FREQUENCY DOMAIN ANALYSIS OF FEEDBACK INTERCONNECTIONS OF STABLE SYSTEMS

A THESIS SUBMITTED TO THE UNIVERSITY OF MANCHESTER
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
IN THE FACULTY OF ENGINEERING AND PHYSICAL SCIENCES

2015

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Abstract

The study of non-linear input-output maps can be summarized by three concepts: Gain, Positivity and Dissipativity. However, in order to make efficient use of these theorems it is necessary to use loop transformations and weightings, or so called ”multipliers”.

The first problem this thesis studies is the feedback interconnection of a Linear Time Invariant system with a memoryless bounded and monotone non-linearity, or so called Absolute Stability problem, for which the test for stability is equivalent to show the existence of a Zames-Falb multiplier. The main advantage of this approach is that Zames–Falb multipliers can be specialized to recover important tools such as Circle criterion and the Popov criterion.

Albeit Zames-Falb multipliers are an efficient way of describing non-linearities in frequency domain, the Fourier transform of the multiplier does not preserve the $\mathcal{L}_1(-\infty,\infty)$ norm. This problem has been addressed by two paradigms: mathematically complex multipliers with exact $\mathcal{L}_1(-\infty,\infty)$ norm and multipliers with mathematically tractable frequency domain properties but approximate $\mathcal{L}_1(-\infty,\infty)$ norm. However, this thesis exposes a third factor that leads to conservative results: causality of Zames-Falb multipliers. This thesis exposes the consequences of narrowing the search Zames-Falb multipliers to causal multipliers, and motivated by this argument, introduces an anti-causal complementary method for the causal multiplier synthesis in [1].

The second subject of this thesis is the feedback interconnection of two bounded systems. The interconnection of two arbitrary systems has been a well understood problem from the point of view of Dissipativity and Passivity. Nonetheless, frequency domain analysis is largely restricted for passive systems by the need of canonically factorizable multipliers, while Dissipativity mostly exploits constant multipliers.

This thesis uses IQC to show the stability of the feedback interconnection of two non-linear systems by introducing an equivalent representation of the IQC Theorem, and then studies formally the conditions that the IQC multipliers need. The result of this analysis is then compared with Passivity and Dissipativity by a series of corollaries.

Keywords

Integral Quadratic Constraints, IQC, Zames-Falb multiplier, Popov multiplier, Linear Matrix Inequalities, LMI, Absolute Stability, Passivity, Dissipativity, Small Gain, Robust Control, Non-linear Control.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Acknowledgements

The author would like record his appreciation for the help of the following:

His parents, Martin Maya-Vasquez and Enriqueta Gonzalez-DeLoza, and brother Andres Maya-Gonzalez, whose sacrifices made possible the opportunity of presenting this work.

Prof. Alexander Lanzon, who supervised this research, and his guidance during the writing of this thesis and research papers.

Dr. William Paul Heath, who offered always an honest judgement of this research.

Dr. Joaquin Carrasco, who offered always useful discussions and kindly reviewed some of the results.

Daniel Harding and Gillian Menichino who patiently proofread this thesis.

This work was sponsored by the Engineering and Physical Sciences Research Council.

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Journal Papers

- Crowther, B., Lanzon, A., Maya-Gonzalez, M., & Langkamp, D. (2011). ”Kinematic Analysis and Control Design for a Nonplanar Multirotor Vehicle”. Journal of Guidance, Control, and Dynamics, 34(4), 1157-1171. doi:10.2514/1.51186.
- Carrasco, J., Maya-Gonzalez, M., Lanzon, A., & Heath, W. P. (2014). ”LMI searches for anticausal and noncausal rational ZamesFalb multipliers”. Systems & Control Letters, 70, 17-22. doi:10.1016/j.sysconle.2014.05.005.

Conference Papers

- Carrasco, J., Maya-Gonzalez, M., Lanzon, A., & Heath, W. P. (2012). ”LMI search for rational anticausal Zames-Falb multipliers”. 51st IEEE Conference on Decision and Control, December 10-13, Maui, Hawaii, USA., 7770-7775.

Glossary

Function Spaces

Notation	Description
$\mathcal{L}_2^n(-\infty,0]$	subspace of $\mathcal{L}_2^n(-\infty,\infty)$ with functions zero for $t>0$
$\mathcal{L}_2^n[0,\infty)$	subspace of $\mathcal{L}_2^n(-\infty,\infty)$ with functions zero for $t<0$
$\mathcal{L}_{2e}^n$	extension of space $\mathcal{L}_2^n[0,\infty)$, i. e. $f\in\mathcal{L}_{2e}^n$ if $P_T f\in\mathcal{L}_2^n[0,\infty)$ for all $T\in\mathbb{R}$, $T>0$
$\mathcal{L}_2^n(-\infty,\infty)$	time domain Hilbert space of square integrable $\mathbb{R}^n$ valued functions $f:\mathbb{R}\to\mathbb{R}^n$
$<x,y>_{\mathcal{L}_2}$	inner product of $x,y\in\mathcal{L}_2^n(-\infty,\infty)$
$<x,y>_T$	denotes $<P_Tx,P_Ty>_{\mathcal{L}_2}$ for $x,y\in\mathcal{L}_2[0,\infty)^n$
$\|\|x\|\|_{\mathcal{L}_2}$	induced norm of $x\in\mathcal{L}_2^n(-\infty,\infty)$
$\|\|F\|\|_{\mathcal{L}_2\to\mathcal{L}_2}$	induced norm of bounded causal operator $F:\mathcal{L}_2^n[0,\infty)\to\mathcal{L}_2^m[0,\infty)$
$\|\|G\|\|_\infty$	induced norm of bounded Linear Time Invariant operator $G\in\mathcal{RL}_\infty^{n\times m}$, i. e. $\|\|G\|\|_{\mathcal{L}_2\to\mathcal{L}_2}=\|\|G\|\|_\infty$
$\mathcal{H}_2$	space of functions in $\mathcal{L}_2(j\mathbb{R})$ that are analytic in $\mathbb{C}_+$ and uniformly square integrable along $Re\{s\}=a$ for all $a\in\mathbb{R}_+$
$<x,y>_{\mathcal{H}_2}$	inner product of $x,y\in\mathcal{H}_2^n$
$\mathcal{L}_1^n(-\infty,\infty)$	time domain Lebesgue space of absolute integrable $\mathbb{R}^n$ valued functions $f:\mathbb{R}\to\mathbb{R}^n$
$\|\|x\|\|_1$	induced norm of $x\in\mathcal{L}_1(-\infty,\infty)$
$\mathcal{L}_2(j\mathbb{R})$	space of square integrable functions on $j\mathbb{R}$ including ∞
$<x,y>_{\mathcal{L}_2(j\mathbb{R})}$	inner product of $x,y\in\mathcal{L}_2(j\mathbb{R})^n$
$\mathcal{RL}_\infty$	set of proper rational transfer functions with real coefficients and no poles on the imaginary axis

Notation	**Description**
$\mathcal{RL}_\infty^{n\times m}$	set of $n\times m$ matrices with elements in $\mathcal{RL}_\infty$
$\mathcal{RH}_\infty$	subset of $\mathcal{RL}_\infty$ consisting of functions without poles on the closed right-half plane
$\mathcal{RH}_\infty^{n\times m}$	set of $n\times m$ matrices with elements in $\mathcal{RH}_\infty$
$\mathcal{RH}_\infty^-$	subset of $\mathcal{RL}_\infty$ consisting of functions without poles on the closed left-half plane
$\mathcal{RH}_\infty^{-(n\times m)}$	set of $n\times m$ matrices with elements in $\mathcal{RH}_\infty^-$
$P_T f$	truncation of signal $f(t)$ at time T, i.e. $f(t)$ is zero for all $t>T$

Field of Numbers

Notation	**Description**
$\mathbb{C}$	complex numbers
$\mathbb{C}^{n\times m}$	complex matrices of dimension $n\times m$
$\mathbb{C}^n$	complex column vector with n entries
j	the imaginary unit, i.e. $j=\sqrt{-1}$
$j\mathbb{R}$	the imaginary axis
$\mathbb{Z}$	integer numbers
$\mathbb{Z}_+$	strictly positive integer numbers
$\mathbb{R}$	real numbers
$\mathbb{R}^{n\times m}$	real matrices of dimension $n\times m$
$\mathbb{R}_+$	strictly positive real numbers
$\mathbb{R}^n$	real column vector with n entries

Matrix Operations

Notation	**Description**
$A=(a_{ij})_{n\times m}$	matrix of dimension $n\times m$, with elements a_{ij}

Notation	Description
A^*	complex conjugate transpose of matrix A
$A \cdot B$	inner product of the vectors $A, B \in \mathbb{R}^m$
A^{-1}	inverse of matrix A
A^{-*}	denotes $(A^{-1})^*$ or equivalently $(A^*)^{-1}$
A^{-T}	denotes $(A^{-1})^T$ or equivalently $(A^T)^{-1}$
A^T	transpose of a matrix A
$dim(A)$	dimension of the matrix A
I	identity matrix with compatible dimensions
I_n	identity matrix of dimension $n \times n$
$diag\{a_1, a_2, ..., a_n\}$	diagonal matrix with elements $a_1, a_2, ..., a_n$ on the main diagonal
$A < B$	denotes $(A - B) < 0$
$A \leq 0$	hermitian matrix $A = A^*$ with non-positive eigenvalues
$A < 0$	hermitian matrix $A = A^*$ with strictly negative eigenvalues
$A \geq 0$	hermitian matrix $A = A^*$ with non-negative eigenvalues
$A > 0$	hermitian matrix $A = A^*$ with strictly positive eigenvalues
0	zero matrix with compatible dimensions

