wedge_* [y_3, y_4]u+&[y_3, y_1] \wedge_* [y_2, y_4]u + [y_2, y_3] \wedge_* [y_1, y_4]u \\ &= w(y_1, y_2, y_3, y_4; u).\end{aligned}$$

The following results in this section are derived from Kuz'min's paper [20], and some of them have been obtained independently in [10]. These results capture the properties which will be important for our purposes in the following sections.

Lemma 3.4 Let $n \geq 5$ be an odd integer, $y_1, \dots, y_n \in X$ and $u = y_5 \dots y_n \in \mathcal{U}$. Then the following holds for the t-element $w = w(y_1, y_2, y_3, y_4; u) \in (M \wedge M) \otimes_U \mathbb{Z}$.

(i) If any two of the elements y_1, y_2, y_3, y_4 are equal, then $w = 0$. In particular, w is antisymmetric in y_1, y_2, y_3, y_4.

(ii) $w(y_1, y_2, y_3, y_4; y_5 y_6 \dots y_n) = w(y_1, y_2, y_3, y_5; y_4 y_6 \dots y_n)$.

Proof. (i) Suppose $y_1 = y_2$. Then by the definition of the t-element w, we have

$$\begin{aligned}&w(y_1, y_2, y_3, y_4; u) \\ =&[y_1, y_2] \wedge_* [y_3, y_4]u + [y_2, y_3] \wedge_* [y_1, y_4]u + [y_3, y_1] \wedge_* [y_2, y_4]u \\ =&[y_1, y_1] \wedge_* [y_3, y_4]u + [y_1, y_3] \wedge_* [y_1, y_4]u + [y_3, y_1] \wedge_* [y_1, y_4]u \\ =&[y_1, y_3] \wedge_* [y_1, y_4]u - [y_1, y_3] \wedge_* [y_1, y_4]u \\ =&0.\end{aligned}$$

The proof for the cases where $y_1 = y_3$ and $y_2 = y_3$ is similar.

For $y_1 = y_3$,

$$\begin{aligned}
&w(y_1, y_2, y_3, y_4; u)\\
=&[y_1, y_2] \wedge_* [y_3, y_4]u + [y_2, y_3] \wedge_* [y_1, y_4]u + [y_3, y_1] \wedge_* [y_2, y_4]u\\
=&[y_1, y_2] \wedge_* [y_1, y_4]u + [y_2, y_1] \wedge_* [y_1, y_4]u + [y_1, y_1] \wedge_* [y_2, y_4]u\\
=&[y_1, y_2] \wedge_* [y_1, y_4]u - [y_1, y_2] \wedge_* [y_1, y_4]u\\
=&0
\end{aligned}$$

and for $y_2 = y_3$,

$$\begin{aligned}
&w(y_1, y_2, y_3, y_4; u)\\
=&[y_1, y_2] \wedge_* [y_3, y_4]u + [y_2, y_3] \wedge_* [y_1, y_4]u + [y_3, y_1] \wedge_* [y_2, y_4]u\\
=&[y_1, y_2] \wedge_* [y_2, y_4]u + [y_2, y_2] \wedge_* [y_1, y_4]u + [y_2, y_1] \wedge_* [y_2, y_4]u\\
=&[y_1, y_2] \wedge_* [y_2, y_4]u - [y_1, y_2] \wedge_* [y_2, y_4]u\\
=&0.
\end{aligned}$$

Hence, w is antisymmetric with respect to the first three entries y_1, y_2, y_3. Namely,

$$w(y_1, y_2, y_3, y_4; u) = -w(y_2, y_1, y_3, y_4; u) = -w(y_1, y_3, y_2, y_4; u) = -w(y_3, y_2, y_1, y_4; u).$$

To complete the proof of part (i) it is now sufficient to show that $w = 0$ if one of y_1, y_2, y_3 is equal to y_4. We may assume that $y_3 = y_4$. We mention that so far we have not used the assumption that n is odd. This condition, however, is required for the case where $y_3 = y_4$. If this holds, we have

$$\begin{aligned}
&w(y_1, y_2, y_3, y_4; u)\\
=&[y_1, y_2] \wedge_* [y_3, y_4]u + [y_2, y_3] \wedge_* [y_1, y_4]u + [y_3, y_1] \wedge_* [y_2, y_4]u\\
=&[y_1, y_2] \wedge_* [y_3, y_3]u + [y_2, y_3] \wedge_* [y_1, y_3]u + [y_3, y_1] \wedge_* [y_2, y_3]u.
\end{aligned}$$

The first element on the right hand side is zero, and by using anti-commutativity and the relation (3.15) the sum of the second and third summands is equal to

$$-[y_2, y_3] \wedge_* [y_3, y_1]u + (-1)^{\deg u}[y_2, y_3] \wedge_* [y_3, y_1]u.$$

Since $\deg u$ is odd, the sum of the second and third summands is equal to zero. Hence,

$$w(y_1, y_2, y_3, y_4; u) = 0.$$

Similarly, when $y_2 = y_4$, we have

$$\begin{aligned}
& w(y_1, y_2, y_3, y_4; u) \\
=& [y_1, y_2] \wedge_* [y_3, y_4]u + [y_2, y_3] \wedge_* [y_1, y_4]u + [y_3, y_1] \wedge_* [y_2, y_4]u \\
=& [y_1, y_4] \wedge_* [y_3, y_4]u + [y_4, y_3] \wedge_* [y_1, y_4]u + [y_3, y_1] \wedge_* [y_4, y_4]u \\
=& -[y_1, y_4] \wedge_* [y_4, y_3]u - [y_1, y_4]u \wedge_* [y_4, y_3] \\
=& -[y_1, y_4] \wedge_* [y_4, y_3]u - (-1)^{\deg u}[y_1, y_4] \wedge_* [y_4, y_3]u.
\end{aligned}$$

Since $\deg u$ is odd, the sum of the first and second summands on the right hand side is equal to zero. Hence,

$$w(y_1, y_2, y_3, y_4; u) = 0.$$

And also choosing $y_1 = y_4$, we obtain

$$\begin{aligned}
& w(y_1, y_2, y_3, y_4; u) \\
=& [y_1, y_2] \wedge_* [y_3, y_4]u + [y_2, y_3] \wedge_* [y_1, y_4]u + [y_3, y_1] \wedge_* [y_2, y_4]u \\
=& [y_4, y_2] \wedge_* [y_3, y_4]u + [y_2, y_3] \wedge_* [y_4, y_4]u + [y_3, y_4] \wedge_* [y_2, y_4]u \\
=& [y_4, y_2] \wedge_* [y_3, y_4]u + [y_4, y_2]u \wedge_* [y_3, y_4] \\
=& [y_4, y_2] \wedge_* [y_3, y_4]u + (-1)^{\deg u}[y_4, y_2] \wedge_* [y_3, y_4]u \\
=& 0.
\end{aligned}$$

Hence, it is easy to see that the entries y_1, y_2, y_3 in w are antisymmetric with y_4.

Namely,

$$w(y_1, y_2, y_3, y_4; u) = -w(y_4, y_2, y_3, y_1; u) = -w(y_1, y_4, y_3, y_2; u) = -w(y_1, y_2, y_4, y_3; u).$$

This proves (i).

(ii) Firstly, we set $u = y_5 v$, where $v = y_6 y_7 \dots y_n$. Then using the relation (3.12)

$$\begin{aligned}
&w(y_1, y_2, y_3, y_4; y_5 v) \\
=&[y_1, y_2] \wedge_* [y_3, y_4] y_5 v + [y_2, y_3] \wedge_* [y_1, y_4] y_5 v + [y_3, y_1] \wedge_* [y_2, y_4] y_5 v \\
=&-[y_1, y_2] \wedge_* [y_4, y_5] y_3 v + [y_1, y_2] \wedge_* [y_3, y_5] y_4 v \\
&-[y_2, y_3] \wedge_* [y_4, y_5] y_1 v + [y_2, y_3] \wedge_* [y_1, y_5] y_4 v \\
&-[y_3, y_1] \wedge_* [y_4, y_5] y_2 v + [y_3, y_1] \wedge_* [y_2, y_5] y_4 v.
\end{aligned}$$

The sum of the second summands in the last three rows is equal to

$$\begin{aligned}
[y_1, y_2] \wedge_* [y_3, y_5] y_4 v + [y_2, y_3] \wedge_* [y_1, y_5] y_4 v + [y_3, y_1] \wedge_* [y_2, y_5] y_4 v \\
= w(y_1, y_2, y_3, y_5; y_4 v),
\end{aligned}$$

and the sum of the first summands in these rows is zero. Indeed, by using (3.14) we get

$$\begin{aligned}
&-[y_1, y_2] \wedge_* [y_4, y_5] y_3 v - [y_2, y_3] \wedge_* [y_4, y_5] y_1 v - [y_3, y_1] \wedge_* [y_4, y_5] y_2 v \\
=&[y_1, y_2] y_3 \wedge_* [y_4, y_5] v + [y_2, y_3] y_1 \wedge_* [y_4, y_5] v + [y_3, y_1] y_2 \wedge_* [y_4, y_5] v \\
=&([y_1, y_2] y_3 + [y_2, y_3] y_1 + [y_3, y_1] y_2) \wedge_* [y_4, y_5] v,
\end{aligned}$$

and by the definition of the Jacobi identity, $[y_1, y_2] y_3 + [y_2, y_3] y_1 + [y_3, y_1] y_2 = 0$. Hence,

$$-[y_1, y_2] \wedge_* [y_4, y_5] y_3 v - [y_2, y_3] \wedge_* [y_4, y_5] y_1 v - [y_3, y_1] \wedge_* [y_4, y_5] y_2 v = 0.$$

Consequently,

$$w(y_1, y_2, y_3, y_4; y_5v) = w(y_1, y_2, y_3, y_5; y_4v).$$

This completes the proof of part (ii).

□

Corollary 3.5 ([20], Lemma 29) For any t-element $w(y_1, y_2, y_3, y_4; u)$ with $u = y_5 \dots y_n$ of odd degree,

$$2w(y_1, y_2, y_3, y_4; u) = 0.$$

Moreover, if any two of the elements $y_1, y_2, \dots, y_n$ are equal, then

$$w(y_1, y_2, y_3, y_4; u) = 0.$$

Proof. Let $u = y_5v$ where $v = y_6 \dots y_n$. By using parts (i) and (ii) of Lemma 3.4, we have

$$\begin{aligned}
w(y_1, y_2, y_3, y_4; y_5v) =& w(y_1, y_2, y_3, y_5; y_4v)\\
=& -w(y_1, y_2, y_5, y_3; y_4v)\\
=& -w(y_1, y_2, y_5, y_4; y_3v)\\
=& w(y_1, y_2, y_4, y_5; y_3v)\\
=& w(y_1, y_2, y_4, y_3; y_5v)\\
=& -w(y_1, y_2, y_3, y_4; y_5v).
\end{aligned}$$

Hence, we have $2w(y_1, y_2, y_3, y_4; y_5v) = 0$, as required.

Finally, we suppose that any two entries of w are equal. Let y_i and y_j be equal in w for $1 \leq i \leq j \leq n$. Then Lemma 3.4 implies that

$$w(y_1, y_2, y_3, y_4; u) = \pm w(y_i, y_j, \dots; u')$$

for some suitable $u' \in \mathcal{U}$. By part (i) of Lemma 3.4, the right hand side is zero. Hence, $w(y_1, y_2, y_3, y_4; u) = 0$.

□

Now we have already defined the generating sets for G''. It remains to show that they are bases for G''.

3.5 The 6-term exact sequence

In this section we introduce a 6-term exact sequence that will be our main tool for working out dimensions of the homogeneous components of $G'' \otimes_U K$, where K is a field. To obtain this 6-term exact sequence, we will use the short exact sequences (3.3) and (3.9).

These two short exact sequences give rise to a chain complex

$$0 \longrightarrow M \wedge M \xrightarrow{\alpha_1} P \wedge P \xrightarrow{\alpha_2} \Delta \otimes P \xrightarrow{\alpha_3} U \circ U \xrightarrow{\alpha_4} U \xrightarrow{\alpha_5} K/2K \longrightarrow 0 \qquad (3.23)$$

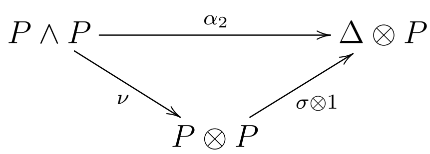
of abelian groups, where the maps are given as follows:

(i) $\alpha_1 : M \wedge M \to P \wedge P$ is given by

$$m_1 \wedge m_2 \mapsto m_1\mu \wedge m_2\mu$$

for $m_1, m_2 \in M$.

(ii) $\alpha_2 : P \wedge P \to \Delta \otimes P$ is the composite of the embedding map $P \wedge P \to P \otimes P$ and $\sigma \otimes 1 : P \otimes P \to \Delta \otimes P$. We have the following diagram



For any $p_1, p_2 \in P$, the image of $p_1 \wedge p_2$ in $P \otimes P$ is $p_1 \otimes p_2 - p_2 \otimes p_1$, then the image of $p_1 \otimes p_2 - p_2 \otimes p_1$ in $\Delta \otimes P$ is

$$\begin{aligned}(p_1 \otimes p_2 - p_2 \otimes p_1)(\sigma \otimes 1) =& (p_1 \otimes p_2)(\sigma \otimes 1) - (p_2 \otimes p_1)(\sigma \otimes 1)\\ =& p_1\sigma \otimes p_2 - p_2\sigma \otimes p_1,\end{aligned}$$

so the map α_2 is given by

$$(p_1 \wedge p_2) \mapsto p_1\sigma \otimes p_2 - p_2\sigma \otimes p_1$$

for $p_1, p_2 \in P$.

