ON THE FINE STRUCTURE OF DYNAMICALLY-DEFINED INVARIANT GRAPHS

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When considering the graphs of arbitrary continuous functions, many people tend to think of smooth functions. However the graph of a continuous function is, typically, nowhere differentiable. In this thesis we consider different families of continuous functions $W : S^1 \to \mathbb{R}$ which arise as invariant repellers for skew-product dynamical systems $F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$. Under certain non-degeneracy assumptions we see that such functions are nowhere differentiable. These functions include the original example of a continuous yet nowhere differentiable function, given by Karl Weierstrass in 1872, where $W(x) = \sum_{j=0}^{\infty} \lambda^j \cos(2\pi b^j x)$ for $\lambda \in (0, 1)$, $b > 1$ and $\lambda b > 1$. We consider more general functions and answer questions regarding the fine structure of their graphs. In particular, when such functions are nowhere differentiable it is expected that their graphs posses a 'dimension' which takes a non-integer value. We study the dimension theory of these graphs, which includes the well known notions of box dimension and Hausdorff dimension. We also study an alternative definition of dimension, known as the $\mathcal{K}$-dimension. In addition we consider singularites of such graphs, in particular what is known as a knot point. The primary tools which we use in our work come from ergodic theory and thermodynamic formalism, a subject area originally derived from statistical mechanics.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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This thesis is dedicated to the memory of Madge and Vincent Naughton.

Chapter 1

Introduction

In this thesis we study graphs of continuous functions as subsets of $\mathbb{R}^2$, which arise as so-called invariant graphs for a certain class of dynamical system. Such graphs will, typically, be continuous yet nowhere differentiable and will exhibit fractal-like behaviour. In particular we study the fine structure of these graphs, and dimension theoretic and ergodic theoretic properties using tools from thermodynamic formalism.

The study of such graphs arises from the study of invariant graphs for forced systems, which can be realised as skew-product type dynamical systems. Stark [St1, St2] studied the application of non-linear dynamics, where the driven system is contracting. Many dynamical systems of both practical and theoretical importance arise in the case where one system is driven by another. One such example of such a system in discrete time is that of a skew-product system (see Chapter 2). If the contraction in the base, or driven direction is sufficiently strong, the set invariant under the system is invariably a normally hyperbolic manifold [HiPu, HiShPu]. In particular, we consider a driving system f, and investigate systems of the form

$$(f, g) : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}.$$

In the context given by Stark [St1], g is contracting in the $\mathbb{R}$ direction, but the rate of contraction of g is bounded by that of f. Moreover, if the strongest contraction in the S^1 direction is weaker than the weakest contraction in the $\mathbb{R}$ direction, then the function ϕ, the graph of which is invariant under (f, g) is smooth. Stark [St2] studied the assumptions under which such invariant graphs exhibit regular, or smooth behaviour. Throughout this thesis, we consider similar invariant graphs, and look into the cases under which such smooth behaviour does not occur. The study of skew-product dynamical systems has attracted interest in the field of dynamical systems, due to its links with the dynamical properties of invariant graphs and the underlying topological properties of such systems. Under sufficient contraction assumptions on the dynamics we can guarantee that there exists [Hut] a unique, non-empty set which is invariant under (f, g), known as the invariant graph for the given system. Under certain regularity assumptions the geometric properties of such graphs can be studied in detail. The study of regularity has been of significant importance, and strict restrictions can be given on the cases in which regularity fails, and typically, fractal behaviour occurs. This is one of the main focuses of our work.

It was originally thought that any continuous function $f : [0, 1] \to \mathbb{R}$ must be differentiable everywhere except for a countable set of points. However in 1872 Karl Weierstrass was the first to publish an example of a continuous function which is nowhere differentiable (see Fig. 1.1). Given an odd integer $b > 1$ and $\lambda \in (0, 1)$ Weierstrass showed that the function $W : [0, 1] \to \mathbb{R}$ given by

$$W(x) = \sum_{j=0}^{\infty} \lambda^j \cos(2\pi b^j x) \tag{1.1}$$

is nowhere differentiable provided that

$$\lambda b > 1 + \frac{3}{2}\pi.$$

Hardy [Ha] in 1916 was able to relax these conditions to $\lambda b > 1$ for all $\lambda \in (0, 1)$ and $b > 1$. Continuous nowhere differentiable functions have been studied in great detail over the last century, and it is now known that ‘typical’ continuous functions are in fact nowhere differentiable. Letting V be the Banach space of all real-valued continuous functions $f : [0, 1] \to \mathbb{R}$ equipped with the supremum norm, the set of nowhere differentiable functions from $[0, 1]$ to $\mathbb{R}$ is dense in V. This is a well known application of the Baire category theorem.

We study the fine structure of graphs of a class of continuous nowhere differentiable functions, which includes the function of Weierstrass. A particular subject of interest is the dimension theory of such subsets of $\mathbb{R}^2$. Two of the most studied notions of dimension are box dimension and Hausdorff dimension (see [Fa1]). The Hausdorff

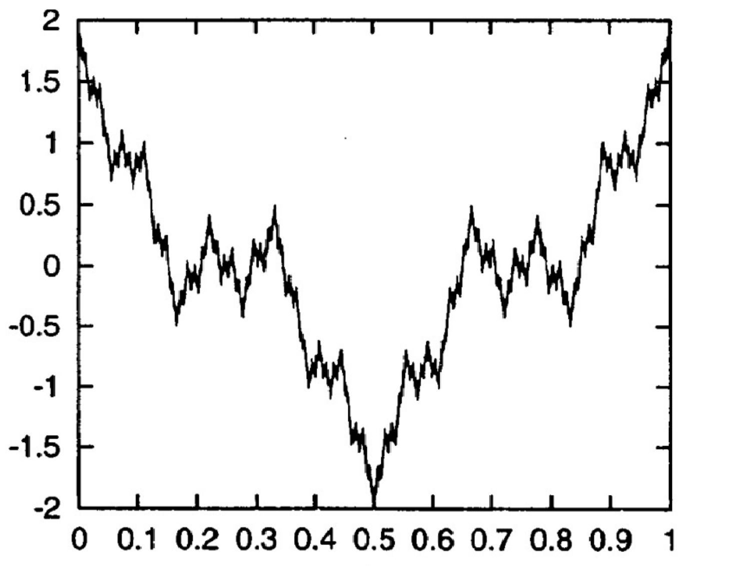


Figure 1.1: The classical Weierstrass function in the case where $\lambda = 0.6$ and $b = 2$.

dimension of subsets of $\mathbb{R}^n$ is always bounded from above by the box dimension, and it is often the case that these values coincide. However it is often far simpler both numerically and theoretically, to calculate the box dimension as opposed to the Hasudorff dimension of sets. There are also known examples [PU, McM] in which the Hausdorff dimension of the graph of a function is strictly less that its box dimension.

Although Hausdorff dimension is strictly a global quantity, the study of pointwise dimension [Yo] and multifractal formalism [Pe, PeWe2] has established a link between the local geometry of a set and global properties, including results regarding the Hausdorff dimension. Given a set $\Lambda \subset \mathbb{R}^n$ and $x \in \Lambda$ the pointwise dimension at x is given by

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}$$

where μ is a suitable probability measure. Here $B(x, r)$ denotes the n-dimensional open ball of radius r centred at x. In the study of multifractal analysis [PeWe2] it can be shown that for suitable measures μ,

$$\dim_H(\Lambda) = d_\mu(x)$$

for μ-almost every $x \in \Lambda$. Such a connection was studied in [LY1, LY2], regarding Pesin's entropy formula, singularities and dimension. We discuss local geometry and pointwise dimension in more detail in Chapter 6.

Billingsley [Bill] was one of the first to study the connection between the dimension theory of dynamical systems and ergodic theory. We consider the thermodynamic formalism, a branch of ergodic theory originally derived from statistical mechanics [Ru]. Dynamical systems possessing some form of hyperbolic behaviour have an uncountable number of invariant measures, and we are interested in choosing measures which reflect interesting and relevent dynamics. One of the main objects of study in our work is that of topological pressure [Bo, Wa]. We introduce several different characterisations of topological pressure in Chapter 2. One such characterisation [Wa] is via the variational principle. The variational principle says that given a continuous function f, the pressure of f is given by

$$P(f) = \sup_{\mu} \left\{ h(\mu) + \int f \, d\mu \right\}$$

where the supremum is taken over all invariant probability measures μ and $h(\mu)$ is the measure theoretic entropy of our dynamical system. If our system is hyperbolic and f is Hölder continuous, there exists [Bo] a unique measure which achieves this supremum, known as the equilibrium state for f.

Bowen [Bo] was the first to establish a remarkable connection between topological pressure and the Hausdorff dimension of dynamically interesting sets. For example, consider a piecewise linear map $T : I_1 \cup I_2 \to [0,1]$, where I_1 and I_2 are closed disjoint subsets of $[0,1]$ and $T(I_i) = [0,1]$ for $i = 1,2$. Such a map is known as a cookie cutter map. It can be shown that for $t > 0$,

$$P(-t \log |T'|) = \log(a_1^{-t} + a_2^{-t})$$

where P denotes topological pressure and a_1, a_2 are the absolute values of the gradients of the maps $T|_{I_1}$ and $T|_{I_2}$ respectively. Define

$$\Lambda = \bigcap_{j=0}^{\infty} T^{-i}[0,1].$$

If $\overline{I}_1 \cup \overline{I}_2 \neq [0,1]$ then the set Λ is a Cantor set. Moreover Λ is a non-empty, compact invariant repeller for T. Using properties of pressure, provided T is expanding there exists a unique solution t to the equation

$$P(-t \log |T'|) = 0.$$

Bowen shows that this solution coincides with the Hausdorff dimension of the set Λ.

Example. Suppose that T is linear and $I_1 = [0, 1/3]$, $I_2 = [2/3, 1]$. Then T has slope 3 and the invariant repeller for T is the middle third Cantor set. Moreover the Bowen equation reduces to

$$P(-t \log 3) = \log(2 \times 3^{-t}) = 0.$$

In particular the unique solution is $t = \frac{\log 2}{\log 3}$, the Hausdorff dimension of the middle third Cantor set.

Example. If one was to consider T linear and $I_1 = [0, 1/3], I_2 = [3/4, 1]$, then Bowens' equation would reduce to $\log(3^{-t} + 4^{-t}) = 0$.

Throughout this thesis we study subsets of $\mathbb{R}^2$ which arise as invariant sets of a different type of dynamical system. We identify the unit interval with the circle $S^1 = \mathbb{R}/\mathbb{Z}$. We use an equation similar to that of Bowen involving topological pressure to prove dimension theoretic results about invariant graphs. Given a continuous function $f : S^1 \to \mathbb{R}$ we study

$$\mathrm{graph}(f) = \left\{(x, f(x)) \mid x \in S^1\right\} \subset S^1 \times \mathbb{R}.$$

If f is smooth then graph(f) is a 1-dimensional manifold, however if f is nowhere-differentiable then graph(f) is typically a fractal curve.

For an integer $b \geq 2$, let $T : S^1 \to S^1$ be the map $T(x) = bx \bmod 1$, let $\lambda \in (0, 1)$ and let $p : S^1 \to \mathbb{R}$ be Lipschitz continuous. Consider the map $F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ given by

$$F(x, y) = (T(x), \lambda^{-1}(y - p(x))).$$

Such an F is known as a skew-product dynamical system. Then [Hut] there exists a function $W : S^1 \to \mathbb{R}$ whose graph is the unique, compact invariant set for F. In particular W can be written in the form

$$W(x) = \sum_{j=0}^{\infty} \lambda^j p(T^j(x)).$$

Remark. If we let $p(x) = \cos(2\pi x)$ then W takes the form of Weierstrass' original example as in (1.1).

Note that the infinite sum defining W converges uniformly, hence W is continuous. Moreover if $\lambda b > 1$ then W is nowhere differentiable. Kaplan, Mallet-Paret an Yorke proved the box dimension result for the classical Weierstrass function [KaMPYo], that is

$$\dim_B(\text{graph}(W)) = 2 + \frac{\log \lambda}{\log b} \in (1, 2).$$

This result was further generalised by Bedford and Urbanski [BeUr] to functions of the form W as above. It is widely conjectured, although as yet unproven that the Hausdorff dimension of such graphs is also equal to this value. Hunt [Hunt] has proven the Hausdorff dimension of the classical Weierstrass function with added random phases. For W as in (1.1), given $b \geq 2$ the result regarding Hausdorff dimension is known for all λ in some sufficiently small interval, and almost every λ in some larger interval [BBR].

More generally we let $T : S^1 \to S^1$ be a uniformly expanding b-to-one map of the circle for $b \geq 2$ and let $\lambda : S^1 \to (0, 1)$ be Lipschitz continuous. Letting $p : S^1 \to \mathbb{R}$ be Lipschitz we consider skew products of the form

$$F(x, y) = (T(x), \lambda(x)^{-1}(y - p(x))).$$

Then graph(W) is invariant under F where

$$W(x) = \sum_{j=0}^{\infty} \lambda(x)\lambda(T(x)) \ldots \lambda(T^{j-1}(x)))p(T^j(x)).$$

Bedford [Be] studied such skew products, and showed that under a certain non-degeneracy assumption the invariant graph for such an F is nowhere differentiable. This was studied in more detail in [HNW], where it was shown that under a certain non-degeneracy assumption such graphs are Hölder continuous of some exponent, but not Lipschitz continuous.

It is shown [Be] that under such a non-degeneracy assumption the box dimension of graph(W) is equal to the unique solution s_0 to the topological pressure equation

$$P((1 - s_0) \log |T'| + \log \lambda) = 0.$$

Moreover by the variational principle there exists a measure μ_0, namely the unique equilibrium state of $(1 - s_0) \log |T'| + \log \lambda$ such that

$$s_0 = 1 + \frac{h_{\mu_0}(T) + \int \log \lambda \, d\mu_0}{\int \log |T'| \, d\mu_0}$$

where $h_{\mu_0}(T)$ denotes the measure-theoretic entropy of T with respect to μ_0.

The fact that the Hausdorff dimension $\dim_{\mathcal{H}}(\text{graph}(W)) \leq s_0$ follows from the result for box dimension. However proving a lower bound for Hausdorff dimension is often [Fa1, Fa2] significantly more difficult. It is widely conjectured however that $\dim_{\mathcal{H}}(\text{graph}(W)) \geq s_0$ so that equality of dimension occurs. It is shown [MoWa] that if one modifies the function p by adding in a random phase shift, then the Hausdorff dimension coincides with such an s_0 as above almost surely.

An alternative definition of dimension for subsets of $\mathbb{R}^2$, known as $\mathcal{K}$-dimension is introduced in [HL]. It is shown that this value is bounded below by Hausdorff dimension and above by box dimension. Therefore it is worth studying this notion of dimension, particularly when the Hausdorff dimension is unknown. Moreover unlike box dimension, the $\mathcal{K}$-dimension is constructed via a family of measures, which act similarly to the Hausdorff measure. Hence properties such as countable additivity hold and therefore, unlike box dimension a countable set will always have dimension zero with respect to the $\mathcal{K}$-dimension.

In [HL] Weierstrass-type functions of the form

$$W(x) = \sum_{j=0}^{\infty} \lambda^j p(T^j(x))$$

are studied, where $T(x) = bx \bmod 1$ and λ is constant. It is shown [HL] that if $\lambda b > 1$ then the $\mathcal{K}$-dimension of graph(W) is equal to $2 - \alpha$ where

$$\alpha = \frac{-\log \lambda}{\log b} \in (0, 1).$$

Moreover such an α is in fact the Hölder exponent of W.

In this thesis we consider a more general skew product dynamical system, and in varying cases of generality study the dimension theory and fine geometric structure of invariant graphs. We consider $F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ where

$$F(x, y) = (T(x), g_x(y)). \tag{1.2}$$

Here T is a uniformly expanding map of the circle and for each $x \in S^1$, $g_x : \mathbb{R} \to \mathbb{R}$ is an expanding diffeomorphism of $\mathbb{R}$. Then (see Chapter 2) there exists a continuous function $W : S^1 \to \mathbb{R}$ whose graph is invariant under F. We impart a partial hyperbolicity assumption on our system. That is we require that the rate of expansion of F in the S^1 direction is strictly greater than the rate of expansion in the $\mathbb{R}$ direction.

Formally, we let $h_x = g_x^{-1}$ and let $\mathcal{D}h$ be the derivative of the map h_x in the $\mathbb{R}$ direction. That is (see Chapter 2) $|\mathcal{D}h|$ denotes the rate of contraction of the fibre map in the $\mathbb{R}$ direction. Then we assume that

$$\inf_{x\in S^1}|T'(x)|\inf_{x,y\in S^1}|\mathcal{D}h_x(y)|>1.$$

In the case as in [Be] where $g_x(y)=\lambda(x)^{-1}(y-p(x))$ it can be seen that $h_x(y)=\lambda(x)y+p(x)$. Then $|\mathcal{D}h_x(y)|=\lambda(x)$ and we assume that $\inf_{x\in S^1}\lambda(x)\inf_{x\in S^1}|T'(x)|>1$. Under a certain non-degeneracy assumption [HNW] as in Chapter 2 such invariant graphs are nowhere differentiable. We will often study such a case, where $g_x(y)=\lambda(x)^{-1}(y-p(x))$ is affine.

The content of this thesis is organised as follows; in Chapter 2 we introduce the relevant tools and definitions required throughout. We formally introduce the class of invariant graphs that we study and the necessary non-degeneracy assumptions. We also introduce symbolic dynamics and cylinder sets, which are intervals constructed via sequences of inverse branches of T. If we let $T(x)=bx \bmod 1$, then a cylinder of rank n is constructed by applying a sequence of n inverse branches of T to S^1. In particular, letting C_n denote a cylinder of rank n, $T^n(C_n)=S^1$ and $\operatorname{diam}(C_n)=b^{-n}$.

We introduce the necessary thermodynamic formalism and dimension theory, in particular the notion of $\mathcal{K}$-dimension [HL]. In addition we take $T:S^1\to S^1$ and $W:S^1\to\mathbb{R}$ and construct a natural lift to $\hat{T}:\mathbb{R}\to\mathbb{R}$ and to $\widehat{W}:\mathbb{R}\to\mathbb{R}$. Here $\hat{T}$ is invertible, and we show that $\widehat{W}$ restricted to $[0,1]$ is W itself.

In Chapter 3 we generalise results in [Be, MoWa] which use the underlying dynamics of the skew-product F to describe the behaviour of graph(W) over cylinders. We study the affine case where $g_x(y)=\lambda(x)^{-1}(y-p(x))$ as in [Be]. Given an arbitrary interval $I\subset[0,1]$, we choose n to be maximal such that $\operatorname{diam}(\hat{T}^n(I))\le 1$. Then we show that the height, or oscillation of the graph over I can be estimatedby the product of λ; the derivative of the fibre map in the $\mathbb{R}$-direction, evaluated along the orbit of points $x\in I$ under T. Formally we show that given $I\subset[0,1]$, $x\in I$ and associated $n\ge 0$ there exists a constant $C>0$ independent of I and x such that, letting $\operatorname{osc}(W,I)$ denote the height of the graph over I

$$C^{-1}\prod_{j=0}^{n-1}\lambda(T^j(x))\le\operatorname{osc}(W,I)\le C\prod_{j=0}^{n-1}\lambda(T^j(x)).$$

In Chapter 4 we consider graph(W), the invariant graph of F as in (1.2) and prove results regarding the $\mathcal{K}$-dimension of the graph. We improve upon the result in [HL] and generalise the method used by Pesin [Pe] to prove results on $\mathcal{K}$-dimension of such graphs using tools from thermodynamic formalism. One such result is the following;

Theorem 1.0.1

Let graph(W) *be invariant under the skew product F as in (1.2) where we assume that* $T(x) = bx \bmod 1$ *and* $g_x(y) = \lambda(x)^{-1}(y - p(x))$*. Under the non-degeneracy assumption (see Chapter 2)*

$$\dim_{\mathcal{K}}(\mathrm{graph}(W)) = s_0$$

where s_0 is the unique solution to the pressure equation

$$P((1 - s_0)\log b + \log \lambda) = 0.$$

Moreover there exists a measure μ_0, the unique equilibrium state for $(1-s_0)\log b + \log \lambda$*, such that*

$$\begin{aligned}\dim_{\mathcal{K}}(\mathrm{graph}(W)) &= 1 + \frac{h_{\mu_0}(T) + \int \log \lambda \, d\mu_0}{\log b} \\ &= 1 + \frac{P(\log \lambda)}{\log b}.\end{aligned}$$

Such a case includes the function of Weierstrass and gives equality of $\mathcal{K}$-dimension when we have a much more general family of mappings in the fibre direction than those studied in [HL]. We also consider the case in which T is a uniformly expanding (not necessarily linear) map, and give upper and lower bounds for $\mathcal{K}$-dimension, making remarks on when we can prove equality of dimension.

In Chapter 5 we let $T : S^1 \to S^1$ be the doubling map $T(x) = 2x \bmod 1$ and let

$$q(x) = \begin{cases} -1 & \text{if } x \in [0, 1/2) \\ 1 & \text{if } x \in [1/2, 1). \end{cases}$$

For $\lambda \in (1/2, 1)$ consider

$$f_\lambda(x) = \sum_{j=0}^{\infty} \lambda^j q(T^j(x)).$$

For certain values of λ it is shown [PU, McM] that the Hausdorff dimension of graph(f_λ) is strictly less than the box dimension of the graph. In this chapter we answer the natural question, what is the $\mathcal{K}$-dimension of such graphs?

In Chapter 6 we introduce a different class of continuous functions as studied in [Oka, McColl] which include the functions of Bourbaki [Bou] and Perkins [Per]. Okamoto studies differentiability properties from a purely analytic perspective, while McCollum studies dimension theoretic results of a one-parameter family of functions, which arise interatively. We generalise the setting in which they work and show that such graphs indeed arise as invariant graphs of a dynamical system, and use thermodynamic formalism and ergodic theory to improve upon their results. We let $T(x) = bx \bmod 1$ and choose parameters $\alpha = (0, \alpha_1, \ldots, \alpha_{b-1}, 1)$ where $\alpha_j \in (0,1)$ for all $j = 1, \ldots, b-1$. Then we let

$$F(x,y) = (T(x), g(x)y)$$

where $g(x)y$ is peicewise linear in y and depends on the choice of values in α. In particular, for $j \in \{0, \ldots, b-1\}$ we define the inverse branches of $g(x)$ to be the contraction mappings

$$g_j(x)y = (\alpha_{j+1} - \alpha_j)y + \alpha_j.$$

Given x with first coordinate in its symbolic coding x_0, define the locally constant function $\beta(x) = (\alpha_{x_0+1} - \alpha_{x_0})$. Thus $|\beta|$ measures the rate of contraction of the fibre map in the $\mathbb{R}$ direction, and we give estimates on the height of the graph in terms of the function β. We prove the following regarding Hausdorff dimension of such graphs;

Theorem 1.0.2

Let $T(x) = bx \bmod 1$ and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{b-1}, \alpha_b)$ be given where $\alpha_0 = 0$ and $\alpha_b = 1$. Let β be as above. Let Γ_α be the invariant graph for F as above. Then

$$\dim_{\mathcal{H}}(\Gamma_\alpha) = s_0$$

where s_0 is the unique solution to the pressure equation

$$P((1 - s_0)\log b + \log|\beta|) = 0.$$

Moreover

$$\dim_{\mathcal{H}}(\Gamma_\alpha) = 1 + \frac{\log \sum_{j=0}^{b-1} |\alpha_{j+1} - \alpha_j|}{\log b}.$$

This generalises the result in [McColl] who only considers the case introduced in [Oka] where $b = 3$ and $\alpha = (0, \alpha_0, 1 - \alpha_0, 1)$ for $\alpha_0 \in (0,1)$. The case in which $\alpha_0 = 2/3$ gives the function of Bourbaki.

We also investigate differentiability properties of the invariant graph, and improve upon results [Oka] for our more general class of functions. We use the Birkhoff ergodic theorem to classify the set of points of differentiability on the graph for given values of α, and then study the multifractal analysis for such ergodic averages. The use of ergodic theory in the study of such invariant graphs allows us to give considerably more detail in results regarding differentiability properties of such graphs. This gives rise to some interesting questions involving the exceptional set for ergodic averages.

Finally in Chapter 7 we investigate functions of the form

$$W(x) = \sum_{j=0}^{\infty} \lambda(x) \ldots \lambda(T^{j-1}(x)) p(T^j(x))$$

where T is a uniformly expanding map of the circle and $\lambda : S^1 \to (0,1)$ is Lipschitz continuous. We study a type of singularity and generalise the results in [HL] for Weierstrass-type functions. Given a continuous function f, we say that x is a knot-point for f if the upper right and upper left derivatives of f at x are ∞ and the lower right and lower left derivatives of f at x are $-\infty$. For example, the function $f(x) = \sin(1/x)$ has a knot-point at zero. It is shown [HL] that if $T(x) = bx \bmod 1$ and $\lambda \in (0,1)$ and $\lambda b > 1$ then the set of knot points for W contains a dense G_δ subset. We improve upon this and show the following;

Theorem 1.0.3

Let W be as above, where T is uniformly expanding (not necessarily linear) and $\lambda \in (0,1)$ is constant. Then there exists a set $G \subset [0,1]$ where

- *G is a dense G_δ subset of $[0,1]$,*
- *G has full measure with respect to any ergodic T-invariant probability measure which takes positive measure on non-empty open sets,*
- *G has Hausdorff dimension 1*

such that for all $x \in G$, x is a knot-point for W.

We also consider the case in which λ is variable and impart extra hypotheses to prove the existence of knot-points.

Chapter 2

Preliminaries

In this chapter we introduce a certain class of continuous functions and their graphs which arise as invariant sets for a class of skew-product dynamical systems. In particular we introduce the tools needed to study such graphs in both a symbolic and geometric setting. We introduce the notions of Hausdorff dimension and box dimension, and an alternative definition of dimension known as $\mathcal{K}$-dimension, first studied in [HL] in which we will be particularly interested. The $\mathcal{K}$-dimension is specifically used to study subsets of $\mathbb{R}^2$ which arise as invariant graphs for the skew-products we introduce. We also consider the thermodynamic formalism required to study the dimension theoretic properties of the graphs and some useful tools which make the study of the geometric properties of such graphs easier throughout later chapters.

2.1 Invariant graphs

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. We are interested in studying the graph of f. That is we look at the subset of $\mathbb{R}^2$ given by

$$\mathrm{graph}(f) = \{(x, f(x)) \mid x \in [0,1]\} \subset [0,1] \times \mathbb{R}.$$

In some cases it will be more convenient for us to identify the unit interval with the circle $S^1 = \mathbb{R}/\mathbb{Z}$.

In general we consider skew-product dynamical systems $F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ of the form

$$F(x, y) = (T(x), g_x(y)). \tag{2.1}$$

Here $T : S^1 \to S^1$ is a $C^{1+\gamma}$ continuous uniformly expanding b-to-one map of the circle, i.e. $\inf_{x\in S^1} |T'(x)| > 1$ and the derivative is Holder of some exponent. We consider $g_x : \mathbb{R} \to \mathbb{R}$ to be a family of continuous mappings of $\mathbb{R}$ where $x \in S^1$. We assume that g_x is a uniformly expanding diffeomorphism of $\mathbb{R}$. For each x we assume that g_x is invertible and denote the mappings $g_x^{-1}(y) = h_x(y)$ for all $y \in \mathbb{R}$. That is we take $h_x : \mathbb{R} \to \mathbb{R}$ such that $h_x \circ g_x(y) = g_x \circ h_x(y) = y$ for all $y \in \mathbb{R}$. In addition since we assume that g_x is a uniformly expanding map of $\mathbb{R}$, h_x is uniformly contracting.

Then [Hut] there exists a function W, such that graph$(W) \subset S^1\times\mathbb{R}$ is compact and invariant under F, that is $F(\text{graph}(W)) = \text{graph}(W)$. Equivalently W is a solution to the functional equation

$$W(T(x)) = g_x(W(x)) \tag{2.2}$$

or alternatively

$$W(x) = h_x(W(T(x))) \tag{2.3}$$

as shown below.

$$\begin{array}{ccc} W(x) & \xrightarrow{\;F\;} & W(T(x)) \\ {\scriptstyle W}\big\uparrow & & \big\uparrow{\scriptstyle W} \\ x & \xrightarrow[F]{} & T(x) \end{array}$$

We express the iterates of F as $F^n(x, y) = (T^n(x), g_x^{(n)}(y))$ where $g_x^{(n)}(y)$ is expressed as a composition of operators acting on $\mathbb{R}$. That is

$$g_x^{(n)}(y) = g_{T^{n-1}(x)} \circ \ldots \circ g_{T(x)} \circ g_x(y).$$

We define

$$h_x^{(n)}(y) = h_x \circ h_{T(x)} \circ \ldots \circ h_{T^{n-1}(x)}(y)$$

so that for all $y \in \mathbb{R}$ it holds that $h_x^{(n)} \circ g_x^{(n)}(y) = y$. By the functional equation (2.2), in order for graph(W) to be invariant under F we require that for all $x \in S^1$

$$\begin{aligned} F(x, W(x)) &= (T(x), W(T(x))) \\ &= (T(x), g_x(W(x))). \end{aligned}$$

In particular we require that for all n

$$W(T^n(x)) = g_x^{(n)}(W(x)).$$

That is

$$W(x) = h_x^{(n)}(W(T^n(x)))$$

for all $n \geq 1$. We can explicitly give a formula for the function W. That is

$$W(x) = \lim_{n\to\infty} h_x^{(n)}(y). \tag{2.4}$$

Such a limit exists and is continuous. To see this let $a_n = h_x^{(n)}(y)$. Consider

$$\begin{aligned}
|a_{n+1} - a_n| &\leq h_x \circ \ldots \circ h_{T^{n-1}(x)} |h_{T^n(x)}(y) - y| \\
&\leq \sup_{x' \in S^1} |h'_{x'}||x| \ldots \sup_{x' \in S^1} |h'_{x'}||T^{n-1}(x)|\, |h_{T^n(x)} - 1|(y) \\
&\leq \left[\sup_{x' \in S^1} |h'_{x'}|\right]^n |x| \ldots |T^{n-1}(x)|\, |y|
\end{aligned}$$

where $y \in \mathbb{R}$. Since $|x|, \ldots, |T^{n-1}(x)| \in S^1$ and since h is a contracting mapping there exists $C_0 > 0$ and $\rho \in (0,1)$ such that

$$|a_{n+1} - a_n| \leq C_0 \rho^n$$

for all $n \geq 1$. Moreover given $m \geq n$,

$$\begin{aligned}
|a_m - a_n| &\leq |a_{n+1} - a_n| + |a_{n+2} - a_{n+1}| \ldots + |a_m - a_{m-1}| \\
&\leq C_0(\rho^n + \rho^{n+1} + \ldots + \rho^m) \\
&= C_0 \sum_{j=n}^{m} \rho^j \\
&= C_0 \frac{\rho^n - \rho^{m+1}}{1-\rho} \\
&\leq C_0 \frac{\rho^n}{1-\rho}.
\end{aligned}$$

Let $\epsilon > 0$ be given. Let $N \geq 1$ be such that $C_0 \frac{\rho^N}{1-\rho} < \epsilon$. Then for all $n \leq m$ such that $n \geq N$, by the above $|a_m - a_n| < \epsilon$. Therefore the sequence a_n is uniformly Cauchy, hence converges uniformly to the limit W, which is therefore continuous.

Moreover for such an invariant graph, given $x \in S^1$, $(x, W(x)) \in \text{graph}(W)$ and letting $y \in \mathbb{R}$

$$\begin{aligned} F(x, W(x)) &= F(x, \lim_{n\to\infty} h_x^{(n)}(y)) \\ &= (T(x), g(x) \circ \lim_{n\to\infty} h_x^{(n)}(y)) \\ &= (T(x), \lim_{n\to\infty} h_{T(x)}^{(n)}(y)) \\ &= (T(x), W(T(x))) \in \mathrm{graph}(W). \end{aligned}$$

That is W, is in fact independent of the choice of $y \in \mathbb{R}$.

Suppose we consider a special case of the above dynamical system. Let $\lambda \in (0,1)$ and let $b \in \mathbb{N}$, $b \geq 2$. Then the invariant set for

$$F(x, y) = (bx \bmod 1, \lambda^{-1}(y - \cos(2\pi x)))$$

is the graph of the well known Weierstrass function

$$W(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x).$$

This sum converges uniformly hence W is a continuous function. Moreover if $\lambda b > 1$ then W is nowhere differentiable [Ha].

Throughout this thesis we consider the case where the fibre map is affine. That is we let T be a $C^{1+\gamma}$ continuous uniformly expanding b-to-one map of the circle (see next section) and let $\lambda : S^1 \to (0,1)$ be a Lipschitz continuous function. Letting $p : S^1 \to \mathbb{R}$ be a Lipschitz continuous function we consider the skew-product $F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ where

$$F(x, y) = (T(x), \lambda(x)^{-1}(y - p(x))). \tag{2.5}$$

That is, we consider $g_x(y) = \lambda(x)^{-1}(y - p(x))$ and it is clear that $h_x(y) = \lambda(x)y + p(x)$ so that $g(x) \circ h(x)y = y$ for all $x \in S^1$ and all $y \in \mathbb{R}$. Then one can show that given $x \in S^1, y \in \mathbb{R}$ and $n \geq 1$,

$$h_x^{(n)}(y) = \lambda^n(x)y + \sum_{j=0}^{n-1} \lambda^j(x) p(T^j(x))$$

where $\lambda^j(x) = \prod_{i=0}^{j-1} \lambda(T^i(x))$ and we use the convention that $\lambda^0(x) = 1$. Therefore the invariant graph for such a system is the graph of the function

$$W(x) = \lim_{n\to\infty} h_x^{(n)}(y)$$

which, since $\lambda : S^1 \to (0,1)$ takes values in $(0,1)$ is independent of y and

$$W(x) = \sum_{j=0}^{\infty} \lambda^j(x) p(T^j(x)).$$

Moreover

$$\begin{aligned} |W(x)| &\leq \sum_{j=0}^{\infty} \lambda^j(x) |p(T^j(x))| \\ &\leq |p|_\infty \sum_{j=0}^{\infty} \left(\sup_{x \in S^1} \lambda(x) \right)^j \\ &= |p|_\infty \frac{1}{1 - \sup_{x \in S^1} \lambda(x)} < \infty. \end{aligned}$$

That is, the infinite sum W converges uniformly, hence W is a continuous function.

Under certain non-degeneracy assumptions and partial hyperbolicity conditions which we will see later, such a function is, generically everywhere continuous yet nowhere differentiable. We shall study such generalised Weierstrass type functions in detail throughout, in the above case where $g_x(y) = \lambda(x)^{-1}(y - p(x))$. Some results can be generalised to the the non-affine case, however we only work in the affine case for notational convenience.

2.2 Expanding maps and symbolic dynamics

Let $T : S^1 \to S^1$ be a continuous uniformly expanding b-to-one $C^{1+\gamma}$ map of the circle for $\gamma \in (0,1)$. In particular the derivative T' is Hölder continuous of some exponent $\gamma \in (0,1)$. We assume that there exists $\rho > 1$ such that $\rho \leq |T'(x)| < \infty$ for all $x \in S^1$.

There exists a partition of S^1 into intervals $I_j = [a_j, a_{j+1}]$ for $0 \leq j \leq b-1$ where, if necessary the intervals are taken modulo 1 such that the restriction $T|_{I_j} : I_j \to S^1$ is a homomorphism of I_j and a diffeomorphism on the interior of I_j. The intervals I_j in fact form a Markov partition of S^1. Let $T_j : S^1 \to I_j$ for $0 \leq j \leq b-1$ denote the inverse branches of T. For example let $b = 2$ and consider T to be a uniformly expanding map with 2 branches, a more general case of the well known doubling map given by $T(x) = 2x$ mod 1. However the slopes could vary, as long as the derivative of T has absolute value bounded away from 1.

Note also that we only assume that $|T'(x)| > 1$ for all $x \in S^1$. That is, we work in the general case in which T can be orientation preserving or reversing. However, without loss of generality we assume that the map is orientation preserving. If this were not true we could replace T with $T \circ T = T^2$ and we are in the orientation preserving case. In Chapter 7 we shall assume the case in which T is orientation preserving, however for the work in earlier chapters, it is sufficient to assume that the absolute values of the slopes are bounded away from 1 and ∞.

Let $\Sigma = \{0, 1, \ldots, b-1\}^{\mathbb{N}}$ be the full one-sided shift on b symbols with associated shift map $\sigma : \Sigma \to \Sigma$. Let $\mathbf{x} \in \Sigma$ and $\mathbf{x} = (x_0, \ldots, x_{n-1}, \ldots)$. Then define the projection $\chi : \Sigma \to S^1$ by $\chi(\mathbf{x}) = \bigcap_{n \geq 0} T_{x_0,\ldots,x_{n-1}}(S^1)$. The inverse branches of T are strict contractions therefore $\chi(\mathbf{x})$ is a singleton for all $\mathbf{x} \in \Sigma$. The map χ is surjective, and is one-to-one except for sequences which end in an infinite tail of zeros or $(b-1)$s.

One can also see that $\chi \circ \sigma = T \circ \chi$, that is T and σ are topologically conjugate. We use the above construction in the following way. Let $\mathbf{x} \in \Sigma$ and $n \geq 1$ be given such that $\mathbf{x} = (x_0, \ldots, x_{n-1}, \ldots)$. Then letting $\chi(\mathbf{x}) = x \in S^1$, we apply the corresponding sequence of inverse branches to the circle as follows;

$$C_n(x) = T_{x_0} \circ \ldots \circ T_{x_{n-1}}(S^1) = T_{x_0,\ldots,x_{n-1}}(S^1).$$

We say that $C_n(x)$ is a cylinder of rank n containing x. One may find it useful to use this symbolic setting in the case when we are given $x \in S^1$ and $n \geq 0$. Then $\chi^{-1}x \in \Sigma$ and this determines what sequence of inverse branches are needed to construct a cylinder of rank n that contains x. In particular given a point $x \in S^1$ we can code this point using the map T, and we say that x has symbolic coding $(x_0, x_1, \ldots)$, where $x_j \in \{0, 1, \ldots, b-1\}$ for all $j \geq 0$.

We also introduce the following notation; given a continuous function f

$$\mathrm{m}(f) = \inf_{x \in S^1} |f(x)| \tag{2.6}$$

and

$$\sup_{x \in S^1} |f(x)| = |f|_\infty. \tag{2.7}$$

Then $\mathrm{m}(T') > 1$ and $|T'|_\infty < \infty$.

Recall that the mean value theorem says that given $f : \mathbb{R} \to \mathbb{R}$ continuous, and $x, y \in \mathbb{R}$ such that $x < y$, there exists $c \in (x, y)$ such that $(y - x)f'(c) = f(y) - f(x)$.

By the chain rule

$$\begin{aligned}\sup_{z\in S^1}|(T^n)'(z)| &= \sup_{z\in S^1}|T'(T^{n-1}(z))T'(T^{n-2}(z))\dots T'(z)| \\ &\leq |T'|_\infty^n.\end{aligned}$$

Therefore

$$\begin{aligned}|T^n(x)-T^n(y)| &\leq \sup_{z\in S^1}|(T^n)'(z)||x-y| \\ &\leq (\sup_{z\in S^1}|T'(z)|)^n|x-y| \\ &= |T'|_\infty^n|x-y|.\end{aligned}$$

Similarly

$$\begin{aligned}|T^n(x)-T^n(y)| &\geq \inf_{z\in S^1}|(T^n)'(z)||x-y| \\ &\geq (\mathrm{m}(T'))^n|x-y|.\end{aligned}$$

2.3 A non-degeneracy and partial hyperbolicity assumption

Throughout we work in the case where $g_x(y)=\lambda(x)^{-1}(y-p(x))$ where $\lambda : S^1 \to (0,1)$ is Lipschitz, and $p : S^1 \to \mathbb{R}$ is Lipschitz. Consider the partial derivative of g_x in the fibre direction. For convenience of notation we denote this

$$\mathcal{D}g_x(y)=\lim_{\epsilon\to 0}\frac{g_x(y+\epsilon)-g_x(y)}{\epsilon}=\lambda(x)^{-1}.$$

We assume that F as in (2.1) is a partially hyperbolic system [Pesin]. That is, we shall always require that the fibre map $g_x(y)$ is an expanding diffeomorphism of $\mathbb{R}$, but that this expansion is bounded above by that of T. Equivalently, we require that

$$1<\sup_{x\in S^1}\lambda(x)^{-1}<\inf_{x\in S^1}|T'(x)|\,.$$

Letting the inverse operator $g_x^{-1} : \mathbb{R}\to\mathbb{R}$ be denoted by h_x we look at the rate of contraction in the $\mathbb{R}$ direction of the fibre map. We shall denote this rate of contraction as

$$\mathcal{D}h_x(y)=\lim_{\epsilon\to 0}\frac{h_x(y+\epsilon)-h_x(y)}{\epsilon}$$

throughout and it can be seen that $\mathcal{D}h_x(y) = \lambda(x)$. Therefore the partial hyperbolicity assumption becomes

$$\inf_{x\in S^1} \lambda(x) \inf_{x\in S^1} |T'(x)| > 1. \tag{2.8}$$

We saw earlier that the function W such that graph(W) is invariant under F converges uniformly, hence is continuous. If a continuous function f is differentiable then graph(f) is a 1-dimensional manifold. However by construction of F and W, graph(W) is typically a fractal curve. For this to happen [HNW] we require an assumption.

Non-Degeneracy Assumption

To assume non-degeneracy we assume that there exists no Lipschitz function $U(x)$ not equal to W which satisfies the equation

$$\lambda(x)U(T(x)) = U(x) - p(x). \tag{2.9}$$

If there exists some Lipschitz continuous solution U to the above equation then $W = U$ and the graph of W is as smooth as U. Generically this is not the case [HNW]. If there exists no such smooth solution to equation (2.9) then W is nowhere differentiable. We shall always consider the case in which this non-degeneracy assumption holds. For such a condition to hold, it is equivalent [HNW] to saying that W is Hölder continuous of some exponent $\alpha \in (0, 1)$, but no better. In particular, there exists a constant $\beta > 0$ such that for all $x, y \in S^1$

$$|W(x) - W(y)| \leq \beta|x - y|^\alpha,$$

but W is not Hölder continuous for any exponent $\gamma > \alpha$. In particular W is not Lipschitz continuous. This will prove important in later work.

2.4 Dimension of subsets of $\mathbb{R}^n$

In the following section we introduce two of the more well-known notions of dimension of subsets of $\mathbb{R}^n$ which will be used throughout (for more detail see [Fa1, Fa2]).

2.4.1 Hausdorff dimension

Suppose that U is some non-empty subset of $\mathbb{R}^n$. We define the diameter of U to be $\text{diam}(U) = \sup\{|x - y| : x, y \in U\}$. We say that a collection of subsets $\{U_i\}$ is a

δ-cover of $E \subset \mathbb{R}^n$ if $E \subset \bigcup_{i=1}^{\infty} U_i$ and $0 < \mathrm{diam}(U_i) \leq \delta$ for all i. Suppose that $s \geq 0$ and $E \subset \mathbb{R}^n$. We define for any $\delta > 0$

$$\mathcal{H}^s_\delta(E) = \inf_{\{U_i\}} \sum_{i=1}^{\infty} \mathrm{diam}(U_i)^s$$

where the infimum is taken over all δ-covers of E. Then define the s-dimensional Hausdorff measure of E to be

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E). \tag{2.10}$$

The *Hausdorff dimension* of a subset $E \subset \mathbb{R}^n$ is defined as [Fa]

$$\dim_{\mathcal{H}}(E) = \inf \{s > 0 \mid \mathcal{H}^s(E) = 0\} = \sup \{s > 0 \mid \mathcal{H}^s(E) = \infty\}. \tag{2.11}$$

2.4.2 Box dimension

Suppose that E is some non-empty bounded subset of $\mathbb{R}^n$, and define $N_\delta(E)$ to be the least number of open balls of radius δ in $\mathbb{R}^n$ needed to cover E. Then we define

$$\underline{\dim}_B(E) = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta},$$

and

$$\overline{\dim}_B(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta},$$

to be the lower and upper box dimension of E respectively.

