TOPOLOGICAL AND SYMBOLIC DYNAMICS OF THE DOUBLING MAP WITH A HOLE

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Engineering and Physical Sciences

2014

Rafael Alcaraz Barrera
School of Mathematics

Contents

Word count 19333

The University of Manchester

Rafael Alcaraz Barrera

Doctor of Philosophy

Topological and Symbolic dynamics of the doubling map with a hole

November 25, 2014

This work motivates the study of open dynamical systems corresponding to the doubling map. In particular, the dynamical properties of the attractor of $2x \mod 1$ when a symmetric, centred open interval (a, b) is removed are studied. Using the arithmetical properties of the binary expansion of the points on the boundary of the removed interval, as suggested in [BKT11], we study properties such as topological transitivity, the specification property and intrinsic ergodicity. The properties of the function that associates to each hole (a, b) the topological entropy of the attractor of the considered dynamical system are also shown. For these purposes, a subshift corresponding to an element of the lexicographic world is associated to each attractor and the mentioned properties are studied symbolically.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

Acknowledgements

I am deeply grateful to my supervisor, Dr. Nikita Sidorov. Firstly, for accepting me as his first PhD. student without knowing anything about me. Secondly, for all his patience with me and with the development of this project, giving me independence and freedom to work. Also, he showed me the beauty of symbolic dynamics and the importance of examples in Mathematics. Finally, I would like to thank him for all the advice and encouragement that I have received from him.

I owe thanks to my examiners, Prof. Paul Glendinning and Dr. Mike Todd for all their usefull remarks on the presentation and the content of this work.

Financial support from the National Council of Science and Technology (CONACyT, Mexico), Scholarship for Doctoral Students no. 213600, during my time as a PhD student, and the EPS-CONACyT PhD Programme are gratefully acknowledged. I would especially like to thank Dr. Teresa Alonso Rasgado for all her support during my PhD.

I am grateful to Prof. Jean Paul Allouche for his hospitality in my visit to Institut de Mathématiques de Jussieu-PRG, Université Pierre et Marie Curie. The visit was sponsored by the Royal Society International Exchanges grant IE130940: Open Dynamical Systems and the Lexicographic World.

I owe thanks to Dr. Simon Baker and Lyndsey Clark for their interest in the ideas presented in this work, for all their comments, for all the stimulating conversations that we had and, most importantly, for their friendship. I would like to thank Dr. Dave Naughton for being such good colleague and friend. I must thank all the people who made the Informal Dynamics Seminar possible. Also, I am deeply grateful to all the lecturers who gave me the opportunity to be their teaching assistant.

I must thank Prof. Sergio Macías and Dr. Aubin Arroyo for also being a big support back in Mexico, not only mathematically. I will always consider them an inspiration to carry on with my career as a mathematician.

I must thank my beloved wife, Paola, for following our dreams and for being with me during these years. Her patience, encouragement and love are invaluable. They are a reason to carry on doing maths.

I would like to dedicate this work to my parents, Patricia and José Luis. This work could not have been done without them. They have supported me in every step of my education and have encouraged me in everything. Also, I will always be grateful to my brothers Goyo and Beto for all their support. I am indebted to Alejandro Valencia Tobón, who has been more than a friend during my studies and to Jair Muñoz for all his support, help and friendship since I arrived in England.

I am grateful to Nikesh and Zoe Solanki, Simon, Rachel and John Dean, John Hosie, Tim Crinion and Paul Grundey for becoming my family in Manchester. Also, I want to thank Citlalitl Nava, Michael Crabb, Javier Hernandez, Kernel Prieto and all the guys of the Turing Machine for making the Alan Turing Building a great place to work. Also, I must thank the guys from the Anthropology department for all the good times during our studies.

I must thank Lenox Green and my mother-in-law Dunia Quiñónez for all their spiritual support and friendship during these four years. Last, but certainly not least, I thank God for giving me the opportunity to carry on with my dreams.

Chapter 1

Introduction

In the following thesis, we study the properties of *open dynamical systems*, colloquially *maps with holes*, corresponding to the doubling map when an interval $(a, b) \subset S^1$ is removed. Through the presented work, we shall call a pair (X, f) where X is a compact metric space and $f : X \to X$ is a continuous transformation a *discrete dynamical system* or simply a *dynamical system*.

The provided definition requires a topological structure on (X, f). The study of dynamical systems with topological structure is called *topological dynamics*. Nonetheless, to study dynamical systems in general, the topological structure of (X, f) is not always required since it is possible to associate to (X, f) a different structure. For example, we can consider $(X, \mathcal{B}(X), \mu)$ a measure space and a transformation $f : X \to X$ to be measurable. The study of the dynamics with measure theoretic structure is known as *Ergodic Theory*.

There is a rich relation between the ergodic properties and the topological properties of a dynamical system. For example, it is possible to show the following statement: Let $(X, \mathcal{B}(X), \mu)$ be a measure space such that $\mathcal{B}$ is the Borel σ-algebra on X and let $f : X \to X$ be a continuous map. Then, if μ is an ergodic measure such that it gives positive value to the open subsets of X then f is topologically transitive [Mañ83, Proposition 2.8]. Nonetheless, the reciprocal property does not hold. This fact gives an example to show that topological properties do not imply ergodic properties necessarily - see [Mañ83, Section 3.7].

In contrast to the previous fact, the main idea of this work is to use topological properties of dynamical systems in order to obtain ergodic properties of such systems.

Let us introduce the reader to the basic set-up. Consider a dynamical system (X, f) such that the topological entropy of f is positive and let U be an open subset of X which we refer as a *hole*. The sets

$$X_U = \{x \in X \mid f^n(x) \notin U \text{ for every } n \in \mathbb{Z}$$

if f is bijective or

$$X_U = \{x \in X \mid f^n(x) \notin U \text{ for every } n \geq 0\}$$

if f is not injective are called the *U-exceptional set* or the *U-survivor set*. Observe that X_U is a compact and forward f-invariant set if f is not injective and f-invariant if f is bijective. Then, it is natural to consider the dynamical system (X_U, f_U) where $f_U = f\mid_{X_U}$. We call (X_U, f_U) an *open dynamical system*.

The notion of open dynamical system was introduced by Pianigiani and Yorke in [PY79] in a slightly different way. The distinctive feature of open dynamical systems is that the orbits on X may eventually escape X through U, whereas in a dynamical system (X, f) the orbits maintain their image on the space for every time n.

Nowadays, open dynamical systems is an active area of research. General properties of this class of dynamical systems have been studied for hyperbolic diffeomorphisms -see [BDM10, BKT11, BY11, CM97a, CM97b, CMT00, DW12, DWY12, KL09] and billiards -see [CMT00, DWY10, DWY12, LM96]. In particular, the properties of the escape rate function have been studied extensively.

However, the answers of some essential and certainly interesting questions about the dynamical properties of (X_U, f_U) are still unknown. In particular, if X_U has positive Hausdorff dimension we may ask if (X_U, f_U) transitive, or if (X_U, f_U) has the specification property. Another natural question to ask is if (X_U, f_U) is intrinsically ergodic [Sid03b].

Another reasonable question to ask is if the topological entropy of these systems change continuously with respect to the holes and if it has dependence on their size or form. There exist studies on this direction on maps with holes in defined in unit interval. Życzkowski and Bollt [ŻB99] showed numerically that if f is a tent map with a centred symmetric hole then the entropy function is a devil's staircase. Besides, they showed numerically that any entropy plateau corresponding to an orbit of length n has an infinite number of adjacent plateaus with the same entropy caused by orbits of period 2^{kn} for every $k \in \mathbb{N}$. Using kneading theory (see [CE80] for the necessary background) Ban, Hsu and Lin in [BHL03, p. 116] show that if f is a strongly transitive unimodal map defined on the unit interval then the entropy function of the family of maps with a hole centred at the turning point c is a devil's staircase. Misiurewicz in [Mis04, Theorem 1.1] considered a one parameter family of exclusion maps f_a with gaps given by $\{x \in [0,1] \mid f(x) \leq a\}$. He showed that if f is a piecewise monotone continuous function then the entropy function is globally constant or a devil's staircase. Besides, it was showed that for any continuous map f of the unit interval, the entropy function is monotone and locally constant (not necessarily continuous).

1.1 The doubling map

A natural candidate to study the posed questions is the dynamical system (S^1, f) where S^1 is the unit circle and f is doubling map - see Section 2.2. We will restrict ourselves to consider connected holes only.

Let us introduce formally our object of study. Let $(a, b) \subset S^1$. Then, the (a, b)-*exceptional set* is given by

$$X_{(a,b)} = \{x \in S^1 \mid f^n(x) \notin (a, b) \text{ for any } n \geq 0\}.$$

Observe that

$$X_{(a,b)} = \bigcap_{n=0}^{\infty} S^1 \setminus f^{-n}((a, b)),$$

and that $X_{(a,b)}$ is an f-invariant set only when $X_{(a,b)} = \emptyset$ or S^1 but $X_{(a,b)}$ is a forward f-invariant set. Let $f_{(a,b)}$ denote the map $f\mid_{X_{(a,b)}}$. Then, *the open dynamical system corresponding to a and b* is the pair $(X_{(a,b)}, f_{(a,b)})$. Since (S^1, f) is an ergodic dynamical system with respect to the Lebesgue measure, $X_{(a,b)}$ has Lebesgue measure 0 by the ergodic theorem.

There is a recent body of works which study particular features of open dynamical systems for the doubling map -see [Det13, GS01, GS14, LGDL14, Nil09, Sid14, Urb86, Urb87]. In particular, a study of the fine structure of the cycles of $(X_{(a,b)}, f_{(a,b)})$ is developed in [ACS09, HS14] and the properties of the Hausdorff dimension of $X_{(a,b)}$ are studied in [GS01, GS14, Sid14]. Also, Urbański in [Urb86] showed if $g : S^1 \to S^1$ is a C^2 uniformly expanding map and the holes are given by intervals of the form $(0, b)$, where 0 denotes the fixed point of g, then the function which associates to each hole the topological entropy (which we shall refer as *entropy function*) of f_b^0 is a devil's staircase. Furthermore, Nilsson in [Nil09] characterised the end points of the entropy function for the doubling map with holes of the form $(0, b)$ and it is shown that for every b, $(X_{(0,b)}, f_b^0)$ is a topologically mixing dynamical system. Moreover, if the removed intervals are of the form $(0, a)$ or $(1, b)$ with $a \in (0, \frac{1}{2})$ and $b \in (\frac{1}{2}, 1)$ then $(X_U, f\mid_{X_U})$ is topologically conjugated to a β-shift - see Section 6.1. This implies that $(X_U, f\mid_{X_U})$ is automatically intrinsically ergodic [Hof78, Wal78]. For this reason, we will consider *centred holes* (i.e. $(a, b) \in S^1$ such that $\frac{1}{2} \in (a, b)$) only. To the best of the knowledge of the author, the dynamical properties of the case when $(a, b) \subset S^1$ does not contain 0 and $\frac{1}{2}$ are still not known -see Section 7.2.

Since the doubling map is semi-conjugate to the full one sided shift in two symbols (Σ_2, σ) (see Section 2.2), it is natural to ask if the mentioned properties will follow from the symbolic representation of the boundary points of the hole -see [BKT11]. Moreover, the set of one sided infinite sequences, Σ_2, has a natural order, *the lexicographic order* $\prec$. Thus, the properties of the attractor of $(X_{(a,b)}, f_{(a,b)})$, denoted by $\Lambda_{(a,b)}$ are determined by a particular symbolic object; *a lexicographic subshift* $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ [1] - see Section 2.2. Therefore, to study the topological and ergodic properties of the considered open dynamical systems it suffices to comprehend those properties symbolically.

An important problem that relates the topological properties with the ergodic properties of a dynamical system is to establish when (X, f) is *intrinsically ergodic*, that is (X, f) has a unique measure of maximal entropy (Definition 2.4.2). In particular, it is

[1] It is worth to mention that the lexicographic subshifts and the lexicographic world were introduced by Gan in [Gan01]. However, this notion was studied previously -see [GW79] among others.

of our interest to determine when a lexicographic subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is intrinsically ergodic. Our approach to this problem is to use some classical results. In [Par64a] Parry has shown that every transitive subshift of finite type is intrinsically ergodic. In [Wei70, Wei73] Weiss obtained the same result for transitive sofic subshifts, and Bowen in [Bow75] proved that subshifts with the specification property are intrinsically ergodic (we summarise these results in Theorem 2.4.3). Also, Gurevič in [Gur72, Gur80] provided a general criterion to determine when a subshift is intrinsically ergodic. Recently, Climenhaga and Thompson in [CT12] described properties that guarantee the intrinsic ergodicity of a subshift. We mention both criteria in Chapter 7.

1.2 Statement of Results

In Chapter 3 we will describe general properties of the open dynamical systems for the doubling map. In particular, in Theorem 3.2.3 we show that the dynamics on every attractor $\Lambda_{(a,b)}$ is conjugate to a lexicographic subshift -see Definition 2.2.1. In Theorem 3.4.2 we show that for almost every (a, b) the associated attractor $\Lambda_{(a,b)}$ is a subshift of finite type which implies that the set of parameters with this property is dense. Also, we describe the function which associates to each pair (a, b) the topological entropy of $(\Lambda_{(a,b)}, f_{(a,b)})$ and in Theorem 3.4.4 is constant almost everywhere.

In Chapter 4 we present the results obtained for the family of open dynamical systems corresponding to a centred symmetric hole, published in [AB14]. In Section 4.1 we study the topological transitivity of symmetric subshifts -see Definition 2.2.4. In Theorem 4.1.7 we characterise the set of sequences which generate a transitive subshift of finite type. Moreover, in Theorem 4.1.3 we determine a lower bound for this set of sequences. In Section 4.2 we show that the entropy function for symmetric subshifts is a devil's staircase. In Theorem 4.2.14 the entropy plateaus are characterised and in Theorem 4.2.13 we characterise the exceptional set. In Section 4.3 the specification property is studied. In Theorem 4.3.2 the set of parameters with the specification property is described. Also, in Theorem 4.3.16 we give a sufficient condition for a subshift to not have the specification property. Also a family of examples with without specification is constructed. Finally in Section 4.4 we describe intrinsically ergodic symmetric subshifts.

In Chapter 5 we characterise the set of pairs (a, b) which $(\Lambda_{(a,b)}, f_{(a,b)})$ is transitive. Also, some partial results on the intrinsic ergodicity of asymmetric subshifts are presented.

In Chapter 6 we establish a connection between our families of open dynamical systems and β-expansions. We state some final remarks in Chapter 7. Some open questions are posed in Section 7.2. Chapter 2 contains all the necessary concepts from symbolic dynamics and ergodic theory used in our study.

Chapter 2

Preliminaries

We restrict ourselves to study symbolic dynamical systems given by an alphabet with only two symbols, namely 0 and 1. For a complete exposition on symbolic dynamics and detailed proofs of some of the presented statements we refer the reader to [LM95]. For further background and a detailed exposition on Ergodic Theory we refer the reader to [Mañ83, Wal82].

In Section 2.1, the *one-sided shift transformation* is introduced and its topological properties are mentioned. Moreover, the dynamical properties of *subshifts*; i.e. forward σ-invariant subsets, are stated. Subshifts provide a model which allow us to code certain dynamical systems, in particular, the doubling map.

In Section 2.2 we mention the dynamical properties of the *doubling map* and we introduce the key object of our research: *lexicographic subshifts*. This class of subshifts provides us with the model used to study the properties of the attractor of the doubling map with a hole.

Section 2.3 includes a brief summary on combinatorics on words necessary to complete our exposition. A detailed analysis on this topic can be found in [Lot05, Chapter 2] or [Vui03].

In Section 2.4 we present some basic concepts in Ergodic Theory. We make particular emphasis on measures of maximal entropy of a dynamical system and the intrinsic ergodicity of such a system. Moreover, we mention the classical results about the intrinsic ergodicity of subshifts used in our work.

Recall that given a dynamical system (X, f), a subset $A \subset X$ is *forward f-invariant* if $f(A) = A$ and we say that A is *f-invariant* if $f^{-1}(A) = A$. Given $n \in \mathbb{N}$, the *n-th iteration of f* is denoted by $f^n = f \circ \ldots \circ f$ n times. Note that f^n is well defined for every $n \in \mathbb{Z}$ if and only if f is bijective. The easiest example of an f-invariant set is a *fixed point for f*, i.e, a point $x \in X$ such that $f(x) = x$. In general we say that a point $x \in X$ is *periodic* if there exists $n \in \mathbb{N}$ such that $f^n(x) = n$. If x is a periodic point, the smallest $n \in \mathbb{N}$ such that $f^n(x) = x$ is the *period of x*. A point x is *preperiodic*, if $f^n(x)$ is a periodic point for some $n \in \mathbb{N}$. We write $Per(f)$ to denote the set of periodic points of f and put

$$Per_n(f) = \{x \in X \mid f^k(x) = x \text{ for some } 1 \leq k \leq n\}.$$

2.1 Symbolic Dynamics

The set $\mathcal{A} = \{0, 1\}$ is called an *alphabet* and whose elements are called *symbols* or *digits*. A finite sequence of symbols ω of $\mathcal{A}$ is called a *word*. Given a word ω, *the length of ω, $\ell(\omega)$* is, by definition, the number of symbols it contains. Given two words $\omega = w_1, \ldots w_{\ell(\omega)}$ and $\nu = u_1 \ldots u_{\ell(u)}$ we write $\omega\nu$ to denote their *concatenation*, i.e, the word $\omega\nu = w_1, \ldots w_{\ell(\omega)} u_1 \ldots u_{\ell(u)}$. Observe that the word $\omega\nu$ has length $\ell(\omega) + \ell(\nu)$. In particular, the notation ω^k means that the word ω is concatenated to itself k times.

Let $\Sigma_2 = \prod_{n=1}^{\infty} \{0, 1\}$, with the distance given by:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} 2^{-j} & \text{if } \boldsymbol{x} \neq \boldsymbol{y}; \quad \text{where } j = \min\{i \mid x_i \neq y_i\} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that Σ_2 is the set of all one sided sequences with symbols in $\mathcal{A}$. Moreover, Σ_2 is a Cantor set with the topology given by d. A sequence $\boldsymbol{x} = (x_i)_{i=1}^{\infty} \in \Sigma_2$ is said to be *finite* if there exists $k > 1$ such that $x_i = 0$ for every $i > k$. Observe that for every word ω there is a natural way associate a finite sequence to it by considering the sequence $\omega 0^{\infty}$. Also, given a finite sequence $\boldsymbol{x} \in \Sigma_2$, the *length of $\boldsymbol{x}$* is, by definition,the number of symbols it contains before the block 0^{∞} occurs in $\boldsymbol{x}$. For simplicity of notation, we denote the length of a finite sequence $\boldsymbol{x}$ by $\ell(\boldsymbol{x})$.

It is of our interest to introduce a natural dynamical system in Σ_2: *the full one sided shift*. Let $\sigma : \Sigma_2 \to \Sigma_2$ be given by $\sigma((x_i)_{i=1}^{\infty}) = (x_{i+1})_{i=1}^{\infty}$. With the topology given by d, σ is a continuous and two to one map. The dynamical system given by (Σ_2, σ) is called the *one sided full shift in two symbols* or simply the *full shift*.

Given $\boldsymbol{x} \in \Sigma_2$ and a word ω we say that ω *is a factor of in* $\boldsymbol{x}$ or ω *occurs in* $\boldsymbol{x}$, if there are coordinates i and j such that $\omega = x_i \dots x_j$. Note that the same definition holds if $\boldsymbol{x}$ is a finite word. For a finite sequence $\boldsymbol{x}$ with length n we say that 0^j occurs in $\boldsymbol{x}$ if and only if 0^j occurs in $x_1 \dots x_n$ and $j < n$.

Consider $\mathcal{F}$ to be a set of words. Let

$$\Sigma_{\mathcal{F}} = \{\boldsymbol{x} \in \Sigma_2 \mid \nu \text{ is not a factor of } \boldsymbol{x} \text{ for any } \nu \in \mathcal{F}\}.$$

We call $\mathcal{F}$ a *forbidden set of factors*. Observe that given a set of words $\mathcal{F}$, the set $\Sigma_{\mathcal{F}}$ is always a closed, forward σ-invariant set. Let $\sigma_{\mathcal{F}}$ denote $\sigma \mid_{\Sigma_{\mathcal{F}}}$. Therefore, $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is a dynamical system in its own right and $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is called a *subshift of* Σ_2 or simply a *subshift*.

Note that given any closed and forward σ-invariant set $A \subset \Sigma_2$ it is possible to consider the dynamical system given by (A, σ_A). It is shown in [BKT11, Theorem 6.1.21] that for every compact and forward σ-invariant set A, there always exists a set of forbidden factors such that $A = \Sigma_{\mathcal{F}}$. Moreover, subshifts are characterized by [LM95, Proposition 6.1.19] and [LM95, Theorem 6.1.21], i.e. (A, σ_A) is a subshift if and only if $A \subset \Sigma_2$ is a closed and forward σ-invariant set.

Let $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ be subshift. For every $n \in \mathbb{N}$, *the set of admissible words of length* n *of* $\Sigma_{\mathcal{F}}$ is given by:

$$B_n(\Sigma_{\mathcal{F}}) = \{\upsilon \in \{0,1\}^n \mid \upsilon \text{ is a factor of } \boldsymbol{x}, \text{ for } \boldsymbol{x} \in \Sigma_{\mathcal{F}}\}.$$

The *set of admissible words* or *the language of* $\Sigma_{\mathcal{F}}$, denoted by $\mathcal{L}(\Sigma_{\mathcal{F}})$, is defined to be

$$\mathcal{L}(\Sigma_{\mathcal{F}}) = \bigcup_{m=1}^{\infty} B_m(\Sigma_{\mathcal{F}}).$$

Given a subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$, *the topological entropy of* $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is defined by

$$h_{top}(\sigma_{\mathcal{F}}) = \lim_{n \to \infty} \frac{1}{n} \log |B_n(\Sigma_{\mathcal{F}})| ,$$

where log is considered to be $\log_2$. It is important to point out that topological entropy is a general notion for dynamical systems. We will not use the definition of topological entropy stated by Bowen (see [LM95, p. 190]). Nevertheless, both definitions are equivalent for subshifts [BS02, Proposition 3.1.1]. The advantage of using the provided definition lies in the fact that it depends only on words and their asymptotic growth.

A forbidden set of words $\mathcal{F}$ may be finite or infinite. Note that $\mathcal{F} = \emptyset$ for the full shift and $\mathcal{F}$ is at most countable for any other subshift. When $\mathcal{F}$ is finite, we say that $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is a *subshift of finite type*.

Given two dynamical systems (X, f) and (Y, g), a continuous function $h : X \to Y$ is said to be an *homomorphism between* (X, f) *and* (Y, g) if $h \circ f = g \circ h$. If h is injective we call h an *embedding*. If h is surjective, then h is known as a *factor map or semi-conjugacy*. In this case, we say that Y is a *factor of* X and we say that (X, f) and (Y, g) are *semi-conjugate*. If h is an homeomorphism, h is called a *topological conjugacy* and we say (X, f) and (Y, g) are *topologically conjugated* or *topologically equivalent*.

A subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is a *sofic shift* if $\Sigma_{\mathcal{F}}$ is a factor of a set $\Sigma \subset \Sigma_2$ and $(\Sigma, \sigma_{\Sigma})$ is subshift of finite type.

Recall that a dynamical system (X, f) is said to be *expansive* if there exists $\varepsilon > 0$ such that if

$$d(f^n(x), f^n(y)) < \varepsilon \text{ for all } n \geq 0 \text{ implies } x = y.$$

By the definition of the distance d on Σ_2, it is clear that (Σ_2, σ) and every subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ are expansive.

Let us introduce the following dynamical properties of subshifts:

Definition 2.1.1. A subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is *(topologically) transitive* or *irreducible* if for every ordered pair of words $\upsilon, \nu \in \mathcal{L}(\Sigma_{\mathcal{F}})$, there is $\omega \in \mathcal{L}(\Sigma_{\mathcal{F}})$, called *bridge*, such that $\upsilon\omega\nu \in \mathcal{L}(\Sigma_{\mathcal{F}})$.

Definition 2.1.2. We say that a subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is *topologically mixing* if for every ordered pair of words $\upsilon, \nu \in \mathcal{L}$, there exists $N \in \mathbb{N}$ such that for each $n \geq N$ there is a bridge $\omega \in B_n(\Sigma)$, such that $\upsilon\omega\nu \in \mathcal{L}(\Sigma)$.

Observe that every topologically mixing subshift is transitive, however not every transitive subshift is mixing. Recall that topological transitivity and topological mixing are invariant under semi-conjugacy.

We recall the following notion introduced by A. Bertrand-Mathis, which can be found in [Boy00, p.66].

Definition 2.1.3. A subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ has *the specification property* or simply *has specification* if there exists $m \in \mathbb{N}$ such that for any $\upsilon, \nu \in \mathcal{L}(\Sigma_{\mathcal{F}})$ there exists $\omega \in B_m(\Sigma_{\mathcal{F}})$ such that $\upsilon\omega\nu \in \mathcal{L}(\Sigma_{\mathcal{F}})$.

Roughly speaking, a subshift has specification if every two words υ and ν can be connected by a word ω of fixed length m. Note that specification property is also clearly preserved under semi-conjugacy. For the purpose of our study, we state the specification property for a subshift as follows: Let $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ be a subshift. Let m_n be given by

$$m_n = \inf\{k \mid \text{for every } \upsilon, \nu \in B_n(\Sigma_{\mathcal{F}}) \text{ there exists } \omega \in B_k(\Sigma_{\mathcal{F}}) \text{ such that } \upsilon\omega\nu \in \mathcal{L}(\Sigma_{\mathcal{F}})\}.$$

Then $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ has the specification property if and only if $\lim_{n\to\infty} m_n < \infty$. We call to $\max_{n\in\mathbb{N}} \{m_n\}_{n\in\mathbb{N}}$ *the specification number of* $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ and it is denoted by s.

Definition 2.1.4. A subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ has *the almost specification property* or simply $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ *has almost specification* if there exists $m \in \mathbb{N}$ such that for any $\upsilon, \nu \in \mathcal{L}(\Sigma_{\mathcal{F}})$ there exists $\omega \in \mathcal{L}(\Sigma_{\mathcal{F}})$ such that $\upsilon\omega\nu \in \mathcal{L}(\Sigma_{\mathcal{F}})$ and $\ell(\omega) \leq m$.

Thus, a subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ has almost specification if every two words υ and ν can be connected by a word ω of at most length m. It is worth pointing out that the specification property and the almost specification property are equivalent if $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is a topologically mixing subshift [Jun11, Lemma 3.1]. Also, Parry showed in [Par64a] that every transitive subshift of finite type has specification.

A subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is said to be a *coded system* if

$$\Sigma_{\mathcal{F}} = \overline{\bigcup_{n=1}^{\infty} \Sigma^n},$$

where each $(\Sigma^n, \sigma_{\Sigma^n})$ is a transitive subshift of finite type and $\Sigma^n \subset \Sigma^{n+1}$ for every $n \in \mathbb{N}$. Observe that in the provided definition the closure $\bigcup_{n=1}^{\infty} \Sigma^n$ corresponds to the closure in Σ_2. It is clear if $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is a coded system then $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is a transitive subshift. An equivalent formulation of a coded system (see, e.g. [LM95, p. 450]) is given by the following statement: A subshift $(\Sigma_{\mathcal{F}}, \sigma_{\sigma})$ is *coded* if there is a countable collection of words $C \subset \mathcal{L}(\Sigma_{\mathcal{F}})$ called *code*, such that $\Sigma_{\mathcal{F}}$ is the closure of the set of sequences obtained by freely concatenating the elements of C. The elements of C are called *generators* -see [CT12, FF01]. Both definitions are equivalent [FF01, Theorem 2.1].

As stated in [Boy00], we have the following list of implications for topologically mixing (transitive) subshifts:

$$\text{Shift of finite type} \Longrightarrow \text{Sofic} \Longrightarrow \text{Specification} \Longrightarrow \text{Almost Specification} \Longrightarrow \text{Coded.}$$

We emphasize that the stated definitions of topological transitivity, topological mixing and specification agree with the classical definitions of this notions for general discrete dynamical systems.

It is easy to show that (Σ_2, σ) is an expansive and topologically mixing dynamical system. Also (Σ_2, σ_2) has positive topological entropy and the specification property. Also, observe that the set of periodic points $Per(\sigma)$ is dense in Σ_2.

2.2 The doubling map and the Lexicographic World

Here and subsequently f will denote the *doubling map* defined as follows. Consider the unit circle S^1 as $S^1 = [0,1]/\mathbb{Z}$ and we set S^1 to have length 1. Also, if $a, b \in S^1$ with $a \neq b$ then by (a, b) we mean the open arc anticlockwise oriented from a to b We call (a, b) an open interval. Let $a, b \in S^1$ be given. We say that $a < b$ if $l((0, a)) < l((0, b))$ where l is the length of the segments (x, y) with $x, y \in S^1$. Let $f : S^1 \to S^1$ given by

$$f(x) = 2x \mod 1 = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}]; \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

The doubling map f satisfies the following properties:

1. f is continuous, surjective but not injective.

2. f is expansive.

3. f is topologically transitive.

4. The set of periodic points $\mathrm{Per}(f)$ is dense in S^1.

5. f is uniformly expanding (i.e. $|f'(x)| > 1$ for every $x \in S^1$).

6. $h_{top}(f)$ is positive.

Let $x \in [0,1]$. We say that a sequence $\boldsymbol{x} = (x_i)_{i=1}^{\infty}$ is a *binary expansion for* x if

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}.$$

The digit x_i is calculated by the following rule: $x_i = 0$ if $f^i(x) \in [0, \frac{1}{2})$ and $x_i = 1$ if $f^i(x) \in [\frac{1}{2}, 1)$.

Let $\pi : \Sigma_2 \to [0,1]$ be *the projection map* given by

$$\pi(\boldsymbol{x}) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

It is clear that π is a semi-conjugacy between f and σ. Observe that π is injective except for the set of sequences ending with 0^∞ or 1^∞. Since f is defined in S^1, we will identify sequences $\boldsymbol{x}, \boldsymbol{y} \in \Sigma_2$ such that $\boldsymbol{x}$ ends with 0^∞, $\boldsymbol{y}$ ends with 1^∞ and $x_1 \dots x_k = y_1 \dots y_k$, $x_{k+1} = 1$, $y_{k+1} = 0$ and $x_i = 0$ for every $i > k+1$ and $y_j = 1$ for every $j > k+1$. Also, π is injective on the complement of this set. Moreover, π is a bi-Lipschitz map where $c_1 = \frac{1}{2}$ and $c_2 = 1$ are the Lipschitz constants for π and π^{-1} respectively.

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} \in \Sigma_2$. We say that $\boldsymbol{\alpha}$ *lexicographically less than* $\boldsymbol{\beta}$, denoted by $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ if there exists $k \in \mathbb{N}$ such that $a_j = b_j$ for $i < k$ and $a_k < b_k$. The lexicographic order $\prec$ on Σ_2 is a total order on Σ_2. Moreover, the lexicographic order is preserved under the projection map π, i.e. $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ if and only if $\pi(\boldsymbol{\alpha}) < \pi(\boldsymbol{\beta})$. Also, it is possible to transfer this order to $B_n(\Sigma)$. In this case $k \leq n$. If $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$, the *lexicographic open interval from* $\boldsymbol{\alpha}$ *to* $\boldsymbol{\beta}$, denoted by $(\boldsymbol{\alpha}, \boldsymbol{\beta})_\prec$ is given by

$$(\boldsymbol{\alpha}, \boldsymbol{\beta})_\prec = \{\boldsymbol{x} \in \Sigma_2 \mid \boldsymbol{\alpha} \prec \boldsymbol{x} \prec \boldsymbol{\beta}\}.$$

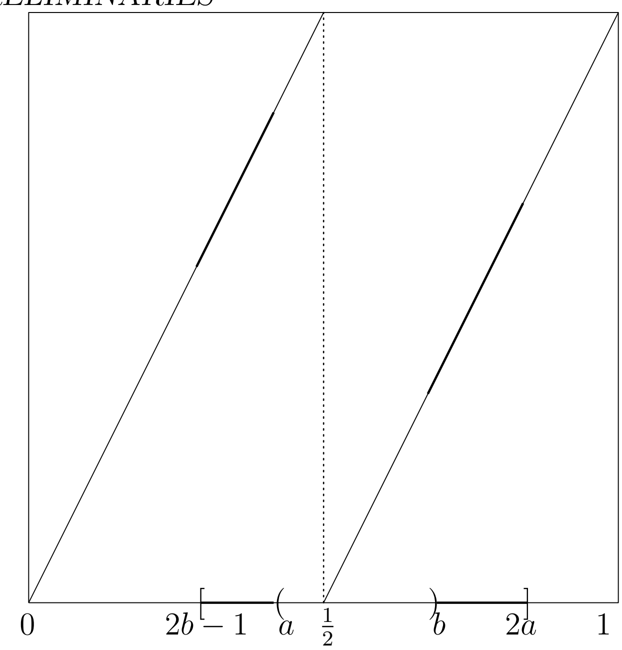

Figure 2.1: The doubling map f with a hole (a,b)

Similarly, we also consider the *lexicographic closed interval from* $\boldsymbol{\alpha}$ *to* $\boldsymbol{\beta}$, denoted by $[\boldsymbol{\alpha},\boldsymbol{\beta}]_\prec$ by changing $\prec$ for $\preccurlyeq$.

Given $\boldsymbol{x}\in\Sigma_2$, *the mirror image of* $\boldsymbol{x}$, denoted by $\bar{\boldsymbol{x}}$, is defined to be the sequence $\bar{\boldsymbol{x}}=(1-x_i)_{i=1}^\infty$, and $1-x_i$ is denoted by $\bar{x}_i$.

Now we introduce one the main objects of our research, the *attractor of an open dynamical system corresponding to* (a,b), $(X_{(a,b)},f_{(a,b)})$ and the *lexicographic world.* Firstly, consider $(a,b)\subset S^1$ such that $\frac{1}{2}\in(a,b)$ and let $(X_{(a,b)},f_{(a,b)})$ be the corresponding open dynamical system. Note $f(a)=2a$ and $f(b)=2b-1$. Then, for every $x\in[0,a]$, $f(x)\in[0,2a]$. Also, if $x\in[b,1]$ then $f(x)\in[2b-1,1]$. Thus, $[2b-1,2a]\cap X_{(a,b)}$ is the *attractor of* $f_{(a,b)}$ and it is denoted by $\Lambda_{(a,b)}$. Observe that $\Lambda_{(a,b)}\supseteq\Lambda_{(a',b')}$ if $(a,b)\subseteq(a',b')$.

We introduce to the reader the notion of lexicographic subshift and the lexicographic world. Consider $\Sigma_2^1=\{\boldsymbol{x}\in\Sigma_2\mid x_1=1\}$ and $\Sigma_2^0=\{\boldsymbol{x}\in\Sigma_2\mid x_1=0\}$. Note that for every $\boldsymbol{x}\in\Sigma_2^0$, $\bar{\boldsymbol{x}}\in\Sigma_2^1$ and vice versa. One of the main ideas of this work is to associate to an interval $(a,b)\subset S^1$ a pair $(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Sigma_2^1\times\Sigma_2^0$. Such association is to associate to a the sequence $\boldsymbol{\alpha}=\pi^{-1}(2a)$ and to b the sequence $\boldsymbol{\beta}=\pi^{-1}(2b-1)$.

Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Sigma_2^1 \times \Sigma_2^0$. Consider the set $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ given by:

$$\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \{\boldsymbol{x} \in \Sigma_2 \mid \boldsymbol{\beta} \preccurlyeq \sigma^n(\boldsymbol{x}) \preccurlyeq \boldsymbol{\alpha} \text{ for every } n \geq 0\}.$$

It is clear that $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ is a closed and forward σ-invariant subset of Σ_2. We denote by $\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ the map $\sigma\mid_{\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}}$.

Definition 2.2.1. The subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is called *the* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$*-lexicographic subshift* or the *lexicographic subshift corresponding to the pair* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

From the definitions of the lexicographic order and lexicographic intervals it is clear that if $(\boldsymbol{\alpha}, \boldsymbol{\beta}), (\boldsymbol{\alpha}', \boldsymbol{\beta}') \in \Sigma_2^1 \times \Sigma_2^0$ satisfy that $\boldsymbol{\alpha}' \preccurlyeq \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \preccurlyeq \boldsymbol{\beta}'$ then $[\boldsymbol{\beta}, \boldsymbol{\alpha}]_\prec \subset [\boldsymbol{\beta}', \boldsymbol{\alpha}']_\prec$ and $\Sigma_{(\boldsymbol{\alpha}',\boldsymbol{\beta}')} \subset \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$.

We will show in Theorems 3.2.2 and 3.2.3 that the dynamics on every attractor $\Lambda_{(a,b)}$ is topologically conjugated to a lexicographic subshift $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$.

The main idea of our study is to undestand the dynamical properties of $(\Lambda_{(a,b)}, f_{(a,b)})$ using the symbolic properties of a and b as it was suggested by Bundfuss et. al. in [BKT11]. In particular, the dynamics of $(\Lambda_{(a,b)}, \sigma_{(a,b)})$ are topologically conjugated to a subshift of finite type if a and b fall into the hole (a, b) [BKT11, Proposition 4.1],[BDJ14, Theorem 2.4]. We describe symbolically the properties of the boundary points of a hole (a, b) containing $\frac{1}{2}$ as follows:

A sequence $\boldsymbol{\alpha} \in \Sigma_2^1$ satisfying $\sigma^n(\boldsymbol{\alpha}) \preccurlyeq \boldsymbol{\alpha}$ for every $n \geq 0$ is called a *Parry sequence*. Set

$$P = \{\boldsymbol{\alpha} \in \Sigma_2^1 \mid \boldsymbol{\alpha} \text{ is a Parry sequence}\},$$

and let

$$\bar{P} = \{\boldsymbol{\beta} \in \Sigma_2^0 \mid \bar{\boldsymbol{\beta}} \in P\}.$$

Observe that $\bar{P}$ coincides with the set M previously introduced by Nilsson in [Nil09, p. 105]. Hence, as a consequence of [Nil09, Theorems 3.7, 3.8], $\pi(P)$ and $\pi(\bar{P})$ are sets of Lebesgue measure zero with $\dim_H \pi(P) = \dim_H \pi(\bar{P}) = 1$ where $\dim_H$ is the Hausdorff dimension.

Let $\boldsymbol{\alpha} \in P$ and $a = \pi(\boldsymbol{\alpha})$. Note that $f^n(a) \notin (a, 1]$ for every $n \geq 0$. Similarly, for every $\boldsymbol{\beta} \in \bar{P}$ and $b = \pi(\beta)$, $f^n(b) \notin [0, b)$. Thus, Parry sequences and their mirror images model points that never fall into a particular interval. With this idea in mind we introduce the notion of *extremal pair*. Roughy speaking, extremal pairs are the symbolic description of the end points of an intervals (a, b) such that at least one of the points lying on its boundary does not avoid (a, b) under iteration.

Definition 2.2.2. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Sigma_2^1 \times \Sigma_2^0$ such that:

i) $\boldsymbol{\alpha} \in P$ and $\boldsymbol{\beta} \in \bar{P}$;

ii) $\sigma^n(\boldsymbol{\alpha}) \not\preccurlyeq \boldsymbol{\beta}$ and $\sigma^n(\boldsymbol{\beta}) \not\succcurlyeq \boldsymbol{\alpha}$ for every $n \in \mathbb{N}$.

A pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Sigma_2^1 \times \Sigma_2^0$ is said to be *extremal* if $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ does not satisfy *i)* or *ii)*.

Note that Definition 2.2.2 also describes symbollically the intervals (a, b) neither a nor b fall into (a, b) under iteration. The family of non-extremal pairs will model the dynamical systems and is called *the lexicographic world* $\mathcal{LW}$.

Definition 2.2.3. The parameter family

$$\mathcal{LW} = \{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in P \times \bar{P} \mid (\boldsymbol{\alpha}, \boldsymbol{\beta}) \text{ is not extremal and } \Sigma_{\boldsymbol{\alpha},\boldsymbol{\beta}} \neq \emptyset\}$$

is called *the lexicographic world* [AG09, Gan01].