(iii) $\alpha_3 : \Delta \otimes P \to U \circ U$ is the composite of $1 \otimes \sigma : \Delta \otimes P \to \Delta \otimes \Delta$ and the natural projection $\Delta \otimes \Delta \to U \circ U$. The following diagram commutes

$$\begin{array}{ccccc} \Delta \otimes P & & \xrightarrow{\alpha_3} & & U \circ U \\ & \searrow^{1\otimes\sigma} & & \nearrow & \\ & & \Delta \otimes \Delta & & \end{array}$$

For $\delta \in \Delta$ and $p \in P$, the image of $\delta \otimes p$ in $\Delta \otimes P$ is

$$(\delta \otimes p)(1 \otimes \sigma) = \delta \otimes p\sigma,$$

then under the natural projection $\Delta \otimes \Delta \to U \circ U$, the image of $\delta \otimes p\sigma$ in $U \circ U$ is $\delta \circ p\sigma$. Therefore, the map α_3 is defined by $(\delta \otimes p) \mapsto \delta \circ p\sigma$ for $\delta \in \Delta$ and $p \in P$.

(iv) $\alpha_4 : U \circ U \to U$ is the composite of the embedding map $U \circ U \to U \otimes U$ and $1 \otimes \varepsilon : U \otimes U \to U \otimes K$, diagrammatically,

$$\begin{array}{ccc} U \circ U & \xrightarrow{\alpha_4} & U \\ \downarrow{\scriptstyle\theta} & & \uparrow \\ U \otimes U & \xrightarrow[1\otimes\varepsilon]{} & U \otimes K. \end{array}$$

For $f, g \in U$, the image of $f \circ g$ in $U \otimes U$ is $f \otimes g + g \otimes f$, then the image of $f \otimes g + g \otimes f$

under $1 \otimes \varepsilon$ calculates as

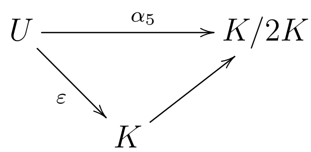
$$\begin{aligned}(f \otimes g + g \otimes f)(1 \otimes \varepsilon) =&(f \otimes g)(1 \otimes \varepsilon) + (g \otimes f)(1 \otimes \varepsilon)\\ =&f \otimes g\varepsilon + g \otimes f\varepsilon.\end{aligned}$$

Since the map $U \otimes K \to U$ is an isomorphism as defined by $f \otimes k \to fk$ for all $f \in U$ and $k \in K$, the image of $f \otimes g\varepsilon + g \otimes f\varepsilon$ in U is $f(g\varepsilon) + g(f\varepsilon)$. Thus, the mapping α_4 is given by

$$(f \circ g) \mapsto (f\varepsilon)g + (g\varepsilon)f$$

for $f, g \in U$.

(v) $\alpha_5 : U \to K/2K$ is the canonical epimorphism defined by $f \mapsto f\varepsilon + 2K$ for $f \in U$. Diagrammatically,



where the relevant maps are given by $f \mapsto f\varepsilon$ and $f\varepsilon \mapsto f\varepsilon + 2K$. Hence, for $f \in U$, the image of f under ε is $f\varepsilon$, then the image of $f\varepsilon$ in $K/2K$ is equal to $f\varepsilon + 2K$.

Lemma 3.6 The chain complex (3.23) is an exact sequence of U-modules.

Proof. We first consider the sequence (3.9)

$$0 \to M \to P \to \Delta \to 0.$$

Since P and Δ are K-free, the sequence (3.9) splits over K. This means that there exists a left inverse to $\sigma : P \to \Delta$, i.e. a homomorphism $\imath : \Delta \to P$ such that $\imath\sigma = 1_\Delta$. The map $\imath$ is called a splitting map. Then P is isomorphic to the direct sum of the image of M in P and the image of Δ in P, namely, $P \cong M\mu \oplus \Delta\imath$ as a K-module. Consequently, for the exterior square of P, $P \wedge P$, we have the direct sum decomposition

$$\begin{aligned} P \wedge P \cong & (M\mu \oplus \Delta\imath) \wedge (M\mu \oplus \Delta\imath) \\ \cong & (M\mu \wedge M\mu) \oplus (\Delta\imath \otimes M\mu) \oplus (\Delta\imath \wedge \Delta\imath) \end{aligned}$$

and also the tensor product $\Delta \otimes P$ has the direct sum decomposition

$$\begin{aligned} \Delta \otimes P \cong & \Delta \otimes (M\mu \oplus \Delta\imath) \\ \cong & (\Delta \otimes M\mu) \oplus (\Delta \otimes \Delta\imath). \end{aligned}$$

Now, the map $\alpha_1 : M \wedge M \to P \wedge P$ maps $M \wedge M$ isomorphically onto the first summand $M\mu \wedge M\mu$ of $P \wedge P$, while the map $\alpha_2 : P \wedge P \to \Delta \otimes P$ maps the second summand $\Delta\imath \otimes M\mu$ of $P \wedge P$ isomorphically onto the direct summand $\Delta \otimes M\mu$ of $\Delta \otimes P$.

Moreover, the map α_2 maps the third summand $\Delta\imath \wedge \Delta\imath$ of $P \wedge P$ injectively into the second summand $\Delta \otimes \Delta\imath$ of $\Delta \otimes P$, and by the definition of α_2 we get

$$\begin{aligned} (\delta_1\imath \wedge \delta_2\imath) \stackrel{\alpha_2}{\longmapsto} & \delta_1 \underbrace{\imath\sigma}_{1_\Delta} \otimes \delta_2\imath - \delta_2 \underbrace{\imath\sigma}_{1_\Delta} \otimes \delta_1\imath \\ & = \delta_1 1_\Delta \otimes \delta_2\imath - \delta_2 1_\Delta \otimes \delta_1\imath \\ & = \delta_1 \otimes \delta_2\imath - \delta_2 \otimes \delta_1\imath \end{aligned}$$

for $\delta_1, \delta_2 \in \Delta$. In fact, it is easy to see that the image of $\Delta\imath \wedge \Delta\imath$ in $\Delta\imath \otimes \Delta\imath$ is exactly the kernel of the canonical projection $\Delta\imath \otimes \Delta\imath \to \Delta \circ \Delta$. Indeed, by definition ν, the image of $\delta_1\imath \wedge \delta_2\imath \in \Delta\imath \wedge \Delta\imath$ in $\Delta\imath \otimes \Delta\imath$ is

$$\delta_1\imath \wedge \delta_2\imath \stackrel{\nu}{\longmapsto} \delta_1\imath \otimes \delta_2\imath - \delta_2\imath \otimes \delta_1\imath,$$

then the canonical projection $\Delta\imath \otimes \Delta\imath \to \Delta \circ \Delta$ is the composite of $\sigma \otimes \sigma : \Delta\imath \otimes \Delta\imath \to$

$\Delta \otimes \Delta$ and the projection map $\Delta \otimes \Delta \to \Delta \circ \Delta$, diagrammatically,

$$\begin{array}{ccccc} \Delta\imath \wedge \Delta\imath & \xrightarrow{\nu} & \Delta\imath \otimes \Delta\imath & & \\ & & {\scriptstyle \sigma\otimes\sigma}\swarrow & \searrow & \\ & \Delta \otimes \Delta & \longrightarrow & & \Delta \circ \Delta. \end{array}$$

Hence, we have that

$$\begin{aligned} \delta_1\imath \wedge \delta_2\imath &\overset{\nu}{\longmapsto} \delta_1\imath \otimes \delta_2\imath - \delta_2\imath \otimes \delta_1\imath \\ &\overset{\sigma\otimes\sigma}{\longmapsto} \delta_1 \underbrace{\imath\sigma}_{1_\Delta} \otimes \delta_2 \underbrace{\imath\sigma}_{1_\Delta} - \delta_2 \underbrace{\imath\sigma}_{1_\Delta} \otimes \delta_1 \underbrace{\imath\sigma}_{1_\Delta} \\ &\quad = \delta_1 1_\Delta \otimes \delta_2 1_\Delta - \delta_2 1_\Delta \otimes \delta_1 1_\Delta \\ &\quad = \delta_1 \otimes \delta_2 - \delta_2 \otimes \delta_1 \\ &\longmapsto \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \\ &\quad = \delta_1 \circ \delta_2 - \delta_1 \circ \delta_2 \\ &\quad = 0 \end{aligned}$$

for all $\delta_1, \delta_2 \in \Delta$.

To summarize, we see that we have the following diagram:

$$\begin{array}{ccccccccc} 0 & & & & & & & & \\ \downarrow & & & & & & & & \\ M \wedge M & & 0 & & 0 & & & & \\ \downarrow & & \downarrow & & \downarrow & & & & \\ M\mu \wedge M\mu & \oplus & \Delta\imath \otimes M\mu & \oplus & \Delta\imath \wedge \Delta\imath & \cong & P \wedge P \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha_2} & & & & \\ 0 & & \Delta \otimes M\mu & \oplus & \Delta \otimes \Delta\imath & \cong & \Delta \otimes P \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & \Delta \circ \Delta & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

This diagram gives rise to the following exact sequence

$$0 \to M \wedge M \to P \wedge P \to \Delta \otimes P \to \Delta \circ \Delta \to 0, \tag{3.24}$$

where the first two maps are as in (3.23) and the third, $\Delta \otimes P \to \Delta \circ \Delta$ is the composite of $1 \otimes \sigma : \Delta \otimes P \to \Delta \otimes \Delta$ and the mapping $\Delta \otimes \Delta \to \Delta \circ \Delta$, is given by

$$\begin{aligned} \delta \otimes p &\overset{1\otimes\sigma}{\longmapsto} \delta \otimes p\sigma \\ &\longmapsto \delta \circ p\sigma, \end{aligned}$$

where $\delta \in \Delta$ and $p \in P$.

Now we consider the sequence (3.3)

$$0 \to \Delta \to U \to K \to 0.$$

Since Δ, U and K are K-free, we have that the short exact sequence (3.3) too splits over K, and hence $U \cong \Delta \oplus K$, where we identify K with the constant polynomials in $U = K[X]$. For the symmetric square of U, $U \circ U$, we have the direct sum decomposition,

$$\begin{aligned} U \circ U \cong& (\Delta \oplus K) \circ (\Delta \oplus K) \\ \cong& (\Delta \circ \Delta) \oplus (\Delta \otimes K) \oplus (K \circ K) \\ \cong& (\Delta \circ \Delta) \oplus \Delta \oplus K. \end{aligned}$$

It is easy to see that the under the map $\alpha_4 : U \circ U \to U$ in (3.23), the image of the first summand $\Delta \circ \Delta$ of $U \circ U$ in U is equal zero. Indeed, for $\delta_1, \delta_2 \in \Delta$ we get

$$\begin{aligned} (\delta_1 \circ \delta_2) &\overset{\alpha_4}{\longmapsto} (\underbrace{\delta_1 \varepsilon}_{=0})\delta_2 + (\underbrace{\delta_2 \varepsilon}_{=0})\delta_1 \\ &= 0. \end{aligned}$$

The map α_4 maps the second summand $\Delta \otimes K$ of $U \circ U$ isomorphically onto the K-direct summand Δ of U. For $\delta \in \Delta$ and $k \in K$, we have

$$\begin{aligned}(\delta \otimes k) \overset{\alpha_4}{\longmapsto} (\underbrace{\delta\varepsilon}_{=0})k + (\underbrace{k\varepsilon}_{=k})\delta \\ &= 0k + k\delta \\ &= k\delta \in \Delta.\end{aligned}$$

Moreover, since we have

$$\begin{aligned}(k_1 \circ k_2) \overset{\alpha_4}{\longmapsto} (\underbrace{k_1\varepsilon}_{=k_1})k_2 + (\underbrace{k_2\varepsilon}_{=k_2})k_1 \\ &= k_1k_2 + k_2k_1 \\ &= 2k_1k_2 \in 2K\end{aligned}$$

for $k_1, k_2 \in K$, the third summand $K \circ K$ of $U \circ U$ is mapped isomorphically onto $2K$.

Now again summarizing we see the following diagram :

$$\begin{array}{ccccccc}
0 & & & & & & \\
\downarrow & & & & & & \\
\Delta \circ \Delta & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\Delta \circ \Delta & \oplus & \Delta \otimes K & \oplus & K \circ K & \cong & U \circ U \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & \Delta & \oplus & K & \cong & U \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & K/2K & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

This diagram yields a 4-term exact sequence

$$0 \to \Delta \circ \Delta \to U \circ U \to U \to K/2K \to 0. \tag{3.25}$$

By combining the exact sequences (3.24) and (3.25), we get the 6-term exact sequence (3.23) which will be our main tool in the following sections

$$0 \longrightarrow M \wedge M \longrightarrow P \wedge P \longrightarrow \Delta \otimes P \longrightarrow U \circ U \xrightarrow{\alpha_4} U \longrightarrow K/2K \longrightarrow 0. \tag{3.26}$$

This exact sequence is of abelian groups, but since all the maps in (3.23) agree with the derivation action, we have actually an exact sequence of U-modules. This completes the proof of the lemma.