Miscellaneous

Notation	Description
$\in$	belongs to
$\square$	end of proof
$\forall$	for all

Notation	**Description**
$\iff$	equivalent to
$\implies$	implies
$\impliedby$	is implied by
$:$	such that
$\exists$	there exists
$\geq$	greater or equal
$>$	greater than
$\leq$	less or equal
$<$	less than
$\gg$	much greater than
$\ll$	much less than
$\lvert x \rvert$	modulus (or magnitude) of $x \in \mathbb{C}$
$x \in [a,b]$	$a \leq x \leq b$ where $a,x,b \in \mathbb{R}$
$x \in [a,b)$	$a \leq x < b$ where $a,x,b \in \mathbb{R}$
$x \in (a,b]$	$a < x \leq b$ where $a,x,b \in \mathbb{R}$
$x \in (a,b)$	$a < x < b$ where $a,x,b \in \mathbb{R}$
$\inf_{x\in X} f(x)$	infimum of the function $f(x)$ over $x \in X$
$\lim_{x\to a} f(x)$	$f(x)$ in the limit as x tends to a
$\lim_{x\uparrow a} f(x)$	$f(x)$ in the limit as x tends to a from below
$\max_{x\in X} f(x)$	maximum of the function $f(x)$ over $x \in X$
$\min_{x\in X} f(x)$	minimum of the function $f(x)$ over $x \in X$
$\sup_{x\in X} f(x)$	supremum of the function $f(x)$ over $x \in X$
$\nabla_x S_u$	gradient of the Function S_u with respect to the variables vector x
$\frac{\partial}{\partial x}$	differentiation operator with respect to x
$[G,C]$	standard positive feedback interconnection of plant P with controller C
$\lvert\lvert x \rvert\rvert$	Euclidean vector norm

Notation	Description

Operations on Systems

Notation	Description
$G^{\sim}$	adjoint of a real-rational system $G \in \mathcal{RL}_\infty^{m\times n}$, that is $G(s)^{\sim} = G(-s)^T = G^T(-s)$
$G(j\omega)^*$	complex conjugate transpose of frequency-response function G, that is $G(j\omega)$ at each frequency ω, that is $G(j\omega)^* = G(jj\omega)^T$
G^{-1}	inverse of real-rational system G, that is $G^{-1}(s) = G(s)^{-1}$
$G^{-\sim}$	denotes $(G^{-1})^{\sim}$ or equivalently $(G^{\sim})^{-1}$
$G(j\omega)^{-*}$	denotes $(G(j\omega)^{-1})^*$ or equivalently $(G(j\omega)^*)^{-1}$
G^{-T}	denotes $(G^{-1})^T$ or equivalently $(G^T)^{-1}$
G^T	transpose of a real-rational system $G \in \mathcal{RL}_\infty^{m\times n}$, that is $G(s)^T = G^T(s)$

Chapter 1

Introduction

1.1 Background and motivation

Feedback control is a tool that can lead to complicated problems because it changes fundamentally the behaviour of a system and, most importantly, its stability. However, there are two reasons for the need of its use, the first is disturbance rejection and the second is to reduce the effect of uncertainty in the model of the system dynamics [2]. The stability of a system is a fundamental problem in control theory. The object of the stability theory is to find the properties of the behaviour of a system without actually solving the differential equations that describe it. Lyapunov was the first mathematician that introduced mathematical tools for the analysis of arbitrary differential equations and proved many of the fundamental theorems using the concept of state space representation [3]. However, Lyapunov stability using state space is not the only tool available to describe the stability or instability of a system; Input-Output stability introduced by Sandberg [4, 5] and Zames [6, 7, 8, 9] is a parallel and not always equivalent tool which describes a system not by using differential equations and state space representations but by input-output mappings that relate an output signal to each input signal. One of the most noticeable differences is that Input-Output stability makes no mention of ”states” [3].

The study of non-linear input-output maps can be summarized by three concepts: Gain, Positivity and Dissipativity.

The work of Zames [6] compiles for first time a study of systems using input-output mapping and assigns a gain, or operator norm to each of these mappings. Using this concept, Zames describes the Small Gain Theorem. However, in practical systems, the gain if this operators is big, therefore, in order to make an efficient use of this theorem, it is necessary to use loop transformations and weightings, or so called ”multipliers”[7].

In parallel to gain, some operators will hold the positivity of an inner product between the input and its image. This concept is known as Passive systems, and was inspired by the study of linear electric circuits [10]. The Passive operator theory was first connected to the stability of closed loop systems by Youla [11] who proved that a passive network in closed loop with a resistor is stable [12]. This concept has been widely covered, usually under the name ”Positive operators” [13, 14, 10].

Dissipativity was first proposed by Popov [15, 10]. Popov himself also extended its application with the aforementioned absolute stability [16] and the use of multipliers [17]. However, the first formalization of the current definition of Dissipativity came from Willems [18, 19]. Thereafter, Hill and Moylan [20, 21] extended the Kalman-Yakubovich-Popov Lemma to deal with certain classes of non-linear systems.

The non-linear control problem is broad and irregular. However, for some specific problems tools already exist. The main subject of this thesis is the feedback interconnection of two bounded systems.

In order to gain an initial understanding of the broad problem, firstly the interconnection of a Linear Time Invariant system with a memoryless non-linearity is addressed. In practice, the Linear Time Invariant system will be studied separately from the non-linear, time varying or uncertain element. In many applications this is a straightforward task because the construction of the closed loop explicitly includes a relay, actuator or sensor non-linearity; other applications are less obvious, but nevertheless by means of tools like loop transformations their alternative representation becomes evident [22, 3].

For the class of non-linearities that admit a frequency description, Popov developed a solution that he called the theory of absolute stability [16], in which stability is defined as the existence of a global uniform asymptotic stable equilibrium point at the origin for all non-linearities in a given sector [22]. The first of these frequency-domain conditions is nowadays known as the Popov Criterion [16]. Several authors developed the Circle Criterion simultaneously in which the class of non-linearities is slope-restricted [9, 5]. Their simplicity and graphical interpretation make these criteria the classical examples of multipliers in modern non-linear control textbooks, e.g. [3, 23, 22, 24].

Most Time Invariant Linear systems and non-linearities do not fulfil the Passivity or the Small Gain Theorems by themselves. However, some of these systems can be corrected to hold the necessary gain or phase conditions by placing a convolution operator or ”multiplier” [14].

For the class of bounded and monotone non-linearities, the celebrated paper [25] by Zames and Falb introduced a general characterization of the multiplier, preserving the positivity of the non-linearity. These multipliers take into account the monotone property to describe more accurately the non-linearity so stability can be determined for bigger sector sizes. Zames–Falb multipliers [26] are a more general tool than Circle criterion and the Popov criterion, as they can be recovered from Zames–Falb multipliers by making specific simplifications. In addition, the conditions for the existence of the multiplier are relaxed if the non-linearity is odd.

Zames-Falb multipliers are an efficient way of deducing stability for the Absolute Stability problem, but the description of the multiplier does not allow the production of a Fourier transform that preserves the $\mathcal{L}_1(-\infty,\infty)$ norm in the frequency domain [27, 25]. This problem has been addressed by two paradigms: mathematically complex multipliers with exact $\mathcal{L}_1(-\infty,\infty)$ norm [28, 29, 30], and multipliers with mathematically tractable frequency domain properties [31, 32, 1, 33] but approximate $\mathcal{L}_1(-\infty,\infty)$ norm. However, little attention has been paid to the effects that causality has on their performance. The first question that this thesis aims to answer is ”Are causal multipliers sufficient to construct Zames-Falb multipliers?”. The answer to this question motivates the construction of an anti-causal complementary method for the only method in the literature that lacks anti-causal multiplier synthesis [1].

An understanding of the multipliers is not restricted to Zames-Falb multipliers. Since the introduction of multipliers by Popov in [16], the idea of describing non-linearities that lie in a sector using a quadratic constraint ”over-bound” has expanded to more classes of non-linearities which has led to the discovery of more frequency domain conditions for absolute stability [34]. The introduction of the Integral Quadratic Constraints and their revolutionary homotopy based argument collects many of these results and formalizes a more general frequency domain condition for the stability of a non-linear operator with a Linear Time Invariant operator [35].

Integral Quadratic Constraints (IQC) is an attractive tool because it comprises many results using one unified framework. The extension that is of most interest to this research is the insight that IQCs bring to results for the positive operators [16, 36, 9, 8, 18, 19, 14] and the dissipative operators [24, 10, 37, 38, 39, 21, 20, 40].

IQC is not the only tool to study non-linear systems with frequency domain descriptions. The interconnection of two arbitrary systems has been a well understood problem from the point of view of Dissipativity and Passivity. Nonetheless, frequency domain analysis is largely restricted to passive systems by canonical factorizations [14] and Dissipativity can be restrictive because it mostly exploits constant multipliers [10].