Observe that if $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$, then neither $\boldsymbol{\alpha}$ nor $\boldsymbol{\beta}$ can have arbitrarily long strings of 0's or 1's unless $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is given by the pair $\boldsymbol{\alpha} = 1^\infty$ and $\boldsymbol{\beta} = 0^\infty$. In this case $(\Sigma_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \sigma_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = (\Sigma_2, \sigma)$.

Given a sequence $\boldsymbol{\alpha} \in \Sigma_2$, consider

$$0_{\boldsymbol{\alpha}} = \max\{n \in \mathbb{N} \mid 0^n \text{ is a factor of } \boldsymbol{\alpha}\},$$

and

$$1_{\boldsymbol{\alpha}} = \max\{n \in \mathbb{N} \mid 1^n \text{ is a factor of } \boldsymbol{\alpha}\}.$$

Observe that $0_{\boldsymbol{\alpha}}$ and $1_{\boldsymbol{\alpha}}$ are well defined if $\boldsymbol{\alpha} \in (P \cup \bar{P}) \setminus \{0^\infty, 1^\infty\}$. Moreover, it is clear that for every $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$, $0_{\boldsymbol{\alpha}} \leq 0_{\boldsymbol{\beta}}$ and $1_{\boldsymbol{\beta}} \leq 1_{\boldsymbol{\alpha}}$.

Now we introduce the families of lexicographic subshifts that are relevant in our study.

Definition 2.2.4. A lexicographic subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is said to be *symmetric* if for every $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, $\bar{\boldsymbol{x}} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ and we say that $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is *asymmetric* if $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is not symmetric.

In Chapter 4, we will study the following family of lexicographic subshifts.

Definition 2.2.5. Let $\boldsymbol{\alpha} \in \Sigma_2^1$. Consider

$$\Sigma_{\boldsymbol{\alpha}} = \{\boldsymbol{x} \in \Sigma_2 \mid \bar{\boldsymbol{\alpha}} \preccurlyeq \sigma^n(\boldsymbol{x}) \preccurlyeq \boldsymbol{\alpha} \text{ for every } n \geq 0\}$$

and $\sigma_{\boldsymbol{\alpha}} = \sigma \mid_{\Sigma_{\boldsymbol{\alpha}}}$. The subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ are called *the symmetric subshift parametrised by* $\boldsymbol{\alpha}$.

It is easy to see that the subshifts given in Definition 2.2.5 are effectively symmetric subshifts. This is shown in the following proposition.

Proposition 2.2.6. *Let* $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ *such that* $(10)^\infty \preccurlyeq \boldsymbol{\alpha}$ *and* $(01)^\infty \succcurlyeq \boldsymbol{\beta}$*. Then, a subshift* $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ *is symmetric if and only if* $\boldsymbol{\beta} = \bar{\boldsymbol{\alpha}}$*.*

Proof. Let $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ be a symmetric subshift. Then, from Definition 2.2.4, $\bar{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\beta}} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Then, $\boldsymbol{\beta} \preccurlyeq \bar{\boldsymbol{\alpha}} \preccurlyeq \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \preccurlyeq \bar{\boldsymbol{\beta}} \preccurlyeq \boldsymbol{\alpha}$ and since $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ then $\boldsymbol{\beta} \preccurlyeq \bar{\boldsymbol{\alpha}} \prec \bar{\boldsymbol{\beta}} \preccurlyeq \boldsymbol{\alpha}$. Assume that $\bar{\boldsymbol{\alpha}} \neq \boldsymbol{\beta}$. Then, there exists $j \in \mathbb{N}$ such that $b_j = 0$ and $\bar{a}_j = 1$. This implies that $\bar{\bar{\boldsymbol{\alpha}}} \prec a$ which is a contradiction. The other implication is clear. □

2.3 Combinatorics on words

It is necessary to introduce some tools from combinatorics on words in order to aid our study. In particularly, we use these tools to explain the set of parameters studied in this work (see Section 3.1).

Let ω be a word. We denote by 0-$\max_\omega$ to the lexicographically largest cyclic permutation of ω starting with 0 and 1-$\min_\omega$ to the lexicographically smallest cyclic permutation of ω starting with 1. Also, $\max_\omega$ denotes the lexicographically largest cyclic permutation of ω and $\min_\omega$ denotes the lexicographically smallest cyclic permutation of ω. It is clear that $\sigma(1\text{-}\min_\omega^\infty) = \min_\omega^\infty$ and $\sigma(0\text{-}\max_\omega^\infty) = \max_\omega^\infty$.

Proposition 2.3.1. *Let* ω *be a word such that* $\omega \neq 1^n$ *and* $\omega \neq 0^m$ *for any* $n, m \in \mathbb{N}$*. Then* $\max_\omega$ *ends with* 0 *and* $\min_\omega$ *ends with* 1.

Proof. If $\max_\omega$ ends with 1 then $1\max_{\omega_1}, \ldots \text{-}\max_{\omega_{\ell(\omega)-1}} \succ \max_\omega$ which contradicts the maximality of $\max_\omega$. Therefore, $\max_\omega$ ends with 0. The proof of $\min_\omega$ ending with 1 is similar. □

Proposition 2.3.2. *Let ω be a word such that $\omega \neq 1^n$ and $\omega \neq 0^m$ for any $n, m \in \mathbb{N}$. Then there exist words u and v such that $u_1 = 0$, $v_1 = 1$, $0\text{-}\max_\omega^\infty = (uv)^\infty$ and $1\text{-}\min_\omega^\infty = (vu)^\infty$*

Proof. Let ω be a word. Observe that $(0\text{-}\max_\omega)^\infty$ and $(1\text{-}\min_\omega)^\infty$ are cyclic permutations of each other. Then, there exist $n < \ell(\omega)$ such that $\sigma^n(0\text{-}\max_\omega^\infty) = (1\text{-}\min_\omega)^\infty$. Then $u = 0\text{-}\max_{\omega 1} \ldots 0\text{-}\max_{\omega n-1}$ and $v = 0\text{-}\max_{\omega n} \ldots 0\text{-}\max_{\omega \ell(\omega)}$. □

2.4 Ergodic Theory

Let X be a compact metric space. Let us denote by $\mathbb{P}(X)$ the set of all probability measures μ defined on the Borel σ-algebra, $\mathcal{B}(X)$. Let (X, f) be a dynamical system. A measure $\mu \in \mathbb{P}(X)$ is said to be *f-invariant* if for every measurable set A, $\mu(f^{-1}(A)) = \mu(A)$ and let

$$\mathbb{M}(f) = \{\mu \in \mathbb{P}(M) \mid \mu \text{ is an } f\text{-invariant measure}\}.$$

$\mathbb{M}(f)$ is a non empty, compact space with the weak*-topology [Wal82, Chapter 6]. We say that $\mu \in \mathbb{M}(f)$ is *ergodic* if for every f-invariant set A, $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Let α be a countable partition of X and $\mu \in \mathbb{M}(f)$. The *entropy of μ relative to a partition α*, denoted by $h_\mu(f, \alpha)$, is defined to be

$$h_\mu(f, \alpha) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} f^{-j}(\alpha)\right),$$

where

$$H(\alpha) = -\sum_{A\in\alpha} \mu(A) \log \mu(A).$$

Then, we define *the measure theoretical entropy of f with respect to μ* or *entropy of μ* to be

$$h_\mu(f) = \sup\{h_\mu(f, \alpha) \mid \alpha \text{ is a countable partition of } X\}.$$

The relationship between the topological entropy and the measure theoretical entropy of a dynamical system (X, f) is established by the following theorem, whose proof can be consulted in [Wal82, Theorem 8.6].

Theorem 2.4.1. *Let (X, f) be a dynamical system. Then*

$$h_{top}(f) = \sup\{h_\mu(f) \mid \mu \text{ is an invariant measure}\}.$$

Theorem 2.4.1 is known as *the variational principle.*

Intrinsic Ergodicity and Classical Results

We are interested in a particular set of measures for a dynamical system (X, f) called *measures of maximal entropy of* (X, f). An invariant measure μ of a dynamical system (X, f) is a *measure of maximal entropy* if $h_{top}(f) = h_\mu(f)$.

Definition 2.4.2. A dynamical system (X, f) is *intrinsically ergodic* if (X, f) has a unique measure of maximal entropy.

Historically, an important problem in Ergodic Theory is to determine which dynamical systems are intrinsically ergodic - see e.g. [Bow75, Bow08, Gur72, Gur80, Hof79, Par64a, Wei70, Wei73]. Recall that if μ is a measure of maximal entropy for a dynamical system (X, f) then μ is ergodic [Mañ83, Proposition 7.7]. Furthermore, if (X, f) is an expanding dynamical system then we can be sure about the existence of measures of maximal entropy for (X, f) [Wal82, Theorem 8.2, Theorem 8.7].

Since every lexicographic subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is expansive then measures of maximal entropy always exist for $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Therefore, as far as we are concerned, the main problem is to show the uniqueness of such a measure for lexicographic subshifts.

An explicit expression for a measure of maximal entropy for a lexicographic subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is given by considering the proof of [Wal82, Theorem 8.6 (2)]. Consider a lexicographic subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Let $\{\mu_n\}_{n=1}^{\infty}$ be the sequence of measures defined by

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1}(\sigma^j_{(\boldsymbol{\alpha},\boldsymbol{\beta})})^*(\nu_n),$$

where

$$\nu_n = \frac{1}{|Per_n(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})|} \sum_{\boldsymbol{x} \in Per_n(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})} \delta_{\boldsymbol{x}}.$$

Since $\mathbb{M}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is compact in the weak*-topology there is a convergent subsequence $\{\mu_{n_j}\}_{j=1}^{\infty}$ of $\{\mu_n\}_{n=1}^{\infty}$. Let μ be the weak* limit such a subsequence. The measure μ is know as a *Parry measure of* $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$.

We can now formulate the classical results used to study the ergodic properties of lexicographic subshifts. The following theorem summarise these results.

Theorem 2.4.3. *Let* $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ *be a subshift. Then* $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ *is intrinsically ergodic if:*

i) $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ *is a transitive subshift of finite type* [Par64a];

ii) $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ *is a transitive sofic subshift* [Wei70, Wei73];

iii) $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ *has the specification property* [Bow75].

It is worth pointing out that intrinsic ergodicity does not follow neither from the topological transitivity nor topological mixing of a subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ - see, e.g. [Gur72], [Hay13], [Pet86].

Chapter 3

The doubling map with holes. General results

During this chapter we formalise the relation between $(\Lambda_{(a,b)}, f_{(a,b)})$ and the lexicographic world $\mathcal{LW}$.

Firstly, in Section 3.1, we mention some results obtained by Glendinning and Sidorov in [GS14] and by Hare and Sidorov in [HS14]. These results are useful since they define the set of parameters where our study is developed.

In Section 3.2 we show in Theorem 3.2.2 that for every $(\Lambda_{(a,b)}, f_{(a,b)})$ such that $(a, b) \in D_1$, where D_1 is the set of parameters $(a, b) \in \left(\frac{1}{4}, \frac{1}{2}\right) \times \left(\frac{1}{2}, \frac{3}{4}\right)$ where $X_{(a,b)}$ is uncountable, there is a lexicographic subshift which describes the dynamics of $(\Lambda_{(a,b)}, f_{(a,b)})$. However it is clear that the binary expansions of $2a$ and $2b - 1$ can be extremal pairs.

In Section 3.3 we deal with this problem. In Lemma 3.3.10 we describe how to associate an element $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ to a pair $(a, b) \in D_1$ such that both a and b fall into (a, b) under iteration. Also, in Lemma 3.3.13 we do the same for pairs $(a, b) \in D_1$ such that either a or b but not both fall into (a, b) under iteration. The idea that we followed was to use directly the symbolic properties of the binary expansion of $2a$ and $2b - 1$. Firstly, we find a continuous way to associate every (a, b) an element $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in P \times \bar{P}$ where P is the set of Parry sequences -see Section 2.2- and then we construct the function I which associate $P \times \bar{P}$ to $\mathcal{LW}$ -see Theorem 3.3.16.

Finally, in Section 3.4 we show that the set of parameters which generates a subshift of finite type are generic. Also, we show that the function that associates to each $(a, b) \in D_1$ the topological entropy of $\Lambda_{(a,b)}$ is locally constant almost everywhere. In contrast to Section 3.3 in Section 3.4 we will use the properties of the doubling map directly rather than work with symbolic dynamics.

3.1 Parameter space

Let $R = (\frac{1}{4}, \frac{1}{2}) \times (\frac{1}{2}, \frac{3}{4})$. We confine ourselves to studying open dynamical systems parametrised by $(a, b) \in R$ based on the following results.

Lemma 3.1.1. [GS14, Lemma 1.1] *Let $a, b \in S^1$ with $a < b$. Then:*

i) If $0 < a < \frac{1}{4}$ and $\frac{1}{2} < b < 1$, or $0 < a < \frac{1}{2}$ and $\frac{3}{4} < b < 1$ then $X_{(a,b)} = \{0\}$.

ii) If $\frac{1}{2} < a < 1$ or $0 < b < \frac{1}{2}$, then $\dim_H X_{(a,b)} > 0$.

Note that if $a \in (\frac{1}{2}, 1)$ and $b = 1$ or $a = 0$ and $b \in (0, \frac{1}{2})$ then $(X_{(a,b)}, f_{(a,b)})$ is topologically conjugated to a β-shift -see Chapter 6. As a direct consequence of this observation, it is possible to obtain a different proof of Lemma 3.1.1 *ii)*.

Let $\phi : (\frac{1}{4}, \frac{1}{2}) \to (\frac{1}{2}, \frac{3}{4})$ and $\chi : (\frac{1}{4}, \frac{1}{2}) \to (\frac{1}{2}, \frac{3}{4})$ be defined by

i) $\phi(a) = \sup\{b \in S^1 \mid X_{(a,b)} \neq \{0\}\}$;

ii) $\chi(a) = \sup\{b \in S^1 \mid X_{(a,b)} \text{ is uncountable}\}$.

The functions ϕ and χ were introduced in [GS14] and [LM06]. It is clear that $\chi(a) \leq \phi(a)$. Moreover, in [GS14, Theorem 2.13] the authors give an explicit formula to calculate them. It is worth pointing out that both ϕ and χ were studied symbolically and that f was considered as a transformation of the unit interval. Nonetheless, the results remain valid for f defined in S^1 if $0 \notin (a, b)$.

It is not our purpose to study ϕ and χ. Nonetheless, they determine the parameter space in question. We denote $D_0 = \{(a, b) \in R \mid b \leq \phi(a)\}$, and $D_1 = \{(a, b) \in R \mid b \leq \chi(a)\}$.

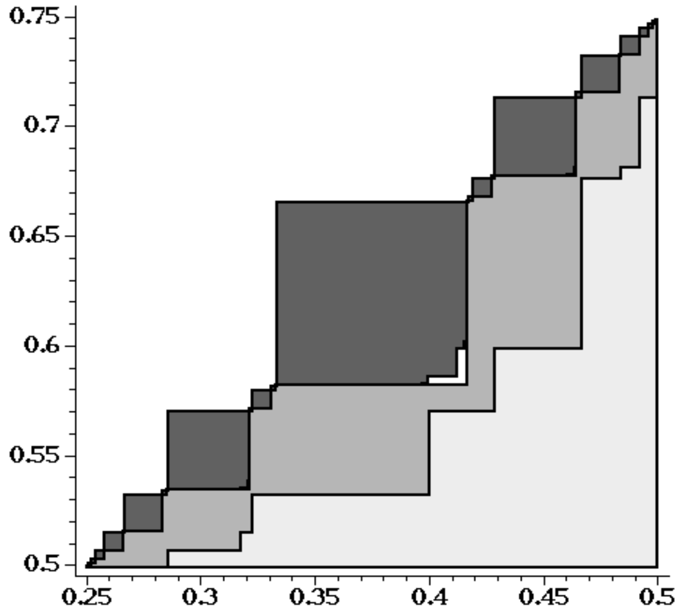

Figure 3.1: The sets D_3 (light grey), D_2 (light grey + grey), D_1 (light grey + grey + white), and D_0 (light grey + grey + white + dark grey) as in [HS14]

Now, we introduce the following definitions as in [HS14]. Let $(a, b) \in D_0$. We call $n \geq 3$ *bad for* (a, b) if every periodic orbit of period n of f intersects (a, b). Let us denote by $B(a, b) = \{n \geq 3 \mid n \text{ is bad for } (a, b)\}$. Let D_2 be given by

$$D_2 = \{(a, b) \in R \mid B(a, b) \text{ is finite } \}$$

and let D_3 be defined by

$$D_3 = \{(a, b) \in R \mid B(a, b) = \emptyset\}.$$

Hare and Sidorov in [HS14] shown that

$$D_3 \subset D_2 \subset D_1 \subset D_0$$

-see figure 3.1. Since

$$\{(a, b) \in R \mid h_{top}(f_{(a,b)}) > 0\} = D_1 \setminus \partial D_1$$

[LM06, Theorem 2] and $X_{(a,b)}$ is countable for $(a, b) \in D_0 \setminus D_1$, we will restrict ourselves studying the pairs $(a, b) \in D_1$. Moreover, in [GS14, Lemma 1.1, Theorem 2.16], the authors characterise ∂D_1 in terms of χ.

3.2 Attractors and the lexicographic world

We start this section with the following claim. Essentially, Proposition 3.2.1 allows us to fix a parameter $a \in (\frac{1}{4}, \frac{1}{2})$ and move the parameter $b \in (\frac{1}{2}, \frac{3}{4})$ and vice versa.

Proposition 3.2.1. *Let $(a,b) \in D_1$ such that $b \leq 1-a$. Then $\Lambda_{(a,b)}$ is topologically conjugated to $\Lambda_{(1-b,1-a)}$.*

Proof. Let $h : S^1 \to S^1$ given by $h(x) = 1-x$. Observe that h is a homeomorphism of the unit circle that reverses orientation. Moreover,

$$h \circ f = f \circ h = \begin{cases} 1-2x & \text{if} \quad x \in [\frac{1}{2}, 1] \\ 2-2x & \text{if} \quad x \in [0, \frac{1}{2}) \end{cases}$$

Let $x \in \Lambda_{(a,b)}$. Then, $2b-1 \leq x \leq 2a$. Note that $1-2a \leq h(x) \leq 2-2b$. Also, since $x \notin (a,b)$, $h(x) \notin (1-b, 1-a)$. Then $h(\Lambda_{(a,b)}) = \Lambda_{(1-b,1-a)}$, and our result follows. □

The following theorem gives a lexicographic characterisation of $\Lambda_{(a,b)}$.

Theorem 3.2.2. *Let $(a,b) \in D_1$. Then*

$$\pi^{-1}(\Lambda_{(a,b)}) = \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$$

where $\boldsymbol{\alpha} = \pi^{-1}(2a)$ and $\boldsymbol{\beta} = \pi^{-1}(2b-1)$.

Proof. Let $\boldsymbol{x} \in \pi^{-1}(\Lambda_{(a,b)})$. Note that $\pi(\boldsymbol{x}) \in \Lambda_{(a,b)}$, i.e.

$$\pi(\boldsymbol{x}) \in [2b-1, 2a] \cap (X_{(a,b)}).$$

Then, $f^n(\pi(\boldsymbol{x})) \notin (a,b)$ and $2b-1 < f^n(\pi(\boldsymbol{x})) < 2a$ for every $n \geq 0$. By substituting $f^n(\pi(\boldsymbol{x}))$ by $\pi(\sigma^n(\boldsymbol{x}))$ it is clear that $\sigma^n(\boldsymbol{x}) \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ for every $n \geq 0$, which implies that

$$\pi^{-1}(\Lambda_{(a,b)}) \subset \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}.$$

Now, consider $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Then $2b-1 < \pi(\sigma^n(\boldsymbol{x})) < 2a$ for every $n \geq 0$. This implies that $f^n(\pi(\boldsymbol{x})) \in [2b-1, 2a]$ for every $n \geq 0$. Suppose that $\pi(\boldsymbol{x}) \notin \Lambda_{(a,b)}$. Then, there exists $n \geq 0$ such that $f^n(\pi(\boldsymbol{x})) \in (a,b)$. Recall that $(a,b) = (a, \frac{1}{2}) \cup [\frac{1}{2}, b)$. Suppose that $f^n(\pi(\boldsymbol{x})) \in (a, \frac{1}{2})$ then $f^{n+1}(\pi(\boldsymbol{x})) \in (2a, 1)$. Therefore $\sigma^{n+1}(\boldsymbol{x})) \succcurlyeq \boldsymbol{\alpha}$ which contradicts our assumption on $\boldsymbol{x}$. If $f^n(\pi(\boldsymbol{x})) \in [\frac{1}{2}, b)$ then $f^{n+1}(\pi(\boldsymbol{x})) \in [0, 2b-1)$. Therefore, $\sigma^{n+1}(\boldsymbol{x})) \preccurlyeq \boldsymbol{\beta}$ which, again, contradicts our assumption on $\boldsymbol{x}$. □

Observe that the pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ where $\boldsymbol{\alpha} = \pi^{-1}(2a)$ and $\boldsymbol{\beta} = \pi^{-1}(2b-1)$ might be extremal. Nonetheless, as a consequence of Theorem 3.2.2 the following theorem is true.

Theorem 3.2.3. *For every $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ there exists $(a, b) \in D_1$ such that $(\Lambda_{(a,b)}, f_{(a,b)})$ is topologically conjugated to $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$.*

3.3 Associating subshifts to extremal pairs

Note that Theorem 3.2.2 implies that for every $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$, $\pi^{-1}(\Lambda_{(\pi(0a),\pi(1b))}) = \Sigma_{(a,b)}$. The rest of the section is devoted to showing that for every $(a, b) \in D_1$ there is a unique element $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ such that $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \pi^{-1}(\Lambda_{(a,b)})$. Observe that Lemma 3.4.1 implies that for almost every $(a, b) \in D_1$, the pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ where $\boldsymbol{\alpha} = \pi^{-1}(2a)$ and $\boldsymbol{\beta} = \sigma(\pi^{-1}(2b-1)))$ is extremal. Geometrically, extremal pairs are elements of $\Sigma_2^1 \times \Sigma_2^0$ corresponding under projection to the end points of $(a, b) \in S$ or $(a, b) \in D_1 \setminus S$ such that there exist $n(a), m(b) \in \mathbb{N}$ such that either $f^{n(a)}(a) \in (a, b)$, or $f^{m(b)}(b) \in (a, b)$.

To every $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Sigma_2^1 \times \Sigma_2^0$, it is natural to associate the pair $(\pi(0\boldsymbol{\alpha}), \pi(1\boldsymbol{\beta})) \in (\frac{1}{4}, \frac{1}{2}) \times (\frac{1}{2}, \frac{3}{4})$. It is clear that every $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in (\Sigma_2^1 \times \Sigma_2^0) \setminus \mathcal{LW}$ is a extremal pair (Definition 2.2.2).

In [LSS14, Theorem 1.3] it is shown that if $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ satisfies that $\left(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right)$ is a subshift of finite type then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are periodic sequences and vice versa. Taking this fact into account, a suitable way is needed to associate an element of the lexicographic world to every $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Sigma_2^1 \times \Sigma_2^0$. The idea is to find a pair of periodic sequences $(\boldsymbol{\alpha}', \boldsymbol{\beta}') \in \mathcal{LW}$ which reflects faithfully the dynamical behaviour of the shift associated to the extremal pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Firstly, we show that every $(a, b) \in D_1$ can be represented by an element of $P \times \bar{P}$ where P is the set of Parry sequences - see Section 2.2.

Proposition 3.3.1. *The set $P \cap Per(\sigma)$ is dense in P.*

Proof. Let $\boldsymbol{\alpha} \in P$ and $\varepsilon > 0$. Firstly, note that if $\boldsymbol{\alpha}$ is a finite sequence, then the sequence $\boldsymbol{\alpha}' = (a_1 \ldots a_{\ell(a)} 0^{[\frac{1}{\varepsilon}]+1})^\infty$ is a Parry sequence with $d(\boldsymbol{\alpha}, \boldsymbol{\alpha}') < \varepsilon$. Assume that $\boldsymbol{\alpha}$ is not finite. Without losing generality we can assume that $\boldsymbol{\alpha} \notin P \cap Per(\sigma)$.

Consider $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$, $a_n = 0$ and $a_{n+1} = 1$. Consider the sequence $\boldsymbol{\alpha}' = (a_1 \dots a_n)^\infty$. Observe that $d(\boldsymbol{\alpha}, \boldsymbol{\alpha}') \leq \frac{1}{2^n} < \varepsilon$ and $\boldsymbol{\alpha}'$. Thus, it suffices to show that $\boldsymbol{\alpha}' \in P$. Assume, on the contrary that $\boldsymbol{\alpha}' \notin P$. Then, there exists $m \in \mathbb{N}$ such that $\sigma^m(\boldsymbol{\alpha}') \succ \boldsymbol{\alpha}'$. Since $\boldsymbol{\alpha}'$ is periodic, we can be sure that $m \in \{1, \dots, n\}$. This implies that $\sigma^m(\boldsymbol{\alpha}') \succ \boldsymbol{\alpha}'$. Therefore, $\sigma^m(\boldsymbol{\alpha}) \succ \boldsymbol{\alpha}$ which is a contradiction. Therefore $\boldsymbol{\alpha}' \in P$ and hence $P \cap Per(\sigma)$ is dense in P. □

Let $N = \Sigma_2 \setminus (P \cup \bar{P})$. As we observed previously, $\pi(N)$ has full Lebesgue measure.

Proposition 3.3.2. *N is an open and dense subset of Σ_2.*

Proof. Firstly, we show that N is an open subset of Σ_2. Let $\boldsymbol{x} \in N$ such that $\boldsymbol{x} \in \Sigma_2^1$. Note that, $\boldsymbol{x} \in [1^n, 1^{n+1}]_\prec$ for some $n \in \mathbb{N}$. Let $j = \min\{i \in \mathbb{N} \mid \sigma^i(\boldsymbol{x}) \succ \boldsymbol{x}\}$ and $n_j = \min\{n \in \mathbb{N} \mid \sigma^j(\boldsymbol{x})_n \succ x_{j+n}\}$. Consider the set

$$N(\boldsymbol{x}) = \{\boldsymbol{y} \in \Sigma_2 \mid y_i = x_i \text{ for every } 1 \leq i \leq n_j\}.$$

Observe that $N(\boldsymbol{x}) \subset N$ and $\boldsymbol{x} \in N(\boldsymbol{x})$. Let us show that $N(\boldsymbol{x})$ is open. Observe that for every $\boldsymbol{\alpha} \in N(\boldsymbol{x})$ $d(\boldsymbol{x}, \boldsymbol{\alpha}) < \frac{1}{2^{n_j+1}}$. Let $\varepsilon > 0$ sufficiently small, that is $\varepsilon < \frac{1}{2^{n_j+1}}$. Then, there exists $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \varepsilon < \frac{1}{2^{n_j+1}}$. Let $N_\varepsilon(\boldsymbol{x}) = \{\boldsymbol{y} \in N(\boldsymbol{x}) \mid y_i = x_i \text{ for every } 1 \leq i \leq k\}$. Then, $d(\boldsymbol{x}, \boldsymbol{y}) < \varepsilon$ for every $\boldsymbol{y} \in N_\varepsilon(\boldsymbol{x})$. Thus, $N(\boldsymbol{x})$ is an open set. Observe for $\boldsymbol{x}$ such that $x \in \Sigma_2^0 \cap N$ we can do a simmilar construction. Then we can be sure that N is an open set.

In order to show that N is dense, let $\boldsymbol{x} \in \Sigma_2^1$ and $\varepsilon > 0$. Observe that $\boldsymbol{x} \in [1^n, 1^{n+1}]_\prec$ for some $n \in \mathbb{N}$ and recall there exists $j \in \mathbb{N}$ such that $\frac{1}{2^j} < \varepsilon$. Then, $\boldsymbol{\alpha} \in N$ given by $\boldsymbol{\alpha} = x_1 \dots x_j 0 1^{n+2} 0^\infty$ satisfies that $d(\boldsymbol{x}, \boldsymbol{\alpha}) < \varepsilon$. A similar construction can be done for $\boldsymbol{x} \in \Sigma_2^0$ such that $x_1 = 0$. Therefore N is a dense subset of Σ_2. □

Note that, by [Nil09, Theorem 3.6], every $\boldsymbol{\alpha} \in P$ is a limit point of P, therefore a perfect set. Since P is totally disconnected and by Proposition 3.3.2 compact, P and $\bar{P}$ are Cantor sets.

Let $N_0 = \{\boldsymbol{x} \in \Sigma_2^0 \cap N\}$ and $N_1 = \{\boldsymbol{x} \in \Sigma_2^1 \cap N\}$. Note that $N = N_0 \cup N_1$. For every $\boldsymbol{\alpha} \in N_1$ consider

$$n_{\boldsymbol{\alpha}} = \min\{n \in \mathbb{N} \mid \sigma^n(\boldsymbol{\alpha}) \succ \boldsymbol{\alpha}\},$$

and for every $\boldsymbol{\beta} \in N_0$ consider

$$m_{\boldsymbol{\beta}} = \min\{n \in \mathbb{N} \mid \sigma^n(\boldsymbol{\beta}) \prec \boldsymbol{\beta}\}.$$

Note that both $n_{\boldsymbol{\alpha}}$ and $n_{\boldsymbol{\beta}}$ exist for every $\boldsymbol{\alpha} \in N_1$ and $\boldsymbol{\beta} \in N_0$ respectively.

We define $\varsigma : \Sigma_2^1 \to \Sigma_2^1$ by

$$\varsigma(\boldsymbol{\alpha}) = \begin{cases} \boldsymbol{\alpha} & \text{if } \boldsymbol{\alpha} \in P \\ (a_1 \dots a_{n_a-1}0)^\infty & \text{otherwise.} \end{cases}$$

Observe that ς is well defined, since $n_{\boldsymbol{\alpha}}$ exists for every $\boldsymbol{\alpha} \in N$. Moreover, $n_{\boldsymbol{\alpha}}$ is unique. Also, $\varsigma(\Sigma_2^1) = P$ and $\varsigma(\boldsymbol{\alpha}) \preccurlyeq \boldsymbol{\alpha}$ for every $\boldsymbol{\alpha} \in \Sigma_2^1$. Note that it is possible to define $\varsigma' : \Sigma_2^0 \to \Sigma_2^0$ as $\varsigma'(\boldsymbol{\beta}) = \overline{\varsigma(\bar{\boldsymbol{\beta}})}$.

Lemma 3.3.3. *Let $\boldsymbol{\alpha} \in P \cap Per(\sigma)$. Then, ς is constant on intervals of the form $[\boldsymbol{\alpha}, a_1 \dots a_n 1^\infty]_\prec$ where n is the period of $\boldsymbol{\alpha}$. Moreover, if $\boldsymbol{\alpha}' \notin [\boldsymbol{\alpha}, a_1 \dots a_n 1^\infty]_\prec$ then $\varsigma(\boldsymbol{\alpha}) \neq \varsigma(\boldsymbol{\alpha}')$.*

Proof. It is clear that $\varsigma(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ and $\varsigma(a_1 \dots a_{n)} 1^\infty) = \boldsymbol{\alpha}$ from the definition of ς. Let $\boldsymbol{\alpha}' \in (\boldsymbol{\alpha}, a_1 \dots a_n 1^\infty)_\prec$. Then there exists $m > n$ such that $a'_m = 1$ and $a_m = 0$. This implies that $\boldsymbol{\alpha}' \notin P$. Consider $\varsigma(\boldsymbol{\alpha}')$ and assume that $\varsigma(\boldsymbol{\alpha}') \neq \boldsymbol{\alpha}$. Suppose first that $\varsigma(\boldsymbol{\alpha}') \prec \boldsymbol{\alpha}$. Then, there exists m' such that $\varsigma(\boldsymbol{\alpha}')_{m'} < a_{m'}$. This implies that $a'_1 \dots a'_{n_{a'}} \prec \boldsymbol{\alpha}$ which contradicts that $\boldsymbol{\alpha} \in (\boldsymbol{\alpha}, a_1 \dots a_n 1^\infty)_\prec$. If $\varsigma(\boldsymbol{\alpha}') \succ \boldsymbol{\alpha}$ then $a'_1 \dots a'_{n_{a'}} \succ \boldsymbol{\alpha}$ which again contradicts that $\boldsymbol{\alpha}' \in (\boldsymbol{\alpha}, a_1 \dots a_n 1^\infty)_\prec$. □

Remark 3.3.4. In the proof of Lemma 3.3.3 we considered the sequence $a_1 \dots a_n 1^\infty$ instead of $a_1 \dots a_{n-1} 10^\infty$. The purpose of this modification is to not change the provided definition of ς. However, if $a_1 \dots a_{n-1} 10^\infty$ is considered, then $\varsigma(a) = (a_1 \dots a_{\ell(a)-1} 0)^\infty$ (see [ACS09, Definition 2.1]).

An immediate consequence of Lemma 3.3.3 is that $\varsigma\ |_P$ is an increasing function with respect to the lexicographic order. Also, note that ς' will satisfy the same properties as ς, except that ς' is an increasing function.

Theorem 3.3.5. *The function ς is continuous. Moreover, $\pi \circ \varsigma \circ \pi^{-1} : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$ is a devil's staircase.*

Proof. From Lemma 3.3.3, ς is continuous and constant on intervals of the form $[\boldsymbol{\alpha}, a_1 \dots a_n 1^\infty]_\prec$ where $n = Per(\boldsymbol{\alpha})$. Observe that if $\boldsymbol{\alpha}' \in \Sigma_2^1 \setminus \bigcup_{\boldsymbol{\alpha} \in P \cap Per(\sigma)} [\boldsymbol{\alpha}, a_1 \dots a_n 1^\infty]_\prec$ then $\varsigma(\boldsymbol{\alpha}') = \boldsymbol{\alpha}'$. Moreover, for every $\boldsymbol{\alpha}' \in \Sigma_2^1 \setminus P$ there exists $\boldsymbol{\alpha} \in P \cap Per(\sigma)$ such that $\boldsymbol{\alpha}' \in [\boldsymbol{\alpha}, a_1 \dots a_n 1^\infty]_\prec$. Then from Proposition 3.3.2 ς is continuous and constant on a dense set of Σ_2^1. Then, it is just needed to show that ς is continuous in $P \setminus P \cap Per(\sigma)$. Let $\boldsymbol{\alpha} \in P \setminus P \cap Per(\sigma)$ and $\{\boldsymbol{\alpha}_i\}_{i=1}^\infty$ such that $\boldsymbol{\alpha}_i \xrightarrow[i \to \infty]{} \alpha$. Let $\varepsilon > 0$. Then, there exist $k \in \mathbb{N}$ such that $d(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}) < \frac{1}{2^{k+1}} < \frac{\varepsilon}{2}$. Recall that $d(\boldsymbol{\alpha}_n, \varsigma(\boldsymbol{\alpha}_i)) \leq \frac{1}{2^{n_{\alpha_i}}}$. Also, since $\boldsymbol{\alpha}_i$ converge to $\boldsymbol{\alpha}$ then $n_{\boldsymbol{\alpha}_i} > k$ for every $i \geq k$. This gives that

$$d(\varsigma(\boldsymbol{\alpha}_i), \varsigma(\boldsymbol{\alpha})) \leq \frac{1}{2^{n_{\boldsymbol{\alpha}_i}}} + \frac{\varepsilon}{2} < \varepsilon.$$

Thus, ς is continuous.

Let $a \in [\frac{1}{2}, 1]$ such that $\pi^{-1}(\{a\})$ has two elements. By Remark 3.3.4 we can consider $\boldsymbol{x} \in \pi^{-1}(\{a\})$ such that $\boldsymbol{x} = x_1 \dots x_n 1^\infty$. Then, $\pi \circ \varsigma \circ \pi^{-1}$ is a well defined function and it is continuous, increasing, constant on $\pi([[\boldsymbol{\alpha}, a_1 \dots a_{\ell(a)} 1^\infty]_\prec])$. Moreover, since $\pi(P)$ is a set of Lebesgue measure zero,

$$\Sigma_2^1 \setminus \bigcup_{\boldsymbol{\alpha} \in P \cap Per(\sigma)} [\boldsymbol{\alpha}, a_1 \dots a_{\ell(a)} 1^\infty]_\prec$$

has Lebesgue measure zero and $\pi \circ \varsigma \circ \pi^{-1}$ is not differentiable in

$$\pi(\Sigma_2^1 \setminus \bigcup_{\boldsymbol{\alpha} \in P \cap Per(\sigma)} [\boldsymbol{\alpha}, a_1 \dots a_{\ell(a)} 1^\infty]_\prec).$$

Thus, $\pi \circ \varsigma \circ \pi^{-1}$ is a devil's staircase. $\square$

Lemma 3.3.6. *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Sigma_2^1 \times \Sigma_2^0$ such that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is extremal, $\boldsymbol{\alpha} \in N$ and $\boldsymbol{\beta} \in \bar{N}$ and $\sigma^n(\boldsymbol{\beta}) \prec \boldsymbol{\alpha}$ and $\sigma^n(\boldsymbol{\alpha}) \succ \boldsymbol{\beta}$ for every $n \in \mathbb{N}$. Then $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \Sigma_{(\varsigma(\boldsymbol{\alpha}),\varsigma'(\boldsymbol{\beta}))}$.*

Proof. Note that it suffices to show our result for $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in N_0 \times N_1$. Observe that $\boldsymbol{\beta} \prec \varsigma'(\boldsymbol{\beta}) \prec \varsigma(\boldsymbol{\alpha}) \prec \boldsymbol{\alpha}$. Then $\Sigma_{(\varsigma(\boldsymbol{\alpha})\varsigma'(\boldsymbol{\beta}))} \subset \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Assume that $\Sigma_{(\varsigma(\boldsymbol{\alpha}),\varsigma'(\boldsymbol{\beta}))} \subsetneq \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Let $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \setminus \Sigma_{(\varsigma(\boldsymbol{\alpha}),\varsigma'(\boldsymbol{\beta}))}$. Then

$$\sigma^n(\boldsymbol{x}) \in [\boldsymbol{\beta}, \varsigma'(\boldsymbol{\beta}))_\prec \cup (\varsigma(\boldsymbol{\alpha}), \boldsymbol{\alpha}]_\prec \text{ for every } n \geq 0.$$

Then there exists m' such that either $\sigma^{m'}(\boldsymbol{x})_{m_b} < (\varsigma'(\boldsymbol{\beta}))_{m_b}$ or $\sigma^{m'}(\boldsymbol{x})_{n_a} > (\varsigma(\boldsymbol{\alpha}))_{n_a}$. This implies that $\sigma^{m_b+j}(\boldsymbol{x}) \prec \boldsymbol{\beta}$ or $\sigma^{n_a+j}(\boldsymbol{x}) \succ \boldsymbol{\alpha}$, which is a contradiction. Then $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \Sigma_{(\varsigma(\boldsymbol{\alpha}),\varsigma'(\boldsymbol{\beta}))}$. $\square$

Observe that given $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$, $(\boldsymbol{\alpha},\boldsymbol{\beta})$ might be extremal. The possible cases of extremal pairs are defined as follows:

Definition 3.3.7. Let $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$.

- We say that $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is *two sided extremal* if:

  i) $\sigma^n(\boldsymbol{\beta}) \succ \boldsymbol{\alpha}$ for some $n \in \mathbb{N}$;

  ii) $\sigma^m(\boldsymbol{\alpha}) \prec \boldsymbol{\beta}$ for some $m \in \mathbb{N}$.

- We say that $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is *right extremal* if:

  i) $\sigma^m(\boldsymbol{\alpha}) \succ \boldsymbol{\beta}$ for every $m \in \mathbb{N}$;

  ii) $\sigma^n(\boldsymbol{\beta}) \succ \boldsymbol{\alpha}$ for some $n \in \mathbb{N}$.

- We say that $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is *left extremal* if:

  i) $\sigma^n(\boldsymbol{\beta}) \prec \boldsymbol{\alpha}$ for every $n \in \mathbb{N}$;

  ii) $\sigma^m(\boldsymbol{\alpha}) \prec \boldsymbol{\beta}$ for some $m \in \mathbb{N}$.