□

3.5.1 Free modules in the 6-term exact sequence

In this subsection we examine the modules in the 6-term exact sequence (3.23)

$$0 \longrightarrow M \wedge M \longrightarrow P \wedge P \longrightarrow \Delta \otimes P \longrightarrow U \circ U \longrightarrow U \longrightarrow K/2K \longrightarrow 0.$$

We know that U itself is a free U-module with free generator 1. It turns out that the same is true for the tensor product $\Delta \otimes P$. We know that P is a free U-module with free generators $e_1, e_2, \ldots, e_r$ and Δ is a K-free U-module. It is a well-known general fact that the tensor product of a K-free U-module and a free U-module under derivation action is always a free module; hence, $\Delta \otimes P$ is a free U-module. Moreover, this general fact holds for both tensor products of modules for groups with diagonal action and tensor products of modules for Lie algebras with derivation action (see [27], Theorem 1.9.4).

To begin, we state the following lemma which is probably known but a precise reference is hard to find, hence we give an elementary proof for the current setting. This proof also provides an explicit free generating set that will be useful later.

Lemma 3.7 Let N be an arbitrary U-module that is free as a K-module with K-basis $\mathcal{N}$. Then the tensor product $N \otimes U$ is a free U-module and the elements $m \otimes 1$ with $m \in \mathcal{N}$ form a free generating set for $N \otimes U$ as a U-module.

Proof. As previously mentioned, $\mathcal{U}$ denotes the K-basis of U consisting of all monomials. Then the elements $n \otimes u$ with $n \in \mathcal{N}$ and $u \in \mathcal{U}$ form a K-basis of $N \otimes U$. To prove that the elements $m \otimes 1$, $m \in \mathcal{N}$ form a generating set of $N \otimes U$ as a U-module, it is sufficient to show that each basis element $n \otimes u$ is a linear combination of those elements with coefficients in U.

We argue by induction on $\deg u$. When $\deg u = 0$, the lemma obviously holds as $u = 1$.

Now let $\deg u = k > 0$. Then suppose that $u = vy$ for some $v \in \mathcal{U}$ with $\deg v = k - 1$ and some $y \in X$, therefore applying the module action we have

$$n \otimes u = n \otimes vy = (n \otimes v)y - ny \otimes v,$$

and by the inductive hypothesis both $n \otimes v$ and $ny \otimes v$ can be expressed as a linear combination of the elements $m \otimes 1$, where $m \in \mathcal{N}$ with coefficients in U. Consequently, $n \otimes vy$ can be expressed in this way. Hence, the set of all such elements $m \otimes 1$ is a generating set for $N \otimes U$.

It therefore remains to show that this set is actually a free generating set. Now we suppose that this is not the case, and argue for a contradiction. If this set is not a free generating set, then there exists a finite subset $\{n_1, n_2, \ldots, n_k\}$ of $\mathcal{N}$ and non-zero polynomials $f_1, f_2, \ldots, f_k \in U$ such that

$$\sum_i (n_i \otimes 1) f_i = 0. \tag{3.27}$$

Observe that for any monomial $u \in \mathcal{U}$ and any $n \in \mathcal{N}$ we have

$$(n \otimes 1)u = n \otimes u + \sum c_{m,v} m \otimes v, \tag{3.28}$$

where the sum runs over some basis elements $m \otimes v$ with $m \in \mathcal{N}$, $v \in \mathcal{U}$ with $\deg v < \deg u$ and $c_{m,v} \in K$. Note that the $\mathcal{U}$-components of the basis elements under the sum in (3.28) are of degree strictly less than the degree of u. Now consider the dependence (3.27). We may assume that f_1 is of maximal degree among the f_i. Let $w \in \mathcal{U}$ be a monomial that occurs with non-zero coefficient $a_{1,w}$ in the leading term of f_1. Then, if we expand the left hand side of (3.27) as a K-linear combination of the basis elements $n \otimes u$, (3.28) implies that the coefficient at the basis element $n_1 \otimes w$ is precisely $a_{1,w}$. Namely,

$$\begin{aligned}\sum_i (n_i \otimes 1) f_i =& (n_1 \otimes 1) f_1 + (n_2 \otimes 1) f_2 + \dots \\ =& (n_1 \otimes 1)(a_{1,w} w + \dots) + (n_2 \otimes 1) f_2 + \dots \\ =& a_{1,w}(n_1 \otimes 1) w + \dots \\ =& a_{1,w}(n_1 \otimes w + \sum c_{m,v} m \otimes v) + \dots \\ =& a_{1,w}(n_1 \otimes w) + \dots .\end{aligned}$$

Since $a_{1,w} \neq 0$, the left hand side of (3.27) is not zero, and the resulting contradiction completes the proof of the lemma.

□

This lemma is the backbone of the proof of following results in this section.

Corollary 3.8 Let N be an arbitrary U-module that is free as a K-module with K-basis $\mathcal{N}$, and let P be a free U-module with free generators $e_1, e_2, \dots, e_r$. Then the tensor product $N \otimes P$ is a free U-module and the elements $m \otimes e_i$ with $m \in \mathcal{N}$ and $i = 1, 2, \dots, r$ form a free generating set $N \otimes P$ as a U-module.

Proof. Since P is a free U-module with free generators $e_1, e_2, \dots, e_r$, we have that

$$P \cong \bigoplus_i e_i U.$$

Then we get

$$\begin{aligned} N \otimes P &\cong N \otimes (\bigoplus_i e_i U) \\ &\cong \bigoplus_i (N \otimes e_i U). \end{aligned}$$

For all $i \in \{1, 2, \dots, r\}$, $e_i U$ is isomorphic to U. Hence,

$$N \otimes P \cong \bigoplus_i (N \otimes U).$$

By Lemma 3.7, $N \otimes U$ is a free U-module. In conclusion, the direct sum of free U-modules is a free U-module.

□

Now we return the sequence (3.23)

$$0 \longrightarrow M \wedge M \xrightarrow{\alpha_1} P \wedge P \xrightarrow{\alpha_2} \Delta \otimes P \xrightarrow{\alpha_3} U \circ U \xrightarrow{\alpha_4} U \xrightarrow{\alpha_5} K/2K \longrightarrow 0.$$

In this sequence, by Lemma 3.7 and Corollary 3.8, the tensor product $\Delta \otimes P$ is a free U-module. However, the exterior and symmetric squares in (3.23) which are $M \wedge M$, $P \wedge P$ and $U \circ U$ are not free U-modules. This generates considerable problems with using (3.23) for obtaining information about the tensor product $(M \wedge M) \otimes_U K$.

3.5.2 An approach to the problem

Let K denote a field of characteristic other than 2.

Proposition 3.9 If K is a field of characteristic other than 2, then the exterior and symmetric squares $U \wedge U$ and $U \circ U$ are free U-modules. The elements $u \wedge 1$ with $u \in \mathcal{U}$ and $\deg u$ odd form a free generating set for $U \wedge U$, and the elements $u \circ 1$ with $u \in \mathcal{U}$ and $\deg u$ even form a free generating set for $U \circ U$.

Proof. The exterior square $U \wedge U$ is a homomorphic image of the tensor square $U \otimes U$ via the projection map $\pi : U \otimes U \to U \wedge U$ given by $f_1 \otimes f_2 \mapsto f_1 \wedge f_2$ for $f_1, f_2 \in U$.

By Lemma 3.7, the elements $u \otimes 1$ with $u \in \mathcal{U}$ form a generating set of $U \otimes U$ as a U-module. The image of the generating set of $U \otimes U$ in $U \wedge U$ consists of the elements $u \wedge 1$ and these elements form a generating set of $U \wedge U$ as a U-module. Consider the trivialization homomorphism $U \wedge U \to (U \wedge U) \otimes_U K$. In the tensor product $(U \wedge U) \otimes_U K$ we have the relations

$$(f_1 y \wedge_* f_2) = -(f_1 \wedge_* f_2 y)$$

for all $f_1, f_2 \in U$ and $y \in X$, and, consequently, for a monomial $u \in \mathcal{U}$ by applying the relations (3.15) and (3.13) respectively, we have

$$u \wedge_* 1 = (-1)^{\deg u}(1 \wedge_* u) = (-1)^{\deg u+1}(u \wedge_* 1).$$

If $\deg u$ is even, we obtain $2(u \wedge_* 1) = 0$. Since the characteristic of the ground field K is not 2, this implies that $(u \wedge_* 1) = 0$ in $(U \wedge U) \otimes_U K$ if $\deg u$ is even. Hence, in this case $u \wedge 1 \in (U \wedge U)\Delta$. But this means that the elements $u \wedge 1$ with u of even degree belong to the submodule of $U \wedge U$ that is generated by the elements $v \wedge 1$ with $\deg v < \deg u$. It follows that these elements can be removed from the generating set of $U \wedge U$ as a U-module. In other words, the elements $u \wedge 1$ with $u \in \mathcal{U}$ and $\deg u$ odd form a generating set of $U \wedge U$ as a U-module.

Now we show that this is actually a free generating set. To this end we consider the images of these generators in $U \otimes U$ under the embedding $\nu : U \wedge U \to U \otimes U$ that is given by $(f_1 \wedge f_2) \mapsto f_1 \otimes f_2 - f_2 \otimes f_1$ for $f_1, f_2 \in U$.

Suppose $u = y_1 y_2 \ldots y_k$. Then

$$\begin{aligned} 1 \otimes u =& 1 \otimes y_1 y_2 \ldots y_k \\ =& (1 \otimes y_1 y_2 \ldots y_{k-1}) y_k - (y_k \otimes y_1 y_2 \ldots y_{k-2}) y_{k-1} \\ &+ (y_{k-1} y_k \otimes y_1 y_2 \ldots y_{k-3}) y_{k-2} - \ldots \\ &\ldots + (-1)^{k-1}(y_2 \ldots y_k \otimes 1) y_1 + (-1)^k y_1 y_2 \ldots y_k \otimes 1. \end{aligned}$$

Consequently, if k, the degree of u, is odd, we have

$$(u \wedge 1)\nu = u \otimes 1 - 1 \otimes u = 2(u \otimes 1) + w, \tag{3.29}$$

where w belongs to the submodule of $U \otimes U$ that is generated by the elements $v \otimes 1$ with $v \in \mathcal{U}$ and $\deg v < \deg u$. Since the elements $u \otimes 1$ as free generators of $U \otimes U$ are linearly independent over U, it follows easily from (3.29) that the elements $u \wedge 1$ with $\deg u$ odd are also linearly independent over U, and hence they are free generators for $U \wedge U$ as a U-module.

The proof for $U \circ U$ is similar with the embedding $\theta : U \circ U \to U \otimes U$ given by $f_1 \circ f_2 \mapsto f_1 \otimes f_2 + f_2 \otimes f_1$ for $f_1, f_2 \in U$ being used instead of ν. The symmetric square $U \circ U$ is a homomorphic image of the tensor square $U \otimes U$ with the map $U \otimes U \to U \circ U$ given by $f_1 \otimes f_2 \mapsto f_1 \circ f_2$. By Lemma 3.7, the elements $u \otimes 1$ with $u \in \mathcal{U}$ form a generating set of $U \otimes U$ as a U-module. The image of the generating set of $U \otimes U$ in $U \circ U$ consists of the elements $u \circ 1$ and these elements form a generating set of $U \circ U$ as a U-module. Consider the trivialization homomorphism $U \circ U \to (U \circ U) \otimes_U K$. In the tensor product $(U \circ U) \otimes_U K$ we have the relations

$$f_1 y \circ_* f_2 = -(f_1 \circ_* f_2 y)$$

for all $f_1, f_2 \in U$ and $y \in X$, and consequently, for a monomial $u \in \mathcal{U}$ we have

$$u \circ_* 1 = (-1)^{\deg u}(1 \circ_* u) = (-1)^{\deg u}(u \circ_* 1).$$

Since the characteristic of the ground field K is not 2, this implies that $(u \circ_* 1) = 0$ in $(U \circ U) \otimes_U K$ if $\deg u$ is odd. Hence, in this case $u \circ 1 \in (U \circ U)\Delta$. But this means that the elements $u \circ 1$ with u of odd degree belong to the submodule of $U \circ U$ that is generated by the elements $v \circ 1$ with $\deg v < \deg u$. It follows that these elements can be removed from the generating set of $U \circ U$ as a U-module. In other words, the elements $u \circ 1$ with $u \in \mathcal{U}$ and $\deg u$ even form a generating set of $U \circ U$ as a U-module.

Now we show that this is actually a free generating set. To this end we consider the images of these generators in $U \otimes U$ under the embedding $\theta : U \circ U \to U \otimes U$ that is given by $(f_1 \circ f_2) \mapsto f_1 \otimes f_2 + f_2 \otimes f_1$ for $f_1, f_2 \in U$.