However, the IQC theory is not exempt from having complications. The IQC theory makes the search for multipliers simple at the cost of having to show the well posedness of "attenuated" versions of the original closed loop in addition to the unattenuated one. In order to make full use of this multiplier simplification, convex optimization tools are the main focus of most research on IQC multipliers because they offer a mathematically tractable procedure to obtain stability results in the form of Linear Matrix Inequalities (LMI) [13, 41, 34]. In order to use this well understood family of tools, the multiplier has to have a state space representation and therefore usually multipliers with this description are the most frequently studied [42].

The second question that this thesis addresses is "Can IQC be used to introduce dynamic multipliers to Dissipativity Theorem?". The use of IQC to show the stability of two non-linear systems which are interconnected has been proposed previously by [43], as a possible extension to the IQC Theorem. The objective of this work is to present formal conditions of the stability of the feedback interconnection of two non-linear systems using IQC alone. The result of this work is then summarized as a series of corollaries that describe classical results, including the construction of IQCs for Output Dissipative systems.

In order to expose the gap between IQC, Dissipativity and Passivity, this research first presents sufficient conditions that IQCs can hold in order to show stability of closed loops using the Small Gain Theorem, the Passivity Theorem and hard IQC factorizations from [42] as multipliers.

1.2 Organization of this thesis

This thesis is organized in 5 Chapters.

Chapter 2: Preliminaries

The objective of this chapter is to familiarize the reader with the notation and standard definitions that are going to be used in the main body of this thesis. The first section presents the definitions and theorems that will become useful from the functional analysis and provides a summary of the main stability theorems derived from the input-output representation of systems. In the second section the search for anti-causal Zames-Falb multipliers is motivated using a simple example. In the third section, the feedback interconnection of two non-linear systems is studied using well known factorizations of positive-negative IQCs, the resulting lemmas then reveal that multipliers for the Small Gain Theorem and Passivity Theorem have fundamental and non-trivial restrictions. For the well known results the proofs are only referenced. When a new result is introduced the proof is described in the appendices for the sake of continuity in the corresponding argument.

Chapter 3: Search for SISO Zames-Falb multipliers

This chapter presents the first result of this thesis. The second section summarizes the state of the art in the search for Zames-Falb multipliers and indicates the advantages and restrictions of each method. In this section an algebraic correction to the LMI search in [32] is proposed. This work is made in order simplify the use of Zames-Falb multipliers for comparison and general application purposes. The third section focuses on the implementation of Zames-Falb multipliers using only causal functions from which this work derives a novel and complementary result in order to also search for anti-causal multipliers. In the same section, an extension using Popov multipliers is deduced, achieving parallelisms with the previous result. The fourth section presents 9 examples compares the performances that all the methods described in this chapter achieve. It concludes by presenting the advantages and restrictions of the novel multiplier synthesis method.

Chapter 4: IQC and Dissipativity

This chapter presents the second result of this thesis. It firstly presents a loop transformation and its stability properties. Then it proceeds to restate the conditions of the IQC Theorem, such that it can now link two non-linear systems by Theorem 4.2.2. In order to emphasize the importance of this reinterpretation, a complete version of the IQC is recovered from Theorem 4.2.2 in Corollary 4.2.4 and Lemma 4.2.5. A discussion on the well posedness of the required closed loop is presented, such that the applicability of the main theorem is ensured when it is used in some special cases. In the third section a series of corollaries are deduced that specialize version of Theorem 4.2.2 to versions of Dissipativity and Passivity for bounded operators. In the final section, a well known example is reinterpreted in the light of the main result of this chapter, extending previous remarks made about delay operators and IQCs to the interconnection of two non-linear systems.

Chapter 5: Conclusions

This chapter presents the conclusions of this thesis and recapitulates the main contributions. In this section are summarized possible directions of future research.

Appendices A.1, A.2 and A.3

This chapter presents the proofs of the minor results presented in the introduction. This material is placed in the appendices for the sake of continuity in the introduction.

Chapter 2

Preliminaries

2.1 Notation and definitions

2.1.1 Function spaces

Let $\mathcal{L}_2^l(-\infty,\infty)$ be the Hilbert space of all $\mathbb{R}^l - valued$ Lebesgue measurable functions $f:(-\infty,\infty)\rightarrow\mathbb{R}^l$ equipped with an inner product $<.,.>_{\mathcal{L}_2}: \mathcal{L}_2^l(-\infty,\infty)\times\mathcal{L}_2^l(-\infty,\infty)\rightarrow\mathbb{R}$, defined as $<f,g>_{\mathcal{L}_2}=<g,f>_{\mathcal{L}_2}=\int_{-\infty}^{\infty}f(t)^Tg(t)dt$ with $f,g\in\mathcal{L}_2^l(-\infty,\infty)$. The induced norm is defined as $||f||_{\mathcal{L}_2}=<f,f>_{\mathcal{L}_2}=\int_{-\infty}^{\infty}||f(t)||^2dt<\infty$.

Definition 2.1.1. *[44, p. 20] Let $M:\mathcal{L}_2^l(-\infty,\infty)\rightarrow\mathcal{L}_2^m(-\infty,\infty)$ be a bounded linear operator. The Hilbert adjoint of M is the operator $M^{\sim}:\mathcal{L}_2^m(-\infty,\infty)\rightarrow\mathcal{L}_2^l(-\infty,\infty)$ such that:*

$$<Mf,g>_{\mathcal{L}_2}=<f,M^{\sim}g>_{\mathcal{L}_2}\ \forall f\in\mathcal{L}_2^l(-\infty,\infty), g\in\mathcal{L}_2^m(-\infty,\infty).$$

An operator M is called self adjoint if, and only if, $M=M^{\sim}$.

The Fourier transform of $f\in\mathcal{L}_2(-\infty,\infty)$ is denoted by

$$\hat{f}(j\omega)=\mathcal{F}\{f(t)\}=\int_{-\infty}^{\infty}e^{-j\omega t}f(t)dt.$$

The Fourier transform of the space $\mathcal{L}_2(-\infty,\infty)$ is the space $\mathcal{L}_2(j\mathbb{R})$, and the Fourier transform of the space $\mathcal{L}_2[0,\infty)$ is the space $\mathcal{H}_2$ [45].

The inner product can also be defined using the Parseval's Theorem [14]. If $f,g\in\mathcal{L}_2(-\infty,\infty)$ and $\hat{f},\hat{g}\in\mathcal{L}_2(j\mathbb{R})$, then:

$$< f,g >_{\mathcal{L}_2}= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(j\omega)^*\hat{g}(j\omega)d\omega =< \hat{f},\hat{g} >_{\mathcal{L}_2(j\mathbb{R})}$$

A truncation of the signal f at time T is given by

$$P_T(f)(t) = (f)_T(t) = \begin{cases} f(t) & \forall t \le T, \\ 0 & \forall t > T. \end{cases}$$

For $\varepsilon > 0$, $t \ge 0$, D_ε denotes the *delay operator on $\mathcal{L}_{2e}$, i.e. $D_\varepsilon x$ is given by*

$$D_\varepsilon f(t) = \begin{cases} f(t-\varepsilon) & \forall t \ge \varepsilon, \\ 0 & \forall t < \varepsilon. \end{cases}$$

In addition, f belongs to the *extended space* $\mathcal{L}_{2e}^l$ if $P_T(f) \in \mathcal{L}_2^l[0,\infty)$ for all $T \in \mathbb{R}, T > 0$. The norm of the signal $f \in \mathcal{L}_{2e}^l$ is $||f||_T = ||P_T f||_{\mathcal{L}_2} = \int_0^T ||f(t)||^2 dt < \infty \quad \forall T > 0$.

Likewise, the inner product of the signals $f,g \in \mathcal{L}_{2e}^l$ is denoted $< f,g >_T =< P_T f, P_T g >_{\mathcal{L}_2}$

The following definitions are obtained from [13], specialized for the $\mathcal{L}_2[0,\infty)$ and $\mathcal{L}_{2e}$ spaces.

Definition 2.1.2. *An operator $F : \mathcal{L}_{2e}^l \to \mathcal{L}_{2e}^m$ is causal if $P_T F P_T = P_T F \quad \forall T \in [0,\infty)$.*

Definition 2.1.3. *An operator $F : \mathcal{L}_{2e}^l \to \mathcal{L}_{2e}^m$ is said to be strongly causal if it is causal and for all $T \in [0,\infty)$, $\varepsilon > 0$, and $T' \in [0,\infty), T' \le T$ there exists a real number $\Delta T > 0$ such that*

$$||P_{T'+\Delta T}(Fx - Fy)||_{\mathcal{L}_2} \le \varepsilon ||P_{T+\Delta T}(x-y)||_{\mathcal{L}_2} \quad \forall x,y \in \mathcal{L}_{2e}^m,$$

with $P_{T'}x = P_{T'}y$.