If these values are equal then we refer to this common value as the *box dimension* of E, namely

$$\dim_B(E) = \lim_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}. \tag{2.12}$$

2.5 An alternative dimension for subsets of $\mathbb{R}^2$

In the following section we introduce an alternative definition of dimension first introduced in [HL] for a subset of $S^1 \times \mathbb{R}$. We will show that it is a refinement of box dimension and introduce some useful properties which will be of use when we study results regarding graphs as introduced above.

2.5.1 The $\mathcal{K}$-dimension

Let f be a continuous function on $[0,1]$ and let $I \subset [0,1]$ be an open interval. We define q_I to be the least number of squares $I \times I'$ of side length $\operatorname{diam}(I)$ needed to cover the graph of f above I. Define $\operatorname{osc}(f,I)$ to be the oscillation or height of f over the interval I,

$$\operatorname{osc}(f,I) = \sup\left\{|f(x) - f(y)| : x, y \in I\right\}.$$

Notice that for any continuous function f and interval $I \subset [0,1]$

$$q_I - 1 \leq \operatorname{osc}(f,I)/\operatorname{diam}(I) \leq q_I. \tag{2.13}$$

Let $\mathcal{I}$ be an open cover of $[0,1]$. Let $\delta > 0$ and $s > 0$ and define $\operatorname{diam}(\mathcal{I}) = \sup\left\{\operatorname{diam}(I) : I \in \mathcal{I}\right\}$. As in [HL] we define

$$\mathcal{K}^s_\delta(\operatorname{graph}(f)) = \inf_{\operatorname{diam}(\mathcal{I}) \leq \delta} \sum_{I \in \mathcal{I}} \operatorname{diam}(I)^s q_I$$

where the infimum is taken over all open covers of $[0,1]$ by intervals of diameter at most δ. The s-dimensional $\mathcal{K}$-measure of $\operatorname{graph}(f)$ is then defined to be

$$\mathcal{K}^s(\operatorname{graph}(f)) = \lim_{\delta \to 0} \mathcal{K}^s_\delta(\operatorname{graph}(f)). \tag{2.14}$$

It is straightforward [HL] to show that $\mathcal{K}^s$ is an outer measure on subsets of $\mathbb{R}^2$. Note that this limit exists, since we are considering all covers of $[0,1]$ by intervals of diameter no more than δ, hence as δ decreases the class of allowed covers will be reduced. Therefore $\mathcal{K}^s_\delta(\operatorname{graph}(f))$ increases and approaches a limit as $\delta \to 0$.

Given $\delta \in (0,1)$, notice that $\mathcal{K}^s_\delta(\operatorname{graph}(f))$ and hence $\mathcal{K}^s(\operatorname{graph}(f))$ are non-negative and non-increasing with s.

Suppose [Fa1, HL] that $t > s$ and $\mathcal{I}$ is an open cover of $[0,1]$ by intervals of diameter at most δ, thus

$$\sum_{I \in \mathcal{I}} \operatorname{diam}(I)^t q_I \leq \delta^{t-s} \sum_{I \in \mathcal{I}} \operatorname{diam}(I)^s q_I.$$

Taking infima

$$\mathcal{K}^t_\delta(\operatorname{graph}(f)) \leq \delta^{t-s} \mathcal{K}^s_\delta(\operatorname{graph}(f)).$$

Now letting $\delta \to 0$, if $\mathcal{K}^s(\operatorname{graph}(f)) < \infty$ then $\mathcal{K}^t(\operatorname{graph}(f)) = 0$ for $t > s$. Thus there exists some value of s where $\mathcal{K}^t(\operatorname{graph}(f)) = 0$ for all $t > s$. Let s_0 denote the point where this jump to zero occurs.

Now consider $t < s_0$. Then in a similar way to that above, for $\delta \in (0,1)$

$$\mathcal{K}^t_\delta(\text{graph}(f)) \geq \delta^{t-s_0}\mathcal{K}^{s_0}_\delta(\text{graph}(f)).$$

As $\delta \to 0$, if $\mathcal{K}^{s_0}(\text{graph}(f)) > 0$ we know that $\mathcal{K}^t(\text{graph}(f)) = \infty$ for all $t < s_0$. Therefore there exists some value of s where $\mathcal{K}^s(\text{graph}(f))$ jumps from ∞ to 0. We call this critical value the $\mathcal{K}$-dimension of graph(f) and write it as $\dim_{\mathcal{K}}(\text{graph}(f))$. Formally

$$\begin{aligned}\dim_{\mathcal{K}}(\text{graph}(f)) &= \sup\{s > 0 \mid \mathcal{K}^s(\text{graph}(f)) = \infty\} \\ &= \inf\{s > 0 \mid \mathcal{K}^s(\text{graph}(f)) = 0\}.\end{aligned}$$

Remark. One can think of the $\mathcal{K}$-dimension as being like Hausdorff dimension in the S^1 direction and box dimension the fibre (y) direction. We are interested in seeing if we can use this notion to provide a better insight into what the Hausdorff dimension of graph(W) is, where W is the invariant graph for F as in (2.1).

2.5.2 Properties of the $\mathcal{K}$-dimension

The following proposition shows that it will in fact be of use to us to study the $\mathcal{K}$-dimension, particularly in the case that the box dimension is known but the Hausdoff dimension is not.

Remark. Suppose that $s \in [0,1)$, then summing lengths of subintervals of $[0,1]$ to a power s one can see that $\mathcal{K}^s(\text{graph}(f)) = \infty$. Additionally if $s > 2$, then looking at any arc of the graph $E \subset \text{graph}(f)$ and taking the limit as $\delta \to 0$, it is the case that $\mathcal{K}^s(E) = 0$. Thus $\dim_{\mathcal{K}}(\text{graph}(W)) \in [1,2]$.

Proposition 2.5.1
Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then

$$\dim_{\mathcal{H}}(\text{graph}(f)) \leq \dim_{\mathcal{K}}(\text{graph}(f)) \leq \underline{\dim}_B(\text{graph}(f)).$$

Proof. By Remark 2.5.2 we assume that $s \in [1,2]$. To see the first inequality let $\delta > 0$ and let $\mathcal{I}$ be an arbitrary open cover of $[0,1]$ by intervals $I_j \in \mathcal{I}$ such that $\text{diam}(I_j) \leq \delta$ for each $j = 1,\ldots,k$ for some positive integer k.

Then let $\mathcal{U} = \bigcup U_{j_i}$, $j = 1,\ldots,k$, $i = 1,\ldots,n_j$ for some $n_j > 0$ be a cover of graph(f) by open balls in $S^1 \times \mathbb{R}$ of diameter at most δ such that the projection of each $U_{j_i} \in \mathcal{U}$ onto the x-axis is the open interval I_j. Then for each I_j and open balls U_{j_i} for $i = 1, \ldots, n_j$, covering the graph above this interval, noting that $q_{I_j} \in \mathbb{N}$ one can see that

$$\sum_{U_{j_i} \in \mathcal{U}} \mathrm{diam}(U_{j_i})^s \leq \sum_{I_j \in \mathcal{I}} \mathrm{diam}(I_j)^s q_{I_j}.$$

Then taking the infimum over respective covers we see that $\mathcal{H}^s_\delta(\mathrm{graph}(f)) \leq \mathcal{K}^s_\delta(\mathrm{graph}(f))$, and letting $\delta \to 0$ the inequality follows.

For the second inequality we prove that

$$\dim_{\mathcal{K}}(\mathrm{graph}(f)) \leq \underline{\dim}_B(\mathrm{graph}(f)) = \liminf_{\delta \to 0} \frac{\log N_\delta(\mathrm{graph}(f))}{-\log \delta}$$

where $N_\delta(\mathrm{graph}(f))$ is the least number of open balls of radius δ needed to cover graph(f). Let s be any number such that $s \geq \underline{\dim}_B(\mathrm{graph}(f))$. We can [Fa1] replace the open balls in the definition of box-dimension by open squares of equal side length δ. Then the projection of these squares onto the x-axis forms a cover of $[0, 1]$ by open intervals I of diameter δ. Define $\hat{N}_\delta(\mathrm{graph}(f))$ to be the smallest number of open squares of side length δ needed to cover graph(f). Then

$$s \geq \liminf_{\delta \to 0} \frac{\log \hat{N}_\delta(\mathrm{graph}(f))}{-\log \delta}$$

and hence

$$\liminf_{\delta \to 0} \hat{N}_\delta(\mathrm{graph}(f))\delta^s \leq 1.$$

Therefore

$$\begin{aligned}
\mathcal{K}^s(\mathrm{graph}(f)) &= \lim_{\delta \to 0} \mathcal{K}^s_\delta(\mathrm{graph}(f)) \\
&\leq \lim_{\delta \to 0} \sum \mathrm{diam}(I)^s q_I \\
&= \liminf_{\delta \to 0} \hat{N}_\delta(\mathrm{graph}(f))\delta^s \\
&\leq 1
\end{aligned}$$

where I the are open intervals of length δ which form a cover of $[0, 1]$ as discussed earlier and the sum is taken over all intervals in this cover. Thus $\mathcal{K}^s(\mathrm{graph}(f)) \leq 1$ implies that $\dim_{\mathcal{K}}(\mathrm{graph}(f)) \leq s$. Recall we took s to be any $s \geq \underline{\dim}_B(\mathrm{graph}(f))$. Hence the proposition follows by letting $\epsilon > 0$ be arbitrary and taking $s = \underline{\dim}_B(\mathrm{graph}(f)) + \epsilon$.

❏

2.6 Topological pressure

In this section we introduce a definition of topological pressure in terms of cylinders, as seen in [Bo, Pe, Wa]. Let $T : S^1 \to S^1$ be a uniformly expanding map of the circle and let $\phi : S^1 \to \mathbb{R}$ be a Hölder continuous function. Let us use the notation

$$\phi^m(x) = \sum_{j=0}^{m-1} \phi(T^j(x)).$$

Then define

$$\Gamma(S^1, \phi, n) = \inf_{\mathcal{C}} \sum_{C \in \mathcal{C}} \exp\left\{\sup_{x \in C} \phi^{n(C)}(x)\right\}$$

where we take the infimum over all covers $\mathcal{C}$ of S^1 consisting of cylinders of rank no less than n, and the sum is taken over all cylinders C in the cover where $C \in \mathcal{C}$ has rank $n(C) \geq n$. Similarly define

$$\Lambda(S^1, \phi, n) = \inf_{C_n} \sum_{C \in C_n} \exp\left\{\sup_{x \in C} \phi^n(x)\right\}$$

where the infimum here is taken over all covers C_n of S^1 consisting of cylinders of rank equal to n. Then we define

$$P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \Gamma(S^1, \phi, n),$$

$$\underline{CP}(\phi) = \liminf_{n \to \infty} \frac{1}{n} \log \Lambda(S^1, \phi, n),$$

$$\overline{CP}(\phi) = \limsup_{n \to \infty} \frac{1}{n} \log \Lambda(S^1, \phi, n),$$

to be the pressure, lower capacity pressure and upper capacity pressure of ϕ respectively. Since S^1 is compact and invariant under the uniformly expanding map T [Pe. Chapter 11] it is true that in our context $P(\phi) = \underline{CP}(\phi) = \overline{CP}(\phi)$.

Thus for a Hölder continuous function ϕ we can define the *topological pressure* of ϕ to be

$$P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \left(\inf_{\mathcal{C}} \sum_{C \in \mathcal{C}} \exp\left\{\sup_{x \in C} \phi^{n(C)}(x)\right\}\right), \tag{2.15}$$

where the infimum is taken over all covers $\mathcal{C}$ of $[0, 1]$ by cylinders of rank no less than n. We can also define $P(\phi)$ by the variational principle. That is

$$P(\phi) = \sup\left\{h_\mu(T) + \int \phi \, d\mu\right\}$$

where the supremum is taken over all T-invariant probability measures μ and $h_\mu(T)$ is the measure theoretic entropy of T. The unique T-invariant measure which achieves this supremum is known as the unique equilibrium state corresponding to the Hölder continuous function ϕ.

We now describe some of the basic properties of the pressure function $P : C(S^1) \to \mathbb{R}$, which follow from the definitions given above (see [Wa]);

1. $P : C(S^1) \to \mathbb{R}$ is monotone increasing. That is, if $f, g \in C(S^1)$ and $f \le g$, then $P(f) \le P(g)$, and if $f < g$ then pressure is strictly monotone.
2. $P : C(S^1) \to \mathbb{R}$ is convex,
3. $P : C(S^1) \to \mathbb{R}$ is Lipschitz continuous.

Given a uniformly expanding map T and Hölder continuous function ϕ, it should be noted that $P(\phi)$ depends on both T and ϕ. By the variational principle, $P(0) = h_{\text{top}}(T) > 0$, since the unique equilibrium state corresponding to Hölder continuous potential zero is the measure of maximal entropy.

As in earlier sections we work in the case where $g_x(y) = \lambda(x)^{-1}(y - p(x))$. Recall that $\lambda(x)$ denotes the rate of contraction of the fibre map in the $\mathbb{R}$ direction for $x \in S^1$. We consider a similar approach to that of Bowen as discussed in Section 1, who considered the thermodynamic formalism for potentials of the form $-s \log |T'|$ for $s \in \mathbb{R}$ and considered maps of the form $s \mapsto P(-s \log |T'|)$.

Letting $s \in \mathbb{R}$ we consider the Hölder continuous potential $\phi : S^1 \to \mathbb{R}$ where $\phi(x) = (1 - s) \log |T'(x)| + \log \lambda(x)$. In particular we consider the function

$$p(s) : \; s \longmapsto P((1 - s) \log |T'| + \log \lambda).$$

Recalling the partial hyperbolicity assumption, we have that $\lambda(x)|T'(x)| > 1$ for all $x \in S^1$ and hence $(\log |T'| + \log \lambda) > 0$. By monotonicity of pressure

$$p\,(0) = P(\log |T'| + \log \lambda) \ge P(0) = h_{\text{top}}(T) > 0.$$

Since T is a uniformly expanding map, (i.e. $|T'(x)| > 1$ for all $x \in S^1$), given $s \ge 0$ we have that there exists $K > 0$ such that $-s \log |T'| \le -sK$. Hence

$$(1 - s) \log |T'| + \log \lambda \le (1 - s)K + \log \lambda.$$

Moreover since $\lambda(x) \in (0,1)$ for all $x \in S^1$

$$(1-s)\log|T'| + \log\lambda \le (1-s)K$$

where $K > 0$. Thus since K is constant and pressure is monotone

$$\begin{aligned} P((1-s)\log|T'| + \log\lambda) &\le P((1-s)K) \\ &= h_{\text{top}}(T) + (1-s)K. \end{aligned}$$

Since $K > 0$ is constant $\lim_{s\to\infty} p(s) = -\infty$.

By continuity there exists some unique value $s_0 \in (0,\infty)$ such that $p(s_0) = 0$. This is the value in which we will be particularly interested.

We also give the following characterisation of topological pressure for a Hölder continuous function as seen in [PaPo] using periodic points of the map T. Letting T be a b-to-one uniformly expanding map of the circle, we can model T as a full shift on b symbols. Let

$$\text{Fix}_n = \left\{ x \in S^1 \mid T^n(x) = x \right\}.$$

Then [PaPo] given a Hölder continuous function ϕ we can express topological pressure as

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in\text{Fix}_n} \exp\phi^n(x) \tag{2.16}$$

where $\phi^n(x) = \sum_{j=0}^{n-1} \phi(T^j(x))$.

Suppose that a function ϕ is locally constant and depends only on the first 2 coordinates of a given point, that is $\phi(x) = \phi(x_0, x_1)$ where $x = (x_0, x_1, x_2, \ldots)$. Then [PaPo]

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x_0\ldots x_{n-1}x_0} e^{\phi(x_0,x_1)} e^{\phi(x_1,x_2)} \ldots e^{\phi(x_{n-1},x_0)}.$$

We study this definition of topological pressure for a particular locally constant function in more detail in Chapter 6.

We also mention the following property of equilibrium states. Given a Hölder continuous function ϕ, let μ_ϕ denote the unique equilibrium state corresponding to ϕ. That is, μ_ϕ achieves the supremum over all invariant measures as in the variational principle. Then [PaPo, Wa] μ_ϕ satisfies a Gibbs property. That is there exists a constant $C > 0$ such that given $x \in S^1$ and $n \ge 1$, and $C_n(x)$ a cylinder of rank n

containing x

$$C^{-1} \leq \frac{\mu_\phi(C_n(x))}{\exp\left\{\sum_{j=0}^{n-1} \phi(T^j(x)) - nP(\phi)\right\}} \leq C. \tag{2.17}$$

2.7 Moran covers

In the following we introduce a specific type of geometric construction, which allows us to construct an optimal cover of S^1 consisting of cylinders as seen in [Pe, PeWe2]. Let T be a continuous uniformly expanding b-to-one map of the circle as before. Then as in Section 2.2 we choose a Markov partition of S^1 with respect to T. Thus we can construct cylinders using the inverse branches of the map T. Let $\delta > 0$ be given. For $x \in S^1$ we let $n(x)$ be the unique positive integer, which depends on x and δ such that the diameter of the cylinder of rank $n(x)$ determined by the inverse branches of the map T which contains x is at most δ, but the diameter of the cylinder of rank $n(x) - 1$ containing x is strictly greater than δ. That is given x and $\delta > 0$ there exists $x_0, \ldots, x_{n(x)-1} \in \{0, \ldots, b-1\}$ such that

$$\operatorname{diam}(T_{x_0,\ldots,x_{n(x)-1}}(S^1)) \leq \delta$$

but

$$\operatorname{diam}(T_{x_0,\ldots,x_{n(x)-2}}(S^1)) > \delta.$$

Let $C_{n(x)}(x)$ denote the cylinder of rank $n(x)$ containing x. As x varies over S^1 [Pe] the sets $C_{n(x)}(x)$ either coincide or have disjoint interior. Let $\mathcal{C}_\delta$ be the finite collection of the cylinders $C_{n(x)}(x)$. These sets comprise a cover of $[0,1]$ consisting of cylinders of diameter at most δ, whose interiors are disjoint. We say that $\mathcal{C}_\delta$ is the *δ-Moran cover of* $[0,1]$. Note that such a Moran cover depends on the choice of Markov partition. We fix the conventional Markov partition as in Section 2.2.

Given $\delta > 0$ we construct the δ-Moran cover $\mathcal{C}_\delta$ of $[0,1]$, a finite cover consisting of cylinders of rank no less than

$$n = \min\{n(x) \mid x \in [0,1]\} \tag{2.18}$$

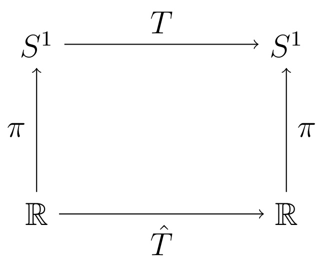
where n depends only on $\delta > 0$.

Given $x \in [0,1]$ and $\delta > 0$ there exists $M > 0$ independent of x and δ such that the number of cylinders in the δ-Moran cover of $[0,1]$ that have non-empty intersection with the open ball $B(x, \delta)$ is bounded above by M. We refer to $M > 0$ as the *Moran multiplicity factor*.

2.8 A natural lift to $\mathbb{R}$

In the following we construct a natural lift of the function W to a function $\widehat{W} : \mathbb{R} \to \mathbb{R}$ and relate results regarding $\widehat{W}$ to those for W. In later work it will indeed prove easier to study the lift $\widehat{W}$ and relate results back to the function W. We work in the affine case (2.5) but such a lift can be generalised for more general fibre maps $g_x(y)$.

Let T be a continuous uniformly expanding b-to-one map of the circle. Without loss of generality we can assume that $T(0) = 0$. If this were not the case then since T is uniformly expanding, there exists x_0, a fixed point of T such that $T(x_0) = x_0$. Letting $\chi(x) = x - x_0$ there exists a map $\overline{T}$ such that $\overline{T}(0) = 0$ and $\chi \circ T = \overline{T} \circ \chi$. Then we take a continuous lift of T to a map $\hat{T} : \mathbb{R} \to \mathbb{R}$. That is, there exists a map $\pi : \mathbb{R} \to S^1$ such that $T \circ \pi = \pi \circ \hat{T}$ as below.



Here $\hat{T}$ is a continuous, invertible uniformly expanding map of $\mathbb{R}$. Moreover we can assume that $\hat{T}(0) = 0$ and $\hat{T}(1) = b$. Therefore for any x, $\hat{T}(x+1) = \hat{T}(x) + b$, and moreover for any $m \in \mathbb{N}$, $\hat{T}(x+m) = \hat{T}(x) + bm$. That is, $\hat{T}(x+m)$ and $\hat{T}(x)$ differ by an integer.

In addition, by construction of the lift, given $x \in S^1$ it can be seen that $\hat{T}(x)$ and $T(x)$ will differ by an integer. In fact given $j \geq 0$, $\hat{T}^j(x)$ and $T^j(x)$ differ by an integer. Since $\hat{T}$ is invertible, $\hat{T}^{-1}(0) = 0$ and $\hat{T}^{-n}(x) \to 0$ for all $x \in \mathbb{R}$ as $n \to \infty$ since $\hat{T}$ is uniformly expanding. Furthermore for all $x \in S^1$, the slope of the lift is the same as the slope of T. That is, $\hat{T}'(x) = T'(x)$ for $x \in S^1$. The important thing here, is that given $x \in \mathbb{R}$, $\hat{T}(x)$ takes a value which differs from $\hat{T}(x_0)$ for some $x_0 = x \bmod 1$ by an integer value, which in turn differs from $T(x_0)$ by some integer value. Thus we can relate $\hat{T}$ to T by working modulo 1.

We take $\hat{p} : \mathbb{R} \to \mathbb{R}$ to be the 1-periodic lift of p, where we extend the continuous circle function p periodically along the real line. Similarly let $\hat{\lambda} : \mathbb{R} \to (0,1)$ be the 1-periodic extension of $\lambda : S^1 \to (0,1)$ to $\mathbb{R}$. Then we can let $\hat{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the continuous lift of the skew-product F, where

$$\hat{F}(x,y) = (\hat{T}(x), \hat{\lambda}(x)^{-1}(y - \hat{p}(x)))$$

and graph$(\widehat{W})$ is invariant under $\hat{F}$ where $\widehat{W} : \mathbb{R} \to \mathbb{R}$ is given by

$$\widehat{W}(x) = \sum_{j=0}^{\infty} \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x)) \tag{2.19}$$

and we use the notation $\hat{\lambda}^j(x) = \hat{\lambda}(x)\ldots\hat{\lambda}(\hat{T}^{j-1}(x))$.

Note that since $\hat{\lambda}$ and $\hat{p}$ are just the one periodic extensions of λ and p, it is clear that $\hat{\lambda}$ and $\hat{p}$ restricted to the unit interval are simply the functions λ and p respectively. Furthermore $\hat{T}(x+1) = \hat{T}(x) + b$ and for all $j \geq 1$ it is easily shown that $\hat{T}^j(x+1) = \hat{T}^j(x) + b^j$. Therefore letting $x \in S^1$ be given

$$\begin{aligned}
\widehat{W}(x+1) &= \sum_{j=0}^{\infty} \hat{\lambda}^j(x+1)\hat{p}(\hat{T}^j(x+1)) \\
&= \sum_{j=0}^{\infty} \hat{\lambda}(x+1)\ldots\hat{\lambda}(\hat{T}^{j-1}(x) + b^{j-1})\hat{p}(\hat{T}^j(x) + b^j) \\
&= \sum_{j=0}^{\infty} \hat{\lambda}(x)\ldots\hat{\lambda}(\hat{T}^{j-1}(x))\hat{p}(\hat{T}^j(x)) \\
&= \widehat{W}(x).
\end{aligned}$$

That is, $\widehat{W}$ is 1-periodic.

In addition, recall that for $x \in S^1$, $\hat{T}(x)$ and $T(x)$ differ only by integer values, thus the same holds for $\hat{T}^j(x)$ and $T^j(x)$ for all $j \geq 1$. Therefore, using periodicity of $\hat{\lambda}$ and $\hat{p}$ and above discussions it can be shown that $\widehat{W}$ restricted to the unit interval is the function W itself. Thus when we study W, it can be useful to study the invertible lift itself, and restrict to $[0,1]$.

Chapter 3

Estimates on the height and width of graph(W) over intervals

3.1 Introduction

In the following chapter we consider graph(W) as introduced in previous sections. We shall consider the case in which the fibre map is affine as in (2.5) for notational convenience, however such estimation results apply to more general fibre maps $g_x(y)$.

We look to use the dynamics of the skew-product F (2.5), under which graph(W) is invariant, to give estimates on the size of the graph restricted to subsets of S^1 in terms of the underlying dynamics. Such estimates will prove important in later chapters.

Considering $x \in S^1$ and $n \geq 0$, as in Section 2.2 we construct $C_n(x)$; the cylinder of rank n containing x. Then $T^n(C_n(x)) = S^1$. We can use the dynamics of the uniformly expanding map T to estimate the diameter of $C_n(x)$ in terms of the derivatives T' along the orbit of x, and the skew-product F to estimate the height of the graph over $C_n(x)$, namely the quantity we denote

$$\operatorname{osc}(W, C_n(x)) = \sup \{|W(x) - W(y)| \mid x, y \in C_n(x)\}.$$

However we wish to work in a more general setting, where we wish to estimate these values over arbitrary intervals $I \subset S^1$. By remarks in Section 2.8, given $I \subset [0,1]$ we can use the lifts $\hat{T}$ and $\widehat{W}$, and since $\widehat{W}$ restricted to $[0,1]$ is simply W, the diameter of I and the height of graph(W) above the interval I will be the same as that for $\widehat{W}$. In particular we study the invertible maps $\hat{F}$ and $\hat{T}$ and recall that $\hat{T}'(x) = T'(x)$ and

$\hat{\lambda} = \lambda(x)$ for all $x \in [0,1]$.

Let $I \subset [0,1]$ be given. Choose $n \geq 1$ to be the largest integer such that $\text{diam}(\hat{T}^n(I)) \leq 1$. We call this interval $\hat{T}^n(I) = J$. One may think of such an n, as defining a fixed and unique rank on arbitrary intervals.

Remark. If the chosen I were a cylinder, then there exists some n, namely the rank of the cylinder such that $\text{diam}(\hat{T}^n(I)) = 1$. In fact $\hat{T}^n(I) = [0,1] \bmod 1$. For arbitrary $I \subset [0,1]$ we choose n such that $\text{diam}(\hat{T}^n(I)) \leq 1$ but $\text{diam}(\hat{T}^{n+1}(I)) > 1$.

We take I as above, and use the dynamics of the lifts $\hat{F}$ and $\hat{T}$ to estimate the diameter of I and the height of graph($\widehat{W}$) above I in terms of the dynamics of $\hat{T}$ and $\hat{\lambda}$, which can then be transferred to estimates on graph(W) by remarks in the previous section.

3.2 Diameters of cylinders

In the following we use the lift $\hat{T}$ and look to get estimates on the diameter of intervals, with respect to the dynamics. We begin with a useful bounded variation estimate.

Lemma 3.2.1

There exists a constant $C_0 > 0$ such that given $I \subset [0,1]$ and n maximal such that $\text{diam}(\hat{T}^n(I)) \leq 1$ *and given points $x \in I$, and $x_j \in I$ for $j = 0, \ldots, n-1$,*

$$C_0^{-1} \prod_{j=0}^{n-1} \left|\hat{T}'(\hat{T}^j(x))\right|^{-1} \leq \prod_{j=0}^{n-1} \left|\hat{T}'(\hat{T}^j(x_j))\right|^{-1} \leq C_0 \prod_{j=0}^{n-1} \left|\hat{T}'(\hat{T}^j(x))\right|^{-1}.$$

Proof. Let I and n be given. Define $\hat{T}^n(I) = J$. Let $x, x_j \in I$ for $0 \leq j \leq n-1$. Recall that T, hence $\hat{T}$ is $C^{1+\gamma}$, thus $\log \hat{T}'$ is Hólder of some exponent $\gamma \in (0,1)$, and therefore

$$\begin{aligned}
&\left|\log\left(\frac{\prod_{j=0}^{n-1}\left|\hat{T}'(\hat{T}^j(x_j))\right|^{-1}}{\prod_{j=0}^{n-1}\left|\hat{T}'(\hat{T}^j(x))\right|^{-1}}\right)\right| \\
\leq\ & \sum_{j=0}^{n-1}\left|\log|\hat{T}'(\hat{T}^j(x_j))|^{-1} - \log|\hat{T}'(\hat{T}^j(x))|^{-1}\right| \\
=\ & \sum_{j=0}^{n-1}\left|\log|\hat{T}'(\hat{T}^j(x))| - \log|\hat{T}'(\hat{T}^j(x_j))|\right| \\
\leq\ & ||\log \hat{T}'||_\gamma \sum_{j=0}^{n-1}\left|\hat{T}^j(x) - \hat{T}^j(x_j)\right|^\gamma \\
=\ & ||\log \hat{T}'||_\gamma \sum_{j=0}^{n-1}\left|\hat{T}^{j-n}\hat{T}^n(x) - \hat{T}^{j-n}\hat{T}^n(x_j)\right|^\gamma .
\end{aligned}$$

Since $||\log \hat{T}'||_\gamma = ||\log T'||_\gamma$, $\mathrm{m}(\hat{T}') = \mathrm{m}(T') > 1$ and since $\hat{T}^n(x), \hat{T}^n(x_j) \in J$ for all $j = 0, \ldots, n-1$,

$$\begin{aligned}
&\left|\log\left(\frac{\prod_{j=0}^{n-1}\left|\hat{T}'(\hat{T}^j(x_j))\right|^{-1}}{\prod_{j=0}^{n-1}\left|\hat{T}'(\hat{T}^j(x))\right|^{-1}}\right)\right| \\
\leq\ & ||\log T'||_\gamma \mathrm{m}(T')^{-n\gamma}\sum_{j=0}^{n-1}\mathrm{m}(T')^{j\gamma}\left|\hat{T}^n(x) - \hat{T}^n(x_j)\right|^\gamma \\
\leq\ & ||\log T'||_\gamma \mathrm{m}(T')^{-n\gamma}\mathrm{diam}(J)^\gamma \frac{\mathrm{m}(T')^{n\gamma}-1}{\mathrm{m}(T')^\gamma - 1} \\
\leq\ & ||\log T'||_\gamma \mathrm{diam}(J)^\gamma \frac{1}{\mathrm{m}(T')^\gamma - 1}.
\end{aligned}$$

Since $\mathrm{diam}(J) \leq 1$ the lemma follows by taking

$$C_1 = \exp\left\{||\log T'||_\gamma \frac{1}{\mathrm{m}(T')^\gamma - 1}\right\}$$

independent of I, n, x and x_j where $\gamma \in (0,1)$ is fixed and depends only on T. ❑

Proposition 3.2.2

There exists a constant $C_1 > 0$ such that given $I \subset [0,1]$ and n chosen to be maximal such that $\mathrm{diam}(\hat{T}^n(I)) \leq 1$ and given $x \in I$

$$C_1^{-1}\prod_{j=0}^{n-1}\left|\hat{T}'(\hat{T}^j(x))\right|^{-1} \leq \mathrm{diam}(I) \leq C_1 \prod_{j=0}^{n-1}\left|\hat{T}'(\hat{T}^j(x))\right|^{-1}.$$

Proof. Let I be given. Choose n maximal such that $\operatorname{diam}(\hat{T}^n(I)) \leq 1$. Define $J = \hat{T}^n(I)$. Suppose firstly that $n = 1$. Then $\hat{T}(I) = J$ and

$$\inf_{y\in I} \left|\hat{T}'(y)\right| \operatorname{diam}(I) \leq \operatorname{diam}(J) \leq \sup_{y\in I} \left|\hat{T}'(y)\right| \operatorname{diam}(I).$$

That is,

$$\left[\sup_{y\in I} \left|\hat{T}'(y)\right|\right]^{-1} \operatorname{diam}(J) \leq \operatorname{diam}(I) \leq \left[\inf_{y\in I} \left|\hat{T}'(y)\right|\right]^{-1} \operatorname{diam}(J)$$

and therefore

$$\inf_{y\in I} \left(\left|\hat{T}'(y)\right|^{-1}\right) \operatorname{diam}(J) \leq \operatorname{diam}(I) \leq \sup_{y\in I} \left(\left|\hat{T}'(y)\right|^{-1}\right) \operatorname{diam}(J).$$

Given $n \geq 1$ a simple induction shows that

$$\prod_{j=0}^{n-1} \inf_{y_j\in\hat{T}^j(I)} \left(\left|\hat{T}'(y_j)\right|^{-1}\right) \operatorname{diam}(J) \leq \operatorname{diam}(I) \leq \prod_{j=0}^{n-1} \sup_{y_j\in\hat{T}^j(I)} \left(\left|\hat{T}'(y_j)\right|^{-1}\right) \operatorname{diam}(J). \tag{3.1}$$

For each $j = 0, 1, \ldots, n-1$ let $y_j^+ \in \hat{T}^j(I)$ be the point which achieves the supremum in (3.1), and let y_j^- be the point which achieves the infimum.

For each $y_j^+ \in \hat{T}^j(I)$, since $\hat{T}$ is invertible we can find a point $x_j^+ \in I$ such that $\hat{T}^j(x_j^+) = y_j^+$ and similarly $x_j^- \in I$ such that $\hat{T}^j(x_j^-) = y_j^-$. Therefore

$$\prod_{j=0}^{n-1} \left|\hat{T}'(\hat{T}^j(x_j^-)))\right|^{-1} \operatorname{diam}(J) \leq \operatorname{diam}(I) \leq \prod_{j=0}^{n-1} \left|\hat{T}'(\hat{T}^j(x_j^+))\right|^{-1} \operatorname{diam}(J).$$

Let $x \in I$ be given. Then there exists $C_1 > 0$ as in Lemma 3.2.1, for x_j as in the statement of the lemma to be both x_j^+ and x_j^- such that

$$C_1^{-1} \prod_{j=0}^{n-1} \left|\hat{T}'(\hat{T}^j(x))\right|^{-1} \operatorname{diam}(J) \leq \operatorname{diam}(I) \leq C_1 \prod_{j=0}^{n-1} \left|\hat{T}'(\hat{T}^j(x))\right|^{-1} \operatorname{diam}(J).$$

By construction of J, $\operatorname{diam}(J) \leq 1 < |\hat{T}'|_\infty = |T'|_\infty$ and $|T'|_\infty \operatorname{diam}(J) \geq \operatorname{diam}(\hat{T}(J)) > 1$, we can see that $|T'|_\infty^{-1} < \operatorname{diam}(J) < |T'|_\infty$. Letting $C_2 = |T'|_\infty C_1$ the proposition follows. ❑

By discussions in previous section about the lift to $\mathbb{R}$, since $\hat{T}'(x) = T'(x)$ for all $x \in [0, 1]$ and remarks on values of $\hat{T}^j(x)$ and $\hat{T}^j(x+m)$ differing by only integers, the derivative remains unchanged and it follows that given $I \subset S^1$ and $x \in I$, the above result holds for $\hat{T}$ replaced by the original map T. We often use the lift $\hat{T}$, then refer back to T itself for ease of working with invertible maps.

3.3 Estimates on the height of the graph

In Sections 3.3, 3.3.1 and 3.3.2 we use methods similar to those found in [Be, MoWa]. We look to give bounds on $\mathrm{osc}(W, I) = \sup_{x,y\in I} |W(x) - W(y)|$ for arbitrary $I \subset [0, 1]$. As in the above work on diameters of intervals we consider the lift $\widehat{W} : \mathbb{R} \to \mathbb{R}$ and make estimates on graph$(\widehat{W})$. Recall the invertible skew-product $\hat{F} : \mathbb{R}^2 \to \mathbb{R}^2$ under which graph$(\widehat{W})$ is invariant; $\hat{F}(x, y) = (\hat{T}(x), \hat{\lambda}(x)^{-1}(y - \hat{p}(x)))$ with inverse

$$\hat{F}^{-1}(x, y) = (\hat{T}^{-1}(x), \hat{\lambda}(y - \hat{p}(x))).$$

The map $\hat{F}$ has derivative of the form

$$\mathcal{J}_{\hat{F}^{-1}(x,y)} = \begin{pmatrix} (\hat{T}')^{-1}(x) & 0 \\ S(x)y & \hat{\lambda}(\hat{T}^{-1}(x)) \end{pmatrix}$$

where $S(x)y$ is the derivative of the fibre map in the x-direction.
Throughout the following we shall denote $\mathcal{S} = \sup_{(x,y)\in\mathrm{graph}(W)|_{[0,1]}} |S(x)y|$. By periodicity of $\widehat{W}$ and by remarks in previous sections on the lift to $\mathbb{R}$ $\mathcal{S} \geq |S(x)y|$ for all $(x, y) \in$ graph$(\widehat{W})$. Moreover, the supremum is taken over points on the graph, and since $\hat{F}^{-1}$ is continuous and has bounded derivative, $\mathcal{S} < \infty$.
Recall the earlier notation that

$$\inf_{x\in S^1} \lambda(x) = \mathrm{m}(\lambda) \tag{3.2}$$

and

$$\sup_{x\in S^1} \lambda(x) = |\lambda|_\infty. \tag{3.3}$$

By properties of the 1-periodic lift $\hat{\lambda}$, it follows that for all $x \in S^1$ and all $n \in \mathbb{Z}$ $\hat{\lambda}(x + n) = \lambda(x)$. We begin with the following bounded variation estimate.

Proposition 3.3.1

There exists $C_2 > 0$ such that given some interval J such that $\mathrm{diam}(J) \leq 1$, and $n \geq 1$ given and given $x, x_j \in I$ such that $\hat{T}^{-n}(J)$ for $j = 0, \dots, n-1$ it holds that

$$C_2^{-1} \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x)) \leq \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x_j)) \leq C_2 \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x)).$$

Proof. Let J and $n \geq 1$ be given. Define $I = \hat{T}^{-n}(J)$. Let $x, x_j \in I$ for $j = 0, \dots, n-1$. Since λ and hence $\log \hat{\lambda}$ is taken to be Lipschitz we let $|\log \hat{\lambda}|_{\mathrm{Lip}}$ be the

Lipschitz constant of $\log\hat{\lambda}$. Since $\hat{T}^n(x), \hat{T}^n(x_j) \in J$, by the Mean Value Theorem we have that

$$\begin{aligned}
&\left|\log\frac{\prod_{j=0}^{n-1}\hat{\lambda}(\hat{T}^j(x_j))}{\prod_{j=0}^{n-1}\hat{\lambda}(\hat{T}^j(x))}\right| \\
\leq\;& \sum_{j=0}^{n-1}\left|\log\hat{\lambda}(\hat{T}^j(x_j)) - \log\hat{\lambda}(\hat{T}^j(x))\right| \\
\leq\;& |\log\hat{\lambda}|_{\mathrm{Lip}}\sum_{j=0}^{n-1}\left|\hat{T}^j(x_j) - \hat{T}^j(x)\right| \\
=\;& |\log\lambda|_{\mathrm{Lip}}\sum_{j=0}^{n-1}\left|\hat{T}^{j-n}\hat{T}^n(x_j) - \hat{T}^{j-n}\hat{T}^n(x)\right| \\
\leq\;& |\log\lambda|_{\mathrm{Lip}}\mathrm{m}(T')^{-n}\sum_{j=0}^{n-1}\mathrm{m}(T')^j\left|\hat{T}^n(x_j) - \hat{T}^n(x)\right| \\
\leq\;& |\log\lambda|_{\mathrm{Lip}}\mathrm{m}(T')^{-n}\mathrm{diam}(J)\sum_{j=0}^{n-1}\mathrm{m}(T')^j \\
\leq\;& \frac{|\log\lambda|_{\mathrm{Lip}}}{\mathrm{m}(T')^n}\times\frac{\mathrm{m}(T')^n-1}{\mathrm{m}(T')-1} \\
\leq\;& \frac{|\log\lambda|_{\mathrm{Lip}}}{\mathrm{m}(T')-1}.
\end{aligned}$$

Therefore letting

$$C_2 = \exp\left\{\frac{|\log\lambda|_{\mathrm{Lip}}}{\mathrm{m}(T')-1}\right\}$$

the result follows. ❏

Given an interval $I \subset [0,1]$ we denote $\mathrm{graph}(\widehat{W}, I) \subset \mathbb{R}^2$ to be the graph of $\widehat{W}$ restricted to I. We are interested in the height of the graph over I, namely $\mathrm{osc}(\widehat{W}, I)$. Recall that $\mathrm{graph}(\widehat{W})$ is invariant under $\hat{F}$, i.e. $\hat{F}(x, \widehat{W}(x)) = (\hat{T}(x), \widehat{W}(\hat{T}(x)))$. We use invariance properties, and the derivative of the map $\hat{F}^{-1}$ as above to give bounds on the height of $\mathrm{graph}(\widehat{W})$.

3.3.1 An upper bound

Let $I \subset [0,1]$ be given. Choose n to be the maximal integer such that $\mathrm{diam}(\hat{T}^n(I)) \leq 1$ and denote $\hat{T}^n(I) = J$. We give an upper bound on the height of $\mathrm{graph}(\widehat{W})$ restricted to I in terms of the derivative of the fibre map in the $\mathbb{R}$-direction. We start with the following useful lemma. Recall that $\mathcal{S} = \sup_{(x,y)\in\mathrm{graph}(W)|_{[0,1]}}|S(x)y|$ is finite.

Lemma 3.3.2

Let $I \subset \mathbb{R}$ be given. Then

$$\operatorname{osc}(\widehat{W}, I) \leq \operatorname{osc}(\widehat{W}, \hat{T}(I)) \sup_{x \in T(I)} \hat{\lambda}(\hat{T}^{-1}(x)) + \mathcal{S} \operatorname{diam}(\hat{T}(I)).$$

Proof. Let $I \subset \mathbb{R}$ be given. We look to relate $\operatorname{osc}(\widehat{W}, I)$ to $\operatorname{osc}(\widehat{W}, \hat{T}(I))$. We let $\gamma(t) = (\gamma_H(t), \gamma_V(t))$ denote a differentiable curve in $\mathbb{R}^2$. Here we let H and V denote the horizontal and vertical directions respectively and π_H and π_V be the corresponding projections. Let $x^+, x^- \in I$ be the points which achieve the height of the graph over I. That is $\operatorname{osc}(\widehat{W}, I) = \widehat{W}(x^+) - \widehat{W}(x^-)$.