Firstly, we will study two sided extremal pairs $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$. To motivate this study we present the following example.

Example 3.3.8. The pair $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$ where

$$\boldsymbol{\alpha} = 11101010000101001011001010000111000 1\ldots$$

and

$$\boldsymbol{\beta} = 000010110111010011011101101011 1\ldots$$

is two sided extremal since $\sigma^7(\boldsymbol{\alpha}) \prec \boldsymbol{\beta}$ and $\sigma^{19}(\boldsymbol{\beta}) \succ \boldsymbol{\alpha}$.

Let $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$ be two sided extremal. Consider

$$M_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = \min\{m \in \mathbb{N} \mid \sigma^m(\boldsymbol{\beta}) \succ \boldsymbol{\alpha}\},$$

and

$$N_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = \min\{n \in \mathbb{N} \mid \sigma^n(\boldsymbol{\alpha}) \prec \boldsymbol{\beta}\}.$$

Then we define

$$\iota(\boldsymbol{\alpha}) = (a_1 \dots a_{N_b(a)-1}0)^\infty$$

and

$$\iota(\boldsymbol{\beta}) = (b_1 \dots b_{M_a(b)-1}1)^\infty.$$

Observe that $\iota(\boldsymbol{\alpha})$ and $\iota(\boldsymbol{\beta})$ are periodic sequences. In Example 3.3.8 $\iota(\boldsymbol{\alpha}) = (1110100)^\infty$ and $\iota(\boldsymbol{\beta}) = (000101101110100111)^\infty$. Also, $\iota(\boldsymbol{\alpha})$ and $\iota(\boldsymbol{\beta})$ satisfy that $\iota(\boldsymbol{\alpha}) \prec \boldsymbol{\alpha}$, $\boldsymbol{\beta} \succ \iota(\boldsymbol{\beta})$, $d(\boldsymbol{\alpha}, \iota(\boldsymbol{\alpha})) \leq \frac{1}{2^{N_{\boldsymbol{\beta}}(\boldsymbol{\alpha})}}$ and $d(\boldsymbol{\beta}, \iota(\boldsymbol{\beta})) \leq \frac{1}{2^{M_{\boldsymbol{\alpha}}(\boldsymbol{\beta})}}$.

Proposition 3.3.9. *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in P \times \bar{P}$. Then $(\iota(\boldsymbol{\alpha}), \iota(\boldsymbol{\beta})) \in \mathcal{LW}$.*

Proof. From the definition of $(\iota(\boldsymbol{\alpha}), \iota(\boldsymbol{\beta}))$ it is clear that $(\iota(\boldsymbol{\alpha}), \iota(\boldsymbol{\beta})) \in P \times \bar{P}$. Assume that $(\iota(\boldsymbol{\alpha}), \iota(\boldsymbol{\beta}))$ is extremal. Firstly, suppose that $\sigma^n(\iota(\boldsymbol{\alpha})) \prec \iota(\boldsymbol{\beta})$ for some $n \in \mathbb{N}$. Consider $N_{\iota(\boldsymbol{\beta})}(\iota(\boldsymbol{\alpha}))$. Since $\iota(\boldsymbol{\alpha})$ is periodic, then $1 < N_{\iota(\boldsymbol{\beta})}(\iota(\boldsymbol{\alpha})) < N_{\boldsymbol{\beta}}(\boldsymbol{\alpha})$. This contradicts the choice of $N_{\boldsymbol{\beta}}(\boldsymbol{\alpha})$. Thus, $\boldsymbol{\beta} \prec \sigma^n(\iota(\boldsymbol{\alpha}))$ for every $n \in \mathbb{N}$. Similarly, $\sigma^n(\boldsymbol{\beta}) \prec \boldsymbol{\alpha}$ for every $n \in \mathbb{N}$. Therefore $(\iota(\boldsymbol{\alpha}), \iota(\boldsymbol{\beta})) \in \mathcal{LW}$. □

Lemma 3.3.10. *If $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in P \times \bar{P}$ is two sided extremal, then $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \Sigma_{(\iota(\boldsymbol{\alpha}),\iota(\boldsymbol{\beta}))}$.*

Proof. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be a two sided extremal pair. Then by [BKT11, Proposition 4.1], $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is a subshift of finite type. Note that [LSS14, Theorem 1.3] implies that $(\Sigma_{(\iota(\boldsymbol{\alpha}),\iota(\boldsymbol{\beta}))}, \sigma_{(\iota(\boldsymbol{\alpha}),\iota(\boldsymbol{\beta}))})$ is a subshift of finite type. Since, $\boldsymbol{\alpha} \prec \iota(\boldsymbol{\alpha})$ and $\iota(\boldsymbol{\beta}) \prec \boldsymbol{\beta}$ then $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \subset \Sigma_{(\iota(\boldsymbol{\alpha}),\iota(\boldsymbol{\beta}))}$. □

Now, we would like to study right and left extremal pairs. For this purpose, the following examples are presented.

Example 3.3.11. The pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ where $\boldsymbol{\alpha} = (11100)^\infty$ and $\boldsymbol{\beta} = (01)^\infty$ is left extremal and the pair $(\boldsymbol{\alpha}', \boldsymbol{\beta}')$ where $\boldsymbol{\alpha}' = (11101010)^\infty$ and $\boldsymbol{\beta}' = (0001011101011)^\infty$ is right extremal.

Then given $\boldsymbol{\alpha} \in P$, we define $\xi_{\boldsymbol{\alpha}} : P' \to P'$ by:

$$\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = \begin{cases} \boldsymbol{\beta} & \text{if } \sigma^n(\boldsymbol{\beta}) \prec \boldsymbol{\alpha} \text{ for every } n \geq 0; \\ (b_1, \dots b_{M_{\boldsymbol{\alpha}}(\boldsymbol{\beta})-1}1)^\infty & \text{otherwise.} \end{cases}$$

Similarly, if $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$ is left extremal, we define $\xi'_{\boldsymbol{\beta}} : P \to P$ by:

$$\xi'_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = \begin{cases} \boldsymbol{\alpha} & \text{if } \sigma^n(\boldsymbol{\alpha}) \succ \boldsymbol{\beta} \text{ for every } n \geq 0; \\ (a_1 \dots a_{N_{\boldsymbol{\beta}(\boldsymbol{\alpha})}-1}0)^\infty & \text{otherwise.} \end{cases}$$

Proposition 3.3.12. *Let* $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$. *Then*

i) $\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) \in \bar{P}$ *if* $(\boldsymbol{\alpha},\boldsymbol{\beta})$ *is right extremal;*

ii) $\xi'_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) \in P$ *if* $(\boldsymbol{\alpha},\boldsymbol{\beta})$ *is left extremal.*

Proof. Let $(\boldsymbol{\alpha},\boldsymbol{\beta})$ be right extremal. Assume that $\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) \notin \bar{P}$. Then there exist $j \in \mathbb{N}$ such that $\sigma^j(\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})) \prec \xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$. Since $\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$ is a periodic sequence of period $M_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$ and $\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})_i = b_i$ for every $i \in \{1,\dots,M_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) - 1$ then $j = M_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$. Then $\sigma^j(\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})) \prec \xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$. This implies that $\sigma^j(\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}))_1 = 0$ which contradicts the definition of $\sigma^j(\xi_{\boldsymbol{\alpha}}$. Then $\sigma^j(\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})) \in \bar{P}$.

The proof of *ii*) is analogous. □

Lemma 3.3.13. *Let* $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$. *Then:*

i) $(\boldsymbol{\alpha},\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})) \in \mathcal{LW}$ *if* $(\boldsymbol{\alpha},\boldsymbol{\beta})$ *is right extremal;*

ii) $(\xi'_{\boldsymbol{\beta}}(\boldsymbol{\alpha}),\boldsymbol{\beta}) \in \mathcal{LW}$ *if* $(\boldsymbol{\alpha},\boldsymbol{\beta})$ *is left extremal.*

Proof. It suffices to show case *i*). Assume that $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is right extremal. To show that $(\boldsymbol{\alpha},\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})) \in \mathcal{LW}$ we need to show that $\sigma^n(\boldsymbol{\alpha}) \succcurlyeq \xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$ and $\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) \preccurlyeq \sigma^m(\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})) \preccurlyeq \boldsymbol{\alpha}$ for every $n, m \in \mathbb{N}$. From Proposition 3.3.12 we have that $\sigma^m(\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})) \succcurlyeq (\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$ for every $m \in \mathbb{N}$. Assume that there is $j \in \mathbb{N}$ such that $\sigma^j(\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})) \succcurlyeq \boldsymbol{\alpha}$. From the definition of $\xi_{\boldsymbol{\alpha}}$ we have that $j = M_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$. Then there is $j' \in \mathbb{N}$ such that $(\sigma^j(\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})))_{j'} > a_{j'}$ which contradicts the minimality of $M_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$.

We need to show now that $\sigma^n(\boldsymbol{\alpha}) \succcurlyeq \xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$ for every $n \in \mathbb{N}$. Suppose that there is $n \in \mathbb{N}$ such that $\sigma^n(\boldsymbol{\alpha}) \prec \xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$. Since $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is right extremal, $\boldsymbol{\beta} \prec \sigma^j(\boldsymbol{\alpha}) \prec \xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$. Recall that $d(\boldsymbol{\beta},\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})) = \frac{1}{2^{M_{\boldsymbol{\alpha}(\boldsymbol{\beta})}}}$. This implies that $\sigma^j(\boldsymbol{\alpha})_{M_{\boldsymbol{\alpha}(\boldsymbol{\beta})}} = 0$ which contradicts again the minimality of $M_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$. □

Lemma 3.3.14. *Let* $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$. *Then:*

i) $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \Sigma_{(\boldsymbol{\alpha},\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}))}$ *if* $(\boldsymbol{\alpha},\boldsymbol{\beta})$ *is right extremal;*

ii) $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \Sigma_{(\xi'_{\boldsymbol{\beta}}(\boldsymbol{\alpha}),\boldsymbol{\beta})}$ *if* $(\boldsymbol{\alpha},\boldsymbol{\beta})$ *is left extremal.*

Proof. It suffices to show *i)* since the proof of *ii)* is analogous.

Observe that $\Sigma_{(\boldsymbol{\alpha},\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}))} \subset \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ since $\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) \succcurlyeq \boldsymbol{\beta}$. We want to show that $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \subset \Sigma_{(\boldsymbol{\alpha},\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})}$. Assume that the inclusion does not hold. Then there exists $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \setminus \Sigma_{(\boldsymbol{\alpha},\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}))}$. This implies that there exists $n \in \mathbb{N}$ such that $\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) \succ \sigma^n(\boldsymbol{x}) \succ \boldsymbol{\beta}$. Then there exists $k_{\boldsymbol{\beta}} \in \mathbb{N}$ such that $\sigma^n(\boldsymbol{x})_{k_{\boldsymbol{\beta}}} < \xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})_{k_{\boldsymbol{\beta}}}$ and $\sigma^n(\boldsymbol{x})_i = \xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})_i$ for every $i < k_{\boldsymbol{\beta}}$. Thus, $\sigma^n(\boldsymbol{x}) \succ \boldsymbol{\alpha}$. This contradicts that $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}))}$. Hence $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \subset \Sigma_{(\boldsymbol{\alpha},\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}))}$. □

Consider the examples given in Example 3.3.11. Firstly, recall that the pair $(\boldsymbol{\alpha},\boldsymbol{\beta})$ given by $\boldsymbol{\alpha} = (11100)^\infty$ and $\boldsymbol{\beta} = (01)^\infty$ is left extremal. Then $\xi'_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = (110)^\infty$. For the right extremal pair $(\boldsymbol{\alpha}',\boldsymbol{\beta}')$ where $\boldsymbol{\alpha}' = (11101010)^\infty$ and $\boldsymbol{\beta}' = (0001011101011)^\infty$, $\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = (00011)^\infty$.

We define $I : P \times \bar{P} \to P \times \bar{P}$ as:

$$I(\boldsymbol{\alpha},\boldsymbol{\beta}) = \begin{cases} (\boldsymbol{\alpha},\boldsymbol{\beta}) & \text{if } (\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{LW}; \\ (\iota(\boldsymbol{\alpha}),\iota(\boldsymbol{\beta})) & \text{if } (\boldsymbol{\alpha},\boldsymbol{\beta}) \text{ is two sided extremal;} \\ (\boldsymbol{\alpha},\xi_{\boldsymbol{\alpha}}(\boldsymbol{\beta})) & \text{if } (\boldsymbol{\alpha},\boldsymbol{\beta}) \text{ is left extremal;} \\ (\xi'_{\boldsymbol{\beta}}(\boldsymbol{\alpha}),\boldsymbol{\beta}) & \text{if } (\boldsymbol{\alpha},\boldsymbol{\beta}) \text{ is right extremal.} \end{cases}$$

The function I provides the sought link between $P \times \bar{P}$ and the lexicographic world.

Theorem 3.3.15. *Let* $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$. *Then* I *is well defined. Moreover,* $I(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{LW}$ *for every* $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$.

Proof. From Lemmas 3.3.9 and 3.3.13, I is well defined and $I(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{LW}$ for every $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$. □

Theorem 3.3.16. *For every* $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$, $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \Sigma_{I(\boldsymbol{\alpha},\boldsymbol{\beta})}$.

Proof. This is a direct consequence of Lemmas 3.3.10 and 3.3.14. □

Using Theorems 3.3.15 and 3.3.16, we obtain the proof of Theorem 3.2.3, i.e. for every $(a,b) \in D_1$, there exists $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{LW}$ such that $(\Lambda_{(a,b)}, f_{(a,b)})$ is topologically conjugated to the lexicographic subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha}.\boldsymbol{\beta})})$.

As a direct consequence of Theorems 3.2.2, 3.3.15 and 3.3.16 we obtain the following results.

Theorem 3.3.17. *For every* $(\boldsymbol{\alpha}', \boldsymbol{\beta}') \in \Sigma_2^1 \times \Sigma_1^0$ *there exists* $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ *such that* $(\Sigma_{(\boldsymbol{\alpha}', \boldsymbol{\beta}')}, \sigma_{(\boldsymbol{\alpha}', \boldsymbol{\beta}')}) = (\Sigma_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha}, \boldsymbol{\beta})})$.

Corollary 3.3.18. *For every* $(a, b) \in D_1$ *there exists* $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ *such that* $(\Lambda_{(a,b)}, f_{(a,b)})$ *is topologically conjugated to* $(\Sigma_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha}, \boldsymbol{\beta})})$.

3.4 Genericity of subshifts of finite type and the entropy function

In this section we describe the pairs $(a, b) \in D_1$ such that the corresponding exclusion subshift is a *subshift of finite type*. Recall that a subshift (A, σ_A) is a *subshift of finite type* if the forbidden set of factors $\mathcal{F}$ corresponding to A is finite. Recall that for every $(a, b) \in D_1$ we associate the pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Sigma_2^1 \times \Sigma_2^0$ where $\boldsymbol{\alpha} = \pi^{-1}(2a)$ and $\boldsymbol{\beta} = \pi^{-1}(2b - 1)$. We show that the set

$$\{(a, b) \in D_1 \mid (\Sigma_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}) \text{ is a subshift of finite type}\}$$

is open and dense (see Theorem 3.4.2).

Let

$$S = \{(a, b) \in D_1 \mid \text{ there exist } n \text{ and } m \text{ such that } f^n(a), f^m(b) \in (a, b)\}.$$

Lemma 3.4.1. *The set* S *is open and dense in* D_1.

Proof. Let $(a, b) \in S$ and recall that f denotes the doubling map. Let

$$n = \min\left\{j \in \mathbb{N} \mid f^j(a) \in (a, b)\right\}$$

and

$$m = \min\left\{i \in \mathbb{N} \mid f^i(b) \in (a, b)\right\}.$$

Recall that f^j is continuous for every $j \geq 0$. Let $\varepsilon' > 0$. Then there exists $\varepsilon_a > 0$ such that if $d(a, a') \leq \varepsilon_a$ then $d(f^n(a), f^n(a')) < \varepsilon'$. Similarly, there exists $\varepsilon_b > 0$ such that if $d(b,b') \leq \varepsilon_b$ then $d(f^m(b), f^m(b')) < \varepsilon'$. Let $\varepsilon = \min\{\varepsilon_a, \varepsilon_b\}$. Then $B_\varepsilon((a,b)) \subsetneq S$. Therefore S is open.

To show that S is dense consider $(a,b) \in D_1 \setminus \partial D_1$. Let $\varepsilon > 0$ such that $\frac{1}{2} \notin (a-\varepsilon, a+\varepsilon)$ and $\frac{1}{2} \notin (b-\varepsilon, b+\varepsilon)$. Moreover, since S is open without losing generality we can assume that $(a-\varepsilon, a+\varepsilon) \times (b-\varepsilon, b+\varepsilon) \subset D_1 \setminus \partial D_1$. Recall that the Lebesgue measure of $(a-\varepsilon, a+\varepsilon)$ and $(b-\varepsilon, b+\varepsilon)$ is positive. Then, by the Poincaré recurrence theorem (see e.g. [Mañ83, Theorem 2.2]), there exists $a' \in (a-\varepsilon, a+\varepsilon)$ and $j \in \mathbb{N}$ such that $f^j(a') \in (a-\varepsilon, a+\varepsilon)$. Similarly, there exists a point $b' \in (b-\varepsilon, b+\varepsilon)$ and $i \in \mathbb{N}$ such that $f^i(b') \in (b-\varepsilon, b+\varepsilon)$. Since f^j is continuous for every $j \geq 0$ there exist $\varepsilon_a > 0$ and $\varepsilon_b > 0$ such that for every $a'' \in (a' - \varepsilon_a, a'_a + \varepsilon)$, $f^j(a'') \in (a' - \varepsilon_a, a' + \varepsilon_a)$ and for every $b'' \in (b' - \varepsilon_b, b'_b + \varepsilon)$, $f^i(b'') \in (b' - \varepsilon_b, b' + \varepsilon_b)$. Then $(a'', b'') \in S$ and $d((a,b),(a'',b'')) < \varepsilon$. Hence, S is dense. □

In [BKT11, Proposition 4.1] and [BDJ14, Theorem 2.4] it is shown that if $(a,b) \in S$ then $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ is a subshift of finite type. We can see inmediately that for every $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in N \times \bar{N}$ there is a pair $(a,b) \in S$ such that $\pi^{-1}(2a) = \boldsymbol{\alpha}$ and $\pi^{-1}(2b-1) = \boldsymbol{\beta}$. However there exist pairs $(\boldsymbol{\alpha},\boldsymbol{\beta})$ such that $(\pi(0\boldsymbol{\alpha}), \pi(1\boldsymbol{\beta}) \in S$. Such pairs are called *two sided extremal* and they will be studied in Section 3.3. On the other hand we are sure that $\mathcal{LW} \subset P \times \bar{P}$. Therefore, from ([Nil09, Theorems 3.7, 3.8]) we are sure that $\pi(\mathcal{LW})$ has Lebesgue measure zero. Observe that Lemma 3.4.1 combined with the fact that $\pi(P)$ and $\pi(\bar{P})$ are sets of Lebesgue measure zero imply that $D_1 \cap S$ is a set of full Lebesgue measure. Thus, we have proven the following theorem.

Theorem 3.4.2. *The set*

$$\{(a,b) \in D_1 \mid (\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \text{ is a subshift of finite type}\}$$

is open and dense. Moreover, it has full Lebesgue measure.

Theorem 3.4.3. *For every $(a,b) \in S$ there is an open set $U \subset D_1$ such that $(a,b) \in U$ and $(X_{(a,b)}, f_{(a,b)}) = (X_{(a',b')}, f^{a'}_{b'})$ for every $(a',b') \in U$* [1].

Proof. The argument is essentially the same as the one of the proof of [Urb86, Theorem 1] which proves the case when $a = 0$. Let $(a,b) \in S$ be fixed. Then, there exist $n = n(a)$

[1] After completion of this proof the author was made aware of the work of Baker, Dajani and Jiang [BDJ14, Theorem 2.4]. Our result follows immediately from the proof of the mentioned theorem.

and $m = m(b) \in \mathbb{N}$ such that $f^n(a), f^m(b) \in (a, b)$. Without loosing generality, assume that such n and m are minimal. Consider

$$C_1 = \left\{ b' \in \left(\frac{1}{2}, \chi(a) \right) \mid f^m(b') \in (a, b) \right\}.$$

Observe that $C_1 \neq \emptyset$ since $b \in C_1$. Since f is continuous, there exists $\varepsilon > 0$ such that for any $b' \in \overline{B_\varepsilon(b)}$, $f^m(b') \in (a, b')$. Note that for any $c \in B_\varepsilon(b)$, $f^m(c) \in (a, b - \frac{\varepsilon}{2})$. Recall that $X_{(a,b+\frac{\varepsilon}{2})} \subset X_{(a,b)} \subset X_{(a,b-\frac{\varepsilon}{2})}$. Let $x \in S^1 \setminus X_{(a,b+\frac{\varepsilon}{2})}$. Then there exists $m \in \mathbb{N}$ such that $f^m(x) \in (a, b + \frac{\varepsilon}{2})$. If $x \in (b - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})$ then there exists $n \in \mathbb{N}$ such that $f^n(f^m(x)) \in (a, b - \frac{\varepsilon}{2})$. This shows that $X_{(a,b-\frac{\varepsilon}{2})} \subset X_{(a,b+\frac{\varepsilon}{2})}$ and

$$X_{(a,b-\frac{\varepsilon}{2})} = X_{(a,b+\frac{\varepsilon}{2})} = X_{(a,b)}.$$

Similarly, we consider

$$C_2 = \left\{ a' \in \left(\frac{1}{4}, \frac{1}{2} \right) \mid f^n(a') \in (a, b) \right\}.$$

Then, we can show in a similar way that there exists $\varepsilon' > 0$ such that $X_{(a-\frac{\varepsilon'}{2},b)} \subset X_{(a+\frac{\varepsilon'}{2},b)}$ and

$$X_{(a-\frac{\varepsilon'}{2},b)} = X_{(a+\frac{\varepsilon'}{2},b)} = X_{(a,b)}.$$

Then, $U = (a - \epsilon, a + \epsilon) \times (b - \epsilon, b + \epsilon)$ where $\epsilon = \min\{\frac{\varepsilon}{2}, \frac{\varepsilon'}{2}\}$. □

For every $(a, b) \in D_1$ it is natural to associate it the topological entropy of the corresponding attractor $\Lambda_{(a,b)}$. To formalize this, let $H : D_1 \to [0, 1]$ given by

$$H((a, b)) = h_{top}(f_{(a,b)}).$$

From [Urb87, Theorem 4] we have that H is a continuous function. Furthermore, note that for every $a \in [0, \frac{1}{2})$, if $b = a$ then $X_{(a,b)} = S^1$ and $h_{top}(f_{(a,b)}) = 1$. Moreover, if $b = \chi(a)$ then $h_{top}(f_{(a,b)}) = 0$ as a direct consequence of [GS14, Theorem 2.16]. An immediate consequence of Theorem 3.4.3 is that H is constant almost everywhere.

Corollary 3.4.4. *The set* $\{(a, b) \in R \mid h_{top}(f_{(a,b)}) \text{ is constant}\}$ *is open and dense. Moreover, it has full measure.*

As a result of Corollary 3.4.4 and the well know formula

$$\dim_H X_{(a,b)} = \frac{h_{top}(f_{(a,b)})}{\lambda},$$

where λ is the Lyapunov exponent of $2x \mod 1$, we obtain the following corollary. The proof is omitted.

Corollary 3.4.5. *The set of pairs* $(a, b) \in R$ *such that* $\dim_H X_{(a,b)}$ *is constant is open and dense with full measure.*

It is worth mentioning that Urbański in [Urb87] considered exclusion systems given by open sets $U = \bigcup_{j=1}^{n} (a_j, b_j)$. It was shown in [Urb87, Theorem 4] that the topological entropy of the exclusion system (X_U, f_U) and the Hausdorff dimension of X_U change continuously if the end points a_j, b_j also vary continuously for every $j \in \{1, \dots n\}$.

Chapter 4

The doubling map with symmetric holes

All the results of this chapter and their proofs are contained in [AB14]. We respect the order stated in the cited article within this chapter.

We would like to remind to the reader the class of subshifts that we are interested to study in this chapter. A lexicographic subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is *symmetric* if $\bar{\boldsymbol{x}} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ for every $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Also, given $\boldsymbol{\alpha} \in \Sigma_2^1$ we define *the symmetric subshift parametrised by* $\boldsymbol{\alpha}$ as

$$\Sigma_{\boldsymbol{\alpha}} = \{\boldsymbol{x} \in \Sigma_2 \mid \bar{\boldsymbol{\alpha}} \preccurlyeq \sigma^n(\boldsymbol{x}) \preccurlyeq \boldsymbol{\alpha}\}.$$

It is clear that $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is symmetric if and only if $\boldsymbol{\beta} = \bar{\boldsymbol{\alpha}}$. Then the dynamical properties of $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ depend only on one parameter.

In the presented proofs we use strongly subshifts defined by finite sequences instead of periodic sequences as in Chapter 3. However, applying the functions ς and ς' to pairs $(\boldsymbol{\alpha}, \bar{\boldsymbol{\alpha}})$ such that $\boldsymbol{\alpha} = \pi^{-1}(2a)$ and $\boldsymbol{\beta} = \pi^{-1}(1-2a)$ are finite sequences we can easily see that the results are equivalent.

In Section 4.1 we study the cases when a symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is transitive. In Theorem 4.1.3 we show that for any $\alpha \prec 11(01)^\infty$ the corresponding symmetric subshift is not transitive. We introduce the notions of *irreducible word* and *irreducible sequence* -see Definition 4.1.4. Using irreducible sequences we show in Theorem 4.1.7 that for every irreducible sequence $\boldsymbol{\omega}$ the corresponding symmetric subshift $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}})$

is transitive.

In Section 4.2 we describe the behaviour of the function $\boldsymbol{\alpha} \to h_{top}(\sigma_\alpha)$. In particular we show that such function is a devil's staircase. In order to show the mentioned result, we introduce the notion of i-irreducible word which is a natural generalisation of the notion of irreducible word. In Theorem 4.2.7 and Lemma 4.2.8 we show that the plateaus of the entropy function are parametrised by i-irreducible words and in Theorem 4.2.13 we characterise the set $\mathcal{E}$ using a natural approximation from below for every symmetric subshift -see Theorem 4.2.10, that is $\Sigma_{\boldsymbol{\alpha}} = \overline{\bigcup_{n=1}^{\infty} \Sigma_{\boldsymbol{\alpha}_n^-}}$ where each $\boldsymbol{\alpha}_n^-$ is a finite sequence and $\boldsymbol{\alpha}_n^- \prec \boldsymbol{\alpha}_{n+1}^-$, thus $(\Sigma_{\boldsymbol{\alpha}_n^-}, \sigma_{\boldsymbol{\alpha}_n^-})$ is a shift of finite type. Also, it is shown in Theorem 4.2.10 that every symmetric subshift can be approximated from above as well, that is $\Sigma_{\boldsymbol{\alpha}} = \bigcap_{n=1}^{\infty} \Sigma_{\boldsymbol{\alpha}_n^+}$ where each $\boldsymbol{\alpha}_n^+$ is a finite sequence and $\boldsymbol{\alpha}_n^- \succ \boldsymbol{\alpha}_{n+1}^-$, thus $(\Sigma_{\boldsymbol{\alpha}_n^+}, \sigma_{\boldsymbol{\alpha}_n^-})$ is a shift of finite type as well.

In Section 4.3 we study the specification property for transtive symmetric subshifts, i.e. transitive symmetric subshifts such that any two words can be connected by a word of bounded length. In Theorem 4.3.2 we describe the cases when we know that a symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has specification. This result depends in two properties: the number of blocks of consecutive 0's occurring in $\boldsymbol{\alpha}$ and in the specification number -see Definition 4.3.1 of each element of the approximation from above mentioned previously. Also, in Theorem 4.3.16 we give a condition for $\boldsymbol{\alpha}$ which guarantee that $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ does not have the specification property and we construct a family of examples where this condition is fulfilled.

Finally, in Section 4.4 we show when a symmetric subshift is intrinsically ergodic. As we mention in Theorem 2.4.3, transitive subshifts of finite type as well as subshifts with the specification property are intrinsically ergodic. In Theorem 4.4.5 we show that for every $\boldsymbol{\alpha}$ belonging to an entropy plateau, the corresponding subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is intrinsically ergodic. the main idea is to extend the intrinsic measure corresponding to the left end point of an entropy plateau and to show that such extension is the unique measure of maximal entropy for such subshift.

4.1 Transitivity

In this section, we put a particular emphasis on finding the arithmetic conditions on a sequence $\boldsymbol{\alpha}$ so that $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is a transitive subshift of finite type (Theorem 4.1.7). Moreover, a lower bound for the parameters corresponding to symmetric transitive subshifts is given in Theorem 4.1.3. The following theorem gives a lower bound for the parameters in $[0, \frac{1}{2}]$ that define symmetric subshifts.

Theorem 4.1.1. [GS01, Theorem 2] *Let* $a \in [0, \frac{1}{2})$ *and* $\boldsymbol{\alpha} = \pi^{-1}(2a)$. *Then:*

i) $\Sigma_{\boldsymbol{\alpha}}$ *is empty if* $a \in [0, \frac{1}{3}]$*;*

ii) $\Sigma_{\boldsymbol{\alpha}} = \{(01)^\infty, (10)^\infty\}$ *if* $a \in (\frac{1}{3}, \frac{2}{5})$*;*

iii) $\Sigma_{\boldsymbol{\alpha}}$ *is countable* $a \in (\frac{2}{5}, \pi(\boldsymbol{\alpha}^*))$*;*

iv) $\Sigma_{\boldsymbol{\alpha}}$ *is uncountable* $a \in [\pi(\boldsymbol{\alpha}^*), \frac{1}{2}]$*.*

The sequence $\boldsymbol{\alpha}^*$ is known as the *Thue-Morse sequence* . We define it as follows. Consider the substitutions defined by $0 \to 01$ and $1 \to 10$, and consider the following sequence of words: Let $a_0 = 0$. Then we use the substitution on a^0 to construct $a^1 = 01$. The word a^n is defined to be the substitution applied to a^{n-1}. The sequence $\{a^n\}_{n=0}^\infty$ converges coordinate-wise to the sequence $\boldsymbol{\alpha}^*$.

Note that, Theorem 4.1.1 provides us the first example of a transitive subshift: if $\boldsymbol{\alpha} \in [11, 1101)_\prec$ then $\Sigma_{\boldsymbol{\alpha}} = \{(01)^\infty, (10)^\infty\}$. Nonetheless, it is just a 2-cycle.

From now on we make the assumption that $\boldsymbol{\alpha} \in \Sigma_2^1$.

Let $\boldsymbol{\alpha}$ be a finite sequence. Then the symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ admits a *maximal (respectively minimal) word of length* $\ell(\boldsymbol{\alpha})$ denoted by $\boldsymbol{\alpha}_{\max}$ (respectively $\boldsymbol{\alpha}_{\min}$), $\boldsymbol{\alpha}_{\max}$ and $\boldsymbol{\alpha}_{\min}$ are given by $\boldsymbol{\alpha}_{\max} = a_1 \dots a_{\ell(a)-1}0$ and $\boldsymbol{\alpha}_{\min} = \bar{\boldsymbol{\alpha}}_{\max}$. Furthermore, for any sequence a and for every $1 \leq n < \ell(\boldsymbol{\alpha})$, the word $\boldsymbol{\alpha}_{\max_n} = a_1 \dots a_n$ is the *maximal word of length* n and the word $\bar{\boldsymbol{\alpha}}_{\max_n}$ is the *minimal word of length* n.

The following proposition provides a useful partition of Σ_2^1.

Proposition 4.1.2. *Let $n \geq 2$ and consider $\boldsymbol{\alpha} \in (1^n, 1^{n+1}]_{\prec}$. Then if $\boldsymbol{x} \in \Sigma_{\boldsymbol{\alpha}}$ then 0^{n+1} and 1^{n+1} does not occur in $\boldsymbol{x}$.*

Proof. Let $n \geq 2$ and $\boldsymbol{\alpha} \in (1^n, 1^{n+1}]_{\prec}$. Note that $\Sigma_{\boldsymbol{\alpha}}$ contains the subshift corresponding to the finite sequence $\boldsymbol{\alpha}' = 1^n$. Since $\boldsymbol{\alpha} \in (1^n, 1^{n+1}]_{\prec}$, the words 0^n and $1^n \in \mathcal{L}(\Sigma_{\boldsymbol{\alpha}})$. Suppose that there exists a point $\boldsymbol{x} \in \Sigma_{\boldsymbol{\alpha}}$ such that the block 1^{n+1} occurs. Let $k \in \mathbb{N}$ be such that $\sigma^k(\boldsymbol{x}) = 1^{n+1}$. Then $\sigma^k(\boldsymbol{x}) \not\succcurlyeq 1^{n+1}$, which is a contradiction. By the symmetry of $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ we can conclude our result. □

Now, we provide a lower bound for symmetric transitive subshifts to occur. The proof of the following theorem also shows that not every symmetric subshift is transitive.

Theorem 4.1.3. *If $\boldsymbol{\alpha} \in ((10)^\infty, 11(01)^\infty)_{\prec}$ then $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is not transitive.*

Proof. Let $\boldsymbol{\alpha} \in ((10)^\infty, 11(01)^\infty)_{\prec}$. Since $11(01)^\infty \prec 111$, $\boldsymbol{\alpha}$ cannot contain the blocks 0^3 and 1^3. Observe that $\boldsymbol{\alpha} = 11(01)^k001\ldots$ for some $k \geq 0$. Then $11(01)^{k+1}$ and $00(10)^{k+1} \notin \mathcal{L}(\Sigma_{\boldsymbol{\alpha}})$. Observe that the periodic orbit $(01)^\infty \in \Sigma_{\boldsymbol{\alpha}}$. This implies that the block $(10)^k$ is admissible for every $k \in \mathbb{N}$. Consider the words $\upsilon = 11$ and $\nu = (10)^{k+1}$. Note that υ and ν cannot be concatenated. Besides, they cannot be connected by 1 because $\upsilon 1 \nu$ will have three consecutive ones, and they cannot be connected by 0 because the block $[\upsilon 0 \nu]$ contains the block $[11(01)^{k+1}]$. Note that υ and ν cannot be connected by 00 (and by any word ending in 00) because the block $[00(10)^{k+1}]$ will occur. Furthermore, υ and ν cannot be connected by 01 (and by any word ending in 01 or 1) because the block $[11(01)^{k+1}]$ will occur. Therefore the only possibility is to consider a word ω ending in 10. Note that if $w_{n-1}w_n = 10$, then $w_{n-3}w_{n-2} = 01$ by the argument shown above. Then ν and ω cannot be concatenated in Σ_a. This shows that $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is not transitive. □

Note that Theorem 4.1.3 does not assure that if $\boldsymbol{\alpha} \succ 1101^\infty$ then $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is a transitive symmetric subshift.

For any word ω starting with 1, we define $t(\omega)$ to be

$$\omega \xrightarrow{t} w_1 \ldots w_{\ell(\omega)} \bar{w}_1 \ldots \bar{w}_{\ell(\omega)-1} 1.$$

If $\omega = 10^\infty$ then $\omega \xrightarrow{t} 101$. We will denote $t(\omega)$ as ω'. In addition, given a word ω starting with 1 consider the following sequences:

i) $\boldsymbol{\omega}'' = \lim_{n\to\infty} t^n(\omega)$. The sequence $\boldsymbol{\omega}''$ is called the *generalised Thue-Morse sequence of the word* ω;

ii) $\boldsymbol{\omega}''' = w_1 \dots w_{\ell(\omega)}(\bar{w}_1 \dots w_{\ell(\omega)-1}1)^\infty$;

For $\boldsymbol{\omega} = 1$, $1''' = 1(01)^\infty$. Observe that if $\boldsymbol{\omega} = 11$ then $\boldsymbol{\omega}''$ coincides with $\sigma(\boldsymbol{\alpha}^*)$. From now on, all the considered sequences $\boldsymbol{\alpha} \in \Sigma_2^1$ and all the considered words ω will satisfy $\boldsymbol{\alpha} \succ \sigma(\boldsymbol{\alpha}^*)$ and $\omega \succ \sigma(\boldsymbol{\alpha}^*)$. This property will imply that the considered symmetric subshifts have positive topological entropy.

Definition 4.1.4. Let ω be a finite word starting with 1. We say that ω is *irreducible* if for any $k < \ell(\omega)$ such that $w_k = 1$, the word $\omega_k = w_1 \dots w_k$ satisfies that $\omega_k''' \prec \omega$. Also, we say that a finite sequence $\boldsymbol{\omega}$ is irreducible if $\boldsymbol{\omega} = \omega 0^\infty$ where ω is an irreducible word.

From the definition of $\boldsymbol{\omega}'''$ we know that 11 is an irreducible word. Observe that it is not true that for any $k \leq \ell(\omega)$ such that $w_k = 1$ the word $\boldsymbol{\omega}_k'''$ belongs to $\Sigma_{\boldsymbol{\omega}}$. For example, the word 11011010011 is an irreducible word, and the word $(1101101001)'''$ is not admissible because

$$(1101101001)''' = 1101101001(0010010111)^\infty.$$

Note that 1101101001 is not an irreducible word.

The motivation for the definition of *irreducible word* will make sense after we show that every symmetric subshift corresponding an irreducible sequence $\boldsymbol{\alpha}$ is transitive. Before stating and proving Theorem 4.1.7, we need to state some technical remarks and lemmas.

Remark 4.1.5. Observe that if $\boldsymbol{\omega}$ is a finite sequence ending with 1, then for every $k \in \mathbb{N}$ and for every $\upsilon \in B_k(\Sigma_\omega)$ there exists $j > k$ such that the word $u_1 \dots u_k(u_{k+1} \dots u_{j-1}1) \in B_j(\Sigma_\omega)$. This is a direct consequence of the symmetry of the subshift and [EJK90, Theorem 1].

Lemma 4.1.6. *If $\boldsymbol{\omega}$ is a finite sequence. Then for every $\nu \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$ such that $v_1 = 0$, the word $1\nu \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$.*

Proof. Let $\nu \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$ and consider the word 1ν. Note that for every $1 \leq j \leq \ell(\nu)$, $\sigma^j(1\nu) \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$. By Theorem 4.1.1 $\omega_i = 1$ for every $i \in \{1, \dots n\}$ for some $2 \leq n \leq \ell(\boldsymbol{\omega})$. This implies that $1\nu \prec \omega$. By symmetry $\bar{\omega} \prec 1\nu$. □

Let $\boldsymbol{\omega} \in [11(01)^\infty, 1^\infty]_\prec$. From Definition 2.2.5 and Lemma 4.1.6 it is true that for any $\nu \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$ such that $v_1 = 1$ the word $0\nu \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$. Thus, by Lemma 4.1.6, for every word starting with 0 (respectively 1) the word $1(01)^k\nu$ is admissible (respectively $0(10)^k\nu$) for any $k \in \mathbb{N}$. Consider $\nu \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$ and $v_1 = 1$. Then from Proposition 4.1.2 and an adapted proof of Lemma 4.1.6 it is true that the word $0^{n-1}\nu$ is admissible if $\boldsymbol{\omega} \in [1^n, 1^{n+1}]$. Moreover, note that it is not always possible to concatenate a finite word υ ending with 1 with $(01)^k$ concatenating it to the left, i.e. $\upsilon(01)^k$ is not necessarily an element of $\mathcal{L}(\Sigma_\omega)$. For example, if $\omega = 11101$ and $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}})$ is the symmetric subshift given by $\boldsymbol{\omega}$ the word $111(01)^2 \succ 11101$. Therefore, $111(01)^2 \notin \mathcal{L}(\Sigma_\omega)$.

Next, we show that the subshift corresponding to an irreducible sequence $\boldsymbol{\omega}$ is transitive. Furthermore, the presented proof gives an upper bound for the specification number of a symmetric subshift of finite type defined in Section 4.3. This bound is essential to prove the results presented in Section 4.3.