For $U \circ U$, the generating set consists of the elements $u \circ 1$ for $u \in \mathcal{U}$ with $\deg u$ even. If the degree of u is even, we have

$$(u \circ 1)\theta = u \otimes 1 + 1 \otimes u = 2(u \otimes 1) + w', \tag{3.30}$$

where w' belongs to the submodule of $U \otimes U$ that is generated by the elements $v \otimes 1$ with $v \in \mathcal{U}$ and $\deg v < \deg u$. Since the elements $u \otimes 1$ as free generators of $U \otimes U$ are linearly independent over U, we shall see from (3.30) that the elements $u \circ 1$ with $\deg u$ even are also linearly independent over U, and hence, they are free generators for $U \circ U$ as a U-module.

□

Corollary 3.10 If K is a field of characteristic other than 2 and P is a free U-module with free generators $e_1, e_2, \ldots e_r$, then

(i) $P \wedge P$ is a free U-module and the elements $e_i u \wedge e_i$ with $i = 1, 2, \ldots, r$, $u \in \mathcal{U}$ and $\deg u$ odd together with the elements $e_i u \wedge e_j$ with $1 \leq i < j \leq r$, $u \in \mathcal{U}$ form a free generating set of $P \wedge P$ as a U-module,

(ii) $P \circ P$ is a free U-module and the elements $e_i u \circ e_i$ with $i = 1, 2, \ldots, r$, $u \in \mathcal{U}$ and $\deg u$ even together with the elements $e_i u \circ e_j$ with $1 \leq i < j \leq r$, $u \in \mathcal{U}$ form a free generating set of $P \circ P$ as a U-module.

Proof. Since P is a free U-module with free generators $e_1, e_2, \ldots e_r$, we have that $P \cong \bigoplus_i e_i U$, and then we get

$$\begin{aligned} P \wedge P \cong & (\bigoplus_i e_i U) \wedge (\bigoplus_i e_i U) \\ & \cong \bigoplus_i (e_i U \wedge e_i U) \oplus \bigoplus_{i<j} (e_i U \otimes e_j U). \end{aligned}$$

As we know, for all $i \in \{1, 2, \ldots, r\}$, e_iU is isomorphic to U. Hence, we obtain

$$P \wedge P \cong \bigoplus_{i}(U \wedge U) \oplus \bigoplus_{i<j}(U \otimes U).$$

By Proposition 3.9, $U \wedge U$ is a free U-module and also the tensor square $U \otimes U$ of the free U-module U is a free U-module. Therefore, the direct sum of free U-modules is a free U-module. Applying a similar argument to the proof for $P \circ P$ yields that

$$\begin{aligned} P \circ P \cong& (\bigoplus_{i} e_iU) \circ (\bigoplus_{i} e_iU) \\ \cong& \bigoplus_{i}(e_iU \circ e_iU) \oplus \bigoplus_{i<j}(e_iU \otimes e_jU) \\ \cong& \bigoplus_{i}(U \circ U) \oplus \bigoplus_{i<j}(U \otimes U). \end{aligned}$$

By Proposition 3.9, $U \circ U$ is a free U-module also the tensor square $U \otimes U$ of the free U-module U is a free U-module. Therefore, the result follows.

□

3.6 The 5-term exact sequence

Let K be a field of characteristic other than 2. Since $K/2K = 0$, the 6-term exact sequence (3.23) turns into

$$0 \to M \wedge M \to P \wedge P \to \Delta \otimes P \to U \circ U \overset{\alpha_4}{\to} U \to 0. \tag{3.31}$$

As we have known, the ring of polynomials over K, $U = K[X]$, is a graded ring and in view of the grading by degree in $U = K[X]$, all the modules in (3.31) have a natural grading. Therefore, we may say that the sequence is an exact sequence of graded modules. Moreover, we know from the previous section that all the U-modules to the right of $M \wedge M$ are free U-modules. In this section we focus on the structure of the U-module $M \wedge M$. To begin, we consider that the sequence (3.31) decomposes into short exact sequences

$$0 \to M \wedge M \to P \wedge P \to P \wedge P/M \wedge M \to 0 \tag{3.32}$$

$$\|$$

$$0 \to P \wedge P/M \wedge M \to \Delta \otimes P \to \Delta \circ \Delta \to 0 \tag{3.33}$$

$$\|$$

$$0 \to \Delta \circ \Delta \to U \circ U \stackrel{\alpha_4}{\to} U \to 0 \tag{3.34}$$

In these sequences, by Corollary 3.10 and Proposition 3.9, $P \wedge P$, $U \circ U$ and U are free U-modules. Since every free module is projective, these modules are projective U-modules. Beginning with the third exact sequence (3.34), in this sequence

$$0 \to \Delta \circ \Delta \to U \circ U \stackrel{\alpha_4}{\to} U \to 0,$$

U is projective, this means that there exists a homomorphism $\beta : U \to U \circ U$ such that $\beta\alpha_4 = 1_U$. The map $\beta : U \to U \circ U$ is a splitting map for the short exact sequence (3.34) and then we must have that $U \circ U$ is isomorphic to the direct sum of $(\Delta \circ \Delta)$ and U, that is,

$$U \circ U \cong (\Delta \circ \Delta) \oplus U.$$

Since $U \circ U$ is free U-module, $U \circ U$ is projective. Hence, this direct sum is projective, by Theorem 2.3, each of direct summands is a projective U-module. As a result, $\Delta \circ \Delta$ is a projective module. By tensoring the direct sum $(\Delta \circ \Delta) \oplus U$ with K, we get

$$((\Delta \circ \Delta) \oplus U) \otimes_U K \cong ((\Delta \circ \Delta) \otimes_U K) \oplus (U \otimes_U K).$$

Consequently,

$$(U \circ U) \otimes_U K \cong ((\Delta \circ \Delta) \otimes_U K) \oplus (U \otimes_U K).$$

It follows that the sequence

$$0 \to (\Delta \circ \Delta) \otimes_U K \to (U \circ U) \otimes_U K \to U \otimes_U K \to 0 \tag{3.35}$$

is exact.

Now we will return to the diagram for sequence (3.33). Similarly, in view of $\Delta \circ \Delta$ being a projective U-module, in (3.33) we have that $\Delta \otimes P$ is isomorphic to the direct sum of $\Delta \circ \Delta$ and $P \wedge P/M \wedge M$, and $P \wedge P/M \wedge M$ is also a projective U-module. By tensoring the direct sum $(\Delta \circ \Delta) \oplus (P \wedge P/M \wedge M)$ with K, we obtain

$$((\Delta \circ \Delta) \oplus (P \wedge P/M \wedge M)) \otimes_U K \cong ((\Delta \circ \Delta) \otimes_U K) \oplus ((P \wedge P/M \wedge M) \otimes_U K).$$

Hence, tensoring the sequence (3.33) yields the exact sequence

$$0 \to (P \wedge P/M \wedge M) \otimes_U K \to (\Delta \otimes P) \otimes_U K \to (\Delta \circ \Delta) \otimes_U K \to 0. \tag{3.36}$$

By applying the same process to the first sequence (3.32) in the diagram

$$0 \to M \wedge M \to P \wedge P \to P \wedge P/M \wedge M \to 0,$$

we shall see that $M \wedge M$ is projective and also by tensoring with K we get

$$0 \to (M \wedge M) \otimes_U K \to (P \wedge P) \otimes_U K \to (P \wedge P/M \wedge M) \otimes_U K \to 0. \tag{3.37}$$

Combining the exact sequences (3.35), (3.36) and (3.37) yields the following exact sequence

$$0 \to (M \wedge M) \otimes_U K \to (P \wedge P) \otimes_U K \to (\Delta \otimes P) \otimes_U K \to (U \circ U) \otimes_U K \to U \otimes_U K \to 0. \tag{3.38}$$

In other words, the exact sequence (3.31) stays exact after tensoring with K. In addition the modules in this exact sequence become vector spaces. This shows that the sequence (3.38) is an exact sequence of graded K-spaces, and it can be used to work out the dimensions of the homogeneous components of $(M \wedge M) \otimes_U K$.

We are now ready to calculate the dimensions of the homogeneous components and fine homogeneous components of $(M \wedge M) \otimes_U K$.

3.7 Dimensions

Our aim in this section is to derive formulae for the dimensions of the homogeneous components and fine homogeneous components of $(M \wedge M) \otimes_U K$. In the previous section we have proved the exactness of the sequence (3.38) of vector spaces over a field K of characteristic other than 2. In view the exactness of (3.38), we have

$$\begin{aligned}\dim((M \wedge M) \otimes_U K)_* - \dim((P \wedge P) \otimes_U K)_* + \dim((\Delta \otimes P) \otimes_U K)_* \\ - \dim((U \circ U) \otimes_U K)_* + \dim(U \otimes_U K)_* = 0.\end{aligned}$$

Thus, the dimension of the homogeneous component $((M \wedge M) \otimes_U K)_*$ is

$$\begin{aligned}\dim((M \wedge M) \otimes_U K)_* = &\dim((P \wedge P) \otimes_U K)_* - \dim((\Delta \otimes P) \otimes_U K)_* \\ &+ \dim((U \circ U) \otimes_U K)_* - \dim(U \otimes_U K)_*. \qquad (3.39)\end{aligned}$$

It remains to work out the terms on the right hand side of (3.39). Recall that if $\mathcal{F}$ is a homogeneous free generating set for a graded U-module F, then $\{f \otimes 1 : f \in \mathcal{F}\}$ is a basis for the tensor product $F \otimes_U K$ as a K-space (see Section 1.1.7 in [1]). Hence, in order to determine the dimensions on the right hand side of (3.39), we need to count the number of generators of free module for a given degree or multidegree. This is carried out in the following three lemmas.

First of all, since U is a free U-module with free generator 1, the term $U \otimes_U K$ is a K-space of dimension one with free generator $(1 \otimes 1)$ of degree 0. Hence, this term does not contribute to homogeneous components of degree other than 0. For the other three terms on the right hand side of (3.39), the modules $(P \wedge P)$, $(\Delta \otimes P)$

and $(U \circ U)$ are free and their free generating sets have been obtained in Section 5 of this chapter.

Lemma 3.11 Let K be a field of characteristic other than 2, then the following holds.

(i) If $n \geq 3$ is odd, then

$$\dim((P \wedge P) \otimes_U K)_n = \binom{r+1}{2}\binom{n+r-3}{n-2}.$$

Moreover, if $q \models n$ is a composition of n in r parts such that k of the parts are non-zero and m of the parts are 1, then

$$\dim((P \wedge P) \otimes_U K)_q = \binom{k}{2} + k - m.$$

(ii) If $n \geq 2$ is even, then

$$\dim((P \wedge P) \otimes_U K)_n = \binom{r}{2}\binom{n+r-3}{n-2}.$$

Moreover, if $q \models n$ is a composition of n in r parts such that k of the parts are non-zero and m of the parts are 1, then

$$\dim((P \wedge P) \otimes_U K)_q = \binom{k}{2}.$$

Proof. If n is odd, Corollary 3.10 implies that the elements $e_i u \wedge e_j$ for $1 \leq i \leq j \leq r$, where $u \in \mathcal{U}$ with $\deg u = n-2$ form a basis of $P \wedge P$. Hence, a basis of $((P \wedge P) \otimes_U K)_n$ consists of the elements $(e_i u \wedge e_j) \otimes 1$. To count these elements, it is sufficient to count the elements $e_i u \wedge e_j$ for $1 \leq i \leq j \leq r$, where $u \in \mathcal{U}$ with $\deg u = n-2$. By (2.2), the number of monomials of degree $(n-2)$ in r variables is $\binom{n+r-3}{n-2}$. As to possible choices of e_i, e_j, there are $\binom{r}{2}$ with $e_i \neq e_j$, and r with $e_i = e_j$. Hence, the number of possible pairs e_i, e_j with $i \leq j$ is equal to

$$\binom{r}{2} + r = \frac{1}{2}r(r-1) + r = \binom{r+1}{2}.$$

Hence, we get the dimension formula for $((P \wedge P) \otimes_U K)_n$ as follows:

$$\dim((P \wedge P) \otimes_U K)_n = \binom{r+1}{2}\binom{n+r-3}{n-2}.$$

In order to determine the dimension of $((P \wedge P) \otimes_U K)_q$, we need to count the number of basis elements of multidegree q. For each possible choice of the pair e_i, e_j in $e_i u \wedge e_j$, there is precisely one such basis element of multidegree q. By (2.2), the number of possible choices of the pair e_i, e_j is $\binom{k}{2}$ with $e_i \neq e_j$, and $k - m$ with $e_i = e_j$. Namely, the number of possible pairs e_i, e_j with $i \leq j$ is equal to

$$\binom{k}{2} + k - m.$$

This yields the dimension formula for $((P \wedge P) \otimes_U K)_q$.