Definition 2.1.4. *An operator $F : \mathcal{L}_2[0,\infty)^l \to \mathcal{L}_2[0,\infty)^m$ is said to be Lipschitz continuous if*

$$\sup_{\substack{x,y \in \mathcal{L}_2[0,\infty) \\ x \ne y}} \frac{||Fx - Fy||_{\mathcal{L}_2}}{||x-y||_{\mathcal{L}_2}} < \infty$$

Definition 2.1.5. *An operator $F : \mathcal{L}_{2e}^l \to \mathcal{L}_{2e}^m$ is said to be locally Lipschitz continuous if $\forall T \in [0,\infty), P_T F P_T$ is Lipschitz continuous on $\mathcal{L}_2[0,\infty)$.*

Definition 2.1.6. *An operator $F : \mathcal{L}_{2e}^l \to \mathcal{L}_{2e}^m$ is said to be strongly causal, uniformly with respect to past inputs, if it is strongly causal operator and $\forall T \in [0,\infty)$, $\varepsilon > 0$, and $T' \in [0,\infty)$, $T' \le T$, there exist real numbers $\Delta T > 0$ and $K < \infty$ such that*

$$||(P_{T'+\Delta T} - P_{T'})(Fx - Fy)||_{\mathcal{L}_2} \le K||P_{T'}(x-y)||_{\mathcal{L}_2} + \varepsilon ||(P_{T'+\Delta T} - P_{T'})(x-y)||_{\mathcal{L}_2} \quad \forall x,y \in \mathcal{L}_{2e}^m.$$

The induced norm of a bounded causal operator $F : \mathcal{L}_2^l[0,\infty) \to \mathcal{L}_2^m[0,\infty)$ is given by

$$||F||_{\mathcal{L}_2 \to \mathcal{L}_2} = \sup\{\frac{||F(u)||_{\mathcal{L}_2}}{||u||_{\mathcal{L}_2}} : u \in \mathcal{L}_2^l[0,\infty), u \neq 0\}.$$

Furthermore, if the operator G is Linear Time Invariant, the induced norm can be calculated as: $||G||_\infty = ||G||_{\mathcal{L}_2 \to \mathcal{L}_2} = \sup_{\omega \in \mathbb{R}} \sigma_{max}(G(j\omega))$, where σ_{max} is the maximum singular value [10].

2.1.2 Feedback Systems

2.1.2.1 Well posedness

Let the Figure 2.1 define a closed loop feedback interconnection between the operators Δ_1 and Δ_2, this will be denoted by the *feedback interconnection* $[\Delta_1, -\Delta_2]$.

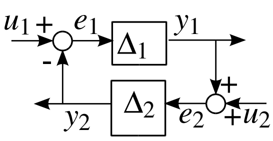


Figure 2.1: Dissipativity feedback loop

The system in Figure 2.1 is represented by the following equations:

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & -I_m \\ I_n & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = u + Hy, \tag{2.1}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = Ge, \tag{2.2}$$

where $G_{ij} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ and $H_{ij} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ are locally Lipschitz causal continuous operators.

For the sake of completeness, here it will be reproduced the standard definitions of well posedness for closed loop systems, taken from [46, 13]. It describes in detail the necessary conditions that a functional description of a physical system has to hold to be valid.

Definition 2.1.7. *[46] The Closed Loop in equations* (2.1) *and* (2.2) *is said to be well posed if it possesses properties 1)-3).*

1. *There exists a pair of causal, locally Lipschitz continuous operators* $E : \mathcal{L}_{2e}^{m+n} \to$

$\mathcal{L}_{2e}^{m+n}$ and $Y : \mathcal{L}_{2e}^{m+n} \to \mathcal{L}_{2e}^{m+n}$ such that the pair (e,y) given by $e = Eu$ and $y = Yu$ is the unique solution to equations (2.1) *and* (2.2).

2. *Let CS_ε be the system described by the equations obtained from equations* (2.1) *and* (2.2) *after replacing G_{ij} and H_{ij} by $D_{\varepsilon_{ij}}G_{ij}$ and $D_{\varepsilon'_{ij}}H_{ij}$, respectively. Then CS_ε satisfies the above property 1).*

3. *Let $(e^\varepsilon, y^\varepsilon)$ be the solution of CS_ε and let*

$$\varepsilon_0 = \max_{i=1,\dots,m+n;\, j=1,\dots,m+n} \{\varepsilon_{ij}, \varepsilon'_{ij}\}$$

Then

$$\lim_{\varepsilon_0 \to 0} P_\tau e^\varepsilon = P_\tau e, \quad \lim_{\varepsilon_0 \to 0} P_\tau y^\varepsilon = P_\tau y \quad \text{for any} \quad \tau \in [0,\infty) \tag{2.3}$$

Now, a set of definitions is introduced that are useful in the description of the well posedness of the feedback loop.

Definition 2.1.8. *[46] Let F be a locally Lipschitz continuous causal operator from $\mathcal{L}_{2e}^m$ to $\mathcal{L}_{2e}^n$. Let $T \in [0,\infty)$ and $\Delta T > 0$. Then, there exists two nonnegative numbers $M_{T,\Delta T}$ and $K_{T,\Delta T}$ such that*

$$\begin{aligned} &||(P_{T+\Delta T} - (P_{T-\Delta T})(Fx - Fy)||_{\mathcal{L}_2} \\ &\le M_{T,\Delta T}||(P_{T+\Delta T} - P_{T-\Delta T})(x-y)||_{\mathcal{L}_2} + K_{T,\Delta T}||P_{T-\Delta T}(x-y)||_{\mathcal{L}_2} \quad \forall x,y \in \mathcal{L}_{2e}^m \end{aligned} \tag{2.4}$$

Definition 2.1.9. *[46] Let $\mathcal{S}$ be the set of the pairs $(M_{T,\Delta T}, K_{T,\Delta T})$ which satisfies inequality* (2.4). *The infimum $M^*_{T,\Delta T}$ of $M_{T,\Delta T}$ over the set $\mathcal{S}$ is defined as the uniform gain of F in the interval $[T-\Delta T, T+\Delta T]$.*

Definition 2.1.10. *[46] The uniform instantaneous gain of F at T is defined as* $\lim_{\Delta T \to 0} M^*_{T,\Delta T}$

The following Definition and Theorem are necessary in order to describe the structure of matrices for multi input, multi output systems.

Theorem 2.1.11. *[46] Let A be a real square matrix with non-positive off-diagonal elements. Then, the next four conditions are mutually equivalent:*

1. *The principal minors of A are all positive.*

2. *The leading principal minors of A are all positive.*

3. *There is a vector x (or y) whose elements are all positive such that the elements of of Ax (or y) are all positive.*

4. *A is non-singular and the elements of A^{-1} are all non-negative.*

Proof. See Theorem B1 in [46]. □

Definition 2.1.12. *[46] The matrix A satisfying the conditions of Theorem 2.1.11 is called an $M-matrix$.*

Using this theorems, the following theorem will determine the well posedness of the closed loop in Figure 2.1.

Theorem 2.1.13. *[46] Let the uniform instantaneous gains of G_{ij} and H_{ji} at $T \in [0,\infty)$ be $a_{ij}(T)$ and $b_{ji}(T)$, respectively. Define the gain-product matrix $\Theta(T) = (\theta_{jj'}(T))$ by*

$$\theta_{jj'}(T) = \sum_{i=1}^{m+n} b_{ji}(T)a_{ij'}(T), \quad j, j' = 1, ..., m+n. \tag{2.5}$$

Then, if the matrix $I - \Theta(T)$ is an $M-matrix$ for all T, the closed loop system defined by equations (2.1) *and* (2.2) *is well posed in the sense of Definition 2.1.7.*

Proof. See Theorem 3 in [46]. □

Note that for the single input, single output case, Theorem 2.1.13 reduces to the case 1 of Theorem 2.1.14 [46], while still shows well posedness in the sense of Definition 2.1.7

Theorem 2.1.14. *The feedback systems described by Figure 2.1 is well posed if either of the following conditions is satisfied for all $T \in [0,\infty)$*

1. *The product of the uniform instantaneous gain of the operators Δ_1 and Δ_2 is less than $\alpha < 1$.*

Proof. See Theorem 4.1 in [13]. □

From this theorem, [13] deduces a series of corollaries for which the closed loop is well posed for particular systems frequently found in engineering.

The first definitions describe the closed loop of systems with no feed-through:

Definition 2.1.15. *Let $F : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ be a causal operator. Then, F is said to delay all inputs if for some $\varepsilon > 0$ the operator F_ε defined as $F_\varepsilon x(t) = Fx(t+\varepsilon)$ is also causal (i.e. F can be cascaded with a predictor and the composition remains causal).*

The following Corollary describes sufficient conditions for the well posedness of the closed loop.

Corollary 2.1.16. *The feedback system described by Figure 2.1 is well posed if the open loop operator $\Delta_1\Delta_2$ delays all inputs.*

Proof. See Corollary 4.1.1 in [13]. □

The next Corollary describes sufficient conditions for the well posedness for a more general class of operators, which include some classes of ordinary differential equations.

Corollary 2.1.17. *The feedback system described by the Figure 2.1 is well posed if the operator Δ_1 is strongly causal, uniformly with respect to past inputs.*

Proof. See Corollary 4.1.2 in [13]. □

2.1.2.2 Stability

The following definition describes the notion of stability of systems in the input-output framework.

Definition 2.1.18. *[24] Let $F : \mathcal{L}_{2e}^l \to \mathcal{L}_{2e}^m$. F is $\mathcal{L}_2[0,\infty)$ stable if $F(u) \in \mathcal{L}_2^m[0,\infty)$ for every $u \in \mathcal{L}_2^l[0,\infty)$. F is $\mathcal{L}_2[0,\infty)$ stable with finite gain and zero bias if there exists $c > 0$ such that*

$$||F(u)||_{\mathcal{L}_2} \leq c||u||_{\mathcal{L}_2} \quad \forall u \in \mathcal{L}_2^l[0,\infty) \tag{2.6}$$

Note that if F is causal, then (2.6) is equivalent to $||F(u)||_T \leq c||u||_T, \quad \forall T > 0, u \in \mathcal{L}_{2e}^l$.