Theorem 4.1.7. *For every irreducible sequence $\boldsymbol{\omega}$, $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}})$ is a transitive subshift of finite type.*

Proof. It suffices to show that for any $\upsilon, \nu \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$ such that $\ell(\upsilon), \ell(\nu) \leq \ell(\omega)$ there exists ϖ such that $\upsilon\varpi\nu \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$. Note that by Lemma 4.1.6 we only need to consider words υ ending with 1 and words ν starting with 1. Moreover, Lemma 4.1.6 also implies that we just need to consider $\upsilon, \nu \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$ such that $\ell(\upsilon) = \ell(\nu) = \ell(\omega)$.

Consider an irreducible sequence $\boldsymbol{\omega}$ corresponding to the irreducible word $\omega = w_1 \dots w_{\ell(\omega)}$ such that $\boldsymbol{\omega} \in [1^n, 1^{n+1})_\prec$ for some $n \geq 2$. Observe that if υ satisfies that the blocks 0^n or 1^n do not occur in υ, then 0^{n-1} is a bridge between υ and ν.

Consider $\upsilon, \nu \in B_{\ell(\omega)}(\Sigma_{\boldsymbol{\omega}})$ such that 0^n and 1^n do not occur simultaneously in υ. Let

$$j = \min\{k \in \{0, \dots \ell(\omega) - n\} \mid \sigma^k(\upsilon) = 0^n 1 \dots u_{\ell(\omega)-1} 1 \text{ or } 1^n 0 \dots u_{\ell(\omega)-1} 1\}.$$

Observe that $\sigma^j(\upsilon) \in B_{\ell(\omega)-j}(\Sigma_\omega)$. Suppose that $\sigma^j(\upsilon)$ starts with 0^n. If $\sigma^j(\upsilon) \succ \omega_{\min_{\ell(\omega)-j}}$, then 0^{n-1} is a bridge between υ and ν. If $\sigma^j(\upsilon) \preccurlyeq \omega_{\min_{\ell(\omega)-j}}$, let $z = z_1, \ldots, z_j$ be such that $z_i = \omega_{\min_{\ell(\omega)_{j+i}}}$ for $i \in \{1, \ldots, j\}$. Then $\sigma^j(\upsilon)z = \omega_{\min}$. Note that $[\omega_{\min}0^{n-1}]$ is always an admissible block. Then the word $\varpi = z0^{n-1}$ is a bridge between υ and ν. Observe that the argument is similar if $\sigma^j(\nu)$ starts with 1^n. In such a case, $z = z_1, \ldots, z_j$ will be such that $z_i = \omega_{\max_{\ell(\omega)_{j+i}}}$. Then the word $\varpi = z10^{n-1}$ is a bridge between υ and ν. If 0^n and 1^n occur simultaneously in υ and $u_{\ell(\upsilon)-j} \neq 1$ for every $j \in \{0, \ldots, n-1\}$ consider

$$j = \max\{j \in \{0, \ldots, \ell(\omega) - 2n\} \mid \sigma^k(\upsilon) = 0^n1 \ldots u_{\ell(\omega)-1}1 \text{ or } 1^n0 \ldots u_{\ell(\omega)-1}1\}.$$

Then we can construct ϖ in a similar way as explained above. Suppose $u_{\ell(\upsilon)-j} \neq 1$ for every $j \in \{0, \ldots, n-1\}$. Then let

$$k = \max_{1 \leq j < \ell(\omega)} \{\omega_k = 1 \text{ and } w_1, \ldots w_k \text{ is irreducible }\}.$$

Let $\omega_k = w_1 \ldots w_k$. Note that such a k exists because $11(01)^\infty \leq \omega$. Observe that $\bar{\omega}_k^{n-1}$ is admissible and $1^n\bar{\omega}_k^{n-1}0^{n-1}$ is an admissible word. Then $\bar{\omega}_k0^{n-1}$ is a bridge from υ to ν. □

An immediate consequence of Theorems 4.1.3 and 4.1.7 is that if $\boldsymbol{\omega}$ is an irreducible sequence, then $\boldsymbol{\omega} \succ 11(01)^\infty$.

From [LM95, Proposition 4.5.10 (4)] and [ACS09, Proposition 2.10, Proposition 2.16] we can obtain [Nil10, Theorem 46] which states that $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}})$ is topologically mixing if $\boldsymbol{\omega}$ is an irreducible sequence.

4.2 The entropy function structure

We say that a continuous function $f : [a, b] \to \mathbb{R}$, is a *Cantor function, singular function* or a *devil's staircase* if f satisfies the following properties:

1. $f(a) < f(b)$ (or $f(a) > f(b)$);
2. f is non-decreasing (or non-increasing) on $[a, b]$;
3. There exists a set $\mathcal{E}$ of Lebesgue measure 0 such that for all $x \in [a, b] \setminus \mathcal{E}$ the derivative of f in x exists and $f'(x) = 0$. We call $\mathcal{E}$ *the exceptional set of* f.

Note that for every $a \in \left[\frac{1}{3}, \frac{1}{2}\right]$ it is natural to associate to each a the topological entropy of the symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$, and also to associate to each $a \in [\frac{1}{3}, \frac{1}{2}]$ the entropy of f restricted to the attractor $\Lambda_{\boldsymbol{\alpha}}$. In general, it is also natural to fix $a \in (\frac{1}{4}, \frac{1}{2})$ and consider the function that associates to every $b \in (\frac{1}{2}, \chi(a))$, $h_{top}(f_{(a,b)})$. We call this function H_a.

From [Urb87, Theorem 4], the entropy function is continuous. Then we devote this section to show that in fact, the entropy function associated to the attractor of the doubling map with a symmetric hole is a devil's staircase. Moreover, we provide a characterisation of the end points of the entropy plateaus (Theorem 4.2.14) and we characterise the exceptional set $\mathcal{E}$.

A lexicographic interval $[\boldsymbol{\alpha}, \boldsymbol{\beta}]_{\prec}$ is an *entropy plateau*, if $h_{top}(\sigma_{\boldsymbol{\zeta}}) = h_{top}(\sigma_{\boldsymbol{\alpha}})$ for every $\boldsymbol{\zeta} \in [\boldsymbol{\alpha}, \boldsymbol{\beta}]$ and $h_{top}(\sigma_{\boldsymbol{\zeta}}) \neq h_{top}(\sigma_{\boldsymbol{\alpha}})$ if $\boldsymbol{\zeta} \notin [\boldsymbol{\alpha}, \boldsymbol{\beta}]_{\prec}$.

Proposition 4.2.1. *Let $\boldsymbol{\omega}, \boldsymbol{v}$ be finite sequences. If $\boldsymbol{v} \in (\boldsymbol{\omega}, \boldsymbol{\omega}''')_{\prec}$, then $(\boldsymbol{v}, \boldsymbol{v}''')_{\prec} \subset (\boldsymbol{\omega}, \boldsymbol{\omega}''')_{\prec}$.*

Proof. Let $\boldsymbol{\omega}, \boldsymbol{v}$ satisfying our hypothesis. Observe that it suffices to show that $\boldsymbol{v}''' \prec \boldsymbol{\omega}'''$. Observe that $v_i = w_i$ for every $i \in \{1, \ldots, \ell(\omega)\}$. Firstly, we suppose that

$$\boldsymbol{v} = w_1 \ldots w_{\ell(\omega)} (\bar{w}_1 \ldots \bar{w}_{\ell(\omega)-1} 1)^k 0^\infty.$$

Then $\boldsymbol{v}'''_{(k+2)\ell(\omega)} = 0$ and $\boldsymbol{\omega}'''_{(k+2)\ell(\omega)} = 1$. This also implies that $\boldsymbol{v}''' \prec \omega_{k+1}$. Hence, $\boldsymbol{v}''' \prec \boldsymbol{\omega}'''$.

Observe that the sequence of finite sequences $\{\boldsymbol{\omega}_k\}_{k \geq 0}$ given by

$$\boldsymbol{\omega}_k = w_1 \ldots w_{\ell(\omega)} (\bar{w}_1 \ldots \bar{w}_{\ell(\omega)-1} 1)^k 0^\infty$$

satisfies the following properties:

1. $\boldsymbol{\omega}_k \prec \boldsymbol{\omega}_{k+1}$ for every $k \geq 0$ and $\lim_{k \to \infty} \boldsymbol{\omega}_k = \boldsymbol{\omega}'''$;
2. $(\boldsymbol{\omega}_k, \boldsymbol{\omega}'''_k)_{\prec} \cap (\boldsymbol{\omega}_{k+1}, \boldsymbol{\omega}'''_{k+1})_{\prec} = \emptyset$.
3. $\boldsymbol{\omega}'' \in [\boldsymbol{\omega}', \boldsymbol{\omega}_2]_{\prec}$.

Assume now that $\boldsymbol{v} \in (\boldsymbol{\omega}_k, \boldsymbol{\omega}_{k+1})_{\prec}$ for some $k \geq 0$. It suffices to show that $\boldsymbol{v}''' \leq \boldsymbol{\omega}'''_{k+1}$. Observe that $\ell(v) > \ell(\omega_k)$ and $v_i = w_i^k$ for every $i \in \{1, \ldots \ell(\omega_k)\}$ where w_i^k is the i-th digit of ω_k. Moreover, $\ell(\upsilon) \geq \ell(\omega_k)+r$, where $r = \min\{1 \leq r \leq \ell(\omega) \mid \omega_r = 0\}$. Consider

$$S = \{j \in \{\ell(\omega_k) + r, \dots, \ell(\omega_{k+1})\} \mid \omega_j^{k+1} = 1\}.$$

If $\ell(\upsilon) > \ell(\omega_{k+1})$ then there exists $j \in S$ such that $\upsilon_j = 0$. Then, it is obvious that $\upsilon''' \prec \omega'''_{k+1}$. If $\ell(\upsilon) \leq \ell(\omega^{k+1})$, we have two cases to consider. If υ is not a truncated word: i.e. if there exists $j \in S$ such that $\upsilon_j = 1$, $\upsilon_{j+1} = 0$, $j+1 \in S$ and $\ell(\upsilon) > j+1$ then the result holds. Suppose that υ is a truncated word: i.e. there exists $j \in S$ such that $\upsilon_j = 1$ and $\ell(\upsilon) = j$. If $j+1 \in S$ then $\upsilon''' \prec \omega_{k+1}$. Suppose that $j+1 \notin S$. Then $\boldsymbol{\upsilon}''' = \omega_1^{k+1} \dots \omega_j^{k+1} (\bar{\omega}_1^{k+1} \dots \bar{\omega}_{j-1}^{k+1} 1)^\infty$. Observe that in this case, it suffices to show that $\bar{\omega}_1 \dots \bar{\omega}_j \leq (\bar{w}_{j+1} \dots \bar{w}_{\ell(\omega)} \bar{w}_1 \dots \bar{w}_j)^\infty$, which is a consequence of Theorem [EJK90, Theorem 1]. □

Let $\sigma(\boldsymbol{\alpha}^*)$ be the image under the shift map of the Thue-Morse sequence - see Section 4.1 p.44. For every $k \in \mathbb{N}$ consider the word $\omega_{a_k^*} = a_1^* \dots a_{2^k}^*$. We say that a sequence a is an *i-sequence* if $\boldsymbol{\omega}_{\boldsymbol{\alpha}_{i+1}^*}''' \prec \boldsymbol{\alpha} \preccurlyeq \boldsymbol{\omega}_{\boldsymbol{\alpha}_i^*}'''$. Observe that $\boldsymbol{\omega}_{\boldsymbol{\alpha}_1^*} = 110^\infty$ and $\boldsymbol{\omega}'''_{a_1^*} = 11(01)^\infty$. For $i = 0$, $\boldsymbol{\alpha}$ is 0-sequence if it satisfies $\boldsymbol{\omega}'''_{\boldsymbol{\alpha}_1^*} \prec \boldsymbol{\alpha} \prec 1^\infty$. A finite i-sequence $\boldsymbol{\omega}$ is *i-irreducible* if for every $2^i < j < \ell(\omega)$ such that $w_j = 1$, $(w_1 ... w_j)''' \prec \boldsymbol{\omega}$. Observe that 0-irreducible sequences are simply irreducible sequences.

Lemma 4.2.2. *For every $k \in \mathbb{N}$,*

$$h_{top}(\sigma_{\omega'''_{\boldsymbol{\alpha}_k^*}}) = \frac{1}{2^k}.$$

Proof. Let $k \in \mathbb{N}$. Recall that $\omega_{\boldsymbol{\alpha}_k^*} \prec \sigma(\boldsymbol{\alpha}^*)$. By Theorem 4.1.1, $\Sigma_{\boldsymbol{\omega}_{\boldsymbol{\alpha}_k^*}}$ is a countable set. Nonetheless $\boldsymbol{\alpha}^* \prec \boldsymbol{\omega}'''_{\boldsymbol{\alpha}_k^*}$, then $h_{top}(\sigma_{\omega'''_{\boldsymbol{\alpha}_k^*}}) > 0$. Henceforth the entropy of $(\Sigma_{\omega'''_{\boldsymbol{\alpha}_k^*}}, \sigma_{\omega'''_{\boldsymbol{\alpha}_k^*}})$ will be given by (Σ_A, σ_A) where

$$A = \{x \in \Sigma_{\omega'''_{\boldsymbol{\alpha}_k^*}} \mid \omega_{\boldsymbol{\alpha}_k^*} \text{ occurs in } \boldsymbol{x}\}.$$

Take $\boldsymbol{x} \in A$. Without loss of generality it can be assumed that $x_i = w^*_{\boldsymbol{\alpha}_k i}$ for every $i \in \{1 \dots 2^k\}$. Note that $x_i = \omega_{\bar{\boldsymbol{\alpha}}_k^* i}$ for every $i \in \{2^k + 1, \dots, 2^{n+1} - 1\}$ and $x_{2^{k+1}}$ can be chosen. This implies that $|B_n(A)| = 2^{\frac{n}{2^k}}$ for every $n \geq 2^k$, which in turn implies that

$$h_{top}(\sigma_{\omega'''_{a_k^*}}) = \frac{1}{2^k}.$$

□

Note that $h_{top}(\sigma_{\boldsymbol{\alpha}}) = \frac{1}{2}$ if $\boldsymbol{\alpha} = 11(01)^\infty$. Observe that from Lemma 4.2.2, given $i \in \mathbb{N}$ we can find that a particular infinite sequence satisfying $h_{top}(\Sigma_{\boldsymbol{\alpha}}) = \frac{1}{2^i}$. Therefore, it is natural to ask ourselves if there is a finite i-sequence $\boldsymbol{\omega}$ such that $\Sigma_{\boldsymbol{\omega}}$ satisfies the same property. However, we will show that this is not possible. This fact is a consequence of Proposition 4.2.13 and Theorem 4.2.14 below. The following lemma gives an estimate for the length of i-irreducible words.

Lemma 4.2.3. *Let $\boldsymbol{\omega}$ be an i-irreducible sequence. If*

$$\frac{1}{2^{i+1}} < h_{top}(\sigma_{\boldsymbol{\omega}}) < \frac{1}{2^i},$$

then $\ell(\boldsymbol{\omega}) \geq 3 \cdot 2^i$.

Proof. Let $i \geq 0$. By Lemma 4.2.2 we obtain that $\boldsymbol{\omega}'''_{a^*_{i+1}} \prec \omega \prec \boldsymbol{\omega}'''_{a^*_i}$. Consider $\boldsymbol{\alpha} = \boldsymbol{\omega}_{\boldsymbol{\alpha}^*_{i+1}}(\bar{\omega}_{\boldsymbol{\alpha}^*_{i+1}})^2$. Note that $\boldsymbol{\omega}'''_{\boldsymbol{\alpha}^*_{i+1}} \prec \boldsymbol{\alpha} \prec \boldsymbol{\omega}'''_{\boldsymbol{\alpha}^*_i}$, and $\ell(a) = 3 \cdot 2^i$.

Suppose that there exists a finite word ω such that $\boldsymbol{\omega}'''_{\boldsymbol{\alpha}^*_{i+1}} \prec \boldsymbol{\omega} \prec \boldsymbol{\omega}'''_{a^*_i}$, and $\ell(\boldsymbol{\alpha}) > \ell(\omega)$. Note that the first $3 \cdot 2^i - 1$ symbols of $\omega'''_{a^*_{i+1}}$ and $\omega'''_{\boldsymbol{\alpha}^*_i}$ coincide, therefore, $w_1 \dots w_{2^{i+1}} = a_1 \dots a_{2^{i+1}}$. This implies that there exists $j \in \{2^{i+1}, \dots, 3 \cdot 2^i\}$ such that $(\omega'''_{a^*_{i-1}})_j = 0$ and $w_j = 1$, therefore, $\boldsymbol{\omega} \succ \boldsymbol{\omega}'''_{a^*_i}$, which is a contradiction. □

Lemma 4.2.4. *Let $\boldsymbol{\omega}, \boldsymbol{\upsilon}$ be finite i-irreducible sequences satisfying that $\boldsymbol{\omega} \neq \boldsymbol{\upsilon}$. Then $(\boldsymbol{\omega}, \boldsymbol{\omega}''')_\prec \cap (\boldsymbol{\upsilon}, \boldsymbol{\upsilon}''')_\prec = \emptyset$.*

Proof. Let $\omega = w_1 \dots w_{\ell(\omega)}$ and $\upsilon = v_1 \dots v_{\ell(\upsilon)}$ be the i-irreducible corresponding to $\boldsymbol{\upsilon}$ and $\boldsymbol{\upsilon}$ respectively words with $i \geq 0$. Without loss of generality we can assume that $\omega \prec \upsilon$. It is already assumed $(\boldsymbol{\omega}, \boldsymbol{\omega}''') \cap (\boldsymbol{\upsilon}, \boldsymbol{\upsilon}''') \neq \emptyset$. Then by Proposition 4.2.1, $(\boldsymbol{\upsilon}, \boldsymbol{\upsilon}''')_\prec \subset (\boldsymbol{\omega}, \boldsymbol{\omega}''')_\prec$. This implies that $\ell(\upsilon) \geq \ell(\omega)$ and that $v_j = w_j$ for every $j \in \{1, \dots, \ell(\omega)\}$. By the i-irreducibility of υ, $\boldsymbol{\omega}''' \prec \boldsymbol{\upsilon}$, which is a contradiction. □

Lemma 4.2.5. *For every finite sequence $\boldsymbol{\upsilon} \succ \boldsymbol{\alpha}^*$ there exists a unique i-irreducible sequence $\boldsymbol{\omega}$ such that $\boldsymbol{\upsilon} \in [\boldsymbol{\omega}, \boldsymbol{\omega}''']$.*

Proof. From Lemma 4.2.2, $\boldsymbol{\upsilon}$ is an i-sequence for some $i \geq 0$. Observe that if $\boldsymbol{\upsilon}$ is an i-irreducible sequence then $\boldsymbol{\omega} = \boldsymbol{\upsilon}$ satisfies the conclusion of the lemma automatically. Consider $\boldsymbol{\upsilon}$ to be an i-sequence for some $i \geq 0$ such that $\boldsymbol{\upsilon}$ is not i-irreducible. Then there exists $2 < j < \ell(\upsilon)$ such that $\boldsymbol{\upsilon}'''_j \geq \boldsymbol{\upsilon}$, where $\boldsymbol{\upsilon}_j = v_1 \dots v_j 0^\infty$ and $v_j = 1$. By Lemma 4.2.1, $\boldsymbol{\upsilon} \in (\boldsymbol{\upsilon}_j, \boldsymbol{\upsilon}'''_j)$. Therefore, if $\boldsymbol{\upsilon}_j$ is i-irreducible then $\boldsymbol{\omega} = \boldsymbol{\upsilon}_j$. Assume that for every $2 < j < \ell(v)$, $\boldsymbol{v}_j$ is not i-irreducible. Observe that $v_{2^{i+1}} = 1$ since if $v_{2^{i+1}} = 0$ then $\boldsymbol{v}$ will not be an i-sequence. Moreover, $v_1 \dots v_{2^{i+1}} = \omega_{a^*_{i+1}}$. This implies that $\boldsymbol{v} \prec (v_1 \dots v_{2^{i+1}})'''$, which contradicts that $\boldsymbol{v}$ is an i-sequence. The uniqueness of the i-irreducible sequence $\boldsymbol{\omega}$ is given by Lemma 4.2.4. □

We will show that the intervals of the form $[\boldsymbol{\omega}, \boldsymbol{\omega}''']_\prec$ satisfy the conditions to be entropy plateaus. For this purpose, we need to show that every symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ contains a unique *transitive component of maximal entropy*. Given a dynamical system (X, f), a subset A of X is a *transitive component* if A is closed, completely invariant (i.e. $f^{-1}(A) = A = f(A)$), $f\mid_A: A \to A$ is topologically transitive and there is no other set A' such that $A \subsetneq A'$ and A' contains a dense orbit [BKT11, p. 1313]. It is known that $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has at most 8 transitive components - see [BKT11, Theorem 6.3]. Moreover, if the $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is a subshift of finite type then it has at most 4 components - see [BKT11, Theorem 6.4]. Moreover, by [BKT11, Theorem 6.4] we can be sure that there exists a transitive component of maximal entropy $A \subset \Sigma_{\boldsymbol{\alpha}}$ for every $\boldsymbol{\alpha}$.

The following lemma provide us an upper bound for the transitive components that do not reach the topological entropy of $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$.

Lemma 4.2.6. *Consider $\boldsymbol{\alpha} \in [\sigma(\boldsymbol{\alpha}^*), 1^\infty]$ and let $\boldsymbol{\omega}$ be an i-irreducible finite sequence for $i \in \mathbb{N}$. If $\boldsymbol{\alpha} \in (\boldsymbol{\omega}, \boldsymbol{\omega}''']$, then $h_{top}(\sigma\mid_A) \leq \frac{1}{\ell(\boldsymbol{\omega})} < h_{top}(\sigma_{\boldsymbol{\omega}})$, where A is any transitive component of $\Sigma_{\boldsymbol{\alpha}}$ such that $h_{top}(A) < h_{top}(\Sigma_{\boldsymbol{\alpha}})$.*

Proof. Let $\boldsymbol{\omega} = w_1 \dots w_{\ell(\boldsymbol{\omega})} 0^\infty$ be an i-irreducible sequence and $\boldsymbol{\alpha} \in (\boldsymbol{\omega}, \boldsymbol{\omega}''']$. By Proposition 4.1.2 $\boldsymbol{\alpha} \in (1^n, 1^{n+1}]$ for some $n \geq 2$. Note that $\Sigma_{\boldsymbol{\omega}} \subset \Sigma_{\boldsymbol{\alpha}}$ and

$$(\Sigma_{\boldsymbol{\alpha}} \setminus \Sigma_{\boldsymbol{\omega}}) = \{x \in \Sigma_{\boldsymbol{\alpha}} \mid \omega \text{ or } \bar{\omega} \text{ occurs in } \boldsymbol{x}\}$$

since the word ω is forbiden in $\Sigma_{\boldsymbol{\omega}}$. Consider $A \subset (\Sigma_{\boldsymbol{\alpha}} \setminus \Sigma_{\boldsymbol{\omega}})$, a transitive component given by [BKT11, Theorem 6.3]. Since A is a transitive component then A must be $\sigma_{\boldsymbol{\alpha}}$-invariant. Therefore, it is only required to consider words such that ω or $\bar{\omega}$ occurs in the first $\ell(\omega)$ positions. Furthermore, $|B_n(A)| \geq 0$ if $n \geq \ell(\omega)$, otherwise is zero. Moreover, $|B_n(A)| \geq 2$ for $n > \ell(\omega)$ and $|B_{\ell(\omega)}(A)| = 2$. Then for every $1 \leq i < \ell(\omega)$, if $v \in B_{\ell(\omega)+i}$ the $\ell(\omega) + j$ entry is fixed for every $j \leq i$ because $\boldsymbol{\alpha} \preccurlyeq \boldsymbol{\omega}'''$. Indeed, if v starts with ω then $u_{\ell(\omega)+i} = \bar{w}_i$. Hence $|B_{\ell(\omega)+i}| \leq 2^{\frac{n}{\ell(\omega)}}$. Notice that the digit $u_{2\ell(\omega)}$

can be chosen from 0 or 1. Then $|B_{2\ell(\omega)}(A)| = 4$. Therefore, $|B_n(A)| \leq 2^{\frac{n}{\ell(\omega)}}$, which implies that

$$h_{top}(A) = \lim_{n\to\infty} \frac{1}{n} \log |B_n(A)| \leq \frac{1}{\ell(\omega)}.$$

Observe that by Lemmas 4.2.2 and 4.2.3 we can conclude that $h_{top}(A) < h_{top}(\Sigma_{\boldsymbol{\omega}})$. □

Theorem 4.2.7. *Let $i \in \mathbb{N}$. Then for every i-irreducible sequence $\boldsymbol{\omega}$ and $\boldsymbol{\alpha} \in [\boldsymbol{\omega}, \boldsymbol{\omega}''']$, $\Sigma_{\boldsymbol{\alpha}}$ contains a unique transitive component A such that $h_{top}(\sigma\,|_A) = h_{top}(\sigma_{\boldsymbol{\omega}})$.*

Proof. Since $\boldsymbol{\omega}$ is an i-irreducible sequence and $i \in \mathbb{N}$, then $\boldsymbol{\omega} = 11(01)^k001\ldots$ for some $k \geq 1$. Recall that the words $\upsilon = [(10)^{k-1}11$, $\nu = [(10)^k]$ and their mirror images are admissible blocks of length 2^k and that there is no $\varpi \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$ such that $\upsilon\varpi\nu \in \mathcal{L}(\Sigma_{\boldsymbol{\omega}})$ -see Theorem 4.1.3. Define

$$X = \{x \in \Sigma_{\boldsymbol{\omega}} \mid \upsilon, \nu, \bar{\upsilon} \text{ or } \bar{\nu} \text{ occurs in } \boldsymbol{x}\}.$$

Note that if υ or $\bar{\upsilon}$ occur in $\boldsymbol{x}$ then there exists $N \geq \ell(\boldsymbol{\omega})$ such that $\sigma^n(\boldsymbol{x}) \in \Sigma_{\boldsymbol{\omega}} \setminus X$ for every $n \geq N$. Besides, if $\nu, \bar{\nu}$ occurs in $\boldsymbol{x}$, then $\boldsymbol{x}$ has to be $(01)^\infty$ or $(10)^\infty$. Therefore $\{(01)^\infty, (10)^\infty\}$ is a transitive component with zero topological entropy, which we denote by A_1. Let A be a transitive component such that $h_{top}(\sigma_A) = h_{top}(\sigma_{\boldsymbol{\omega}})$. Note that by [BS02, Proposition 2.5.5],

$$h_{top}(\Sigma_\omega) = \max\{h_{top}(A_j)\} = h_{top}(\sigma_A)$$

where A_j are the transitive components of $\Sigma_{\boldsymbol{\omega}}$. Suppose that there exists another transitive component A_2 such that $A_2 \cap A = \emptyset$ and $h_{top}(\sigma_{A_2}) = h_{top}(\sigma_{\boldsymbol{\omega}})$. Then $A_2 \neq \{(01)^\infty, (10)^\infty\}$ and $A_2 \subset X$. Since $A_2 \neq \{(01)^\infty, (10)^\infty\}$ then for every $\boldsymbol{x} \in A_2$, υ or $\bar{\upsilon}$ occur in $\boldsymbol{x}$, therefore A is not σ-invariant, then A is not a transitive component. □

The unique transitive component of a non-transitive subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is called *the main component of* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$. From Lemma 4.2.5 and Lemma 4.2.6 the conclusion of Theorem 4.2.7 also hold for $\boldsymbol{\alpha} \succ 11(01)^\infty$. In this case, the main component of $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}}$ is parametrised by an irreducible sequence $\boldsymbol{\omega}$, and by Theorem 4.1.7 $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}})$ is a transitive subshift of finite type.

In the proof of the following lemma, we need the following auxiliary result: If (Σ, σ) is a transitive sofic subshift and (A, σ_A) is a proper subshift of (Σ, σ) then $h_{top}(\sigma) > h_{top}(\sigma_A)$ [LM95, Corollary 4.4.9].

Lemma 4.2.8. *Let $\boldsymbol{\omega} \in [\sigma(\boldsymbol{\alpha}^*), 1^\infty]_\prec$ and let $\boldsymbol{\omega}$ be an i-irreducible sequence for $i \geq 0$. If $\boldsymbol{\alpha} \notin (\boldsymbol{\omega}, \boldsymbol{\omega}''')$, then $h_{top}(\sigma_{\boldsymbol{\alpha}}) \neq h_{top}(\sigma_{\boldsymbol{\omega}})$.*

Proof. Without losing generality suppose that $\boldsymbol{\omega}''' \prec \boldsymbol{\alpha}$. Then there exists $k \in \mathbb{N}$ such that $\boldsymbol{\omega}'''_k = 0$ and $a_k = 1$. Let $\boldsymbol{\nu} = a_1 \dots a_k 0^\infty$. Recall that

$$h_{top}(\sigma_{\boldsymbol{\omega}}) \leq h_{top}(\sigma_{\boldsymbol{\nu}}) \leq h_{top}(\sigma_{\boldsymbol{\alpha}}),$$

since $\Sigma_{\boldsymbol{\omega}'''} \subset \Sigma_{\boldsymbol{\nu}} \subset \Sigma_{\boldsymbol{\alpha}}$.

Note that $\boldsymbol{\nu} \notin (\boldsymbol{\omega}, \boldsymbol{\omega}''')_\prec$. Let A_ω and A_ν be the main components of $\Sigma_{\boldsymbol{\omega}}$ and $\Sigma_{\boldsymbol{\nu}}$ respectively. Observe that $\nu_{\min}$ and $\nu_{\max} \in \mathcal{L}(A_\nu)$. This implies for any $u, v \in \mathcal{L}(A_\omega)$, there exist $w, z \in \mathcal{L}(\Sigma_\nu)$ such that $uw[a_1 \dots a_{k-1}0]^j zv \in \mathcal{L}(A_\nu)$ for any $j \in \mathbb{N}$. Moreover,

$$uw[a_1 \dots a_{k-1}0]^j zv \in \mathcal{L}(A_\nu \setminus A_\omega).$$

Furthermore, $(a_1 \dots a_{k-1}0)^\infty$ and $\overline{(a_1 \dots a_{k-1}0)^\infty} \in A_\nu \setminus A_\omega$. This implies that A_ω is a proper subshift of A_ν and (A_ν, σ_{A_ν}) is a transitive subshift of finite type, thus (A_ν, σ_{A_ν}) is sofic and transitive. Then by Theorem 4.2.7 and [LM95, Corollary 4.4.9] we conclude $h_{top}(\sigma_{\boldsymbol{\alpha}}) > h_{top}(\sigma_{\boldsymbol{\omega}})$. □

Approximation properties and the exceptional set

We introduce two different ways to approximate subshifts in terms of shifts of finite type as follows.

Definition 4.2.9. Let (Σ, σ) be a subshift. We say that (Σ, σ) is *approximated from below* if there exists a sequence of subshifts $\{(\Sigma^n, \sigma_n)\}$ such that for every $n \in \mathbb{N}$:

i) $\Sigma^n \subset \Sigma^{n+1}$;

ii) $\Sigma = \overline{\bigcup_{n=1}^{\infty} \Sigma^n}$ where the closure is taken in Σ_2;

iii) there exists a homeomorphic copy X^n of Σ^n contained in Σ^{n+1} such that $\sigma_{n+1} |_{X^n} = \sigma_n$;

iv) there exists a homeomorphic copy Y^n of Σ^n contained in Σ such that $\sigma |_{Y^n} = \sigma_n$.

Furthermore, we say that (Σ, σ) is *approximated from above*, if there exists a sequence of subshifts $\{(\Sigma^n, \sigma_n)\}$ such that for every $n \in \mathbb{N}$:

i') $\Sigma^n \supset \Sigma^{n+1}$ and $\sigma_n \mid_{\Sigma^{n+1}} = \sigma_{n+1}$;

ii') $\Sigma = \bigcap_{n=1}^{\infty} \Sigma^n$ and;

iii') $\sigma = \sigma_n \mid_{\Sigma}$.

We consider homeomorphic copies of Σ^n, X^n and Y^n for the approximation from below since we cannot be sure that Σ^n is an invariant set neither under σ_{n+1} nor σ.

Theorem 4.2.10. *For any $\boldsymbol{\alpha} \in [1(0)^\infty, 1^\infty)_\prec$, there exist sequences $\{\boldsymbol{\alpha}_n^+\}_{n=1}^\infty$ and $\{\boldsymbol{\alpha}_n^-\}_{n=1}^\infty$ of finite sequences such that:*

1. *$(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is approximated from below by $(\Sigma_{\boldsymbol{\alpha}_n^-}, \sigma_{\boldsymbol{\alpha}_n^-})$;*

2. *$(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is approximated from above by $(\Sigma_{\boldsymbol{\alpha}_n^+}, \sigma_{\boldsymbol{\alpha}_n^+})$.*

Proof. Note that if $\boldsymbol{\alpha}$ is a finite sequence it suffices to consider the sequence $\boldsymbol{\alpha}_n = \boldsymbol{\alpha}$ and the sequence of subshifts $(\Sigma_{\boldsymbol{\alpha}_n} \sigma_{\boldsymbol{\alpha}_n}) = (\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ for every $n \in \mathbb{N}$ to approximate from below and from above $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$.

Firstly, we will prove item *(1)*. Consider an infinite sequence $\boldsymbol{\alpha}$. By Theorem 4.1.1, we may assume that $a_1 = a_2 = 1$, and by Proposition 4.1.2 we can assume that $a_1 = \ldots = a_n = 1$ for some $n \geq 2$. Consider $\boldsymbol{\alpha}_1^- = 1^n$. Let $k_2 > n$ be the first index such that $a_k = 1$ and $a_j = 0$ for every $n < j < k$. Such k exists because $\boldsymbol{\alpha}$ is an infinite sequence. Let $\boldsymbol{\alpha}_2^- = a_1 \ldots a_{k_2}$. Define k_3 as the first $k > k_2$ such that $a_k = 1$ and $\boldsymbol{\alpha}_3^- = a_1 \ldots a_{k_3}$. Then we can define a sequence k_n to be the first index k such that $k > k_{n-1}$ and $a_k = 1$, and $\boldsymbol{\alpha}_n^-$ to be $a_1 \ldots a_{k_n}$. Note that $\boldsymbol{\alpha}_n^- \underset{n\to\infty}{\longrightarrow} \boldsymbol{\alpha}$.

Note that $\boldsymbol{\alpha}_n^-$ is an increasing sequence of finite sequences such that $\boldsymbol{\alpha}_n^- \prec \boldsymbol{\alpha}_{n+1}^-$ for every $n \in \mathbb{N}$. Therefore, $\Sigma_{\boldsymbol{\alpha}_n^-} \subset \Sigma_{\boldsymbol{\alpha}_{n+1}^-}$ for every $n \in \mathbb{N}$, which gives $i)$ of Definition 4.2.9.

Note that Definition 2.2.5 and the fact that $\Sigma_{\boldsymbol{\alpha}}$ is a closed set imply

$$\overline{\bigcup_{n=1}^{\infty} \Sigma_{\boldsymbol{\alpha}_n^-}} \subset \Sigma_{\boldsymbol{\alpha}}.$$

Consider a cylinder $[\omega] \subset \Sigma_{\boldsymbol{\alpha}}$, that is $[\omega] = \{\boldsymbol{x} \in \Sigma_{\boldsymbol{\alpha}} \mid \omega \text{occurs in } \boldsymbol{x}\}$. Recall that cylinder sets generate the Borel σ-algebra for symbolic spaces, thus, every admissible cylinder set is measurable.Also, recall that there exists a one to one correspondence between cylinders of length n and admissible words of length n. Therefore, we can consider $[\omega]$ as an admissible word of length $\ell(\omega)$. Let $n \in \mathbb{N}$ such that $\ell(\boldsymbol{\alpha}_n^-) \le \ell(\omega) < \ell(\boldsymbol{\alpha}_{n+1}^-)$. Then $\omega \in B_{\ell(\omega)}(\Sigma_{\boldsymbol{\alpha}_{n+1}^-})$. Therefore, there exists a point $\boldsymbol{x} \in \Sigma_{\boldsymbol{\alpha}_{n+1}^-}$ such that $\boldsymbol{x} \in [\omega]$, hence $\bigcup_{n=1}^{\infty} \Sigma_{\boldsymbol{\alpha}_n^-}$ is dense on $\Sigma_{\boldsymbol{\alpha}}$, which gives us *ii*).

Let $n \in \mathbb{N}$ and consider $\boldsymbol{\alpha}_n$ and $\boldsymbol{\alpha}_{n+1}$ and their associated subshifts $\Sigma_{\boldsymbol{\alpha}_n^-}$ and $\Sigma_{\boldsymbol{\alpha}_{n+1}^-}$. Let

$$X^n = \{x \in \Sigma_{\boldsymbol{\alpha}_{n+1}^-} \mid a_n^- \text{ or } \bar{a}_n^- \text{ does not occur in } x\}.$$

Note that this set is $\sigma_{\boldsymbol{\alpha}_{n+1}}$ invariant and it is in bijective correspondence with $\Sigma_{\boldsymbol{\alpha}_n^-}$ by associating to each sequence $\boldsymbol{x} \in \Sigma_{\boldsymbol{\alpha}_n^-}$ the same sequence $\boldsymbol{x}$ in X^n. This gives us item *iii*) of Definition 4.2.9. Using the same argument we can construct Y^n in Σ_a, satisfying the properties required by Definition 4.2.9 *iv*).

To show *(2)*, note that the construction of the sequence is similar to item *(1)* but we have to take certain considerations. Consider $\boldsymbol{\alpha}_n^-$ defined above. Let $\boldsymbol{\alpha}_1^+ = \boldsymbol{\alpha}_{k_j}^- 1$, where k_n is the first index such that $a_{k_j+1} = 0$ and $k_j > n$, where n is given by $\boldsymbol{\alpha}_1^-$. Then let $\boldsymbol{\alpha}_i^+ = \boldsymbol{\alpha}_{k_i}^- 10^\infty$, where i satisfies that k_i is the first index such that $a_{k_i+1} = 0$ and $k_i > k_{i-1}$. Note that $\boldsymbol{\alpha}_n^+ \underset{n\to\infty}{\longrightarrow} \boldsymbol{\alpha}$. Observe that $\boldsymbol{\alpha}_n^+$ is a decreasing sequence such that $\boldsymbol{\alpha}_{n+1}^+ \prec \boldsymbol{\alpha}_n^-$ for every $n \in \mathbb{N}$, then $\Sigma_{\boldsymbol{\alpha}_{n+1}^+} \subset \Sigma^{\boldsymbol{\alpha}_n^+}$ for every $n \in \mathbb{N}$. Given $\boldsymbol{x} \in \Sigma_{\boldsymbol{\alpha}_{n+1}^+}$, $\sigma_n(\boldsymbol{x}) \in \Sigma_{\boldsymbol{\alpha}_{n+1}^+}$, this shows that $\sigma_{\boldsymbol{\alpha}_n}(\boldsymbol{x}) = \sigma_{\boldsymbol{\alpha}_{n+1}}(\boldsymbol{x})$, proving *i′*).

By construction $\boldsymbol{\alpha} \prec \boldsymbol{\alpha}_n^+$ for every $n \in \mathbb{N}$, then $\Sigma_{\boldsymbol{\alpha}} \subset \Sigma_{\boldsymbol{\alpha}_n^+}$. This fact implies that $\Sigma_{\boldsymbol{\alpha}} \subset \bigcap_{n=1}^{\infty} \Sigma_{\boldsymbol{\alpha}_n^+}$. Let $\boldsymbol{x} \in \bigcap_{n=1}^{\infty} \Sigma_{\boldsymbol{\alpha}_n^+}$. Then $\bar{\boldsymbol{\alpha}}_n^+ \prec \sigma_{\boldsymbol{\alpha}_n^+}^j(\boldsymbol{x}) \prec \boldsymbol{\alpha}_n^+$ for every $j, n \ge 0$. Therefore, $\bar{\boldsymbol{\alpha}} \preccurlyeq \sigma^j(\boldsymbol{x}) \preccurlyeq \boldsymbol{\alpha}$ for every $j \ge 0$.