Let γ_0 denote the line segment joining $\widehat{W}(x^+)$ to $\widehat{W}(x^-)$. That is, γ_0 is a line whose horizontal projection $\pi_H(\gamma_0) \subseteq I$. Recall the skew-product $\hat{F}$ under which $\widehat{W}$ is invariant, with inverse $\hat{F}^{-1}$ and derivative as in the previous section. Since $\gamma_0 \subset \mathbb{R}^2$, we denote $\gamma = \hat{F}(\gamma_0)$ and $\pi_H(\gamma) \subseteq \hat{T}(I)$. It should be noted that, since the graph is invariant under $\hat{F}$, and since the endpoints of γ_0 are points on the graph and achieve the height of the graph over I, the endpoints of γ lie on the graph above the interval $\hat{T}(I)$. By construction of such curves, and by properties of the derivative of the map $\hat{F}^{-1}$ it can be seen that

$$\begin{aligned}
\operatorname{osc}(\widehat{W}, I) &= \operatorname{osc}(\widehat{W}, \pi_H(\hat{F}^{-1}(\gamma))) \\
&\leq \int |\pi_V(\hat{F}^{-1}(\gamma))'(t)| \, dt \\
&\leq \int \hat{\lambda}(\hat{T}^{-1}(\gamma_H'(t))) |\gamma_V'(t)| \, dt + \int |S(\gamma(t))| |\gamma_H'(t)| \, dt \\
&= \int A_1(t) \, dt + \int A_2(t) \, dt.
\end{aligned}$$

Now the x coordinate of the curve γ, denoted by $\gamma_H'(t)$ as t varies is contained within the interval $\hat{T}(I)$. Thus we can bound the first integrand

$$A_1(t) \leq \sup_{x \in \hat{T}(I)} \hat{\lambda}(\hat{T}^{-1}(x)) |\gamma_V'(t)|.$$

Now γ is the arc of a curve, whose projection to the x-axis is contained in $\hat{T}(I)$, and by invariance of the graph, the endpoints of this curve lie on graph$(\widehat{W})$. Therefore since $\int |\gamma_V'(t)| \, dt$ is the height of this arc, then

$$\begin{aligned}
\int A_1(t) \, dt &\leq \sup_{x \in \hat{T}(I)} \hat{\lambda}(\hat{T}^{-1}(x)) \int |\gamma_V'(t)| \, dt \\
&\leq \sup_{x \in \hat{T}(I)} \hat{\lambda}(\hat{T}^{-1}(x)) \operatorname{osc}(\widehat{W}, \hat{T}(I)).
\end{aligned}$$

In addition $A_2(t) \leq \mathcal{S}|\gamma_H'(t)|$ where $\mathcal{S} = \sup_{(x,y)\in\text{graph}(\widehat{W})|_{[0,1]}} |S(x)y|$ and by previous discussions, and properties of the lift to $\mathbb{R}$ and $\hat{F}^{-1}$ having bounded derivative, such an $\mathcal{S}$ is and finite. Moreover $\int |\gamma_H'(t)|\,dt$ is the width of the arc γ, whose projection to the x-axis is contained within $\hat{T}(I)$, therefore

$$\int A_2(t)\,dt \leq \mathcal{S}\,\text{diam}(\hat{T}(I)).$$

Therefore

$$\text{osc}(\widehat{W}, I) \leq \sup_{x\in\hat{T}(I)} \hat{\lambda}(\hat{T}^{-1}(x))\text{osc}(\widehat{W}, \hat{T}(I)) + \mathcal{S}\,\text{diam}(\hat{T}(I))$$

as required. ❑

The following lemma can be easily proven using Lemma 3.3.2 inductively.

Lemma 3.3.3

Let $I \subset \mathbb{R}$ be given, and let n be such that $\text{diam}(\hat{T}^n(I)) \leq 1$. *Then*

$$\begin{aligned}\text{osc}(\widehat{W}, I) &\leq \text{osc}(\widehat{W}, \hat{T}^n(I)) \prod_{j=0}^{n-1} \sup_{x\in\hat{T}^{j+1}(I)} \hat{\lambda}(\hat{T}^{-1}(x)) \\ &+ \sum_{i=0}^{n-1} \mathcal{S}\,\text{diam}(\hat{T}^{i+1}(I)) \left(\prod_{j=0}^{i-1} \sup_{x\in\hat{T}^{j+1}(I)} \hat{\lambda}(\hat{T}^{-1}(x)) \right)\end{aligned}$$

where $\mathcal{S} = \sup_{(x,y)\in\text{graph}(\widehat{W})} |S(x)y|$ *and by previous remarks is finite. We use the convention that* $\prod_{j=0}^{-1}(\cdot) = 1$.

Using Lemma 3.3.3 and Proposition 3.3.1 we can now give an upper bound on the height of the graph over arbitrary intervals.

Proposition 3.3.4

There exists $C_3 > 0$ such that given $I \subset [0,1]$ and n maximal such that $\text{diam}(\hat{T}^n(I)) \leq 1$, *given $x \in I$*

$$\text{osc}(\widehat{W}, I) \leq C_3 \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x)).$$

Proof. Let $I \subset [0,1]$ and $n \geq 0$ be given. Let $J = \hat{T}^n(I)$. By Lemma 3.3.3

$$\begin{aligned}\text{osc}(\widehat{W}, I) &\leq \text{osc}(\widehat{W}, J) \prod_{j=0}^{n-1} \sup_{x\in\hat{T}^{j+1}(I)} \hat{\lambda}(\hat{T}^{-1}(x)) \\ &+ \sum_{i=0}^{n-1} \mathcal{S}\,\text{diam}(\hat{T}^{i+1}(I)) \left(\prod_{j=0}^{i-1} \sup_{x\in\hat{T}^{j+1}(I)} \hat{\lambda}(\hat{T}^{-1}(x)) \right)\end{aligned}$$

where $\mathcal{S} = \sup_{(x,y)\in \mathrm{graph}(\widehat{W})} |S(x)y|$ is finite. Then for each given $x \in \hat{T}^{j+1}(I)$ it is true that $\hat{\lambda}(\hat{T}^{-1}(x)) = \hat{\lambda}(x')$ for some $x' \in \hat{T}^j(I)$. Hence taking suprema over points in $\hat{T}^j(I)$ the above becomes

$$\begin{aligned}\mathrm{osc}(\widehat{W}, I) \;&\leq\; \mathrm{osc}(\widehat{W}, J)\prod_{j=0}^{n-1}\sup_{x'\in\hat{T}^j(I)}\hat{\lambda}(x')\\ &+\; \sum_{i=0}^{n-1}\mathcal{S}\,\mathrm{diam}(\hat{T}^{i+1}(I))\left(\prod_{j=0}^{i-1}\sup_{x'\in\hat{T}^j(I)}\hat{\lambda}(x')\right).\end{aligned}$$

However for each point $x' \in \hat{T}^j(I)$ which achieves the suprema in the above inequality, there exists some $x_j \in I$ such that $x' = \hat{T}^j(x_j)$ and $\hat{\lambda}(x) = \hat{\lambda}(\hat{T}^j(x_j))$. Thus the suprema become $\sup_{x_j\in I}\hat{\lambda}(\hat{T}^j(x_j))$. We let x_j for $j = 0,\ldots,n-1$ be the point in I which achieves the supremum of $\hat{\lambda}(\hat{T}^j(x_j))$. Hence for such points which achieve the suprema, the inequality becomes

$$\begin{aligned}\mathrm{osc}(\widehat{W}, I) \;&\leq\; \mathrm{osc}(\widehat{W}, J)\prod_{j=0}^{n-1}\hat{\lambda}(\hat{T}^j(x_j))\\ &+\; \sum_{i=0}^{n-1}\mathcal{S}\,\mathrm{diam}(\hat{T}^{i+1}(I))\left(\prod_{j=0}^{i-1}\hat{\lambda}(\hat{T}^j(x_j))\right).\end{aligned}$$

Then given $x \in I$, we let x_j as above and apply Proposition 3.3.1. Since $\mathrm{diam}(J) \leq 1$ there exists $C_2 > 0$, independent of I, n, x and x_j such that for each x_j, applying Proposition 3.3.1 to such J and I such that $\hat{T}^n(I) = J$ we obtain

$$\begin{aligned}\mathrm{osc}(\widehat{W}, I) \;&\leq\; C_2\mathrm{osc}(\widehat{W}, J)\prod_{j=0}^{n-1}\hat{\lambda}(\hat{T}^j(x))\\ &+\; C_2\sum_{i=0}^{n-1}\mathcal{S}\,\mathrm{diam}(\hat{T}^{i+1}(I))\prod_{j=0}^{i-1}\hat{\lambda}(\hat{T}^j(x)).\end{aligned}$$

It can then be shown that for given $x \in I$,

$$\begin{aligned}\mathrm{osc}(\widehat{W}, I) \;&\leq\; \prod_{j=0}^{n-1}\hat{\lambda}(\hat{T}^j(x))\\ &\times\; \left[C_2\mathrm{osc}(\widehat{W}, J) + C_2\sum_{i=0}^{n-1}\mathcal{S}\,\mathrm{diam}(\hat{T}^{i+1}(I))\left(\hat{\lambda}(\hat{T}^i(x))\ldots\hat{\lambda}(\hat{T}^{n-1}(x))\right)^{-1}\right].\end{aligned}$$

Moreover for $i = 0,\ldots,n-1$ we have that $\hat{T}^{i+1}(I) = \hat{T}^{-(n-1)+i}(J)$ and so

$$\mathrm{diam}(\hat{T}^{-(n-1)+i}(J)) \leq \mathrm{m}(T')^{-(n-1)+i}\mathrm{diam}(J).$$

Recalling our partial hyperbolicity assumption

$$\begin{aligned}
& C_2\mathrm{osc}(\widehat{W},J)+C_2\sum_{i=0}^{n-1}\mathcal{S}\,\mathrm{diam}(\hat{T}^{i+1}(I))\left(\hat{\lambda}(\hat{T}^i(x))\ldots\hat{\lambda}(\hat{T}^{n-1}(x))\right)^{-1} \\
\leq\ & C_2\mathrm{osc}(\widehat{W},J)+C_2\sum_{i=0}^{n-1}\mathcal{S}\,\mathrm{m}(T')^{-(n-1)+i}\mathrm{diam}(J)\mathrm{m}(\lambda)^{-n+i} \\
\leq\ & C_2\mathrm{osc}(\widehat{W},J)+C_2\mathcal{S}\,\mathrm{m}(T')\mathrm{diam}(J)(\mathrm{m}(T')\mathrm{m}(\lambda))^{-n}\sum_{i=0}^{n-1}(\mathrm{m}(T')\mathrm{m}(\lambda))^i \\
=\ & C_2\mathrm{osc}(\widehat{W},J)+\frac{C_2\mathcal{S}\,\mathrm{diam}(J)\mathrm{m}(T')}{\mathrm{m}(T')\mathrm{m}(\lambda)-1}\times\frac{(\mathrm{m}(T')\mathrm{m}(\lambda))^n-1}{(\mathrm{m}(T')\mathrm{m}(\lambda))^n} \\
\leq\ & C_2\mathrm{osc}(\widehat{W},J)+\frac{C_2\mathcal{S}\,\mathrm{diam}(J)\mathrm{m}(T')}{\mathrm{m}(T')\mathrm{m}(\lambda)-1} \\
\leq\ & C_2\mathrm{osc}(\widehat{W},[0,1])+\frac{C_2\mathcal{S}\,\mathrm{m}(T')}{\mathrm{m}(T')\mathrm{m}(\lambda)-1}
\end{aligned}$$

since $\mathrm{diam}(J)\leq 1$ and $\widehat{W}$ is 1-periodic. Therefore letting

$$C_3=C_2\mathrm{osc}(\widehat{W},[0,1])+\frac{C_2\mathcal{S}\,\mathrm{m}(T')}{\mathrm{m}(T')\mathrm{m}(\lambda)-1}>0$$

a constant independent of I, n and x the result follows. ❑

3.3.2 A lower bound

We now use a similar method to the above to give a lower bound on the height of the graph. Firstly we consider some $J\subset\mathbb{R}$ such that $\mathrm{diam}(J)\leq 1$ and take $n\geq 1$. We denote $I=\hat{T}^{-n}(J)$. We come back to the required result for arbitrary $I\subset[0,1]$ at a later stage. We firstly require the following estimates.

Lemma 3.3.5

Let $I\subset\mathbb{R}$ be a given interval. Then

$$\mathrm{osc}(\widehat{W},I)\geq\mathrm{osc}(\widehat{W},\hat{T}(I))\inf_{x\in\hat{T}(I)}\hat{\lambda}(\hat{T}^{-1}(x))-\mathcal{S}\mathrm{diam}(\hat{T}(I))$$

where $\mathcal{S}=\sup_{(x,y)\in\mathrm{graph}(\widehat{W})|_{[0,1]}}|S(x)y|$.

Proof. Let I be given. We look at the arc of the graph of $\widehat{W}$ over I and over $\hat{T}(I)$. By construction of the map $\hat{F}$, and since the graph is invariant under this map, $\mathrm{graph}(\widehat{W},I)=\hat{F}^{-1}(\mathrm{graph}(\widehat{W},\hat{T}(I)))$.

Let $x^+,x^-\in\hat{T}(I)$ be the points which achieve the height of the graph of $\widehat{W}$ above the interval $\hat{T}(I)$. That is $\text{osc}(\widehat{W}, \hat{T}(I)) = \widehat{W}(x^+) - \widehat{W}(x^-)$.

We now look to construct a smooth curve γ, the path of which we denote $\gamma(t) = (\gamma_H(t), \gamma_V(t))$, where H and V denote the horizontal and vertical directions respectively, and π_H and π_V the corresponding projection maps. In particular, we construct γ which approximates the arc of the graph between $\widehat{W}(x^+)$ and $\widehat{W}(x^-)$.

Let $\epsilon > 0$ be given. Therefore we can find $\delta > 0$ such that we construct such a path γ, a smooth curve which δ-approximates the graph between $\widehat{W}(x^+)$ and $\widehat{W}(x^-)$. In particular since $\hat{F}^{-1}$ is continuous, it is true that we can find such a δ-approximation γ, so that $\hat{F}^{-1}$ is an ϵ-approximation of the graph between $\hat{F}^{-1}(x, \widehat{W}(x^+))$ and $\hat{F}^{-1}(x, \widehat{W}(x^-))$ over I. We denote $\gamma_0 = \hat{F}^{-1}(\gamma)$ and the projection to the x-axis of this curve is contained within I. Then

$$\begin{aligned}
\text{osc}(\widehat{W}, I) &\geq \text{osc}(\widehat{W}, \pi_H(\gamma_0)) \\
&\geq \text{height}(\gamma_0) - \epsilon \\
&= \text{height}(\hat{F}^{-1}(\gamma)) - \epsilon.
\end{aligned}$$

Thus by the above construction, and using properties of the derivative of $\hat{F}^{-1}$,

$$\begin{aligned}
\text{height}(\gamma_0) &= \text{height}(\hat{F}^{-1}(\gamma)) \\
&= \int \left| \pi_V(\hat{F}^{-1}(\gamma))'(t) \right| \, dt \\
&\geq \int \hat{\lambda}(\hat{T}^{-1}(\gamma_H(t))) |\gamma_V'(t)| \, dt \\
&- \int |S(\gamma(t))| |\gamma_H'(t)| \, dt \\
&= \int B_1(t) \, dt - \int B_2(t) \, dt.
\end{aligned}$$

The projections to the x-axis of the arc $\gamma(t)$, is contained within $\hat{T}(I)$ and therefore

$$\hat{\lambda}(\hat{T}^{-1}(\gamma_H(t))) \geq \inf_{x \in \hat{T}(I)} \hat{\lambda}(\hat{T}^{-1}(x))$$

where as t varies, $\gamma_H(t)$ takes values in $\hat{T}(I)$. In addition $|\gamma_V'(t)|$ as t varies, is γ projected to the y axis. In particular, by construction of γ the projection of this arc to the y axis, is precisely the height of the graph over $\hat{T}(I)$. That is

$$\begin{aligned}
\int B_1(t) \, dt &\geq \inf_{x \in \hat{T}(I)} \hat{\lambda}(\hat{T}^{-1}(x)) \int |\gamma_V'(t)| \, dt \\
&= \inf_{x \in \hat{T}(I)} \hat{\lambda}(\hat{T}^{-1}(x)) \text{osc}(\widehat{W}, \hat{T}(I)).
\end{aligned}$$

In a similar way to the proof of Proposition 3.3.4 in the previous section

$$\int B_2(t)\,dt \le \mathcal{S}\int |\gamma_H'(t)|\,dt$$

where $\mathcal{S} = \sup_{(x,y)\in \mathrm{graph}(W)} |S(x)y|$. Now γ_H is the projection to the x-axis, which is contained within $\hat{T}(I)$, so the above integral is the width of this line segment. Therefore $\int B_2(t)\,dt \le \mathcal{S}\,\mathrm{diam}(\hat{T}(I))$.

Therefore, we have constructed γ, a sufficiently close approximation to the graph, so that under the continuous differentiable map $\hat{F}^{-1}$, $\gamma_0 = \hat{F}^{-1}(\gamma)$ approximates an arc of the graph over I arbitrarily closely. In particular

$$\mathrm{osc}(\widehat{W}, I) \ge \inf_{x\in\hat{T}(I)} \hat{\lambda}(\hat{T}^{-1}(x))\mathrm{osc}(\widehat{W}, \hat{T}(I)) - \mathcal{S}\,\mathrm{diam}(\hat{T}(I)) - \epsilon.$$

Since $\epsilon > 0$ was chosen to be arbitrary, the required result holds. ❑

Using Lemma 3.3.5 inductively, it is easy to show the following.

Lemma 3.3.6

Let I be given and $n \ge 0$ chosen to be the maximal integer such that $\mathrm{diam}(\hat{T}^n(I)) \le 1$. Then

$$\begin{aligned}\mathrm{osc}(\widehat{W}, I) &\ge \mathrm{osc}(\widehat{W}, \hat{T}^n(I)) \prod_{j=0}^{n-1} \inf_{x\in\hat{T}^{i+1}(I)} \hat{\lambda}(\hat{T}^{-1}(x)) \\ &- \sum_{i=0}^{n-1} \mathcal{S}\mathrm{diam}(\hat{T}^{i+1}(I)) \left(\prod_{j=0}^{i-1} \inf_{x\in\hat{T}^{j+1}(I)} \hat{\lambda}(\hat{T}^{-1}(x))\right)\end{aligned}$$

where $\prod_0^{-1}(\cdot) = 1$ and $\mathcal{S}$ is fixed and finite.

Using Lemma 3.4 and bounded variation estimates in Proposition 3.3.1 given J and associated n and I we have the following useful estimate.

Proposition 3.3.7

There exists constants $C_4, C_5 > 0$ such that given an interval J such that $\mathrm{diam}(J) \le 1$ and $n \ge 0$, we denote $I = \hat{T}^{-n}(J)$ and letting $x \in I$ be given

$$\mathrm{osc}(\widehat{W}, I) \ge \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x)) \left[C_4 \mathrm{osc}(\widehat{W}, J) - C_5 \mathrm{diam}(J)\right].$$

Proof. Let J be given and let I be as above and $x \in I$ be given. By Lemma 3.4

$$\begin{aligned}\mathrm{osc}(\widehat{W}, I) &\ge \mathrm{osc}(\widehat{W}, J) \prod_{j=0}^{n-1} \inf_{x\in\hat{T}^{i+1}(I)} \hat{\lambda}(\hat{T}^{-1}(x)) \\ &- \sum_{i=0}^{n-1} \mathcal{S}\,\mathrm{diam}(\hat{T}^{i+1}(I)) \left(\prod_{j=0}^{i-1} \inf_{x\in\hat{T}^{j+1}(I)} \hat{\lambda}(\hat{T}^{-1}(x))\right)\end{aligned}$$

where $\prod_0^{-1}(\cdot) = 1$ and by discussions in the previous section $\mathcal{S}$ is fixed and finite. Working analogously to the proof of the upper bound, for $x \in \hat{T}^{j+1}(I)$ we can find $x' \in \hat{T}^j(I)$ so that $\hat{\lambda}(T^{-1}(x')) = \hat{\lambda}(x)$. Moreover for each $x' \in \hat{T}^j(I)$ which achieves the infimum in the above inequality there exists a point $x_j \in I$ such that $x' = \hat{T}^j(x_j)$ and so $\hat{\lambda}(x) = \hat{\lambda}(\hat{T}^j(x_j))$. Then these infima become $\inf_{x_j \in I} \hat{\lambda}(\hat{T}^j(x_j))$. For each $j = 0, \ldots, n-1$ let $x_j \in I$ be the point which achieves the infimum of $\hat{\lambda}(\hat{T}^j(x_j))$. For such points we have that

$$\begin{aligned} \operatorname{osc}(\widehat{W}, I) \;&\geq\; \operatorname{osc}(\widehat{W}, J) \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x_j)) \\ &\quad - \sum_{i=0}^{n-1} \mathcal{S}\, \operatorname{diam}(\hat{T}^{i+1}(I)) \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x_j)). \end{aligned}$$

Given $x \in I$ and letting $x_j \in I$ as above, by Proposition 3.3.1 there exists $C_2 > 0$ independent of J, n, x and x_j such that

$$\begin{aligned} \operatorname{osc}(\widehat{W}, I) \;&\geq\; C_2^{-1} \operatorname{osc}(\widehat{W}, J) \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x)) \\ &\quad - C_2 \sum_{i=0}^{n-1} \mathcal{S}\, \operatorname{diam}(\hat{T}^{i+1}(I)) \prod_{j=0}^{i-1} \hat{\lambda}(\hat{T}^j(x)). \end{aligned}$$

Then we can show that

$$\begin{aligned} \operatorname{osc}(\widehat{W}, I) \;&\geq\; \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x)) \\ &\quad \times \left[C_2^{-1} \operatorname{osc}(\widehat{W}, j) - C_2 \sum_{i=0}^{n-1} \mathcal{S} \operatorname{diam}(\hat{T}^{i+1}(I)) \left(\hat{\lambda}(\hat{T}^i(x)) \ldots \hat{\lambda}(\hat{T}^{n-1}(x)) \right)^{-1} \right]. \end{aligned}$$

Since $i = 0, \ldots, n-1$, $\operatorname{diam}(\hat{T}^{i+1}(I)) = \operatorname{diam}(\hat{T}^{-(n-1)+i}(J)) \leq \mathrm{m}(T')^{-(n-1)+i} \operatorname{diam}(J)$. Moreover

$$\left(\hat{\lambda}(\hat{T}^i(x)) \ldots \hat{\lambda}(\hat{T}^{n-1}(x)) \right)^{-1} \leq \mathrm{m}(\lambda)^{-n+i}.$$

Then by our partial hyperbolicity assumption we have that

$$\begin{aligned} & C_2^{-1} \operatorname{osc}(\widehat{W}, J) - C_2 \sum_{i=0}^{n-1} \mathcal{S} \operatorname{diam}(\hat{T}^{i+1}(I)) \left(\hat{\lambda}(\hat{T}^i(x)) \ldots \hat{\lambda}(\hat{T}^{n-1}(x)) \right)^{-1} \\ \geq\;& C_2^{-1} \operatorname{osc}(\widehat{W}, J) - C_2 \mathcal{S} \mathrm{m}(T') \operatorname{diam}(J) (\mathrm{m}(T')\mathrm{m}(\lambda))^{-n} \sum_{i=0}^{n-1} (\mathrm{m}(T')\mathrm{m}(\lambda))^i \\ \geq\;& C_2^{-1} \operatorname{osc}(\widehat{W}, J) - \frac{C_2 \mathcal{S} \mathrm{m}(T') \operatorname{diam}(J)}{\mathrm{m}(T')\mathrm{m}(\lambda) - 1}. \end{aligned}$$

Therefore setting $C_4 = C_2^{-1}$ and $C_5 = \frac{C_2 \mathcal{S} \mathrm{m}(T')}{\mathrm{m}(T')\mathrm{m}(\lambda) - 1}$ we have our result. ❑

We now use the above estimate, which holds for all J to give a lower bound on the height of the graph over arbitrary intervals.

Proposition 3.3.8

There exists a constant $C_6 > 0$ such that given $I \subset [0,1]$ and n maximal such that $\mathrm{diam}(\hat{T}^n(I)) \leq 1$ and for $x \in I$ given, provided the non-degeneracy assumption as in Section 2.3 it is true that

$$\mathrm{osc}(\widehat{W}, I) \geq C_6 \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x)).$$

Proof. Let $x \in I$ be given. Define $\hat{T}^n(I) = J$. Since the non-degeneracy assumption holds, $\widehat{W}$ is nowhere differentiable. Therefore given $x \in I$, $\hat{T}^n(x) \in J$ and $\widehat{W}$ is not differentiable at $\hat{T}^n(x)$. Therefore there exists some neighbourhood of $\hat{T}^n(x)$, which we call $J' \subset J$ such that $\mathrm{osc}(\widehat{W}, J')/\mathrm{diam}(J')$ is as large as we like. That is, choose J' such that $\mathrm{osc}(\widehat{W}, J')/\mathrm{diam}(J') > C_5/C_4$, where $C_4, C_5 > 0$ as in Proposition 3.3.7 are independent of J'. Then we can let $C_4 \mathrm{osc}(\widehat{W}, J') - C_5 \mathrm{diam}(J') = C_6 > 0$.

Take $I' = \hat{T}^{-n} J'$ so that $x \in I'$ and $I' \subset I$. Then applying Proposition 3.3.7 to J' and associated I' to see that we can find such a constant $C_6 > 0$ independent of I, n and $x \in I$ such that

$$\mathrm{osc}(\widehat{W}, I') \geq C_6 \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x))$$

where $x \in I$. Moreover $I' \subset I$ and therefore

$$\mathrm{osc}(\widehat{W}, I) \geq C_6 \prod_{j=0}^{n-1} \hat{\lambda}(\hat{T}^j(x))$$

as required. ❏

We can now give the following full bound which we will refer to throughout later work. Note that we shall use the easier notation in the following, where constants are numbered for ease. The only important fact about such constants is that they are independent of the given interval, and of the chosen point in such an interval.

Proposition 3.3.9

There exists a constant $C_2 > 0$ such that given $I \subset [0,1]$, n chosen maximal such that $\mathrm{diam}(\hat{T}^n(I)) \leq 1$ and $x \in I$ given

$$C_2^{-1} \prod_{j=0}^{n-1} \lambda(T^j(x)) \leq \mathrm{osc}(W, I) \leq C_2 \prod_{j=0}^{n-1} \lambda(T^j(x)).$$

Proof. By Proposition 3.3.4 and Proposition 3.3.8, letting $C_2 \geq \max(C_3, C_6^{-1})$ and by discussions on the lift in Section 2.8 the result follows. ❏

Chapter 4

$\mathcal{K}$-dimension of invariant graphs

4.1 Introduction

In this chapter we study invariant graphs of skew products as introduced in Chapter 2 under different levels of generality, and use techniques in thermodynamic formalism and ergodic theory to prove dimension theoretic results for such graphs. In particular we consider the notion of $\mathcal{K}$-dimension, as introduced in Chapter 2 and look to give results regarding the $\mathcal{K}$-dimension for the class of invariant graphs introduced. In particular, we look to relate the dimension to the zero of a certain equation involving topological pressure. We firstly give an upper bound for box dimension of the graph, as in [Be, Walk], which will then yield an upper bound for the $\mathcal{K}$-dimension. We then use estimates on the height of the graph to give a lower bound, and investigate cases in which equality of dimension holds. The particular value for dimension in which we are interested arises as a solution to a pressure equation, and this value is also conjectured to co-incide with the Hausdorff dimension of such graphs, although as discussed in Chapter 1 proving this is significantly more difficult.

We study the invariant graphs of skew-products F, as in the affine case given in (2.5) and use the estimates on the height and diameter of the graph as given by Proposition 3.2.2 and Proposition 3.3.9. As in Section 2.5 we choose a Markov partition with respect to the b-to-one uniformly expanding map T. That is, given a cylinder of rank n, namely C_n it holds that $\mathrm{diam}(\hat{T}^n(C_n)) = 1$. In particular, $T^n(C_n) = S^1$. By Proposition 3.3.4 there exists $C_1 > 0$ such that given $n \geq 0$ and a cylinder of rank n

denoted C_n, and given a point $x \in C_n$,

$$C_1^{-1} \prod_{j=0}^{n-1} \left|T'(T^j(x))\right|^{-1} \leq \operatorname{diam}(C_n) \leq C_1 \prod_{j=0}^{n-1} \left|T'(T^j(x))\right|^{-1}.$$

Alternatively if $x \in C_n$, a point in a cylinder of rank n, has symbolic coding $(x_0, \ldots, x_{n-1}, \ldots)$, we construct the cylinder of rank n containing x by applying the corresponding sequence of inverse branches to the unit circle, so that $C_n = T_{x_0,\ldots,x_{n-1}}(S^1)$. It follows that

$$C_1^{-1} \prod_{j=0}^{n-1} \left|T'_{x_j}(T^j(x))\right| \leq \operatorname{diam}(C_n) \leq C_1 \prod_{j=0}^{n-1} \left|T'_{x_j}(T^j(x))\right|.$$

We use both these equivalent estimates throughout.

Recall that given $\delta > 0$, we can construct the δ-Moran cover of S^1, which is a cover consisting of cylinders of rank no less than some $n(\delta)$, which have diameter at most δ. However for such cylinders in the cover, the cylinder of rank one less, has diameter greater than δ. In the following we use the circle map T and function of the form $W : S^1 \to \mathbb{R}$ where graph(W) is invariant under skew-products of the form (2.1). We also recall that given a cylinder in some δ-Moran cover, the number of cylinders in such a cover which have non-empty intersection with a ball of radius δ is bounded above by some constant M, which is independent of δ. We call this the Moran multiplicity factor.

4.2 The box dimension of graph(W)

In this section we firstly establish an upper bound for the box dimension of graph(W) invariant under (2.1), which will give us an upper bound for the $\mathcal{K}$-dimension. We then give some results [Be, Walk, Pe] which give equality of box dimension for any open set of the graph. We begin with the following useful lemma, as in [Pe Theorem 13.1].

Lemma 4.2.1

Let $\overline{d} = \overline{\dim}_B(\text{graph}(W)$ and let $\epsilon > 0$ be given. Then there exists some $N \in \mathbb{N}$ such that for all $\delta > 0$ sufficiently small, where $\mathcal{C}_\delta$ is the δ-Moran cover of S^1,

$$\text{card } \left\{ j \mid C^{(j)} \in \mathcal{C}_\delta,\ \text{rank}(C^{(j)}) = N \right\} \geq \delta^{2\epsilon - \overline{d}}.$$

Remark. The above lemma simply says that given a δ-Moran cover of S^1 we know that if δ is sufficiently small, there exists some positive integer N such that the number of cylinders in the cover which have rank N is bounded from below.

Proof. Let $\epsilon > 0$ be given. By definition of upper box dimension, there exists some $\delta > 0$ sufficiently small such that

$$N_\delta(\text{graph}(W)) \geq \delta^{\epsilon - \overline{d}}.$$

Now take the δ-Moran cover $\mathcal{C}_\delta$ of S^1, consisting of cylinders $C^{(j)}$ of rank $n(j)$ for $j = 1, \ldots, N(\delta)$. Notice that this cover need not be optimal. In particular $N(\delta) \geq N_\delta(\text{graph}(W))$.

By construction of the Moran cover, for each $j = 1, \ldots, N(\delta)$

$$\frac{\delta}{|T'|_\infty} \leq \text{diam}(C^{(j)}) \leq \delta.$$

Then by Proposition 3.2.2 there exists $C_1 > 0$ such that for $x \in C^{(j)}$ $x_0, \ldots, x_{n(j)-1} \in \{0, 1, \ldots, k-1\}$ such that $C^{(j)} = T_{x_0, \ldots, x_{n(j)-1}}(S^1)$ it holds that

$$\frac{\delta}{C_1 |T'|_\infty} \leq \prod_{j=0}^{n(j)-1} \left| T'_{x_j}(T^j(x)) \right| \leq C_1 \delta.$$

Alternatively

$$\frac{1}{C_1 \delta} \leq \prod_{j=0}^{n(j)-1} \left| T'(T^j(x)) \right| \leq \frac{C_1 |T'|_\infty}{\delta} \tag{4.1}$$

for each $j = 1, \ldots, N(\delta)$.

Therefore

$$\frac{1}{C_1 \delta} \leq |T'|_\infty^{n(j)} \quad \text{and} \quad (\text{m}(T'))^{n(j)} \leq \frac{C_1 |T'|_\infty}{\delta}.$$

Taking logarithms

$$\frac{\log(1/C_1\delta)}{\log |T'|_\infty} \leq n(j) \leq \frac{\log(C_1 |T'|_\infty / \delta)}{\log(\text{m}(T'))}.$$

For ease of notation we denote the above inequalities as $B' \leq n(j) \leq B''$ where $B', B'' > 0$ and δ is sufficiently small. In particular the integer $n(j)$ can take on at most some $B = \lfloor B'' \rfloor - \lceil B' \rceil$ possible values.

Consider the $N(\delta)$ cylinders in $\mathcal{C}_\delta$, the ranks of which take one of B values. Then there exists some value of $n(j)$, such that at least $N(\delta)/B$ cylinders in our cover have rank $n(j)$. Thus there exists some positive integer $N \in [B', B'']$ such that

$$\text{card}\,\{j \mid n(j) = N\} \geq \frac{N(\delta)}{B} \geq \frac{N_\delta(\text{graph}(W))}{B} \geq \frac{\delta^{\epsilon - \overline{d}}}{B''},$$

where

$$B'' = \frac{\log(C_1|T'|_\infty/\delta)}{\log(\mathrm{m}(T'))}.$$

Hence take δ sufficiently small such that B'' is large enough so that

$$\mathrm{card}\,\{j \mid n(j) = N\} \geq \delta^{2\epsilon-\overline{d}}$$

and the result holds. ❑

Recall the notion of topological pressure introduced in Section 2.6. In the following we study the topological pressure involving Hölder potentials and maps of the form

$$s \mapsto P\left((1-s)\log|T'| + \log\lambda\right).$$

By discussions in previous sections and in [Bo,Walk] such a map is decreasing in s, and there exists a unique value of s such that this map is zero.

Proposition 4.2.2

Suppose that s_0 is the unique solution to the pressure equation $P((1-s_0)\log|T'| + \log\lambda) = 0$. Then

$$\overline{\dim}_B(\mathrm{graph}(W)) \leq s_0.$$

Proof. We work in a way similar to [Pe]. Let $\overline{d} = \overline{\dim}_B(\mathrm{graph}(W))$. We show that $\overline{d} \leq s_0$. Let $\epsilon > 0$ be given and choose $\delta > 0$ sufficiently small such that Lemma 4.2.1 holds and such that

$$N_\delta(\mathrm{graph}(W)) \geq \delta^{\epsilon-\overline{d}}.$$

Let $\mathcal{C}_\delta$ be the δ-Moran cover of S^1 consisting of cylinders $C^{(j)}$ each having rank $n(j)$. By Lemma 4.2.1 there exists a positive integer N, such that

$$\mathrm{card}\left\{j \mid C^{(j)} \in \mathcal{C},\ n(j) = N\right\} \geq \delta^{2\epsilon-\overline{d}}.$$

Let $\mathcal{C}$ be an arbitrary cover of S^1 by cylinders of rank N. Then

$$\sum_{C\in\mathcal{C}} \sup_{x\in C}\left[\left(\prod_{j=0}^{N-1}\left|T'(T^j(x))\right|\right)^{1-(\overline{d}-2\epsilon)} \prod_{j=0}^{N-1}\lambda(T^j(x))\right] \tag{4.2}$$

$$\geq \sum_{C\in\mathcal{C}} \sup_{x\in C}\left[\left(\prod_{j=0}^{N-1}\left|T'(T^j(x))\right|\right)^{-(\overline{d}-2\epsilon)} \mathrm{m}(T')^N\mathrm{m}(\lambda)^N\right]$$

$$\geq \sum_{j:n(j)=N} \sup_{x\in C^{(j)}}\left[\left(\prod_{j=0}^{N-1}\left|T'(T^j(x))\right|\right)^{-(\overline{d}-2\epsilon)} \left(\mathrm{m}(T')\mathrm{m}(\lambda)\right)^N\right].$$

Then by (4.1) and using Lemma 4.2.1

$$
\begin{aligned}
(4.2) \quad &\geq \sum_{j:n(j)=N} \left(\frac{\delta}{C_1|T'|_\infty}\right)^{(\overline{d}-2\epsilon)} (\mathrm{m}(T')\mathrm{m}(\lambda))^N \\
&\geq \left(\frac{1}{C_1|T'|_\infty}\right)^{(\overline{d}-2\epsilon)} \delta^{\overline{d}-2\epsilon}\delta^{2\epsilon-\overline{d}} (\mathrm{m}(T')\mathrm{m}(\lambda))^N \\
&\geq C_3 (\mathrm{m}(T')\mathrm{m}(\lambda))^N
\end{aligned}
$$

where $C_3 > 0$ is a constant and δ is sufficiently small.

That is, given $\delta > 0$ sufficiently small, we have a δ-Moran cover consisting of cylinders of rank no less than some n, and for $N \geq n$, recalling the definition of topological pressure in Section 2.6 and by the above inequalities for $\phi = ((1-(\overline{d} - 2\epsilon)) \log |T'| + \log \lambda)$, one can see that

$$\Lambda(S^1, \phi, N) \geq C_3 (\mathrm{m}(T')\mathrm{m}(\lambda))^N$$

where $C_3 > 0$ is independent of n. Note that we took $\mathcal{C}$ to be some arbitrary cover of $[0,1]$ consisting of cylinders of rank $N \geq n$. Therefore by the above and by remarks in Chapter 2 regarding topological pressure [Pe, Chapter 11], and since the partial hyperbolicity assumption gives that $(\mathrm{m}(T')\mathrm{m}(\lambda)) > 1$,

$$
\begin{aligned}
& P((1-(\overline{d}-2\epsilon)) \log |T'| + \log \lambda) \\
= \; & \overline{CP}((1-(\overline{d}-2\epsilon)) \log |T'| + \log \lambda) \\
= \; & \limsup_{n\to\infty} \frac{1}{n} \log \Lambda(S^1, (1-(\overline{d}-2\epsilon)) \log |T'| + \log \lambda, N) \\
\geq \; & \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{C\in\mathcal{C}} \sup_{x\in C} \left[\left(\prod_{j=0}^{N-1} |T'(T^j(x))| \right)^{1-(\overline{d}-2\epsilon)} \prod_{j=0}^{N-1} \lambda(T^j(x)) \right] \right) \\
\geq \; & \lim_{n\to\infty} \frac{\log C_3 (\mathrm{m}(T')\mathrm{m}(\lambda))^N}{n} \\
= \; & \lim_{n\to\infty} \frac{\log C_3 + \log (\mathrm{m}(T')\mathrm{m}(\lambda))^N}{n} \\
> \; & \lim_{n\to\infty} \frac{\log C_4}{n} \\
= \; & 0
\end{aligned}
$$

where $C_4 > 1$ and is independent of n.

Recall that s_0 is the unique solution to the pressure equation $P((1-s_0)\log|T'| + \log \lambda) = 0$. Pressure is strictly decreasing, hence we have shown that $\overline{d} - 2\epsilon < s_0$. Since $\epsilon > 0$ is arbitrary it follows that $\overline{d} \leq s_0$. ❏

We now give a result on the equality of box dimension of the graph restricted to any open subset of the graph. The following theorem also gives equality of the upper and lower box dimensions of the graph.

Theorem 4.2.3

Suppose that U is a non-empty open subset of S^1. Then

$$\dim_B(\text{graph}(W|_U)) = \dim_B(\text{graph}(W)) = s_0$$

where s_0 is the unique solution to the pressure equation $P((1-s_0)\log|T'|+\log\lambda) = 0$.

Remark. Let $U \subset S^1$ be a non-empty open subset. Clearly $\text{graph}(W|_U) \subset \text{graph}(W)$ and so $\overline{\dim}_B(\text{graph}(W|_U)) \leq \overline{\dim}_B(\text{graph}(W))$ and $\underline{\dim}_B(\text{graph}(W|_U)) \leq \underline{\dim}_B(\text{graph}(W))$. Then by Proposition 4.2.2

$$\underline{\dim}_B(\text{graph}(W|_U)) \leq \overline{\dim}_B(\text{graph}(W|_U)) \leq s_0$$

and

$$\underline{\dim}_B(\text{graph}(W)) \leq \overline{\dim}_B(\text{graph}(W)) \leq s_0.$$

Thus to prove the proposition it will be sufficient to show that for any non-empty open set $U \subset S^1$,

$$\underline{\dim}_B(\text{graph}(W|_U)) \geq s_0.$$

Proof. Let $U \subset [0,1]$ be a non-empty open subset. Let $\underline{d} = \underline{\dim}_B(\text{graph}(W|_U))$. Let s_0 be the unique solution to the pressure equation $P((1-s_0)\log|T'|+\log\lambda) = 0$. We show that $s_0 \leq \underline{d}$. Fix $\epsilon > 0$. By definition of lower box dimension, there exists some $\delta > 0$ sufficiently small such that

$$N_\delta(\text{graph}(W|_U)) \leq \delta^{-(\underline{d}+\epsilon)}. \tag{4.3}$$

Recall that we can take $N_\delta(\text{graph}(W|_U))$ to be the least number of squares of side length δ required to cover the graph above U. The x-projection of these squares will form an open cover $\mathcal{I}$ of U consisting of open intervals I_ℓ where $\ell = 1, \ldots, N_\delta(\text{graph}(W|_U))$ and the diameter of each of the I_ℓ is δ.

For such δ we take the δ-Moran cover of U, denoted $\mathcal{C}_\delta$, consisting of cylinders $C^{(j)}$ of diameter $\delta_j \leq \delta$ and rank $n(j) \geq n(\delta)$. Now for each I_ℓ there exists at most M cylinders in $\mathcal{C}_\delta$ which have non empty intersection with each I_ℓ, where $M > 0$ is the

Moran multiplicity factor. By construction of such a Moran cover and by Propositions 3.2.2 and 3.3.9 there exists $C_1, C_2 > 0$ such that

$$\sum_{C^{(j)}\in\mathcal{C}_\delta} \sup_{x\in C^{(j)}} \left[\left(\prod_{i=0}^{n(j)-1} |T'(T^i(x))| \right)^{1-(\underline{d}+\epsilon)} \prod_{i=0}^{n(j)-1} \lambda(T^i(x)) \right] \tag{4.4}$$

$$\begin{aligned} &\leq (C_1)^{\underline{d}+\epsilon-1} C_2 \sum_{C^{(j)}\in\mathcal{C}_\delta} (\mathrm{diam}(C^{(j)}))^{\underline{d}+\epsilon-1} \mathrm{osc}(W|_U, C^{(j)}) \\ &\leq (C_1)^{\underline{d}+\epsilon-1} C_2\, \delta^{\underline{d}+\epsilon} \sum_{C^{(j)}\in\mathcal{C}_\delta} \frac{\mathrm{osc}(W|_U, C^{(j)})}{\mathrm{diam}(C^{(j)})}. \end{aligned}$$

Recall that $\mathcal{I}$ is an open cover of U consisting of intervals I_ℓ, each of which is covered by at most M cylinders in the Moran cover. If we consider the height of the graph over a cylinder, this is bounded from above by the sum of the heights of the intervals which have non-empty intersection with such a cylinder. Moreover, as we consider each cylinder, when summing such heights, each interval is counted at most $M > 0$ times, by the property of Moran covers.

In addition, by construction of the δ-Moran cover, for each $C^{(j)}$, $\mathrm{diam}(C^{(j)}) > |T'|_\infty^{-1}\delta$, and since $\mathrm{diam}(I_\ell) = \delta$ for each ℓ, we have that

$$\mathrm{diam}(C^{(j)}) > |T'|_\infty^{-1} \mathrm{diam}(I_\ell)$$

for all i and all ℓ.

In particular since each interval is counted at most $M > 0$ times when summing over intervals with non-empty intersection with each cylinder it can be seen that

$$\begin{aligned} \sum_{C^{(j)}\in\mathcal{C}_\delta} \frac{\mathrm{osc}(W|_U, C^{(j)})}{\mathrm{diam}(C^{(j)})} &\leq M|T'|_\infty \sum_{I_\ell\in\mathcal{I}} \frac{\mathrm{osc}(W|_U, I_\ell)}{\mathrm{diam}(I_\ell)} \\ &\leq M|T'|_\infty \sum_{I_\ell\in\mathcal{I}} q_{I_\ell} \end{aligned}$$

where the last step comes from (2.13).