If n is even, Corollary 3.10 implies that the elements $e_i u \wedge e_j$ for $1 \leq i < j \leq r$ where $u \in \mathcal{U}$ with $\deg u = n-2$ form a basis of $P \wedge P$ and so the elements $(e_i u \wedge e_j) \otimes 1$ for $1 \leq i < j \leq r$ where $u \in \mathcal{U}$ with $\deg u = n-2$ form a basis of $((P \wedge P) \otimes_U K)_n$. The number of monomials of degree $(n-2)$ in r variables is $\binom{n+r-3}{n-2}$ and the number of possible pairs e_i, e_j is $\binom{r}{2}$ with $e_i < e_j$. Therefore, the number of the elements $e_i u \wedge e_j$ for $1 \leq i < j \leq r$, where $u \in \mathcal{U}$ with $\deg u = n-2$ is equal to

$$\binom{r}{2}\binom{n+r-3}{n-2}.$$

This gives the dimension formula for $((P \wedge P) \otimes_U K)_n$.

To determine the dimension of $((P \wedge P) \otimes_U K)_q$, we need to count the number of basis elements of multidegree q. For each possible choice of the pair e_i, e_j in $e_i u \wedge e_j$, there is precisely one such basis element of multidegree q. The number of possible pairs e_i, e_j with $i < j$ is $\binom{k}{2}$. This gives the corresponding dimension formula for $((P \wedge P) \otimes_U K)_q$.

□

Lemma 3.12 For any field K and for all $n \geq 2$,

$$\dim((\Delta \otimes P) \otimes_U K)_n = r\binom{n+r-2}{n-1}.$$

Moreover, if $q \models n$ is a composition of n in r parts such that k of the parts are non-zero, then

$$\dim((\Delta \otimes P) \otimes_U K)_q = k.$$

Proof. Corollary 3.8 implies that the elements $u \otimes e_i$ with $i = 1, 2, \ldots, r$ and $u \in \mathcal{U}$, where $\deg u = n - 1$ form a basis of $\Delta \otimes P$. The number of monomials of degree $(n-1)$ in r variables is $\binom{n+r-2}{n-1}$ and the number of possible choice e_i is r. Therefore, the number of elements $u \otimes e_i$ is equal to

$$\binom{n+r-2}{n-1} r.$$

This gives the dimension formula for $((\Delta \otimes P) \otimes_U K)_n$.

For such basis elements of multidegree q, the number of possible e_i is k and the number of monomials of multidegree q is one. This yields the dimension formula for $((\Delta \otimes P) \otimes_U K)_q$.

□

Lemma 3.13 Let K be a field of characteristic other than 2, then the following holds.

(i) If $n \geq 1$ is odd, then

$$((U \circ U) \otimes_U K)_n = \{0\}.$$

(ii) If $n \geq 2$ is even, then

$$\dim((U \circ U) \otimes_U K)_n = \binom{n+r-1}{n}.$$

Moreover, if $q \models n$ is a composition of n in r parts, then

$$\dim((U \circ U) \otimes_U K)_q = 1.$$

Proof. By a similar argument to those used in the proofs of the previous two lemmas, the results follow from Proposition 3.9. According to Proposition 3.9, the elements $u \circ 1$ with $u \in \mathcal{U}$ where $\deg u = n$ even form a basis of $U \circ U$. The number of monomials of degree n in r variables is $\binom{n+r-1}{n}$. This yields the dimension formula for $((U \circ U) \otimes_U K)_n$.

There is one such basis element of multidegree q.

□

We can now state the main result of this section, which gives the formulae for the dimensions of the homogeneous components and fine homogeneous components of the second derived algebra G''.

Theorem 3.14 *Let G be the free centre-by-metabelian Lie algebra of rank $r > 1$ over a field K of characteristic other than 2. Then the dimensions of the homogeneous components and the fine homogeneous components of the second derived algebra G'' are as follows:*

(i) If $n \geq 5$ is odd, then

$$\dim(G''_n) = \frac{1}{2} r(n-3) \binom{n+r-3}{n-1}.$$

Moreover, if $q \models n$ is a composition of n in r parts such that k of the parts are non-zero and m of the parts are 1, then

$$\dim(G''_q) = \binom{k}{2} - m.$$

(ii) If $n \geq 6$ is even, then

$$\dim(G''_n) = \binom{n-1}{2} \binom{n+r-3}{n}.$$

Moreover, if $q \models n$ is a composition of n in r parts such that k of the parts are non-zero, then

$$\dim(G''_q) = \binom{k-1}{2}.$$

Proof. To prove this theorem, we first use the isomorphism (3.7)

$$G'' \cong (M \wedge M) \otimes_U K.$$

Then we substitute the formulae in the previous three lemmas into the formula (3.39)

$$\begin{aligned}\dim((M \wedge M) \otimes_U K)_* = &\dim((P \wedge P) \otimes_U K)_* - \dim((\Delta \otimes P) \otimes_U K)_* \\ &+ \dim((U \circ U) \otimes_U K)_* - \dim(U \otimes_U K)_*.\end{aligned}$$

(i) Suppose that n is odd and greater than 4. By Lemma 3.7, Lemma 3.11 and Lemma 3.12, the formula for the dimension of the homogeneous component of $((M \wedge M) \otimes_U K)_n$ is

$$\dim((M \wedge M) \otimes_U K)_n = \binom{r+1}{2}\binom{n+r-3}{n-2} - r\binom{n+r-2}{n-1}.$$

Recall that for a and b are non-negative integers, the value of the binomial coefficient $\binom{a}{b}$ is calculated by the formula

$$\binom{a}{b} = \frac{a!}{b!(a-b)!}.$$

By using the formula of the binomial coefficient and doing some elementary calculations, we have

$$\begin{aligned}\dim((M\wedge M)\otimes_U K)_n =&\binom{r+1}{2}\binom{n+r-3}{n-2} - r\binom{n+r-2}{n-1}\\ =&\frac{(r+1)r}{2}\frac{(n+r-3)!}{(r-1)!(n-2)!} - r\frac{(n+r-2)!}{(r-1)!(n-1)!}\\ =&\frac{(r+1)r(n+r-3)!(n-1) - 2(n+r-2)(n+r-3)!}{2(r-1)!(n-1)!}\\ =&\frac{r(n+r-3)!}{2(r-1)!(n-1)!}\Big((r+1)(n-1) - 2(n+r-2)\Big)\\ =&\frac{r(n+r-3)!}{2(r-1)!(n-1)!}(rn-r+n-1-2n-2r+4)\\ =&\frac{r(n+r-3)!}{2(r-1)!(n-1)!}(rn-n-3r+3)\\ =&\frac{r(n+r-3)!}{2(r-1)!(n-1)!}\Big(n(r-1)-3(r-1)\Big)\\ =&\frac{r(n+r-3)!}{2(r-1)!(n-1)!}(r-1)(n-3)\\ =&\frac{1}{2}r(n-3)\cdot\frac{(n+r-3)!}{(r-2)!(n-1)!}\\ =&\frac{1}{2}r(n-3)\binom{n+r-3}{n-1}.\end{aligned}$$

By the isomorphism (3.7),

$$\dim(G''_n) = \frac{1}{2}r(n-3)\binom{n+r-3}{n-1}.$$

Let $q \models n$ be a composition of n in r parts. By Lemma 3.7, Lemma 3.11 and Lemma 3.12, the formula for the dimension of the fine homogeneous component of $((M\wedge M)\otimes_U K)_q$ is

$$\begin{aligned}\dim((M\wedge M)\otimes_U K)_q =&\binom{k}{2} + k - m - k\\ =&\binom{k}{2} - m.\end{aligned}$$

Namely,

$$\dim(G_q'') = \binom{k}{2} - m.$$

(ii) Suppose that n is even and greater than 5. We use a similar method to that used in the proof of (i).

By substituting the formulae Lemma 3.7, Lemma 3.11 and Lemma 3.12 into the formula (3.39) for the dimension of $((M \wedge M) \otimes_U K)_n$, we get

$$\dim((M \wedge M) \otimes_U K)_n = \binom{r}{2}\binom{n+r-3}{n-2} - r\binom{n+r-2}{n-1} + \binom{n+r-1}{n}.$$

By doing some elementary calculations, we obtain

$$\begin{aligned}
&\binom{r}{2}\binom{n+r-3}{n-2} - r\binom{n+r-2}{n-1} + \binom{n+r-1}{n} \\
=&\frac{1}{2}(r-1)r\frac{(n+r-3)!}{(r-1)!(n-2)!} - r\frac{(n+r-2)!}{(r-1)!(n-1)!} + \frac{(n+r-1)!}{(r-1)!n!} \\
=&\frac{1}{2}(r-1)r\frac{(n+r-3)!}{(r-1)!(n-2)!} - r\frac{(n+r-2)(n+r-3)!}{(r-1)!(n-1)(n-2)!} \\
&+ \frac{(n+r-1)(n+r-2)(n+r-3)!}{(r-1)!n(n-1)(n-2)!} \\
=&\frac{(n+r-3)!}{2(r-1)!n!}\Big(r(r-1)n(n-1) - 2rn(n+r-2) + 2(n+r-1)(n+r-2)\Big) \\
=&\frac{(n+r-3)!}{2(r-1)!n!}(r^2n^2 - 3r^2n - 3rn^2 + 9nr + 2n^2 + 2r^2 - 6n - 6r + 4) \\
=&\frac{(n+r-3)!}{2(r-1)!n!}\Big(r^2(n^2-3n+2) - 3r(n^2-3n+2) + 2(n^2-3n+2)\Big) \\
=&\frac{(n+r-3)!}{2(r-1)!n!}\Big((r^2-3r+2)(n^2-3n+2)\Big) \\
=&\frac{(n+r-3)!}{2(r-1)!n!}\Big((n-1)(n-2)(r-2)(r-1)\Big) \\
=&\binom{n-1}{2}\frac{(n+r-3)!}{n!(r-3)!} \\
=&\binom{n-1}{2}\binom{n+r-3}{n}.
\end{aligned}$$

For a composition q of n, we have

$$\begin{aligned}\dim((M \wedge M) \otimes_U K)_q =& \dim((P \wedge P) \otimes_U K)_q - \dim((\Delta \otimes P) \otimes_U K)_q \\ & + \dim((U \circ U) \otimes_U K)_q - \dim(U \otimes_U K)_q \\ =& \binom{k}{2} - k + 1 \\ =& \frac{1}{2}k(k-1) - (k-1) \\ =& \frac{1}{2}(k-1)(k-2) \\ =& \binom{k-1}{2}.\end{aligned}$$

□

3.8 The Basis Theorem

In this section we return to the free centre-by-metabelian Lie ring, that is $K = \mathbb{Z}$. We will give the main theorem here, but before this, we need to explain the following lemma.

Lemma 3.15 Let $X = \{x_1, x_2, \ldots, x_r\}$ with $r \geq 2$ and let n be a positive integer with $n \geq 5$.

(i) The number of Kuz'min elements of degree n with entries in X is

$$k(n, r) = \frac{1}{2}r(n-3)\binom{n+r-3}{n-1}.$$

(ii) If $n = r$, the number of multilinear Kuz'min elements of degree n, that is Kuz'min elements of multidegree $(1, 1, \ldots, 1)$ with entries in X, is

$$\tilde{k}(n) = \frac{1}{2}n(n-3).$$

Proof. For part (i) we use induction on r. The assertion is true for $r = 2$ as the only Kuz'min elements in this case are of the form $[x_2, x_1] \wedge_* [x_2, x_1]x_1^i x_2^{n-4-i}$ for $i = 0, 1, 2, \ldots, n-4$. Hence, there are $(n-3)$ of them, which is the required number.

Now let $r > 2$. By induction, the number of Kuz'min polynomials of degree n in X that do not involve x_1 is $k(n, r-1)$. To this we need to add the number of Kuz'min polynomials that do involve x_1. If x_1 is present, we must have that $y_4 = x_1$. Hence, these polynomials are of the form

$$[y_1, y_2] \wedge_* [y_3, x_1]y_5 \ldots y_n \tag{3.40}$$

with

$$y_1 \geq y_3 > x_1,\ x_1 \leq y_2 \leq y_5 \leq \ldots \leq y_n \tag{3.41}$$

and

$$y_1 > y_2. \tag{3.42}$$

First we count the polynomials satisfying the condition (3.41). In these polynomials (y_1, y_3) can be any pair of elements in $X \backslash \{x_1\}$ with $y_1 \geq y_3$. The number of such pairs is

$$\frac{1}{2}(r-1)r.$$

The entries $y_2, y_5, \ldots, y_n$ can be any elements of X with $y_2 \leq y_5 \leq \ldots \leq y_n$. The number of such sequences of $(n-3)$ elements is $\binom{n+r-4}{n-3}$. Hence, the number of polynomials (3.40) satisfying the conditions (3.41) is

$$\frac{1}{2}(r-1)r\binom{n+r-4}{n-3}. \tag{3.43}$$

In order to find the number of Kuz'min polynomials involving x_1, we need to subtract from (3.43) the number of polynomials satisfying the conditions (3.41) but not (3.42). These are precisely the polynomials (3.40) where the entries $y_1, y_2, y_3, y_5, \ldots, y_n$ satisfy the condition $x_1 < y_3 \leq y_1 \leq y_2 \leq y_5 \leq \ldots \leq y_n$. The number of such sequences of $(n-1)$ elements is $\binom{n+r-3}{n-1}$. Thus the number of Kuz'min polynomials involving x_1 is

$$\frac{1}{2}(r-1)r\binom{n+r-4}{n-3} - \binom{n+r-3}{n-1}. \tag{3.44}$$

Now we get the total number of Kuz'min polynomials in X by adding (3.44) to $k(n, r-1)$ as follows:

$$\begin{aligned}
&k(n,r)\\
=&k(n-1,r) + \frac{1}{2}(r-1)r\binom{n+r-4}{n-3} - \binom{n+r-3}{n-1}\\
=&\frac{(r-1)(n-3)}{2}\binom{n+r-4}{n-1} + \frac{1}{2}(r-1)r\binom{n+r-4}{n-3} - \binom{n+r-3}{n-1}\\
=&\frac{(r-1)(n-3)}{2}\frac{(n+r-4)!}{(r-3)!(n-1)!} + \frac{1}{2}(r-1)r\frac{(n+r-4)!}{(r-1)!(n-3)!} - \frac{(n+r-3)!}{(r-2)!(n-1)!}\\
=&\frac{(r-1)(n-3)}{2}\frac{(n+r-4)!}{(r-3)!(n-1)(n-2)(n-3)!}\\
&+\frac{(r-1)r}{2}\frac{(n+r-4)!}{(r-1)(r-2)(r-3)!(n-3)!}\\
&-\frac{(n+r-3)(n+r-4)!}{(r-2)(r-3)!(n-1)(n-2)(n-3)!}\\
=&\frac{(n+r-4)!(r-1)}{2(r-1)!(n-1)!}\Big((n-3)(r^2-3r+2) + r(n^2-3n+2) - 2(n+r-3)\Big)\\
=&\frac{(n+r-4)!}{2(r-2)!(n-1)!}(r^2n - 3r^2 - 3rn + 9r + 2n - 6 + rn^2 - 3nr + 2r - 2n - 2r + 6)\\
=&\frac{(n+r-4)!}{2(r-2)!(n-1)!}(nr^2 - 6rn - 3r^2 + 9r + n^2r)\\
=&\frac{(n+r-4)!}{2(r-2)!(n-1)!}\Big(r(n-3)(r+n-3)\Big)\\
=&\frac{(n+r-3)!}{(r-2)!(n-1)!}\frac{r(n-3)}{2}\\
=&\frac{1}{2}r(n-3)\binom{n+r-3}{n-1}.
\end{aligned}$$

This elementary calculation shows that this is equal to the number given in the part (i) of the lemma.

For part (ii), observe first that if an element of the form (3.16) is Kuz'min and of multidegree $(1, 1, \ldots, 1)$, then we must have that $y_4 = x_1$, and either $y_2 = x_2$ or $y_3 = x_2$.

Case 1. We assume that $y_4 = x_1$ and $y_2 = x_2$. In this case the element is of the form

$$[y_1, x_2] \wedge_* [y_3, x_1]y_5 \ldots y_n.$$

Such elements are Kuz'min if and only if $y_1 > y_3$, and there are precisely $\binom{n-2}{2}$ such elements of multidegree $(1, 1, \ldots, 1)$.

Case 2. In this case we assume that $y_4 = x_1$ and $y_3 = x_2$. We must have $y_2 = x_3$, so the element is of the form

$$[y_1, x_3] \wedge_* [x_2, x_1]y_5 \ldots y_n.$$

All multilinear elements of this form are Kuz'min elements, and hence, there are $(n - 3)$ of them.

Thus, putting all this altogether gives the desired expression

$$\tilde{k}(n) = \binom{n-2}{2} + n - 3 = \frac{1}{2}n(n-3).$$

This completes the proof of the lemma.

□

Now we are ready for our main theorem. We find it convenient to state it in the terms of Lie rings. In this context the relevant elements are Lie monomials in X of the form

$$[y_1, y_2] \wedge_* [y_3, y_4]y_5 \ldots y_n = [[y_1, y_2], [y_3, y_4, y_5, \ldots, y_n]] \tag{3.45}$$

for all $y_i \in X$ with $i = 1, 2, \ldots, n$. These correspond under the isomorphism (3.7) to the Kuz'min elements and t-elements introduced in Section 1 and Section 4 of this chapter. Recall that Kuz'min elements are Lie monomials of the form (3.45) such that

$$y_1 > y_2,\ y_3 > y_4,\ y_1 \geq y_3,\ y_4 \leq y_2 \leq y_5 \leq \ldots \leq y_n,$$

and t-elements are defined as

$$w(y_1, y_2, y_3, y_4; y_5 \ldots y_n) = [y_1, y_2] \wedge_* [y_3, y_4] y_5 \ldots y_n + [y_2, y_3] \wedge_* [y_1, y_4] y_5 \ldots y_n + [y_3, y_1] \wedge_* [y_2, y_4] y_5 \ldots y_n$$

for all $y_i \in X$ with $i = 1, 2, \ldots, n$.

Theorem 3.16 *Let G be the free centre-by-metabelian Lie ring of rank $r > 1$ on a free generating set $X = \{x_1, x_2, \ldots, x_r\}$, let $q = (q_1, q_2, \ldots, q_r) \models n$ be a composition of $n \geq 5$, and let G''_q denote the fine homogeneous component of multidegree q of the second derived ideal $G'' \subseteq G$.*

(i) Suppose that $q = (q_1, q_2, \ldots, q_r) \models n$ is multilinear with $q_i = 1$ for $i = i_1, i_2, \ldots, i_n$, where $1 \leq i_1 \leq \ldots \leq i_n \leq r$. Then,

(a) if n is odd, G''_q is generated by the Kuz'min elements of multidegree q and the t-element $w(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}; x_{i_5} \ldots x_{i_n})$. The former freely generate a free abelian group of rank $\frac{1}{2}n(n-3)$ and the latter generates a cyclic group of order at most 2,

(b) if n is even, then G''_q is a free abelian group of rank $\binom{n-1}{2}$, and the Kuz'min elements of multidegree q together with the element $[x_{i_3}, x_{i_2}] \wedge_* [x_{i_4}, x_{i_1}] x_{i_5} x_{i_6} \ldots x_{i_n}$ form a free generating set for it.

(ii) Suppose that $q = (q_1, q_2, \ldots, q_r) \models n$ is a composition of n with k non-zero parts such that $q_i \geq 2$ for some i with $1 \leq i \leq n$, and m of the parts of q are 1. Then

(a) if n is odd, G''_q is a free abelian group of rank $\binom{k}{2} - m$, and the elements (3.45) of multidegree q with $y_2 = y_4 = x_i$, $y_1 \geq y_3$ and $y_1, y_3 \neq x_i$, form a free generating set for it,

(b) if n is even, then G''_q is a direct sum of a free abelian group of rank $\binom{k-1}{2}$ that is freely generated by the elements (3.45) of multidegree q with $y_2 = y_4 = x_i$, $y_1 > y_3$ and $y_1, y_3 \neq x_i$, and an elementary abelian 2-group generated by the elements (3.45) of multidegree q such that $y_2 = y_4 = x_i$, $y_1 = y_3 \neq x_i$. If all parts of q are even, then all of the latter elements are zero, and the torsion subgroup of G''_q is trivial.

Proof. In each of the cases under consideration we will show that G''_q has a generating set (as a $\mathbb{Z}$-module) such that some of its elements generate a (possibly trivial) elementary abelian 2-group, while the remaining elements freely generate a free $\mathbb{Z}$-module, i.e. the decomposition $G''_q = F_q \oplus T_q$ will be established separately in each of these cases. Since fine homogeneous components corresponding to equivalent compositions are isomorphic (see Section 2), it is sufficient to prove the part (i) in the case where $r = n$ and $q = (1, 1, \ldots, 1)$. Then, by Lemma 3.3, the Kuz'min elements of multidegree q together with the element $h = [x_3, x_2] \wedge_* [x_4, x_1]x_5x_6 \ldots x_n$ form a generating set of G''_q as a $\mathbb{Z}$-module. Assume that n is odd. Consider the t-element

$$
\begin{aligned}
&w(x_1, x_2, x_3, x_4; x_5x_6 \ldots x_n) \\
&\quad = [x_1, x_2] \wedge_* [x_3, x_4]x_5x_6 \ldots x_n + [x_2, x_3] \wedge_* [x_1, x_4]x_5x_6 \ldots x_n \\
&\qquad + [x_3, x_1] \wedge_* [x_2, x_4]x_5x_6 \ldots x_n.
\end{aligned}
$$

The second summand on the right hand side is equal to h, and both of the first and third summands on the right hand side are (up to sign) equal to Kuz'min elements. It is easy to show that by using (3.11), (3.13) and (3.15)

$$
\begin{aligned}
[x_1, x_2] \wedge_* [x_3, x_4]x_5x_6 \ldots x_n &= [x_2, x_1] \wedge_* [x_4, x_3]x_5x_6 \ldots x_n \\
&= -[x_4, x_3]x_5x_6 \ldots x_n \wedge_* [x_2, x_1] \\
&= -(-1)^n [x_4, x_3] \wedge_* [x_2, x_1]x_5x_6 \ldots x_n \\
&= [x_4, x_3] \wedge_* [x_2, x_1]x_5x_6 \ldots x_n
\end{aligned}
$$

and

$$\begin{aligned}[x_3,x_1]\wedge_*[x_2,x_4]x_5x_6\dots x_n &= -[x_3,x_1]\wedge_*[x_4,x_2]x_5x_6\dots x_n\\ &= [x_4,x_2]x_5x_6\dots x_n\wedge_*[x_3,x_1]\\ &= (-1)^n[x_4,x_2]\wedge_*[x_3,x_1]x_5x_6\dots x_n\\ &= -[x_4,x_2]\wedge_*[x_3,x_1]x_5x_6\dots x_n.\end{aligned}$$

It follows that the element h in our generating set for G''_q can be replaced by the t-element $w(x_1,x_2,x_3,x_4;x_5x_6\dots x_n)$. In other words, the multilinear fine homogeneous component G''_q is generated by the Kuz'min elements of multidegree q and the single t-element $w(x_1,x_2,x_3,x_4;x_5x_6\dots x_n)$. By Corollary 3.5, this t-element is a torsion element that is annihilated by 2.

On the other hand, by Lemma 3.15 (ii), the number of Kuz'min elements of multidegree q is $\frac{1}{2}n(n-3)$. By Theorem 3.14 (i), applied to the case where $r=n=m$,

$$\binom{m}{2}-m=\frac{1}{2}m(m-1)-m=\frac{1}{2}m(m-3),$$

this is exactly the dimension of $G''_q\otimes K$, where K is a field of characteristic other than 2. It follows that the Kuz'min elements of multidegree q in G''_q freely generate a free abelian group of rank $\frac{1}{2}n(n-3)$. Hence, G''_q is direct sum of this free abelian group and the torsion subgroup generated by w. This completes the proof of (i.a).

Assume that n is even. By Lemma 3.3, the Kuz'min elements of multidegree q together with the element $h=[x_3,x_2]\wedge_*[x_4,x_1]x_5x_6\dots x_n$ form a generating set of G''_q as a $\mathbb{Z}$-module. By Lemma 3.15 (ii), the number of Kuz'min elements of multidegree q is $\frac{1}{2}n(n-3)$. Now we get the number of such elements in our generating set for G''_q by adding this number to 1:

$$\tilde{k}(n)+1=\frac{1}{2}n(n-3)+1=\binom{n-1}{2}.$$

By Theorem 3.14 (ii), applied to the case $r=n=k$, this is precisely the dimension of $G''_q\otimes K$, where K is a field of characteristic other than 2. This means that the

Kuz'min elements of multidegree q together with the element h in G''_q freely generate a free abelian group F_q of rank $\binom{n-1}{2}$. Since the free abelian group F_q is generated by all elements in our generating set G''_q, the torsion subgroup T_q of G''_q is equal to zero. Therefore, the fine homogeneous component G''_q coincides with the free abelian group F_q. This completes the proof of (i.b).

It now remains to prove the part (ii) of the theorem for the case where $q = (q_1, q_2, \ldots, q_r) \models n$ with k non-zero parts, and m of the non-zero parts of q are greater than 1. By Lemma 3.2, the elements (3.45) of multidegree q with $y_2 = y_4 = x_i$, $y_1 \geq y_3$ and $y_1, y_3 \neq x_i$, form a generating set of G''_q. The number of such elements is $\binom{k}{2} - m$.

If n is odd, this is precisely the dimension of $G''_q \otimes K$, where K is a field of characteristic other than 2 (see Theorem 3.14 (i)). It follows that these elements freely generate a free abelian group of rank $\binom{k}{2} - m$ in G''_q. There is no torsion part in this fine homogeneous component.