Note that for every $\sigma_{\boldsymbol{\alpha}}(\boldsymbol{x}) = i_n(\sigma_{\boldsymbol{\alpha}}(\boldsymbol{x})) = \sigma_{\boldsymbol{\alpha}_n}(i_n(x))$, where $i_n : \Sigma_{\boldsymbol{\alpha}} \to \Sigma_{\boldsymbol{\alpha}_n^+}$ is the nth inclusion map. This proves item *iii′*). □

Note that if $\boldsymbol{\alpha} \in (11(01)^\infty, 1^\infty)$ and $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is a transitive subshift, then by Theorem 4.2.9 $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is a coded system. Moreover, Theorem 4.2.10 gives us a tool to understand the behaviour of subshifts that are not defined by finite sequences using subshifts of finite type. One of its consequences is the following theorem that allows to calculate $B_n(\Sigma_{\boldsymbol{\alpha}})$ for an infinite sequence $\boldsymbol{\alpha}$. Notice that if $\boldsymbol{\alpha}$ is a finite sequence, $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is approximated by itself. We will call the approximations (1) and (2) given by Theorem 4.2.10 *canonical approximations from below and above*, respectively.

Theorem 4.2.11. *Let* $\boldsymbol{\alpha} \in [(10)^\infty, 1^\infty]_\prec$ *be an infinite sequence. Then for every*

$m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $B_m(\Sigma_{\boldsymbol{\alpha}}) = B_m(\Sigma_{\boldsymbol{\alpha}_n}) = B_m(\Sigma_{\boldsymbol{\alpha}_N})$ for every $n \geq N$, where $\boldsymbol{\alpha}_n$ is a sequence such that $\boldsymbol{\alpha}_n \preccurlyeq \boldsymbol{\alpha}_{n+1}$ and $\boldsymbol{\alpha}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{\alpha}$.

Proof. Let $\boldsymbol{\alpha}$ be an infinite sequence and $\boldsymbol{\alpha}_n$ be the sequence given by Theorem 4.2.10(1). Note that this sequence satisfies $\boldsymbol{\alpha}_n \preccurlyeq \boldsymbol{\alpha}_{n+1}$, $\boldsymbol{\alpha}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_n$ is a finite sequence for every $n \in \mathbb{N}$. Recall that for every $\boldsymbol{\alpha}_n$, $a_i = 1$ for every $1 \leq i \leq k$ for some $k \in \mathbb{N}$. If $m < k$ then $B_m(\Sigma_{\boldsymbol{\alpha}_n})$ is determined by $B_m(X_{\mathcal{F}})$, where $\mathcal{F} = \{0^k, 1^k\}$ and $B_m(\Sigma_{\boldsymbol{\alpha}_n}) = B_m(\Sigma_{\boldsymbol{\alpha}})$ for every $n \in \mathbb{N}$. Besides, if $m = k$ then

$$B_m(\Sigma_{\boldsymbol{\alpha}}) = B_m(\Sigma_{\boldsymbol{\alpha}_n}) = B_m(X_{\mathcal{F}'}),$$

where $\mathcal{F}' = \{0^{k+1}, 1^{k+1}\}$. Suppose that $m > k$. Let $N \in \mathbb{N}$ such that $\boldsymbol{\alpha}_N$ is an element of $\{\boldsymbol{\alpha}_n\}_{n=1}^{\infty}$ such that $\ell(\boldsymbol{\alpha}_{N-1}) \leq m < \ell(\boldsymbol{\alpha}_N)$. It suffices to show that $B_m(\Sigma_{\boldsymbol{\alpha}_N}) = B_m(\Sigma_{\boldsymbol{\alpha}_{N+1}})$. Suppose that $B_m(\Sigma_{\boldsymbol{\alpha}_N}) \neq B_m(\Sigma_{\boldsymbol{\alpha}_{N+1}})$, then by Definition 2.2.5 $|B_m(\Sigma_{\boldsymbol{\alpha}_N})| < |B_m(\Sigma_{\boldsymbol{\alpha}_{N+1}})|$. This implies that there exists $\omega \in B_m(\Sigma_{\boldsymbol{\alpha}_{N+1}})$ such that $\omega \succ v$ and $\bar{\omega} \prec v$ for every $v \in B_m(\Sigma_{\boldsymbol{\alpha}_N})$. Let $\nu = v_1 \dots v_m$ be the maximal element of $B_m(\Sigma_{\boldsymbol{\alpha}_N})$. This implies that there exists $k \leq m$ such that $w_k = 1$ and $v_k = 0$. Note that the first m terms of the word associated to $\boldsymbol{\alpha}_{N+1}$, α_{N+1} are equal to v_i. Then $v \succcurlyeq \alpha_{N+1}$, which is a contradiction. □

The following lemma is essentially a corollary of [LM95, Exercise 1.5.10].

Lemma 4.2.12. *Let $\boldsymbol{\alpha} \in [(10)^\infty, 1^\infty]_\prec$ be an infinite sequence. Then for every $m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $B_m(\Sigma_{\boldsymbol{\alpha}}) = B_m(\Sigma_{\boldsymbol{\alpha}_n})$ for every $n \geq N$, where $\boldsymbol{\alpha}_n$ is a sequence such that $\boldsymbol{\alpha}_n \succcurlyeq \boldsymbol{\alpha}_{n+1}$ and $\boldsymbol{\alpha}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{\alpha}$.*

Proof. Let $\boldsymbol{\alpha}$ be an infinite sequence and $\boldsymbol{\alpha}_n$ be the sequence given by item (2) of Theorem 4.2.10. Let $m \in \mathbb{N}$. Let $N \in \mathbb{N}$ such that $\ell(\boldsymbol{\alpha}_{N-1}) < m \leq \ell(\boldsymbol{\alpha}_N)$. Note that $B_m(\Sigma_{\boldsymbol{\alpha}})$ is determined by the first m symbols of $\boldsymbol{\alpha}$, also, $\boldsymbol{\alpha}_n$ coincides with $\boldsymbol{\alpha}$ in the first $\ell(\boldsymbol{\alpha}_N)$ symbols for every $n \geq N$, then $B_m(\Sigma_{\boldsymbol{\alpha}_n}) = B_m(\Sigma_{\boldsymbol{\alpha}})$ for every $n \geq N$. □

The following theorem gives us a characterisation of the sequences $\boldsymbol{\alpha} \in \mathcal{E}$. We show that a necessary and suficient condition for $\boldsymbol{\alpha} \in \mathcal{E}$ is that $\boldsymbol{\alpha} \notin (\boldsymbol{\omega}, \boldsymbol{\omega}''')_\prec$ for all i-irreducible sequences $\boldsymbol{\omega}$.

Theorem 4.2.13. *A sequence $\boldsymbol{\alpha} \in \mathcal{E}$ if and only if $\boldsymbol{\alpha}$ is an infinite sequence such that it can be approximated from below by subshifts of finite type given by i-irreducible*

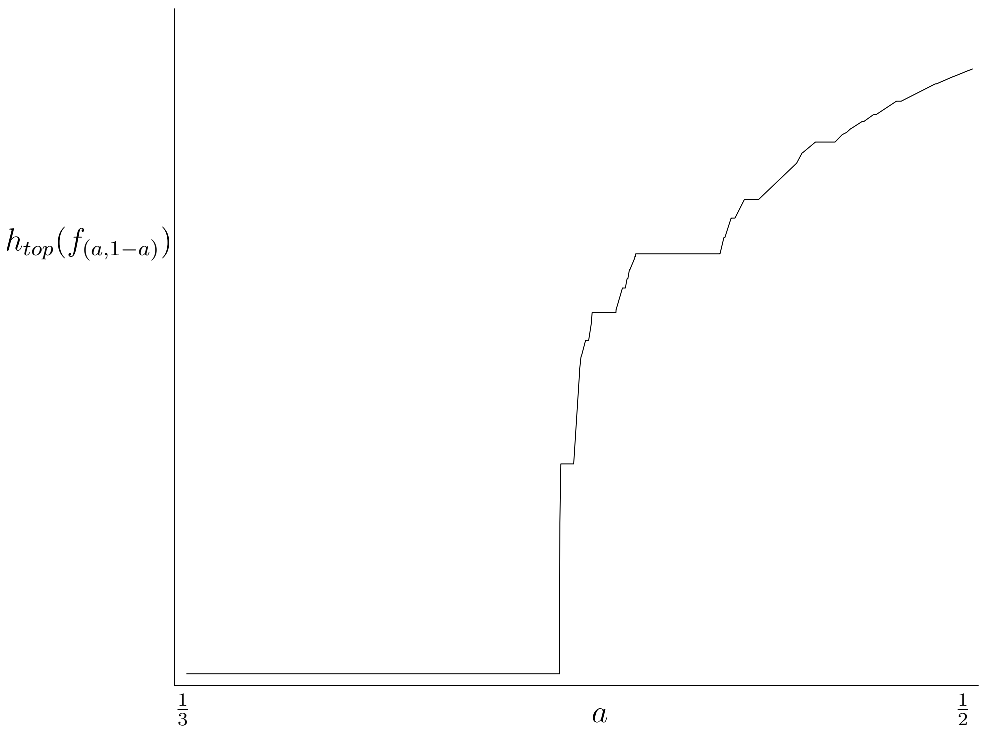

Figure 4.1: The entropy function for symmetric subshifts.

words, i.e, there exists a sequence $\{\boldsymbol{\alpha}_n^-\}$ *of* i*-irreducible sequences such that* $(\Sigma_{\boldsymbol{\alpha}_n^-}, \sigma_{\boldsymbol{\alpha}_n^-})$ *is a subshift of finite type for every* n *and* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *is approximated from below by* $(\Sigma_{\boldsymbol{\alpha}_n^-}, \sigma_{\boldsymbol{\alpha}_n^-})$ *.*

Proof. Let $\boldsymbol{\alpha}$ be an infinite sequence approximated from below by $\{\boldsymbol{\alpha}_n^-\}_{n=1}^{\infty}$, where α_n^- is the i-irreducible word corresponding to $\boldsymbol{\alpha}_n^-$. Suppose that there exists an i-irreducible sequence $\boldsymbol{\omega}$ such that $\boldsymbol{\alpha} \in (\boldsymbol{\omega}, \boldsymbol{\omega}''')_{\prec}$. Let $N \in \mathbb{N}$ be sufficiently large such that $\ell(\boldsymbol{\omega}) < \ell(\alpha_N^-)$. Note that $\boldsymbol{\omega} \prec \boldsymbol{\alpha}_N^- \prec \boldsymbol{\alpha}$. This implies that $a_N^- \in (\omega, \omega''')_{\prec}$, which contradicts Lemma 4.2.4. Therefore, $\boldsymbol{\alpha} \in \mathcal{E}$.

Suppose that $\boldsymbol{\alpha} \in \mathcal{E}$. Note that $\boldsymbol{\alpha}$ is an i-sequence for some $i \geq 0$. Let $\{n_k\}$ be a sequence of positive integers such that $n_k < n_{k+1}$ and $a_{n_k} = 1$ for every $k \in \mathbb{N}$ where a_{n_k} is the n_k symbol of $\boldsymbol{\alpha}$. From Lemma 4.2.4, there exists a finite sequence $\boldsymbol{\alpha}_j^-$ such that the finite sequence $a_1 \dots a_{n_j} \in (\boldsymbol{\alpha}_j^-, \boldsymbol{\alpha}_j^{-'''})$. The sequence $\boldsymbol{\alpha}_j^-$ satisfies Definition 4.2.9. Moreover, since $\boldsymbol{\alpha}_j^- \underset{n\to\infty}{\longrightarrow} \boldsymbol{\alpha}_j^-$ we can choose $\boldsymbol{\alpha}_j^-$ to be i-irreducible for every $j \in \mathbb{N}$ $\square$

Summing up from Proposition 4.2.1 to Lemma 4.2.13 we obtain the following theorem.

Theorem 4.2.14. *Every entropy plateau is of the form* $[\boldsymbol{\omega}, \boldsymbol{\omega}''']_{\prec}$*, where* $\boldsymbol{\omega}$ *is an i-irreducible finite sequence.*

To conclude that the entropy function associated to symmetric subshifts is a devil's staircase, note that from Proposition 3.3.2 and [Nil09, Theorem 3.6], $\mathcal{E} \subset P$. Moreover, since P has Lebesgue measure zero so does $\mathcal{E}$.

4.3 Specification

During this section we will study the dynamics of symmetric subshifts $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ corresponding to parameters $\boldsymbol{\alpha}$ satisfying that $\boldsymbol{\alpha} \in \mathcal{E}$ and $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is transitive. From [LSS14, Theorem 1.3], we know that $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ will not be a subshift of finite type for every $\boldsymbol{\alpha} \in \mathcal{E}$. However, by Therorem 4.2.10 we can always approximate from above and from below $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ by transitive subshifts of finite type.

The property that we would like to be satisfied by $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is *the specification property*. As we mentioned in Section 2.1, we say that a symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has *specification* if there exists $m \in \mathbb{N}$ such that for any $\upsilon, \nu \in \mathcal{L}(\Sigma_{\boldsymbol{\alpha}})$ there exists $\varpi \in B_m(\Sigma)$ such that $\upsilon\varpi\nu \in \mathcal{L}(\Sigma_{\boldsymbol{\alpha}})$. That is, every two words υ and ν can be connected by a word ϖ of fixed lenght m and such length does not depend on υ nor ν. Also, we say that a symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has *almost specification* if there exists $m \in \mathbb{N}$ such that that for any $\upsilon, \nu \in \mathcal{L}(\Sigma_{\boldsymbol{\alpha}})$ there exists $\varpi \in \mathcal{L}(\Sigma_{\boldsymbol{\alpha}})$ such that $\upsilon\varpi\nu \in \mathcal{L}(\Sigma_{\boldsymbol{\alpha}})$ and $\ell(\varpi) \leq m$, i.e. every two words υ and ν can be connected by a word of bounded lenght m and such bound does not depend on $\ell(\upsilon)$ nor $\ell(\nu)$. We point out that the specification property and the almost specification property are equivalent if $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is topologically mixing (see [Jun11, Lemma 3.1]), which is our case. As we mentioned in Theorem 2.4.3 every subshift with the specification property is intrinsically ergodic.

Note that every subshift with specification is transitive. We will use the following approach to show when a symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has specification: Let $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$

be a symmetric subshift and consider

$$m_n = \inf\{k \mid \text{for every } \upsilon, \nu \in B_n(\Sigma_{\boldsymbol{\alpha}}) \text{ there exists } \varpi \in B_k(\Sigma_{\boldsymbol{\alpha}}) \text{ such that } \upsilon\varpi\nu \in \mathcal{L}(\Sigma_{\boldsymbol{\alpha}})\}.$$

Then $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has specification if and only if $\lim_{n\to\infty} m_n < \infty$.

Definition 4.3.1. The number $\max_{n\in\mathbb{N}} \{m_n\}$ (when it exists) is called *the specification number of* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ and it is denoted by $s_{\boldsymbol{\alpha}}$.

Since every transitive subshift of finite type has specification [Par64a] we will denote the specification number of a symmetric subshift $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}})$ given by an irreducible sequence $\boldsymbol{\omega}$ by $s_{\boldsymbol{\omega}}$.

The main idea used to describe the specification property for symmetric subshift is to use the canonical approximations given by Theorem 4.2.10 and the specification number of each element of the approximation from above. We would like to find an upper bound for $\{m_n\}_{n=1}^{\infty}$ when possible. Moreover, if such bound exist, it should not depend on the length of the irreducible sequences used to approximate $\boldsymbol{\alpha} \in \mathcal{E}$. Recall that for every $\boldsymbol{\alpha} \in \mathcal{E}$ there exists $n \geq 2$ such that $\boldsymbol{\alpha} \in [1^n, 1^{n+1}]_{\prec}$. In Lemmas 4.3.3, 4.3.5, 4.3.7, 4.3.9 and 4.3.11 we find such bounds for transitive symmetric subshift of finite type and we observed that such bounds depend in the number of occurences of the block 0^n and where this blocks occur in $\boldsymbol{\alpha}$. Such n is determined by the interval $[1^n, 1^{n+1}]_{\prec}$ which $\boldsymbol{\alpha}$ belongs to. In Theorem 4.3.6 we show that $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has specification provided that 0^n does not occur in $\boldsymbol{\alpha}$. In Theorem 4.3.13 we show that $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has specification provided that 0^n occur finitely many times $\boldsymbol{\alpha}$. Thus, the condition that 0^n occurs infinitely many times in $\boldsymbol{\alpha}$ is necessary to show that the corresponding symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ does not have the specification property. However, as a consequence of Lemma 4.3.12 we show in Theorem 4.3.15 that if 0^n occurs infinitely many times in $\boldsymbol{\alpha}$ is not sufficient to guarantee that $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ does not have specification. We state now the main theorem of the section.

Theorem 4.3.2. *Let* $\boldsymbol{\alpha} \in \mathcal{E} \cap [11(01)^{\infty}, 1^{\infty})_{\prec}$ *and* $\boldsymbol{\alpha} \in (1^n, 1^{n+1})_{\prec}$ *for some* $n \geq 2$. *Then*

1. *If* 0^n *does not occur in* $\boldsymbol{\alpha}$ *then* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *has specification;*

2. *If* 0^n *occurs finite times, then* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *has specification;*

3. *If for every* $r \in \mathbb{N}$*, the canonical approximation from above* $\{\boldsymbol{\alpha}_r^+\}_{r=1}^{\infty}$ *satisfies*

$$\frac{1}{2^{2\ell(\boldsymbol{\alpha}_r^+)}} < d({\boldsymbol{\alpha}_{r-1}^+}', \boldsymbol{\alpha}_r^+) \leq \frac{1}{2^{\ell(\boldsymbol{\alpha}_r^+)+n}},$$

then 0^n *occurs in* $\boldsymbol{\alpha}$ *infinitely many times and* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *has specification.*

As it was mentioned in Section 4.1, from the proof of Theorem 4.1.7, $s_{\boldsymbol{\omega}} \leq 2\ell(\boldsymbol{\omega})$. Moreover, observe that for every $n \geq 3$ an irreducible sequence $\boldsymbol{\omega} \in (1^n, 1^{n+1})_{\prec}$ has at least length $n+2$ and 5 if $\boldsymbol{\omega} \in (11(01)^{\infty}, 111)_{\prec}$. Also, as a consequence of Lemma 4.2.4, there are no irreducible words of the form $1^n 0^{n-1} 1 0^n 1 \ldots$ for every $n \geq 3$, and by Theorem 4.1.3, the same is true for $n = 2$.

Recall that a subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is *coded* if

$$\Sigma_{\boldsymbol{\alpha}} = \overline{\bigcup_{n=1}^{\infty} \Sigma_n},$$

where each (Σ_n, σ_n) is a transitive subshift of finite type. Note that, by definition, coded systems are transitive subshifts. We say that a subshift Σ is an *almost sofic system* if for every $\varepsilon > 0$ there exists a subshift of finite type $\Sigma_{\varepsilon} \subset \Sigma$ such that $h_{top}(\Sigma) > h_{top}(\Sigma_{\varepsilon}) - \varepsilon$. It is worth pointing out now that from Definition 4.2.9, and the continuity of the entropy function, it is clear that for any $\boldsymbol{\alpha} \in [(10)^{\infty}, 1^{\infty}]$, $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is an almost sofic system. And if $\boldsymbol{\alpha} \in \mathcal{E}$ and $\boldsymbol{\alpha} \succ 11(01)^{\infty}$ then $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is a topologically mixing coded system.

In the following lemma, we give an upper bound of the specification number, which depends not on the length of the irreducible sequence $\boldsymbol{\omega}$ but on the occurrence of strings of zeroes.

Lemma 4.3.3. *Let* $\boldsymbol{\omega} \in (1^n, 1^{n+1})_{\prec}$ *be an irreducible sequence with* $n \geq 3$*. If the blocks* 0^{n-1} *or* 1^{n-1} *do not occur in* $\boldsymbol{\omega}$*, then* $s_{\boldsymbol{\omega}} \leq 2(n-1)$*.*

Proof. Let $\boldsymbol{\omega}$ be an irreducible sequence satisfying the hypothesis. Without losing generality it suffices to consider words υ and ν such that υ ends with 1, ν starts with 1 and $\ell(\upsilon) = \ell(\nu) = \ell(\boldsymbol{\omega})$. Note that for any ν starting with 1, the word $0^{n-1}\nu$ is admissible, and if υ contains neither the blocks 0^n nor 1^n then 0^{n-1} is a bridge between υ and ν of length $n-1$. Suppose that 0^n or 1^n occurs in υ. Let

$$j = \max\{k \in \{0, \dots \ell(\omega) - 2n\} \mid \sigma^k(u) = 0^n 1 \dots u_{\ell(\omega)-1} 1 \text{ or } 1^n 0 \dots u_{\ell(\omega)-1} 1\}.$$

If $\sigma^j(\upsilon) = 1^n 0 \dots u_{\ell(\omega)-1)} 1$, then 0^{n-1} is a bridge between υ and ν of length $n-1$. If $\sigma^j(\upsilon) = 0^n 1 \dots 1$, then $\sigma^j(\upsilon) \succ \omega_{\min_{\ell(\omega)-j}}$ or $\sigma^j(\upsilon) = \omega_{\min_{\ell(\omega)-j}}$. If $\sigma^j(\upsilon) \succ \omega_{\min_{\ell(\omega)-j}}$ then 0^{n-1} is a bridge between υ and ν of length $n-1$. If $\sigma^j(\upsilon) = \omega_{\min_{\ell(\omega)-j}}$, then there exists $1 \leq k \leq n-2$ such that $\upsilon 1^k$ is admissible. Then $1^k 0^{n-1}$ is a bridge between υ and ν with length at most $2n-2$. □

Proposition 4.3.4. *Let $\boldsymbol{\alpha} \in \mathcal{E}$ such that $\boldsymbol{\alpha} \in [1^n, 1^{n+1}]_{\prec}$ for some $n \geq 3$. If 0^{n-1} does not occur in $\boldsymbol{\alpha}$ then $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has specification.*

Proof. Let $\boldsymbol{\alpha}$ be an infinite sequence satisfying the hypotheses of the proposition. Let $\boldsymbol{\alpha}_r^+$ be given by Theorem 4.2.10 (2). Observe that the block 0^{n-1} does not occur in $\boldsymbol{\alpha}_n^+$ for any $n \in \mathbb{N}$. By Lemma 4.2.12 there exist $N(r) \in \mathbb{N}$, $m_r = s_{\boldsymbol{\alpha}_{N(r)}^+}$. By Lemma 4.3.3, $m_r \leq 2(n-1)$ for every n, hence

$$\lim_{r \to \infty} m_r < 2(n-1).$$

Therefore, $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has specification. □

The presence of the blocks $0^{n-1}, 1^{n-1}, 0^n$ and 1^n in an infinite sequence $\boldsymbol{\alpha} \in (1^n, 1^{n+1})$ will increase the specification number of each member of the canonical approximation given by Theorem 4.2.10 (2), in certain situations. Firstly, we show that the occurrence of 0^{n-1} will have no mayor effect on the specification number of $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}})$ if $d(\boldsymbol{\omega}, 1^n(0^{n-1}1)^\infty)$ is bounded away from zero.

Lemma 4.3.5. *Let $\boldsymbol{\omega} \in (1^n, 1^{n+1})_{\prec}$ be an irreducible sequence with $n \geq 3$. If $d(\boldsymbol{\omega}, 1^n(0^{n-1}1)^\infty) \geq \frac{1}{2^{2n-1}}$ and 0^n or 1^n does not occur in $\boldsymbol{\omega}$, then $s_{\boldsymbol{\omega}} \leq 2n$.*

Proof. The argument of this proof is similar to the one used for Lemma 4.3.3. Let $\boldsymbol{\omega}$ be an irreducible sequence satisfying the hypothesis. Observe that $d(\boldsymbol{\omega}, 1^n(0^{n-1}1)^\infty) \geq \frac{1}{2^{2n-1}}$, implying that the block $[(0^{n-1}1)]^k$ does not occur after 1^n for any $k \in \mathbb{N}$. Let υ and ν be admissible words such that 0^n or 1^n occurs in υ and $\ell(\upsilon) = \ell(\nu) = \ell(\boldsymbol{\omega})$. Let

$$j = \max\{k \in \{0, \dots \ell(\boldsymbol{\omega}) - 2n\} \mid \sigma^k(\upsilon) = 0^n 1 \dots u_{\ell(\boldsymbol{\omega})-1} 1 \text{ or } 1^n 0 \dots u_{\ell(\omega)-1} 1\}.$$

If $\sigma^j(\upsilon) = 1^n 0 \dots u_{\ell(\omega)-1)}1$, then 0^{n-1} is a bridge between u and υ with length $n-1$. If $\sigma^j(u) = 0^n 1 \dots 1$, then $\sigma^j(\upsilon) \succ \omega_{\min_{\ell(\boldsymbol{\omega})-j}}$ or $\sigma^j(\upsilon) = \omega_{\min_{\ell(\boldsymbol{\omega})-j}}$. If $\sigma^j(\upsilon) \succ \omega_{\min_{\ell(\boldsymbol{\omega})-j}}$ then 0^{n-1} is a bridge between υ and ν of length $n-1$. If $\sigma^j(\upsilon) = \omega_{\min_{\ell(\omega)-j}}$, then there exists $1 \leq k \leq n-2$ such that $\upsilon 1^k$ is admissible. Then $1^k 0^{n-1}$ is a bridge between υ and ν with length at most $2n$. □

Observe that the proof of Proposition 4.3.4 also holds for infinite sequences that can be approximated from above by words satisfying the hypothesis of Lemma 4.3.5. Then the following theorem is true. The proof is left to the reader.

Theorem 4.3.6. *Let $\boldsymbol{\alpha} \in \mathcal{E}$ such that $\boldsymbol{\alpha} \in (1^n, 1^{n+1})_{\prec}$ for some $n \geq 2$. If*

$$d(\boldsymbol{\alpha}, 1^n(0^{n-1}1)^\infty) \geq \frac{1}{2^{2n-1}}$$

and 0^n or 1^n does not occur in $\boldsymbol{\alpha}$ then $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has specification.

Given an irreducible sequence $\boldsymbol{\omega}$, we show that the specification number of $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}})$ will depend on the distance between $\boldsymbol{\omega}$ and $1^n(0^{n-1}1)^\infty$.

Lemma 4.3.7. *Let $\boldsymbol{\omega} \in (1^n, 1^{n+1})_{\prec}$ be an irreducible sequence with $n \geq 2$. If $0 < d(\boldsymbol{\omega}, 1^n(0^{n-1}1)^\infty) \leq \frac{1}{2^{n(k+1)+1}}$ for $k \in \mathbb{N}$ and 0^n or 1^n does not occur in ω, then $nk < s_{\boldsymbol{\omega}} < n(k+2)$.*

Proof. Let $k \in \mathbb{N}$ and $\boldsymbol{\omega}$ be an irreducible sequence such that

$$d(\boldsymbol{\omega}, 1^n(0^{n-1}1)^\infty) \leq \frac{1}{2^{n(k+1)+1}}.$$

Then the first $n(k+1)$ symbols of $\boldsymbol{\omega}$ coincide with $1^n(0^{n-1}1)^k$. Let υ and ν be admissible words such that 0^n or 1^n occurs in υ and $\ell(\upsilon) = \ell(\nu) = \ell(\boldsymbol{\omega})$. Let

$$j = \max\{k \in \{0, \dots \ell(\boldsymbol{\omega}) - 2n\} \mid \sigma^k(\upsilon) = 0^n 1 \dots u_{\ell(\boldsymbol{\omega})-1}1 \text{ or } 1^n 0 \dots u_{\ell(\boldsymbol{\omega})-1}1\}.$$

By our choice of j, if $\sigma^j(\upsilon)$ starts with 0^n, whence $\sigma^j(\upsilon) = 0^n(1^{n-1}0)^r 1^i$, with $0 \leq r < k$ and $1 \leq i \leq n-1$. By symmetry, if $\sigma^j(\upsilon)$ starts with 1^n then $\sigma^j(u) = 1^n(0^{n-1}1)^r$, with $0 \leq r < k$ and $1 \leq i \leq n-1$. Let ϖ be such that $\upsilon\varpi$ is an admissible word. Then the first $n-1-i$ symbols of ϖ are forced to be 1 if $\sigma^j(\upsilon)$ starts with 0^n and to be 0 if $\sigma^j(\upsilon)$ starts with 1^n. The symbol $n-i$ is free. Note that for any choice of the symbol $n-i$, the following $s(n-1)$ symbols will be forced for $1 \leq s \leq k$. Then nk is a lower bound for s_ω. To compute the upper bound note that the symbol $sn+l$ is free for some $2 \leq l \leq n-1$. Then $s_{\boldsymbol{\omega}} \leq nk + l + n - 1 \leq nk + 2(n-1) < n(k+2)$. □

Observe that Lemma 4.3.7 shows that the specification number depends only on the first $n(k+1)$ digits of $\boldsymbol{\omega}$ if

$$d(\boldsymbol{\omega}, 1^n(0^{n-1}1)^\infty) \leq \frac{1}{2^{n(k+1)+1}}.$$

Theorem 4.3.8. *Let* $\boldsymbol{\alpha} \in \mathcal{E} \cap (1^n, 1^{n+1})_\prec$ $n \geq 3$. *If*

$$0 < d(\boldsymbol{\alpha}, 1^n(0^{n-1}1)^\infty) \leq \frac{1}{2^{n(k+1)+1}}$$

for some $k \in \mathbb{N}$ *and* 0^n *or* 1^n *do not occur in* $\boldsymbol{\alpha}$, *then* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *has specification.*

Proof. Let $\boldsymbol{\alpha} \in \mathcal{E} \cap (1^n, 1^{n+1})$. Then there exists $k \in \mathbb{N}$ such that

$$\frac{1}{2^{n(k+2)+1}} < d(\boldsymbol{\alpha}, 1^n(0^{n-1}1)^\infty) \leq \frac{1}{2^{n(k+1)+1}}.$$

Let $\boldsymbol{\alpha}_r^+$ be the canonical approximation from above given by Theorem 4.2.10. Then for every $r \in \mathbb{N}$, the first kn symbols of $\boldsymbol{\alpha}_r^+$ are determined by $1^n(0^{n-1}1)^k$. This implies that $\boldsymbol{\alpha}_r^+$ satisfies the hypothesis of Lemma 4.3.7 for every r, i.e.

$$nk < s_{\boldsymbol{\alpha}^+_{N(r)}} < n(k+2).$$

By Lemma 4.2.12, $m_r = s_{\boldsymbol{\alpha}^+_{N(r)}}$. Therefore,

$$\lim_{r\to\infty} m_r < n(k+2).$$

□

Now, it is needed to study the specification number of symmetric subshifts associated to irreducible sequences $\boldsymbol{\omega}$ when the block 0^n occurs in $\boldsymbol{\omega}$ (Lemmas 4.3.9, 4.3.11 and 4.3.12) in order to prove Theorem 4.3.2. Recall that given a finite sequence $\boldsymbol{\omega}$ starting with 11, the sequence $\boldsymbol{\omega}' = w_1 \dots w_{\ell(\boldsymbol{\omega})}\bar{w}_1 \dots w_{\ell(\bar{\boldsymbol{\omega}}-1)}10^\infty$. Notice that for every irreducible sequence $\boldsymbol{\omega}$ such that $1^n \prec \boldsymbol{\omega} \prec 1^{n+1}$ the block 0^n occurs in $\boldsymbol{\omega}'$ at least once. The situation explained in Lemma 4.3.7 is not limited only to irreducible sequences close to $1^n(0^{n-1}1)^\infty$.

Lemma 4.3.9. *Let* $\boldsymbol{\upsilon}$ *be an irreducible sequence such that* $\boldsymbol{\upsilon} \in (1^n, 1^{n+1})_\prec$ *for a fixed* $n \geq 2$. *If there exists an irreducible sequence* $\boldsymbol{\omega} \in (1^n, 1^{n+1})_\prec$ *such that* $\boldsymbol{\omega}$ *does not contain the block* 0^n *and*

$$\frac{1}{2^{(k+1)\ell(\boldsymbol{\omega})+n}} \leq d(\boldsymbol{\omega}''', \boldsymbol{\upsilon}) \leq \frac{1}{2^{(k+1)\ell(\boldsymbol{\omega})}}$$

for some $k \in \mathbb{N}$ *and* 0^n *does not occur in* $\boldsymbol{\upsilon}$ *after* $(k+1)\ell(\omega)$ *digits, then*

$$k\ell(\boldsymbol{\omega}) \leq s_{\boldsymbol{v}} \leq k\ell(\boldsymbol{\omega}) + n(k+2).$$

Proof. Let $\boldsymbol{\omega}$ and $\boldsymbol{v}$ be irreducible words satisfying the hypothesis. Then $\boldsymbol{v}$ contains the block 0^n at least k times, and by Lemma 4.2.4 there is no $\ell(\boldsymbol{\omega}) < j < (k+1)\ell(\boldsymbol{\omega})$ such that the word $u_1 \dots u_j$ is irreducible. Also, $\ell(\boldsymbol{\omega}) < \ell(\boldsymbol{v})$. Indeed, $\ell(\boldsymbol{v}) > (k+1)\ell(\boldsymbol{\omega})$. By Lemmas 4.3.3, 4.3.5 and 4.3.7 we may confine ourselves to considering sequences which lie in $\Sigma_{\boldsymbol{v}} \setminus \Sigma_{\boldsymbol{\omega}}$ to compute bounds for $s_{\boldsymbol{v}}$.

Consider admissible words ν, τ such that $\ell(\nu) = \ell(\tau) = \ell(\boldsymbol{v})$, τ starts with 1 and

$$v_{\ell(\boldsymbol{v})-(\ell(\boldsymbol{\omega})+1)} \dots v_{\ell(\nu)} = \bar{\omega}1,$$

where ω is the associated irreducible word of $\boldsymbol{\omega}$. Observe that if z is a bridge between ν and τ then $z_1 = 1$. If $z_1 = 0$ then $\sigma^{\ell(\boldsymbol{v})} - (\ell(\boldsymbol{\omega})+1)0 \prec \bar{\boldsymbol{v}}$. The same argument holds for z_j for $j \leq n-1$. Observe that $z_n = 0$. If $z_n = 1$ then νz will contain the block 1^{n+1} which is a contradiction. Therefore, z will satisfy that

$$\sigma^{\ell(\boldsymbol{v})-(\ell(\boldsymbol{\omega})+1)}(u)z = \bar{\omega}\omega_1 \dots \omega_{\ell(\omega)-1},$$

then z has at least length $\ell(\omega)-1$. Observe that the following digit is free. If 1 is chosen then $z_1 \dots z_{\ell(\omega)} = \omega$, which will force the following $\ell(\omega)-1$ digits to coincide with $\bar{\omega}$. If 0 is chosen then $\sigma^{\ell(\boldsymbol{v})-(\ell(\boldsymbol{\omega})+1)}z = \bar{v}$ where v is the irreducible word associated to $\boldsymbol{v}$, therefore, the next $\ell(\omega)-1$ digits of z are forced by the previous argument. It has been shown that z has a length of at least $k(\ell(\omega))-1$ and that after $\ell(\omega)-1$ steps we will have a choice. By Theorem 4.1.7, the minimum number of steps to get from $\sigma^{\ell(v)-(\ell(\omega)+1)}(u)$ to $v_{\min_{(k+1)\ell(\omega)}}$ by z is $\ell(v)-(\ell(\omega)+1)$. Substituting $\ell(v) > (k+1)\ell(\omega)$ in the minimum number of steps we obtain the desired lower bound for $s_{\boldsymbol{v}}$.

To obtain the upper bound, let ν be an admissible word with length $\ell(v)$ such that $\nu \notin B_{\ell(v)}(\Sigma_{\boldsymbol{\omega}})$ such that ν ends with 1. By hypothesis,

$$\frac{1}{2^{(k+1)\ell(\boldsymbol{\omega})+n}} \leq d(\boldsymbol{\omega}''', \boldsymbol{v}) \leq \frac{1}{2^{(k+1\ell(\omega))}}.$$

Therefore, there exists an index $(k+1)\ell(\omega) < j < (k+1)\ell(\omega)+n$ such that $\boldsymbol{v}_i = 0$ for $\ell(\boldsymbol{v}) \leq i < j$ and $v_j = 1$. This implies that the following $j-1$ symbols are forced to be 1 and the symbol j is free. If one is chosen then $z = 1^j$ with $0 \leq j \leq n-1$. By Lemma 4.3.7, z has at most $n(k+2)$ digits forced after its first $k\ell(\omega)$ digits. Therefore, the desired upper bound is obtained. ☐

Note that Lemma 4.3.9 deals with a particular family of subshifts where 0^n does not occur in its defining word; namely, irreducible words such that they are close to a particular entropy plateau defined by a word without 0^n occurring in it.

Proposition 4.3.10. *For every irreducible sequence $\boldsymbol{\upsilon}$ such that 0^n occurs in υ there exists a word ω such that the associated irreducible sequences $\boldsymbol{\omega}$ and $\boldsymbol{\upsilon}$ satisfy the distance condition of Lemma 4.3.9 and ω does not contain 0^n.*

Proof. The result is just a consequence of Proposition 4.2.1 and Lemma 4.2.4 applied to $\boldsymbol{\omega} = 1^n0^\infty$ and the given $\boldsymbol{\upsilon}$. □

It is natural to ask how the specification number $s_{\boldsymbol{\upsilon}}$ will change if 0^n occurs in $\boldsymbol{\upsilon}$. As it will be shown, everything depends on the longest irreducible word, which does not contain 0^n occurring in υ and the way that 0^n is placed. As shown in the following generalisation of Lemma 4.3.9, the specification number for an irreducible word depends entirely on the specification number of the irreducible subwords occurring in $\boldsymbol{\upsilon}$. The proof is essentially the same. In the following theorem τ will play the role of ω on the proof of Lemma 4.3.9.

Lemma 4.3.11. *Let $\boldsymbol{\upsilon}$ be an irreducible sequence such that $\boldsymbol{\upsilon} \in (1^n, 1^{n+1})_\prec$ for a fixed $n \geq 2$, and 0^n occurs in $\boldsymbol{\upsilon}$. If there exists an irreducible word τ such that*

$$\frac{1}{2^{(k+1)\ell(\tau)+n}} \leq d(\boldsymbol{\tau}''', \boldsymbol{\upsilon}) \leq \frac{1}{2^{(k+1)\ell(\tau)}}$$

for some $k \geq 0$, the associated irreducible sequence of τ, $\boldsymbol{\tau}$ satisfies Lemma 4.3.9 and for any $j \leq |(\ell(\boldsymbol{\upsilon}) - 1) - k\ell(\tau)|$ such that $\upsilon_j = 1$ the word $\upsilon_1 \dots \upsilon_j$ is not irreducible, then

$$k\ell(\tau) \leq s_{\boldsymbol{\upsilon}} \leq k\ell(\tau) + n(k+2).$$

Proof. Let $\boldsymbol{\upsilon}$ be an irreducible sequence and suppose that there exists an irreducible word τ satisfying the hypothesis. By Lemmas 4.3.3, 4.3.5, 4.3.7 and 4.3.9 we only need to consider sequences that lie in $\Sigma_{\boldsymbol{\upsilon}} \setminus \Sigma_{\boldsymbol{\tau}}$ to compute bounds for $s_{\boldsymbol{\upsilon}}$.

In this proof we will words with arabic characters. Let u, v be admissible words such that $\ell(u) = \ell(v) = \ell(\boldsymbol{\upsilon})$, v starts with 1 and

$$u_{\ell(\boldsymbol{\upsilon})-(\ell(\tau)+1)} \dots u_{\ell(u)} = \bar{\tau}1.$$

Let z be a bridge between u and v. Applying the same construction as in Lemma 4.3.9 it is shown that z has a length of at least $k(\ell(\tau)) - 1$. Applying Theorem 4.1.7, the minimum number of steps to get from $\sigma^{\ell(\boldsymbol{v})-(\ell(\tau)+1)}(u)$ to $v_{\min_{(k+1)\ell(\tau)}}$ by z is $\ell(\boldsymbol{v}) - (\ell(\tau)+1)$. Substituting $\ell(\boldsymbol{v}) > (k+1)\ell(\tau)$ in the minimum number of steps, we obtain the desired lower bound for $s_{\boldsymbol{v}}$.