By construction of $\mathcal{I}$

$$q_{I_\ell} = N_\delta(\mathrm{graph}(W|_{I_\ell}))$$

for each $\ell = 1, \ldots, M_\delta(\mathrm{graph}(W|_U))$. Hence by construction of the cover $\mathcal{I}$ one can see that

$$\sum_{I_\ell\in\mathcal{I}} N_\delta(\mathrm{graph}(W|_{I_\ell})) = N_\delta(\mathrm{graph}(W|_U)).$$

Therefore

$$\sum_{C^{(j)}\in\mathcal{C}_\delta} \frac{\mathrm{osc}(W|_U, C^{(j)})}{\mathrm{diam}(C^{(j)})} \le M|T'|_\infty N_\delta(\mathrm{graph}(W|_U)).$$

Hence by (4.3) and the above,

$$(4.4) \le (C_1)^{\underline{d}+\epsilon-1} C_2 M|T'|_\infty\, \delta^{\underline{d}+\epsilon}\, \delta^{-(\underline{d}+\epsilon)}.$$

Therefore for $\delta > 0$ sufficiently small and $n = n(\delta)$ consequently sufficiently large as in the construction of the Moran cover, there exists $C_5 = \left[(C_1)^{\underline{d}+\epsilon-1} C_2 M|T'|_\infty\right] > 0$ such that

$$\Gamma(U, (1 - (\underline{d} + \epsilon)) \log|T'| + \log\lambda, n) \le C_5.$$

Moreover

$$P((1 - (\underline{d} + \epsilon)) \log|T'| + \log\lambda) \le \lim_{n\to\infty} \frac{\log C_5}{n} = 0$$

where $n = n(\delta)$ in the construction of the δ-Moran cover, and δ is suffiently small. That is $n \to \infty$ uniformly as we let $\delta \to 0$. Hence $P((1 - (\underline{d} + \epsilon)) \log|T'| + \log\lambda) \le 0$. Since pressure is strictly decreasing in s one can see that $d + \epsilon \ge s_0$, and since $\epsilon > 0$ is arbitrary, the result follows. ❏

4.3 A lower bound for the $\mathcal{K}$-dimension of $\mathrm{graph}(W)$

In the following section we consider lower bounds for the $\mathcal{K}$-dimension of the graph. We firstly show that equality of dimension occurs in a simpler setting, and then give a different lower bound in the more general case. In the next section we make some remarks on where equality of dimension occurs in such a general setting.

4.3.1 On the case when $T(x) = bx \bmod 1$

Consider first the case in which $T(x) = bx \bmod 1$ and $\lambda(x) \in (0, 1)$ for all $x \in S^1$. That is we consider the graph invariant under skew products of the form (2.5), but restrict ourselves to the case in which $T(x) = bx \bmod 1$. Then our partial hyperbolicity assumption becomes $\mathrm{m}(\lambda)b > 1$.

Given $x \in S^1$ and $n \ge 0$ it is clear that $\mathrm{diam}(C_n(x)) = b^{-n}$. In addition there exists $C_2 > 0$ such that given $I \subset [0, 1]$ there exists n such that given $x \in I$, by Proposition

3.3.9

$$C_2^{-1}\prod_{j=0}^{n-1}\lambda(T^j(x)) \leq \operatorname{osc}(W,I) \leq C_2\prod_{j=0}^{n-1}\lambda(T^j(x)).$$

Let $I \subset [0,1]$. Then we require n such that $\operatorname{diam}(b^n(I)) \leq 1$ but $\operatorname{diam}(b^{n+1}(I)) > 1$. In particular for such n

$$\operatorname{diam}(I) \leq b^{-n}$$

and

$$\operatorname{diam}(I) > b^{-(n+1)}.$$

Thus any cylinder of rank $n+1$ has diameter less than I but a cylinder of rank n has diameter at least that of I. We are now ready to give a lower bound for $\mathcal{K}$-dimension in the case where the gradient of the map T is constant.

Proposition 4.3.1

Let $\operatorname{graph}(W)$ *be the invariant graph of* F *as in (2.5) where* $T(x) = bx \bmod 1$. *Suppose that the non-degeneracy assumption from Section 2.3 holds. Let* s_0 *be the unique solution to the pressure equation* $P((1-s_0)\log b + \log\lambda) = 0$. *Then*

$$\dim_{\mathcal{K}}(\operatorname{graph}(W)) \geq s_0.$$

Proof. Let $d = \dim_{\mathcal{K}}(\operatorname{graph}(W))$. Let $\epsilon > 0$ be given arbitrarily. It will then be sufficient to show that $d + \epsilon \geq s_0$. By monotonicity of pressure as in Section 2.6 this is equivalent to showing that

$$P((1-(d+\epsilon))\log b + \log\lambda) \leq 0.$$

Given $\epsilon > 0$, by definition of $\mathcal{K}$-dimension there exists $\delta > 0$ sufficiently small and a finite cover $\{I_\ell\}_{\ell=1}^{K}$ of S^1 such that $|I_\ell| = \delta_\ell \leq \delta$ for all ℓ and

$$\sum_{\ell=1}^{K}\operatorname{diam}(I_\ell)^{d+\epsilon}q_{I_\ell} \leq 1. \tag{4.5}$$

For each I_ℓ an interval of diameter δ_ℓ in the cover, choose $n(\delta_\ell)$ such that

$$\operatorname{diam}(I_\ell) \leq b^{-n(\delta_\ell)}$$

and

$$\operatorname{diam}(I_\ell) > b^{-(n(\delta_\ell)+1)}$$

as in the above discussions. Then for each such I_ℓ consider the cover of S^1 consisting of cylinders of rank $n(\delta_\ell)+1$. Thus these cylinders all have diameter less than the diameter of I_ℓ. Let $\mathcal{C}_{\delta_\ell} = \left\{ C_\ell^{(j)} \mid j = 1, \ldots, m(\ell) \right\}$ be the cylinders in such a cover of rank $n(\delta_\ell)+1$ which have non-empty intersection with I_ℓ. That is $\mathcal{C}_{\delta_\ell}$ forms a cover of I_ℓ consisting of cylinders each of rank $n(\delta_\ell)+1$. However note that there exists $M > 0$ independent of δ_ℓ such that $m(\ell) \leq M$ for each $\ell = 1, \ldots, K$. Then let

$$\mathcal{C}_\delta = \left\{ C_\ell^{(j)} \mid j = 1, \ldots, m(\ell)\, \ell = 1, \ldots, K \right\}.$$

This forms a cover of S^1 consisting of cylinders $C_\ell^{(j)}$ of rank $n(\delta_\ell)$ and diameter

$$\mathrm{diam}(C_\ell^{(j)}) = b^{-(n(\delta_\ell)+1)} < |I_\ell|.$$

Now let $\phi = (1-(d+\epsilon))\log b + \log \lambda$. Given $\epsilon > 0$ choose $\delta > 0$ such that (4.5) holds. Moreover given n, choose δ such that $n(\delta_\ell) > n$ for all $\delta_\ell \leq \delta$ as in the above construction. Then as in the definition of topological pressure in Section 2.6, for $\mathcal{C}_\delta$ the cover as above of cylinders of rank $n(\delta_\ell)+1$

$$\begin{aligned} \Gamma(S^1, \phi, n) &\leq \sum_{C_\ell^{(j)} \in \mathcal{C}_\delta} \exp\left\{ \sup_{x \in C_\ell^{(j)}} \sum_{j=0}^{n(\delta_\ell)} \phi(T^j(x)) \right\} \\ &\leq \sum_{\ell=1}^{K} \sum_{j=1}^{m(\ell)} \exp\left\{ \sup_{x \in C_\ell^{(j)}} \log \prod_{j=0}^{n(\delta_\ell)} (b^{-1})^{d+\epsilon-1} \prod_{j=0}^{n(\delta_\ell)} \lambda(T^j(x)) \right\}. \end{aligned}$$

Since $x \in C_\ell^{(j)}$, a point in a cylinder of rank $n(\delta_\ell)+1$, by Proposition (3.3.9) there exists $C_2 > 0$ such that

$$\Gamma(S^1, \phi, n) \leq C_2 \sum_{\ell=1}^{K} \sum_{j=1}^{m(\ell)} \left(b^{-(n(\delta_\ell)+1)} \right)^{d+\epsilon-1} \mathrm{osc}(W, C_\ell^{(j)}).$$

Note that by construction of the cover $\mathcal{C}_\delta$, for each cylinder $\mathrm{diam}(C_\ell^{(j)}) = b^{-(n(\delta_\ell)+1)}$. Therefore

$$\begin{aligned} &\sum_{j=1}^{m(\ell)} \left(b^{-(n(\delta)+1)} \right)^{d+\epsilon-1} \mathrm{osc}(W, C_\ell^{(j)}) \\ \leq\; &\sum_{j=1}^{m(\ell)} \mathrm{diam}(C_\ell^{(j)})^{d+\epsilon} \frac{\mathrm{osc}(W, C_\ell^{(j)})}{\mathrm{diam}(C_\ell^{(j)})}. \end{aligned}$$

It is clear that $\mathrm{diam}(C_\ell^{(j)}) < \mathrm{diam}(I_\ell)$ and by construction of the cover

$$\mathrm{diam}(C_\ell^{(j)}) \geq b^{-1} \mathrm{diam}(I_\ell)$$

for each $j = 1, \ldots, m(\ell)$ and $\ell = 1, \ldots, K$.

Moreover for each I_ℓ and $C_\ell^{(j)}$ with non-empty intersection, by use of Proposition 3.3.9 twice, there exists $x \in C_\ell^{(j)} \cap I_\ell$ and independent constants $C_2' C_2'' > 0$ such that

$$\begin{aligned}
\operatorname{osc}(W, C_\ell^{(j)}) &\leq C_2' \prod_{j=0}^{n(\delta_\ell)} \lambda(T^j(x)) \\
&\leq C_2' |\lambda|_\infty \prod_{j=0}^{n(\delta_\ell)-1} \lambda(T^j(x)) \\
&\leq C_2' C_2'' |\lambda|_\infty \operatorname{osc}(W, I_\ell).
\end{aligned}$$

Hence for each $j = 1, \ldots, m(\ell)$

$$\begin{aligned}
\operatorname{diam}(C_\ell^{(j)})^{d+\epsilon} \frac{\operatorname{osc}(W, C_\ell^{(j)})}{\operatorname{diam}(C_\ell^{(j)})} &\leq C_2' C_2'' |\lambda|_\infty b \operatorname{diam}(I_\ell)^{d+\epsilon} \frac{\operatorname{osc}(W, I_\ell)}{\operatorname{diam}(I_\ell)} \\
&\leq C_2' C_2'' |\lambda|_\infty b \operatorname{diam}(I_\ell)^{d+\epsilon} q_{I_\ell}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\Gamma(S^1, \phi, n) &\leq C_2 \sum_{\ell=1}^{K} \sum_{j=1}^{m(\ell)} C_2' C_2'' |\lambda|_\infty \, b \operatorname{diam}(I_\ell)^{d+\epsilon} q_{I_\ell} \\
&\leq M C_2 C_2' C_2'' |\lambda|_\infty \, b \sum_{\ell=1}^{K} \operatorname{diam}(I_\ell)^{d+\epsilon} q_{I_\ell} \\
&\leq M C_2 C_2' C_2'' |\lambda|_\infty \, b
\end{aligned}$$

by (4.5).

Hence as $n \to \infty$, $\delta \to 0$ uniformly and so by the definition of topological pressure

$$\begin{aligned}
P((1 - (d+\epsilon)) \log b + \log \lambda) &= \lim_{n\to\infty} \frac{1}{n} \log \Gamma(S^1, \phi, n) \\
&\leq \lim_{n\to\infty} \frac{\log M C_2 C_2' C_2'' |\lambda|_\infty \, b}{n} \\
&= 0.
\end{aligned}$$

Therefore by properties of pressure as in Section 2.6 we have shown that $d + \epsilon \geq s_0$. Since ϵ was chosen arbitrarily the proposition follows. ❑

4.3.2 On the general case

We now return to the general case in which T is a uniformly expanding map of the circle, whose slope is not assumed to be constant. In the following, we consider invariant graphs graph(W) of a skew product of the form

$$F(x,y) = (T(x), \lambda(x)^{-1}(y - p(x)))$$

as in previous sections, where we assume the non-degeneracy assumption as in Section 2.3 and partial hyperbolicity assumption that $\mathrm{m}(T')\mathrm{m}(\lambda) > 1$. We can now give a lower bound on the $\mathcal{K}$-dimension of graph W.

Given $I \subset [0,1]$ there exists a maximal n such that $\hat{T}^n(I) = J$ and $\mathrm{diam}(J) \leq 1$. Thus Propositions 3.3.4 and 3.3.9 hold. In addition W is Hölder continuous of some exponent $\alpha \in (0,1)$, thus there exists $C_3 > 0$ independent of I such that $\mathrm{osc}(W, I) \leq C_3 \,\mathrm{diam}(I)^\alpha$ for all $I \subset [0,1]$.

If we firstly consider the case in which $\lambda \in (0,1)$ is constant, $T(x) = bx \bmod 1$ and p is Lipschitz continuous, then $W(x) = \sum_{j=0}^{\infty} \lambda^j p(T^j(x))$. For W to be Hölder continuous of some $\alpha \in (0,1)$ we require that there exists $C_3 > 0$ such that for all $x, y \in S^1$

$$|W(x) - W(y)| \leq C_3 |x - y|^\alpha.$$

Moreover since p is Lipschitz, then it is α-Hölder for any $\alpha \in (0,1]$. Therefore for α as above

$$\begin{aligned} |W(x) - W(y)| &\leq ||p||_\alpha \sum_{j=0}^{\infty} \lambda^j |T^j(x) - T^j(y)|^\alpha \\ &\leq ||p||_\alpha |x - y|^\alpha \sum_{j=0}^{\infty} (\lambda b^\alpha)^j. \end{aligned}$$

In particular we require that $\lambda b^\alpha < 1$ for this sum to converge. Moreover this requires that

$$\alpha < \frac{-\log \lambda}{\log b}.$$

In the more general case, for W to be α-Hölder we require that

$$|\lambda|_\infty (|T'|_\infty)^\alpha < 1.$$

Letting α be the Hölder exponent of W, recall that W is in fact C^α and no better [HNW]. Since W is not Lipschitz and by our partial hyperbolicity assumption, it is possible to choose $\beta \in (0,1)$, possibly close to 1, to be the minimal value such that

$$\mathrm{m}(\lambda)(\mathrm{m}(T'))^\beta \geq 1. \tag{4.6}$$

Then given $I \subset [0,1]$ and n to be maximal such that $\mathrm{diam}(\hat{T}^n(I)) \leq 1$ as in previous sections, we can use our assumptions on the diameter of I and the height of the graph over I to see that, for some $x \in I$ and $\beta \in (\alpha, 1)$ as above

$$\begin{aligned} \mathrm{osc}(W, I) &\geq C_2^{-1} \prod_{j=0}^{n-1} \lambda(T^j(x)) \\ &\geq C_2^{-1} (\mathrm{m}(\lambda))^n \\ &\geq C_2^{-1} (\mathrm{m}(T'))^{-n\beta} \\ &\geq C_2^{-1} \left(\prod_{j=0}^{n-1} |T'(T^j(x))|^{-1} \right)^\beta \\ &\geq C_2^{-1} C_1^\beta \mathrm{diam}(I)^\beta. \end{aligned}$$

The following observation will prove useful in our proof of the lower bound. Throughout we assume that the non-degeneracy condition from Section 2.3 holds, so that we have the lower bound on the height of the graph over intervals proven in Proposition 3.3.9

Lemma 4.3.2

Let $\epsilon > 0$ be given. Let $\delta > 0$ and $I \subset [0,1]$ such that $\mathrm{diam}(I) = \delta$. Suppose that C_δ is a cylinder of some rank $n(\delta)$ in a δ-Moran Cover of $[0,1]$ with non-empty intersection with I. Let $\alpha \in (0,1)$ be the Hölder exponent of W and $\beta \in (\alpha, 1)$ chosen such that $\mathrm{m}(\lambda)\mathrm{m}(T')^\beta \geq 1$. Fix $d \geq 0$. Then there exists C_4 independent of I and δ such that

$$\begin{aligned} \sup_{x \in C_\delta} & \sum_{j=0}^{n(\delta)-1} \left(\log |T'(T^j(x))|^{1-(d+\epsilon+(\beta-\alpha))} + \log \lambda(T^j(x)) \right) \\ \leq & \quad \log \left(C_4 \, \mathrm{diam}(I)^{d+\epsilon} \frac{\mathrm{osc}(W, I)}{\mathrm{diam}(I)} \right). \end{aligned}$$

Proof. Let I and C_δ be given. Then by construction of the Moran cover as in the statement of the lemma

$$\mathrm{diam}(C_\delta) \leq \mathrm{diam}(I). \tag{4.7}$$

However $T(C_\delta)$ is a cylinder of rank $n(\delta) - 1$ and has diameter strictly greater than that of I. In particular

$$\mathrm{diam}(C_\delta) > |T'|_\infty^{-1} \mathrm{diam}(I). \tag{4.8}$$

Let $\alpha \leq \beta$ be as in the statement of the lemma. Then as in the discussion above there exists $C_1, C_2 > 0$ independent of I such that for all $I \subset [0,1]$

$$\mathrm{osc}(W, I) \geq C_1^\beta C_2^{-1} \mathrm{diam}(I)^\beta. \tag{4.9}$$

Since $\mathrm{diam}(C_\delta) \leq \mathrm{diam}(I)$ and C_δ and I have non-empty intersection, there exists $J \subset [0,1]$ such that $I \cup C_\delta \subseteq J$ and $\mathrm{diam}(J) \leq 2\,\mathrm{diam}(I)$. By the Hölder property there exists $C_3 > 0$ such that for J given as in the above we have that $\mathrm{osc}(W,J) \leq C_3 \mathrm{diam}(J)^\alpha$. Then

$$\begin{aligned} \mathrm{osc}(W, C_\delta) &\leq \mathrm{osc}(W,J) \\ &\leq C_3\,\mathrm{diam}(J)^\alpha \\ &\leq C_3\,2^\alpha\,\mathrm{diam}(I)^\alpha. \end{aligned}$$

By this observation, Propositions 3.3.4 and 3.3.9, and using (4.7), (4.8) and (4.8) it can be shown that

$$\begin{aligned} &\sup_{x\in C_\delta} \sum_{j=0}^{n(\delta)-1} \left(\log |T'(T^j(x))|^{1-(d+\epsilon+(\beta-\alpha))} + \log \lambda(T^j(x))\right) \\ = \;& \sup_{x\in C_\delta} \log \left[\prod_{j=0}^{n(\delta)-1} \left(|T'(T^j(x))|^{-1}\right)^{d+\epsilon-1} \prod_{j=0}^{n(\delta)-1} \left(|T'(T^j(x))|^{-1}\right)^{\beta-\alpha} \prod_{j=0}^{n(\delta)-1} \lambda(T^j(x)) \right] \\ \leq \;& \sup_{x\in C_\delta} \log \left[(C_1^{-1})^{d+\epsilon-1} (C_1^{-1})^{\beta-\alpha} C_2^{-1} \mathrm{diam}(C_\delta)^{d+\epsilon-1} \mathrm{diam}(C_\delta)^{\beta-\alpha} \mathrm{osc}(W, C_\delta) \right] \\ \leq \;& \log \left[(C_1^{-1})^{d+\epsilon-1} (C_1^{-1})^{\beta-\alpha} C_2^{-1} |T'|_\infty C_3\, 2^\alpha \mathrm{diam}(I)^{d+\epsilon} \mathrm{diam}(I)^{-1} \mathrm{diam}(I)^{\beta-\alpha} \mathrm{diam}(I)^{\alpha} \right] \\ = \;& \log \left[(C_1^{-1})^{d+\epsilon-1} (C_1^{-1})^{\beta-\alpha} C_2^{-1} |T'|_\infty C_3\, 2^\alpha \mathrm{diam}(I)^{d+\epsilon} \frac{\mathrm{diam}(I)^\beta}{\mathrm{diam}(I)} \right] \\ \leq \;& \log \left[(C_1^{-1})^{d+\epsilon-1} (C_1^{-1})^{\beta-\alpha} C_2^{-1} |T'|_\infty C_3\, 2^\alpha C_1^{-\beta} C_2\, \mathrm{diam}(I)^{d+\epsilon} \frac{\mathrm{osc}(W,I)}{\mathrm{diam}(I)} \right] \end{aligned}$$

and letting $C_4 = (C_1^{-1})^{d+\epsilon-1}(C_1^{-1})^{\beta-\alpha} C_2^{-1} |T'|_\infty C_3\, 2^\alpha C_1^{-\beta} C_2 > 0$ the lemma follows. ❑

We can now give the following lower bound for the $\mathcal{K}$-dimension.

Proposition 4.3.3

Let s_0 be the unique solution to the pressure equation

$$P((1-s_0)\log|T'| + \log\lambda) = 0.$$

Suppose that the non-degeneracy assumption as in Section 2.3 holds. Then for $\alpha \leq \beta < 1$ as in Lemma 4.3.2 we have that

$$\dim_{\mathcal{K}}(\mathrm{graph}(W)) \geq s_0 - (\beta - \alpha).$$

Proof. We let $d = \dim_{\mathcal{K}}(\mathrm{graph}(W))$ and show that $s_0 \leq d + (\beta-\alpha)$. Let $\epsilon > 0$ be arbitrary. It will be sufficient to show that $s_0 \leq d+\epsilon+(\beta-\alpha)$. In particular by choice of s_0 and properties of pressure we show that

$$P((1-(d+\epsilon+(\beta-\alpha)))\log|T'|+\log\lambda)\leq 0$$

and the required result follows.

In particular we take $\epsilon > 0$, and use Lemma 4.3.2 for a certain cover, which for d chosen as above, and α, β as in the statement of the lemma, the required result follows. Given $\epsilon > 0$, by definition of the $\mathcal{K}$-dimension, there exists some $\delta > 0$ and a finite cover of $[0,1]$ by open intervals I_ℓ of length $\delta_\ell \leq \delta$ for $\ell = 1, 2, \ldots, K$ such that

$$\sum_{\ell=1}^{K} \mathrm{diam}(I_\ell)^{d+\epsilon} q_{I_\ell} \leq 1. \tag{4.10}$$

For each I_ℓ we consider a δ_ℓ-Moran cover of $[0,1]$. Then for each δ_ℓ-Moran cover, choose the $m(\ell)$ cylinders (which have rank no less that $n(\delta_\ell)$) in this cover which have non-empty intersection with I_ℓ. For each $\ell = 1, 2, \ldots, K$ denote the cylinders in a δ_ℓ-Moran cover of $[0,1]$ which intersect I_ℓ to be $C_\ell^{(1)}, C_\ell^{(2)}, \ldots, C_\ell^{(m(\ell))}$ for some $m(\ell) \geq 1$. These $C_\ell^{(j)}$ are cylinders of rank $n(\delta_\ell, j)$ and have diameter at most $\delta_\ell = \mathrm{diam}(I_\ell)$.

Therefore $\mathcal{C}_{\delta_\ell} = \left\{C_\ell^{(1)}, \ldots, C_\ell^{(m(\ell))}\right\}$ is a cover of I_ℓ consisting of cylinders of rank no less than some $n(\delta_\ell) \leq n(\delta_\ell, j)$ for all ℓ and j. By earlier discussions $m(\ell) \leq M$ for all ℓ, where $M > 0$ is the Moran multiplicity factor and is independent of ℓ.

We have now been able to construct a cover of $[0,1]$ which, given $\delta > 0$ we denote

$$\mathcal{C}_\delta = \left\{C_\ell^{(j)} \mid j = 1, \ldots, m(\ell),\ \ell = 1, \ldots, K\right\}$$

consisting of cylinders of diameter at most δ.

Let $\phi = (1-(d+\epsilon+(\beta-\alpha)))\log|T'|+\log\lambda$. Given $\epsilon > 0$ we choose δ such that (4.10) holds. Moreover given n, we can choose such a δ sufficiently small such that it also holds that $n(\delta_\ell, j) \geq n$ for all ℓ and j and construct such a cover $\mathcal{C}_\delta$ as above. In particular, for ϵ and d as given, we choose δ so that the hypotheses in Lemma 4.3.2 hold. That is for each I_ℓ and $C_\ell^{(j)}$ which have non-empty intersection, we can use the lemma. As in the definition of topological pressure

$$\Gamma(S^1, \phi, n) \;\leq\; \sum_{C_\ell^{(j)} \in \mathcal{C}_\delta} \exp\left\{ \sup_{x \in C_\ell^{(j)}} \sum_{j=0}^{n(\delta_\ell, j)-1} \phi(T^j(x)) \right\}.$$

Now since each interval I_ℓ has non-empty intersection with $m(\ell) \leq M$ cylinders in the cover $\mathcal{C}_\delta$ and by (2.13) and Lemma 4.3.2

$$\begin{aligned}\Gamma(S^1, \phi, n) &\leq \sum_{C_\ell^{(j)} \in \mathcal{C}_\delta} C_4 \mathrm{diam}(I_\ell)^{d+\epsilon} \frac{\mathrm{osc}(W, I_\ell)}{\mathrm{diam}(I_\ell)} \\ &\leq C_4 \sum_{\ell=1}^{K} \sum_{j=1}^{m(\ell)} \mathrm{diam}(I_\ell)^{d+\epsilon} q_{I_\ell} \\ &\leq MC_4 \sum_{\ell=1}^{K} \mathrm{diam}(I_\ell)^{d+\epsilon} q_{I_\ell} \\ &\leq MC_4\end{aligned}$$

using (4.10).

As n tends to infinity, δ tends to zero uniformly, thus by definition of topological pressure

$$\begin{aligned}P((1-(d+\epsilon+(\beta-\alpha)))\log|T'|+\log\lambda) &= \lim_{n\to\infty} \frac{1}{n} \log \Gamma(S^1, \phi, n) \\ &\leq \lim_{n\to\infty} \frac{\log MC_4}{n} \\ &= 0.\end{aligned}$$

Then the lower bound follows. ❑

4.4 On equality of dimension

In Theorem 4.3.1 we show that if the slope of the map T is constant, and W is the invariant graph of the skew-product as in (2.1), assuming that our non-degeneracy assumption as in Section 2.3 holds, equality of dimension occurs, following from the upper bound for box dimension and properties of $\mathcal{K}$-dimension. In particular that $\dim_{\mathcal{K}}(\mathrm{graph}(W)) = s_0$ where s_0 is the unique solution to the pressure equation $P((1-s_0)\log b + \log\lambda) = 0$. Moreover by the variational principle and since $\log|T'| = \log b$ is constant it can be shown that

$$\begin{aligned}s_0 &= 1 + \frac{h_{\mu_0}(T) + \int \log\lambda \, d\mu_0}{\log b} \\ &= 1 + \frac{P(\log\lambda)}{\log b}\end{aligned}$$

where, since $\log b$ is constant μ_0 is the unique equilibrium state corresponding to potential $\log\lambda$.

Moreover if the function $\log \lambda$ is equal to a coboundary plus a constant, the the equilibrium state μ_0 is Lebesgue measure, and thus $h_{\mu_0}(T) = \log b$, and so $s_0 = 2 + \frac{\int \log \lambda \, d\mu_0}{\log b}$.

If we consider the case in which $F(x, y) = (T(x), \lambda^{-1}(y - p(x)))$ where p is Lipschitz, $T(x) = bx \bmod 1$ and $\lambda \in (0, 1)$ is constant, then we have equality in dimension where $\dim_{\mathcal{K}}(\text{graph}(W)) = 2 - \alpha$ and $\alpha = \frac{-\log \lambda}{\log b}$ and α is the Hölder exponent of W, and graph(W) is the unique invariant graph under F.

However if we consider the general case as in (2.5) and T has non-constant slope, we cannot give equality in dimension. We have s_0 as the upper bound for $\mathcal{K}$-dimension and the lower bound given in Section 4.3.2. We can however give a technical condition, which if we assume holds, then equality occurs. We give a discussion on why we are unable to give a lower bound without such an assumption in the final chapter of this thesis.

4.4.1 A technical condition

In the following section we consider a technical assuption on the Hölder regularity of W which will yield equality in our previous dimension theoretic results. The following result by [PU] gives a lower bound on the Hausdorff dimension of the graph.

Theorem 4.4.1

Suppose there exists $\alpha \in (0, 1)$ such that a continous function $f : S^1 \to \mathbb{R}$ satisfies the following two condtions:

- *There exists a constant $C_5 > 0$ such that for all $x, y \in S^1$*

$$|f(x) - f(y)| \leq C_5 |x - y|^{\alpha}. \tag{4.11}$$

- *There exists a constant $C_6 > 0$ such that for all subintervals $I \subset S^1$*

$$\text{osc}(f, I) \geq C_6 \text{diam}(I)^{\alpha}. \tag{4.12}$$

Then there exists a constant $C(\alpha, C_6/C_5)$ which depends only on α and C_6/C_5 such that

$$\dim_{\mathcal{H}}(\text{graph}(f)) \geq C(\alpha, C_6/C_5) > 1.$$

Proposition 4.4.2

Suppose that graph(W) *is the invariant graph of a skew product as in previous sections. Let α be the Hölder exponent of W. Then Condition 4.11 clearly holds. If, for the same α Condition 4.12 holds, then for s_0 the unique solution to the pressure equation* $P((1-s_0)\log|T'| + \log\lambda) = 0$,

$$\dim_{\mathcal{K}}(\mathrm{graph}(W)) = s_0.$$

We can see that if such an assumption holds, then α and β as in Lemma 4.3.2 and Proposition 4.3.3 can be taken to be equal. That is, since (4.12) holds, then (4.9) holds with α in place of β. Moreover $\beta - \alpha = 0$ and the result is clear.

If such a condition holds we have equality of dimension, where s_0 is now the unique solution to the pressure equation $P((1-s_0)\log|T'| + \log\lambda) = 0$. Given a Hölder continuous potential ϕ, by the variational principle

$$P(\phi) = \sup_{\mu\in\mathcal{M}(T)} \left\{ h_\mu(T) + \int \phi d\,\mu \right\}$$

where the supremum is taken over all T-invariant probability measures μ, and $h_\mu(T)$ is the measure theoretic entropy of T with respect to the invariant measure μ. Then there exists a unique measure μ_0 which attains the above supremum. This is the unique equilibrium state corresponding to Hölder continuous potential ϕ. Then

$$P((1-s_0)\log|T'| + \log\lambda) = h_{\mu_0}(T) + \int (1-s_0)\log|T'| + \log\lambda\, d\mu_0 = 0.$$

Hence

$$h_{\mu_0}(T) + \int \log|T'|\, d\mu_0 + \int \log\lambda\, d\mu_0 = s_0 \int \log|T'|\, d\mu_0$$

and so

$$\dim_{\mathcal{K}}(\mathrm{graph}(W)) = 1 + \frac{h_{\mu_0}(T) + \int \log\lambda\, d\mu_0}{\int \log|T'|\, d\mu_0}$$

where μ_0 is the unique equilibrium state for $(1-s_0)\log|T'| + \log\lambda$.

It is shown [PU] that the technical conditions in Theorem 4.4.1 hold for the graphs of certain functions of the form $W(x) = \sum_{j=0}^{\infty} \lambda^j p(T^j(x))$. In particular the authors [PU] assume that p is monotone on $[0,1/2]$, $T(x) = bx \bmod 1$ and $\lambda \in (0,1)$. They also show that such conditions holds under other assumptions, yet require assumptions on b and λ being sufficiently large. However these cases are all included in the case considered in Proposition 4.3.1. That is, if the above technical assumption holds, then we have equality in dimension, although we have not been able to show that such a condition holds in the more general setting, although we conjecture that this is indeed the case. We remark further on this in the final Chapter.

Chapter 5

On a limiting Rademacher function

In the following, we consider the notion of $\mathcal{K}$-dimension as in Section 2.5 and answer a natural question which arises in the study of the dimension theory of invariant graphs and fractal sets in $\mathbb{R}^2$. In particular we address the question regarding the $\mathcal{K}$-dimension of a certain graph, and whether this coincides with box dimension and Hausdorff dimension in certain circumstances. Such a graph was studied in [McM, PU] and it was found that the Hausdorff dimension of the graph is strictly less that the box dimension. We ask the natural question, what is the $\mathcal{K}$-dimension of such a graph?

5.1 Preliminaries

We study a function known as a limiting Rademacher function. This was studied in [PU,McM] who were able to show that in certain cases the Hausdorff dimension is strictly less than the box dimension of the graphs of such functions.
Let $\lambda \in (1/2, 1)$ and let $q(x)$ be the Rademacher function given by

$$q(x) = \begin{cases} -1 & \text{if } x \in [0, 1/2) \\ 1 & \text{if } x \in [1/2, 1). \end{cases}$$

Let $T : S^1 \to S^1$ be the doubling map given by $T(x) = 2x \bmod 1$. Consider the skew product $F_\lambda : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ defined by

$$F_\lambda(x, y) = (T(x), \lambda^{-1}(y - q(x))).$$

Then there exists a function f_λ, the graph of which is invariant under F_λ. That is $F_\lambda(\text{graph}(f_\lambda)) = \text{graph}(f_\lambda)$. Moreover as in Section 2.1 the function $f_\lambda : S^1 \to \mathbb{R}$ can be written in the form

$$f_\lambda(x) = \sum_{k=0}^{\infty} \lambda^k q(T^k(x)). \tag{5.1}$$

We call f_λ a limiting Rademacher function.

For all $\lambda \in (1/2, 1)$ it is shown [McM, PU] that $\dim_B(\text{graph}(f_\lambda)) = 2 - \alpha$ where $\alpha = -\log\lambda/\log 2$. Equivalently the box dimension of the graph is given by s_0, the unique solution to the pressure equation $P((1-s_0)\log 2 + \log\lambda) = 0$. The explicit value for s_0 then follows from the variational principle and coincides with $2-\alpha$.

However, the interesting result given in [PU] and [McM] occurs when we study the behaviour of the graph for different values of λ. It can be shown that for all $\lambda \in (1/2, 1)$, $\dim_{\mathcal{H}}(\text{graph}(f_\lambda)) > 1$. In fact for different values of λ, the upper bound for Hausdorff dimension can vary.

A conjecture of Erdös [Er,Ga] says that for almost every λ sufficiently close to 1, the probability distribution given by f_λ is absolutely continuous with respect to Lebesgue measure. This conjecture was settled by Solomyak [Sol] in 1995 and a simpler proof was given in [PerSol]. For the graph f_λ it is shown that equality of Hausdorff and box dimension occurs. However [PU] show that for certain exceptional λ, namely Pisot numbers it is fact the case that such a probability distribution is singular with respect to Lebesgue measure, and they prove that $\dim_{\mathcal{H}}(\text{graph}(f_\lambda)) < 2 - \alpha$.

A natural question would be to ask what is the $\mathcal{K}$-dimension of such a graph for different values of λ?

5.2 On the height of the graph over cylinders

Let $x \in S^1$ and let $\Sigma = \{0,1\}^{\mathbb{N}}$ be the one-sided full shift on 2 symbols, such that Σ consists of sequences $\mathbf{x} = (x_0, x_1, \ldots)$ where $x_j \in \{0,1\}$ for all $j \geq 0$. Let $\chi : \Sigma \to S^1$ be the map such that $\chi(\mathbf{x}) = x$ and for $\mathbf{x} = (x_0, \ldots, x_{n-1}, \ldots)$

$$x = \bigcap_{n \geq 0} T_{x_0,\ldots,x_{n-1}}(S^1) = \bigcap_{n\geq 0} T_{x_0} \circ \ldots \circ T_{x_{n-1}}(S^1).$$

Recall that we call $T_{x_0,\ldots,x_{m-1}}(S^1) = C_m(x)$ a cylinder of rank m containing x. Here $T_0(x) = x/2$ and $T_1(x) = (x+1)/2$ are the inverse branches of the base transformation

$T(x) = 2x \bmod 1$.

Let $m \geq 0$ and let C_m denote a dyadic cylinder of rank m. Clearly $\mathrm{diam}(C_m) = 2^{-m}$. Given $x, y \in C_m$, they can be coded by $\mathbf{x}, \mathbf{y} \in \Sigma$ so that $\chi(\mathbf{x}) = x$ and $\chi(\mathbf{y}) = y$ and have coding

$$\begin{aligned} \mathbf{x} &= (i_0, \ldots, i_{m-1}, x_0, \ldots) \\ \mathbf{y} &= (i_0, \ldots, i_{m-1}, y_0, \ldots). \end{aligned} \tag{5.2}$$

That is for x, y in a cylinder of rank m, their symbolic coding agrees in at least the first m symbols. We consider the height of the graph over cylinders;

$$\mathrm{osc}(f_\lambda, C_m) = \sup_{x,y \in C_m} |f_\lambda(x) - f_\lambda(y)|. \tag{5.3}$$

That is, we wish to maximise

$$\left| \sum_{k=0}^{\infty} \lambda^k q(T^k(x)) - \sum_{k=0}^{\infty} \lambda^k q(T^k(y)) \right|$$

over $x, y \in C_m$.

Note that the map q only depends on the first coordinate in the symbolic coding of a point. That is $q(x) = -1$ if $x_0 = 0$ and 1 if $x_0 = 1$. Hence for $0 \leq k \leq m-1$, $q(T^k(x)) = q(T^k(y))$ for $x, y \in C_m$. Take $x, y \in C_m$, whence they agree in the first m places of their symbolic coding. After this, let x_j as in (5.2) be such that $x_j = 1$ for all $j \geq 0$ and let $y_j = 0$ for all $j \geq 0$. Then it can be seen that such x, y acheive the supremum over all points in such a cylinder of rank m in (5.3).

Moreover

$$\begin{aligned} \mathrm{osc}(f_\lambda, C_m) &= |f_\lambda(x) - f_\lambda(y)| \\ &= \sum_{k=0}^{\infty} \lambda^k \left| q(T^k(x)) - q(T^k(y)) \right| \\ &= \lambda^m \sum_{k=0}^{\infty} \lambda^k \left| q(T^{k+m}(x)) - q(T^{k+m}(y)) \right| \\ &= \lambda^m \sum_{k=0}^{\infty} 2\lambda^k \\ &= \frac{2\lambda^m}{1-\lambda}. \end{aligned}$$

Remark. By the above, it is true that the height of the graph over any cylinder of rank m is the same, and is given by $\frac{2\lambda^m}{1-\lambda}$. In particular, the height of the graph over cylinders depends only on the rank. Therefore given a cylinder of rank m denoted C_m,

$$\lambda^m = \frac{1-\lambda}{2} \operatorname{osc}(f_\lambda, C_m). \tag{5.4}$$

5.3 On the graph over intervals

In the following we give some useful results which relate the behaviour of the graph over cylinders to that over arbitrary intervals. In particular we use the notion of Moran covers and relate heights over cylinders in such covers to heights over intervals, which allows us to show that the $\mathcal{K}$-dimension and box dimension of such graphs are equal.

We firstly consider the projection of the graph to the real axis, the limit set of which is proven in [HL2, Feng1] to be an entire closed interval. The distribution of f_λ in fact induces a Bernoulli convolution on S^1 which is studied in [Feng1, Feng2, LePo1, LePo2]. We only require the following result however.

Let $I \subset [0,1]$ be any non-empty open interval. Let $\mathcal{B}_{\lambda,I}$ denote the projection to the $\mathbb{R}$ axis of the graph above I;

$$\mathcal{B}_{\lambda,I} = \{f_\lambda(x) \mid x \in I\} \subset \mathbb{R}.$$

Lemma 5.3.1
Let $I \subset [0,1]$ be given. Then $\mathcal{B}_{\lambda,I}$ takes values inside a compact interval $[\ell', \ell'']$, is dense in such an interval and takes no values outside $[\ell', \ell'']$.

Proof. Clearly $\mathcal{B}_{\lambda,I} \subseteq [\ell', \ell'']$, where $\ell' = \inf_{x\in I} f_\lambda(x)$ and $\ell'' = \sup_{x\in I} f_\lambda(x)$ and $|\ell'|, |\ell''| < \infty$. We show that given any point in $\mathcal{B}_{\lambda,I}$, we can find another point in the projection to the real axis arbitrarily close.

Let $\epsilon > 0$ be given and let $x \in I$ such that $f_\lambda(x) \in \mathcal{B}_{\lambda,I}$. We show that there exists $y \in I$ such that $|f_\lambda(x) - f_\lambda(y)| < \epsilon$. Suppose that $x \in I$ has coding $(x_0, \ldots, x_n, \ldots)$ where $x_j \in \{0,1\}$. Now choose l such that $\frac{2\lambda^l}{1-\lambda} < \epsilon$. Then we choose a point y, such that x and y are sufficiently close, so that since I is open we have that $y \in I$, and that x and y agree on at least the first l symbols in their symbolic coding. That is $y \in I$ and has coding $(x_0, \ldots, x_{l-1}, y_l, \ldots)$ where $y_l, y_{l+1}, \ldots \in \{0,1\}$.

Then it can be seen that for all $0 \le j \le l-1$

$$q(T^j(x)) = q(T^j(y))$$

and $|q(T^j(x)) - q(T^j(y))| \leq 2$ for all $j \geq l$. Hence

$$\begin{aligned} |f_\lambda(x) - f_\lambda(y)| &= \sum_{j=0}^{\infty} \lambda^j \left|q(T^j(x)) - q(T^j(y))\right| \\ &= \sum_{j=l}^{\infty} \lambda^j \left|q(T^j(x)) - q(T^j(y))\right| \\ &= \lambda^l \sum_{j=0}^{\infty} \lambda^j \left|q(T^{j+l}(x)) - q(T^{j+1}(y))\right| \\ &\leq \frac{2\lambda^l}{1-\lambda} < \epsilon \end{aligned}$$

by choice of l. Thus $\mathcal{B}_{\lambda,I}$ is dense in the interval $[\ell', \ell'']$. ❑

Remark. Given any open interval $I \subset S^1$ the projection to $\mathbb{R}$ of the graph above I is dense in some interval, and the endpoints of I achieve the infimum and supremum of the function restricted to I. In particular for each I

$$q_I - 1 \leq \frac{\operatorname{osc}(f_\lambda, I)}{\operatorname{diam}(I)} \leq q_I. \tag{5.5}$$

We now prove a lemma which will be an important step in proving the $\mathcal{K}$-dimension of graph(f_λ) in the context which we need.

Proposition 5.3.2
Let $I \subset [0,1]$ be an interval of diameter $\delta > 0$. Suppose that $C^{(j)}$ for $j = 1, \ldots, m$ are the cylinders in a δ-Moran cover of S^1 which have non-empty intersection with I. Then for each cylinder $C^{(j)}$ there exists $C_0 > 0$ independent of I and δ such that

$$\operatorname{osc}(f_\lambda, C^{(j)}) \leq C_0 \operatorname{osc}(f_\lambda, I).$$

Proof. Let $I \subset [0,1]$ such that $\operatorname{diam}(I) = \delta$. Let $C^{(j)}$, for $j = 1, \ldots, m$ be the cylinders in the δ-Moran cover of S^1 such that $C^{(j)} \cap I \neq \emptyset$. Note that $m \leq M$ where $M > 0$ is the Moran multiplicity factor. In addition $I \subset \bigcup_{j=1}^m C^{(j)}$. Note also that by construction of the Moran cover, each $C^{(j)}$ has rank $n = n(\delta)$, the smallest integer such that

$$\operatorname{diam}(C^{(j)}) = 2^{-n} \leq \delta$$

and $2^{-(n-1)} > \delta$. In addition, since each cylinder has the same rank and diameter, the height of the graph(f_λ) over each cylinder is given by $\frac{2\lambda^n}{1-\lambda}$. Note also that by construction, the cylinders have disjoint interior and are in fact dyadic intervals.