If n is even, then we split the elements (3.45) of multidegree q with $y_2 = y_4 = x_i$, $y_1 \geq y_3$ and $y_1, y_3 \neq x_i$ into the disjoint union of those with $y_1 = y_3$ and those with $y_1 > y_3$. The former generate an elementary abelian 2-group. Indeed, in view of (3.15), translated into the setting of G'', we have

$$\begin{aligned}[y_1, x_i] \wedge_* [y_1, x_i] y_5 y_6 \ldots y_n &= (-1)^n [y_1, x_i] y_5 y_6 \ldots y_n \wedge_* [y_1, x_i] \ (n \textit{ is even}) \\ &= [y_1, x_i] y_5 y_6 \ldots y_n \wedge_* [y_1, x_i] \ (\textit{anti-commutativity}) \\ &= -[y_1, x_i] \wedge_* [y_1, x_i] y_5 y_6 \ldots y_n,\end{aligned}$$

and hence,

$$2([y_1, x_i] \wedge_* [y_1, x_i] y_5 y_6 \ldots y_n) = 0.$$

So the elements (3.45) with $y_2 = y_4 = x_i$ and $y_1 = y_3$ generate a torsion group. If all free generators involved in such an element occur with even multiplicity, the images of such elements in $(M \wedge M) \otimes_U \mathbb{Z}$ are of the form $[y_1, x_i] \wedge_* [y_1, x_i](z_1^2 \ldots z_{(n-4)/2}^2)$ for some $z_1, \ldots, z_{(n-4)/2} \in X$. But then, by using (3.14),

$$[y_1, x_i] \wedge_* [y_1, x_i](z_1^2 \ldots z_{(n-4)/2}^2) =$$
$$\pm [y_1, x_i](z_1 \ldots z_{(n-4)/2}) \wedge_* [y_1, x_i](z_1 \ldots z_{(n-4)/2}) = 0,$$

and hence, we can delete these elements from our generating set. The number of elements with $y_1 > y_3$ among the elements (3.45) is $\binom{k-1}{2}$. By Theorem 3.14 (ii), this is exactly the dimension of $G''_q \otimes K$, where K is a field of characteristic other than 2. It follows that these elements in our generating set freely generate a free $\mathbb{Z}$-module of rank $\binom{k-1}{2}$. This completes the proof of the theorem.

□

3.8.1 Consequences of the main theorem

Our theorem asserts that each of the homogeneous components G''_n for $n \geq 5$ is a direct sum of a free abelian group and a (possibly trivial) elementary abelian 2-group, that is the direct decomposition (1.1)

$$G''_n = F_n \oplus T_n$$

which has been mentioned in Chapter 1.

Corollary 3.17 For each $n \geq 5$, G''_n is a direct sum of a free abelian group F_n and a (possibly trivial) elementary abelian 2-group T_n. The rank of F_n is equal to the dimension of $G''_n \otimes K$, where K is a field of characteristic other than two. A formula for this dimension is given in Theorem 3.14.

□

For certain compositions q, the generating sets described in part (ii) of the main theorem turn out to be empty, and then the corresponding parts of G''_q are zero. For example, for the composition $q = (4, 1)$ of degree 5, $G''_{(4,1)}$ is zero. For the composition

$q = (3, 1, 1, 1)$ of degree 6, $G''_{(3,1,1,1)}$ is torsion-free, namely $T_{(3,1,1,1)} = 0$. Since the elements of generating set for $F_{(3,1,1,1)}$ are $[x_3, x_1] \wedge_* [x_2, x_1]x_1x_4$, $[x_4, x_1] \wedge_* [x_3, x_1]x_1x_2$, $[x_4, x_1] \wedge_* [x_2, x_1]x_1x_3$, the elements of generating set for $G''_{(3,1,1,1)}$ are $[x_3, x_1] \wedge_* [x_2, x_1]x_1x_4$, $[x_4, x_1] \wedge_* [x_3, x_1]x_1x_2$ and $[x_4, x_1] \wedge_* [x_2, x_1]x_1x_3$. Hence, $G''_{(3,1,1,1)} = F_{(3,1,1,1)}$. Another example is $G''_{(3,3)}$. Its torsion part $T_{(3,3)}$ is generated by the element $[x_2, x_1] \wedge_* [x_2, x_1]x_1x_2$. Hence, $G''_{(3,3)}$ is a torsion group.

According to Kuz'min's paper [20], the t-element in part (i.a) is a non-trivial element of order 2. This is confirmed in [11], which in its turn relies on [[10], Lemma 3.8] for the crucial fact that this element is not zero. In fact, in [10] the relevant part of Lemma 3.8 is attributed to Hurley (unpublished). According to Zerck's paper [45], the torsion elements described in part (ii.b) are not only non-zero, but form a basis of T_q as a $\mathbb{Z}/2\mathbb{Z}$-module. However, no proof is given, and instead the author says that this can be established by the methods used in [20]. The latter is hardly satisfactory since we know that some of the arguments in [20] are flawed. However, the torsion part of G'' is beyond the scope of this work. We focus exclusively on the determination of the ranks and the construction of explicit $\mathbb{Z}$-bases of the free abelian groups F_q. Our results contradict what is claimed in the parts 3) and 4) for even degree of Theorem 4 in [20].

For instance, for $r = 3$ and $n = 6$, Kuz'min's theorem asserts that the elements $[[x_3, x_2], [x_3, x_1, x_2, x_3]]$ and $[[x_3, x_2], [x_2, x_1, x_3, x_3]]$ are linearly independent over $\mathbb{Z}$ in $G''_{(1,2,3)}$, while Theorem 3.16 (ii.b) says that $G''_{(1,2,3)}$ modulo its torsion subgroup is an infinite cyclic group. In other words,

$$[x_3, x_2] \wedge_* [x_3, x_2]x_1x_3 = [x_3, x_2] \wedge_* [x_3, x_1]x_2x_3 - [x_3, x_2] \wedge_* [x_2, x_1]x_3x_3.$$

Our results are in keeping with the parts 1) and 2) for odd degree of Theorem 4 in [20]. Moreover, we can confirm the assertions of this theorem regarding the free abelian part of G''_n for odd n.

Theorem 3.18 *(Parts 1 and 2 of Theorem 4, [20]) For G as in Theorem 3.16, and* *all odd $n \geq 5$, the homogeneous component G''_n is a direct sum*

$$G''_n = F_n \oplus T_n,$$

where F_n is a free abelian group that is freely generated by Kuz'min elements (3.45) of degree n and T_n is an elementary abelian 2-group generated by the multilinear t-elements $w(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}; x_{i_5} \ldots x_{i_n})$ with $x_{i_1}, x_{i_2}, \ldots, x_{i_n} \in X$ and $i_1 < i_2 < \ldots < i_n$.

Proof. Let n be with $n \geq 5$. It is sufficient to prove the theorem for the case where $n = r$. In the proof of Theorem 3.16 (i.a) we have seen that the multilinear component $G''_{(1,1,\ldots,1)}$ is generated by the multilinear Kuz'min elements and the t-element $w(x_1, x_2, x_3, x_4; x_5 \ldots x_n)$ of degree n. Since n is odd, Corollary 3.5 tells us that the t-element has order at most 2. Any other fine homogeneous component G''_q of total degree n is a homomorphic image of the multilinear one under a suitable endomorphism of the form π_f. Moreover, this endomorphism can be chosen in such a way that it preserves the order of the free generators. In this case, the images of Kuz'min elements under π_f will be either Kuz'min elements themselves or zero. Moreover, every Kuz'min element of multidegree q is a homomorphic image of a multilinear Kuz'min element. Furthermore, the image of the multilinear t-element will be a t-element. But the latter will have at least two equal entries, and hence, by Corollary 3.5, it will be zero. It follows that for odd $n \geq 5$ every fine homogeneous component G''_q with $q \models n$ and $q \neq (1, 1, \ldots, 1)$ is generated by the Kuz'min elements multidegree q. By Lemma 3.15 (i) and Theorem 3.16 (i), the total number of Kuz'min elements of degree n is equal to the dimension of $G''_n \otimes K$, where K is an arbitrary field of characteristic other than 2. It follows that the Kuz'min elements of odd degree are linearly independent over $\mathbb{Z}$, and freely generate a free abelian group F_n.

□

Our results imply that the generating sets for the free abelian part of the lower central quotients $\gamma_n(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G})$ of the free centre-by-metabelian group $\mathfrak{G}$ are optimal. We use the notation introduced in Chapter 1.

Corollary 3.19 The generating sets for $\mathfrak{F}_n$, the free abelian part of the group

$$\mathfrak{G}''_n = (\gamma_n(\mathfrak{G}) \cap \mathfrak{G}'')\gamma_{n+1}(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G})$$

with $n \geq 5$ given in ([11], Theorem 1 and Theorem 4) are optimal, i.e. linearly independent over $\mathbb{Z}$.

Proof. By ([21], Theorem 1), we have that $G''_n \otimes \mathbb{Q} \cong \mathfrak{G}''_n \otimes \mathbb{Q}$, where $\mathbb{Q}$ is the field of rationals. In view of this, the result for odd degree n follows immediately from Theorem 3.16 since the generating set in Theorem 1 of [11] is exactly the set of Kuz'min commutators. For even degree n, it is not hard to verify that the number of commutators of a given multidegree $q \models n$ in the generating sets in Theorem 4 of [11] is equal to dimension of $G''_q \otimes \mathbb{Q}$ as given in Theorem 3.14. The result follows.

□

3.9 The torsion subgroup of G'' over $\mathbb{Z}$

Our main theorem, Theorem 3.16, does not address the question of whether or not the torsion subgroups featuring in parts (i.a) and (ii.b) are actually non-trivial. This has been answered affirmatively in the recent paper [19]. The authors determined the dimensions of the fine homogeneous components of G'' over a field of characteristic 2.

Theorem 3.20 *(L.G. Kovács and R. Stöhr, [19]) Let G be the free centre-by-metabelian Lie algebra of rank $r > 1$ over a field K of characteristic 2, and let $q \models n$ with $n \geq 5$ be a composition of n in r parts such that k of the parts are non-zero and m of the parts are equal to 1.*

(i) If q is multilinear, then

$$\dim(G''_q) = \binom{k-1}{2}.$$

(ii) If at least one of the parts of q is greater than 1, then

$$\dim(G''_q) = \begin{cases} \binom{k-1}{2}, & \text{if all parts of } q \text{ are even;} \\ \binom{k}{2} - m, & \text{otherwise.} \end{cases}$$

□

By combining this with our results in Chapter 3, we obtain a complete description of the additive structure of the second derived ideal in the free centre-by-metabelian Lie ring.

Theorem 3.21 *(L.G. Kovács and R. Stöhr, [19]) Let G be the free centre-by-metabelian Lie ring of rank $r > 1$ on a free generating set $X = \{x_1, x_2, \dots, x_r\}$, let $q = (q_1, q_2, \dots, q_r) \models n$ be a composition of $n \geq 5$, and let G''_q denote the fine homogeneous component of multidegree q of the second derived ideal $G'' \subseteq G$.*

(i) Suppose that $q = (q_1, q_2, \dots, q_r) \models n$ is multilinear with $q_i = 1$ for $i = i_1, i_2, \dots, i_n$, where $1 \leq i_1 \leq \cdots \leq i_n \leq r$. Then,

(a) if n is odd, G''_q is generated by the Kuz'min elements of multidegree q and the t-element $w(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}; x_{i_5} \dots x_{i_n})$. The former freely generate a free abelian group of rank $\frac{1}{2}n(n-3)$ and the latter generates a cyclic group of order 2,

(b) if n is even, then G''_q is a free abelian group of rank $\binom{n-1}{2}$, and the Kuz'min elements of multidegree q together with the element $[x_{i_3}, x_{i_2}] \wedge_* [x_{i_4}, x_{i_1}] x_{i_5} x_{i_6} \dots x_{i_n}$ form a free generating set for it.

(ii) Suppose that $q = (q_1, q_2, \dots, q_r) \models n$ is a composition of n with k non-zero parts such that $q_i \geq 2$ for some i with $1 \leq i \leq n$, and m of the parts of q are 1. Then

(a) if n is odd, G''_q is a free abelian group of rank $\binom{k}{2} - m$, and the elements (3.45) of multidegree q with $y_2 = y_4 = x_i$, $y_1 \geq y_3$ and $y_1, y_3 \neq x_i$, form a free generating set for it,

(b) if n is even, then G''_q is a direct sum of a free abelian group of rank $\binom{k-1}{2}$ that is freely generated by the elements (3.45) of multidegree q with $y_2 = y_4 = x_i$, $y_1 > y_3$ and $y_1, y_3 \neq x_i$, and an elementary abelian 2-group. If at least one of the parts of q is odd, then this 2-group is of rank $k - 1 - m$, and it is freely generated,

as a $\mathbb{Z}/2\mathbb{Z}$-module, by the elements (3.45) of multidegree q such that $y_2 = y_4 = x_i$, $y_1 = y_3 \neq x_i$. If all parts of q are even, then the torsion subgroup of G''_q is zero.

Proof. Most of this theorem was established in Theorem 3.16 in Chapter 3. We proved that the torsion subgroup T_q is generated by the t-element $w(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}; x_{i_5} \dots x_{i_n})$ in part (i.a) of Theorem 3.16 in Chapter 3. The only open question remaining was whether or not this element is zero. Now we can give an answer for this question. If we reduce G''_q modulo 2, then by Theorem 3.20 in [19], we obtain an elementary abelian 2-group of rank

$$\binom{n-1}{2} = \frac{1}{2}n(n-3) + 1.$$

We already know that by Theorem 3.16, F_q is a free abelian group of rank $\frac{1}{2}n(n-3)$. Therefore, the torsion subgroup must be non-trivial.