The following theorems present the required properties to develop the proof of the interconnection for two non-linear systems. These two theorems are the main foundation for the stability analysis using the function analysis procedure. Note that in all these theorems, well posedness is first required as a separate test in order to show stability.

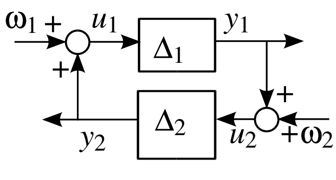


Figure 2.2: Positive feedback interconnection

Theorem 2.1.19. *[22, p. 217] (Small Gain Theorem) Assume that the positive feedback interconnection in Figure 2.2 is well posed and the operators hold $\Delta_1 : \mathcal{L}_{2e}^l \to \mathcal{L}_{2e}^m$, $\Delta_2 : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^l$ for all $T \geq 0$ are $\mathcal{L}_2[0,\infty)$ stable, that is*

- $||\Delta_1(u_1)||_T \leq \gamma_1 ||u_1||_T \quad \forall u_1 \in \mathcal{L}_{2e}^l, \forall T \geq 0$
- $||\Delta_2(u_2)||_T \leq \gamma_2 ||u_2||_T \quad \forall u_2 \in \mathcal{L}_{2e}^m, \forall T \geq 0$

for some scalars $\gamma_1, \gamma_2 \geq 0$. *If*

$$\gamma_1\gamma_2 < 1,$$

then, the closed loop is $\mathcal{L}_2[0,\infty)$ *stable.*

Proof. See Theorem 5.6 in [22]. □

Theorem 2.1.20. *[10, p. 257](Passivity Theorem) Assume that both* Δ_1, Δ_2 *are pseudo Very Strictly Passive, i.e.*

$$< \Delta_1(e_1), e_1 >_T + \beta_1 \geq \delta_1 ||\Delta_1(e_1)||_T^2 + \varepsilon_1 ||e_1||_T^2, \quad \forall e_1 \in \mathcal{L}_{2e}, T \geq 0,$$

$$< \Delta_2(e_2), e_2 >_T + \beta_2 \geq \delta_2 ||\Delta_2(e_2)||_T^2 + \varepsilon_2 ||e_2||_T^2, \quad \forall e_2 \in \mathcal{L}_{2e}, T \geq 0.$$

Assume the negative feedback in Figure 2.3 is well posed.

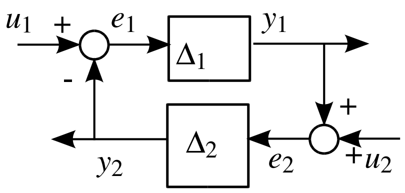


Figure 2.3: Negative feedback interconnection

Then, if

- $\varepsilon_1 + \delta_2 > 0.$
- $\varepsilon_2 + \delta_1 > 0.$

the feedback system is $\mathcal{L}_2[0,\infty)$ *stable.*

Proof. See Theorem 5.4 in [10]. □

2.2 Motivation for Zames-Falb multipliers

2.2.1 Absolute stability problem

The study of stability for non-linear systems is a complicated matter and most of the stability results are only found for special cases. The instance that is the interest of this work is the absolute stability problem, which studies feedback interconnection of a Linear Time Invariant operator with a memoryless non-linear operator. Such interconnection is presented in Figure 2.4. The feedback interconnection is $\mathcal{L}_2[0,\infty)$ stable if for any $r \in \mathcal{L}_2[0,\infty)$, the internal signals hold $Gu \in \mathcal{L}_2[0,\infty)$ and $\phi(y) \in \mathcal{L}_2[0,\infty)$. Attention in this thesis is restricted to rational stable LTI systems, i.e. $G(s) \in \mathcal{RH}_\infty$.

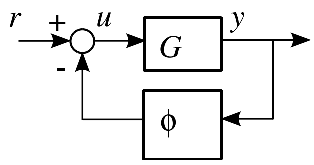


Figure 2.4: Absolute stability

Assume that the non-linearity is now restricted in the following manner

Definition 2.2.1. *A memoryless function $\phi : [0,\infty) \times \mathbb{R}^m \to \mathbb{R}^m$ is said to belong to the sector*

- $[0,\infty]$ *if* $u^T\phi(t,u) \geq 0 \quad \forall t \in [0,\infty), u \in \mathbb{R}^m$.
- $[K_1,\infty]$ *if* $u^T[\phi(t,u) - K_1u] \geq 0 \quad \forall t \in [0,\infty), u \in \mathbb{R}^m$.
- $[0,K_2]$ *with* $K_2 = K_2^T > 0$ *if* $\phi(t,u)^T[\phi(t,u) - K_2u] \leq 0 \quad \forall t \in [0,\infty), u \in \mathbb{R}^m$.
- $[K_1,K_2]$ *with* $K = K_2 - K_1 = K^T > 0$ *if* $[\phi(t,u) - K_1u]^T[\phi(t,u) - K_2u] \leq 0 \quad \forall t \in [0,\infty), u \in \mathbb{R}^m$.

The system in Figure 2.4 is said to be absolutely stable if it has a global uniform asymptotic stable equilibrium point at the origin for all non-linearities given in a sector; that is to say, if the system in Figure 2.4 can be represented by the following equations:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$u = -\phi(t,y)$$

where $x \in \mathbb{R}^n$, $u,y \in \mathbb{R}^m$, (A,B) is controllable, (A,C) is observable and $\phi : [0,\infty) \times \mathbb{R}^m \to \mathbb{R}^m$.

The well posedness of the loop is always assured when $D = 0$ [22]. When this is not the case, conditions for well posedness are readily available throughout Theorem 2.1.13.

In the literature there exist some solutions to this problem based on frequency analysis. The circle criterion [9, 22] is the most simple and thus it presents the worst performance that can be expected from any algorithm.

Theorem 2.2.2. *[22] The closed loop in Figure 2.4 is absolutely stable if:*

- *ϕ belongs to the sector $[K_1, \infty]$ and $G(s)[I + K_1 G(s)]^{-1}$ is strictly positive real, or*
- *ϕ belongs to the sector $[K_1, K_2]$, with $K = K_2 - K_1 = K^T > 0$, and $[I + K_2 G(s)][I + K_1 G(s)]^{-1}$ is strictly positive real.*

Proof. See Theorem 7.1 in [22]. □

For the scalar case, the circle criterion is stated as follows[9]:

Theorem 2.2.3. *[9] Assume that the feedback interconnection in Figure 2.4 is well posed and ϕ belongs to the sector $[k_1 + \gamma, k_2 - \gamma]$, $k_1 \leq k_2, k_2 > 0$ with offset $\delta \geq 0$, then if $G(j\omega)$ holds any of the following conditions:*

- *Case 1A. If $k_1 > 0$, then*

$$\left| G(j\omega) + \frac{1}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \right| \geq \frac{1}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) + \delta \quad \forall \omega \in \mathbb{R}$$

  and the Nyquist diagram of $G(j\omega)$ does not encircle the point $-\frac{1}{2}(1/k_1 + 1/k_2)$,

- *Case 1B. If $k_1 < 0$, then*

$$\left| G(j\omega) + \frac{1}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \right| \leq \frac{1}{2}\left(\frac{1}{k_2} - \frac{1}{k_1}\right) - \delta \quad \forall \omega \in \mathbb{R}.$$

- *Case 2. If $k_1 = 0$, then*

$$Re\{G(j\omega)\} \geq -(1/k_2) + \delta \quad \forall \omega \in \mathbb{R}.$$

for some non-negative δ and γ, where at least one of them is greater than zero, then the closed loop in Figure 2.4 is $\mathcal{L}_2[0, \infty)$ bounded.

Proof. See A Circle Theorem in [9]. □

If the non-linearity $\phi : \mathbb{R} \to \mathbb{R}$ is restricted furthermore to be a local Lipschitz, memoryless single input, single output operator, then the closed loop in Figure 2.4 describes an autonomous system. For this problem there exists the Popov Criterion [16, 22].

Theorem 2.2.4. *Consider the system given by the following equation:*

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u = -\phi(y)$$

where $G(s) = C(sI - A)^{-1}B$, and:

- *(A, B) is controllable,*
- *(A, C) is observable,*
- *$\phi : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz memoryless single input, single output operator such that ϕ belongs to the sector $[0, K]$, i.e.*

$$0 < y\phi(y) < K$$

then, the system is absolutely stable if there exists a constant $\gamma \geq 0$, with $1 + \lambda_n\gamma \neq 0$ for every eigenvalue λ_n of A, such that $\frac{1}{K} + (I + s\gamma)G(s)$ is strictly positive real.

Proof. Assume all systems are single input, single output and see Theorem 7.3 in [22]. □

Although this summary is not exhaustive, it presents the background that inspires this thesis. The following section introduces the Zames-Falb multipliers, the subject of study if the third Chapter of this thesis.