We now consider three cases involving the cylinders $C^{(1)}, \ldots, C^{(m)}$ which intersect I.

- Suppose firstly that $m = 1$. Then $\operatorname{diam}(C^{(1)}) \leq \operatorname{diam}(I)$ and covers I. Therefore $C^{(1)} = I$ and the proposition follows letting $C_0 = 1$.

- Suppose that $m > 2$. Then $C^{(1)}, \ldots, C^{(m)}$ cover I and have disjoint interior, so there exists some $1 \leq k \leq m$ such that $C^{(k)} \subseteq I$. Moreoever for every $C^{(j)}$ has rank n, thus

$$\begin{aligned}\operatorname{osc}(W, C^{(j)}) &= \frac{2\lambda^n}{1-\lambda} \\ &= \operatorname{osc}(f_\lambda, C^{(k)}) \\ &\leq \operatorname{osc}(f_\lambda, I)\end{aligned}$$

  and the proposition follows letting $C_0 = 1$.

- Finally suppose that $m = 2$. Thus $I \subset C^{(1)} \cup C^{(2)}$ but neither $C^{(1)}$ nor $C^{(2)}$ are contained within I. However note that both $C^{(1)}$ and $C^{(2)}$ are dyadic intervals of diameter 2^n. That is there exists $j \in \{1, 2\}$ such that $\operatorname{diam}(C^{(j)} \cap I) \geq \delta/2$. In particular I contains a dyadic interval of diameter $2^{-(n+1)}$. Moreoever I contains a cylinder of rank $n+1$. Denote this cylinder $C_{n+1} \subset I$. Therefore for each $C^{(j)}$ of rank n, for $j = 1, 2$ we have that

$$\begin{aligned}\operatorname{osc}(f_\lambda, C^{(j)}) &= \frac{2\lambda^n}{1-\lambda} \\ &= \lambda^{-1}\frac{2\lambda^{n+1}}{1-\lambda} \\ &= \lambda^{-1}\operatorname{osc}(f_\lambda, C_{n+1}) \\ &\leq \lambda^{-1}\operatorname{osc}(f_\lambda, I).\end{aligned}$$

Thus the proposition holds taking $C_0 = \lambda^{-1}$. Since $\lambda \in (0, 1)$, the proposition holds taking $C_0 = \lambda^{-1}$, which is therefore independent of I. ❏

5.4 On the $\mathcal{K}$-dimension of $\operatorname{graph}(f_\lambda)$.

In this section we show that for any $\lambda \in (1/2, 1)$ it is in fact the case that $\dim_{\mathcal{K}}(\operatorname{graph}(f_\lambda)) = \dim_B(\operatorname{graph}(f_\lambda))$. It is shown [PU] that $\dim_B(\operatorname{graph}(f_\lambda)) = 2 + \frac{\log \lambda}{\log 2} = s_0$ where, by the variational principle s_0 is the solution to the pressure equation $P((1-s_0)\log 2+\log\lambda) = 0$. By Proposition 2.5.1 it will be sufficient to give a lower bound for $\mathcal{K}$-dimension.

Theorem 5.4.1

Let s_0 be the unique solution to the pressure equation

$$P((1-s_0)\log 2+\log\lambda) = 0.$$

Then

$$\dim_{\mathcal{K}}(\mathrm{graph}(f_\lambda)) = s_0 = 2 + \frac{\log\lambda}{\log b}.$$

Proof. By remarks above we only need to prove a lower bound for $\mathcal{K}$-dimension. We let $d = \dim_{\mathcal{K}}(\mathrm{graph}(f_\lambda))$ and show that $s_0 \le d$.

Fix $\epsilon > 0$. By construction of the $\mathcal{K}$-dimension, there exists some $\delta > 0$ and a finite cover of S^1 by open intervals I_ℓ of length $\delta_\ell \le \delta$ for $\ell = 1, 2, \ldots, K$ such that

$$\sum_{\ell=1}^{K} \mathrm{diam}(I_\ell)^{d+\epsilon} q_{I_\ell} \le 1. \tag{5.6}$$

For each ℓ we consider a δ_ℓ-Moran cover of S^1. Then for each δ_ℓ-Moran cover, choose the cylinders in this cover, each of rank $n(\delta_\ell)$ which have non-empty intersection with I_ℓ. For each $\ell = 1, 2, \ldots, K$ denote the cylinders in a δ_ℓ-Moran cover of S^1 which intersect I_ℓ as $C_\ell^{(1)}, C_\ell^{(2)}, \ldots, C_\ell^{(m(\ell))}$ for some $m(\ell) \ge 1$. These cylinders each have rank $n(\delta_\ell)$ and diameter $2^{-n(\delta_\ell)} \le \delta_\ell$.

Therefore $\mathcal{C}_{\delta_\ell} = \left\{C_\ell^{(1)}, \ldots, C_\ell^{(m(\ell))}\right\}$ is a cover of I_ℓ consisting of cylinders each of rank $n(\delta_\ell)$ and diameter $2^{-n(\delta_\ell)}$ for all ℓ. By construction of the Moran cover $m(\ell) \le M$ for all ℓ, where $M > 0$ is the Moran multiplicity factor and is independent of ℓ and δ.

We have now been able to construct a cover of S^1 which, given $\delta > 0$ we denote

$$\mathcal{C}_\delta = \left\{C_\ell^{(j)} \mid j = 1, \ldots, m(\ell),\ \ell = 1, \ldots, K\right\}$$

consisting of cylinders of diameter at most δ and rank no less than $n = n(\delta)$ where

$$n(\delta) = \min_{\ell}\left\{n(\delta_\ell) \mid 2^{-n(\delta_\ell)} \le \delta_\ell\right\}. \tag{5.7}$$

Let $\mathcal{C}_\delta$ be the cover of S^1 as above, by construction of the Moran cover

$$\sum_{C_\ell^{(j)} \in \mathcal{C}_\delta} \left[\left(2^{n(\delta_\ell)}\right)^{1-(d+\epsilon)} \lambda^{n(\delta_\ell)}\right] \tag{5.8}$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{k}\sum_{j=1}^{m(\ell)}\left[\operatorname{diam}\left(C_\ell^{(j)}\right)^{(d+\epsilon)-1}\lambda^{n(\delta_\ell)}\right] \\
&= \frac{1-\lambda}{2}\sum_{\ell=1}^{k}\sum_{j=1}^{m(\ell)}\operatorname{diam}\left(C_\ell^{(j)}\right)^{(d+\epsilon)-1}\operatorname{osc}(f_\lambda, C_\ell^{(j)})
\end{aligned}
$$

using (5.4). Now by construction of the cover, $\operatorname{diam}\left(C_\ell^{(j)}\right) \le \operatorname{diam}(I_\ell)$ and $\operatorname{diam}\left(C_\ell^{(j)}\right) > \frac{1}{2}\operatorname{diam}(I_\ell)$. Thus

$$\operatorname{diam}\left(C_\ell^{(j)}\right)^{-1} < 2\operatorname{diam}(I_\ell)^{-1}.$$

Then we apply Proposition 5.3.2 to each cylinder $C_\ell^{(j)}$ and associated I_ℓ so that for each cylinder $C_\ell^{(j)}$ there exists $C_0 > 0$ independent of δ and ℓ such that

$$\operatorname{osc}(f_\lambda, C_\ell^{(j)}) \le C_0 \operatorname{osc}(f_\lambda, I_\ell).$$

Therefore

$$
\begin{aligned}
(5.8) \quad &\le C_0(1-\lambda)\sum_{\ell=1}^{K}\sum_{j=1}^{m(\ell)}\operatorname{diam}(I_\ell)^{d+\epsilon}\frac{\operatorname{osc}(f_\lambda, I_\ell)}{\operatorname{diam}(I_\ell)} \\
&\le C_0(1-\lambda)M\sum_{\ell=1}^{K}\operatorname{diam}(I_\ell)^{d+\epsilon}q_{I_\ell} \\
&\le C_0(1-\lambda)M.
\end{aligned}
$$

by (5.5), (5.6) and properties of Moran covers.

Now given $\delta > 0$ we choose n as in (5.7). Letting $\phi = ((1-(d+\epsilon))\log 2 + \log\lambda)$ as in [Pe], since $\mathcal{C}_\delta$ is a cover of S^1 by cylinders of rank no less than n we have that

$$
\begin{aligned}
\Gamma(S^1, \phi, n) &\le \sum_{C_\ell^{(j)}\in\mathcal{C}_\delta}\exp\left(\sup_{x\in C_\ell^{(j)}}\sum_{k=0}^{n(\delta_\ell)-1}\phi(T^k(x))\right) \\
&\le \sum_{C_\ell^{(j)}\in\mathcal{C}_\delta}\exp\left(\log\left(\left(2^{n(\delta_\ell)}\right)^{1-(d+\epsilon)}\lambda^{n(\delta_\ell)}\right)\right) \\
&= \sum_{C_\ell^{(j)}\in\mathcal{C}_\delta}\left(\left(2^{n(\delta_\ell)}\right)^{1-(d+\epsilon)}\lambda^{n(\delta_\ell)}\right) \\
&\le C_0(1-\lambda)M.
\end{aligned}
$$

Now as $\delta \to 0$, $n \to \infty$ uniformly, thus by definition of topological pressure

$$
\begin{aligned}
P((1-(d+\epsilon))\log 2 + \log\lambda) &= \lim_{n\to\infty}\frac{1}{n}\log\Gamma(S^1, \phi, n) \\
&\le \lim_{n\to\infty}\frac{\log(C_0(1-\lambda)M)}{n} \\
&= 0.
\end{aligned}
$$

Hence we have shown that $P((1-(d+\epsilon))\log 2+\log\lambda)\leq 0$.

Now $p(s)=P((1-s)\log 2+\log\lambda)$ is strictly decreasing as s increases, so $s_0\leq d+\epsilon$. Since $\epsilon>0$ was chosen arbitrarily we conclude that $s_0\leq d$ as required.

Moreover since λ is constant, the unique equilibrium state corresponding to $(1-s_0)\log 2+\log\lambda$ is Lebesgue measure, thus by the variational principle it follows that

$$\dim_{\mathcal{K}}(\mathrm{graph}(f_\lambda))=2+\frac{\log\lambda}{\log b}.$$

❑

Chapter 6

On a parameterized class of non-differentiable functions

In this chapter we introduce a class of continuous functions which arise as a parameterised family of functions. These were first introduced by Okatmoto [Oka] and studied by McCollom [McColl] and Kobayashi [Kob] and include the functions of Bourbaki [Bou] and Perkins [Per], as well as the Cantor-Lebesgue singular functions. We consider a more general family of functions, which include these examples, and which can be constructed iteratively; they also arise as invariant sets for a skew-product dynamical system. We look at different constructions of such graphs, and we use the underlying dynamics to provide estimates on the behaviour of the graph which in turn allow us to prove dimension theoretic results using tools from thermodynamic formalism. In addition we elaborate on some of the results in [Oka, McColl] regarding differentiability properties of the graphs for different parameter values. These properties give rise to many interesting questions in this field of research, therefore we aim to give an overview of some of the results and examples where different behaviour occurs, highlighting how ergodic theory can be used to study such graphs.

6.1 Preliminaries

Let $T : S^1 \to S^1$ be a b-to-one uniformly expanding map of the circle as in Section 2.2. Choosing a Markov partition as in Section 2.2 we can model the dynamics of T as a full shift on b symbols. That is we can code points $x \in S^1$ as $x = (x_0, x_1, \ldots)$

where $x_j \in \{0, 1, \ldots, b-1\}$ and T acts as a shift map. Moreover given $x \in S^1$ we define a cylinder of rank n containing x to be the set obtained by applying a sequence of inverse branches of T to S^1, where the inverse branches are determined by the symbolic coding of x. By Proposition 3.2.2 we can give estimates on the diameter of cylinders. If $T(x) = bx \bmod 1$ then it is clear that the diameter of any cylinder of rank n is simply b^{-n}.

Given such a map $T(x) = bx \bmod 1$ we take a set of parameters $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_b)$ where we assume that $\alpha_0 = 0, \alpha_b = 1$ and $\alpha_i \in (0, 1)$ for all $1 \leq i \leq b-1$. If $\alpha_i \neq \alpha_{i+1}$ for all $i = 1, \ldots, b-1$ we can define the following.

Given x with symbolic coding $(x_0, x_1, \ldots)$, where x_0 is the first coordinate of x we define the mapping

$$g(x)y = (\alpha_{x_0+1} - \alpha_{x_0})^{-1}(y - \alpha_{x_0}).$$

Then we define the skew product dynamical system $F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ as

$$F(x, y) = (T(x), g(x)y). \tag{6.1}$$

Moreover given $j \in \{0, 1, \ldots, b-1\}$ define the inverse branches of $g(x)y$ to be the contraction mappings

$$g_j(x)y = (\alpha_{j+1} - \alpha_j)y + \alpha_j. \tag{6.2}$$

We assume that for all $x \in S^1$ $g_j(x)0 = \alpha_j$, $g_j(x)1 = \alpha_{j+1}$ and that $g_j(x)y$ is uniformly contracting. That is, the derivative of the inverse branches of g in the $\mathbb{R}$ direction is strictly less than one. In particular, letting $\mathcal{D}g_j$ denote the derivatives of the inverse branch mappings g_j,

$$|\mathcal{D}g_j| = |(\alpha_{j+1} - \alpha_j)| < 1.$$

Then we define the inverse branches of F as

$$F_j(x, y) = (T_j(x), g_j(x)y) \tag{6.3}$$

where $j \in \{0, 1, \ldots, b-1\}$ and T_j are inverse branches of T as in Chapter 2. Then [Hut] there exists a unique compact invariant subset of $S^1 \times \mathbb{R}$ for the iterated function system defined by $F_0, \ldots, F_{b-1}$ which we call Γ_α. That is Γ_α is an invariant graph for the skew product F as above, i.e. $\Gamma_\alpha = \bigcup_{j=0}^{b-1} F_j(\Gamma_\alpha)$.

We now introduce an alternative construction of Γ_α which will prove useful in later work. In this construction we consider only the inverse branches of g. Therefore if there exists j such that $\alpha_j = \alpha_{j+1}$ then $g_j(x)y = \alpha_j$ and $|\mathcal{D}g_j| = 0$. We make remarks on what happens in such a case at a later stage. Let $\Lambda_0 = S^1 \times S^1$. Then define $\Lambda_1 = \bigcup_{j=0}^{b-1} F_j(\Lambda_0)$ and

$$\Lambda_{n+1} = \bigcup_{j=0}^{b-1} F_j(\Lambda_n). \tag{6.4}$$

Then it can be seen that $\Lambda_{n+1} \subset \Lambda_n$ for all $n \geq 1$.

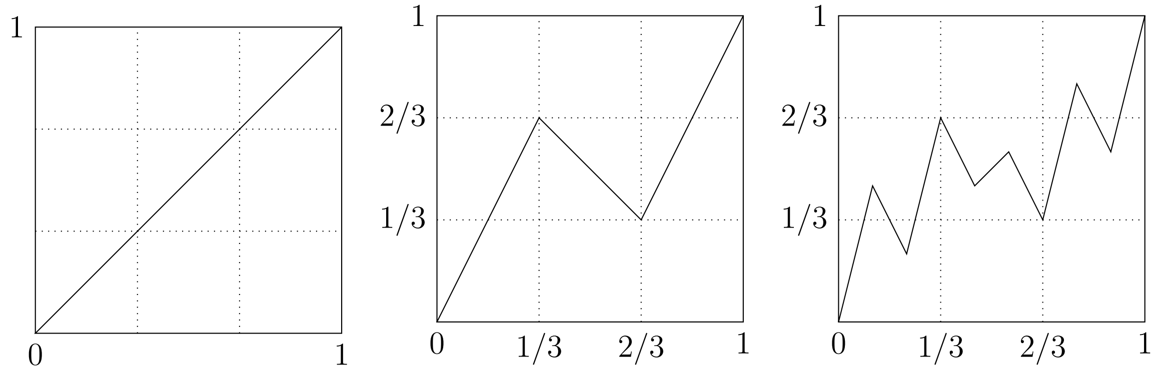
Note that Λ_1 is a union of b rectangles. Moreover Λ_n is a union of b^n rectangles, and we can use Proposition 3.2.2 to estimate the widths of such rectangles. Moreover the projection of these rectangles to the x-axis forms a cover of S^1 consisting of cylinders of rank n. Letting

$$\Gamma_\alpha = \bigcap_{n=0}^{\infty} \Lambda_n$$

it can be seen that Γ_α is indeed the same invariant graph as constructed above for the skew product F as in (6.1). This type of construction (see [Kats]) was modified by McCollum [McColl] so that such an invariant set is the same set introduced as the limit of an iterative process in [Oka]. In our more general setting we let $f_0(x) = x$ and define $f_1 = F(f_0)$ and $f_{n+1} = F(f_n)$. In particular it should be noted that the values of f_n at endpoints of cylinders of rank n are fixed, and f_{n+1} takes the same values at such points for all n. In addition, $\Gamma_\alpha = \lim_{n\to\infty} f_n$ takes the same values as f_n for each n on endpoints of cylinders of rank at most n and is indeed the invariant graph for F. That is, f_n is a union of line segments, the projection to the x-axis of each of these is a cylinder of rank n.

In particular the set Γ_α can be constructed in an iterative way as above as in [Oka], by using rectangles as an invariant set for an iterated function system [McColl] or as the invariant graph of a skew product as we constructed above. Such constructions allow us to study the graph both geometrically and dynamically. It is interesting to note that given a cylinder of rank n the diameter of the cylinder is the same as the diameter of the above rectangle in the level n construction [McColl]. Moreover the graph attains its height over a cylinder at the endpoints and the height of the graph over a cylinder of rank n, is equal to the height of the above rectangle. In addition, the arc of the line segments in the construction f_n over a cylinder of rank n is a straight line segment and the height of the graph over a cylinder of rank n is the height of the line segment of f_n restricted to such a cylinder. It will be useful to use all such constructions in later work.

Example. In the papers [Oka, Kob, McColl] a specific case of the above is studied. In particular let $T(x) = bx \bmod 1$ and let $\alpha = (0, \alpha_1, 1 - \alpha_1, 1)$. Below we illustrate the first 3 iterates, f_0, f_1 and f_2 as in [Oka], where in this case we take $\alpha_1 = 2/3$.



The iterates f_0, f_1 and f_2 respectively, when $\alpha_1 = 2/3$.

It is often easier to visualise the graphs Γ_α as in the iterative process. If $\alpha = 2/3$ as above then Γ_α is the well known function of Bourbaki [Bou].

6.2 On the height of the graph

Given $x \in S^1$ with symbolic coding $x = (x_0, x_1, \ldots, x_n, \ldots)$ with respect to the uniformly expanding b-to-one map T, we study $\operatorname{osc}(\Gamma_\alpha, C_n(x))$; the height of the graph over a given cylinder of rank n. We can use Proposition 3.2.2 to estimate the diameter of cylinders containing a point x in terms of the values of T' evaluated along the orbit of x. Let us firstly define a function which measures the rates of contraction in the $\mathbb{R}$ direction and will allow us to give the height of the graph Γ_α over cylinders. Given $x \in S^1$ where $x = (x_0, x_1, \ldots)$, let

$$\beta(x) = (\alpha_{j+1} - \alpha_j) \text{ if } x_0 = j. \tag{6.5}$$

Then β is a contracting map in the fibre direction of g_j which measures the derivative in the $\mathbb{R}$ direction of the fibre map $g_j(x)y$. Moreover β is locally constant. In fact β depends only on the first coordinate in the symbolic coding of x.

Given $x \in S^1$ and $n \geq 0$ we denote $\beta^n(x) = \beta(x)\beta(T(x))\ldots\beta(T^{n-1}(x))$. Note that $\beta^n(x)$ depends only on the first n coordinates of x. Given a cylinder of rank n, it can be seen that $\beta^n(x)$ is constant on points x in such a cylinder.

In the following we give a result on equality of $\text{osc}(\Gamma_\alpha, C_n(x))$; the height of the graph over $C_n(x)$.

Proposition 6.2.1

Let x be given with symbolic coding $(x_0, \ldots, x_{n-1}, \ldots)$ and $n \geq 0$ be given. Let $C_n(x)$ denote the cylinder of rank n containing x. Then

$$\text{osc}(\Gamma_\alpha, C_n(x)) = \beta^n(x).$$

Proof. Let $C_n(x) = T_{x_0} \circ \ldots \circ T_{x_{n-1}}(S^1)$. Clearly $\text{osc}(\Gamma_\alpha, S^1) = 1$.

Now $T_{x_{n-1}}(S^1)$ has rank 1 and the height of the graph over this cylinder is equal to the height of the rectangle above $T_{x_{n-1}}(S^1)$ as in the construction in [McColl]. Moreover this height is achieved by the endpoints of the cylinder and is given by $|g_{x_{n-1}}(1)1 - g_{x_{n-1}}(0)0| = \left|\alpha_{x_{n-1}+1} - \alpha_{x_{n-1}}\right| = |\beta(y^{(1)})|$ for some $y^{(1)} \in T_{x_{n-1}}(S^1)$. In fact since β depends only on the first coordinate

$$\text{osc}(\Gamma_\alpha, T_{x_{n-1}}(S^1)) = |\beta(y^{(1)}|$$

for any $y^{(1)} \in T_{x_{n-1}}(S^1)$. By a similar argument

$$\begin{aligned} \text{osc}(\Gamma_\alpha, T_{x_{n-2},x_{n-1}}(S^1)) &= |\beta(y^{(2)})|\text{osc}(\Gamma_\alpha, T_{x_{n-1}}(S^1)) \\ &= |\beta(y^{(2)})||\beta(y^{(1)})| \end{aligned}$$

for any $y^{(2)} \in T_{x_{n-2},x_{n-1}}(S^1)$ and any $y^{(1)} \in T_{x_{n-1}}(S^1)$.

Therefore given $x = (x_0, x_1, \ldots)$ and $n \geq 0$

$$\begin{aligned} \text{osc}(\Gamma_\alpha, C_n(x)) &= |\beta(y^{(n)})| \ldots |\beta(y^{(1)})| \\ &= \prod_{j=1}^{n} |\beta(y^{(j)}| \end{aligned}$$

for any $y^{(j)} \in T_{x_{n-j},\ldots,x_{n-1}}(S^1)$.

That is, $\text{osc}(\Gamma_\alpha, C_n(x))$ is given by the product of β evaluated at points $x^{(j)}$, $j = 1, \ldots, n$ where $x^{(j)}$ is any point in a cylinder of rank j. Since $x \in C_n(x)$ is given, $T^j(x)$ is a point in a cylinder of rank $n - j$, and since β depends only on the first coordinate of a point, it can be seen that

$$\begin{aligned} \text{osc}(\Gamma_\alpha, C_n(x)) &= \prod_{j=0}^{n-1} \left|\beta(T^j(x))\right| \\ &= |\beta^n(x)| \end{aligned}$$

as required. ❑

Consider the construction of Γ_α as in [McColl] by a union of rectangles Λ_n as in (6.4). Recall that

$$\Gamma_\alpha = \bigcap_{n=0}^{\infty} \Lambda_n.$$

Recall that Λ_n is a union of b^n rectangles whose projection to the x-axis forms a cover of S^1 by cylinders of rank n. In addition, the height of the graph over such cylinders is equal to the height of the rectangle in Λ_n over such a cylinder.

Given $x \in S^1$ and $n \geq 0$ we denote $\Lambda_n \mid_{C_n(x)}$ to be the rectangle in the construction of Λ_n over the cylinder $C_n(x)$. Hence we define the area of such a rectangle to be

$$a(\Lambda_n \mid_{C_n(x)}) = \operatorname{diam}(C_n(x)) \times \operatorname{osc}(\Gamma_\alpha, C_n(x)).$$

Therefore by Proposition 3.2.2 and Proposition 6.2.1 there exists $C_1 > 0$ such that given $x \in S^1$ and $n \geq 0$

$$C_1^{-1} \prod_{j=0}^{n-1} |T'(T^j(x))|^{-1} \prod_{j=0}^{n-1} |\beta(T^j(x))| \leq a(\Lambda_n \mid_{C_n(x)}) \leq C_1 \prod_{j=0}^{n-1} |T'(T^j(x))|^{-1} \prod_{j=0}^{n-1} |\beta(T^j(x))|. \tag{6.6}$$

In addition given $x \in C_n(x)$, we can estimate the slope of the graph at x using the iterative process of Γ_α as in [Oka] by a union of line segments f_n. Given $x \in S^1$ and $n \geq 0$, the stage n construction f_n of the graph Γ_α over the cylinder $C_n(x)$ is a straight line segment. In particular, if such a limit exists

$$|\Gamma'_\alpha(x)| = \lim_{n\to\infty} |\operatorname{grad}(f_n, x)| = \lim_{n\to\infty} \frac{\operatorname{osc}(\Gamma_\alpha, C_n(x))}{\operatorname{diam}(C_n(x))} \tag{6.7}$$

where graph(f_n, x) is the slope of the line segment of f_n in the construction [Oka] over the cylinder $C_n(x)$. These observations will prove useful in later sections.

6.3 On the differentiability of Γ_α

In the following section, we study the graph constructed as above, and consider the values of parameters α under which points on the graph possess different differentiability properties. We let $T : S^1 \to S^1$ be a uniformly (not necessarily linear) expanding map of the circle as in Chapter 2, and let $\alpha = (0, \alpha_1, \ldots, \alpha_{b-1}, 1)$ and β be given as in Section 6.1. In the following we give examples of choices of parameter values α, under which the resulting graph Γ_α exhibits different behaviour. We make such conditions more precise later on.

Remark. Note that if T and β are such that $f(\alpha_j) = \alpha_j$ for each $j = 1, \ldots, b-1$, then $\Gamma_\alpha(x) = x$ and such a case is not particularly interesting. In addition if $\alpha_1 = \alpha_2 = \ldots = \alpha_{b-1} = 1/2$ then Γ_α is a Cantor-Lebesgue singular function.

It is often easier to visualize such examples as in the case studied in [Oka], where $T(x) = bx \bmod 1$ and $\alpha = (0, \alpha_1, 1-\alpha_1, 1)$. If $\alpha_1 = 1/3$ then clearly $\Gamma_\alpha(x) = x$. If $\alpha_1 = 1/2$ then Γ_α is the devils staircase constructed on the middle third cantor set.

The following result is clear by inspection of the construction of the graph as in [Oka], and is a consequence of the Lebesgue Differentiation Theorem [Leb].

Proposition 6.3.1

Suppose that $\alpha_0 < \alpha_1 < \ldots < \alpha_b$. *Then* $\beta(x) > 0$ *for all* $x \in S^1$ *and so* Γ_α *is non-decreasing and therefore almost-everywhere differentiable.*

In the case of Okamoto, the above restriction is equivalent to requiring that $\alpha_1 \leq 1/2$. We remark further on this later.

Proposition 6.3.2

Suppose that the partial hyperbolicity assumption on F *holds for our skew-product* F. *In particular we assume that*

$$\inf_{x \in S^1} |\beta(x)| \inf_{x \in S^1} |T'(x)| > 1. \tag{6.8}$$

Then Γ_α *is nowhere differentiable.*

Proof. Let $x \in S^1$. Given $n \geq 0$ we let $C_n(x)$ denote the cylinder of rank n containing x. Firstly suppose that x does not lie on the boundary of any cylinder. That is, there exists no such $N \geq 1$ such that x is an endpoint of a cylinder of rank N. By construction of f_n as in [Oka], we know that Γ_α takes the same values as f_n does on the endpoints of cylinders of rank n. In addition for x as above and by Proposition 3.2.2 and Proposition 6.2.1 there exists $C_1 > 0$ such that

$$\begin{aligned} \frac{\operatorname{osc}(\Gamma_\alpha, C_n(x))}{\operatorname{diam}(C_n(x))} &\geq \frac{C_1^{-1}|\beta^n(x)|}{\prod_{j=0}^{n-1} |T'(T^j(x))|^{-1}} \\ &\geq C_1^{-1} \left(\inf_{x \in S^1} |\beta(x)| \inf_{x \in S^1} |T'(x)| \right)^n. \end{aligned}$$

Moreover since x lies in the interior a cylinder of rank n for all n,

$$\begin{aligned}
|\Gamma'_\alpha(x)| &= \lim_{n\to\infty} |\mathrm{grad}(f_n(x))| \\
&= \lim_{n\to\infty} \frac{\mathrm{osc}(\Gamma_\alpha, C_n(x))}{\mathrm{diam}(C_n(x))} \\
&\geq \lim_{n\to\infty} C_1^{-1}\left(\inf_{x\in S^1}|\beta(x)| \inf_{x\in S^1}|T'(x)|\right)^n \\
&= \infty
\end{aligned}$$

by (6.8).

Suppose now that there exists $N \geq 0$ such that x is an endpoint of some cylinder of rank N. Then x is the endpoint of two cylinders, which have disjoint interior. Then one can show that the absolute value of the right hand derivative at x is infinite, using the partial hyperbolicity assumption. The left hand derivative case follows similarly.

❏

The partial hyperbolicity assumption, in the case of Okamoto is equivalent to requiring that $\alpha > 2/3$.

An interesting question is what happens when Γ_α is not non-decreasing, yet the partial hyperbolicity assumption does not hold. In particular there may exists points of differentiability and non-differentiability on the graph Γ_α.

Suppose that $\alpha = (0, \alpha_1, \ldots, \alpha_{b-1}, 1)$ is given such that there exists $1 \leq K < b-1$ such that $\alpha_K > \alpha_{K+1}$. In the case of [Oka] this requires that $\alpha_1 > 1/2$, thus Γ_α is not non-decreasing. Then suppose that $x \in S^1$ is given to be an endpoint of some cylinder, and that the symbol K appears in the symbolic coding of the point x. Inspection [see Oka] shows that both left and right derivatives at x have absolute value ∞. In particular, if there exists K as in the above, there exists infinitely many points at which Γ_α is non-differentiable. We look to classify this in more detail.

If x is given, and is the endpoint of a cylinder, then it lies on the endpoints of two cylinders, whose interiors are disjoint. Then one can calculate both the left and right derivatives at such x, which may differ.

Let $x \in S^1$ be given. Given $n \geq 0$, as in the construction in [Oka], we consider the line segment of the graph f_n over the cylinder $C_n(x)$. Then

$$|f'_n(x)| = \frac{\mathrm{osc}(\Gamma_\alpha, C_n(x))}{\mathrm{diam}(C_n(x))}.$$

Note that for any x in a cylinder of rank n, the slope of f_n over this cylinder is constant, and is equal to $f_n'(x)$ for any x in the interior of a cylinder.

Moreover by Proposition 3.2.2 and Proposition 6.2.1 there exists $C_1 > 0$ such that

$$C_1^{-1} \prod_{j=0}^{n-1} |T'(T^j(x))| \beta(T^j(x))| \leq |f_n'(x)| \leq C_1 \prod_{j=0}^{n-1} |T'(T^j(x))| \beta(T^j(x))|.$$

We are interested in whether Γ_α is differentiable at x. The function Γ_α is differentiable at x if and only if there exists a finite limit ℓ such that

$$\lim_{n\to\infty} |f_n'(x)| = |\ell|.$$

That is, if

$$\lim_{n\to\infty} \prod_{j=0}^{n-1} |T'(T^j(x))||\beta(T^j(x))| = |\ell|.$$

In particular for this limit to be finite we require that that rate of growth

$$\lim_{n\to\infty} \left| \prod_{j=0}^{n-1} |T'(T^j(x))| \, |\beta(T^j(x))| \right|^{1/n} < 1.$$

Takings logarithms, x is a point of differentiability of Γ_α under assumptions on T and β if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |T'(T^j(x))| + \log |\beta(T^j(x))| < 0. \tag{6.9}$$

By Birkhoff's Ergodic Theorem, for any ergodic T-invariant probability measure μ, Γ_α is differentiable μ-almost everywhere if

$$\int \log |T'| + \log |\beta| \; d\mu < 0.$$

Moreover if

$$\int \log |T'| + \log |\beta| \; d\mu \geq 0$$

then for μ-almost every point $x \in S^1$, $\Gamma_\alpha(x)$ is a point of non-differentiability of the graph. That is, we have a tipping point, where the μ-measure, of points of differentiability, for any suitable probability measure, of the graph jumps from 1 to zero, as the integral above becomes positive for different values of T' and β, where β depends on the parameters α.

Example. Suppose that $T(x) = bx \bmod 1$. Then Γ_α is almost everywhere differentiable with respect to some T-invariant ergodic probability measure μ provided that

$$\int \log|\beta|\, d\mu < -\log b.$$

Suppose that the partial hyperbolicity assumption (6.8) holds. Given any $x \in S^1$ and for any $n \geq 0$ it is clear that

$$\begin{aligned} \frac{1}{n}\sum_{j=0}^{n-1} \log|T'(T^j(x))| + \log|\beta(T^j(x))| &\geq \frac{\log\left(\inf_{x\in S^1}|T'(x)| \inf_{x\in S^1}|\beta(x)\right)^n}{n} \\ &= \log\left(\inf_{x\in S^1}|T'(x)| \inf_{x\in S^1}|\beta(x)\right) \\ &> 0. \end{aligned}$$

That is, for any $x \in S^1$ the limit as in (6.9) is positive. Hence as in Proposition 6.3.2, if the partial hyperbolicity assumption holds, then the graph is indeed nowhere differentiable.

6.4 On the non-decreasing case of Γ_α

In the following section we let $T : S^1 \to S^1$ be a uniformly expanding map of the circle, and let $\beta(x) > 0$ for all $x \in S^1$; that is $\alpha = (0, \alpha_1, \ldots, \alpha_{b-1}, 1)$ where $\alpha_j < \alpha_{j+1}$ for all j. Then it is clear by construction of the graph as in [Oka] that Γ_α is non-decreasing, and by such an observation we can prove the following dimension theoretic result.

Proposition 6.4.1

Suppose that T is a uniformly expanding map of the circle and that $\beta(x) > 0$ for all x. Then Γ_α is non-decreasing and

$$\dim_{\mathcal{H}}(\Gamma_\alpha) = 1.$$

Proof. Showing that $\dim_H(\Gamma_\alpha) = 1$ is equivalent to showing that the length, equivalently the 1-dimensional Hausdorff measure of Γ_α is positive and finite.

Recall the construction of Γ_α as in [Oka] where $\Gamma_\alpha = \lim_{n\to\infty} f_n$ where f_n is a finite union of line segments. Since Γ_α is non-decreasing, each line segment in f_n is non-decreasing. Moreover the arc of f_n over a cylinder of rank n is a straight line, the height of which is the same as the height of the graph of the cylinder.

Given $x \in S^1$ and $n \geq 0$ let $l_n(x)$ denote the length of the arc of f_n over $C_n(x)$. Then it is clear than

$$l_n(x) \leq \text{diam}(C_n(x)) + \text{osc}(\Gamma_\alpha, C_n(x)).$$

Since Γ_α is non-decreasing, letting $\mathcal{C}_n$ be the cover of S^1 consisting of cylinders of rank n, which have disjoint interior

$$\sum_{\mathcal{C}_n} \text{diam}(C_n(x)) = 1$$

and

$$\sum_{\mathcal{C}_n} \text{osc}(\Gamma_\alpha, C_n(x)) = 1.$$

Thus given n, and summing over cylinders $C_n(x)$ in a cover $\mathcal{C}_n$ of S^1 by cylinders of rank n we can give an upper bound on the length of f_n. That is

$$\begin{aligned}
\text{length}(f_n) &= \sum_{\mathcal{C}_n} f_n(x) \\
&\leq \sum_{\mathcal{C}_n} \text{diam}(C_n(x)) + \sum_{\mathcal{C}_n} \text{osc}(\Gamma_\alpha, C_n(x)) \\
&= 2.
\end{aligned}$$

In particular the 1-dimensional Hausdorff measure

$$\mathcal{H}^1(\Gamma_\alpha) = \lim_{n\to\infty} \text{length}(f_n) < \infty$$

and hence $\dim_{\mathcal{H}}(\Gamma_\alpha) \leq 1$.

Recall the construction of $\Gamma_\alpha = \lim_{n\to\infty} f_n$ and that Γ_α increases from 0 to 1. By construction $\text{length}(f_{n+1}) \geq \text{length}(f_n)$. It is easy to see that $\text{length}(\Gamma_\alpha) \geq \text{length}(f_0) = \sqrt{2}$. In particular $\mathcal{H}^1(\Gamma_\alpha) > 0$ and so $\dim_{\mathcal{H}}(\Gamma_\alpha) \geq 1$ and the result follows. ❑

6.5 An upper bound for box dimension

In the following we restrict ourselves to case in which $T(x) = bx \bmod 1$. In addition we assume that $\alpha \neq (1, 1/b, \ldots, 1/b^{n-1}, 1)$. If it were the case that $\alpha = (1, 1/b, \ldots, 1/b^{n-1}, 1)$ it is clear that $\Gamma_\alpha(x) = (x)$, and since the graph is linear it is clear that $\dim_B(\Gamma_\alpha) =$

$\dim_{\mathcal{H}}(\Gamma_\alpha) = 1$.

Recall that we define the upper box-dimension of Γ_α to be

$$\overline{\dim}_B(\Gamma_\alpha) = \limsup_{\delta \to 0} \frac{\log N_\delta(\Gamma_\alpha)}{-\log \delta},$$

where $N_\delta(\Gamma_\alpha)$ is the least number of boxes (squares) of side length δ required to cover the graph Γ_α. If we consider the cover of S^1 consisting of cylinders of rank n, we consider coverings of Γ_α by boxes of side length b^{-n} above these cylinders. In particular let

$$\overline{d} = \limsup_{n \to \infty} \frac{\log N_{b^{-n}}(\Gamma_\alpha)}{-\log b^{-n}}. \tag{6.10}$$

Here $N_{b^{-n}}(\Gamma_\alpha)$ is the least number of boxes of side length b^{-n} required to cover Γ_α. Note that the projection to the x-axis of these boxes forms a cover of S^1 by cylinders of rank n. It is also worth noting that $\overline{\dim}_B(\Gamma_\alpha) \leq \overline{d}$. Before giving an upper bound for box dimension, we have the following technical lemma.

Lemma 6.5.1

Let α be such that $\Gamma_\alpha \neq x$. We consider $N_{b^{-n}}(\Gamma_\alpha)$ to be the number of boxes of side length b^{-n} needed to cover Γ_α, where the projection of these boxes to the x-axis forms a cover $\mathcal{C}_n$ of S^1 consisting of cylinders of rank n. Then there exists $C_2 > 0$ such that for all $n \geq 1$

$$N_{b^{-n}}(\Gamma_\alpha) - b^n \geq C_2 N_{b^{-n}}(\Gamma_\alpha).$$

Proof. Since Γ_α is continuous an increases from 0 to 1, for each n there exists at least 1 cylinder in the cover of rank n, such that the number of boxes of side length b^{-n} directly above such a cylinder required to cover the graph is strictly greater than 1. Recall that by construction of the cover, the projection of these boxes to the x-axis forms a cover of S^1 consisting of cylinders of rank n. Thus for each n, we have b^n cylinders in a cover of rank n and we know that $N_{b^{-n}}(\Gamma_\alpha) > b^n$. That is, for all $n \geq 0$

$$\frac{b^n}{N_{b^{-n}}(\Gamma_\alpha)} \in (0, 1).$$

Moreover when covering the continuous function Γ_α by boxes, the projection of which forms a cover of S^1 by cylinders of rank n it is clear that

$$N_{b^{-(n+1)}}(\Gamma_\alpha) \geq N_{b^{-n}}(\Gamma_\alpha) \times b.$$

Therefore for each n,

$$\begin{aligned}\frac{b^n}{N_{b^{-n}}(\Gamma_\alpha)} &\geq \frac{b^{-1}b^{n+1}}{N_{b^{-(n+1)}}(\Gamma_\alpha)b^{-1}} \\ &= \frac{b^{n+1}}{N_{b^{-(n+1)}}(\Gamma_\alpha)}.\end{aligned}$$

Hence

$$0 \leq 1 - \frac{b^n}{N_{b^{-n}}(\Gamma_\alpha)} \leq 1 - \frac{b^{n+1}}{N_{b^{-(n+1)}}(\Gamma_\alpha)}.$$

Moreover

$$\frac{N_{b^{-n}}(\Gamma_\alpha) - b^n}{N_{b^{-n}}(\Gamma_\alpha)} \geq \frac{N_{b^{-(n-1)}}(\Gamma_\alpha) - b^{n-1}}{N_{b^{-(n-1)}}(\Gamma_\alpha)}$$

for each $n \geq 1$.

Therefore, letting

$$C_2 = \frac{N_{b^{-1}}(\Gamma_\alpha) - b}{N_{b^{-1}}(\Gamma_\alpha)} > 0,$$

it is true that for all n

$$\frac{N_{b^{-n}}(\Gamma_\alpha) - b^n}{N_{b^{-n}}(\Gamma_\alpha)} \geq C_2$$

and the result follows. ❏

Given a Hölder continuous function ϕ, recall that we can define the topological pressure of ϕ as in Section 2.6;

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \inf_{\mathcal{C}_n} \sum_{C\in\mathcal{C}_n} \exp\left\{\sup_{x\in C} \phi^n(x)\right\}$$

where $\phi^n(x) = \sum_{j=0}^{n-1} \phi(T^j(x))$ and the infimum is taken over covers of S^1 by cylinders of rank n. In the following we always assume that the map T is given by $T(x) = bx \bmod 1$. We can now prove an upper bound for the box dimension of Γ_α in terms of the unique solution to a a Bowen type equation.

Remark. We consider potentials of the form $\phi(x) = (1 - s_0)\log b + \log|\beta(x)|$ and consider the unique solution to the equation $P(\phi) = 0$. Since Γ_α increases from 0 to 1 and using Proposition 6.2.1 it can be shown that $P(\log|\beta|) \geq 0$. Moreover since $T(x) = bx \bmod 1$ is uniformly expanding it is clear that $P(\log b + \log|\beta|) > P(\log|\beta|) \geq 0$. That is when $s_0 = 0$ we have that $P(\phi) > 0$. In addition arguments similar to that in Section 2.6 on can show that $P(\phi) \to -\infty$ as $s_0 \to \infty$. That is, by monotonicity of topological pressure there exists a unique s_0 such that $P(\phi) = 0$.