To obtain the upper bound, let u be an admissible word with length $\ell(\boldsymbol{v})$ such that $u \notin B_{\ell(\boldsymbol{v})}(\Sigma_{\boldsymbol{\tau}})$ and u ends with 1. By hypothesis

$$\frac{1}{2^{(k+1)\ell(\tau)+n}} \leq d(\omega''', v) \leq \frac{1}{2^{(k+1\ell(\tau))}}.$$

Therefore, there exists an index $(k+1)\ell(\tau) < j < (k+1)\ell(\tau) + n$ such that $v_i = 0$ for $\ell(v) \leq i < j$ and $v_j = 1$. This implies that the following $j-1$ symbols are forced to be 1 and the symbol j is free. If one is chosen then $z = 1^j$ with $0 \leq j \leq n-1$. By Lemma 4.3.9, z has at most $n(k+2)$ digits forced after its first $k\ell(\tau)$ digits. Therefore, the desired upper bound is obtained. □

Recall that given a finite word ω starting with 1 and ending with 1,

$$\omega' = w_1 \dots w_{\ell(\omega)} \bar{w}_1 \dots w_{\ell(\bar{\omega})-1} 1.$$

For the convenience of the reader we write the statement for Lemma 4.3.12.

Lemma 4.3.12. *Let $n \geq 2$ and $\boldsymbol{v} \in (1^n, 1^{n+1})_{\prec}$ be an irreducible word such that for every irreducible subword ω^i, 0^n occurs in ω^i, and*

$$\frac{1}{2^{2\ell(\omega^i)}} < d(\omega^{i'}, \boldsymbol{v}) \leq \frac{1}{2^{\ell(\omega^i)+n}}.$$

Then

$$s_{\boldsymbol{v}} \leq \ell(\omega) + n,$$

where ω is the longest irreducible word such that 0^n does not occur in $\boldsymbol{v}$.

Proof. Let $\omega^1 = \omega$. By hypothesis

$$\frac{1}{2^{2\ell(\omega^1)}} < d(\omega^{1'}, \boldsymbol{v}) \leq \frac{1}{2^{\ell(\omega^1)+n}}.$$

This implies that there exists $\ell(\omega^1)+(n+1) < j < 2\ell(\omega^1)$ such that $\omega'_j = 0$ and $v_j = 1$. Moreover, $j \leq \ell(\boldsymbol{v})$ and $\tau = v_1 \dots v_j$ is an irreducible word such that for any $j' < j$

satisfying $\upsilon_{j'} = 1$, $\upsilon_1 \dots \upsilon_{j'}$ is irreducible. Firstly, we will calculate the specification number for $\Sigma_{\boldsymbol{\tau}}$.

Let $u \in B_{\ell(\tau)}(\Sigma_{\boldsymbol{\tau}}) \setminus B_{\ell(\tau)}(\Sigma_\omega)$ such that u ends with 1 and $v \in B_{\ell(\tau)}(\Sigma_{\boldsymbol{\tau}})$ such that v starts with 1. Then the words ω or $\bar{\omega}$ occur in u. Let $1 \leq k \leq \ell(\tau) - \ell(\omega)$ such that $\sigma^k(u)$ starts with ω or $\bar{\omega}$. Without losing generality suppose that $\sigma^k(u)$ starts with $\bar{\omega}$. Observe that the word $z0^s$, where $z_i = \bar{\tau}_{\ell(\omega)+i}$ for $1 \leq i \leq (j - \ell(\omega))$ and $1 \leq s \leq n-1$, is a bridge between u and v. Then $s_\tau \leq j - \ell(\omega^1) + n \leq \ell(\omega) + n$.

Inductively, suppose that for every irreducible word, every irreducible subword ω^i is such that

$$\frac{1}{2^{2\ell(\omega^i)}} < d(\omega^{i'}, \upsilon) \leq \frac{1}{2^{\ell(\omega^i)+n}},$$

then

$$s_{\boldsymbol{\upsilon}} \leq \ell(\omega) + n.$$

Let ω^{i+1} be the next irreducible word such that

$$\frac{1}{2^{2\ell(\omega^{i+1})}} < d(\omega^{i+1'}, \boldsymbol{\upsilon}) \leq \frac{1}{2^{\ell(\omega^{i+1})+n}}.$$

Then $\ell(\omega^2)+(n+1) < \ell(\omega^3) < 2\ell(\omega^2)$, $\omega^{2'}_j = 0$ and $\upsilon_{\ell(\omega^3)} = 1$. Observe that 0^n occurs in ω^{i+1} at least twice. Let $u \in B_{\ell(\omega^{i+1})}(\Sigma_{\omega^{i+1}}) \setminus B_{\ell(\omega^{i+1})}(\Sigma_{\omega^i})$ such that u ends with 1 and $v \in B_{\ell(\omega^{i+1})}(\Sigma_{\omega^i})$ such that v starts with 1. Let $k \in \{0, \dots, \ell(\omega^{i+1}) - \ell(\omega^i)\}$ such that $\sigma^k(u)$ starts with ω^i or $\bar{\omega}^i$. Without losing generality suppose that $\sigma^k(u)$ starts with $\bar{\omega}^i$. Observe that the word $z0^s$, where $z_i = \omega_i$ for $1 \leq i \leq (j - \ell(\omega^1))$ and $1 \leq s \leq n-1$, is a bridge between u and v. Then $s_{\omega^{i+1}} \leq j - \ell(\omega^1) + n \leq \ell(\omega^1) + n$.

Observe that there exists $I \leq \ell(\boldsymbol{\upsilon})$ such that ω^I satisfies the distance condition stated above and for any $i \geq I$, 0^n does not occur in ω^i. Then $\ell(\omega) + n \leq s_{\boldsymbol{\upsilon}}$. If $\ell(\boldsymbol{\upsilon}) = I$ then $s_{\boldsymbol{\upsilon}} = \ell(\omega) + n$. Suppose that $\ell(\upsilon) > I$. To show that $s_{\boldsymbol{\upsilon}}$ is bounded from above, consider $u \in B_{\ell(\boldsymbol{\upsilon})}(\Sigma_{\boldsymbol{\upsilon}}) \setminus B_{\ell(\boldsymbol{\upsilon})}(\Sigma_I)$ and $v \in B_{\ell(\boldsymbol{\upsilon})}(\Sigma_{\boldsymbol{\upsilon}})$ with u ending with 1 and v starting with 1. Then the block $s(\bar{\omega})^I$ occurs in u. Note that $\upsilon_{I+1} = 0$. If $\upsilon_{I+1} = 1$ then $\upsilon_1 \dots \upsilon_{I+1}$ is an irreducible word which contains 0^n and will satisfy the required distance condition, which is a contradiction to the choice of I. Let $t \leq \ell(\boldsymbol{\upsilon})$ such that $\upsilon_1 \dots \upsilon_t$ is irreducible. Note that $t \leq I + n - 1$. If $t \leq I + n$ then 0^n occurs in υ_t after I symbols. Therefore there exists $1 \leq j \leq n-1$ such that the following j symbols after $I + n$ symbols of $\bar{\upsilon}$ are forced to be 0. Then $z = \bar{\upsilon}_{I+j} \dots \bar{\upsilon}_{t-1}10^{n-1}$ is a bridge between u and v with $\ell(z) \leq 2n \leq \ell(\omega) + n$. Then $s_{\boldsymbol{\upsilon}} = \ell(\omega) + n$. □

Theorem 4.3.13. *If $\boldsymbol{\alpha} \in \mathcal{E}$ with $\boldsymbol{\alpha} \succ 11(01)^\infty$ and $\boldsymbol{\alpha}$ contains the block 0^n a finite number of times, then $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has specification.*

Proof. Let $\boldsymbol{\alpha} \in \mathcal{E}$ such that 0^n occurs l times. Let $\{\boldsymbol{\alpha}_r^+\}_{r=1}^\infty$ be the canonical approximation from above given by Theorem 4.2.10 (2). By Lemma 4.2.12 $m_r = s_{a^+_{N(r)}}$. By hypothesis, there exists $N \in \mathbb{N}$ such that for every $j \geq N$ the block 0^n occurs l times. Then:

(i) For every $k \leq N$,

$$\frac{1}{2^{2\ell(\boldsymbol{\alpha}_k^+)}} < d(\boldsymbol{\alpha}_k^{+\prime}, \boldsymbol{\alpha}_{k+1}^+) \leq \frac{1}{2^{\ell(\boldsymbol{\alpha}_N^+)+n}}.$$

In this case, by Lemma 4.3.12 for every $k \geq N$, $s_{\boldsymbol{\alpha}_k^+} = \ell(\boldsymbol{\alpha}_i^+) + n$, where i satisfies that for every $j \leq i$, $\boldsymbol{\alpha}_j^+$ does not contain 0^n.

(ii) For every $k > N$,

$$\frac{1}{2^{\ell(\boldsymbol{\alpha}_N^+)+n}} \leq d(\boldsymbol{\alpha}_N^{+\prime\prime\prime}, \boldsymbol{\alpha}_k^+) \leq \frac{1}{2^{\ell(\boldsymbol{\alpha}_k^+)}}.$$

By Lemma 4.3.11,

$$\ell(\boldsymbol{\alpha}_N^+) \leq s_{\boldsymbol{\alpha}_k^+} \leq \ell(\boldsymbol{\alpha}_{N+1}^+) + 3n$$

for every $k > n$.

This implies that

$$\lim_{r\to\infty} m_r < \ell(\boldsymbol{\alpha}_{N+1}^+) + 3n.$$

Therefore, $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has specification. □

To illustrate Theorem 4.3.13 we will study an specific example.

Example 4.3.14. Let

$$\omega_n = 11110100100001011100001100001110 01(01)^2(001)^3 \ldots (01)^{n+1}(001)^{n+2},$$

and $\boldsymbol{\alpha} = \lim_{n\to\infty} \omega_n$.

Observe that ω_n is an irreducible word for every $n \in \mathbb{N}$ and $\omega_n \prec \omega_{n+1}$. Then $\boldsymbol{\alpha} \in \mathcal{E}$, it is aperiodic and $\boldsymbol{\alpha}$ contains the block 0^4 three times. Moreover, $\boldsymbol{\alpha}$ satisfies the hypothesis of Theorem 4.3.2. Nonetheless we show this explicitly.

Consider the canonical approximation from above of $\boldsymbol{\alpha}$, $\{\boldsymbol{\alpha}_k^+\}_{n=1}^\infty$. We write down the first 10 associated irreducible words of the elements of the approximation:

$\alpha_1^+ = 1111011;$

$\alpha_2^+ = 1111010011;$

$\alpha_3^+ = 111101001000011;$

$\alpha_4^+ = 1111010010000101111;$

$\alpha_5^+ = 1111010010000101110000111;$

$\alpha_6^+ = 1111010010000101110000110000111 1;$

$\alpha_7^+ = 1111010010000101110000110000111 0011;$

$\alpha_8^+ = 1111010010000101110000110000111 001011;$

$\alpha_9^+ = 1111010010000101110000110000111 00101011;$

$\alpha_{10}^+ = 1111010010000101110000110000111 0010101 0011.$

Note that the word 111101001 is the longest irreducible word occurring in $\boldsymbol{\alpha}$ such that 0^4 does not occur in it. On the other hand, for any $n \geq 6$, α_n^+ contains the block 0^4 three times.

Note that

$\alpha_1^{+\prime} = 1111011(0000101);$

$\alpha_2^{+\prime} = 1111010011(0000101101);$

$\alpha_3^{+\prime} = 111101001000011(000010110111101);$

$\alpha_4^{+\prime} = 1111010010000101111(0000101101111010001);$

$\alpha_5^{+\prime} = 1111010010000101110000111(0000101101111010001111 0011);$

$\alpha_6^{+\prime} = 11110100100001011100001100001111(00001011011110100011110011110001).$

Following the proof of Theorem 4.3.13 we have that for every $k < 6$,

$$d(\alpha_k^{+\prime}, \alpha_{k+1}^+) = \frac{1}{2^{\ell(\alpha_k^+)}}.$$

Then by Lemma 4.3.12,

$$s_{\alpha_6^+} \leq \ell(\alpha_1^+) + 4 = 11.$$

Observe that

$$\alpha_6^{+'''} = 111101001000010111000011000011111(00001011011110100011110011110001)^\infty.$$

Then for every $n \geq 6$, $d(\alpha_6^{+'''}, \alpha_n^+) = \frac{1}{2^{\ell(\alpha_6^+)}}$. By Lemma 4.3.11,

$$s_{\alpha_n^+} \leq \ell(\alpha_6^+) + 12 = 44.$$

An immediate consequence of Theorem 4.3.13 is that for any $\boldsymbol{\alpha} \in \mathcal{E}$ such that $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ does not have specification, the block 0^n occurs in $\boldsymbol{\alpha}$ infinitely many times. However, it will be shown that this condition is not sufficient.

Theorem 4.3.15. *Let $n \geq 2$ and $\boldsymbol{\alpha} \in \mathcal{E} \cap (1^n, 1^{n+1})_\prec$. If for every $r \in \mathbb{N}$, the canonical approximation from above $\{\boldsymbol{\alpha}_r^+\}_{r=1}^\infty$ satisfies*

$$\frac{1}{2^{2\ell(\boldsymbol{\alpha}_r^+)}} < d({\boldsymbol{\alpha}_{r-1}^+}', \boldsymbol{\alpha}_r^+) \leq \frac{1}{2^{\ell(\boldsymbol{\alpha}_r^+)+n}},$$

then 0^n occurs in $\boldsymbol{\alpha}$ infinitely many times and $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ has specification.

Proof. Let $n \geq 2$ and $\boldsymbol{\alpha} \in \mathcal{E} \cap (1^n, 1^{n+1})_\prec$. Let N be such that for any $N \leq r$, $\boldsymbol{\alpha}_r^+$ does not contain 0^n and for any $r' \leq N$, $\boldsymbol{\alpha}_{r'}^+$ contains 0^n. By Lemma 4.3.12 $s_{\boldsymbol{\alpha}_r^+} = \ell(\boldsymbol{\alpha}_N^+) + n$. Therefore $\lim_{r \to \infty} m_r = \ell(\boldsymbol{\alpha}_N^+) + n$. □

Summing up Theorems 4.3.4, 4.3.6, 4.3.8, 4.3.13 and 4.3.15 we obtain the proof of Theorem 4.3.2.

Non-Specification and The Thue-Morse Family

Let

$$S(\mathcal{E}) = \{\boldsymbol{\alpha} \in \mathcal{E} \mid (\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}}) \text{ has the specification property}\},$$

and $NS(\mathcal{E}) = \mathcal{E} \backslash S(\mathcal{E})$. The following theorem will show that $NS(\mathcal{E}) \neq \emptyset$. We will show that if the length of the defining words of the elements of the canonical approximation from above increases exponentially, then the limit subshift will not have specification.

Theorem 4.3.16. *Let $n \geq 2$ be fixed. Let $\boldsymbol{\alpha} \in \mathcal{E}$ such that $\boldsymbol{\alpha} \in (1^n, 1^{n+1})_\prec$, $\boldsymbol{\alpha} \succ 11(01)^\infty$, 0^n occurs in $\boldsymbol{\alpha}$ infinitely many times and let $\boldsymbol{\alpha}_r^+$ be the approximation given* *by Theorem 4.2.10 (2). If there exists an increasing sequence $\{r_i\}_{i=1}^{\infty} \subset \mathbb{N}$ and $R \in \mathbb{N}$ such that for every $r_i \geq R$ $\boldsymbol{\alpha}_{r_i}^+$ satisfies*

$$\ell(\boldsymbol{\alpha}_{r_{i-1}}^+(\overline{a_{r_{i-1}1}^+} \dots \overline{a_{r_{i-1}\ell(\boldsymbol{\alpha}_{r_{i-1}}^+)-1}^+}1)^{k_{r_i}}) \leq \ell(\boldsymbol{\alpha}_{r_i}^+)$$

and

$$\frac{1}{2^{(k_{r_i}+1)\ell(\boldsymbol{\alpha}_{r_i}^+)+n}} \leq d(\boldsymbol{\alpha}_{r_i}^{+\prime\prime\prime}, \boldsymbol{\alpha}) \leq \frac{1}{2^{(k_{r_i}+1)\ell(\boldsymbol{\alpha}_{r_i}^+)}}$$

for some $k_{r_i} \geq 1$, then $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ does not have specification.

Proof. Choose $\boldsymbol{\alpha} \in \mathcal{E}$ satisfying $\boldsymbol{\alpha}$ the hypothesis of the statement. Let $\{r_i\}_{i=1}^{\infty} \subset \mathbb{N}$ such that for every $r_i \geq R$, $\boldsymbol{\alpha}_{r_i}^+$ satisfies that

$$\ell(\boldsymbol{\alpha}_{r_{i-1}}^+(\overline{a_{r_{i-1}1}^+} \dots \overline{a_{r_{i-1}\ell(\boldsymbol{\alpha}_{r_{i-1}}^+)-1}^+}1)^{k_{r_i}}) \leq \ell(\boldsymbol{\alpha}_{r_i}^+)$$

for some $k_{r_{i-1}} \geq 1$. By Proposition 4.3.10, there exists r_1 such that $\boldsymbol{\alpha}_{r_1}^+$ does not contain 0^n and for every $r_i \geq r_1$

$$\frac{1}{2^{(k_1+1)\ell(\boldsymbol{\alpha}_{r_1}^+)+n}} \leq d(\boldsymbol{\alpha}_{r_1}^{+\prime\prime\prime}, \boldsymbol{\alpha}_{r_i}) \leq \frac{1}{2^{(k_1+1)\ell(\boldsymbol{\alpha}_{r_1}^+)}}$$

for some $k_1 \in \mathbb{N}$. Then by Lemma 4.3.9, $k_1\ell(\boldsymbol{\alpha}_{r_1}^+) \leq s_{\boldsymbol{\alpha}_{r_i}^+}$ for every $i \geq 2$. Let $r_1 < r_2$. By hypothesis

$$\ell(\boldsymbol{\alpha}_{r_1}^+(\overline{a_{r_11}^+} \dots \overline{a_{r_1\ell(\boldsymbol{\alpha}_{r_1}^+)-1}^+}1)^{k_{r_2}}) \leq \ell(\boldsymbol{\alpha}_{r_2}^+)$$

for some $k_{r_2} \geq 1$, and $\frac{1}{2^{(k_2+1)\ell(\boldsymbol{\alpha}_{r_1}^+)+n}} \leq d(\boldsymbol{\alpha}_{r_1}, \boldsymbol{\alpha}_{r_2}) \leq \frac{1}{2^{(k_2+1)\ell(\boldsymbol{\alpha}_{r_1}^+)}}$. Therefore, by Lemma 4.3.11 and Theorem 4.1.7

$$k_2k_1(s_{\boldsymbol{\alpha}_{r_1}^+}) \leq k_2k_1\ell(\boldsymbol{\alpha}_{r_1}^+) \leq k_2(s_{\boldsymbol{\alpha}_{r_1}^+}) \leq k_2(\ell(\boldsymbol{\alpha}_{r_1}^+)) \leq s_{\boldsymbol{\alpha}_{r_2}^+}.$$

Inductively,

$$(\prod_{i=1}^{l} k_i)(s_{\boldsymbol{\alpha}_{r_1}^+}) \leq (\prod_{i=1}^{l} k_i)\ell(\boldsymbol{\alpha}_{r_1}^+) \leq s_{\boldsymbol{\alpha}_{r_{l+1}}^+},$$

which implies that $\lim_{r\to\infty} m_r$ is not bounded. Therefore $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ does not have specification. □

To illustrate Theorem 4.3.16 a specific family of symmetric subshifts without specification will be constructed. Let ω be an irreducible word and $\omega \in (1^n, 1^{n+1})_{\prec}$ for $n \geq 2$ such that 0^n does not occur in ω. Let A be the block $[0^{n-1}1]$. Define $\omega^0 = \omega$, $\omega^1 = \omega^{0\prime}A$ and for every $r \in \mathbb{N}$, $\omega^r = \omega^{r-1\prime}A$. Note that by Proposition 4.3.10, ω^r is an irreducible word for every $r \geq 0$. Let $\boldsymbol{\omega}^* = \lim_{r\to\infty} \omega^r$. Then by Theorem 4.2.13, $\boldsymbol{\omega}^* \in \mathcal{E}$. Note that for any r the last n digits of ω^r are not n consecutive 1's. Therefore, the sequence $\boldsymbol{\alpha}_k^+ = \omega^k 10^\infty$ approximates ω^* from above. Note that the construction assures that 0^n occurs infinitely many times in $\boldsymbol{\omega}^*$. Note also that for every $r \in \mathbb{N}$ $\boldsymbol{\alpha}_r^+$ the conditions of Theorem 4.3.16 are satisfied. Therefore $\Sigma_{\boldsymbol{\omega}^*}$ does not have specification.

We call this family *the Thue-Morse non-specification family* and it is denoted by $T(\mathcal{E})$. Note that the growth of the specification number $s_{\boldsymbol{\alpha}_r^+}$ is exponential, that is $2^r(s_\omega) \leq s_{\boldsymbol{\alpha}_r^+}$ for every $r \in \mathbb{N}$. Observe that this can be done for every irreducible word ω and it can be modified considering $A_j = 0^j1$ with $j \in \{0, \dots, n-1\}$, and considering the sequence

$\omega^0 = \omega$;

$\omega^1 = \omega(\bar{\omega}_1 \dots \bar{\omega}_{\ell(\omega)-1}1)_1^k (A_j)_1$;

and for any $r \geq 2$, $\omega^r = \omega_{r-1}(\bar{\omega}_1 \dots \bar{\omega}_{\ell(\omega^r)-1}1)^{k_r}(A_j)_r$.

Also, observe that for the proposed families the sequence $\{r_i\}_{i=1}^\infty \subset \mathbb{N}$ stated in Theorem 4.3.16 satisfy that $r_i = i$ for every i.

4.4 Intrinsic Ergodicity

We devote this section to study intrinsic ergodicity for symmetric subshifts. Our aim is to prove the following theorem.

Theorem 4.4.1. (Λ_a, f_a) *is intrinsically ergodic for almost every* $a \in (\pi(\sigma(\boldsymbol{\alpha}^*), \frac{1}{2})$.

The proof of Theorem 4.4.1 will be done in steps. From Theorem 4.2.14 it is clear that for almost every $a \in (\pi(\sigma(\boldsymbol{\alpha}^*), \frac{1}{2})$, $\pi^{-1}(2a)$ belongs to an entropy plateau parametrized by an i-irreducible word for some $i \geq 0$. Theorems 4.4.4 and 4.4.5 imply that for every parameter $\boldsymbol{\alpha}$ in an entropy plateau, the associated symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is intrinsically ergodic, which implies our result. Also, as a consequence of Theorem 4.1.7 and [AS07, Theorem 3.5] we obtain the following result.

Theorem 4.4.2. *Let* $\boldsymbol{\alpha} \succ 11(01)^\infty$. *Then*

1. *If* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *is given by an irreducible sequence, then* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *is intrinsically ergodic.*

2. *If* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *is given by a pre-periodic sequence and* $\boldsymbol{\alpha}$ *and* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *is a transitive subshift then* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *is intrinsically ergodic.*

Furthermore, observe that the families of symmetric subshifts described in Theorems 4.3.4, 4.3.6, 4.3.8 have specification, thus they are intrinsically ergodic. We summarise this result as follows.

Theorem 4.4.3. *Let* $\boldsymbol{\alpha} \in \mathcal{E} \cap [11(01)^\infty, 1^\infty]$. *Then:*

1. *If* 0^n *does not occur in* $\boldsymbol{\alpha}$ *then* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *is intrinsically ergodic;*

2. *If* 0^n *occurs in* $\boldsymbol{\alpha}$ *finitely many times then* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *is intrinsically ergodic;*

3. *If* 0^n *occurs in* $\boldsymbol{\alpha}$ *infinitely many times and* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *has specification, then* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *is intrinsically ergodic.*

It is not clear if it is possible to state under which conditions a general non-transitive subshift of finite type can be intrinsically ergodic. Nonetheless, it is conjectured that is possible to show that every symmetric subshift in two symbols is intrinsically ergodic. Moreover, we conjecture that almost every lexicographic subshift is intrinsically ergodic.

Theorem 4.4.4. *If* $\boldsymbol{\alpha} \in \Sigma_2$ *satisfies* $\boldsymbol{\alpha}^* \prec \boldsymbol{\alpha} \prec 11(01)^\infty$ *and* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *is a shift of finite type, then* $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ *is intrinsically ergodic.*

Proof. Let A be the main component of $\Sigma_{\boldsymbol{\alpha}}$ given by Theorem 4.2.7, and $X = \Sigma_{\boldsymbol{\alpha}} \setminus A$. Let $[w]$ be a cylinder such that it is totally contained in X. Then

$$\nu_n([w]) = \frac{1}{|Per(n)|} \sum_{\boldsymbol{x} \in Per(n)} \delta_{\boldsymbol{x}}([w]) = \frac{|Per(n) \cap [w]\,|}{|Per(n)|}.$$

Note that $\nu_n([w]) \geq 0$ if and only if $w = u$ or $w = \bar{u}$, where $u = (01)^k$ for $k \geq \ell(\omega)$. In this case

$$\nu_n([w]) \leq \frac{2}{|Per(n)|}.$$

Observe that $\sigma^{-j}([w])$ occurs in $(01)^\infty$. Then

$$\mu_n([w]) = \frac{2}{|Per(n)|},$$

which tends to zero when n tends to ∞ because $h_{top}(\sigma_{\boldsymbol{\alpha}}) > 0$. This shows that the support of μ is $\Sigma_{\boldsymbol{\alpha}} \setminus X$.

Suppose that $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is not intrinsically ergodic. Then there exists m such that $h_\mu = h_m$ and $\mu \neq m$. In [Bow71, Bow75] it was shown that μ is ergodic. Then the support of m has to be contained in X. Because the support of m is a σ invariant subset, it has to be $\{(01)^\infty, (10)^\infty\}$, then $h_m = 0$, which is a contradiction. □

This argument can be adapted to show the intrinsic ergodicity of any symmetric subshift given by a parameter $\boldsymbol{\alpha}$ lying on an entropy plateau, irrespective of whether $\boldsymbol{\alpha} \prec 11(01)^\infty$ or $\boldsymbol{\alpha} \succ 11(01)^\infty$.

Theorem 4.4.5. *If $\boldsymbol{\alpha} \in [\boldsymbol{\omega}, \boldsymbol{\omega}''']_\prec$ and $[\boldsymbol{\omega}, \boldsymbol{\omega}''']_\prec$ is an entroply plateau, then $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is intrinsically ergodic.*

Proof. Suppose first that $\boldsymbol{\alpha} \succ 11(01)^\infty$ and $\boldsymbol{\alpha}$ belongs to an entropy plateau $[\boldsymbol{\omega}, \boldsymbol{\omega}''']$. Then $\boldsymbol{\omega}$ is an irreducible sequence, therefore, $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\omega}})$ is a transitive subshift of finite type. By Theorem 4.4.2, $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is intrinsically ergodic. Let μ_ω be the unique measure of maximal entropy of $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$. Note that we can extend μ_ω to a measure $\mu_{\boldsymbol{\alpha}}$ on $\Sigma_{\boldsymbol{\alpha}}$ as follows:

$$\mu_{\boldsymbol{\alpha}}(B) = \mu_\omega(B \cap \Sigma_\omega).$$

Clearly, $\mu_{\boldsymbol{\alpha}}$ is an ergodic measure of maximal entropy for $\Sigma_{\boldsymbol{\alpha}}$, satisfying

$$\operatorname{supp}(\mu_{\boldsymbol{\alpha}}) = \operatorname{supp}(\mu_\omega).$$

Suppose that there exists another measure $\nu_{\boldsymbol{\alpha}}$ of maximal entropy for (Σ_a, σ_a). By [Wal82, Theorem 8.7] we can consider $\nu_{\boldsymbol{\alpha}}$ as an ergodic measure for $\Sigma_{\boldsymbol{\alpha}}$. By [Wal82, Theorem 6.10] they are mutually singular. Let A be the set such that $\mu_{\boldsymbol{\alpha}}(A) = 1 = \nu_{\boldsymbol{\alpha}}(\Sigma_{\boldsymbol{\alpha}} \setminus A)$. Recall that

$$\mu_{\boldsymbol{\alpha}}(\operatorname{supp}(\mu_{\boldsymbol{\alpha}}) \cap \operatorname{supp}(\nu_{\boldsymbol{\alpha}})) = \nu(\operatorname{supp}(\mu_{\boldsymbol{\alpha}}) \cap \operatorname{supp}(\nu_{\boldsymbol{\alpha}})) = 0.$$

Note that for any measurable set $B \subset A$, $\nu_{\boldsymbol{\alpha}}(B)\log(\nu_{\boldsymbol{\alpha}}(B)) = 0$. Then $h_{\nu_{\boldsymbol{\alpha}}} = h_{top}(\sigma_{\boldsymbol{\alpha}} \mid_{\Sigma_{\boldsymbol{\alpha}} \setminus A})$. Note that $\Sigma_{\boldsymbol{\alpha}} \setminus A = \Sigma_a \setminus \Sigma_\omega \mod 0$. Then $h_{\nu_a} = h_{top}(\Sigma_a \setminus \Sigma_{\boldsymbol{\omega}})$.

Therefore, by Lemma 4.2.6

$$h_{\nu_{\boldsymbol{\alpha}}} = h_{top}(\Sigma_{\boldsymbol{\alpha}} \setminus \Sigma_{\boldsymbol{\omega}}) \leq \frac{1}{\ell(\omega)},$$

and by By Proposition 4.2.2 and Lemma 4.2.3

$$h_{\nu_{\boldsymbol{\alpha}}} < h_{\mu_{\boldsymbol{\alpha}}},$$

which contradicts that $\nu_{\boldsymbol{\alpha}}$ is a measure of maximal entropy. Therefore $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is intrinsically ergodic.

Suppose now that $\boldsymbol{\alpha} \prec 11(01)^{\infty}$. Let $\boldsymbol{\omega}$ be the left end point of the entropy plateau which $\boldsymbol{\alpha}$ belongs to. By Theorem 4.4.4, $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}})$ is intrinsically ergodic. Let μ_{ω} be the unique measure of maximal entropy of $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}})$. Note that μ_{ω} is supported in the main component of $(\Sigma_{\boldsymbol{\omega}}, \sigma_{\boldsymbol{\omega}})$, denoted by A. Let

$$\mu_{\boldsymbol{\alpha}}(B) = \mu_{\omega}(B \cap A).$$

Using the same argument as in the transitive case, we can conclude that μ_a is the unique measure of maximal entropy for $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$. □

As a consequence of Theorems 4.4.2, 4.3.2, 4.4.5 and the fact that $NS(\mathcal{E}) \subset \mathcal{E}$ Theorem 4.4.1 follows. Note that it is possible to construct examples of transitive subshifts with two transitive components with the same topological entropy (see [Hay13]), then the argument shown in the proof of Theorem 4.4.5 will not hold.

Chapter 5

The doubling map with asymmetrical holes

In this chapter we present some partial results about the dynamics of asymmetric subshifts. Also, we present some ideas in order to provide a full classification of the dynamics of asymmetric subshifts. Recall that a lexicographic subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is *asymmetric* if $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ is not symmetric, i.e, there exist $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ such that $\bar{\boldsymbol{x}} \notin \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$.

In Section 5.1 we give a very brief summary about Lorenz maps and their kneading invariants. Also we make a connection between them and the attractor of the doubling map with an asymmetric hole, that is for every expanding Lorenz map $g : [0, 1] \to [0, 1]$ there exists $(a, b) \in D_1$ such that $([0, 1], g)$ is a factor of $(\Lambda_{(a,b)}, f_{(a,b)})$.

In Section 5.2 we use the notion of renormalisation of the kneading invariant of a Lorenz map - see [Gle90]- in order to characterise transitive asymmetric subshifts. We show in Theorem 5.2.9 and Theorem 5.2.13 that every assymetric subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is transitive if and only if $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is not renormalisable. It is worth to mention that this result is a well known result for Lorenz maps. However, the presented proof is merely symbolic.

In Section 5.3 we present a family of assymetric subshifts introduced by Hofbauer in [Hof79] and Glendinning and Hall in [GH96] and it is parametrised by pairs $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$. We show that every element of the mentioned family can not have an intrinsic measure.

5.1 Expanding Lorenz Maps

During this section we formalise the connection between Lorenz maps and the studied attractors. It is worth to mention that in [GS93, 6.2] and [HS90, 3] it was mentioned that Lorenz maps can be studied as open dynamical systems of the doubling map.

We say that a map $g : [0,1] \to [0,1]$ is a *topologically expanding Lorenz map* if there exists $c \in (0,1)$ such that:

i) $g\mid_{[0.c)}$ and $g\mid_{(c,1]}$ are continuous and strictly increasing;

ii) $\lim\limits_{x^+ \to c} g(x) = 1$ and $\lim\limits_{x^- \to c} g(x) = 0$;

iii) $\overline{I_c} = [0,1]$ where $I_c = \bigcup\limits_{n=0}^{\infty} g^{-n}(c)$.

We call the pair $([0,1], g)$ an *expanding Lorenz dynamical system.*

For these maps, it is also natural to associate to them a symbolic space as follows: let g be an expanding Lorenz map and $x \in [0,1] \setminus I_c$. Then the *kneading sequence of* x, denoted by $k_g(x)$, is the sequence whose terms $k_i(x)$ are given by

$$k_i(x) = \begin{cases} 0 & \text{if } g^i(x) \in [0,c); \\ 1 & \text{if } g^i(x) \in (c,1]. \end{cases}$$

If $x \in I_c$ we define the *upper kneading sequence of* x as $k_g^+(x) = \lim\limits_{y \to x^+} k_g(y)$ and the *lower kneading sequence of* x as $k_g^-(x) = \lim\limits_{y \to x^-} k_g(y)$. Observe that if $x \in [0,1] \setminus I_c$, $k_g^+(x) = k_g^-(x) = k_g(x)$. Finally, we define the *kneading invariant of* g to be the pair $(k_g^-(c), k_g^+(c))$.

Hubbard and Sparrow have shown in [HS90, Theorem 1] that given an expanding Lorenz map g, $(\sigma(k_g^+(c)), \sigma(k_g^-(c))) \in \mathcal{LW}$. Moreover, every element $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ determine a unique Lorenz map g up to topological conjugacy. Also, in [HS90, Theorem 2] it is shown that $\Sigma_{(\sigma(k_g^+(c)),\sigma(k_g^-(c)))}$ is semi-conjugate to g, and the set of points such that the semi-conjugacy is not injective is precisely I_c. Therefore, by Theorem 3.2.3, the following statement is true.

Proposition 5.1.1. *For every expanding Lorenz map g there exists $(a,b) \in D_1$ such that $(\Lambda_{(a,b)}, f_{(a,b)})$ is semi-conjugate to $([0,1], g)$.*

5.2 Transitivity and Renormalisation

In this section we introduce the notion of *renormalisability* of a pair $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{LW}$. Renormalisability has a dynamical interpretation for Lorenz maps. The dynamical properties of Lorenz maps have been extensively studied - see [Gle90, GS93, HS90, LSS14] and references therein. The aim of this section is to characterise transitive subshifts $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ in terms of renormalisability. It is worth pointing out that it is a well known result that if $g : [0,1] \to [0,1]$ is an expanding Lorenz map, then $([0,1], g)$ is not transitive if and only if the kneading invariant $(k_g^-(c), k_g^+(c))$ is renormalisable. Nonetheless, since $([0,1], g)$ is merely a factor of $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ we cannot be sure if this property is transferred to $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ automatically. Then we will give a symbolic proof of this statement.

Firstly, we prove some easy claims which provide a useful partition of $\mathcal{LW}$. Also, Proposition 5.2.3 generalises Proposition 4.1.2.

Proposition 5.2.1. *Let $n \in \mathbb{N}$ such that $n \geq 2$. Then for every*

$$\boldsymbol{\alpha} \in ((1^{n-2}0)^\infty, (1^{n-1}0)^\infty]_\prec,$$

1^n is not a factor for any $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ for every $\boldsymbol{\beta} \in \bar{P}$.

Proof. Let $n \geq 2$ and $\boldsymbol{\alpha} \in ((1^{n-2}0)^\infty, (1^{n-1}0)^\infty]_\prec$. Then $a_i = 1$ for every $i \in \{1, \dots n-1\}$. Let $\boldsymbol{\beta} = 0^\infty$. Assume that there exist $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ such that 1^n is a factor of $\boldsymbol{x}$. Let

$$j = \min\{k \in \mathbb{N} \mid (\sigma^k(\boldsymbol{x}))_i = 1 \text{ for } i \in \{1, \dots n\}\}.$$

Then $\sigma^j(\boldsymbol{x}) \succ \boldsymbol{\alpha}$ which is a contradiction. Since $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \subset \Sigma_{(\boldsymbol{\alpha},0^\infty)}$ for every $\boldsymbol{\beta} \in (0^\infty, \pi^{-1}(\chi(a)))_\prec$ then 1^n is not a factor of any $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. □

Observe that it is possible to show an analogous result interchanging 1's to 0's which is stated below. The proof is essentially the same as the one for Proposition 5.2.1, so it is omitted.

Proposition 5.2.2. *Let $n \in \mathbb{N}$ such that $n \geq 3$. Then for every*

$$\boldsymbol{\beta} \in ((0^{n-1}1)^\infty, (0^{n-2}1)^\infty],$$

0^n *is not a factor for any* $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$.

Proposition 5.2.3. *Let $j, k \geq 2$, then the subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ where $\mathcal{F} = \{0^j, 1^k\}$ is transitive. Moreover, this subshift induces the lexicographic subshift of finite type given by $\boldsymbol{\alpha} = (1^{k-1}0)^\infty$ and $\boldsymbol{\beta} = (0^{j-1}1)^\infty$.*

Proof. Let $n = \max\{j, k\}$. Assume that $n = j$, then $\mathcal{F} = \{0^n, 1^k\}$. Let $u, v \in \mathcal{L}(\Sigma_{\mathcal{F}})$. If $u_{\ell(u)} = 1$ and $v_1 = 1$ then the word 0^{n-1} is a bridge between u and v. For $u_{\ell(u)} = 0$ and $v_1 = 0$ the word 1^{k-1} is a bridge between u and v. If $u_{\ell(u)} = 1$ and $v_1 = 0$ then the word $0^{n-1}1$ is a bridge between u and v. Finally, if $u_{\ell(u)} = 0$ and $v_1 = 1$ the word 10^{n-1} connect u and v. If $n = k$, then $\mathcal{F} = \{0^j, 1^n\}$. For this case similar bridges can be constructed. By [LSS14, Theorem 1.3] the subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ corresponding to $\boldsymbol{\alpha} = (1^{k-1}0)^\infty$ and $\boldsymbol{\beta} = (0^{j-1}1)^\infty$ is a subshift of finite type. Observe that $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \neq \emptyset$ since $(0^{j-1}1^{k-1})^\infty \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Also, note that for every $\boldsymbol{x} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, neither 0^j nor 1^n are factors of $\boldsymbol{x}$, which implies that $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \subset \Sigma_{\mathcal{F}}$. Let $\boldsymbol{x} \in \Sigma_{\mathcal{F}}$ and assume that $\boldsymbol{x} \notin \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Then there exists $n \in \mathbb{N}$ such that $\sigma^n(\boldsymbol{x}) \prec \boldsymbol{\beta}$ or $\sigma^n(\boldsymbol{x}) \succ \boldsymbol{\alpha}$. Without losing generality assume that implies that $\sigma^n(\boldsymbol{x}) \prec \boldsymbol{\beta}$. Then there exists $j' \in \mathbb{N}$ such that $(\sigma^n(\boldsymbol{x}))_{j'} = 0$ and $b_{j'} = 1$, which implies that 0^j is a factor of x, a contradiction. Therefore $\Sigma_{\mathcal{F}} = \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ and $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is a subshift of finite type. □

Now we introduce the notion of renormalisation for $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$.