2.2.2 Zames-Falb Multipliers

Although they were put forward more than 40 years ago, Zames–Falb multipliers remain attractive as a solution for different problems. For example, in [27] conditions for their application to multivariable non-linearities are established; their application to repeated non-linearities is described in [47] and [48]; in [49], a less-conservative bound on the $\mathcal{L}_2[0, \infty)$ norm of a system including saturations is given using a Zames-Falb multiplier, with a correction in the methodology published in [50], and most recently in [51] a subclass of the Zames–Falb multiplier is proposed in order to remove some constraints on the non-linear system stability analysis.

Following [25], Zames–Falb multipliers are used when the non-linearity is a real-valued function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying the following properties: (*i*) $\phi(0) = 0$, (*ii*) ϕ is bounded by $C > 0$, i.e. $|\phi(x)| \leq C|x|$, and (*iii*) ϕ is monotone, i.e. $(x-y)(\phi(x)-\phi(y)) \geq 0 \forall x,y \in \mathbb{R}$.

The classical loop transformation (for instance, see [22, 52]) allows the generalization to slope-restricted non-linearities in the sector $[\alpha, \beta]$ as follows:

$$\alpha \leq \frac{\phi(x)-\phi(y)}{x-y} \leq \beta$$

for all $x \neq y$, as shown in Figure 2.5.

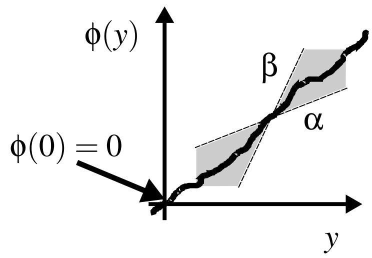


Figure 2.5: Slope-restricted monotone non-linearity in the sector $[\alpha, \beta]$

The first stability result is presented in [22] and uses only the sector boundaries and a loop transformation to find sufficient conditions for the stability of the closed loop.

Corollary 2.2.5. *[22] Let $\phi : \mathbb{R} \to \mathbb{R}$ be a bounded and sector-restricted non-linearity in the sector $[\alpha, \beta]$ with $\phi(0) = 0$. Then, the feedback interconnection in Figure 2.4 is stable if, and only if, the feedback interconnection between*

$$\tilde{G}(s) = \frac{1+\beta G(s)}{1+\alpha G(s)}$$

and a bounded monotone non-linearity $\tilde{\phi}(x)$ in Figure 2.6 with $\tilde{\phi}(0) = 0$, is stable.

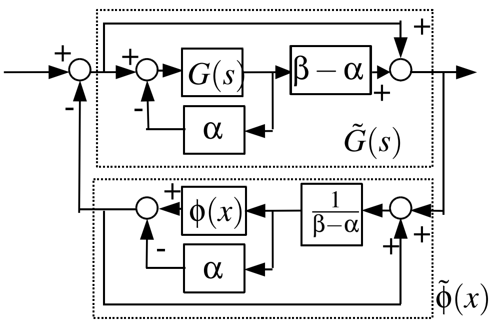


Figure 2.6: Loop transformation

Then, using the loop transformations in Figure 2.7, a set of less conservative stability conditions is derived, using a Zames-Falb multiplier. Note that the sector condition for the non-linearity is now restricted to $[0,k]$.

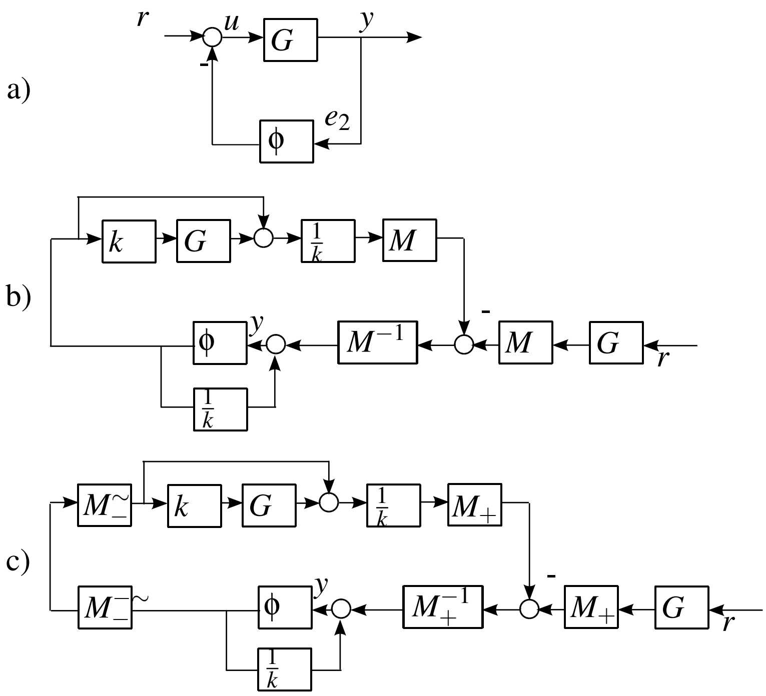


Figure 2.7: Turner loop transformation[1]

Theorem 2.2.6. *[14] Consider a feedback system from Figure 2.4, with equations*

$$u = r - \phi(y)$$
$$y = Gu$$

where $G \in \mathcal{RH}_\infty$ is a Linear Time Invariant operator represented by convolution

$$y(t) = G * u(t) = \sum_{i=1}^{\infty} g_i u(t-\tau_i) + \int_0^{\infty} g(\tau)u(t-\tau)d\tau$$

with the following properties

- $g \in \mathcal{L}_1[0,\infty)$, *i.e.* $\int_0^\infty |g(\tau)|d\tau < \infty$.
- $\{\tau_i\}$ *is a sequence in* $[0,\infty)$ *and* g_i *is a sequence in* l_1, *i.e.* $\sum_{i=1}^\infty |g_i| < \infty$.

and $\phi : \mathbb{R} \to \mathbb{R}$ is a monotone memoryless operator such that ϕ has the following properties:

- $\phi(0) = 0$;
- $[\phi(a) - \phi(b)](a - b) \geq 0 \quad \forall a, b \in \mathbb{R}$;
- *and for some* $K < \infty$, ϕ *belongs to the sector* $\phi \in [0, K]$.

Assume the feedback interconnection is well posed. Suppose that there is a non-causal convolution operator M whose impulse response is of the form

$$m(t) = \delta(t) + z(t) = \delta(t) + \sum_{-\infty}^{\infty} z_i \delta(t - t_i) - z_a(t) \tag{2.7}$$

where

$$\sum |z_i| < \infty, \quad \int_{-\infty}^{\infty} |z_a(\tau)| d\tau < \infty \quad \text{and} \quad t_i \in \mathbb{R}$$

Assume that

$$||M||_1 = \sum_{-\infty}^{\infty} |z_i| + \int_{-\infty}^{\infty} |z_a(\tau)| d\tau < 1 \tag{2.8}$$

and that there is a $\delta > 0$ *such that*

$$Re\{M(j\omega)G(j\omega)\} \geq \delta > 0, \quad \forall \omega \in \mathbb{R} \tag{2.9}$$

where $M(j\omega) = \mathcal{F}\{m(t)\}$. *Then, if either*

$$z_a(t) \geq 0 \quad \text{and} \quad z_i \geq 0, \quad \forall i \tag{2.10}$$

or

$$\phi \quad \text{is an odd function,} \tag{2.11}$$

then the system is $\mathcal{L}_2[0, \infty)$ *stable.*

Proof. See Theorem VI.9.20 in [14]. □

From now on, Equation (2.8) is refereed to as the $\mathcal{L}_1(-\infty, \infty)$ norm of the multiplier and it will be denoted for the rational transfer functions as

$$||z_a(t)||_1 = \int_{-\infty}^{\infty} |z_a(t)| dt < \infty.$$

For the summation of $\delta(.)$, Equation (2.8) will be denoted in similar fashion:

$$||z(t)||_1 = \sum_{i=-\infty}^{\infty} |z_i| < \infty.$$

Using conditions in Theorem 2.2.6, two classes of Zames-Falb multipliers can be defined: the general class when the non-linearity is odd and a subclass when the non-linearity is non-odd.

Definition 2.2.7. *Let M be a multiplier such that its impulse response $m(t)$ is defined by* (2.7)*. Then M is a Zames–Falb multiplier for odd monotone non-linearities, i.e. $M \in \mathcal{M}_{odd}$, if $m(t)$ satisfies* (2.8) *and ϕ satisfies* (2.11).

Definition 2.2.8. *Let M be a multiplier such that its impulse response $m(t)$ is defined by* (2.7)*. Then M is a Zames–Falb multiplier for monotone non-linearities, i.e. $M \in \mathcal{M}$, if $m(t)$ satisfies* (2.8) *and* (2.10).

Remark 2.2.9. *$\mathcal{M}$ is a subset of $\mathcal{M}_{odd}$.*

Remark 2.2.10. *The set of odd monotone non-linearities is a subset of the monotone non-linearities.*

The main limitation of the Zames–Falb multipliers is that they do not have an equivalent frequency-domain characterization. When the multiplier approach is used, two steps must be completed. The first step consists of finding a class of multipliers preserving the positivity of the class of non-linearities. The second step is to fulfil the open loop frequency conditions for the linear system, as stated in the IQC theorem [35, 14, 13].