Theorem 6.5.2

Let $T(x) = bx \bmod 1$ *and let* α *be such that* $\Gamma_\alpha(x) \neq x$. *Let* $\overline{d}$ *be as in (6.10) and let* s_0 *be the unique solution to* $P((1-s_0)\log b + \log|\beta|) = 0$. *Then*

$$\overline{\dim}_B(\Gamma_\alpha) \leq \overline{d} \leq s_0.$$

Proof. Let $\epsilon > 0$ be arbitrary. It will be sufficient to show that $\overline{d} - \epsilon \leq s_0$. Since P is continuous and decreasing we show that

$$P((1-(\overline{d}-\epsilon))\log b + \log|\beta|) > 0.$$

Given ϵ, choose n sufficiently large such that

$$\overline{d} - \epsilon \leq \frac{\log N_{b^{-n}}(\Gamma_\alpha)}{-\log b^{-n}} \leq \overline{d} + \epsilon.$$

In particular, for $N_{b^{-n}}(\Gamma_\alpha)$ the number of boxes in the cover as in (6.10)

$$N_{b^{-n}}(\Gamma_\alpha) \geq (b^{-n})^{\epsilon - \overline{d}}. \tag{6.11}$$

Consider the cover of S^1 by cylinders of rank n, which we call $\mathcal{C}_n$. Taking $\phi = (1-(\overline{d}-\epsilon))\log b + \log|\beta|$ we consider the sum $\sum_{C_n(x)\in\mathcal{C}_n} \exp\phi^n(x)$. That is we take the sum over the cylinders $C_n(x)$ in the cover, where x is some arbitrary point in such cylinders. In particular since $\beta^n(x) = \prod_{j=0}^{n-1}\beta(T^j(x))$ is constant on points in a cylinder of rank n, by Proposition 6.2.1

$$\begin{aligned}\sum_{x\in C_n(x)\in\mathcal{C}_n} \exp\phi^n(x) &= \sum_{x\in C_n(x)\in\mathcal{C}_n} b^{n(1-(\overline{d}-\epsilon))}|\beta^n(x)| \\ &= \sum_{x\in C_n(x)\in\mathcal{C}_n} b^{n(1-(\overline{d}-\epsilon))}\mathrm{osc}(\Gamma_\alpha, C_n(x)).\end{aligned}$$

We let $N_{b^{-n}}(\Gamma_\alpha|_{C_n(x)})$ denote the number of boxes of side length b^{-n} needed to stack up above one another in order to cover the graph directly above $C_n(x)$. Then as we defined $\overline{d}$ as in (6.10) it can be seen that

$$\sum_{\mathcal{C}_n} N_{b^{-n}}(\Gamma_\alpha|_{C_n(x)}) = N_{b^{-n}}(\Gamma_\alpha).$$

Then it is easy to see that

$$b^n\mathrm{osc}(\Gamma_\alpha, C_n(x)) \leq N_{b^{-n}}(\Gamma_\alpha|_{C_n(x)}) < b^n\mathrm{osc}(\Gamma_\alpha, C_n(x)) + 1.$$

Therefore by the above

$$\sum_{x\in C_n(x)\in\mathcal{C}_n} \exp\phi^n(x) \;>\; \sum_{x\in C_n(x)\in\mathcal{C}_n} b^{-n(\overline{d}-\epsilon)}\left(N_{b^{-n}}(\Gamma_\alpha\mid_{C_n(x)})-1\right).$$

Thus there exists C_3 as in Lemma 6.5.1 such that

$$\begin{aligned} P(\phi) &> \lim_{n\to} \frac{1}{n}\log\left[b^{n(\epsilon-\overline{d})}\sum_{\mathcal{C}_n}\left(N_{b^{-n}}(\Gamma_\alpha\mid_{C_n(x)})-1\right)\right] \\ &= \lim_{n\to} \frac{1}{n}\log b^{n(\epsilon-\overline{d})}\left(N_{b^{-n}}(\Gamma_\alpha)-b^n\right) \\ &\geq \lim_{n\to} \frac{1}{n}\log C_3 b^{n(\epsilon-\overline{d})}N_{b^{-n}}(\Gamma_\alpha). \end{aligned}$$

By (6.11) we see that $b^{n(\epsilon-\overline{d})}N_{b^{-n}}(\Gamma_\alpha)\geq 1$ and so for $C_3>0$ as above,

$$\begin{aligned} P(\phi) &> \lim_{n\to\infty}\frac{1}{n}\log C_3 \\ &= 0. \end{aligned}$$

Since $\epsilon>0$ was chosen arbitrarily we have shown that $\overline{d}\leq s_0$ and hence $\overline{\dim}_B(\Gamma_\alpha)\leq s_0$ and the upper bound follows. ❑

Therefore by the above proposition, for s_0 the unique solution to $P((1-s_0)\log b+\log|\beta|)=0$, $\dim_H(\Gamma_\alpha)\leq s_0$.

6.6 A lower bound for Hausdorff dimension

In the following we give a lower bound for the Hausdorff dimension of Γ_α. An integral tool for our proof is the mass distribution principle, see [Fa1]. In the following we always assume that $T: S^1\to S^1$ is the map $T(x)=bx \bmod 1$.

Proposition 6.6.1 (Mass Distribution Principle)
Let μ be a mass distribution on Γ. Suppose that there exists $s>0$, $\delta>0$ and $c>0$ such that

$$\mu(U)\leq c\,\mathrm{diam}(U)^s$$

for all open sets U such that $\mathrm{diam}(U)\leq\delta$. Then $\dim_{\mathcal{H}}(\Gamma)\geq s$.

Recall the construction of Γ_α as in [McColl] by a union of rectangles Λ_n as in (6.4). Recall that

$$\Gamma_\alpha = \bigcap_{n=0}^{\infty} \Lambda_n.$$

Recall that since $T(x) = bx \bmod 1$, Λ_n is a union of b^n rectangles each of horizontal diameter b^{-n} whose projection to the x-axis forms a cover of S^1 by cylinders of rank n. Recall that the height of the graph over these cylinders is precisely the height of the level n rectangles in our construction.

Given $n \geq 0$ we consider the construction Λ_n as above. Then let $\mathcal{C}_n$ be the corresponding cover by cylinders of rank n. Given a cylinder $C_n(x) \in \mathcal{C}_n$, the area of the rectangle in Λ_n above $C_n(x)$ is given by

$$\begin{aligned} a(\Lambda_n |_{C_n(x)}) &= \operatorname{diam}(C_n(x))\operatorname{osc}(\Gamma_\alpha, C_n(x)) \\ &= b^{-n}|\beta^n(x)|. \end{aligned}$$

Note that since $\beta^n(x)$ is constant on points in cylinders of rank n, the area is independent of choice of point x. Hence the total area of the stage n construction

$$a(\Lambda_n) = \sum_{x \in C_n(x) \in \mathcal{C}_n} b^{-n}|\beta^n(x)|. \tag{6.12}$$

Using this construction we show that by the mass distribution principle the Hausdorff dimension of Γ_α is bounded from below by the unique value of s_0 such that $P((1 - s_0)\log b + \log|\beta|) = 0$, and by Theorem 6.5.2 we attain equality of dimension.

Theorem 6.6.2

Let $T(x) = bx \bmod 1$ and α given such that $\Gamma_\alpha(x) \neq x$. Let s_0 be the unique solution to the pressure equation $P((1 - s_0)\log b + \log|\beta|) = 0$. Then $\dim_{\mathcal{H}}(\Gamma_\alpha) \geq s_0$.

Proof. Let $s < s_0$ be chosen arbitrarily. Then by our assumption and properties of topological pressure as in Section 2.6 we know that $P((1 - s)\log b + \log|\beta|) > 0$. We let μ be the natural mass distribution on Γ_α. As in [McColl] we start with unit mass on Λ_0 and repeatedly spread this mass over the total area of each Λ_n. That is we have a sequence of probability measures μ_n where at stage n of the construction we have that

$$\mu_n(\Lambda_n |_{C_n(x)}) = \frac{a(\Lambda_n |_{C_n(x)})}{a(\Lambda_n)},$$

and thus $\mu_n(\Lambda_n) = 1$ for all $n \geq 0$. Moreover $\mu_n \rightharpoonup \mu$ where μ is the natural mass distribution on Γ_α, in the weak-$*$ sense.

Let U be any open set satisfying $\mathrm{diam}(U) < 1$. Choose $n \geq 1$ such that

$$b^{-(n+1)} \leq \mathrm{diam}(U) < b^{-n}. \tag{6.13}$$

Furthermore, given any U we can find n such that U can be contained within some open square of side length b^{-n} and has non-empty intersection with at most 2 rectangles in the level n construction Λ_n. In particular

$$a(U) \leq b^{-2n}. \tag{6.14}$$

Given $\epsilon > 0$ choose n sufficiently large such that, by our assumption on $s < s_0$

$$\frac{1}{n} \log \sum_{x \in C_n(x) \in \mathcal{C}_n} \exp\left\{ \sum_{j=0}^{n-1} (1-s)\log b + \log|\beta(T^j(x))| \right\} > \epsilon$$

where $\mathcal{C}_n$ is a cover of S^1 consisting of cylinders of rank n. In particular since $\beta^n(x)$ is constant across all points in $C_n(x)$,

$$\frac{1}{n} \log \sum_{\mathcal{C}_n} b^{n(1-s)} |\beta^n(x)| > \epsilon$$

and so

$$\sum_{\mathcal{C}_n} b^{n(1-s)} |\beta^n(x)| > e^{n\epsilon}. \tag{6.15}$$

There exists some $\delta > 0$ such that for any $\mathrm{diam}(U) \leq \delta$ we can choose n sufficiently large such that both (6.13) and (6.15) hold.

Given U as above, U is contained within some open square of side length b^{-n}. By construction of the mass distribution μ and discussions above, for such $n \geq 1$ sufficiently large it can be shown that

$$\begin{aligned}
\mu(U) &\leq \frac{a(U \cap \Lambda_n)}{a(\Lambda_n)} \\
&\leq \frac{a(U)}{a(\Lambda_n)} \\
&\leq \frac{b^{-2n}}{\sum_{\mathcal{C}_n} b^{-n} |\beta^n(x)|} \\
&= \frac{1}{b^{2n} \sum_{\mathcal{C}_n} b^{-n} |\beta^n(x)|} \\
&= \left(b^n \sum_{\mathcal{C}_n} |\beta^n(x)| \right)^{-1}.
\end{aligned}$$

Now by (6.15)

$$b^n \sum_{\mathcal{C}_n} |\beta^n(x)| > e^{n\epsilon} b^{ns}.$$

Therefore

$$\mu(U) \leq e^{-n\epsilon} b^{-ns}.$$

By our assumptions on U, $b^{-(n+1)} \leq \mathrm{diam}(U)$ and hence $b^{-n} \leq b\,\mathrm{diam}(U)$. Therefore since we took $\epsilon > 0$ and took $s > 0$ to be fixed and bounded, we have shown that

$$\begin{aligned} \mu(U) &\leq e^{-n\epsilon}(b\,\mathrm{diam}(U))^s \\ &\leq b^s \mathrm{diam}(U)^s. \end{aligned}$$

In particular, by the Mass Distribution principle and since $s < s_0$ was chosen to be arbitrary we have shown that $\dim_H(\Gamma_\alpha) \geq s_0$ as required. ❏

6.7 An explicit formula for $\dim_{\mathcal{H}}(\Gamma_\alpha)$

Let $T(x) = bx \bmod 1$ and $\alpha = (0, \alpha, \ldots, \alpha_{b-1}, 1)$ be given as in previous sections. Recall that β is a locally constant function which depends solely on the first coordinate of a given point. That is, as in Section 6.1

$$\beta(x) = \alpha_{j+1} - \alpha_j$$

where x_0, the first coordinate in the symbolic coding of x is $x_0 = j \in \{0, \ldots, b-1\}$.

If $x = (x_0, x_1, \ldots, x_{n-1}, \ldots)$, then it will be easier for us to write $\beta(x) = \beta(x_0)$. Moreover since β depends only on the first coordinate of x, we have that

$$\beta^n(x) = \prod_{j=0}^{n-1} \beta(T^j(x)) = \beta(x_0)\beta(x_1)\ldots\beta(x_{n-1}),$$

which depends only on the first n coordinates of a point.

By Theorem 6.5.2 and Theorem 6.6.2 we know that $\dim_{\mathcal{H}}(\Gamma_\alpha) = s_0$ where s_0 is the unique solution to the pressure equation $P((1-s_0)\log b + \log|\beta|) = 0$. Moreoever by the variational principle there exists a probability measure μ_0, the unique equilibrium state for $(1-s_0)\log b + \log|\beta|$ such that

$$\dim_{\mathcal{H}}(\Gamma_\alpha) = 1 + \frac{h_{\mu_0}(T) + \int \log|\beta|\, d\mu_0}{\log b}.$$

However since $\log b$ is constant, μ_0 is also the unique equilibrium state for $\log|\beta|$. Moreover the variational principle gives us that $P(\log|\beta|) = h_{\mu_0}(T) + \int \log|\beta|\, d\mu_0$. In particular

$$\dim_{\mathcal{H}}(\Gamma_\alpha) = 1 + \frac{P(\log|\beta|)}{\log b}.$$

In the following we give an explicit formular for $P(\log|\beta|)$, which allows us to give an explicit formula for the Hausdorff dimension of the graph. This allows us to see how the dimension changes for different parameter values α. We relate our results to those in [McColl].

Recall the characterisation of topological pressure by periodic points as in Section 2.6 as in (2.16). Supposing that a function ϕ is Hölder continuous and depends on the first two coordinates of a given point x, then [PaPo]

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x_0, x_1, \ldots, x_{n-1}, x_0} \exp\left\{\phi(x_0, x_1) + \ldots + \phi(x_{n-1}, x_0)\right\}.$$

Note that since T can be modelled by the full (one-sided) shift on b symbols, the corresponding adjacency matrix is given by the $b \times b$ matrix A where $A_{(i,j)} = 1$ for all $1 \leq i, j \leq b$.

For ϕ, depending on the first two coordinates of a given point we define the matrix A_ϕ where

$$(A_\phi)_{(i,j)} = \exp\phi(i,j).$$

Then it is shown [PaPo Chapter 5] that

$$\begin{aligned} \sum_{x_0, x_1, \ldots, x_{n-1}, x_0} \exp\left\{\phi(x_0, x_1) + \ldots + \phi(x_{n-1}, x_0)\right\} &= \mathrm{Trace} A_\phi^n \\ &= e^{nP(\phi)} + \lambda_2^n + \ldots + \lambda_b^n \end{aligned}$$

where $e^{P(\phi)}, \lambda_2, \ldots, \lambda_k$ are the eigenvalues of the matrix A_ϕ. In addition $e^{P(\phi)}$ is the maximal eigenvalue. That is $P(\phi) = \log\lambda_1$ where λ_1 is the maximal eigenvalue of the matrix A_ϕ. Moreover the sum of the eigenvalues of A_ϕ is the trace of the matrix A_ϕ.

In our context recall the function $\beta(x) = \beta(x_0)$ where β depends solely on the first coordinate in the symbolic coding of $x = (x_0, x_1, \ldots)$. We use the above construction [PaPo] to give an explicit formula for $P(\log|\beta|)$. Since T is modelled as a full shift on b symbols, and since the potential $\log\beta$ depends only on the first coordinate of a given point, the corresponding matrix $A_{\log|\beta|}$ is given by

$$(A_{\log|\beta|})_{(i,j)} = e^{\log|\beta(x_{i-1})|}$$

where $1 \le i \le b$. That is, $A_{\log|\beta|}$ is the $b \times b$ matrix with identical columns, where the entry on each column of row i takes value $|\beta(i-1)|$, i.e. $A_{\log|\beta|}$ is the matrix where each column is the same, and takes the value of all possible values of the locally constant function β. In particular, working as in [PaPo] it can be seen that $e^{P(\log|\beta|)}$ is the maximal eigenvalue of the matrix $A_{\log|\beta|}$. Now since $A_{\log|\beta|}$ has identical columns, all eigenvalues are zero except for one. Thus $e^{P(\log|\beta|)}$ is equal to the sum of all eigenvalues, namely the trace of the matrix $A_{\log|\beta|}$, or alternatively the sum of the entries in any columns of $A_{\log|\beta|}$. Therefore it can be seen that

$$P(\log|\beta|) = \log \sum_{j=0}^{b-1} |\beta(j)|. \tag{6.16}$$

Recalling that β was defined for all $j \in \{0, \ldots, b-1\}$ as $\beta(j) = \alpha_{j+1} - \alpha_j$, we have that

$$P(\log|\beta|) = \log \sum_{j=0}^{b-1} |\alpha_{j+1} - \alpha_j|\,.$$

Therefore given $T(x) = bx \bmod 1$ we have the explicit formula

$$\dim_{\mathcal{H}}(\Gamma_\alpha) = 1 + \frac{\log \sum_{j=0}^{b-1} |\alpha_{j+1} - \alpha_j|}{\log b}$$

where $\alpha_0 = 1$ and $\alpha_b = 1$ as in Section 6.1.

Remark. Suppose that as in [McColl] we have $T(x) = 3x \bmod 1$ and $\alpha = (0, \alpha_1, \alpha_2, 1) = (0, \alpha - 1, 1 - \alpha_1, 1)$ where $\alpha_1 > 1/2$. Then

$$\sum_{j=0}^{b-1} |\alpha_{j+1} - \alpha_j| = \alpha_1 + |2\alpha_1 - 1| + \alpha_1.$$

Since $\alpha_1 > 1/2$ it is clear that

$$\log \sum_{j=0}^{b-1} |\alpha_{j+1} - \alpha_j| = 4\alpha_1 - 1$$

and hence

$$\begin{aligned} \dim_{\mathcal{H}}(\Gamma_\alpha) &= 1 + \frac{\log(4\alpha_1 - 1)}{\log 3} \\ &= \frac{\log(12\alpha_1 - 3)}{\log 3} \end{aligned}$$

as in [McColl].

In the case of the function of Bourbaki [Bou] $\alpha_1 = 2/3$, thus the Hausdorff dimension of Bourbaki's function $\dim_{\mathcal{H}}(\Gamma_\alpha) = \frac{\log 5}{\log 3}$.

Suppose that we are in the case of Proposition 6.4.1, where $\alpha = (0, \alpha_1, \ldots, \alpha_{b-1}, 1)$ and $\alpha_j < \alpha_{j+1}$ for all j. Then $|\alpha_{j+1} - \alpha_j| = \alpha_{j+1} - \alpha_j$. Hence

$$\begin{aligned}\sum_{j=0}^{b-1} |\alpha_{j+1} - \alpha_j| &= \sum_{j=1}^{b-1} \alpha_{j+1} - \alpha_j \\ &= \alpha_b - \alpha_0 \\ &= 1.\end{aligned}$$

Therefore $P(\log|\beta|) = 0$, thus if we are in the case where Γ_α is non-decreasing, it is clear from our formula, that $\dim_{\mathcal{H}}(\Gamma_\alpha) = 1$.

Finally, if we let $T(x) = 3x \bmod 1$ and $\alpha = (0, 1/2, 1/2, 1)$ then as in remarks made in earlier sections, the resulting graph Γ_α is the devils staircase constructed on the middle third Cantor set. Thus $P(\log|\beta|) = 0$ and the Hausdorff dimension of the graph is 1.

6.8 Multifractal analysis of Birkhoff averages

Recall that in Section 6.3 we discussed restrictions under which points of the graph Γ_α are differentiable or not. In particular, we found that given a map T and α and β as in previous sections the graph is differentiable at a point x if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (\log|T'| + \log|\beta|)(T^j(x)) < 0.$$

By Birkhoff's Ergodic Theorem, given any T-invariant ergodic probability measure μ, Γ_α is differentiable μ-almost everywhere if and only if

$$\int \log|T'| + \log|\beta| \, d\mu < 0.$$

For such invariant measures, we can describe the behaviour of almost every point, however we can use the so called multifractal formalism or analysis to describe the fine structure of the expectional set for the above ergodic averages. In particular we ask what other values can such limits take, and whether the limits even exist. This demonstrates the so called multifractal miracle [PeWe], and we study the decomposition of the phase space into level sets of ergodic averages. This analysis can be found in [Pe, PeWe1, PeWe2, PeWe3]. In the following, we give a brief description of the multifractal formalism, and some of the results which follow. A full description of the decomposition into level sets and properties of the Birkhoff spectrum, notions which we introduce in the following, can be found in [Pe]. In particular, we desribe the range of values which this limit can take, and study the relationships of ergodic averages with pointwise dimension, and how this in turn relates to properties of different equilibrium states. We first introduce the following tools.

Throughout the analysis we always consider the case in which $T : S^1 \to S^1$ is the map $T(x) = bx \bmod 1$. Let $\phi = \log b + \log|\beta|$ and let $\overline{\phi}$ denote the limit

$$\overline{\phi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j(x)). \tag{6.17}$$

Thus Γ_α is differentiable at a point x if and only if $\overline{\phi}(x) < 0$.

Given an ergodic T-invariant probability measure μ then $\overline{\phi}(x) = \int \phi \; d\mu$ for μ-almost every $x \in S^1$.

Let us denote the levels sets $\mathcal{B}_\alpha = \left\{x \in S^1 \mid \overline{\phi}(x) = \alpha\right\}$ and

$$\hat{\mathcal{B}} = \left\{x \in S^1 \mid \text{ the limit } \overline{\phi}(x) \text{ does not exist } \right\}.$$

Then we have the decomposition of the phase space into level sets $S^1 = \bigcup_\alpha \mathcal{B}_\alpha \cup \hat{\mathcal{B}}$ where the (possibly uncountable) union is taken over the set of all possible limits α. It is clear that for any ergodic probability measure μ, $\mu(\mathcal{B}_{\int \phi\, d\mu}) = 1$. However we wish to ask the natural questions, regarding other sets in this decomposition. In particular, how many other level sets of measure zero are there, and what possible values do such limits take. The multifractal analysis allows us to do this.

For a Hölder continuous potential ϕ, we let μ_ϕ denote the unique equilibrium state corresponding to ϕ. Given $\alpha \in \mathbb{R}$ we define the Birkhoff spectrum to be

$$b_\phi(\alpha) = \dim_{\mathcal{H}}(\mathcal{B}_\alpha).$$

We study the multifractal analysis for the Birkhoff Spectrum of a given potential ϕ. That is, we look at the values taken by $\overline{\phi}$ for the function ϕ which measures whether a point x is a point of differentiability for Γ_α or not. Our analysis follows that in [PeWe1].

In order to do this we introduce the notion of pointwise dimension. Given a probability measure μ we define the pointwise or local dimension at x to be

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

We denote the level sets of pointwise dimension to be $\mathcal{K}_\alpha = \{x \in S^1 \mid d_\mu(x) = \alpha\}$. Letting

$$\hat{\mathcal{K}} = \left\{x \in S^1 \mid \text{the limit } d_\mu(x) \text{ does not exist}\right\}$$

we have the decomposition of the phase space into level sets of pointwise dimension

$$S^1 = \bigcup_\alpha \mathcal{K}_\alpha \cup \hat{\mathcal{K}}.$$

Define the dimension spectrum

$$f_\mu(\alpha) = \dim_{\mathcal{H}}(\mathcal{K}_\alpha).$$

We now have the following [PeWe1 Proposition 1] which relates pointwise dimension to Birkhoff averages. Given $x \in S^1$ and $n \geq 0$ let $C_n(x)$ denote the cylinder of rank n containing x. Since $T(x) = bx \bmod 1$ it is clear that any cylinder of rank n has diameter b^{-n}.

Proposition 6.8.1

Let $T(x) = bx \bmod 1$, and let ϕ be a Hölder continuous function on S^1 with corresponding equilibrium state μ_ϕ. Then

$$\overline{\phi}(x) = P(\phi) - \log b \; d_{\mu_\phi}(x).$$

and consequently

$$b_\phi(\alpha) = f_{\mu_\phi}\left(\frac{P(\phi) - \alpha}{\log b}\right).$$

Proof. Notice firstly that since μ_ϕ is an equilibrium state, it satisfies a Gibbs property as in (2.17). That is there exists $C_4 > 0$ such that given a cylinder $C_n(x)$

$$C_4^{-1} \leq \frac{\mu_\phi(C_n(x))}{\exp\left\{\sum_{j=0}^{n-1} \phi(T^j(x)) - nP(\phi)\right\}} \leq C_4.$$

Thus

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\phi(C_n(x)) = \overline{\phi}(x) - P(\phi).$$

Let $\delta \le b^{-n}$ so that $\mu_\phi(B_\delta(x)) \le 2\mu_\phi(C_n(x))$. Therefore

$$\begin{aligned} d_{\mu_\phi}(x) &= \lim_{\delta\to 0} \frac{\log \mu_\phi(B_\delta(x))}{\log \operatorname{diam}(B_\delta(x))} \\ &= \lim_{n\to\infty} \frac{\log \mu_\phi(C_n(x))}{\log \operatorname{diam}(C_n(x))} \\ &= \lim_{n\to\infty} \frac{\log \mu_\phi(C_n(x))}{n} \frac{n}{\log b^{-n}} \\ &= \frac{\overline{\phi}(x) - P(\phi)}{-\log b} \end{aligned}$$

and the first result follows. Then

$$\begin{aligned} b_\phi(\alpha) &= \dim_{\mathcal{H}} \left\{x \mid \overline{\phi}(x) = \alpha\right\} \\ &= \dim_{\mathcal{H}} \left\{x \mid d_{\mu_\phi}(x) = \frac{\alpha - P(\phi)}{-\log b}\right\} \\ &= f_{\mu_\phi}\left(\frac{P(\phi) - \alpha}{\log b}\right). \end{aligned}$$

❑

We give the following multifractal analysis for the dimension spectrum, using methods from [Pe][PeWe1]. Let m denote the measure of full dimension. That is, m is the unique equilibrium state corresponding to potential $-d\log b$ where $P(-d\log b) = 0$ and by Bowens equation $d = \dim_H(S^1) = 1$. Moreover since $\log b$ is constant the measure m is equal to Lebesgue measure on S^1.

Theorem 6.8.2

Let ϕ be the Hölder potential $\phi(x) = \log b + \log|\beta|$ where $T(x) = bx \bmod 1$ with corresponding equilibrium state μ_ϕ. Then

1. *If $m \ne \mu_\phi$, the Birkhoff spectrum $b_\phi(\alpha)$ is real analytic and strictly convex on some open interval (a,b). Thus $\overline{\phi}$ attains an interval of values.*

2. *For each $\alpha \in [a,b]$ each $\mathcal{B}_\alpha$ is an uncountable dense subset of S^1.*

We use the multifractal analysis (see [Pe]) for pointwise dimension, to fully describe the dimension spectrum. Then one can use Proposition 6.8.1 to relate the interval of values attained by $\overline{\phi}$ to those found in the analysis for d_{μ_ϕ} below. We give an outline of the proof, from which we can extract information about the values achieved by the limits for points in the exceptional set. The method follows in the standard way.

Let ϕ and μ_ϕ be as in the statement of the theorem. We define the function $\log\psi = \phi - P(\phi)$. Then $\log\psi$ is a Hölder continuous function and μ_ϕ is the unique equilibrium state for $\log\psi$ and it is clear that $P(\log\psi) = 0$.

For a parameter $q \in (-\infty, \infty)$ we define the one-parameter Hölder continuous family of potentials on S^1 to be

$$\varphi_q(x) = -T(q)\log b + q\log\psi(x)$$

where $T(q)$ is chosen such that $P(\varphi_q) = 0$.

Let μ_{φ_q} be the unique equilibrium state corresponding to potential φ_q. By choice of zero pressure potentials $\log\psi$ and $-d\log b$ as above and Bowens equation it can be seen that $T(0) = d = 1$ and $T(1) = 0$.

Remark. By the previous proposition and Birkhoff's Ergodic Theorem we know that μ_ϕ almost everywhere $\overline{\phi}(x) = \int \log\phi \, d\mu_\phi$ and

$$d_{\mu_\phi}(x) = \frac{P(\phi) - \int \log\phi \, d\mu_\phi}{\log b}.$$

We now prove the theorem.

Proof. We use the notation $\log\psi^n(x) = \prod_{j=0}^{n-1} \log\psi(T^j(x))$ where $\log\psi = \phi - P(\phi)$ as above.

By Proposition 6.8.1 note that

$$\begin{aligned} d_{\mu_\phi}(x) &= \frac{\overline{\phi}(x) - P(\phi)}{-\log b} \\ &= \frac{\lim_{n\to\infty}\left(\frac{1}{n}\sum_{j=0}^{n-1}\phi(T^j(x))\right) - P(\phi)}{-\log b} \\ &= \lim_{n\to\infty} \frac{\frac{1}{n}\left(\sum_{j=0}^{n-1}\phi(T^j(x)) - nP(\phi)\right)}{-\log b} \\ &= \lim_{n\to\infty} \frac{\frac{1}{n}\log\psi^n(x)}{-\log b}. \end{aligned}$$

Therefore we define the level sets of pointwise dimension to be

$$\mathcal{K}_\alpha = \left\{ x \in S^1 \mid \lim_{n\to\infty} \frac{\frac{1}{n}\log\psi^n(x)}{-\log b} = \alpha \right\}.$$

Letting

$$\alpha(q) = \frac{\int \log\psi \, d\mu_{\varphi_q}}{-\log b} \tag{6.18}$$

by Birkhoff's Ergodic Theorem it is clear that

$$\mu_{\varphi_q}(\mathcal{K}_{\alpha(q)}) = 1. \tag{6.19}$$

Now given $x \in \mathcal{K}_{\alpha(q)}$ we have that

$$\lim_{n\to\infty} \frac{\frac{1}{n}\log\psi^n(x)}{-\log b} = \alpha(q).$$

For Hölder continuous φ_q with corresponding equilibrium states μ_{φ_q} where $P(\varphi_q) = 0$ we define the limit (if it exists)

$$\overline{\varphi}_q(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi_q(T^j(x)).$$

By Proposition 6.8.1 it can be shown that

$$\begin{aligned} d_{\mu_{\varphi_q}}(x) &= \frac{\overline{\varphi}_q(x) - P(\varphi_q)}{-\log b} \\ &= \frac{\lim_{n\to\infty}\left(\frac{1}{n}\log(b^n)^{-T(q)}\log\psi^n(x)^q\right)}{-\log b}. \end{aligned}$$

Let $\epsilon > 0$ be given. Since $x \in \mathcal{K}_{\alpha(q)}$ we can choose $N \geq 1$ sufficiently large such that for all $n \geq N$

$$\alpha(q) - \epsilon \leq \frac{\frac{1}{n}\log\psi^n(x)}{-\log b} \leq \alpha(q) + \epsilon. \tag{6.20}$$

Therefore

$$\begin{aligned} d_{\mu_{\varphi_q}}(x) &= \lim_{n\to\infty} \frac{\frac{1}{n}\log(b^n)^{-T(q)}\log\psi^n(x)^q}{-\log b} \\ &\leq \lim_{n\to\infty} \frac{\frac{1}{n}\log(b^n)^{-T(q)-q(\alpha(q)+\epsilon)}}{-\log b} \\ &= \lim_{n\to\infty} \frac{\frac{1}{n}\log(b^{-n})^{T(q)+q(\alpha(q)+\epsilon)}}{-\log b} \\ &= T(q) + q(\alpha(q)+\epsilon). \end{aligned}$$

Then since $\epsilon > 0$ was chosen arbitrarily, we have shown that

$$d_{\mu_{\varphi_q}}(x) \leq T(q) + q\alpha(q).$$

In a similar way we use the reverse inequality in (6.20) to show that $d_{\mu_{\varphi_q}}(x) \geq T(q) + q\alpha(q)$. In particular for all $x \in \mathcal{K}_{\alpha(q)}$

$$d_{\mu_{\varphi_q}}(x) = T(q) + q\alpha(q).$$

Then as in [Pe] it can be shown that

$$\dim_{\mathcal{H}}(\mathcal{K}_{\alpha(q)}) = T(q) + q\alpha(q).$$

Using the fact that pressure is real analytic and by the inverse function theorem, along with Ruelles' formula for the derivative of pressure, it can be shown that $T(q)$ is real analytic, convex and strictly convex if and only if $\mu_\phi \neq m$. That is, if μ_ϕ is not Lebesgue measure.

In addition $\alpha(q) = -T'(q)$. To see this [PeWe2] we let $\varphi_{q,r}(x) = -r\log b + q\log\psi$. Now $P(\varphi_q) = 0$ and therefore

$$\frac{d}{dq}P(\varphi_q) = \frac{\partial P(\varphi_{q,r})}{\partial q} + \frac{\partial P(\varphi_{q,r})}{\partial r}\mid_{T(q)} T'(q) = 0.$$

In particular it can be shown that, by the above and the formula for the derivative of pressure,

$$\begin{aligned} T'(q) &= -\frac{\int \log\psi d\mu_{\varphi_q}}{-\log b} \\ &= -\alpha(q). \end{aligned}$$

Moreover $\alpha(q)$ is analytic and $\alpha'(q) = -T''(q) < 0$ so [Pe] the range of $\alpha(q)$ contains an interval. That is $\alpha(q)$ takes values in $[\alpha_1, \alpha_2]$ where $0 \leq \alpha_1 < \alpha_2 < \infty$. Moreover

$$f_{\mu_\phi}(\alpha(q)) = T(q) + q\alpha(q).$$

Since $\mu_{\varphi_q}(K_{\alpha(q)}) = 1$ and equilibrium states take positive measure on non empty open sets, K_α is a dense subset of S^1 when $\alpha \in [\alpha_1, \alpha_2]$. In addition if $\mu_\phi \neq m$, then $f_{\mu_\phi}(\alpha)$ and $T(q)$ are strictly convex and form a Legendre transform pair.

We can now use the above and relate our results to those for $\overline{\phi}$ and $b_\phi(\alpha)$. By Proposition 6.8.1 we can see that $K_{\alpha(q)} = B_{P(\phi)-\log b\alpha}$ thus B_α forms an uncountable dense subset of S^1. It can also be shown that $b_\phi(\alpha)$ takes values in some interval, which we can calculate in terms of $\alpha(q)$.

In particular

$$\begin{aligned} b_\phi(\alpha) &= f_{\mu_\phi}\left(\frac{P(\phi)-\alpha}{\log b}\right) \\ &= T(q) + q\left(\frac{P(\phi)-\alpha(q)}{\log b}\right). \end{aligned}$$

❏

Remark. Schmelling [Sch] has shown that for a Hölder continuous function ϕ, $\overline{\phi}$ takes no values outside the interval $[a, b]$ as in the statement of the theorem. Barreira and Schmelling schowed that [BaSch] provided $\mu_\phi \neq m$ (i.e. μ_ϕ is not lebesgue measure) then the Hausdorff dimension of the set of points where the limit $\overline{\phi}(x)$ does not exists is the same as the Hausdorff dimension of S^1 itself.

Chapter 7

Singularities of Weierstrass type functions

In the following chapter we introduce a specific type of singularity as studied in [HL] for continuous functions of the form W, invariant under skew-products of the form F introduced in Section 2.1. We investigate conditions under which such functions contain large sets of these singularities. We look at different degrees of difficulty and the restrictions required in order for our results to hold.

7.1 Introduction

Let $T : S^1 \to S^1$ be a uniformly expanding b-to-one continuous map of the circle. Without loss of generality we assume that T is orientation preserving. If this were not the case we replace T with $T^2 = T \circ T$, which is orientation preserving and proceed in the same way. Let $\lambda : S^1 \to (0, 1)$ be a Lipschitz continuous function. Since S^1 is a compact space we then have that $\log \lambda$ is also Lipschitz continuous. In addition let $p : S^1 \to \mathbb{R}$ be Lipschitz continuous. Without loss of generality we assume that $p(0) = 0$. If this were not the case then since p is well defined there exists c such that $p(0) = c$. By subtracting a constant, take $\overline{p}(x) = p(x) - c$ so that $\overline{p}(0) = 0$.

Consider the skew-product $F(x, y) = (T(x), \lambda(x)^{-1}(y - p(x)))$ as in (2.5). Then as in Section 2.1 there exists a continuous function W whose graph is invariant under F.

We can write $W : S^1 \to \mathbb{R}$ as

$$W(x) = \sum_{j=0}^{\infty} \lambda^j(x) p(T^j(x)) \tag{7.1}$$

where $\lambda^j(x) = \lambda(x) \dots \lambda(T^{j-1}(x))$. Recall the notation $\sup_{x \in S^1} |T'(x)| = |T'|_\infty < \infty$ and $\inf_{x \in S^1} |T'(x)| = \mathrm{m}(T') > 1$. Denote $\sup_{x \in S^1} \lambda(x) = |\lambda|_\infty < 1$ and $\inf_{x \in S^1} \lambda(x) = \mathrm{m}(\lambda) > 0$. Throughout this chapter we make the partial-hyperbolicity assumption that

$$\mathrm{m}(T')\mathrm{m}(\lambda) > 1. \tag{7.2}$$

Under the non-degeneracy assumption in Section 2.3 we assume that there exists no such Lipschitz function $u : S^1 \to \mathbb{R}$ that solves

$$u(x) - p(x) = \lambda(x) u(T(x)). \tag{7.3}$$

Then W is everywhere continuous but nowhere differentiable. Typically [HNW] this is the case. In particular recall that W is Hölder continuous of some exponent $\alpha \in (0, 1)$ but is not Lipschitz continuous.

In the following we study a particular type of singularity. Given a continuous function f, we define the upper right and lower right derivatives of f at x respectively to be

$$D^+ f(x) = \limsup_{y^+ \to x} \frac{f(y^+) - f(x)}{y^+ - x}$$

and

$$D_+ f(x) = \liminf_{y_+ \to x} \frac{f(y_+) - f(x)}{y_+ - x}$$

where $y^+, y_+ > x$ and approach from the right, i.e. $y^+, y_+ > x$ and $y^+, y_+ \to x$. We define the upper left and lower left derivatives at x respectively to be

$$D^- f(x) = \limsup_{y^- \to x} \frac{f(y^-) - f(x)}{y^- - x}$$

and

$$D_- f(x) = \liminf_{y_- \to x} \frac{f(y_-) - f(x)}{y_- - x}$$

where $y^-, y_- < x$ and approach from the left, i.e. $y^-, y_- < x$ and $y^-, y_- \to x$.

Definition. Let f be a continuous function. We say that x is a knot point for f if

$$D^+ f(x) = D^- f(x) = \infty$$

and

$$D_- f(x) = D_+ f(x) = -\infty.$$

Under certain conditions, we show that graph(W) contains a large set of knot points. We characterise what we mean by large in a toplogical sense, as well as in a measure theoretic and dimension theoretic setting.

7.2 On W lifted to $\mathbb{R}$

In the following we recall the natural lift of W to a function $\widehat{W} : \mathbb{R} \to \mathbb{R}$ as in Section 2.8. That is $\widehat{W}$ takes the form of (2.19). By discussions in Section 2.8 $\widehat{W}$ restricted to the unit interval is equal to W. We study the lift $\widehat{W}$ and relate results regarding knot points of $\widehat{W}$ to those for W itself. Recalling Section 2.8 we study the natural lift of the skew-product F, where the fibre map is given by $g(x)y = \lambda(x)^{-1}(y - p(x))$.

Remark. By discussions in Section 2.8 the derivative of the continuous lift of T remains unchanged. That is $m(\hat{T}') = m(T')$ and $|\hat{T}'|_\infty = |T'|_\infty$. Since $\hat{p}$ is the 1-periodic lift of the Lipschitz continuous function p to $\mathbb{R}$, then $|\hat{p}|_{\text{Lip}} = |p|_{\text{Lip}}$ and $|\hat{p}|_\infty = |p|_\infty$. In the same way $|\hat{\lambda}|_\infty = |\lambda|_\infty$, $\text{m}(\hat{\lambda}) = \text{m}(\lambda)$, $|\hat{\lambda}|_{\text{Lip}} = |\lambda|_{\text{Lip}}$ and $|\log \hat{\lambda}|_{\text{Lip}} = |\log \lambda|_{\text{Lip}}$.

By the above remark we retain the partial hyperbolicity assumption (7.2). Furthermore since $\hat{T}$ is an invertible, uniformly expanding continuous map of $\mathbb{R}$ from Section 2.8 without loss of generality we can assume that $\hat{T}(0) = 0$. Moreover $\hat{T}^j(0) = 0$ for all $j \in \mathbb{Z}$ and $\hat{T}^{-n}(x) \to 0$ for all $x \in \mathbb{R}$ as $n \to \infty$.

Since $\hat{T}$ is invertible, given $j \geq 1$ we define the inverse of $\hat{\lambda}^j(x)$ to be

$$\hat{\lambda}^{-j}(x) = \left(\hat{\lambda}(\hat{T}^{-1}(x)) \dots \hat{\lambda}(\hat{T}^{-j}(x))\right)^{-1}. \tag{7.4}$$

Since we can assume that $\hat{p}(0) = 0$ and $\hat{T}^j(0) = 0$ for all $j \in \mathbb{Z}$, we define

$$\hat{V}(x) = \sum_{j=-\infty}^{\infty} \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x))$$

such that $\hat{V}(0) = 0$.

We show that $\hat{V}$ is a well defined continuous function. Note that

$$\left|\hat{V}(x)\right| \leq \left|\widehat{W}(x)\right| + \sum_{j=1}^{\infty} \left|\hat{\lambda}^{-j}(x)\right| \left|\hat{p}(\hat{T}^{-j}(x))\right|.$$

Since $\hat{p}$ is Lipschitz continuous and $\hat{p}(0) = 0$, and since $\hat{T}^{-j}(0) = 0$ for all $j \geq 1$ given

$$\begin{aligned}\left|\hat{p}(\hat{T}^{-j}(x))\right| &= \left|\hat{p}(\hat{T}^{-j}(x)) - \hat{p}(\hat{T}^{-j}(0))\right| \\ &\leq |p|_{\mathrm{Lip}}\left|\hat{T}^{-j}(x) - \hat{T}^{-j}(0)\right| \\ &\leq |p|_{\mathrm{Lip}}\mathrm{m}(T')^{-j}\left|x\right|.\end{aligned}$$

Hence by (7.2)

$$\begin{aligned}\sum_{j=1}^{\infty}\left|\hat{\lambda}^{-j}(x)\right|\left|\hat{p}(\hat{T}^{-j}(x))\right| &\leq |p|_{\mathrm{Lip}}|x|\sum_{j=1}^{\infty}(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-j} \\ &= \frac{|p|_{\mathrm{Lip}}|x|(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}} \\ &< \infty\end{aligned}$$

by (7.2). Also note that

$$\begin{aligned}|\widehat{W}(x)| &\leq \sum_{j=0}^{\infty}\hat{\lambda}^{j}(x)|\hat{p}(\hat{T}^{j}(x))| \\ &\leq |p|_{\infty}\sum_{j=0}^{\infty}|\lambda|_{\infty}^{j} \\ &< \infty.\end{aligned}$$

Thus $|\hat{V}(x)| < \infty$ and converges uniformly, hence is continuous.

We require the following bounded variation result.

Lemma 7.2.1

Given $j \in \mathbb{N}$ and $x, y \in [0,1]$ there exists $C_0 > 1$ independent of j such that

$$C_0^{-1} \leq \frac{\hat{\lambda}^{-j}(x)}{\hat{\lambda}^{-j}(y)} \leq C_0.$$

Proof. Let $x, y \in [0,1]$ and $j \in \mathbb{N}$ be given. Since $\hat{\lambda}$ is the one periodic lift of the Lipschitz function λ, it is Lipschitz. Thus $\log\hat{\lambda}$ is Lipschitz continuous and recalling

(7.4) we have that

$$\begin{aligned}
\left|\log \frac{\hat{\lambda}^{-j}(x)}{\hat{\lambda}^{-j}(y)}\right| &= \left|\log \frac{(\hat{\lambda}(\hat{T}^{-1}(x))\ldots\hat{\lambda}(\hat{T}^{-j}(x)))^{-1}}{(\hat{\lambda}(\hat{T}^{-1}(y))\ldots\hat{\lambda}(\hat{T}^{-j}(y)))^{-1}}\right| \\
&= \left|\log \frac{\hat{\lambda}(\hat{T}^{-1}(y))\ldots\hat{\lambda}(\hat{T}^{-j}(y))}{\hat{\lambda}(\hat{T}^{-1}(x))\ldots\hat{\lambda}(\hat{T}^{-j}(x))}\right| \\
&\leq \sum_{k=1}^{j}\left|\log\hat{\lambda}(\hat{T}^{-k}(y)) - \log\hat{\lambda}(\hat{T}^{-k}(x))\right| \\
&\leq |\log\lambda|_{\mathrm{Lip}}\sum_{k=1}^{j}\left|\hat{T}^{-k}(y) - \hat{T}^{-k}(x)\right| \\
&\leq |\log\lambda|_{\mathrm{Lip}}\sum_{k=1}^{\infty}\mathrm{m}(T')^{-k}|y-x| \\
&\leq \frac{|\log\lambda|_{\mathrm{Lip}}\mathrm{m}(T')^{-1}}{1-\mathrm{m}(T')^{-1}}|y-x|.
\end{aligned}$$

The result follows letting

$$C_0 = \exp\left\{\frac{|\log\lambda|_{\mathrm{Lip}}\mathrm{m}(T')^{-1}}{1-\mathrm{m}(T')^{-1}}|y-x|\right\} > 1. \tag{7.5}$$

❑

The following gives us some non-degeneracy conditions on the regularity of the functions $\widehat{W}$ and $\hat{V}$.