Moreover, L.G. Kovács and R. Stöhr proved that the elements in part (ii.b) of Theorem 3.16 are non-trivial and linearly independent over $\mathbb{Z}/2\mathbb{Z}$. By using Theorem 3.20, we have an elementary abelian 2-group of rank $\binom{k}{2} - m$. Since by Theorem 3.16 G''_q/T_q is a free abelian group of rank $\binom{k-1}{2}$, the torsion subgroup T_q must be non-trivial and its rank is $(k-1-m)$. This number is equal to the number of elements (3.45) such that $y_2 = y_4 = x_i$, $y_1 = y_3 \neq x_i$. Consequently, these elements are independent over $\mathbb{Z}/2\mathbb{Z}$.

□

3.10 A Direct Decomposition

The decomposition of each homogeneous component and fine homogeneous component of G'' into a direct sum of a free abelian group and an elementary abelian 2-group was a by-product of the proof of Theorem 3.16. In conclusion we give a short direct proof of this result. This is actually a special case of a far more general result proved in Zerck's paper [45]. However, since this paper is not easily accessible, and since our proof is short and does not require the full force of the arguments used in [45], we felt it is justified to include it for completeness.

Theorem 3.22 *The second derived ideal G'' of the free centre-by-metabelian Lie ring G of rank $r \geq 2$ is a direct sum of a free abelian group and an elementary abelian 2-group.*

Proof. In view of the isomorphism (3.7), that is $(M \wedge M) \otimes_U \mathbb{Z} \cong G''$, it is sufficient to prove the result for the tensor product $(M \wedge M) \otimes_U \mathbb{Z}$. For the exterior square $M \wedge M$, there is an embedding $\nu : M \wedge M \to M \otimes M$ given by $m_1 \wedge m_2 \mapsto m_1 \otimes m_2 - m_2 \otimes m_1$ for $m_1, m_2 \in M$. Moreover, there is the epimorphism $\pi : M \otimes M \to M \wedge M$ given by $m_1 \otimes m_2 \mapsto m_1 \wedge m_2$. The composite $\nu\pi$

$$M \wedge M \xrightarrow{\nu} M \otimes M \xrightarrow{\pi} M \wedge M$$

amounts to multiplication by 2 on $M \wedge M$. After tensoring this sequence by $\mathbb{Z}$, it is easy to see that the composite $\nu\pi \otimes 1$,

$$(M \wedge M) \otimes_U \mathbb{Z} \xrightarrow{\nu \otimes 1} (M \otimes M) \otimes_U \mathbb{Z} \xrightarrow{\pi \otimes 1} (M \wedge M) \otimes_U \mathbb{Z},$$

too amounts to multiplication by 2 on $(M \wedge M) \otimes_U \mathbb{Z}$. Indeed, for $m_1, m_2 \in M$, the image of $(m_1 \wedge m_2) \otimes 1$ under $\nu \otimes 1$ is $(m_1 \otimes m_2) \otimes 1 - (m_2 \otimes m_1) \otimes 1$ and then this image of $(m_1 \otimes m_2) \otimes 1 - (m_2 \otimes m_1) \otimes 1$ under $\pi \otimes 1$ is equal to

$$\begin{aligned}(m_1 \wedge m_2) \otimes 1 - (m_2 \wedge m_1) \otimes 1 =& (m_1 \wedge m_2) \otimes 1 + (m_1 \wedge m_2) \otimes 1 \\ =& 2((m_1 \wedge m_2) \otimes 1).\end{aligned}$$

It follows that the kernel of $\nu\pi \otimes 1$ is annihilated by 2, i.e. it is an elementary abelian 2-group.

We claim that the tensor product $(M \otimes M) \otimes_U \mathbb{Z}$ is a free abelian group. Once established, this will prove the theorem, as $(M \wedge M) \otimes_U \mathbb{Z}$ will be the direct sum of the kernel of $\nu\pi \otimes 1$ and the image of $\nu \otimes 1$ in $(M \otimes M) \otimes_U \mathbb{Z}$.

Tensoring the short exact sequence (3.9)

$$0 \to M \to P \to \Delta \to 0$$

with M yields an exact sequence

$$0 \to M \otimes M \to M \otimes P \to M \otimes \Delta \to 0.$$

By tensoring this sequence with $\mathbb{Z}$, we get a long exact sequence of the form

$$\begin{aligned} &\ldots \to Tor_1^U(M \otimes P, \mathbb{Z}) \to Tor_1^U(M \otimes \Delta, \mathbb{Z}) \\ &\to (M \otimes M) \otimes_U \mathbb{Z} \to (M \otimes P) \otimes_U \mathbb{Z} \to (M \otimes \Delta) \otimes_U \mathbb{Z} \to 0. \end{aligned}$$

Recall that $M \otimes P$ is a free U-module (see Corollary 3.8). Then part of the long exact sequence associated with that short exact sequence looks as follows:

$$0 \to Tor_1^U(M \otimes \Delta, \mathbb{Z}) \to (M \otimes M) \otimes_U \mathbb{Z} \to (M \otimes P) \otimes_U \mathbb{Z} \to (M \otimes \Delta) \otimes_U \mathbb{Z} \to 0.$$

Here $(M \otimes P) \otimes_U \mathbb{Z}$, the trivialization of a free U-module, is a free abelian group. But $Tor_1^U(M \otimes \Delta, \mathbb{Z})$ too is a free abelian group. Indeed, tensoring the sequence (3.9) with Δ yields an exact sequence

$$0 \to M \otimes \Delta \to P \otimes \Delta \to \Delta \otimes \Delta \to 0.$$

By Corollary 3.8, $P \otimes \Delta$ is a free U-module. Therefore we have

$$Tor_1^U(M \otimes \Delta, \mathbb{Z}) \cong Tor_2^U(\Delta \otimes \Delta, \mathbb{Z}).$$

By tensoring the short exact sequence (3.3) with Δ, we get

$$0 \to \Delta \otimes \Delta \to \Delta \otimes U \to \underbrace{\Delta \otimes \mathbb{Z}}_{\cong \Delta} \to 0.$$

Here, by Lemma 3.7, $\Delta \otimes U$ is free U-module; therefore,

$$Tor_2^U(\Delta \otimes \Delta, \mathbb{Z}) \cong Tor_3^U(\underbrace{\Delta \otimes \mathbb{Z}}_{\cong \Delta}, \mathbb{Z}).$$

Similarly, by tensoring the short exact sequence (3.3) with $\mathbb{Z}$, we get

$$0 \to \Delta \otimes \mathbb{Z} \to U \otimes \mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Z} \to 0.$$

The long exact sequence of this sequence shows that

$$Tor_3^U(\Delta, \mathbb{Z}) \cong Tor_4^U(\underbrace{\mathbb{Z} \otimes \mathbb{Z}}_{\cong \mathbb{Z}}, \mathbb{Z}).$$

Therefore, dimension shifting using these sequences gives

$$Tor_1^U(M \otimes \Delta, \mathbb{Z}) \cong Tor_4^U(\mathbb{Z}, \mathbb{Z}).$$

Furthermore, $Tor_4^U(\mathbb{Z}, \mathbb{Z})$ is a free abelian group of rank $\binom{r}{4}$. The proof of this fact can be found in Section VII.2 of [22]. Consequently, the long exact homology sequence gives that $(M \otimes M) \otimes_U \mathbb{Z}$ is free abelian, and this completes the proof of the theorem.

$\square$

Chapter 4

Products of three homogeneous components

4.1 Introduction

Let L be a free Lie algebra of finite rank over a field K and let L_n denote the degree n homogeneous component of L. Recall that the dimension of L_n is given by Witt's formula

$$\dim L_n = f(n,r) = \frac{1}{n}\sum_{d|n}\mu(d)r^{\frac{n}{d}},$$

where μ is the Möbius function (see Section 2.2.1 in Chapter 2). Formulae for the dimensions of the subspaces $[L_n, L_m]$ for all n and m were obtained by Ralph Stöhr and Michael Vaughan-Lee in [39].

Theorem 4.1 *(Ralph Stöhr and M. Vaughan Lee, [39]) If $n > m$ and $m \nmid n$, then*

$$\dim([L_n, L_m]) = \dim L_n \dim L_m. \tag{4.1}$$

If $n = sm$ with $s \geq 1$, then

$$\dim([L_n, L_m]) = (\dim L_n - f(s, \dim L_m)) \dim L_m + f(s+1, \dim L_m). \tag{4.2}$$

□

In the author's masters thesis [24] and the subsequent paper [25] were aimed at deriving similar formulae for triple products of the form $[L_n, L_m, L_k]$. Under certain conditions on n, m and k, explicit formula for the dimension of such products were obtained (see [[25], Theorem 3.1]). The most intriguing result emerging from thesis studies, however, was the observation that the dimension of a triple product $[L_n, L_m, L_k]$ depends not only on n, m and k, but also on the field K. This happens for the product $[L_2, L_2, L_1]$. For $r \geq 5$, its dimension over a field of characteristic 2 is different from the dimension over fields of other characteristic. This is an immediate consequence of results on free centre-by-metabelian Lie algebras in Chapter 3.

4.2 The Dimension of $[L_2, L_2, L_1]$

Lemma 4.2 Over any field K, let G_5'' be the degree 5 homogeneous component of the second derived ideal G''. Then

$$\dim[L_2, L_2, L_1] = \dim[L_3, L_2] - \dim G_5''.$$

Proof. Recall that the free centre-by-metabelian Lie algebra G is the quotient $L/[L'', L]$, where L'' is the second derived ideal of L. Then G is a graded algebra, and we denote its degree n homogeneous component by G_n. Here $G_n \cong L_n/(L_n \cap [L'', L])$. Moreover, the second derived ideal of G is the quotient $G'' = L''/[L'', L]$. As we have seen in Chapter 1, $G_n'' = G'' \cap G_n$. We are interested in $G_5 \cap G''$.

As we have seen in Chapter 2, the second derived ideal of L can be expressed as $[L_2, L_2] \oplus [L_3, L_2] \oplus ([L_4, L_2] + [L_3, L_3]) \oplus \ldots$. Hence, we have

$$\begin{aligned} [L'', L] =& [[L_2, L_2] \oplus [L_3, L_2] \oplus \ldots, L_1 \oplus L_2 \oplus \ldots] \\ =& [L_2, L_2, L_1] \oplus [L_3, L_2, L_1] \oplus \ldots. \end{aligned}$$

For degree 5, we have

$$\begin{aligned} G''_5 =& G_5 \cap G'' \\ \cong& (L_5/(L_5 \cap [L'', L]) \cap L''/[L'', L] \\ \cong& (L_5 \cap L'')/(L_5 \cap [L'', L]). \end{aligned}$$

Since L'' has only the subspace $[L_3, L_2]$ and $[L'', L]$ has only the subspace $[L_2, L_2, L_1]$ for degree 5, we have $L_5 \cap L'' = [L_3, L_2]$ and $L_5 \cap [L'', L] = [L_2, L_2, L_1]$. Hence,

$$G''_5 \cong [L_3, L_2]/[L_2, L_2, L_1].$$

As a result, we obtain

$$\dim G''_5 = \dim[L_3, L_2] - \dim[L_2, L_2, L_1]$$

or

$$\dim[L_2, L_2, L_1] = \dim[L_3, L_2] - \dim G''_5.$$

This completes the proof of the lemma.

□

Theorem 4.3 *Let L be the free Lie algebra of rank r over a field K. If $r \geq 5$, then the dimension of $[L_2, L_2, L_1]$ over a field of characteristic 2 is strictly less than the dimension of $[L_2, L_2, L_1]$ over a field of characteristic other than 2.*

Proof. Let $q \models 5$ be a composition of 5 in r parts such that k of the parts are non-zero and m of the parts are 1. The homogeneous component of G''_5 is the sum of the fine homogeneous components G''_q, namely,

$$G''_5 = \bigoplus_{q \models 5} G''_q.$$

Suppose that K is the field of characteristic other than 2. According to Theorem 3.14, we have

$$\dim(G''_q) = \binom{k}{2} - m.$$

If q is multilinear, namely, $m = k$,

$$\dim(G''_q) = \binom{k}{2} - k = \frac{1}{2}k(k-1) - k = \binom{k-1}{2} - 1.$$

Suppose that Char$K$=2. According to Theorem 3.20, if q is multilinear, then

$$\dim(G''_q) = \binom{k-1}{2}.$$

If at least one of the parts of q is greater than 1, then

$$\dim(G''_q) = \binom{k}{2} - m.$$

We can show the formulae of dimensions for G''_q in the following diagram:

	Char$K = 2$	Char$K \neq 2$
q multilinear	$\binom{k-1}{2}$	$\binom{k-1}{2} - \mathbf{1}$
q non-multilinear	$\binom{k}{2} - m$	$\binom{k}{2} - m$

By this diagram, it is easy to see that for q multilinear composition of 5, the dimension of G''_q over a field of characteristic 2 is more by 1 than the dimension of G''_q over a field of characteristic other than 2. Therefore, since the dimension of G''_5 is the sum of the dimensions of the fine homogeneous components G''_q, the dimension of G''_5 over a field of characteristic 2 is greater than the dimension of G''_q over a field of characteristic other than 2.

By Lemma 4.2, we have

$$\dim[L_2, L_2, L_1] = \dim[L_3, L_2] - \dim G_5''.$$

Therefore, it is clear to see that the dimension of $[L_2, L_2, L_1]$ over a field of characteristic 2 is strictly less than the dimension of $[L_2, L_2, L_1]$ over a field of characteristic other than 2.

□

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