Since the introduction of the Zames–Falb multipliers [25], great efforts have been made to characterize the $\mathcal{L}_1(-\infty,\infty)$ norm of the multiplier, often trading simplicity for the calculation of the norm by the complexity of the problem to the test of equation (2.9), either allowing non rational multipliers [28, 26], or using only rational multipliers with limited structure [29, 27, 30]. However, [1] showed that this was not the only source of conservatism; the use of causal multipliers can be as restrictive. In Chapter 3 this restriction will be studied in more detail.

For now, an example that illustrates the problem can be shown. In order to know how conservative a method can be, in the literature it is common to use the Kalman conjecture as the maximum slope for which the feedback interconnection of Figure 2.4 can be stable [28, 26]. Using the definition of the Nyquist value:

Definition 2.2.11. *Given a stable linear plant $G \in \mathcal{RH}_\infty$, the Nyquist value, k_k is the supremum of the values k such that kG satisfies the Nyquist criterion for all $k \in [0,K]$, i.e.*

$$k_k = \sup\{k \in \mathbb{R}^+ : \inf_{\omega\in\mathbb{R}}\{|1+kG(j\omega)|\} > 0 \quad \forall k \in [0,K]\}$$

then, the Kalman conjecture states:

Conjecture 2.2.12. *The feedback interconnection of a strictly proper plant $G \in \mathcal{RH}_\infty$, and the sector non-linearity $\phi_k : \mathbb{R} \to \mathbb{R}$ belonging to the sector $[0, k_k]$ is stable if the feedback loop $[G, k]$ is stable for all $0 \leq k < k_k$.*

This is known to be true for systems of order $n \leq 3$, but is false in general [53]. However, this result suggests that for systems of order $n \leq 3$, the feedback loop in Figure 2.4 is stable and consequently it should be possible to find a Zames-Falb multiplier that reaches that Nyquist value. The following Lemma formalizes the existence of a first order Zames-Falb multiplier:

Lemma 2.2.13. *Given a strictly proper plant $G \in \mathcal{RH}_\infty$ with order 3 or less, and $k < k_k$, then there exists a first order Zames-Falb multiplier M such that $Re\{M(j\omega)(1 + kG(j\omega))\} > 0 \quad \forall \omega \in \mathbb{R}$.*

Proof. From Theorem 1 in [53], it is known that for all $G \in \mathcal{RH}_\infty$ of order 3 or less, there exists a multiplier M of the form

$$M = \{M(s) = \tau + \theta s - \xi s^2, \tau \geq 0, \xi \geq 0, \tau + \xi + |\theta| > 0\}$$

This multiplier belongs to the set of Park multipliers when $\xi \neq 0$:

$$M_P = \{M_P(s) = a^2 + bs - s^2, a, b \in \mathbb{R}\}$$

Then, using Lemma 5.3 in [54], the result follows. □

The following example exploits this property to show how conservative a causal multiplier can be.

Take Example 1 in [1]

$$G(s) = \frac{s^2 - 0.2s - 0.1}{s^3 + 2s^2 + s + 1} \tag{2.12}$$

It is evident that for the frequency $\omega = \sqrt{\frac{13-\sqrt{129}}{20}}$, the Nyquist plot is closer to the Nyquist point when $k_k = 4.5895$. Then, if a Zames-Falb multiplier exists, it will be restricted to the sector $k \in [0, 4.5895)$. Using Lemma 2.2.13 it is known that there exists a first order Zames-Falb multiplier such that $Re\{M(j\omega)(1 + kG(j\omega))\} > 0 \quad \forall \omega \in \mathbb{R}$. The Zames-Falb multiplier $M(j\omega)$ needs to compensate for the phase response at any frequency where $1 + kG(j\omega)$ exceeds $[-90, 90]$. In the case of Example 1, Figure 2.8 shows that the phase excess is 75°.

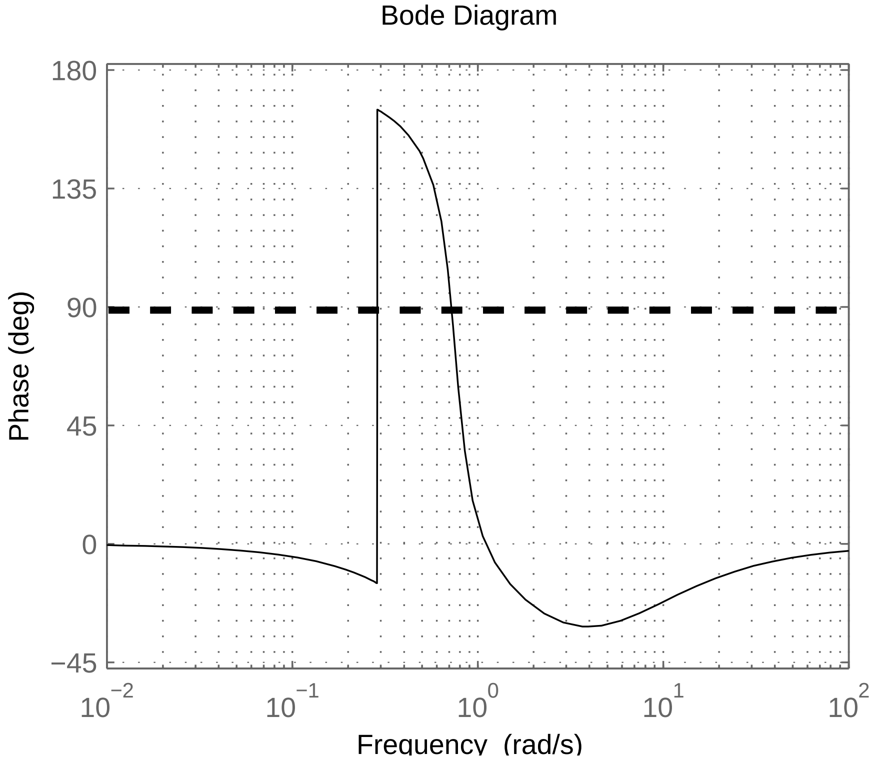


Figure 2.8: Excess of phase lead for k=4.5895

It was first in [55] that the artificial limits introduced by the choice of causal and anti-causal multipliers were noted. Given that the Zames-Falb multiplier is of order 1, the maximum phase compensation introduced by them can be calculated. Firstly note that causal multipliers have little effect on the phase lag.

Lemma 2.2.14. *Let $M(j\omega)$ be a causal Zames-Falb multiplier of order 1, there then exists a multiplier such that its phase lag reaches at most $\angle(M(j\omega)) > -\arctan(\sqrt{2}/4)$.*

Proof. Take the multiplier $M(j\omega) = 1 + Z(j\omega)$, where $Z(j\omega) = \frac{a_c}{j\omega + b_c}$ and where $a_c, b_c > 0$. It is easy to show that in order to make $M(j\omega)$ hold the $\mathcal{L}_1(-\infty,\infty)$ norm in equation (2.8), $a_c < b_c$. Then, taking the phase of the multiplier $M(j\omega)$:

$$\angle(M(j\omega)) = \angle\left(1 + \frac{a_c}{j\omega + b_c}\right) = arctan\left(-\frac{a_c\omega}{\omega^2 + b_c^2 + a_c b_c}\right),$$

it shows that the maximum of this function happens at $\omega_0 = \sqrt{b_c^2 + b_c a_c}$. Note then that

$\angle(M(j\omega_0)) = \arctan\left(-\frac{a_c}{2\sqrt{b_c^2+a_cb_c}}\right)$ and

$$-\arctan\left(\frac{a_c}{2\sqrt{b_c^2+a_c^2}}\right) < \angle(M(j\omega_0)) < -\arctan\left(\frac{a_c}{2\sqrt{2}b_c}\right).$$

Taking the limits when $a_c \to b_c$, then $\angle(M(j\omega_0)) \to -\arctan\left(\sqrt{2}/4\right)$. □

However, when the multiplier needs phase lag, an anti-causal multiplier can introduce the necessary compensation.

Lemma 2.2.15. *Let $M(j\omega)$ be an anti-causal Zames-Falb multiplier of order 1, there then exists a multiplier such that its phase lag reaches at most $\angle(M(j\omega)) > -90°$.*

Proof. Similar to Lemma 2.2.14. □

Likewise, note that anti-causal multipliers have little effect on the phase lead.

Lemma 2.2.16. *Let $M(j\omega)$ be an anti-causal Zames-Falb multiplier of order 1, there then exists a multiplier such that its phase lead reaches at most $\angle(M(j\omega)) < \arctan(\sqrt{2}/4)$.*

Proof. Similar to Lemma 2.2.14. □

But when the multiplier needs phase lead, a causal multiplier can introduce maximum compensation.