Proposition 7.2.2

The following statements are equivalent

1. $\hat{V} \not\equiv 0$,

2. *There exists no Lipschitz continuous function* $\hat{u} : \mathbb{R} \to \mathbb{R}$ *such that*

$$\hat{u}(x) - \hat{p}(x) = \hat{\lambda}(x)\hat{u}(\hat{T}(x)). \tag{7.6}$$

Proof. We firstly prove that Statement 2 implies Statement 1 by contrapositive. Suppose firstly that $\hat{V} \equiv 0$. That is,

$$\widehat{W}(x) = -\sum_{j=1}^{\infty}\hat{\lambda}^{-j}(x)\hat{p}(\hat{T}^{-j}(x)).$$

Given $x, y \in [0,1]$,

$$\begin{aligned}\left|\widehat{W}(x)-\widehat{W}(y)\right| &\leq \sum_{j=1}^{\infty}\left|\hat{\lambda}^{-j}(x)\hat{p}(\hat{T}^{-j}(x))-\hat{\lambda}^{-j}(y)\hat{p}(\hat{T}^{-j}(y))\right| \\ &\leq \sum_{j=1}^{\infty}\hat{\lambda}^{-j}(x)\left|\hat{p}(\hat{T}^{-j}(x))-\hat{p}(\hat{T}^{-j}(y))\right| \\ &\quad + \sum_{j=1}^{\infty}\left|\hat{\lambda}^{-j}(x)-\hat{\lambda}^{-j}(y)\right|\left|\hat{p}(\hat{T}^{-j}(y))\right| \\ &= \mathcal{A}_1+\mathcal{A}_2.\end{aligned}$$

Firstly by the partial hyperbolicity assumption (7.2) and by the Lipschitz property of $\hat{p}$ we have that

$$\begin{aligned}\mathcal{A}_1 &= \sum_{j=1}^{\infty}\hat{\lambda}^{-j}(x)\left|\hat{p}(\hat{T}^{-j}(x))-\hat{p}(\hat{T}^{-j}(y))\right| \\ &\leq |p|_{\mathrm{Lip}}\sum_{j=1}^{\infty}\mathrm{m}(\lambda)^{-j}\mathrm{m}(T')^{-j}|x-y| \\ &\leq \frac{|p|_{\mathrm{Lip}}(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}\left|x-y\right|.\end{aligned}$$

Secondly by Lemma 7.2.1 there exists $C_0 > 1$ as in (7.5) such that

$$|\hat{\lambda}^{-j}(x)-\hat{\lambda}^{-j}(y)| \leq |C_0-1|\hat{\lambda}^{-j}(y).$$

Hence

$$\mathcal{A}_2 \leq |C_0-1|\sum_{j=1}^{\infty}\hat{\lambda}^{-j}(y)|\hat{p}(\hat{T}^{-j}(y))|.$$

Recalling C_0 as in (7.5), since the exponential function is Lipschitz continuous there exists $\beta > 0$ such that

$$\begin{aligned}|C_0-1| &= \left|\exp\left\{\frac{|\log\lambda|_{\mathrm{Lip}}\mathrm{m}(T')^{-1}}{1-\mathrm{m}(T')^{-1}}|y-x|\right\}-\exp\{0\}\right| \\ &\leq \beta\left|\frac{|\log\lambda|_{\mathrm{Lip}}\mathrm{m}(T')^{-1}}{1-\mathrm{m}(T')^{-1}}|y-x|-0\right| \\ &\leq \beta\frac{|\log\lambda|_{\mathrm{Lip}}\mathrm{m}(T')^{-1}}{1-\mathrm{m}(T')^{-1}}|x-y|.\end{aligned}$$

Since $\hat{p}(0) = 0$, for $x, y \in [0,1]$ we have that

$$\begin{aligned}\mathcal{A}_2 &\leq \beta\frac{|\log\lambda|_{\mathrm{Lip}}\mathrm{m}(T')^{-1}}{1-\mathrm{m}(T')^{-1}}\sum_{j=1}^{\infty}\hat{\lambda}^{-j}(y)\left|\hat{p}(\hat{T}^{-j}(y))-\hat{p}(\hat{T}^{-j}(0))\right||x-y| \\ &\leq \beta\frac{|\log\lambda|_{\mathrm{Lip}}\mathrm{m}(T')^{-1}}{1-\mathrm{m}(T')^{-1}}|p|_{\mathrm{Lip}}\frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}|x-y|.\end{aligned}$$

In particular we have shown that

$$\begin{aligned} |\widehat{W}(x)-\widehat{W}(y)| &\leq \mathcal{A}_1+\mathcal{A}_2 \\ &\leq \left(1+\beta\frac{|\log\lambda|_{\mathrm{Lip}}\mathrm{m}(T')^{-1}}{1-\mathrm{m}(T')^{-1}}\right)|p|_{\mathrm{Lip}}\frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}|x-y| \\ &= C_1|x-y| \end{aligned}$$

where $C_1>0$ and is independent of $x,y\in[0,1]$. Thus W is Lipschitz continuous on $[0,1]$. Since the lift to $\mathbb{R}$ $\widehat{W}$ is 1-periodic as in Section 2.8, $\widehat{W}$ is globally Lipschitz.

Moreover

$$\begin{aligned} \widehat{W}(x)-\hat{p}(x) &= \sum_{j=0}^{\infty}\hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x))-\hat{p}(x) \\ &= \sum_{j=1}^{\infty}\hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x)) \\ &= \hat{\lambda}(x)\sum_{j=0}^{\infty}\hat{\lambda}^j(T(x))\hat{p}(\hat{T}^{j+1}(x)) \\ &= \hat{\lambda}(x)\widehat{W}(\hat{T}(x)). \end{aligned}$$

That is if $\hat{V}\equiv 0$, then we have shown that $\widehat{W}$ itself is a Lipschitz continuous solution to the equation (7.6).

For the second part we show that Statement 1 implies Statement 2, again by proving the contrapositive. Suppose that there exists a Lipschitz function $\hat{u}$ such that $\hat{u}(x)-\hat{p}(x)=\hat{\lambda}(x)\hat{u}(\hat{T}(x))$. Letting $x=0$ see that $\hat{u}(0)-\hat{p}(0)=\hat{\lambda}(0)\hat{u}(\hat{T}(0))$. Since $\hat{p}(0)=0$ and $\hat{T}(0)=0$ we have that $\hat{u}(0)=\hat{\lambda}(0)\hat{u}(0)$, and since λ takes values in $(0,1)$ we have that $\hat{u}(0)=0$.

Given x define

$$\widehat{W}_k(x) = \sum_{j=0}^{k}\hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x)).$$

For each $j=0,1,\ldots,k$ by our assumption we have that

$$\hat{p}(\hat{T}^j(x))=\hat{u}(\hat{T}^j(x))-\hat{\lambda}(\hat{T}^j(x))\hat{u}(\hat{T}^{j+1}(x)).$$

Hence terms in the sum $\widehat{W}_k$ will cancel and one can show that substituting in the above into $\widehat{W}_k(x)$ for each $\hat{p}(\hat{T}^j(x))$,

$$\widehat{W}_k(x)=\hat{u}(x)-\hat{\lambda}^k(x)\hat{u}(\hat{T}^{k+1}(x)).$$

Similarly given $l \geq 0$ for $j = 1, \ldots, l$ by our assumption

$$\hat{p}(\hat{T}^{-l}(x)) = \hat{u}(\hat{T}^{-l}(x)) - \hat{\lambda}(\hat{T}^{-l}(x))\hat{u}(\hat{T}^{-(l-1)}(x)).$$

Then substituting in the above for $\hat{p}(\hat{T}^{-l}(x))$ terms in the below sum cancel

$$\sum_{j=1}^{l} \hat{\lambda}^{-j}(x)\hat{p}(\hat{T}^{-j}(x)) = \hat{\lambda}^{-l}(x)\hat{u}(\hat{T}^{-l}(x)) - \hat{u}(x).$$

Then by the above observations

$$\begin{aligned}
\hat{V}(x) &= \lim_{k\to\infty} \widehat{W}_k(x) + \lim_{l\to\infty} \sum_{j=1}^{l} \hat{\lambda}^{-j}\hat{p}(\hat{T}^{-j}(x)) \\
&= \lim_{k\to\infty} -\hat{\lambda}^k(x)\hat{u}(T^{k+1}(x)) + \hat{u}(x) + \lim_{l\to\infty} \hat{\lambda}^{-l}(x)\hat{u}(\hat{T}^{-l}(x)) - \hat{u}(x) \\
&= \lim_{k\to\infty} -\hat{\lambda}^k(x)\hat{u}(T^{k+1}(x)) + \lim_{l\to\infty} \hat{\lambda}^{-l}(x)\hat{u}(\hat{T}^{-l}(x)).
\end{aligned}$$

Since $\hat{u}$ is Lipschitz and $\hat{u}(0) = 0$ and since $\hat{T}^{-1}(0) = 0$;

$$\begin{aligned}
|\hat{V}(x)| &\leq \lim_{k\to\infty} |\hat{u}|_\infty |\hat{\lambda}^k(x)| + \lim_{l\to\infty} |\hat{\lambda}^{-l}(x)||\hat{u}(\hat{T}^{-l}(x))| \\
&\leq \lim_{k\to\infty} |\hat{u}|_\infty |\hat{\lambda}^k(x)| + \lim_{l\to\infty} |\hat{\lambda}^{-l}(x)||\hat{u}(\hat{T}^{-l}(x)) - \hat{u}(\hat{T}^{-l}(0))| \\
&\leq \lim_{k\to\infty} |\hat{u}|_\infty |\lambda|_\infty^k + |\hat{u}|_{\text{Lip}} \lim_{l\to\infty} \text{m}(\lambda)^{-l} |\hat{T}^{-l}(x) - \hat{T}^{-l}(0)| \\
&\leq |\hat{u}|_\infty \lim_{k\to\infty} |\lambda|_\infty^k + |\hat{u}|_{\text{Lip}} |x| \lim_{l\to\infty} (\text{m}(T')\text{m}(\lambda))^{-l} \\
&= 0
\end{aligned}$$

since $\hat{\lambda}(x) \in (0, 1)$ for all x and $\text{m}(T')\text{m}(\lambda) > 1$.
Hence $\hat{V} \equiv 0$ and the result follows. ❑

Remark. If $\hat{V} \not\equiv 0$, then in [HL] it is shown that $\widehat{W}$ is continuous but nowhere differentiable. By Proposition 7.2.2 this is equivalent to the non-degeneracy assumption as in Section 2.3. Generically [HNW], this is the case. In addition $\hat{V}$ is also nowhere differentiable, and $\widehat{W}$ and $\hat{V}$ are Hölder continuous of some exponent $\alpha \in (0, 1)$ but no better. In particular, they are not Lipschitz.

Throughout we always assume that our non-degeneracy assumption from Section 2.3 holds.

Theorem 7.2.3 (Lebesgue Differentiation Theorem [Leb])
If f is a monotone real valued function on an interval $[a, b]$, then the derivative $f'(x)$ exists and is finite at (Lebesgue) almost every point on $[a, b]$.

Taking the contrapositive, if a function is nowhere differentiable, then it is nowhere monotone. That is, it is not monotone on any open interval. Since we are assuming our non-degeneracy assumption $\widehat{W}$ and $\hat{V}$ are nowhere monotone. By nowhere-monotonicity of $\hat{V}$ there exists a positive constant $a > 0$ and points $y^-, y_- < z < y^+, y_+ \in [0,1]$ such that

$$\begin{aligned}\hat{V}(y^+) - \hat{V}(z) &> 8a,\\ \hat{V}(y^-) - \hat{V}(z) &> 8a,\\ \hat{V}(y_+) - \hat{V}(z) &< -8a,\\ \hat{V}(y_-) - \hat{V}(z) &< -8a.\end{aligned}$$

The above observation will prove vital in our later work.

7.3 Periodicity and recurrence

Given $r \geq 1$ we define the partial sums

$$\widehat{W}_r(x) = \sum_{j=0}^{r} \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x))$$

and

$$\hat{V}_r(x) = \sum_{j=-r}^{r} \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x)). \tag{7.7}$$

Clearly $\widehat{W}_r(x) \to \widehat{W}(x)$ and $\hat{V}_r(x) \to \hat{V}(x)$ uniformly as $r \to \infty$. Moreover it can be shown that $\hat{V}_r$ is Lipschitz. We denote the Lipschitz constant of $\hat{V}_r$ as $|V_r|_{\text{Lip}}$.

Proposition 7.3.1

Let $\hat{V}_r$ be the partial sum as in (7.7). Then $\hat{V}_r$ is periodic of period b^r. That is, for all $x \in \mathbb{R}$ and all $k \in \mathbb{Z}$

$$\hat{V}_r(x + kb^r) = \hat{V}_r(x).$$

Proof. Recall that without loss of generality as in Section 2.8 we can assume that $\hat{T}(0) = 0$. By construction of the lift $\hat{T}(1) = b$ and $\hat{T}(x+1) = \hat{T}(x) + b$. Moreover for $n \geq 1$,

$$\hat{T}^n(x+m) = \hat{T}^n(x) + b^n m \tag{7.8}$$

and

$$\hat{T}^{-n}(x+m) = \hat{T}^{-n}(x) + b^{-n}m. \tag{7.9}$$

Given $x \in \mathbb{R}$, for each $1 \le j \le r$, by (7.8) and since $\hat{\lambda}$ is 1-periodic,

$$\begin{aligned}\hat{\lambda}^j(x+1) &= \hat{\lambda}(x+1)\dots\hat{\lambda}(\hat{T}^{j-1}(x+1)) \\ &= \hat{\lambda}(x+1)\dots\hat{\lambda}(\hat{T}^{j-1}(x)+b^{j-1}) \\ &= \hat{\lambda}^j(x).\end{aligned}$$

By 1-periodicity of $\hat{p}$ and since $b^j \in \mathbb{Z}$ for all $1 \le j \le r$

$$\begin{aligned}\widehat{W}_r(x+1) &= \sum_{j=0}^{r} \hat{\lambda}^j(x+1)\hat{p}(\hat{T}^j(x+1)) \\ &= \sum_{j=0}^{r} \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x)+b^j) \\ &= \widehat{W}_r(x).\end{aligned}$$

Hence $\widehat{W}_r$ is 1-periodic for all $r \ge 1$. Thus the limit $\widehat{W}$ is 1-periodic, as was shown in Chapter 2. By (7.9) and since $b^{r-j} \in \mathbb{N}$ for all $1 \le j \le r$, by 1-periodicity of $\hat{\lambda}$ we have

$$\begin{aligned}\hat{\lambda}^{-j}(x+b^r) &= \hat{\lambda}(\hat{T}^{-1}(x+b^r))^{-1}\dots\hat{\lambda}(\hat{T}^{-j}(x+b^r))^{-1} \\ &= \hat{\lambda}(\hat{T}^{-1}(x)+b^{r-1})^{-1}\dots\hat{\lambda}(\hat{T}^{-j}(x)+b^{r-j})^{-1} \\ &= \hat{\lambda}(\hat{T}^{-1}(x))^{-1}\dots\hat{\lambda}(\hat{T}^{-j}(x))^{-1} \\ &= \hat{\lambda}^{-j}(x).\end{aligned}$$

By the above, using the 1-periodicity of $\hat{p}$

$$\begin{aligned}\sum_{j=1}^{r} \hat{\lambda}^{-j}(x+b^r)\hat{p}(\hat{T}^{-j}(x+b^r)) &= \sum_{j=1}^{r} \hat{\lambda}^{-j}(x)\hat{p}(\hat{T}^{-j}(x)+b^{r-j}) \\ &= \sum_{j=1}^{r} \hat{\lambda}^{-j}(x)\hat{p}(\hat{T}^{-j}(x)).\end{aligned}$$

Therefore

$$\begin{aligned}\hat{V}_r(x+b^r) &= \widehat{W}_r(x+b^r) + \sum_{j=1}^{r} \hat{\lambda}^{-j}(x+b^r)\hat{p}(\hat{T}^{-j}(x+b^r)) \\ &= \widehat{W}_r(x) + \sum_{j=1}^{r} \hat{\lambda}^{-j}(x)\hat{p}(\hat{T}^{-j}(x)) \\ &= \hat{V}_r(x)\end{aligned}$$

as required. However note that for all $1 \le j \le r$ and any $K < b^r$

$$\begin{aligned}\hat{\lambda}^{-j}(x+K) &= \hat{\lambda}(\hat{T}^{-1}(x+K))^{-1}\dots\hat{\lambda}(\hat{T}^{-j}(x+K))^{-1} \\ &= \hat{\lambda}(\hat{T}^{-1}(x)+K/b^r)^{-1}\dots\hat{\lambda}(\hat{T}^{-j}(x)+b^jK/b^r)^{-1} \\ &\neq \hat{\lambda}^{-j}(x)\end{aligned}$$

since $K/b^r \notin \mathbb{Z}$. In particular $\hat{V}_r$ is not K-periodic for any $K < b^r$. Moreover the limit $\hat{V} = \lim_{r\to\infty} \hat{V}_r$ is not periodic. ❑

The following results allow us to use recurrent properties of the map $\hat{T}$ in order to characterise what we mean by a large set of singularities. We give other characterisations of such a set of points later. We also relate this set of points to their behaviour under the original circle map T.

Lemma 7.3.2

Let $z \in [0,1], r \geq 1$ and $\epsilon > 0$ be given. Let $\mathcal{U} = \bigcup_{k=0}^{\infty}(kb^r - \epsilon, kb^r + \epsilon)$. Then there exists a dense-G_δ subset $G \subset [0,1]$ such that for all $x \in G$

$$\hat{T}^n(x) \in z + \mathcal{U}$$

for infinitely many $n \geq 1$.

Proof. Let z and $\mathcal{U}$ be given as in the statement of the lemma. Let I be an open interval with $\mathrm{diam}(I) < \infty$ such that I contains at least one point of $z + \mathcal{U}$. For each positive integer $p \geq 1$, let $B_p = (\hat{T}^p(I)) \cap (z + \mathcal{U})$ be the set of points in $z + \mathcal{U}$ which are also in the pth interate of I. Then let $A_p = \hat{T}^{-p}(B_p)$ denote the set of points in I which under the pth iterate of $\hat{T}$, land in $z + \mathcal{U}$. Given $k \geq 1$, consider $\bigcup_{p=k}^{\infty} A_p$; the set of points in I which are in $z + \mathcal{U}$ after p or more iterations. Note that $\bigcup_{p=k}^{\infty} A_p \supset \bigcup_{p=k+1}^{\infty} A_p$ and $\bigcup_{p=k}^{\infty} A_p = \hat{T}^{-k} \bigcup_{p=0}^{\infty} A_p$. Since $\hat{T}$ is uniformly expanding, then $\bigcup_{p=k}^{\infty} A_p$ is open and dense in I.

Let $G = \bigcap_{k\geq 1} \bigcup_{p\geq k} A_p$. Then G is a dense-G_δ subset of I, that is, a countable intersection of dense open sets. Taking $I = [0,1]$ we have constructed a dense G_δ subset of I where points in G are recurrent to $z + \mathcal{U}$ infinitely often under the map $\hat{T}$ as required. ❑

Remark. Given z, r and $\epsilon > 0$ we have a dense G_δ subset $G \subset [0,1]$ such that for all $x \in G$ the orbit of x under $\hat{T}$ comes within distance 2ϵ of some translation of z by a multiple of b^r infinitely often. Moreover by construction of the lift $\hat{T}$ of T to $\mathbb{R}$, for $x \in G$ the orbit of x under the circle map T comes arbitrarily close to $z \in [0,1]$ infinitely often.

We now look to characterise this with respect to the dynamics of T.

Definition. Let $T : S^1 \to S^1$ be a dynamical system and let μ be a T-invariant ergodic probability measure supported on S^1. We say that T is totally ergodic with respect to μ if T is ergodic, and any positive power of T is ergodic with respect to μ.

Consider our uniformly expanding map $T : S^1 \to S^1$. Let μ be any T-invariant ergodic probability measure supported on S^1 which takes positive measure on non-empty open sets. Since T is $C^{1+\gamma}$ for some $\gamma \in (0, 1)$ then T is weak mixing. Moreover [Sil] it is therefore true that T is totally ergodic with respect to μ. If μ is the unique equilibrium state for some Hölder continuous function, then it is indeed totally ergodic. We use this notion of total ergodicity to give a characterisation of the set G in a measure theoretic setting.

Lemma 7.3.3

Given $z \in [0, 1]$, $r \geq 1$ and $\epsilon > 0$ let G be as in Lemma 7.3.2. Then G takes full measure with respect to any ergodic T-invariant probability measure which takes positive measure on non-empty open sets.

Proof. Let z, r and ϵ be given, now fixed. Let G be as in the statement of the lemma. Given $x \in G$ there exists an infinite sequence $\{n_j\}_{j=1}^{\infty}$ such that for each $n \in \{n_j\}_{j=1}^{\infty}$ there exists $k = k(n)$ such that

$$\left|\hat{T}^n(x) - (z + kb^r)\right| < 2\epsilon.$$

Moreover in terms of the circle map T, for such $n \in \{n_j\}_{j=1}^{\infty}$, since $x \in G$

$$\left|T^{nb^r}(x) - z\right| < 2\epsilon.$$

That is, G is the set of points whose orbit under T^{b^r} comes within an ϵ-neighbourhood of z infinitely often.

Let $\mathcal{A} = (z - \epsilon, z + \epsilon)$. Let $f(x) = 1_{\mathcal{A}}(x)$ be the indicator function of the ϵ-neighbourhood of z. Then $f(T^{nb^r}(x))$ takes value 1 if $T^{nb^r}(x)$ lands in $\mathcal{A}$ and zero otherwise. Then $\sum_{j=0}^{n-1} f(T^{jb^r}(x))$ is the number of times that the orbit of x under T^{b^r} lands in $\mathcal{A}$ after the first n iterates. Moreover

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^{jb^r}(x))$$

is the frequency with which the orbit of x under T^{b^r} lands in $\mathcal{A}$ after n iterates. Therefore G is precisely the set of points x where the limiting frequency

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^{jb^r}(x)) > 0,$$

i.e. the set of points whose orbit under T^{b^r} lands in $\mathcal{A}$ infinitely often.

Let μ be any ergodic T-invariant probability measure supported on $[0,1]$ which takes positive measure on non-empty open sets. Since T is $C^{1+\gamma}$, the remarks before the statement of the lemma [Sil] give that T is totally ergodic with respect to the measure μ. Therefore by the Birkhoff Ergodic theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^{jb^r}(x)) \to \int f\, d\mu = \mu(\mathcal{A}) > 0$$

for μ-almost every $x \in [0,1]$. That is, for μ an ergodic T-invariant probability measure which takes positive measure on non-empty open sets, then μ-almost every point has positive limiting frequency, and thus lands in $\mathcal{A}$ infinitely often. In particular, $G \subset [0,1]$ takes full measure with respect to such a measure μ. ❏

Corollary 7.3.4

Let z, r, ϵ be given as above. Let G be constructed as above. Then we can characterise G as follows;

- *G is a dense G_δ subset of $[0,1]$*
- *G has full measure with respect to any ergodic T-invariant probability measure which has positive measure on non-empty open sets*
- *G has Hausdorff dimension 1.*

Proof. It remains to prove the final statement. Note that $G \subset [0,1]$ has full measure with respect to any ergodic T-Invariant probability measaure. In particular, G has full measure with respect to the measure μ_0, the unique equilibrium state of $-\log|T'|$. Thus μ_0 is a SRB measure, which is an absolutely continuous invariant measure with respect to Lebesgue. That is G has Lebesgue measure 1, and it follows [Fa1] that the one-dimension Hausdorff measure of G is positive and finite. Moreover $\dim_{\mathcal{H}}(G) = 1$. ❏

In [HL] it is proven that for a set G as in Lemma 7.3.2, the set of knot points of W contains G, where

$$W(x) = \sum_{j=0}^{\infty} \lambda^j p(T^j(x))$$

where $T(x) = bx \bmod 1$ and $\lambda b > 1$.

7.4 The case where λ is constant

We now consider a case of $\widehat{W}$, in which we look to prove that for $x \in G \subset [0,1]$; the set of points as constructed in Lemma 7.3.2, x is in fact a knot point for $\widehat{W}$, hence by remarks on the continuous lift as in Section 2.8 is a knot point for W. In the following we take $\hat{T}$ to be a natural lift to $\mathbb{R}$ of a uniformly expanding map of the circle and let $\lambda \in (0,1)$ be fixed. This strengthens the result in [HL]. Our partial hyperbolicity assumption becomes

$$\lambda \mathrm{m}(T') > 1. \tag{7.10}$$

We consider singularities of functions of the form $\widehat{W}(x) = \sum_{j=0}^{\infty} \lambda^j \hat{p}(\hat{T}^j(x))$.

We always assume that our non-degeneracy assumption as in Section 2.3 holds. Recall that as in Section 7.2, by non-monotonicity of $\hat{V}$ there exists $a > 0$ and points $y^-, y_- < z < y^+, y_+ \in [0,1]$ such that

$$\begin{aligned}
\hat{V}(y^+) - \hat{V}(z) &> 8a, \\
\hat{V}(y_-) - \hat{V}(z) &> 8a, \\
\hat{V}(y_+) - \hat{V}(z) &< -8a, \\
\hat{V}(y^-) - \hat{V}(z) &< -8a.
\end{aligned}$$

Such $a > 0$ and points in $[0,1]$ are now fixed so that the above estimates hold.

Since $\hat{V}_r \to \hat{V}$ uniformly as $r \to \infty$, choose r sufficiently large such that

$$\begin{aligned}
\hat{V}_r(y^+) - \hat{V}_r(z) &> 4a, \\
\hat{V}_r(y_-) - \hat{V}_r(z) &> 4a, \\
\hat{V}_r(y_+) - \hat{V}_r(z) &< -4a, \\
\hat{V}_r(y^-) - \hat{V}_r(z) &< -4a.
\end{aligned}$$

Moreover for $a > 0$ now fixed we choose $r \geq 1$ sufficiently large such that it also holds such that

$$\frac{2|p|_{\text{Lip}}}{\lambda m(T') - 1}(\lambda m(T'))^{-r} \leq a \tag{7.11}$$

and

$$\frac{2|p|_{\infty}}{1-\lambda}\lambda^{r+1} \leq a. \tag{7.12}$$

Note that we can always choose such an r sufficiently large, by (7.10) and since $\lambda \in (0,1)$.

Fix $a > 0$ as above and let r be as in the above. Choose $\epsilon \in (0, 1/2]$ to be such that

$$|\hat{V}_r|_{\text{Lip}}(2\epsilon) \leq a \tag{7.13}$$

where $|\hat{V}_r|_{\text{Lip}}$ is the Lipschitz constant of $\hat{V}_r$.

Therefore we now have $z \in [0,1]$ and $a > 0$, $\epsilon > 0$ and $r \geq 1$ all fixed as in the above setup, so that above estimates hold. As in the proof of Lemma 7.3.2 we construct the set G, which is the set of points such that for each $x \in G$ there exists infinitely many n, and some $k = k(n)$ such that $\hat{T}^n(x)$ lies within an ϵ-neighbourhood of $z + kb^r$.

7.4.1 The upper right derivative

In following we use the above estimates, to construct a set G as above, and given $x \in G$ we show that the upper right derivative of $\widehat{W}$ at x is ∞. We make remarks on the other three derivatives at a later stage, although the method follows in a similar way.

For ease of notation, assuming as always that our non-degeneracy assumption holds, there exists $a > 0$ as above and $y > z$ so that r is chosen sufficiently large such that (7.11) and (7.12) hold and

$$\hat{V}_r(y) - \hat{V}_r(z) > 4a. \tag{7.14}$$

Such y, z, r and a are now fixed. For $\epsilon > 0$ such that (7.13) holds we construct the corresponding set G as in Lemma 7.3.2 and show that given $x \in G$, $D^+W(x) = \infty$. Since $G \subset [0,1]$ and $\widehat{W}\mid_{[0,1]}$ this is equivalent to showing that for $x \in G$, $D^+\widehat{W}(x) = \infty$.

By construction of G, given $x \in G$ there exists an infinite sequence $\{n_j\}_{j=1}^{\infty}$ such that for each $n \in \{n_j\}_{j=1}^{\infty}$, there exists $k = k(n) \in \mathbb{N}$ such that

$$\hat{T}^n(x) \in (z + kb^r - \epsilon, z + kb^r + \epsilon).$$

Since $y > z$, for each $n \in \{n_j\}_{j=1}^{\infty}$ consider $y + kb^r > z + kb^r$. Recalling that $\{n_j\}_{j=1}^{\infty} \to \infty$, then we can choose infinitely many $n \in \{n_j\}_{j=1}^{\infty}$ such that $n > r$ and for each n define $y_n = \hat{T}^{-n}(y + kb^r)$. That is, we have a sequence of $n \to \infty$, and then construct y_n as above so that $y_n \to x$ as $n \to \infty$.

Recalling that $y, z \in [0, 1]$ we have that $|(y + kb^r) - (z + kb^r)| \leq 1$ and so $|\hat{T}^{-1}(y + kb^r) - \hat{T}^{-1}(z + kb^r)| \leq m(\hat{T}')^{-1}$. Moreover since $y_n = \hat{T}^{-n}(y + kb^r)$

$$\begin{aligned} |y_n - x| &= |\hat{T}^{-n}(y + kb^r) - x + \hat{T}^{-n}(z + kb^r) - \hat{T}^{-n}(z + kb^r)| \\ &\leq |\hat{T}^{-n}(y + kb^r) - \hat{T}^{-n}(z + kb^r)| + |\hat{T}^{-n}(z + kb^r) - \hat{T}^{-n}(\hat{T}^n(x))| \\ &\leq \mathrm{m}(T')^{-n}(1 + 2\epsilon) \\ &\leq 2\mathrm{m}(T')^{-n}. \end{aligned}$$

With the above, we can now prove the main result used to establish that the upper right derivative at x is infinite.

Proposition 7.4.1

Let $y > z$ and a, r and ϵ be fixed as in the above. Let G be as in Lemma 7.3.2. Given $x \in G$ there exists infinitely many $n > r$ and sequence of $y_n \to x$ where $y_n = \hat{T}^{-n}(y + kb^r)$ and k depends on n. Then for all such $n > r$

$$\widehat{W}(y_n) - \widehat{W}(x) > a\lambda^n.$$

Proof. Let r be sufficiently large as above. Then there exists infinitely many $n > r$ where $n \in \{n_j\}_{j=1}^{\infty}$ as in the above discussion, and $y_n > x$ as constructed above. For each such $n > r$ we write

$$\begin{aligned} \widehat{W}(y_n) - \widehat{W}(x) &= \sum_{j=0}^{\infty} \lambda^j \left[\hat{p}(\hat{T}^j(y_n)) - \hat{p}(\hat{T}^j(x))\right] \\ &= \left(\sum_{j=0}^{n-r-1} + \sum_{j=n-r}^{n+r} + \sum_{j=n+r+1}^{\infty}\right) \lambda^j \left[\hat{p}(\hat{T}^j(y_n)) - \hat{p}(\hat{T}^j(x))\right] \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3 \\ &\geq \Sigma_2 - |\Sigma_1| - |\Sigma_3|. \end{aligned}$$

Claim 1

We show that $\Sigma_2 > 3a\lambda^n$. Note that

$$\begin{aligned}\Sigma_2 &= \sum_{j=n-r}^{n+r} \lambda^j \left[\hat{p}(\hat{T}^j(y_n)) - \hat{p}(\hat{T}^j(x))\right] \\ &= \sum_{j=-r}^{r} \lambda^{n+j} \left[\hat{p}(\hat{T}^{n+j}(y_n)) - \hat{p}(\hat{T}^{n+j}(x))\right] \\ &= \lambda^n \left[\sum_{j=-r}^{r} \lambda^j \hat{p}(\hat{T}^{n+j}(y_n)) - \sum_{j=-r}^{r} \lambda^j \hat{p}(\hat{T}^{n+j}(x))\right] \\ &= \lambda^n \left[\hat{V}_r(\hat{T}^n(y_n)) - \hat{V}_r(\hat{T}^n(x))\right].\end{aligned}$$

Since $r < n$ is sufficiently large such that (7.14) holds

$$\begin{aligned}\hat{V}_r(\hat{T}^n(y_n)) - \hat{V}_r(\hat{T}^n(x)) &= \hat{V}_r(y) - \hat{V}_r(z) + \hat{V}_r(z) - \hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(y) + \hat{V}_r(\hat{T}^n(y_n)) \\ &\geq \hat{V}_r(y) - \hat{V}_r(z) - |\hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(z)| - |\hat{V}_r(y) - \hat{V}_r(\hat{T}^n(y_n))| \\ &> 4a - |\hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(z)| - |\hat{V}_r(y) - \hat{V}_r(\hat{T}^n(y_n))|.\end{aligned}$$

Recall that $\hat{V}_r$ is b^r-periodic, and since $x \in G$ and $n \in \{n_j\}_{j=1}^{\infty}$ there exists $k = k(n)$ such that $\hat{T}^n(x) \in (z + kb^r - \epsilon, z + kb^r + \epsilon)$. Using the Lipschitz property of $\hat{V}_r$ and periodicity

$$\begin{aligned}|\hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(z)| &= |\hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(z + kb^r)| \\ &\leq |\hat{V}_r|_{\mathrm{Lip}} |\hat{T}^n(x) - (z + kb^r)|.\end{aligned}$$

By choice of ϵ as in (7.13)

$$\begin{aligned}|\hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(z)| &\leq |\hat{V}_r|_{\mathrm{Lip}}(2\epsilon) \\ &\leq a.\end{aligned}$$

Similarly, by periodicity of $\hat{V}_r$ and construction of y_n

$$|\hat{V}_r(y) - \hat{V}_r(\hat{T}^n(y_n))| = |\hat{V}_r(y + kb^r) - \hat{V}_r(\hat{T}^n(y_n))| = 0.$$

Therefore

$$\hat{V}_r(\hat{T}^n(y_n)) - \hat{V}_r(\hat{T}^n(x)) > 3a$$

and so $\Sigma_2 > 3a\lambda^n$.

Claim 2

We now show that for all $n \in \{n_j\}_{j=1}^{\infty}$ such that $n > r$, $|\Sigma_1| \leq a\lambda^n$. Recall that

$$\begin{aligned} |\hat{T}^n(y_n) - \hat{T}^n(x)| &\leq |(y + kb^r) - (z + kb^r)| + |(z + kb^r) - \hat{T}^n(x)| \\ &\leq 1 + 2\epsilon \\ &\leq 2 \end{aligned}$$

and hence $|y_n - x| \leq 2\mathrm{m}(T')^{-n}$. Moreover it can be seen that $|\hat{T}^{n-j}(y_n) - \hat{T}^{n-j}(x)| \leq 2\mathrm{m}(T')^{-j}$.

Since $\hat{p}$ is Lipschitz continuous and $j < n$

$$\begin{aligned} |\Sigma_1| &\leq \sum_{j=0}^{n-r-1} \lambda^j |\hat{p}(\hat{T}^j(y_n)) - \hat{p}(\hat{T}^j(x))| \\ &\leq |p|_{\mathrm{Lip}} \sum_{j=0}^{n-r-1} \lambda^j |\hat{T}^j(y_n) - \hat{T}^j(x)| \\ &\leq |p|_{\mathrm{Lip}} \sum_{j=0}^{n-r-1} \lambda^j |\hat{T}^{j-n}(\hat{T}^n(y_n)) - \hat{T}^{j-n}(\hat{T}^n(x))| \\ &\leq |p|_{\mathrm{Lip}} \sum_{j=0}^{n-r-1} \lambda^j \mathrm{m}(T')^{j-n} |\hat{T}^n(y_n) - \hat{T}^n(x)| \\ &\leq 2|p|_{\mathrm{Lip}} \mathrm{m}(T')^{-n} \sum_{j=0}^{n-r-1} (\lambda \mathrm{m}(T'))^j \\ &= 2|p|_{\mathrm{Lip}} \mathrm{m}(T')^{-n} \left(\frac{(\lambda \mathrm{m}(T'))^{n-r} - 1}{\lambda \mathrm{m}(T') - 1} \right) \\ &\leq 2|p|_{\mathrm{Lip}} \mathrm{m}(T')^{-n} \left(\frac{(\lambda \mathrm{m}(T'))^{n-r}}{\lambda \mathrm{m}(T') - 1} \right) \\ &= \frac{2|p|_{\mathrm{Lip}} \lambda^n}{\lambda \mathrm{m}(T') - 1} (\lambda \mathrm{m}(T'))^{-r} \\ &\leq a\lambda^n \end{aligned}$$

as required, by choice of r as in (7.11).

Claim 3

Finally we show that $|\Sigma_3| \leq a\lambda^n$. Since $0 < |p|_\infty < \infty$,

$$\begin{aligned} |\Sigma_3| &\leq \sum_{j=n+r+1}^{\infty} \lambda^j \left| \hat{p}(\hat{T}^j(y_n)) - \hat{p}(\hat{T}^j(x)) \right| \\ &\leq \sum_{j=n+r+1}^{\infty} \lambda^j \left(\left| \hat{p}(\hat{T}^j(y_n)) \right| + \left| \hat{p}(\hat{T}^j(x)) \right| \right) \\ &\leq 2|p|_\infty \sum_{j=n+r+1}^{\infty} \lambda^j \\ &= 2|p|_\infty \left(\frac{1}{1-\lambda} - \frac{1-\lambda^{n+r+1}}{1-\lambda} \right) \\ &= \lambda^n \frac{2|p|_\infty \lambda^{r+1}}{1-\lambda} \\ &\leq a\lambda^n \end{aligned}$$

by choice of r as in (7.12).

Then by Claims 1, 2 and 3 we have that there exists $a > 0$ and a sequence $\{n_j\} \to \infty$ such that for each $n \in \{n_j\}$ sufficiently large it holds that

$$\widehat{W}(y_n) - \widehat{W}(x) \geq 3a\lambda^n - a\lambda^n - a\lambda^n = a\lambda^n.$$

❑

Theorem 7.4.2

Let G be as in Lemma 7.3.2. Given $x \in G$ there exists a sequence of points $y_{n_j} \to x$ such that

$$D^+\widehat{W}(x) = \limsup_{n_j \to \infty} \frac{\widehat{W}(y_{n_j}) - \widehat{W}(x)}{y_{n_j} - x} = \infty.$$

Proof. By Proposition 7.4.1 there exists $a > 0$, a sequence $\{n_j\} \to \infty$ and a sequence $y_{n_j} \to x$ such that for all n_j sufficiently large $\widehat{W}(y_{n_j}) - \widehat{W}(x) > a\lambda^{n_j}$. In addition $y_{n_j} > x$ and $y_{n_j} - x \leq 2m(\hat{T}')^{-n_j}$. That is

$$\frac{\widehat{W}(y_{n_j}) - \widehat{W}(x)}{y_{n_j} - x} > \frac{a}{2}(\lambda m(T'))^{n_j}$$

for all n_j sufficiently large. In particular, by the partial hyperbolicity assumption (7.10) we have that

$$D^+\widehat{W}(x) > \lim_{n_j \to \infty} \frac{a}{2}(\lambda m(T'))^{n_j} = \infty$$

as required. ❑

Therefore we have constructed a set $G \subset [0,1]$ such that for each $x \in G$, the upper right derivative of $\widehat{W}$ at x is infinite. In particular by Corollary 7.3.4 G is a dense G_δ subset of $[0,1]$ and has full measure with respect to any ergodic T-invariant measure μ which takes positive measure on non-empty open sets. In addition the set of knot-points contains a subset of $[0,1]$ of Hausdorff dimension 1.

7.4.2 The three other derivatives

In the previous section we have fixed suitable z, a, ϵ and r and constructed a set G such that for all $x \in G$ the upper right derivative of $\widehat{W}$ at x is infinite. We fix the same z, a, ϵ and r as in the previous subsection, and the corresponding set G, and give a sketch of the three remaining cases required in order to prove that given $x \in G$, x is a knot point for $\widehat{W}$, hence by Section 2.8 for W too. In particular, such G is the same set, and for $x \in G$, x is in fact a knot point for W. Note this is only the case if $\lambda \in (0,1)$ is constant (see Section 7.5). The proofs that the upper left derivative is ∞ and the lower right and left derivatives are $-\infty$ are similar to the method in the previous subsection for the upper right derivative.

For the lower right case, for $y > z$ as in the previous setup (for ease of notation we use y in place of y^+), by the nowhere monotonicity of $\hat{V}$ there exists $a > 0$ and $r \geq 1$ sufficiently large such that (7.11) and (7.12) hold and

$$\hat{V}_r(y) - \hat{V}_r(z) < -4a.$$

Given $\epsilon > 0$ as in (7.13) we let G be the same set as in the previous case, and given $x \in G$ we take a sequence of infinitely many $n > r$ and construct y_n in the same way. Then we note that

$$\widehat{W}(y_n) - \widehat{W}(x) \leq \Sigma_2 + |\Sigma_1| + |\Sigma_3|.$$

Furthermore one can show that, by above observations and similar analysis $\Sigma_2 < -3a\lambda^n$, $\Sigma_1 \leq a\lambda^n$ and $\Sigma_3 \leq a\lambda^n$. That is $\widehat{W}(y_n) - \widehat{W}(x) < -a\lambda^n$ and since $y_n > x$ and $y_n - x \leq 2\mathrm{m}(T')^{-n}$ it is clear that

$$\frac{\widehat{W}(y_n) - \widehat{W}(x)}{y_n - x} < -\frac{1}{2}a(\lambda \mathrm{m}(T'))^n.$$

Therefore in the same way as in the proof of Theorem 7.4.2 and by our partial hyperbolocity assumption

$$\begin{aligned} D_+\widehat{W}(x) &= \liminf_{y_n \to x} \frac{\widehat{W}(y_n) - \widehat{W}(x)}{y_n - x} \\ &< \lim_{n\to\infty} -\frac{1}{2}a(\lambda \mathrm{m}(T'))^n \\ &= -\infty. \end{aligned}$$

In order to prove that the upper left derivative is ∞ we use the observation that there exists $a > 0$ and $y < z$ such that

$$\hat{V}(y) - \hat{V}(z) < -8a.$$

In a similar way to the previous cases, letting $r \geq 1$ be sufficiently large we use a similar method and show that for $x \in G$, there exists infinitely many $n > r$ and construct a sequence of $y_n \to x$ such that $\widehat{W}(y_n) - \widehat{W}(x) < -a\lambda^n$. Moreover $y_n < x$ and so $0 \geq y_n - x \geq -2\mathrm{m}(T')^{-n}$. In particular, for infinitely many n, we can show that

$$\frac{\widehat{W}(y_n) - \widehat{W}(x)}{y_n - x} > \frac{-a\lambda^n}{-2\mathrm{m}(T')^{-n}} = \frac{1}{2}a(\lambda \mathrm{m}(T'))^n.$$

Therefore working in the same way to the previous cases

$$\begin{aligned} &\geq \lim_{n\to\infty} \frac{1}{2}a(\lambda \mathrm{m}(T'))^n \\ &= \infty. \end{aligned}$$

One can show the remaining case for the lower left derivative in a similar way.