Definition 5.2.4. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$. We say that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ is *renormalisable* if there exist two words ω and ν and sequences $\{n_i^\omega\}_{i=1}^\infty$ $\{n_i^\nu\}_{i=1}^\infty$, $\{m_j^\omega\}_{j=1}^\infty$ and $\{m_j^\nu\}_{j=1}^\infty \subset \mathbb{N} \cup \{\infty\}$ such that $\omega = 0\text{-max}_\omega$, $\nu = 1\text{-min}_\nu$, $\ell(\omega\nu) \geq 3$ and

$$0\boldsymbol{\alpha} = \omega\nu^{n_1^\nu}\omega^{n_1^\omega}\nu^{n_2^\nu}\omega^{n_2^\omega}\nu^{n_3^\nu}\dots$$

and

$$1\boldsymbol{\beta} = \nu\omega^{m_1^\omega}\nu^{m_1^\nu}\omega^{m_2^\omega}\nu^{m_2^\nu}\omega^{n_3^\omega}\dots$$

If $\ell(\omega\nu) = 3$ we say that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is trivially renormalisable.

Remark 5.2.5. We will also call a pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ renormalisable if ω or ν is an infinite sequence. In this case, $0\boldsymbol{\alpha} = \omega\nu$ and $1\boldsymbol{\beta} = \nu$ if ν is an infinite sequence and $0\boldsymbol{\alpha} = \omega$ and $1\boldsymbol{\beta} = \nu\omega$ if ω is an infinite sequence. In this case, we say that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is *renormalisable by an infinite sequence.*

Observe that, Definition 5.2.4 can be stated considering directly fot the binary expansion of $\pi^{-1}(a)$ and $\pi^{-1}(b)$ when $(a, b) \in \left(\frac{1}{4}, \frac{1}{2}\right) \times \left(\frac{1}{2}, \frac{3}{4}\right)$ and $(\sigma(\pi^{-1}(a)), \sigma(\pi^{-1}(b))) \in \mathcal{LW}$. Since we will state and prove the results of this chapter symbolically, we will state our results in terms of pairs $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$. Firstly, let us prove some technical facts about renormalisable pairs $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Lemma 5.2.6. *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ be renormalisable by ω and ν. Then the sequences $\{n_i^\omega\}_{i=1}^\infty$, $\{n_i^\nu\}_{i=1}^\infty$, $\{m_j^\omega\}_{j=1}^\infty$ and $\{m_j^\nu\}_{j=1}^\infty$ are bounded if neither $n_i = \infty$ nor $m_j = \infty$ for $i, j \in \mathbb{N}$. Moreover,*

$$\max\{m_j^\omega\}_{j=1}^\infty \leq \max\{n_i^\omega\}_{i=1}^\infty = n_1^\omega$$

and

$$\max\{n_i^\nu\}_{i=1}^\infty \leq \max\{m_j^\nu\}_{j=1}^\infty = m_1^\nu.$$

Proof. Firstly, assume that $\{n_i^\nu\}_{i=1}^\infty$ is not bounded. Then for every i there exists i' such that $n_i^\nu < n_{i'}^\nu$. In particular, there exists i_1' such that $n_1 < n_{i_1'}$. Take n sufficiently large such that $\sigma^n(\boldsymbol{\alpha}) = \omega\nu^{n_{i_1'}^\nu}$. Observe that $a_{(\ell(\omega)-1)n_1\ell(\nu)+1} = 0$ and $\sigma^n(\boldsymbol{\alpha})_{(\ell(\omega)-1)n_1\ell(\nu)+1} = 1$, which implies that $\boldsymbol{\alpha} \notin P$ which is a contradiction. Therefore $\{n_i^\nu\}_{i=1}^\infty$ is bounded and $\max\{n_i^\nu\}_{i=1}^\infty = n_1^\nu$.

Secondly, assume that $\{m_j^\omega\}_{j=1}^\infty$ is not bounded. Similarly, for every j there exists j' such that $m_j^\omega < m_{j'}^\omega$. Let j_1' be such that $m_1 < n_{j_1'}$. Take m sufficiently large such that $\sigma^m(a) = \nu\omega^{m_{j_1'}^\omega}$. Observe that $b_{(\ell(\nu)-1)m_1\ell(\omega)+1} = 0$ and $\sigma^m(\boldsymbol{\alpha})_{(\ell(\nu)-1)m_1\ell(\omega)+1} = 0$, which implies that $\boldsymbol{\beta} \notin \bar{P}$ which is a contradiction. Therefore $\{m_j^\omega\}_{j=1}^\infty$ is bounded and $\max\{m_j^\omega\}_{j=1}^\infty = m_1^\omega$.

Now, assume that $\{n_i^\omega\}_{i=1}^\infty$ is not bounded. Since $\{m_j^\omega\}_{j=1}^\infty$ is bounded there exists j such that $m_1^\omega < n_j^\omega$. Since (a, b) is renormalisable, there exists n' such that $\sigma^{n'}(\boldsymbol{\alpha}) = \nu\omega^{n_j^\omega}\nu\ldots$. Then $\sigma^{n'}(\boldsymbol{\alpha}) \prec \boldsymbol{\beta}$ which contradicts that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$. Therefore $\{n_i^\omega\}_{i=1}^\infty$ is bounded by m_1^ω. Similarly, $\{m_j^\nu\}_{j=1}^\infty$ is bounded by n_1^ν. The proof is omitted. $\square$

Observe that $n_i^\nu = \infty$ for some $i \in \mathbb{N}$ if and only if $i = 1$. Also, $m_j^\omega = \infty$ for some $j \in \mathbb{N}$ if and only if $j = 1$. This is a direct corollary of Lemma 5.2.6.

In [GS93, Lemma 2] the authors proved that n_1^ω and m_1^ν cannot be simultaneously equal to 1 based in a more restrictive definition of the lexicographic world $\mathcal{LW}$. Using Definition 2.2.3, it is possible to consider such a case. Next, we introduce the following notion.

Definition 5.2.7. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$. We say that $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ is a *cyclic subshift* if there exists $\omega \in \mathcal{L}(\Sigma_2)$, such that $\boldsymbol{\alpha} = \sigma(0\text{-max}_\omega^\infty)$ and $\boldsymbol{\beta} = \sigma(1\text{-min}_\omega^\infty)$.

From [LSS14, Theorem 1.3] we know that every cyclic subshift is a subshift of finite type. Also, it is clear that for every cyclic subshift $0_b = 0_a$ and $1_b = 1_a$.

Proposition 5.2.8. *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ be renormalisable by ω and ν. Then $\omega^\infty, \nu^\infty \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$.*

Proof. Firstly, note that if $0\boldsymbol{\alpha} = \omega\nu^\infty$ and $1\boldsymbol{\beta} = \nu\omega^\infty$ the result is automatically true. Let

$$0\boldsymbol{\alpha} = \omega\nu^{n_1^\nu}\omega^{n_1^\omega}\nu^{n_2^\nu}\omega^{n_2^\omega}\nu^{n_3^\nu}\ldots \text{ and } 1\boldsymbol{\beta} = \nu\omega^{m_1^\omega}\nu^{m_1^\nu}\omega^{m_2^\omega}\nu^{m_2^\nu}\omega^{n_3^\omega}\ldots.$$

It is clear that $0\boldsymbol{\alpha}, 1\boldsymbol{\beta} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ since $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ and $\omega\nu, \nu\omega \in \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Note that $\omega^2_{\ell(\omega)+1} = 0$ and $\omega\nu_{\ell(\omega)+1} = 1$. Also, $\nu^2_{\ell(\nu)+1} = 1$ and $\nu\omega_{\ell(nu)+1} = 0$. This implies that $\boldsymbol{\beta} \prec \omega^\infty \prec \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \prec \nu^\infty \prec \boldsymbol{\alpha}$. Since $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ it is clear that $\boldsymbol{\beta} \prec \sigma^n(\omega^\infty) \prec \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \prec \sigma^n(\nu^\infty) \prec \boldsymbol{\alpha}$. Thus $\omega^\infty, \nu^\infty \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Note that if

$$0\boldsymbol{\alpha} = \omega\nu^\infty \text{ and } 1\boldsymbol{\beta} = \nu\omega^{m_1^\omega}\nu^{m_1^\nu}\omega^{m_2^\omega}\nu^{m_2^\nu}\omega^{n_3^\omega}\ldots,$$

or

$$0\boldsymbol{\alpha} = \omega\nu^{n_1^\nu}\omega^{n_1^\omega}\nu^{n_2^\nu}\omega^{n_2^\omega}\nu^{n_3^\nu}\ldots \text{ and } 1\boldsymbol{\beta} = \nu\omega^\infty$$

using the same argument we can show that $\omega^\infty, \nu^\infty \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ for both cases. □

Theorem 5.2.9. *If $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ is renormalisable by ω and ν and $\ell(\omega) + \ell(\nu) > 4$ then $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is not transitive.*

Proof. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ be renormalisable by ω and ν. By Proposition 5.2.8, ω^∞ and $\nu^\infty \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Also, $(\omega\nu)^m$ and $(\nu\omega)^m \in \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ for every $m \in \mathbb{N}$. Note that $\sigma(\omega)1 \in B_{\ell(\omega)}$ is the maximal admissible word of length $\ell(\omega)$ and $\sigma(\nu)0 \in B_{\ell(\nu)}$ is the minimal admissible word of length $\ell(\nu)$. It is needed to consider three cases.

Case 1) Assume that

$$0\boldsymbol{\alpha} = \omega\nu^{n_1^\nu}\omega^{n_1^\omega}\nu^{n_2^\nu}\omega^{n_2^\omega}\nu^{n_3^\nu}\dots \text{ and } 1\boldsymbol{\beta} = \nu\omega^{m_1^\omega}\nu^{m_1^\nu}\omega^{m_2^\omega}\nu^{m_2^\nu}\omega^{n_3^\omega}\dots.$$

From Proposition 5.2.8, $\omega^{m_1^\omega+1}$ and $\nu^{n_1^\nu+1} \in \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. We want to show that there are no bridges neither between $\omega 1$ and $\nu^{n_1^\nu+1}$ nor $\nu 0$ and $\omega^{m_1^\omega+1}$. Note that Lemma 5.2.6 implies that $\omega 1\nu^{n_1^\nu+1} \notin \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Suppose that there exists a bridge u such that $\omega 1 u\nu^{n_1^\nu+1} \in \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Since $\sigma(\omega)1$ is the maximal admissible word of length $\ell(\omega)$ the first $\ell(\nu)-2$ digits of u satisfy that $u_i = \nu_{i+1}$ and the following digit is free. If $u_{\ell(\nu)-1} \neq \nu_{\ell(\nu)}$ then $\sigma(\omega)1u \succ \boldsymbol{\alpha}$ or $1u \prec \nu$ which is a contradiction. Then $u_{\ell(\nu)-1} = u_{\ell(\nu)}$. Also, if $\ell(u) = \ell(\nu) - 1$ then $\sigma(\omega)1u\nu^{n_1^\nu+1} \succ a$ which is a contradiction. This implies that $\ell(u) > \ell(\nu) - 1$ and that $u_1 \dots u_{\ell(\nu)-1} = \sigma(\nu)$. Then $u_{\ell(\nu)} = 0$ then the following $\ell(\omega)$ digits coincide with ω and we reach a contradiction. If $u_{\ell(\nu)} = 0$ then the following $\ell(\nu)$ digits will coincide with ν. This implies that $\ell(u) = \infty$ which is a contradiction. Then $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is not transitive.

Case 2) Assume that $n_1^\nu = \infty$ or $m_1^\omega = \infty$ but not both. Without losing generality assume that $n_1^\nu = \infty$. Then $0\boldsymbol{\alpha} = \omega\nu^\infty$ and $1\boldsymbol{\beta} = \nu\omega^{m_1^\omega}\nu^{m_1^\nu}\omega^{m_2^\omega}\nu^{m_2^\nu}\omega^{n_3^\omega}\dots$. We observe that there are no bridges between $\nu 0$ and $\omega^{m_1^\omega+1}$ as follows. Firstly, from Lemma 5.2.6 we now that $\nu 0\omega^{m_1^\omega+1} \notin \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Assume that the bridge u exists. Since $\sigma(\nu)0$ is the minimal admissible word of length $\ell(\nu)$ then u satisfies that $u_i = \omega_{i+1}$ for $i \in \{1, \dots \ell(\omega) - 1\}$ and the following digit is free. If $u_{\ell(\omega)} = 1$ then $u = \sigma(\omega)1$, this implies that $u_{\ell(\omega)+i} = \nu_{i+1}$ for $i \in \{1 \dots \ell(\nu) - 1\}$. Note that if $u_{\ell(\omega)+\ell(\nu)} = 0$ then we fall in our starting case. If $u_{\ell(\omega)+\ell(\nu)} = 1$ then $u_{\ell(\omega)+\ell(\nu)+i} = \nu_i$. Then $u_{\ell(\omega)} = 0$. If $u_{\ell(\omega)} = 0$ then $u_{\ell(\omega)+i} = \omega_{i+1}$ for $i \in \{1 \dots \ell(\omega) - 1\}$. Note that $u_{2\ell(\omega)} = 1$ we get the same conclusion from the case when $u_{\ell(\omega)} = 1$. Thus, $u_{2\ell(\omega)} = 0$. This implies that $u_1 \dots u_{m_1^\omega(\ell(\omega))-1} = \sigma^{(\omega)^{m_1^\omega}}$ and $u_{m_1^\omega \ell(\omega)} = 1$ which takes us to the starting case. Then $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is not transitive.

Case 3) Assume that $n_1^\nu = \infty$ and $m_1^\omega = \infty$. Then $0\boldsymbol{\alpha} = \omega\nu^\infty$ and $1\boldsymbol{\beta} = \nu\omega^\infty$. From Proposition 5.2.8 we know that ω^∞ and $\nu^\infty \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Then $\max(\omega)$ and $\max(\nu) \in \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Since $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{LW}$ then $\max(\nu)^\infty \prec \max(\omega)^\infty$. Note that $(\max(\omega)\max(\nu))^\infty \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ since $\max(\omega) \prec \boldsymbol{\alpha}$, $\max(\nu) \prec \boldsymbol{\alpha}$, $\max(\omega)_{\ell(\omega)} = 0$ and $\max(\nu)_{\ell(\nu)} = 1$. Moreover,

$$\sigma^n((\max(\omega)\max(\nu))^\infty) \succ \nu^\infty \text{ and } \sigma^n((\max(\omega)\max(\nu))^\infty) \prec \omega^\infty$$

for every $n \in \mathbb{N}$. Note that

$$\nu_2 \dots \nu_{\ell(\nu)} 0 \max(\omega)\max(\nu) \notin \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \tag{5.1}$$

since

$$\nu_2 \dots \nu_{\ell(\nu)} 0 \max(\omega)\max(\nu) \prec \boldsymbol{\beta}.$$

Finally, we want to show that there is no bridge between $a_1 \dots a_{\ell(\omega)+\ell(\nu)-1}1$ and $(\max(\omega)\max(\nu))^2$. Assume that such a bridge v exists, i.e.

$$a_1 \dots a_{\ell(\omega)+\ell(\nu)-1} 1 v (\max(\omega)\max(\nu))^2 \in \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$$

and $v \in \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Observe that $v_1 \dots v_{\ell(\nu)-1} = \nu_2 \dots \nu_{\ell(\nu)}$ since if $v_i \neq \nu_{i-1}$ then

$$a_1 \dots a_{\ell(\omega)+\ell(\nu)-1)} 1 v_1 \dots v_i \succ \boldsymbol{\alpha}$$

if $v_i = 1$ or $v_1 \dots v_{\ell(\nu)-1} \prec b$ if $v_i = 0$. Then it is necessary to consider two sub cases. Assume that $v_{\ell(\nu)} = 1$. If $v_{\ell(\nu)-1} = 1$ then

$$a_1 \dots a_{\ell(\omega)+\ell(\nu)-1} 1 v_1 \dots v_{\ell(\nu)-1} = a_1 \dots a_{\ell(\omega)+2\ell(\nu)-1} 1,$$

which implies $v_{i-1} = (\nu)_i^\infty$ for every $i \in \{1 \dots 2\ell(\nu) - 1\}$. Then $v_{2\ell(\nu)}$ is either 0 or 1. If $v_{2\ell(\nu)} = 1$ we get the previous case. Therefore $v_{2\ell(\nu)} = 0$. Note that $v_{2\ell(\nu)} = 0$ is equivalent to considering $v_{\ell(\nu)} = 0$. Consider now that $v_{\ell(\nu)} = 0$. From (5.1), $v_1 \dots v_{\ell(\nu)} \max(\omega)\max(\nu)$ is not admissible. Then $v_1 \dots v_{\ell(\nu)+\ell(\omega)-1} = b_1 \dots b_{\ell(\nu)}$. Then $v_{\ell(\nu)+\ell(\omega)} \dots v_{\ell(\nu)+\ell(\omega)} = \omega_1 \dots \omega_{\ell(\omega)-1}$. Then if $v_{\ell(\nu)+\ell(\omega)+1} = 1$ then $a_1 \dots a_{\ell(\nu)+\ell(\omega)+1} \prec a_1 \dots a_{\ell(\omega)+\ell(\nu)-1} 1 v$ which is a contradiction.

□

Observe that in the proof of Theorem 5.2.9 we avoid the cases where $3 \leq \ell(\omega) + \ell(\nu) \leq 4$. Also, we avoid the cases when $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is trivially renormalisable.

Corollary 5.2.10. *If $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is a cyclic subshift, then $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is not transitive.*

Proof. Let $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ be a cyclic subshift. Observe that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are cyclic permutations of each other. Then there exist $u, v \in \mathcal{L}(\Sigma_2)$ such that u starts with 0, v starts with 1, $0\boldsymbol{\alpha} = (uv)^\infty$ and $1\boldsymbol{\beta} = (vu)^\infty$ which implies that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is renormalisable. Then by Theorem 5.2.9, $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is not transitive. □

To characterise transitivity via the renormalisablility of $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ it is necessary to show that $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is transitive provided that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is not renormalisable. To do this we will split the proof of the theorem as follows.

Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$. We say that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is *essential* if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are periodic sequences and $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is not renormalisable. From [LSS14, Theorem 1.3] $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is a subshift of finite type provided that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is an essential pair. From Proposition 2.3.2 we obtain that every for essential pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ there exists a pair of finite words (ω, ν) such that $\omega = 0\text{-max}_\omega$, $\nu = 1\text{-min}_\nu$ and $0\boldsymbol{\alpha} = \omega^\infty$ and $1\boldsymbol{\beta} = \nu^\infty$. The pair (ω, ν) is called *the associated pair.*

Theorem 5.2.11. *If $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ is an essential pair then $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is transitive.*

Proof. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ be an essential pair with associated pair (ω, ν). Consider the pair $(1\text{-min}_\omega, 0\text{-max}_\nu)$. Observe that $(\omega^\infty, 1\text{-min}_\omega)$ and $(0\text{-max}_\nu^\infty, \nu^\infty)$ are cyclic subshifts. Then we observe that there exist words $\omega_{\boldsymbol{\alpha}}, \omega_{\boldsymbol{\beta}}, \nu_{\boldsymbol{\alpha}}, \nu_{\boldsymbol{\beta}}$ such that $(\omega, \nu) = (\omega_{\boldsymbol{\alpha}}\nu_{\boldsymbol{\alpha}}, \nu_{\boldsymbol{\beta}}\omega_{\boldsymbol{\beta}})$ and $(1\text{-min}_\omega, 0\text{-max}_\nu) = (\nu_{\boldsymbol{\alpha}}\omega_{\boldsymbol{\alpha}}, \omega_{\boldsymbol{\beta}}\nu_{\boldsymbol{\beta}})$. Since $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is an essential pair $\omega_{\boldsymbol{\alpha}} \neq \omega_{\boldsymbol{\beta}}$ or $\nu_{\boldsymbol{\alpha}} \neq \nu_{\boldsymbol{\beta}}$. Let $p_1 = \min\{\omega_{\boldsymbol{\alpha}}, \omega_{\boldsymbol{\beta}}\}$, $p_1' = \max\{\omega_{\boldsymbol{\alpha}}, \omega_{\boldsymbol{\beta}}\}$, $p_2 = \max\{\nu_{\boldsymbol{\alpha}}, \nu_{\boldsymbol{\beta}}\}$, and $p_2' = \min\{\nu_{\boldsymbol{\alpha}}, \nu_{\boldsymbol{\beta}}\}$. Observe that $(p_1p_2)^\infty \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ since $(p_1p_2)^\infty \prec \omega^\infty$, $\boldsymbol{\beta} \prec (p_2p_1)^\infty$ and from the election of p_1 and p_2.

Note that $\Sigma_\mathcal{F} \subset \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ where $\mathcal{F} = \{0^{0\alpha}, 1^{1\beta}\}$. From Proposition 5.2.3, $(\Sigma_\mathcal{F}, \sigma_\mathcal{F})$ is a transitive subshift of finite type. Then we just need to show that for $u \in \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \setminus \mathcal{L}(\Sigma_\mathcal{F})$ and $v \in \mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ there exist a bridge w from u to v. Without losing generality, we can assume that $v_1 = 1$. Suppose that $1_{\boldsymbol{\alpha}}$ occurs in u and $0_{\boldsymbol{\beta}}$ does not. Let

$$n = \max\{j \in \{1, \ldots, \ell(u)\} \mid \sigma^{n-1+1_{\boldsymbol{\alpha}}}(u) = 10 \ldots u_{\ell(u)}\}.$$

Observe that $\sigma^{n-1+1_{\boldsymbol{\alpha}}}(u) \succ (p_{21} \ldots p_{2\ell(u)-(n-1)})$. If $\sigma^{n-1+1_{\boldsymbol{\alpha}}}(u)_{\ell(u)-(n-1)} = 0$ then $w = p_2p_1$ is a bridge between u and v and if $\sigma^{n-1+1_{\boldsymbol{\alpha}}}(u)_{\ell(u)-(n-1)} = 1$ then $w = p_1p_2$ is a bridge between u and v.

Similarly, suppose that $0_{\boldsymbol{\beta}}$ occurs in u and $1_{\boldsymbol{\alpha}}$ does not. Let

$$n' = \max\{j \in \{1, \ldots, \ell(u)\} \mid \sigma^{n-1+0_{\boldsymbol{\beta}}}(u) = 01 \ldots u_{\ell(u)}\}.$$

Note that $\sigma^{n-1+1_{\boldsymbol{\alpha}}}(u) \prec (p_{11} \ldots p_{1\ell(u)-(n-1)})$. If $\sigma^{n-1+0_{\boldsymbol{\beta}}}(u)_{\ell(u)-(n-1)} = 0$ then $w = p_2p_1$ is a bridge between u and v and if $\sigma^{n-1+0_{\boldsymbol{\beta}}}(u)_{\ell(u)-(n-1)} = 1$ then $w = p_1p_2$ is a bridge between u and v.

If $1_{\boldsymbol{\alpha}}$ and $0_{\boldsymbol{\beta}}$ then let $m = \max\{n, n'\}$. If $m = n$ then the proof is the same as the case when $1_{\boldsymbol{\alpha}}$ occurs in u but $0_{\boldsymbol{\beta}}$ does not. Similarly for $m = n'$. □

To finish this section, we prove that every non renormalisable pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ corresponds to a transitive lexicographic subshift. It is clear that given a sequence $\{(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i)\}_{i=1} \subset \mathcal{LW}$ such that for every $i \in \mathbb{N}$, $\boldsymbol{\alpha}_{i+1} \prec \boldsymbol{\alpha}_i$, $\boldsymbol{\beta}_{i+1} \succ \boldsymbol{\beta}_i$, $\boldsymbol{\alpha}_i \to \boldsymbol{\alpha}$, $\boldsymbol{\beta}_i \to \boldsymbol{\beta}$ and $(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i)$ is an essential pair then $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is transitive. In particular, $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is coded. We show now that every non renormalisable pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ is a coded system.

Lemma 5.2.12. *If $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ is non renormalisable, then $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is coded.*

Proof. From Theorem 5.2.11 we just need to show the case when $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ are not periodic sequences. Firstly, assume that $\boldsymbol{\alpha}$ is not a periodic sequence and $\boldsymbol{\beta}$ is. Let i_1 such that $i_1 > 1_{\boldsymbol{\alpha}}$ and $a_{i_1} = 1$. Let $i_2 > i_1$ such that $a_{i_2} = 1$. Then we define inductively the sequence $\{i_m\}_{m=1}^{\infty}$ as $i_m > i_{m-1}$ and $a_{i_m} = 1$. We define $\{\boldsymbol{\alpha}_m\}_{m=1}^{\infty}$ as $\boldsymbol{\alpha}_m = (a_1 \ldots a_{i_m-1}0)^{\infty}$. From the construction it is clear that $\{\boldsymbol{\alpha}_m\}_{m=1}^{\infty} \in Per(\sigma) \cap P$, $\sigma^n(\boldsymbol{\alpha}_m) \prec \boldsymbol{\alpha}$ for every $n, m \in \mathbb{N}$, $\boldsymbol{\alpha}_m \prec \boldsymbol{\alpha}_{m+1}$ for every $m \in \mathbb{N}$, and $\boldsymbol{\alpha}_m \underset{m\to\infty}{\longrightarrow} \boldsymbol{\alpha}$. Let $\{\boldsymbol{\alpha}_{m_k}\}_{k=1}^{\infty} \subset \{\boldsymbol{\alpha}_m\}_{m=1}^{\infty}$ such that $\boldsymbol{\alpha}_{m_k} \in \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ for every $k \in \mathbb{N}$ and $\boldsymbol{\alpha}_{m_k} \underset{k\to\infty}{\longrightarrow} \boldsymbol{\alpha}$. Then $(\boldsymbol{\alpha}_{m_k}, \boldsymbol{\beta}) \in \mathcal{LW}$. We claim that there exist $K \in \mathbb{N}$ such that, for every $k \geq$

K, $(\Sigma_{(\boldsymbol{\alpha}_{m_k},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha}_{m_k},\boldsymbol{\beta})})$ is transitive. The claim implies that $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \overline{\bigcup_{k=K}^{\infty} \Sigma_{(\boldsymbol{\alpha}_{m_k},\boldsymbol{\beta})}}$ and hence, $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is coded. To prove the claim, assume that it is not true. This implies that there exist infinitely many $k \in \mathbb{N}$ such that $(\Sigma_{(\boldsymbol{\alpha}_{m_k},\boldsymbol{\beta}}, \sigma_{(\boldsymbol{\alpha}_{m_k},\boldsymbol{\beta})})$ is not transitive for infinitely many $k \in \mathbb{N}$. Then from Theorem 5.2.11, $(\boldsymbol{\alpha}_{m_k}, \boldsymbol{\beta})$ is renormalisable. Then $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is renormalisable, which is a contradiction.

Note that if $\boldsymbol{\beta}$ is not a periodic sequence and $\boldsymbol{\alpha}$ is, it is possible to develop a similar construction to the previous one considering the sequence $\{j_n\}_{n=1}^{\infty}$ defined as follows: Let j_1 such that $j_1 > 0_b$ and $b_{j_1} = 0$. Let $j_2 > j_1$ such that $b_{j_2} = 0$. Inductively, we define $\{j_n\}_{n=1}^{\infty}$ as $j_n > j_{n-1}$ and $b_{j_n} = 0$. Then $\boldsymbol{\beta}_n = (b_1 \ldots b_{j_n} 1)^{\infty}$. Then $\{\boldsymbol{\beta}_n\}_{n=1}^{\infty} \in Per(\sigma) \cap \bar{P}$, $\sigma^m(\boldsymbol{\beta}_n) \succ \boldsymbol{\beta}$ for every $n, m \in \mathbb{N}$, $\boldsymbol{\beta}_n < \boldsymbol{\beta}_{n-1}$ and $\boldsymbol{\beta}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{\beta}$.

Assume now that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are not periodic. Observe that the sequences $\{j_m\}_{m=1}^{\infty}$ and $\{i_n\}_{n=1}^{\infty}$ can be defined as we showed before. Then there exist a sequence $\{\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k\}_{k=1}^{\infty} \subset \{\boldsymbol{\alpha}_m, \boldsymbol{\beta}_n\}_{n,m\in\mathbb{N}}$ such that $(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k) \in \mathcal{LW}$, $(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k) \underset{n\to\infty}{\longrightarrow} (\boldsymbol{\alpha}, \boldsymbol{\beta})$, $(\Sigma_{(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)}, \sigma_{(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)})$ is a transitive subshift of finite type and $\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \overline{\bigcup_{k=1}^{\infty} \Sigma_{(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)}}$. □

Theorem 5.2.13. *If $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is not renormalisable then $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is transitive.*

Proof. This is an immediate consequence of Theorem 5.2.11 and Theorem 5.2.12. □

From [LM95, Proposition 4.5.10 (4)], [HS14, Proposition 2.6] and Theorem 3.2.3 we obtain that if $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ is not renormalisable and the associated attractor $(\Lambda_{(a,b)}, f_{(a,b)})$ satisfies that $(a, b) \in D_3$ then $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is topologically mixing.

5.3 Intrinsic Ergodicity

In this section we describe a countable family of non transitive asymmetric subshifts due to Hofbauer [Hof79] and Glendinning and Hall [GH96]. In Theorem 5.3.2 we show that none of the elements of the mentioned family is intrinsically ergodic.

Let $\omega = 01$ and $\nu_k = 100(10)^k$ for $k \geq 0$. Let $(0\boldsymbol{\alpha}, 1\boldsymbol{\beta})$ be the pair given by $0\boldsymbol{\alpha} = \omega(\nu_k)^{\infty}$ and $1\boldsymbol{\beta} = \nu_k(\omega)^{\infty}$. The case when $k = 0$ was introduced in [Hof79] and it was extended to the general case in [GH96]. By Theorem 5.2.9, $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is not a transitive subshift.

Definition 5.3.1. Let $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{LW}$. We define *the entropy formula for* $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ by

$$K_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(t) = \sum_{i=0}^{\infty}(b_i - a_i)t^i$$

where $a_0 = 0$ and $b_0 = 1$.

The entropy formula was introduced in [GH96] for Lorenz maps. Recently, Barnsley, Steiner and Vince in [BSV12] give a symbolic proof showing that the smallest positive root of $K(t)$, denoted by κ, satisfies that $\kappa = \left(\lim_{n\to\infty} \sqrt[n]{|B_n(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})|}\right)^{-1}$. Thus, $\log(\frac{1}{\kappa}) = h_{top}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$.

To prove the following theorem, we use some results corresponding to Lorenz maps. Firstly, recall that given a dynamical system (X, f) a point $x \in X$ is *non wandering* if for every open set $U \subset X$ such that $x \in U$ there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. We call $\Omega(f) = \{x \in X \mid x \text{ is non wandering } \}$ the *non-wandering set of* (X, f) [BS02, p. 29]. In [GH96, Corollary 10] is showed that if a Lorenz map g has as kneading invariant $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{LW}$ satisfying that $0\boldsymbol{\alpha} = \omega\nu_k^{\infty}$ and $1\boldsymbol{\beta} = \nu_k\omega^{\infty}$ then there are two basic components of the non-wandering set A and B such that $A \cap B = \emptyset$ and $h_{top}(g \mid_A) = h_{top}(g \mid_b) = h_{top}(g)$. As we show in the following theorem, this result imply that $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is not intrinsically ergodic.

Theorem 5.3.2. *If* $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{LW}$ *satisfies that* $0\boldsymbol{\alpha} = \omega\nu_k^{\infty}$ *and* $1\boldsymbol{\beta} = \nu_k\omega^{\infty}$ *then* $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ *is not intrinsically ergodic.*

Proof. Let $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{LW}$ satisfying our hypothesis. From [HS90, Theorem 1] there exists an expanding Lorenz map $g_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ such that $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is semi-conjugated to $([0,1], g_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ by a semi-conjugacy h. Let $\Omega_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ be the non-wandering set of $([0,1], g_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Then by [GH96, Corollary 10] there exist two $g_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$-invariant sets $A, B \subset \Omega_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ such that

$$h_{top}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \mid_A) = h_{top}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \mid_B) = h_{top}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})}).$$

By [GS93, Theorem 2], $A \cap B = \emptyset$, whence $h^{-1}(A) \cap h^{-1}(B) = \emptyset$. Moreover, by [GH96, Theorem 3], $h_{top}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) = \log(\frac{1}{\kappa})$. Then by [BSV12, Lemma 3], $h_{top}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(h^{-1}(A))) =$

$h_{top}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(h^{-1}(B))) = h_{top}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. This implies that there exist two $\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$-invariant measures μ_A and μ_B such that $\text{supp}(\mu_A) \subset A$, $\text{supp}(\mu_B) \subset B$ and

$$h_{\mu_A} = h_{\mu_B} = h_{top}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}).$$

Thus, $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is not intrinsically ergodic. □

Note that none of the cases in Theorem 5.3.2 is a subshift of finite type or symmetric. Nonetheless, the argument used in the proof of Theorem 5.3.2 may fail for a general non transitive asymmetric subshift. The main technical issue to deal with is the fact that given $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$, $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta}),\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}})$ is semi-conjugated to $([0,1], g_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ where $g_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ is a Lorenz map by a factor map h as we explain as follows.

Let $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta}),\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}})$ be a non transitive lexicographic subshift with

$$(0\boldsymbol{\alpha}, 1\boldsymbol{\beta}) \neq (\omega\nu_k^\infty, \nu_k\omega^\infty)$$

and let $([0,1], g_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ be the associated expanding Lorenz dynamical system. From Theorems 5.2.9 and 5.2.13 we know that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is renormalisable. Then it is well known that the non-wandering set

$$\Omega(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) = \bigcup_{i=0}^{p}(\Omega_i)$$

-see e.g. [GS93, Theorem 2]. By Theorem [BS02, Proposition 2.5.5] it is true

$$h_{top}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) = \max_{0\le i\le p}\{h_{top}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \mid_{\Omega_i})\}.$$

By [GH96, Corollary 10], there exists a unique $i \in \{0, \dots, p\}$ such that

$$h_{top}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \mid_{\Omega_i}) = h_{top}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})}).$$

Nonetheless, this does not guarantee that

$$h_{top}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \mid_{h^{-1}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\mid_{\Omega_j})}) < h_{top}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}).$$

Specifically, since h is a factor map not a conjugacy then

$$h_{top}(h^{-1}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \mid_{\Omega_j})) \le h_{top}(h^{-1}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \mid_{\Omega_i})).$$

However, the author believes that it is possible to show that

$$h_{top}(h^{-1}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\mid_{\Omega_j})) < h_{top}(h^{-1}(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\mid_{\Omega_i})).$$

A possible strategy to do this is, firstly, to generalise Theorem 4.2.7 to the asymmetric case, that is, show that every non transitive lexicographic subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ has a unique transitive component of maximal entropy A. Moreover, the author believes that there exists a correspondence between the transitive components of $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ and the basic sets of $\Omega(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Furthermore, if such correspondence exists, Proposition 5.1.1 and [BKT11, Theorem 6.3] would imply that $\Omega(g_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ has at most 4 basic components. Then it will be necessary to show that (A, σ_A) is intrinsically ergodic. It is known from [BKT11, Proposition 7.1] that the transitive components of $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ are coded, but it is still not known that the main component has the specification property.

Chapter 6

β-expansions and the Lexicographic world

In this Chapter we include some applications of our results to contexts different from the doubling map with holes.

In Section 6.1 we show the correspondence between greedy β-expansions with the doubling map with a hole given by $(a, 1)$ with $a \in (\frac{1}{2}, 1)$ or $(0, b)$ with $b \in (0, \frac{1}{2})$. In Section 6.2 we show that attractor the doubling map with a symmetric hole corresponding to $a \in [\frac{1}{3}, \frac{1}{2})$ corresponds to the set of unique β-expansions with $\beta \in (\varphi, 2)$ where φ denotes the golden ratio. Finally in Section 6.3 we mention the relation between the lexicographic world $\mathcal{LW}$ and intermediate β-expansions.

6.1 The doubling map and β-expansions

Consider $\beta \in \mathbb{R} \setminus \mathbb{Z}$ with $\beta > 1$ and $x \geq 0$. A sequence $\{b_n\}_{n=1}^{\infty}$ satisfying

$$x = \sum_{n=1}^{\infty} b_n \beta^{-n},$$

where $b_n \in \{0, \dots [\beta]\}$ or $\{0, \dots, \lceil \beta \rceil - 1\}$ is said to be *an expansion on base β for x* or simply *a β-expansion for x*. This notion was introduced by Parry and Rényi in [Par60, Rén57] and it is a generalisation of the expansions with integer basis. We will concentrate our attention on $\beta \in (1, 2)$ only.

An algorithm to find one of such expansions for $x \in [0,1]$ is provided by the β-*transformation*. Given $\beta \in (1,2)$, the β-*transformation* is the map $\tau_\beta : [0,1] \to [0,1]$ defined by $\tau_\beta(x) = \beta x \mod 1$. Note that if $\beta = 2$ then $\tau_\beta = 2x \mod 1$. Using τ_β, it is possible to construct a β-expansion for $x \in [0,1)$, namely, $b_n = [\beta \tau_\beta^{n-1}(x)]$ for $n \in \mathbb{N}$. The β-expansion obtained by applying this algorithm to x is called *the greedy expansion of* x and we will refer to the algorithm in question as *the greedy algorithm.*

Let $X_\beta \subset \Sigma_2$ be the set of all sequences obtained by applying the greedy algorithm. Define $\sigma_\beta : X_\beta \to X_\beta$ by $\sigma_\beta = \sigma\mid_{X_\beta}$. Then it is possible to show that (X_β, σ_β) is topologically conjugated to $([0,1], \tau_\beta)$ [Sid03b]. Moreover, it is shown in [Par60] that

$$X_\beta = \{x \in \Sigma_2 \mid \sigma^n(x) \prec 1_\beta \text{ for every } n \in \mathbb{N}\},$$

where $1_\beta = (d_i)_{i=1}^\infty$ is the greedy expansion of 1 if 1_β is not a finite sequence and $1_\beta = (d_1, \ldots d_{k-1}0)^\infty$ if the greedy expansion of 1 is finite and has length k.This expansion is called *quasi-greedy expansion of* 1. Moreover, in [Par60] the set of sequences that are greedy β-expansions of 1 is characterised lexicographically. Formally: a sequence $d = (d_i)_{i=1}^\infty \in \Sigma_2^1$ is the greedy expansion of 1 for some $\beta \in (1,2)$ if and only if $\sigma^n(d) \preccurlyeq d$ for every $n \in \mathbb{N}$. Observe that the lexicographic condition satisfied by Parry sequences defined on Section 2.2 is the same as the sequences which are β-expansions of 1. We shall call the subshift (X_β, σ_β) *the usual* β*-shift.*

The properties of the usual β-shift have been extensively studied. In particular, for every $\beta \in (1,\infty)$, the usual β-shift is a topologically mixing subshift and $h_{top}(\sigma_\beta) = \log(\beta)$ [Sid03b]. Also, (X_β, σ_β) is a subshift of finite type if and only if the greedy expansion of 1 is periodic or finite and (X_β, σ_β) is sofic if and only if the greedy expansion of 1 is preperiodic - see [Sid03b, Theorem 2.2]. Moreover, (X_β, σ_β) has the specification property if and only if the greedy expansion of 1 does not contain blocks of consecutive 0's of arbitrary length [BM86]. In addition, it is shown in [BM86] that the set

$$\{\beta \in (1,2) \mid (X_\beta, \sigma_\beta) \text{ has specification}\}$$

has Lebesgue measure zero. Finally, it is known that the usual β-shift is intrinsically ergodic for every β [Hof78].

In [BKT11, Theorem 3.5], Bundfuss et. al. showed that (X_β, σ_β) is topologically conjugated to $(X_{(a,1)}, f_{(a,1)})$ with $a \in (\frac{1}{2}, 1)$. Also, Nilsson in [Nil09] proved the same result for open dynamical systems of the form $(X_{(0,b)}, f_{(0,b)})$ with $b \in (0, \frac{1}{2})$. Moreover, in [Nil09] it is shown that the entropy function is a devil's staircase and a characterisation of the end points of the entropy plateaus is provided.