Hence given z as in our previous observations, we construct a set $G \subset [0,1]$ such that for all $x \in G$, the above arguments show that x is indeed a knot point for $\widehat{W}$. In addition, since $\widehat{W}\,|_{[0,1]}= W$, for all $x \in G$ x is a knot point for W itself. In particular, we have constructed a set G, such that for all $x \in G$, x is a knot-point for W, and G satisfies Corollary 7.3.4. In particular we have proven Theorem 1.0.3.

7.5 The case where λ is variable

In the following we return to the more general case where $\lambda : S^1 \to (0,1)$ is a Lipschitz continuous function and $\hat{\lambda} : \mathbb{R} \to \mathbb{R}$ is a 1-periodic lift of λ to $\mathbb{R}$. We still assume that $\hat{T}$ is a continuous lift to $\mathbb{R}$ of a uniformly expanding map of the circle. Let $\widehat{W}(x) = \sum_{j=0}^{\infty} \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x))$. We look to prove a similar result regarding knot points of $\widehat{W}$, hence of W as in Section 7.4.

Ideally we would like to prove the existence of a dense G_δ subset $G \subset [0,1]$ which also takes full measure with respect to any T-invariant probability measure supported on $[0,1]$ which takes positive measure on non-empty open sets, such that given $x \in G$, under the non degeneracy assumption as in Section 2.3 and the partial hyperbolicity assumption that $\mathrm{m}(\lambda)\mathrm{m}(T') > 1$, then x is a knot-point for W. However, in order to deal with the fact that $\hat{\lambda}$ is no longer constant, we need extra assumptions on our underlying dynamics, and the sets G which we construct, will differ in each of the four cases. We make remarks on how our results differ slightly in the previous section.

Additional Hypotheses

In the following we impart extra requirements in order for the results to hold. In particular we require that there exists points $z_1 < z_2 < z_3 \in [0,1]$ such that either

$$\begin{aligned} \hat{V}(z_1) > 0,\ \ \hat{V}(z_3) &> 0 \text{ and } \hat{V}(z_2) < 0 \text{ or} \\ \hat{V}(z_1) > 0,\ \ \hat{V}(z_3) &< 0 \text{ and } \hat{V}(z_2) > 0. \end{aligned}$$

We call this requirement Cond(V). This condition will be vital in order to show that our knot points result holds in the case which such a condition is true. We first investigate the cases in which Cond(V) holds.

Remark. If such a condition holds then clearly $\hat{V} \not\equiv 0$ and by Proposition 7.2.2 and earlier discussions as in previous sections, $\hat{V}$ is nowhere differentiable and nowhere monotone.

Proposition 7.5.1

Let $\hat{p}$ be a non-constant 1-periodic Lipschitz function such that $\hat{p}$ crosses the x-axis at least twice on $[0,1]$. Let $C_2 \geq 1$ be the least value such that there exists points $z_1 < z_2 < z_3 \in [0,1]$ such that either

$$C_2\hat{p}(z_1) \;\geq\; |p|_{\mathrm{Lip}},\ C_2\hat{p}(z_2) \leq -|p|_{\mathrm{Lip}},\ C_2\hat{p}(z_3) \geq |p|_{\mathrm{Lip}}$$

or

$$C_2\hat{p}(z_1) \;\leq\; -|p|_{\mathrm{Lip}},\ C_2\hat{p}(z_2) \geq |p|_{\mathrm{Lip}},\ C_2\hat{p}(z_3) \leq -|p|_{\mathrm{Lip}}.$$

Then if $\hat{\lambda}$ *and* $\hat{T}$ *are such that*

$$C_2\left(\frac{|\lambda|_\infty}{1-|\lambda|_\infty}+\frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}\right)<1 \tag{7.15}$$

then $\mathrm{Cond}(V)$ *holds.*

Proof. Let $\hat{p}$ be such that it crosses the x-axis at least twice in $[0,1]$ as in the statement of the proposition. Then there exists either at least 2 arcs of $\hat{p}$ above the axis and one below, or 2 arcs of $\hat{p}$ below the axis and one above. Without loss of generality we assume the former. The case of the latter follows in the same way. That is we assume that there exists $z_1 < z_2 < z_3$ such that $\hat{p}(z_1), \hat{p}(z_3) > 0$ and $\hat{p}(z_2) < 0$. Choose $C_2 \geq 1$ to be the least possible value such that $C_2\hat{p}(z_1) \geq |p|_{\mathrm{Lip}}$, $C_2\hat{p}(z_2) \leq -|p|_{\mathrm{Lip}}$, and $C_2\hat{p}(z_3) \geq |p|_{\mathrm{Lip}}$.

Firstly note that

$$\hat{V}(z_1) \geq \hat{p}(z_1) - \left|\sum_{j=1}^{\infty}\hat{\lambda}^j(z_1)\hat{p}(\hat{T}^j(z_1)) + \sum_{j=1}^{\infty}\hat{\lambda}^{-j}(z_1)\hat{p}(\hat{T}^{-j}(z_1))\right|.$$

Noting that since $\hat{p}$ is 1-periodic, $|p|_\infty = |\hat{p}|_\infty = \sup_{x,y\in[0,1]}|\hat{p}(x)-\hat{p}(y)|$. Letting x', y' be the points which achieve this supremum is it clear that

$$\begin{aligned} |p|_\infty &= |\hat{p}(x')-\hat{p}(y')| \\ &\leq |p|_{\mathrm{Lip}}|x'-y'| \\ &\leq |p|_{\mathrm{Lip}}. \end{aligned}$$

Therefore since $\hat{T}(0)=0$ and $\hat{p}(0)=0$ and by the assumptions as in the statement of the proposition

$$\begin{aligned} &\left|\sum_{j=1}^{\infty}\hat{\lambda}^j(z_1)\hat{p}(\hat{T}^j(z_1)) + \sum_{j=1}^{\infty}\hat{\lambda}^{-j}(z_1)\hat{p}(\hat{T}^{-j}(z_1))\right| \\ \leq\ & |p|_\infty\frac{|\lambda|_\infty}{1-|\lambda|_\infty} + |p|_{\mathrm{Lip}}\sum_{j=1}^{\infty}\mathrm{m}(\lambda)^{-j}\mathrm{m}(T')^{-j} \\ \leq\ & |p|_{\mathrm{Lip}}\left(\frac{|\lambda|_\infty}{1-|\lambda|_\infty}+\frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}\right) \\ \leq\ & C_2\hat{p}(z_1)\left(\frac{|\lambda|_\infty}{1-|\lambda|_\infty}+\frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}\right). \end{aligned}$$

Moreover this shows that

$$\hat{V}(z_1) \geq \hat{p}(z_1)\left(1 - C_2\left(\frac{|\lambda|_\infty}{1-|\lambda|_\infty}+\frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}\right)\right) > 0$$

by choice of $\hat{\lambda}$ and $\hat{T}$ as in the statement of the proposition. The same method shows that under our assumptions $\hat{V}(z_3) > 0$.

Consider z_2 such that $\hat{p}(z_2) < 0$ and $C_2\hat{p}(z_2) \leq -|p|_{\text{Lip}} < 0$. In a similar method to the above, since $\hat{p}(z_2) < 0$ and $|\hat{p}|_{\text{Lip}} \leq |C_2\hat{p}(z_2)|$

$$\begin{aligned}
\hat{V}(z_2) &\leq \hat{p}(z_2) + |p|_{\text{Lip}}\left(\frac{|\lambda|_\infty}{1-|\lambda|_\infty} + \frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}\right) \\
&\leq \hat{p}(z_2) + C_2|\hat{p}(z_2)|\left(\frac{|\lambda|_\infty}{1-|\lambda|_\infty} + \frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}\right) \\
&\leq |\hat{p}(z_2)|\left(-1 + C_2\left(\frac{|\lambda|_\infty}{1-|\lambda|_\infty} + \frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}\right)\right) \\
&< 0.
\end{aligned}$$

Thus Cond(V) holds. ❑

Working in a similar manner it is straightforward to show that Cond(V) holds if we have $z_1 < z_2 < z_3 \in [0,1]$ such that $\hat{p}(z_1)$, $\hat{p}(z_3) < 0$ and $\hat{p}(z_2) > 0$.

Example. We give an example of the additional conditions required for Cond(V) to hold. Consider the function $\hat{p}$ to be

$$\hat{p}(x) = \sin(3\pi x).$$

Note that $\hat{p}$ crosses the x axis twice on $[0,1]$. By Proposition 7.5.1 under extra assumptions on $\hat{\lambda}$ and $\hat{T}$ we can guarantee that Cond(V) holds. We can take $z_1 = 1/6$, $z_2 = 1/2$ and $z_3 = 5/6$ such that $z_1 < z_2 < z_3$ and $\hat{p}(z_1) = 1 = \hat{p}(z_3)$ and $\hat{p}(z_2) = -1$.

In addition $\hat{p}'(x) = 3\pi\cos(3\pi x)$ and therefore $|p|_{\text{Lip}} = 3\pi$. That is, for all $x, y \in [0,1]$ $|\hat{p}(x) - \hat{p}(y)| \leq 3\pi|x-y|$. Therefore C_2 as in the statement of the proposition can be taken as $C_2 = 3\pi$. Hence, for $\hat{p}$ as given above, by Proposition 7.5.1 Cond(V) will hold if

$$\left(\frac{|\lambda|_\infty}{1-|\lambda|_\infty} + \frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}\right) < \frac{1}{3\pi}.$$

For example, we would require $|\lambda|_\infty$ sufficiently small such that

$$\frac{|\lambda|_\infty}{1-|\lambda|_\infty} < \frac{1}{6\pi}.$$

That is

$$|\lambda|_\infty < \frac{1}{(1+6\pi)}.$$

Then for Cond(V) to hold we would require that $\mathrm{m}(T')$ is sufficiently large and $\mathrm{m}(\lambda)$ is sufficiently small such that

$$\frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}}{1-(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-1}} < \frac{1}{6\pi}.$$

In particular, we would require that, for the above example

$$(\mathrm{m}(\lambda)\mathrm{m}(T')) > 1+6\pi.$$

If $\lambda : S^1 \to (0,1)$ was such that $|\lambda|_\infty < 1/25$ then for Cond(V) to hold we would require that λ and T are such that $\mathrm{m}(\lambda)\mathrm{m}(T') > 25$.

By the above, we have been able to show that given a suitable function $\hat{p}$, we can then calculate the restrictions needed on λ and T' in order for Cond(V) to hold. Although it is not ideal to have to impart such incredibly strong partial hyperbolicity conditions, we have illustrated that in certain cases (i.e. given suitable $\hat{p}$) we can guarantee that Cond(V) holds. It is only required, as we see in the next section that Cond(V) holds in order for us to prove our result for knot-points. However we conjecture that such restrictions can be weakened, and that Cond(V) holds in other cases. Such restrictions have been given to illustrate that the result is non-trivial and there exists functions where it is true. In the following sections we shall always assume that Cond(V) holds.

7.5.1 The upper right derivative

From now on we assume that Cond(V) holds. Without loss of generality we assume that there exists $z_1 < z_2 < z_3 \in [0,1]$ such that $\hat{V}(z_1), \hat{V}(z_3) > 0$ and $V(z_2) < 0$. In particular, in order to construct the argument for the upper right derivative we look at $z_3 > z_2$ such that $V(z_3) > 0$ and $V(z_2) < 0$ in order to do this case. If it were in fact the case that there exists $z_1 < z_2 < z_3$ such that $\hat{V}(z_1), \hat{V}(z_3) < 0$ and $V(z_2) > 0$ then we would look at $z_2 > z_1$ and continue in a similar vein.

Let C_3 denote the constant

$$C_3 = \exp\left\{\frac{2|\log\lambda|_{\mathrm{Lip}}}{\mathrm{m}(T')-1}\right\} > 1. \tag{7.16}$$

Since Cond(V) holds, in particular in this case we consider $z_3 > z_2$ such that $\hat{V}(z_3) > 0$ and $\hat{V}(z_2) < 0$. Then given any $t \in [C_3^{-1}, C_3]$ since $\hat{V}(z_3) > 0$ and $\hat{V}(z_2) < 0$ and since $C_3 > 1$ it can be seen that

$$\hat{V}(z_3) - t\hat{V}(z_2) > 0.$$

Moreover there exists some positive constant $a \geq |\hat{V}(z_3)|$ such that for each $t \in [C_3^{-1}, C_3]$

$$\hat{V}(z_3) - tV(z_2) > a.$$

Let such an $a > 0$ be fixed.

Since $\hat{V}_r$ converges uniformly to $\hat{V}$, we choose r sufficiently large such that

$$\hat{V}_r(z_3) - t\hat{V}_r(z_2) > \frac{a}{2}. \tag{7.17}$$

In addition we take r to be sufficiently large such that for C_3 as in (7.16)

$$\frac{2|p|_{\text{Lip}}}{\text{m}(\lambda)\text{m}(T') - 1}(\text{m}(\lambda)\text{m}(T'))^{-r} \leq \frac{a}{16} \tag{7.18}$$

$$\frac{2C_3|\lambda|_{\text{Lip}}|p|_\infty}{((\text{m}(\lambda)\text{m}(T')) - 1)^2}(\text{m}(\lambda)\text{m}(T'))^{-r} \leq \frac{a}{16} \tag{7.19}$$

$$\left(\frac{|p|_\infty(C_3+1)}{1-|\lambda|_\infty}\right)|\lambda|_\infty^{r+1} \leq \frac{a}{8} \tag{7.20}$$

Given r as above, let $\epsilon \in (0, 1/2]$ be sufficiently small such that

$$C_3|\hat{V}_r|_{\text{Lip}}(2\epsilon) \leq \frac{a}{8} \tag{7.21}$$

where C_3 is as in (7.16).

Now we are given $z_2 \in [0,1]$, $r \geq 1$ and $a > 0$ fixed such that (7.17), (7.18), (7.19), and (7.20) hold. Given $\epsilon > 0$ such that (7.21) holds, we can now construct the set G as in Lemma 7.3.2. That is G is now the set of points whose orbit under $\hat{T}$ lands in $z_2 + \bigcup_{k\geq 1}(kb^r - \epsilon, kb^r + \epsilon)$ infinitely often, where b^r is the period of $\hat{V}_r$. Moreover G is the set of points, such that for each $x \in G$, there exists an infinite sequence $\{n_j\}_{j=1}^\infty$ such that for $n \in \{n_j\}_{j=1}^\infty$, there exists $k = k(n) \in \mathbb{N}$ such that $\hat{T}^n(x) \in (z_2 + kb^r - \epsilon, z_2 + kb^r + \epsilon)$.

Let $n > r$, $n \in \{n_j\}_{j=1}^\infty$ be given as above. Thus we have constructed a sequence of infinitely many n suffiently large. Consider $z_3 + kb^r > z_2 + kb^r$. Since $\{n_j\}_{j=1}^\infty \to \infty$ we can take the sequence of n and construct a sequence of $y_n = \hat{T}^{-n}(z_3 + kb^r)$. Since $\hat{T}^{-1}$ is contracting, then $y_n \to x$ as $n \to \infty$.

In particular, since $z_3, z_2 \in [0,1]$ we have that $|(z_3 + kb^r) - (z_2 + kb^r)| \leq 1$ and so $|\hat{T}^{-1}(z_3 + kb^r) - \hat{T}^{-1}(z_2 + kb^r)| \leq m(\hat{T}')^{-1}$.

Moreover as in Section 7.4.1, by choice of $\epsilon \in (0, 1/2]$ and since $y_n = \hat{T}^{-n}(z_3 + kb^r)$

$$\begin{aligned} |y_n - x| &= |\hat{T}^{-n}(z_3 + kb^r) - x + \hat{T}^{-n}(z_2 + kb^r) - \hat{T}^{-n}(z_2 + kb^r)| \\ &\leq \mathrm{m}(T')^{-n}(1 + 2\epsilon) \\ &\leq 2\mathrm{m}(T')^{-n}. \end{aligned}$$

Proposition 7.5.2

Let z_2, a, r and ϵ be as in the above. Construct G as in Lemma 7.3.2. Recall that $z_3 > z_2$ are chosen such that (7.19) holds. Given $x \in G$, and $n \in \{n_j\}_{j=1}^{\infty}$ as in the above where $n > r$, construct the sequence of $y_n = \hat{T}^{-n}(z_3 + kb^r)$. Then there exists $C_3 > 0$ as in (7.16), independent of n, x and y_n such that

$$C_3^{-1} \leq \frac{\hat{\lambda}^n(x)}{\hat{\lambda}^n(y_n)} \leq C_3.$$

Proof. Let $x \in G$ as constructed above. Let n be given and $y_n = \hat{T}^{-n}(z_3 + kb^r)$. Then since $\hat{\lambda}$ is Lipschitz and so is $\log \hat{\lambda}$ we have that

$$\begin{aligned} \left|\log \frac{\hat{\lambda}^n(x)}{\hat{\lambda}^n(y_n)}\right| &\leq \sum_{j=0}^{n-1} \left|\log \hat{\lambda}(\hat{T}^j(x)) - \log \hat{\lambda}(\hat{T}^j(y_n))\right| \\ &= |\log \hat{\lambda}|_{\mathrm{Lip}} \sum_{j=0}^{n-1} \left|\hat{T}^{j-n}(\hat{T}^n x) - \hat{T}^{j-n}(\hat{T}^n y_n)\right| \\ &\leq |\log \hat{\lambda}|_{\mathrm{Lip}} \mathrm{m}(T')^{-n} \sum_{j=0}^{n-1} \mathrm{m}(T')^j |\hat{T}^n(x) - \hat{T}^n(y_n)|. \end{aligned}$$

Recall that by construction and as $z_3, z_2 \in [0, 1]$, $|\hat{T}^n(x) - \hat{T}^n(y_n)| \leq |\hat{T}^n(x) - (z_2 + kb^r)| + |(z_2 + kb^r) - (z_3 + kb^r)| \leq 2$. Hence

$$\begin{aligned} \left|\log \frac{\hat{\lambda}^n(x)}{\hat{\lambda}^n(y_n)}\right| &\leq 2|\log \hat{\lambda}|_{\mathrm{Lip}} \mathrm{m}(T')^{-n} \frac{\mathrm{m}(T')^n - 1}{\mathrm{m}(T') - 1} \\ &\leq \frac{2|\log \hat{\lambda}|_{\mathrm{Lip}}}{\mathrm{m}(T') - 1}. \end{aligned}$$

Letting

$$C_3 = \exp\left\{\frac{2|\log \lambda|_{\mathrm{Lip}}}{\mathrm{m}(T') - 1}\right\}$$

as in (7.16) the result follows. ❏

By the above setup, we are now able to prove the following.

Proposition 7.5.3

Suppose that Cond(V) *holds. Then let $z_3 > z_2$, $a > 0$ and $r \geq 1$ be such that (7.17), (7.18), (7.19) and (7.20) hold. Let $\epsilon > 0$ be so that (7.21) holds.*

For such z_2, a, r and ϵ construct G as in Lemma 7.3.2, the set of points such that for all $x \in G$ there exists a sequence $\{n_j\}_{j=1}^{\infty}$ and for each $n \in \{n_j\}_{j=1}^{\infty}$ there exists $k = k(n)$ such that $|\hat{T}^n(x) - (z_2 + kb^r)| < 2\epsilon$. Given x for each such $n > r$ let $y_n = \hat{T}^{-n}(z_3 + kb^r) > x$. Then

$$\widehat{W}(y_n) - \widehat{W}(x) > \frac{a}{8}\hat{\lambda}^n(y_n).$$

Proof. Let x and y_n be as in the statement of the proposition. Then

$$\begin{aligned}
\widehat{W}(y_n) - \widehat{W}(x) &= \sum_{j=0}^{\infty} \hat{\lambda}^j(y_n)\hat{p}(\hat{T}^j(y_n)) - \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x)) \\
&= \left(\sum_{j=0}^{n-r-1} + \sum_{j=n-r}^{n+r} + \sum_{j=n+r+1}^{\infty} \right) \hat{\lambda}^j(y_n)\hat{p}(\hat{T}^j(y_n)) - \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x)) \\
&= \Sigma_1 + \Sigma_2 + \Sigma_3 \\
&\geq \Sigma_2 - |\Sigma_1| - |\Sigma_3|.
\end{aligned}$$

Claim 1

We show that $\Sigma_2 > \frac{3a}{8}\hat{\lambda}^n(y_n)$. Note that

$$\begin{aligned}
\Sigma_2 &= \sum_{j=n-r}^{n+r} \hat{\lambda}^j(y_n)\hat{p}(\hat{T}^j(y_n)) - \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x)) \\
&= \sum_{j=-r}^{r} \hat{\lambda}^{n+j}(y_n)\hat{p}(\hat{T}^{n+j}(y_n)) - \hat{\lambda}^{n+j}(x)\hat{p}(\hat{T}^{n+j}(x)) \\
&= \sum_{j=-r}^{r} \hat{\lambda}^n(y_n)\hat{\lambda}^j(\hat{T}^n(y_n))\hat{p}(\hat{T}^{n+j}(y_n)) - \hat{\lambda}^n(x)\hat{\lambda}^j(\hat{T}^n(x))\hat{p}(\hat{T}^{n+j}(x)) \\
&= \hat{\lambda}^n(y_n)\hat{V}_r(\hat{T}^n(y_n)) - \hat{\lambda}^n(x)\hat{V}_r(\hat{T}^n(x)).
\end{aligned}$$

By Proposition 7.5.2 there exists $t \in [C_3^{-1}, C_3]$ where C_3 is as in (7.16) such that $\hat{\lambda}^n(x) = t\hat{\lambda}^n(y_n)$. That is for some $t \in [C_3^{-1}, C_3]$

$$\Sigma_2 = \hat{\lambda}^n(y_n)\left[\hat{V}_r(\hat{T}^n(y_n)) - t\hat{V}_r(\hat{T}^n(x))\right].$$

Since $\hat{V}_r$ is b^r periodic and by choice of $n > r$ such that (7.17) holds

$$\begin{aligned}
&\hat{V}_r(\hat{T}^n(y_n)) - t\hat{V}_r(\hat{T}^n(x)) \\
&= \hat{V}_r(z_3) - t\hat{V}_r(z_2) + t\hat{V}_r(z_2) - t\hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(z_3) + \hat{V}_r(\hat{T}^n(y_n)) \\
&\geq \hat{V}_r(z_3) - t\hat{V}_r(z_2) - t|\hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(z_2)| - |\hat{V}_r(z_3) - \hat{V}_r(\hat{T}^n(y_n))| \\
&> \frac{a}{2} - t|\hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(z_2 + kb^r)| - |\hat{V}_r(z_3 + kb^r) - \hat{V}_r(\hat{T}^n(y_n))| \\
&= \frac{a}{2} - t|\hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(z_2 + kb^r)|.
\end{aligned}$$

Since $x \in G$, $|\hat{T}^n(x) - (z_2 + kb^r)| \leq 2\epsilon$, and by construction of y_n,

$$t|\hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(z_2 + kb^r)| \leq t|\hat{V}_r|_{\text{Lip}}(2\epsilon).$$

Since $t \leq C_3$ as in (7.16) and by choice of ϵ as in (7.21)

$$t|\hat{V}_r(\hat{T}^n(x)) - \hat{V}_r(z_2 + kb^r)| \leq \frac{a}{8}.$$

Therefore we have shown that

$$\hat{V}_r(\hat{T}^n(y_n)) - \hat{V}_r(\hat{T}^n(x)) > \frac{a}{2} - \frac{a}{8}$$

and so

$$\Sigma_2 > \frac{3}{8} a\hat{\lambda}^n(y_n).$$

Claim 2

We now show that for such $n > r$, $|\Sigma_1| \leq \frac{a}{8}\hat{\lambda}^n(y_n)$. Recall that since $y_n = \hat{T}^{-n}(z_3 + kb^r)$, using the triangle inequality $|\hat{T}^n(y_n) - \hat{T}^n(x)| \leq |\hat{T}^n(y_n) - (z_3 + kb^r)| + |(z_3 + kb^r) - (z_2 + kb^r)| + |(z_2 + kb^r) - \hat{T}^n(x)| \leq 2$. In addition $|y_n - x| \leq 2\text{m}(\hat{T}')^{-n}$ and $|\hat{T}^{n-j}(y_n) - \hat{T}^{n-j}(x)| \leq 2\text{m}(T')^{-j}$.

Note that

$$\begin{aligned}
|\Sigma_1| &= \left| \sum_{j=0}^{n-r-1} \hat{\lambda}^j(y_n)\hat{p}(\hat{T}^j(y_n)) - \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x)) \right| \\
&\leq \left| \sum_{j=0}^{n-r-1} \hat{\lambda}^j(y_n)\hat{p}(\hat{T}^j(y_n)) - \hat{\lambda}^j(y_n)\hat{p}(\hat{T}^j(x)) \right| \\
&\quad + \left| \sum_{j=0}^{n-r-1} \hat{\lambda}^j(y_n)\hat{p}(\hat{T}^j(x)) - \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x)) \right| \\
&= |\Sigma_1'| + |\Sigma_1''|.
\end{aligned}$$

Firstly since $j < n$ and $\hat{p}$ is Lipchitz continuous

$$\begin{aligned} |\Sigma_1'| &\le \sum_{j=0}^{n-r-1} \hat{\lambda}^j(y_n)|\hat{p}(\hat{T}^j(y_n)) - \hat{p}(\hat{T}^j(x))| \\ &= \sum_{j=0}^{n-r-1} \hat{\lambda}^n(y_n)\hat{\lambda}^{j-n}(\hat{T}^n(y_n))|\hat{p}(\hat{T}^j(y_n)) - \hat{p}(\hat{T}^j(x))| \\ &\le \hat{\lambda}^n(y_n)|p|_{\text{Lip}}\mathrm{m}(\lambda)^{j-n}|\hat{T}^{j-n}(\hat{T}^n(y_n)) - \hat{T}^{j-n}(\hat{T}^n(x))| \\ &\le \hat{\lambda}^n(y_n)|p|_{\text{Lip}}(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-n} \sum_{j=0}^{n-r-1} (\mathrm{m}(\lambda)\mathrm{m}(T'))^j|\hat{T}^n(y_n) - \hat{T}^n(x)|. \end{aligned}$$

Recall that $|\hat{T}^n(y_n) - \hat{T}^n(x)| \le 2$. Therefore

$$\begin{aligned} |\Sigma_1'| &\le 2\hat{\lambda}^n(y_n)|p|_{\text{Lip}}(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-n} \left(\frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{n-r}}{\mathrm{m}(\lambda)\mathrm{m}(T') - 1}\right) \\ &= \hat{\lambda}^n(y_n)\frac{2|p|_{\text{Lip}}}{\mathrm{m}(\lambda)\mathrm{m}(T') - 1}(\mathrm{m}(\lambda)\mathrm{m}(T'))^{-r} \\ &\le \frac{a}{16}\hat{\lambda}^n(y_n) \end{aligned}$$

by choice of r as in (7.18).

Secondly

$$\begin{aligned} |\Sigma_1''| &\le \sum_{j=0}^{n-r-1} \left|\hat{\lambda}^j(y_n)\hat{p}(\hat{T}^j(x)) - \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x))\right| \\ &\le |p|_\infty \sum_{j=0}^{n-r-1} \left|\hat{\lambda}^j(y_n) - \hat{\lambda}^j(x)\right|. \end{aligned}$$

For each $0 \le j \le n-r-1$ a repeated application of the triangle inequality shows that

$$\begin{aligned} &\left|\hat{\lambda}^j(y_n) - \hat{\lambda}^j(x)\right| \\ &= \left|\sum_{i=0}^{j-1} \hat{\lambda}^i(x)\hat{\lambda}^{j-i-1}(\hat{T}^{i+1}(y_n)) \left[\hat{\lambda}(\hat{T}^i(y_n)) - \hat{\lambda}(\hat{T}^i(x))\right]\right|. \end{aligned}$$

Now for $i < n$, bounded variation arguments show that there exists $C_3 > 1$ as in (7.16) such that $\hat{\lambda}^i(x) \le C_3\hat{\lambda}^i(y_n)$. Hence

$$\left|\hat{\lambda}^j(y_n) - \hat{\lambda}^j(x)\right| \le C_3 \sum_{i=0}^{j-1} \hat{\lambda}^i(y_n)\hat{\lambda}^{j-i-1}(\hat{T}^{i+1}(y_n)) \left|\hat{\lambda}(\hat{T}^i(y_n)) - \hat{\lambda}(\hat{T}^i(x))\right|$$

Note that for each $0 \le i \le j-1$ since $\hat{\lambda} : \mathbb{R} \to (0,1)$, $\hat{\lambda}^{j-i-1}(\hat{T}^{i+1}(y_n)) < 1$ and for $i < n$, $\hat{\lambda}^i(y_n) = \hat{\lambda}^n(y_n)\hat{\lambda}^{i-n}(\hat{T}^n(y_n))$. Thus

$$|\Sigma_1''| \le C_3|p|_\infty \sum_{j=0}^{n-r-1} \sum_{i=0}^{j-1} \hat{\lambda}^n(y_n)\hat{\lambda}^{i-n}(\hat{T}^n(y_n)) \left|\hat{\lambda}(\hat{T}^i(y_n)) - \hat{\lambda}(\hat{T}^i(x))\right|.$$

Now since $i < n$ and $\hat{\lambda}$ is Lipschitz

$$\begin{aligned}
& \sum_{i=0}^{j-1} \hat{\lambda}^{i-n}(\hat{T}^n(y_n)) \left| \hat{\lambda}(\hat{T}^i(y_n)) - \hat{\lambda}(\hat{T}^i(x)) \right| \\
\leq \; & \sum_{i=0}^{j-1} \mathrm{m}(\lambda)^{i-n} |\lambda|_{\mathrm{Lip}} \left| \hat{T}^i(y_n) - \hat{T}^i(x) \right| \\
\leq \; & \sum_{i=0}^{j-1} \mathrm{m}(\lambda)^{i-n} |\lambda|_{\mathrm{Lip}} \left| \hat{T}^{i-n}\hat{T}^n(y_n) - \hat{T}^{i-n}\hat{T}^n(x) \right| \\
\leq \; & |\lambda|_{\mathrm{Lip}} \sum_{i=0}^{j-1} \mathrm{m}(\lambda)^{i-n} \mathrm{m}(T')^{i-n} \left| \hat{T}^n(y_n) - \hat{T}^n(x) \right| \\
\leq \; & 2|\lambda|_{\mathrm{Lip}} \sum_{i=0}^{j-1} (\mathrm{m}(\lambda)\mathrm{m}(T'))^{i-n}
\end{aligned}$$

since $|\hat{T}^n(y_n) - \hat{T}^n(x)| \leq 2$.

Now

$$\sum_{i=0}^{j-1} (\mathrm{m}(\lambda)\mathrm{m}(T'))^{i-n} \leq (\mathrm{m}(\lambda)\mathrm{m}(T'))^{-n} \frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^j}{(\mathrm{m}(\lambda)\mathrm{m}(T')) - 1}.$$

Hence

$$\begin{aligned}
|\Sigma_1''| \; & \leq \; 2C_3 |\lambda|_{\mathrm{Lip}} |p|_\infty \hat{\lambda}^n(y_n) \sum_{j=0}^{n-r-1} (\mathrm{m}(\lambda)\mathrm{m}(T'))^{-n} \frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^j}{(\mathrm{m}(\lambda)\mathrm{m}(T')) - 1} \\
& \leq \; 2C_3 |\lambda|_{\mathrm{Lip}} |p|_\infty \hat{\lambda}^n(y_n) (\mathrm{m}(\lambda)\mathrm{m}(T'))^{-n} \frac{(\mathrm{m}(\lambda)\mathrm{m}(T'))^{n-r}}{((\mathrm{m}(\lambda)\mathrm{m}(T')) - 1)^2} \\
& = \; \hat{\lambda}^n(y_n) \frac{2C_3 |\lambda|_{\mathrm{Lip}} |p|_\infty}{((\mathrm{m}(\lambda)\mathrm{m}(T')) - 1)^2} (\mathrm{m}(\lambda)\mathrm{m}(T'))^{-r} \\
& \leq \; \frac{a}{16} \hat{\lambda}^n(y_n)
\end{aligned}$$

by choice of r as in (7.19).

Hence

$$|\Sigma_1| \leq |\Sigma_1'| + |\Sigma_1''| \leq \frac{a}{8} \hat{\lambda}^n(y_n).$$

Claim 3

Finally we show that $|\Sigma_3| \leq \frac{a}{8} \hat{\lambda}^n(y_n)$. Let $y_n > x$ be as above. Since $j < n$, $\hat{\lambda}^j(x) =$

$\hat{\lambda}^n(x)\hat{\lambda}^{j-n}(\hat{T}^n(y_n))$ and so

$$\begin{aligned}
|\Sigma_3| &\leq \sum_{j=n+r+1}^{\infty} \left|\hat{\lambda}^j(y_n)\hat{p}(\hat{T}^j(y_n)) - \hat{\lambda}^j(x)\hat{p}(\hat{T}^j(x))\right| \\
&\leq \sum_{j=n+r+1}^{\infty} \hat{\lambda}^j(y_n)\left|\hat{p}(\hat{T}^j(y_n))\right| + \hat{\lambda}^j(x)\left|\hat{p}(\hat{T}^j(x))\right| \\
&\leq |p|_\infty \sum_{j=n+r+1}^{\infty} \hat{\lambda}^j(y_n) + \hat{\lambda}^j(x) \\
&= |p|_\infty \sum_{j=n+r+1}^{\infty} \hat{\lambda}^n(y_n)\hat{\lambda}^{j-n}(\hat{T}^n(y_n)) + \hat{\lambda}^n(x)\hat{\lambda}^{j-n}(\hat{T}^n(x)) \\
&= |p|_\infty \sum_{j=r+1}^{\infty} \hat{\lambda}^n(y_n)\hat{\lambda}^j(\hat{T}^n(y_n)) + \hat{\lambda}^n(x)\hat{\lambda}^j(\hat{T}^n(x)) \\
&\leq |p|_\infty \sum_{j=r+1}^{\infty} \hat{\lambda}^n(y_n)|\lambda|_\infty^j + \hat{\lambda}^n(x)|\lambda|_\infty^j.
\end{aligned}$$

Then by Proposition 7.5.2 there exists $C_3 > 0$ as in (7.16) independent of n such that

$$\begin{aligned}
|\Sigma_3| &\leq |p|_\infty \sum_{j=r+1}^{\infty} |\lambda|_\infty^j (C_3+1)\hat{\lambda}^n(y_n) \\
&= \hat{\lambda}^n(y_n)|p|_\infty (C_3+1)\left(\frac{1}{1-|\lambda|_\infty} - \frac{1-{|\lambda|_\infty}^{r+1}}{1-|\lambda|_\infty}\right) \\
&= \hat{\lambda}^n(y_n)\left(\frac{|p|_\infty(C_3+1)}{1-|\lambda|_\infty}\right)|\lambda|_\infty^{r+1} \\
&\leq \frac{a}{8}\hat{\lambda}^n(y_n)
\end{aligned}$$

by choice of r as in (7.20).

Then by claims 1, 2 and 3 we have that

$$\begin{aligned}
\widehat{W}(y_n) - \widehat{W}(x) &> \frac{3a}{8}\hat{\lambda}^n(y_n) - \frac{a}{8}\hat{\lambda}^n(y_n) - \frac{a}{8}\hat{\lambda}^n(y_n) \\
&= \frac{a}{8}\hat{\lambda}^n(y_n)
\end{aligned}$$

for infinitely many $n > r$. ❏

Theorem 7.5.4

There exists a dense G_δ subset $G \subset [0,1]$, which also takes full measure with respect to any T invariant probability measure which takes postive measure on non-empty open sets, such that given $x \in G$ there exists a sequence of points $y_{n_j} \to x$ such that

$$D^+\widehat{W}(x) = \limsup_{n_j\to\infty} \frac{\widehat{W}(y_{n_j}) - \widehat{W}(x)}{y_{n_j} - x} = \infty.$$

Proof. We construct G as above, and by Corollary 7.3.4 it can be G satisfies the required properties of the theorem. By Proposition 7.5.3, there exists $a > 0$ and a sequence $\{n_j\}_{j=1}^{\infty}$ such that for all n_j, $\widehat{W}(y_{n_j}) - \widehat{W}(x) > \frac{a}{8}\hat{\lambda}^n(y_{n_j}) > \frac{a}{8}\mathrm{m}(\lambda)^n$. In addition $y_{n_j} > x$ and $y_{n_j} - x \leq 2\mathrm{m}(\hat{T}')^{-n_j}$. That is

$$\frac{\widehat{W}(y_{n_j}) - \widehat{W}(x)}{y_{n_j} - x} > \frac{a}{8}(\mathrm{m}(\lambda)m(T'))^{n_j}$$

for all n_j. In particular, by the partial hyperbolicity assumption,

$$D^+\widehat{W}(x) > \lim_{n_j \to \infty} \frac{a}{16}(\mathrm{m}(\lambda)m(T'))^{n_j} = \infty$$

as required. ❑

By the above, we have shown that the set of points for which $\widehat{W}$, hence (see Section 2.8) W, has upper right derivative ∞, contains a set $G \subset [0,1]$, which by Corollary 7.3.4 is a dense G_δ set, has full measure with respect to any ergodic T-invariant probability measure supported on $[0,1]$ which takes positive measure on non-empty open sets and has Hausdorff dimension 1.

7.5.2 On the other three cases

In the following, we outline the method for the remaining three derivatives. In the previous section we constructed a set which we now call $G^+ \subset [0,1]$, such that for all $x \in G^+$, $D^+W(x) = \infty$.

We can also, using the above method, show that there exists sets, different to the above G^+, however which are still constructed as in Lemma 7.3.2, of sets of points where the lower right and upper and lower derivatives are $-\infty, \infty$ and $-\infty$ respectively.

Once again, we are always assuming that Cond(V) holds. In order to show the upper left derivative case, we take the same $z_3 > z_2$ as in the upper right derivative case. However we use z_3, r, a and ϵ (as opposed to z_2) to construct a different set G^-, and show the required result for $x \in G^-$ in a similar way, where z_2 takes the role of z_3 in the upper right case, and z_3 takes the role of z_2, (i.e. we approach the point x from the left hand side). In addition since Cond(V) holds, we consider $z_1 < z_2$ and proceed in the same way to show the required result for the lower right and left derivatives.

That is, we can show that there exists four subsets of the unit interval, which we call G^+, G_+, G^- and G_-, which are sets of points where $D^+\widehat{W}(x) = \infty$, $D_+\widehat{W}(x) = -\infty$,

$D^-\widehat{W}(x) = \infty$, and $D_-\widehat{W}(x) = -\infty$ respectively. Moreover all such points satisfy Corollary 7.3.4.

Then denote

$$G = G^+ \cap G_+ \cap G^- \cap G_-.$$

Note that G is a finite intersection of dense G_δ subsets of $[0,1]$ which all satisfy Corollary 7.3.4 and G is a dense G_δ subset of $[0,1]$ itself. Moreover G is an intersection of 4 sets which have full measure with respect to any ergodic T-invariant probability measure. Therefore it can be seen that such a set G satisfies Corollary 7.3.4. In conclusion, we have proven the following theorem;

Theorem 7.5.5

Let T be a uniformly expanding map of the circle, let $p : S^1 \to \mathbb{R}$ be Lipschitz and let $\lambda : S^1 \to (0,1)$ be Lipschitz. Assume that the non-degeneracy and partial hyperbolicity assumptions hold. In additon assume that Cond(V) *holds. Then for the function $W : S^1 \to \mathbb{R}$ given by*

$$W(x) = \sum_{j=0}^{\infty} \lambda^j(x) p(T^j(x)),$$

the set of knot-points for W contains a subset G which satisfies Corollary 7.3.4.

Remark. We have proven the knot-point result in the above, assuming that Cond(V) holds true. In earlier section we gave hypotheses which guarantee such a condition to hold. However in the example given, it may be the case that we have to impart strong hyperbolicity conditions in order for Cond(V) to hold. We conjecture that these conditions can be weakened, and that other weaker hypotheses can be given in order to for Cond(V) to hold.

Moreover we conjecture that the statement as in Theorem 7.5.5 holds true if one removes the requirement of Cond(V), and simply assumes partial hyperbolicity and the non-degeneracy assumption, however we have unfortunately been unable to prove this.

Chapter 8

Conclusion and further research

In Chapter 1, we introduced skew-products of the form

$$F(x,y) = (T(x), g_x(y))$$

where T is a uniformly expanding map of the circle, and g_x is a continuous diffeomorphism of $\mathbb{R}$. Throughout this thesis, we predominantly restricted ourselves to the purely affine case, where $g_x(y) = \lambda(x)^{-1}(y - p(x))$ as in (2.5). However the estimation results in Chapter 3 hold for the more general skew-product case.

In Chapter 4 we proved that the $\mathcal{K}$-dimension of the invariant graph of (2.5), when $T(x) = bx \bmod 1$, is given by the unique solution to the topological pressure equation

$$P((1 - s_0)\log b + \log \lambda) = 0.$$

We were also able to give a technical assumption, such that such a result will work for the case in which T is simply a non-uniformly expanding map of S^1. It is conjectured by the author, that such a technical assumption is not necessary, however problems arise in proving the lower bound for $\mathcal{K}$-dimension, as using Moran covers does not take into account the height of the graph over cylinders of the same diameter, which due to the non-linearity of the map T, may have significantly different ranks.

An interesting question to generalise such work, arises when one considers the case in which T is no longer unformly expanding, but there exists an indifferent fixed point for the map. Such an example is that of the Manneville-Pomeau map [MaPo], where the hyperbolicity is non-uniform, and the slope of the map is 1 at the fixed point 0. Sarig [Sa] studied such cases, where one can induce the non-uniformly hyperbolic system to a countable state system, that is, one where the dynamics can be modelled by a one-sided shift on countably infinite many symbols. Sarig was able to give an analogous result of Bowen's equation in such a case, however one must work with a different, potentially non-analytic notion of pressure, due to Gurevic [Gur]. It would be of particular interest to consider invariant graphs when our base map is non-uniformly hyperbolic, and induce to a countable state system, in order to prove similar dimension theoretic results.

In Chapter 5 we consider a special case of an invariant graph whose Hausdorff dimension is strictly less that the box dimension. An interesting question from our standpoint would be, does there exists an example of a graph whose $\mathcal{K}$-dimension is strictly less that the box dimension?

In Chapter 6, using the multifractal analysis, it would be interesting to consider the exceptional set and to consider the values taken by the derivative of the graph at such points. It would also be of particular interest to give a full classification of the values taken by the derivative of the graph for all points for different values of T' and α. In addition it would be beneficial to consider dimension theoretic and multifractal notions, where T is no longer linear, but a uniformly expanding map of the circle. However in looking at dimension theoretic estimates, the case in which T is non linear poses greater problems, in which the mass distribution principle cannot be used in proving a lower bound for the Hausdorff dimension of the graph.

Another question of particular interest involves the restrictions under α where the measure of the set of points of non-differentiability jumps from 0 to 1. That is, in the case in which the measure of the set of points of non-differentiability is zero, we may wish to ask what the Hausdorff dimension of the set of such points is. That is we are interested in

$$\dim_{\mathcal{H}}\left(\bigcup_{\gamma>0}\left\{x \mid \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\big(\log|T'|+\log|\beta|)(T^j(x))\big)=\gamma\right\}\right).$$

However, since we are considering the Hausdorff dimension of an uncountable union, there doesn't seem to be an appropriate way to approach such a problem.

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