We devote the rest of the section to showing that the same results hold for the attractors corresponding to holes $(a, \frac{1}{2})$ with $a \in (0, \frac{1}{2})$. Henceforth, $a \in (0, \frac{1}{2})$ and the binary expansion of $\frac{1}{2}$ will be considered to be 10^∞. It is clear that $\Lambda_{(a,\frac{1}{2})} = X_{(a,\frac{1}{2})} \cap [0, 2a]$. Let $\boldsymbol{\alpha} = \pi^{-1}(2a)$. Then $\Sigma_{(\boldsymbol{\alpha},0^\infty)} = \{x \in \Sigma_2 \mid 0^\infty \preccurlyeq \sigma^n(x) \preccurlyeq \boldsymbol{\alpha}\}$.

Note that if $\boldsymbol{\alpha} \in P$ then $\Sigma_{(\boldsymbol{\alpha},0^\infty)}$ coincides with the greedy $\beta(a)$-shift, where $\beta(a)$ is determined by the unique positive solution to the equation

$$1 = \sum_{n=1}^{\infty} \frac{a_i}{\beta(a)^i},$$

up to a countable set of sequences given by

$$\left(\bigcup_{n=0}^{\infty} \sigma^{-n}(0) \right) \cap \Sigma_{(a,\frac{1}{2})}.$$

In order to describe a way to associate a greedy β-shift to each attractor $\Lambda_{(a,\frac{1}{2})}$ when $a \in (0, \frac{1}{2})$, it is necessary to deal with parameters such that $\boldsymbol{\alpha} \in N$. Nonetheless, if $\boldsymbol{\alpha} \in N$ we associate a greedy β-shift by applying ς to $\boldsymbol{\alpha}$. This is summarised in the following lemma.

Lemma 6.1.1. *For every $\boldsymbol{\alpha} \in \Sigma_2^1$, $(\Lambda_{(a,\frac{1}{2})}, \sigma_{(a,\frac{1}{2})})$ and $(X_{\beta(\varsigma(\boldsymbol{\alpha}))}, \sigma_{\beta(\varsigma(\boldsymbol{\alpha}))})$ are topologically conjugated.*

Proof. Note that if $\boldsymbol{\alpha} \in P$ then $\pi^{-1}(\Lambda_{(a,\frac{1}{2})}) = X_{\beta(\varsigma(\boldsymbol{\alpha}))}$. Therefore the result holds.

Consider $a \in (0, \frac{1}{2})$ such that $\boldsymbol{\alpha} \in N$. Observe that $\varsigma(\boldsymbol{\alpha}) \prec \pi^{-1}(2a)$, then $\Sigma_{\beta(\varsigma(\boldsymbol{\alpha}))} \subset \pi^{-1}(\Lambda_{(a,\frac{1}{2})})$. Assume that there exists $\boldsymbol{x} \in \pi^{-1}(\Lambda_{(a,\frac{1}{2})} \setminus X_{\beta(\varsigma(\boldsymbol{\alpha}))})$. This implies that $x_i = \varsigma(\boldsymbol{\alpha})_i$ at least for every $1 \leq i \leq n_{\boldsymbol{\alpha}}$. Let $k = \min\{j \geq n_{\boldsymbol{\alpha}} \mid \varsigma(\boldsymbol{\alpha})_j < x_j\}$. Observe that $x_k \leq a_j$. Since $\boldsymbol{x} \in \pi^{-1}(X_{(a,\frac{1}{2})})$ then $\sigma^n(\boldsymbol{x}) \prec \boldsymbol{\alpha}$ for every $n \geq 0$. Consider $\sigma^{n_{\boldsymbol{\alpha}}}(\boldsymbol{x})$. Note that $\sigma^{n_{\boldsymbol{\alpha}}}(\boldsymbol{x})_{j-n_{\boldsymbol{\alpha}}} \succ \varsigma(\boldsymbol{\alpha})_{j-n_{\boldsymbol{\alpha}}}$. This implies that $\sigma^{n_{\boldsymbol{\alpha}}}(\boldsymbol{x})_{j-n_{\boldsymbol{\alpha}}} > \boldsymbol{\alpha}_{j-n_a}$, that is $\boldsymbol{x} \notin \pi^{-1}(\Lambda_{(a,\frac{1}{2})})$, which is a contradiction. Therefore $\pi^{-1}(\Lambda_{(a,\frac{1}{2})}) = X_{\beta(\varsigma(\boldsymbol{\alpha}))}$. Then $\pi^{-1} \circ \pi_{\beta(\varsigma(\boldsymbol{\alpha}))}$ is a topological conjugation between $(\Lambda_{(a,\frac{1}{2})}, f_{(a,\frac{1}{2})})$ and $(X_{\beta(\varsigma(\boldsymbol{\alpha}))}, \sigma_{\beta(\varsigma(\boldsymbol{\alpha}))})$. □

As an easy consequence of Lemma 6.1.1 we obtain the following result, which was stated in a different way in [Nil09, Urb86]. The proof is omitted.

Theorem 6.1.2. *The functions* $h : [0, \frac{1}{2}] \to [0, 1]$ *and* $\Delta : [0, \frac{1}{2}] \to [0, 1]$ *given by* $h(a) = h_{top}(f_{(a,\frac{1}{2})})$ *and* $\Delta(a) = \dim_H \Lambda_{(a,\frac{1}{2})}$ *respectively are devil's staircases.*

As a consequence of an easy modification of Proposition 3.2.1 Theorem 6.1.2 also holds for attractors corresponding to holes of the form $(b, 1)$ with $b \in (\frac{1}{2}, 1)$.

6.2 Unique β-expansions

As we showed in Section 6.1, every $x \in [0, 1]$ has at least one β-expansion. Also, as we mentioned in Section 2.2 if $\beta = 2$ then there exists a countable set of points $D \subset [0, 1]$ with two different expansions. A significant difference between the case when $\beta = 2$ and $\beta \in (1, 2)$ is for almost every $x \in \left[0, \frac{1}{\beta-1}\right]$, x has uncountably many β-expansions [EJK90, Sid03a].

Let $\beta \in (1, 2)$. Showing that for every $x \in [0, \frac{1}{\beta-1}]$ there is at least one expansion for x is relatively easy. In fact, a β-expansion for $x \in [0, \frac{1}{\beta-1}]$ can be obtained in a similar way to a binary expansion for $x \in [0, 1]$. Let $T_\beta : [0, \frac{1}{\beta-1}] \to [0, \frac{1}{\beta-1}]$ be the multivalued map given by:

$$T_\beta = \begin{cases} \beta x, & \text{if } x \in [0, \frac{1}{\beta}]; \\ \beta x, \text{ or } \beta x - 1 & \text{if } x \in (\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}); \\ \beta x - 1, & \text{if } x \in [\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}]. \end{cases}$$

Note that if $x \in [\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}]$ it is possible to make a choice between two different branches of T_β. This gives a different choice on the symbols of a β-expansion for x. We shall call the interval $(\frac{1}{\beta}, \frac{1}{\beta(\beta-1)})$ *the switching region of* T_β. Note that 0 and $\frac{1}{\beta-1}$ have unique expansions given by 0^∞ and 1^∞ respectively since they are fixed points of T_β. Then a natural question to ask is whenever a point $x \in [0, \frac{1}{\beta-1}]$ has a *unique β-expansion*. The properties of the mentioned set are well studied, see, *e.g.*, [Bak14, Par64b, Par60, GS01, Sid03a]. Erdös et al. [EJK90] showed that every $x \in (0, \frac{1}{\beta-1})$ has uncountably many β-expansions provided that $\beta < \varphi$ where $\varphi = \frac{1+\sqrt{5}}{2}$.

In contrast to this, Glendinning and Sidorov showed in [GS01] that the set

$$\mathcal{U}_\beta = \left\{ x \in \left(0, \frac{1}{\beta - 1}\right) \mid x \text{ has a unique } \beta\text{-expansion} \right\} \neq \emptyset$$

for $\beta \in (\varphi, 2)$.

Recall that a β-expansion of x, $b = \{b_n\}_{n=1}^{\infty}$ is said to be *lazy* if $\bar{b}$ is greedy. For $\beta \in (\varphi, 2)$ fixed, let

$$\Sigma_\beta = \left\{ b \in \Sigma_2 \mid b \text{ is the unique expansion for } x \in \left(0, \frac{1}{\beta - 1}\right) \right\}.$$

Notice that not every sequence $b \in \Sigma_2$ is the unique β expansion of x. In fact, in [EJK90, Lemma 2] the lexicographic conditions of a sequence $b \in \Sigma_\beta$ are given; that is $b \in \Sigma_\beta$ if and only if b is greedy and lazy. Moreover, the unique β-expansion of 1 is characterised as follows.

Theorem 6.2.1. [EJK90, Theorem 1] *Let $\beta \in (1, 2)$ and consider a β-expansion of 1, 1_β:*

$$1 = \sum_{n=1}^{\infty} b_n \beta^{-n}.$$

Then:

i) 1_β is the greedy expansion of 1 if and only if $\sigma^n(1_\beta) \prec 1_\beta$ for every $n \in \mathbb{N}$;

ii) 1_β is the unique expansion of 1 if and only if i) is satisfied and

$$\overline{\sigma^n(1_\beta)} \prec 1_\beta$$

for every $k \in \mathbb{N}$.

Let $\pi_\beta : \Sigma_\beta \to \mathcal{U}_\beta$ be the projection map given by

$$\pi_\beta(b) = \sum_{n=1}^{\infty} b_n \beta^{-n}.$$

Let $b \in \Sigma_\beta$. Notice that if $b_1 = 0$ then $\pi_\beta \in [0, \frac{1}{\beta}]$ and if $b_1 = 1$ then $\pi_\beta \in [\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}]$. This implies that $U_\beta \cap (\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}) = \emptyset$ [Sid03b]. Also, for every $x \leq 1$ the greedy expansion of x, satisfies that $\sigma^k((b)) \prec d$ for every $k \geq 0$, where d is the greedy expansion of 1 [Par60]. Then if $b \in \Sigma_\beta$ then $\overline{\sigma^k(b)} = \sigma^k(\bar{b})_{n=1}^\infty \prec (d_n)_{n=1}^\infty$. This implies that Σ_β is σ-invariant subset of Σ_2 and

$$\Sigma_\beta = \{x \in \Sigma_2 \mid \sigma^n(x) \prec (d_i)_{i=1}^\infty \text{ and } \overline{\sigma^n(x)} \prec (d_i)_{i=1}^\infty \text{ for every } n \in \mathbb{N}\}.$$

That is, $(\Sigma_\beta, \sigma_\beta)$ is a symmetric lexicographic subshift. Moreover, given the unique β-expansion of 1, $(d_n)_{n=1}^\infty$ where $\varphi < \beta < 2$, there is $a \in (\frac{1}{3}, \frac{1}{2})$ such that $\pi^{-1}(2a) = (d_n)_{n=1}^\infty$.

Thus, we conclude that the set of unique β-expansions for $\beta \in (\varphi, 2)$ has the same dynamics as an attractor Λ_a where $a \in (\frac{1}{3}, \frac{1}{2})$.

6.3 Intermediate β-expansions

A particular set of Lorenz transformations were introduced by Parry in [Par66]. Consider $\beta \in (1, 2)$ and $\alpha \in (0, 2 - \beta)$. Define

$$g_{\beta,\alpha}(x) = \begin{cases} \beta x + \alpha, & \text{if } x \in [0, \frac{1-\alpha}{\beta}]; \\ \beta x + \alpha - 1 & \text{if } x \in [\frac{1-\alpha}{\beta}, 1]. \end{cases}$$

Note that $g_{\beta,\alpha}$ is a linear and expanding Lorenz transformation often called *linear* mod 1 *transformations*. It is well known that $h_{top}(g_{\beta,\alpha}) = \log(\beta)$ [FL96, p. 452]. Also, Glendinning in [Gle90, Theorem 1] gave the precise conditions under which an expanding Lorenz map is topologically conjugated to $g_{\beta,\alpha}$ for $\beta \in (1, 2)$ and $\alpha \in (0, 2 - \beta)$. Let $(a, b) \in \mathcal{LW}$ such that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\sigma(k_g^+(\frac{1-\alpha}{\beta})), \sigma(k_g^-(\frac{1-\alpha}{\beta})))$. Then $\pi_{\beta,\alpha} : \Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \to [0, 1]$ given by $\pi_{\beta,\alpha}(\boldsymbol{x}) = \frac{\alpha}{\beta-1} + \sum_{n=1}^{\infty} \frac{x_n}{\beta^n}$ is a semi-conjugacy between $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ and $([0, 1], g_{\beta,\alpha})$.

In [DK02] the authors suggested the notion of *intermediate β-expansions* or (β, α)-*expansions*. Let $\alpha \in (0, \frac{2-\beta}{\beta-1})$, and consider the map:

$$T_{\beta,\alpha} = \begin{cases} \beta x, & \text{if } x \in [\alpha, \frac{\alpha+1}{\beta}); \\ \beta x - 1 & \text{if } x \in (\frac{\alpha+1}{\beta}, 1 + \alpha]. \end{cases}$$

Note that we defined the map $T_{\beta,\alpha}$ in $[\alpha, 1 + \alpha]$ since this set is an attractor for the extended map defined in $[0, \frac{1}{\beta-1}]$ [DK02]. Considering the homeomorphism $\psi : [0, 1] \to$

$[\alpha, 1+\alpha]$ given by $\psi(x) = x + \alpha$ it is easy to show -see [DK02, Sid03b] that $T_{\beta,\alpha}$ is topologically conjugated to the map g_{β,α^*} given by

$$g_{\beta,\alpha^*} = \begin{cases} \beta x + \alpha^*, & \text{if } x \in [0, \frac{1-\alpha^*}{\beta}); \\ \beta x + \alpha^* - 1 & \text{if } x \in (\frac{1-\alpha^*}{\beta}, 1]. \end{cases}$$

Then as a consequence of Proposition 5.1.1 we obtain the following result.

Proposition 6.3.1. *For every $\beta \in (1,2)$ and $\alpha \in (0, 2-\beta)$ the set of (β,α)-expansions is a factor of an attractor $\Lambda_{(a,b)}$ with $(a,b) \in D_1$.*

Chapter 7

Final Remarks

In Section 7.1 we describe two criteria for a subshift to be intrinsically ergodic. Firstly we describe an adapted version of the criterion introduced by Gurevič in [Gur72]. We will explain the obstructions to use this criterion in our study. In particular, we will concentrate on the obstructions for the family of symmetric subshifts studied in Chapter 4. We will also discuss the criterion introduced by Climenhaga and Thompson in [CT12] and technical obstructions for its applications.

In Section 7.2 we will pose some questions regarding open dynamical systems. To the best of our knowledge, the posed questions are still open.

7.1 Intrinsic Ergodicity criteria

Gurevič criterion

Gurevič in [Gur72, p 571] stated a sufficient condition for the intrinsic ergodicity for subshifts that can be approximated from above. These notions will be adapted to our context.

Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW}$ such that $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is approximated from above by subshifts of finite type $(\Sigma_{(\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n)}, \sigma_{(\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n)})$ (Definition 4.2.9). Let $\rho_n = h_{top}(\sigma_{(\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n)}) - h_{top}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$. Observe that $\rho_n \to 0$ when $n \to \infty$ since the entropy function H is continuous [Urb87, Theorem 4]. Let $s_{(\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n)}$ be the specification number of $(\Sigma_{(\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n)}, \sigma_{(\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n)})$.

Put

$$R = -\limsup_{n\to\infty} \frac{\log(\rho_n)}{n}$$

and

$$S = \limsup_{n\to\infty} \frac{s_{(\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n)}}{n}.$$

Theorem 7.1.1. [Gur72, Theorem 1.1] *Let $((\boldsymbol{\alpha},\boldsymbol{\beta})) \in \mathcal{LW}$ such that $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is transitive. If*

$$\frac{R}{(16+\gamma)h_{top}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})} \geq S$$

for some $\gamma > 0$ then $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is intrinsically ergodic.

Note that, to use this criterion, it is needed to show that ρ_n decreases exponentially fast. Using the formula introduced by Glendinning and Hall in [GH96] and Barnsley, Steiner and Vince in [BSV12] (Section 5.3) it is possible to calculate ρ_n explicitly. For a symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ where $\boldsymbol{\alpha} \in (1^k, 1^{k+1})_{\prec}$ for a fixed $k \geq 2$ it is possible to give a rough bound by substituting $h_{top}(\sigma_{\boldsymbol{\alpha}})$ by $h_{top}(\sigma_{1^{k+1}}) = \log \varphi_k$ where φ_k is the k-th multinacci number [1]. Then it suffices to show that

$$\rho_n \leq (\varphi_k)^{-(16+\gamma)\ell(\boldsymbol{\alpha}_n^+)} \leq (\varphi_k)^{-(16+\gamma)s_{\boldsymbol{\alpha}_n}},$$

for infinitely many $n \in \mathbb{N}$.

Climenhaga-Thompson criterion

Recall that a subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is said to be a *coded system* if

$$\Sigma_{\mathcal{F}} = \overline{\bigcup_{n=1}^{\infty} \Sigma^n},$$

where each $(\Sigma^n, \sigma_{\Sigma^n})$ is a transitive subshift of finite type. As it was mentioned in Chapter 2, a subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is coded if there exists a countable collection of finite words called *generators*, such that X is the closure of the set of sequences obtained by freely concatenating generators [CT12, LM95, Bla89]. All coded systems are transitive [Bla89].

Climenhaga and Thompson introduced a weaker form of the specification property (Definition 2.1.3) as follows. Let $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ be a transitive subshift and $\mathcal{L}(\Sigma_{\mathcal{F}})$ its

[1] φ_k is the unique root of $x^k = \sum_{n=0}^{k-1} x^n$ which lies in $(1, 2)$.

language. Let $\mathcal{G}(\Sigma_{\mathcal{F}})$ be a subset of $\mathcal{L}(\Sigma_{\mathcal{F}})$ and let $m \in \mathbb{N}$ be fixed. We say that $\mathcal{G}(\Sigma_{\mathcal{F}})$ has *the S-specification property with gap size m* if for every $n \in \mathbb{N}$ and $u_1, \dots, u_n \in \mathcal{G}(\Sigma_{\mathcal{F}})$ there exist $v_1, \dots, v_{n-1} \in \mathcal{L}(\Sigma_{\mathcal{F}})$ such that $\ell(v_i) = m$ for every $i \in \{1, \dots, n-1\}$ and the word $w = u_1 v_1 u_2 \dots v_{n-1} u_n \in \mathcal{L}(\Sigma_{\mathcal{F}})$. Besides, we say that $\mathcal{G}(\Sigma_{\mathcal{F}})$ satisfies *the W-specification property with gap size m* if for every $n \in \mathbb{N}$ and $u_1, \dots, u_n \in \mathcal{G}(\Sigma_{\mathcal{F}})$ there exist $v_1, \dots, v_{n-1} \in \mathcal{L}(\Sigma_{\mathcal{F}})$ such that $\ell(v_i) \leq m$ and $w = u_1 v_1 u_2 \dots v_{n-1} u_n \in \mathcal{L}(\Sigma_{\mathcal{F}})$. Note that if $\mathcal{G}(\Sigma_F) = \mathcal{L}(\Sigma_F)$ then $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ has the specification property if $\mathcal{G}(\Sigma_F)$ has the S-specification property and $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ has the almost specification property if $\mathcal{G}(\Sigma_F)$ has the W-specification property.

Given a subshift $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ we say that $\mathcal{L}(\Sigma_{\mathcal{F}})$ admits a *prefix-core-suffix decomposition* if there exist collections of words $\mathcal{P}(\Sigma_{\mathcal{F}}), \mathcal{G}(\Sigma_{\mathcal{F}}), \mathcal{S}(\Sigma_{\mathcal{F}}) \subset \mathcal{L}(\Sigma_F)$ such that every word $\omega \in \mathcal{L}(\Sigma_F)$, $\omega = uvw$ with $u \in \mathcal{P}, v \in \mathcal{G}, w \in \mathcal{S}$. The collections $\mathcal{P}(\Sigma_{\mathcal{F}}), \mathcal{G}(\Sigma_{\mathcal{F}})$, and $\mathcal{S}(\Sigma_{\mathcal{F}})$ will be called *prefix set, core, and suffix set* respectively. Note that a prefix-core-suffix decomposition does not have to be unique.

If $\mathcal{L}(\Sigma_F)$ admits a prefix-core-suffix decomposition, we define

$$\mathcal{G}_n(\Sigma_F) = \{uvw \mid u \in \mathcal{P}(\Sigma_{\mathcal{F}}), v \in \mathcal{G}(\Sigma_{\mathcal{F}}), w \in \mathcal{S}(\Sigma_{\mathcal{F}}), \ell(u) \leq n, \ell(w) \leq n\}.$$

Note that $\bigcup_{n=1}^{\infty} \mathcal{G}_n(\Sigma_{\mathcal{F}}) = \mathcal{L}(\Sigma_{\mathcal{F}})$.

Theorem 7.1.2. [CT12, Theorem C] *Let $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ be a subshift and let $\mathcal{L}(\Sigma_F)$ be its language. If $\mathcal{L}(\Sigma_{\mathcal{F}})$ admits a prefix-core-suffix decomposition satisfying:*

1. *$\mathcal{G}(\Sigma_{\mathcal{F}})$ has S-specification;*
2. *$h_{top}(\mathcal{P}(\Sigma_{\mathcal{F}}) \cup \mathcal{S}(\Sigma_{\mathcal{F}})) < h_{top}(\sigma_{\mathcal{F}})$;*
3. *For every $n \in \mathbb{N}$, there exists m such that for every $\omega \in \mathcal{G}_n(\Sigma_{\mathcal{F}})$, there exist words u, v with $\ell(u) \leq m, \ell(v) \leq m$ such that $u\omega v \in \mathcal{G}(\Sigma_{\mathcal{F}})$,*

Then $(\Sigma_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is intrinsically ergodic.

Intuitively, Theorem 7.1.2 suggests that a lexicographic subshift is intrinsically ergodic if there is a σ-invariant subset X of $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}))$ such that X has specification (or almost specification) and $h_{top}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} |_{\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \setminus X})$ is bounded away from $h_{top}(\sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$.

It was showed in [CT12, Section 3.1] that greedy β-shifts satisfy the conditions of Theorem 7.1.2. The strategy followed by Climenhaga and Thompson was to use the graph representation of a β-shift. Then it is natural to conjecture that a transitive lexicographic subshift $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ without the specification property will satisfy Theorem 7.1.2. However, it is not clear how to construct the graph representation for a general lexicographic subshift and using the graph presentation of $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ in order to find the prefix-core-suffix decomposition of $\mathcal{L}(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ and prove the properties of Theorem 7.1.2 is a challenging problem.

it is worth pointing out that during the completition of this work, Kwietniak, Oprocha, Rams in [KOR14] and Pavlov in [Pav14] studied weaker forms of specification in order to determine when a subshift is intrinsically ergodic.

7.2 Open problems

In [HS90, Figure 3], an approximate graphic illustration of $\mathcal{LW}$ is presented. Moreover, it is suggested that $\mathcal{LW}$ is a self similar set. So, it is natural to ask the following question.

Question 7.2.1. *What is* $\dim_H \mathcal{LW}$*?*

It is worth pointing out that the structure of lexicographic subshifts has been studied in detail. In fact, as it was mentioned in Section 3.4, it is known that $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ is a subshift of finite type if and only if $\boldsymbol{\alpha}$ and be $\boldsymbol{\beta}$ are both finite or periodic. Also, in [AS07, Theorem 3.5] it was shown that a symmetric subshift $(\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}})$ is sofic if and only if $\boldsymbol{\alpha}$ is a pre-periodic sequence. However, a full classification of the structure of these systems is unknown. For instance, it is unknown when an asymmetrical lexicographic subshift is sofic.

Conjecture 7.2.2. *Let* $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{LW}$*. Then* $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ *is sofic if and only if* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *are both preperiodic.*

Question 7.2.3. *Let* $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in P \times \bar{P}$ *be a left (right) extremal pair. When is* $(\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})})$ *sofic?*

There still are some open questions of symmetric subshifts as well. Recall that $\mathcal{E}$ is the exceptional set of the entropy function for symmetric subshifts. Recall that

$$S(\mathcal{E}) = \{\boldsymbol{\alpha} \in \mathcal{E} \mid (\Sigma_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}}) \text{ has the specification property}\},$$

and $NS(\mathcal{E}) = \mathcal{E} \backslash S(\mathcal{E})$. Then Theorem 4.3.16 allows us to state the following conjecture.

Conjecture 7.2.4.

$$\dim_H \mathcal{E} = \dim_H S(\mathcal{E}) = \dim_H NS(\mathcal{E}) = 1.$$

It is worth pointing out that a similar result is proved in [Urb87, Theorem 5]. Moreover, we can ask a more general question than Conjecture 7.2.4.

Question 7.2.5. *Let*

$$\mathcal{E}_{\mathcal{LW}} = \{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{LW} \mid (\Sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \sigma_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \text{ has the specification property}\}.$$

Is it true that $\dim_H \mathcal{E}_{\mathcal{LW}} = \dim_H \mathcal{LW}$*?*

In the present thesis, we have studied centred holes only. We introduce now the following notion: A hole $(a, b) \subset S^1$ is *one-sided hole* if $[a, b] \subsetneq (0, \frac{1}{2})$ or $[a, b] \subsetneq (\frac{1}{2}, 1)$. From Lemma 3.1.1 it is clear that $h_{top}(f_{(a,b)}) > 0$ if (a, b) is a one sided hole. Then it is possible to ask the following questions:

Question 7.2.6. *Let (a, b) be a one-sided hole. Then*

1. *Is $(X_{(a,b)}, f_{(a,b)})$ topologically transitive? topologically mixing?*
2. *Does $(X_{(a,b)}, f_{(a,b)})$ have the specification property?*
3. *Is $(X_{(a,b)}, f_{(a,b)})$ intrinsically ergodic?*

Observe that it is also natural to ask the same questions for holes $U = \bigcup_{j=0}^{k} (a_j, b_j)$. Furthermore, it would be interesting to describe the properties of the binary expansions of a and b such that $(X_{(a,b)}, f_{(a,b)})$ is conjugate to a subshift of finite type when a and b never fall into the hole, that is $f^n(a) \notin (a, b)$ and $f^m(b) \notin (a, b)$ for every $n, m \in \mathbb{N}$.

In [BKT11, Theorem 4.3], the authors proved a result similar to Theorem 3.4.2 for diffeomorphisms of two-dimensional manifolds. However, it is unknown whether a similar result for diffeomorphisms in higher dimensional manifolds holds. Moreover, there are no similar results even for the doubling map when a hole U has more than one connected component.

Observe that the questions posed in Question 7.2.6 can be asked for linear expanding maps of the circle of the form $nx \mod 1$. A good starting point to deal with these questions is to consider $nx \mod 1$ with $n-1$ holes containing $\frac{k}{n}$ with $k \in \{1, \ldots n-1\}$. Recently, in [Bak14, Theorem 4.4] Baker has proved a generalisation of Theorem 4.1.1 for $\beta > 2$, placing particular emphasis on the differences when the number of symbols of the symmetric subshift associated to β is odd or even. Then it is natural to ask if the results obtained so far can be extended to lexicographic subshifts for larger alphabets. Another interesting question is if for $\beta > 2$ the set of unique β-expansions corresponds to one of the afore mentioned open dynamical systems. The Hausdorff dimension of such sets was studied recently by Kong and Li [KL14].

Finally, it is interesting to study the topological properties of open dynamical systems in higher dimensions. Possibly, a good starting point would be to study the properties of open dynamical systems corresponding to the Fibonacci automorphism in the two dimensional torus $\mathbb{T}^2$ and the exclusion sets corresponding to the baker's map.

Bibliography

[AB14] R. Alcaraz Barrera. Topological and ergodic properties of symmetric subshifts. *Discrete Contin. Dyn. Syst.*, 34(11):4459–4486, 2014.

[ACS09] J. Allouche, M. Clarke, and N. Sidorov. Periodic unique beta-expansions: the Sharkovskiĭ ordering. *Ergodic Theory Dynam. Systems*, 29(4):1055–1074, 2009.

[AG09] J.P. Allouche and A. Glen. Distribution modulo 1 and the lexicographic world. *Ann. Sci. Math. Québec*, 33(2):125–143, 2009.

[AS07] S. Akiyama and K. Scheicher. Symmetric shift radix systems and finite expansions. *Math. Pannon.*, 18(1):101–124, 2007.

[Bak14] S. Baker. Generalised golden ratios over integer alphabets. *Integers*, 14:Paper No. A15, 28, 2014.

[BDJ14] S. Baker, K. Dajani, and K. Jiang. On univoque points for self-similar sets. *ArXiv e-prints*, June 2014.

[BDM10] H. Bruin, M. Demers, and I. Melbourne. Existence and convergence properties of physical measures for certain dynamical systems with holes. *Ergodic Theory Dynam. Systems*, 30(3):687–728, 2010.

[BHL03] J.C. Ban, C.H. Hsu, and S.S. Lin. Devil's staircase of gap maps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 13(1):115–122, 2003.

[BKT11] S. Bundfuss, T. Krüger, and S. Troubetzkoy. Topological and symbolic dynamics for hyperbolic systems with holes. *Ergodic Theory Dynam. Systems*, 31(5):1305–1323, 2011.

[Bla89] F. Blanchard. β-expansions and symbolic dynamics. *Theoret. Comput. Sci.*, 65(2):131–141, 1989.

[BM86] A. Bertrand-Mathis. Développement en base θ; répartition modulo un de la suite $(x\theta^n)_{n\geq 0}$; langages codés et θ-shift. *Bull. Soc. Math. France*, 114(3):271–323, 1986.

[Bow71] R. Bowen. Periodic points and measures for Axiom A diffeomorphisms. *Trans. Amer. Math. Soc.*, 154:377–397, 1971.

[Bow08] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, revised edition, 2008. With a preface by David Ruelle, Edited by Jean-René Chazottes.

[Bow75] R. Bowen. Some systems with unique equilibrium states. *Math. Systems Theory*, 8(3):193–202, 1974/75.

[Boy00] M. Boyle. Algebraic aspects of symbolic dynamics. In *Topics in symbolic dynamics and applications (Temuco, 1997)*, volume 279 of *London Math. Soc. Lecture Note Ser.*, pages 57–88. Cambridge Univ. Press, Cambridge, 2000.

[BS02] M. Brin and G. Stuck. *Introduction to dynamical systems.* Cambridge University Press, Cambridge, 2002.

[BSV12] M. Barnsley, W. Steiner, and A. Vince. A Combinatorial Characterization of the Critical Itineraries of a Uniform Dynamical System. *ArXiv e-prints*, http://arxiv.org/abs/1205.5902, May 2012.

[BY11] L. A. Bunimovich and A. Yurchenko. Where to place a hole to achieve a maximal escape rate. *Israel J. Math.*, 182:229–252, 2011.

[CE80] P. Collet and J. P. Eckmann. *Iterated maps on the interval as dynamical systems*, volume 1 of *Progress in Physics*. Birkhäuser Boston, Mass., 1980.

[CM97a] N. Chernov and R. Markarian. Anosov maps with rectangular holes. Nonergodic cases. *Bol. Soc. Brasil. Mat. (N.S.)*, 28(2):315–342, 1997.

[CM97b] N. Chernov and R. Markarian. Ergodic properties of Anosov maps with rectangular holes. *Bol. Soc. Brasil. Mat. (N.S.)*, 28(2):271–314, 1997.

[CMT00] N. Chernov, R. Markarian, and S. Troubetzkoy. Invariant measures for Anosov maps with small holes. *Ergodic Theory Dynam. Systems*, 20(4):1007–1044, 2000.

[CT12] V. Climenhaga and D. J. Thompson. Intrinsic ergodicity beyond specification: β-shifts, S-gap shifts, and their factors. *Israel J. Math.*, 192:785–817, 2012.

[Det13] C Dettmann. Open circle maps: small hole asymptotics. *Nonlinearity*, 26(1):307–317, 2013.

[DK02] K. Dajani and C. Kraaikamp. From greedy to lazy expansions and their driving dynamics. *Expo. Math.*, 20(4):315–327, 2002.

[DW12] M. F. Demers and P. Wright. Behaviour of the escape rate function in hyperbolic dynamical systems. *Nonlinearity*, 25(7):2133–2150, 2012.

[DWY10] M. Demers, P. Wright, and L.S. Young. Escape rates and physically relevant measures for billiards with small holes. *Comm. Math. Phys.*, 294(2):353–388, 2010.

[DWY12] M. F. Demers, P. Wright, and L.S. Young. Entropy, Lyapunov exponents and escape rates in open systems. *Ergodic Theory Dynam. Systems*, 32(4):1270–1301, 2012.

[EJK90] P. Erdös, I. Joó, and V. Komornik. Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems. *Bull. Soc. Math. France*, 118(3):377–390, 1990.

[FF01] D. Fiebig and U. Fiebig. Invariants for subshifts via nested sequences of shifts of finite type. *Ergodic Theory Dynam. Systems*, 21(6):1731–1758, 2001.

[FL96] L. Flatto and J. C. Lagarias. The lap-counting function for linear mod one transformations i: explicit formulas and renormalizability. *Ergodic Theory and Dynamical Systems*, 16:451–491, 6 1996.

[Gan01] S. Gan. Sturmian sequences and the lexicographic world. *Proc. Amer. Math. Soc.*, 129(5):1445–1451 (electronic), 2001.

[GH96] P. Glendinning and T Hall. Zeros of the kneading invariant and topological entropy for Lorenz maps. *Nonlinearity*, 9(4):999–1014, 1996.

[Gle90] P. Glendinning. Topological conjugation of Lorenz maps by β-transformations. *Math. Proc. Cambridge Philos. Soc.*, 107(2):401–413, 1990.

[GS93] P. Glendinning and C. Sparrow. Prime and renormalisable kneading invariants and the dynamics of expanding Lorenz maps. *Phys. D*, 62(1-4):22–50, 1993. Homoclinic chaos (Brussels, 1991).

[GS01] P. Glendinning and N. Sidorov. Unique representations of real numbers in non-integer bases. *Math. Res. Lett.*, 8(4):535–543, 2001.

[GS14] P. Glendinning and N. Sidorov. The doubling map with asymmetrical holes. *Ergodic Theory and Dynamical Systems*, FirstView:1–21, 3 2014.

[Gur72] B. M. Gurevič. Uniqueness of the measure with maximal entropy for symbolic dynamical systems that are close to Markov ones. *Dokl. Akad. Nauk SSSR*, 204:15–17, 1972.

[Gur80] B. M. Gurevich. Stationary random sequences of maximal entropy. In *Multicomponent random systems*, volume 6 of *Adv. Probab. Related Topics*, pages 327–380. Dekker, New York, 1980.

[GW79] J. Guckenheimer and R. F. Williams. Structural stability of Lorenz attractors. *Inst. Hautes Études Sci. Publ. Math.*, (50):59–72, 1979.

[Hay13] N. T. A. Haydn. Phase transitions in one-dimensional subshifts. *Discrete Contin. Dyn. Syst.*, 33(5):1965–1973, 2013.

[Hof78] F. Hofbauer. β-shifts have unique maximal measure. *Monatsh. Math.*, 85(3):189–198, 1978.

[Hof79] F. Hofbauer. On intrinsic ergodicity of piecewise monotonic transformations with positive entropy. *Israel J. Math.*, 34(3):213–237 (1980), 1979.

[HS90] J. H. Hubbard and C. T. Sparrow. The classification of topologically expansive Lorenz maps. *Comm. Pure Appl. Math.*, 43(4):431–443, 1990.

[HS14] K. Hare and N. Sidorov. On cycles for the doubling map which are disjoint from an interval. *Monatshefte fr Mathematik*, 175(3):347–365, 2014.

[Jun11] U Jung. On the existence of open and bi-continuing codes. *Trans. Amer. Math. Soc.*, 363(3):1399–1417, 2011.

[KL09] G. Keller and C. Liverani. Rare events, escape rates and quasistationarity: some exact formulae. *J. Stat. Phys.*, 135(3):519–534, 2009.

[KL14] D. Kong and W. Li. On the Hausdorff dimension of unique beta expansions. *ArXiv e-prints*, http://arxiv.org/abs/1401.6473,January 2014.

[KOR14] D. Kwietniak, P. Oprocha, and M. Rams. On entropy of dynamical systems with almost specification. *ArXiv e-prints*, http://arxiv.org/pdf/1411.1989, November 2014.

[LGDL14] A. L. P. Livorati, O. Georgiou, C. P. Dettmann, and E. D. Leonel. Escape through a time-dependent hole in the doubling map. *Phys. Rev. E*, 89:052913, May 2014.

[LM95] D. Lind and B. Marcus. *An introduction to symbolic dynamics and coding.* Cambridge University Press, Cambridge, 1995.

[LM96] A. Lopes and R. Markarian. Open billiards: invariant and conditionally invariant probabilities on Cantor sets. *SIAM J. Appl. Math.*, 56(2):651–680, 1996.

[LM06] R. Labarca and C. Moreira. Essential dynamics for Lorenz maps on the real line and the lexicographical world. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(5):683–694, 2006.

[Lot05] M. Lothaire. *Applied combinatorics on words*, volume 105 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005.

[LSS14] B. Li, T. Sahlsten, and T. Samuel. Intermediate β-shifts of finite type. *ArXiv e-prints*, http://arxiv.org/abs/1401.7027 January 2014.

[Mañ83] R. Mañé. *Introdução à teoria ergódica*, volume 14 of *Projeto Euclides [Euclid Project]*. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1983.

[Mis04] M. Misiurewicz. Entropy of maps with horizontal gaps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 14(4):1489–1492, 2004.

[Nil09] J. Nilsson. On numbers badly approximable by dyadic rationals. *Israel J. Math.*, 171:93–110, 2009.

[Nil10] J. Nilsson. The fine structure of Dyadically badly approximable numbers. *ArXiv e-prints*, http://arxiv.org/abs/1002.4614 February 2010.

[Par60] W. Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.*, 11:401–416, 1960.

[Par64a] W. Parry. Intrinsic Markov chains. *Trans. Amer. Math. Soc.*, 112:55–66, 1964.

[Par64b] W. Parry. Representations for real numbers. *Acta Math. Acad. Sci. Hungar.*, 15:95–105, 1964.

[Par66] W. Parry. Symbolic dynamics and transformations of the unit interval. *Trans. Amer. Math. Soc.*, 122:368–378, 1966.

[Pav14] R. Pavlov. On intrinsic ergodicity and weakenings of the specification property. *ArXiv e-prints*, http://arxiv.org/pdf/1411.2077 November 2014.

[Pet86] K. Petersen. Chains, entropy, coding. *Ergodic Theory Dynam. Systems*, 6(3):415–448, 1986.

[PY79] G. Pianigiani and J. Yorke. Expanding maps on sets which are almost invariant. Decay and chaos. *Trans. Amer. Math. Soc.*, 252:351–366, 1979.

[Rén57] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar*, 8:477–493, 1957.

[Sid03a] N. Sidorov. Almost every number has a continuum of β-expansions. *Amer. Math. Monthly*, 110(9):838–842, 2003.

[Sid03b] N. Sidorov. Arithmetic dynamics. In *Topics in dynamics and ergodic theory*, volume 310 of *London Math. Soc. Lecture Note Ser.*, pages 145–189. Cambridge Univ. Press, Cambridge, 2003.

[Sid14] N. Sidorov. Supercritical holes for the doubling map. *Acta Mathematica Hungarica*, 143(2):298–312, 2014.

[Urb86] M. Urbański. On Hausdorff dimension of invariant sets for expanding maps of a circle. *Ergodic Theory Dynam. Systems*, 6(2):295–309, 1986.

[Urb87] M. Urbański. Invariant subsets of expanding mappings of the circle. *Ergodic Theory Dynam. Systems*, 7(4):627–645, 1987.

[Vui03] L. Vuillon. Balanced words. *Bull. Belg. Math. Soc. Simon Stevin*, 10(suppl.):787–805, 2003.

[Wal78] P. Walters. Equilibrium states for β-transformations and related transformations. *Math. Z.*, 159(1):65–88, 1978.

[Wal82] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[Wei70] B. Weiss. Intrinsically ergodic systems. *Bull. Amer. Math. Soc.*, 76:1266–1269, 1970.

[Wei73] B. Weiss. Subshifts of finite type and sofic systems. *Monatsh. Math.*, 77:462–474, 1973.

[ŻB99] K. Życzkowski and E. M. Bollt. On the entropy devil's staircase in a family of gap-tent maps. *Phys. D*, 132(3):392–410, 1999.