Lemma 2.2.17. *Let $M(j\omega)$ be a causal Zames-Falb multiplier of order 1, there then exists a multiplier such that its phase lead reaches at most $\angle(M(j\omega)) < 90°$.*

Proof. Similar to Lemma 2.2.14. □

In summary, these lemmas bound the phase of first order causal multipliers to $(-\arctan(\sqrt{2}/4), 90°)$ and anti-causal multipliers to $(-90°, \arctan(\sqrt{2}/4))$. Therefore, to show stability for $0 \leq K \leq 1.243$, the circle criterion is enough given that the phase remains within $(-90°, 90°)$. For the interval $1.243 < k \leq 1.805$ a causal multiplier can provide the necessary $-\arctan(\sqrt{2}/4)$, but no more. In contrast, only an anti-causal multiplier can produce multipliers with the necessary $75°$ to reach the Nyquist interval $1.243 < k \leq 4.5894$ because its phase is not constrained by its $\mathcal{L}_1(-\infty,\infty)$ norm. One of these multipliers is found by inspection:

$$Z(s) = 1 + \frac{2.1342}{s - 2.1722}. \tag{2.13}$$

Multiplier	Maximum slope k
Causal order 1	1.805
Causal order 3, Tuner[1]	2.3418
Causal High Order (order 9) [32]	2.4655
Causal Irrational [29, 26]	2.5938
Causal order 3 plus Popov multiplier [33]	3.9689
Non-causal order 1, equation (2.13)	4.5894
Nyquist Value	4.5894

Table 2.1: Maximum slope for different classes of multipliers

It is easy to see that:

- the $\mathcal{L}_1(-\infty,\infty)$ norm of the dynamic part of $Z(s)$ is

$$\left\|\frac{2.1342}{s-2.1722}\right\|_1 = 0.9825 < 1 = M(\infty)$$

- $Re\{M(j\omega)(1+4.5849G(j\omega))\} > 0 \quad \forall \omega \in \mathbb{R}.$

There are other methods in the literature, such as Turner [1, 50], where a third order Zames-Falb multiplier is found, but it only allows a maximum slope of $k \leq 2.3418$. This improvement seems to suggest that the limitations on the Zames-Falb multiplier are not only due to the order of such a multiplier but is also due its causality.

Table 2.1 shows the results of implementing different methods in the literature, artificially restricted to search only causal multipliers. Methods in [32, 29, 26] are originally designed for anti-causal multipliers as well, but by restricting the search to causal multipliers, the convergence of the methods remains unaltered, while giving an insight into the effect of the causality restriction. Although these methods deliver superior results to those of Turner [1] it is clear that when the artificial condition of causality is introduced, they are still unable to reach the maximum theoretical slope defined by Conjecture 2.2.12.

Note that if the non-linearity is restricted even more, such that Popov Theorem 2.2.4 can be applied to allow the use of the method in [33], the causal search is still unable to reach the Nyquist value.

Chapter 3 will study a complement to the method presented in [1] in order to extend its applicability when an anti-causal multiplier is required.

2.3 Motivation for the IQC Theorem

The successful results presented in Chapter 3 inspired the work in Chapter 4. The inclusion of non-causal Zames-Falb multipliers was shown to remove conservatism in the test for absolute stability. The objective of Chapter 4 is to find the conditions under which dynamic multipliers can be used with the Dissipativity Theorem.

The objective of this analysis is to present sufficient conditions for $\mathcal{L}_2[0,\infty)$ stability for the system presented in Figure 2.9, where $\Delta_1 : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^l$ and $\Delta_2 : \mathcal{L}_{2e}^l \to \mathcal{L}_{2e}^m$ are non-linear causal operators. Among the existing solutions to this problem are the Dissipativity Theorem, Passivity Theorem and Small Gain Theorem. These successful tools present a stability condition for feedback closed loop systems using the open loop properties of the interconnected systems [39, 21, 20, 38, 13, 14].

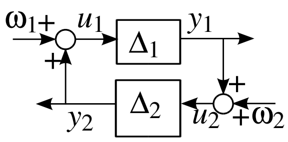


Figure 2.9: Closed loop system

$$
\begin{aligned}
y_1 = \Delta_1(u_1), y_2 = \Delta_2(u_2),\\
u_1 = \omega_1 + y_2,\\
u_2 = \omega_2 + y_1.
\end{aligned}
\tag{2.14}
$$

However, the use of dynamic multipliers with these theorems is not a trivial task. This section will illustrate the most common ways multipliers are introduced into the Small Gain Theorem and the Passivity Theorem, and the fundamental restrictions demanded for any multiplier candidate.

The interconnection of two non-linear systems is well understood in the Dissipativity theory. The seminal work on Dissipativity [39], and its extension presented as cyclo-dissipative systems [37], give an extensive study on stability using constant and dynamic causal multipliers. However, the use of causal multipliers is a necessary restriction because the analysis is made using causal time domain inner products [52, 24]. Passivity does allow to use dynamical anti-causal multipliers, but those multipliers have to hold a so called canonical factorization [14]. Note that the Dissipativity Theorem and Passivity Theorem are defined only for negative feedback, in contrast to the Small Gain Theorem which allows positive and negative feedback.

The following definition of Dissipativity exemplifies the class of multipliers allowed.

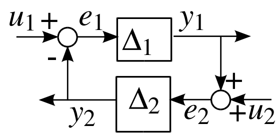


Figure 2.10: Dissipativity feedback loop

Theorem 2.3.1. *[24] Let $\Delta_1 : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^l$ and $\Delta_2 : \mathcal{L}_{2e}^l \to \mathcal{L}_{2e}^m$ be two causal operators such that the feedback interconnection given by Figure 2.10 is well posed. Furthermore, let $Q_1, R_2 : \mathcal{L}_{2e}^l \to \mathcal{L}_{2e}^l$, $Q_2, R_1 : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^m$, and $S_1, S_2^{\sim} : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^l$ be bounded causal operators, Q_1, Q_2, R_1, R_2 are self adjoint, scalars $\sigma, \varepsilon > 0$ exist and*

$$\hat{Q} = \begin{bmatrix} Q_1 + \sigma R_2 & -S_1 + \sigma S_2^T \\ -S_1^T + \sigma S_2 & R_1 + \sigma Q_2 \end{bmatrix} \tag{2.15}$$

satisfies $< y, \hat{Q} y >_T \leq -\varepsilon < y, y >_T \quad \forall y \in \mathcal{L}_{2e}^{l+m}$. If:

$$\left\langle \begin{bmatrix} \Delta_1(u_1) \\ u_1 \end{bmatrix}, \begin{bmatrix} Q_1 & S_1 \\ S_1^T & R_1 \end{bmatrix} \begin{bmatrix} \Delta_1(u_1) \\ u_1 \end{bmatrix} \right\rangle_T \geq 0 \quad \forall u_1 \in \mathcal{L}_{2e}^m \tag{2.16}$$

and

$$\left\langle \begin{bmatrix} u_2 \\ \Delta_2(u_2) \end{bmatrix}, \begin{bmatrix} R_2 & S_2^T \\ S_2 & Q_2 \end{bmatrix} \begin{bmatrix} u_2 \\ \Delta_2(u_2) \end{bmatrix} \right\rangle_T \geq 0 \quad \forall u_2 \in \mathcal{L}_{2e}^l \tag{2.17}$$

then, the feedback interconnection of Δ_1 and Δ_2 given in Figure 2.10 is $\mathcal{L}_2[0, \infty)$ stable.

The conditions for the previous theorem apparently allow for the use of dynamic multipliers Q_1, Q_2, R_1, R_2. However, when these multipliers are set to belong to $\mathcal{RL}_\infty$, the only kind of allowed multiplier that is causal and self-adjoint at the same time is a constant matrix. Furthermore, when the proof for Theorem 2.3.1 is studied in detail, it is also necessary to show that the adjoint of $S_1, S_2^{\sim}$ is stable, and again, if the set is restricted to $\mathcal{RL}_\infty$, this means that they are constant matrices.

Passivity (Theorem 2.1.20), alternatively, presents a way of taking advantage of non-causal multipliers. Non-causal multipliers are specially suitable for the description of memoryless non-linear operators and sometimes they are necessary to avoid conservatism, as presented in Section 2.2. However, Passivity has the disadvantage that it requires the strict factorization of the selected multipliers because it shows stability exists using the construction of an equivalent system [14].

The following is the definition of an IQC. This mathematical tool allows one to characterize a class of non-linearities using frequency domain conditions.

Definition 2.3.2. *[35] Suppose* $\Pi : j\mathbb{R} \to \mathbb{C}^{(l+m)\times(l+m)}$ *is a bounded measurable Hermitian function and* $\Delta : \mathcal{L}_{2e}^{l} \to \mathcal{L}_{2e}^{m}$ *is a causal bounded operator. The operator* Δ *is said to satisfy the IQC defined by* Π *if*

$$\left\langle \begin{bmatrix} U(j\omega) \\ V(j\omega) \end{bmatrix}, \Pi(j\omega) \begin{bmatrix} U(j\omega) \\ V(j\omega) \end{bmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0 \quad \forall U \in \mathcal{H}_2^l, V = \mathcal{F}\{\Delta(\mathcal{F}^{-1}\{U\})\}. \quad (2.18)$$

Using this frequency condition, [35] presents the celebrated IQC Theorem. This theorem shows the stability of the feedback interconnection of a linear system with a class of non-linearities that can be memoryless or dynamic; the only limitation is that this non-linearity has to be $\mathcal{L}_2$ bounded. Note that once it is established that a non-linearity belongs to a class of IQC, the only information that is required to show stability is the multiplier $\Pi(j\omega)$ from Definition 2.3.2. Loosely speaking, the IQC Theorem uses well-known frequency domain conditions and presents a set of requirements under which frequency domain analysis is sufficient to show the stability of the non-linear interconnection in Figure 2.9.

Theorem 2.3.3. *(IQC Theorem [35]) Let* $G \in \mathcal{RH}_\infty^{l\times m}$*, and let* $\Delta : \mathcal{L}_{2e}^{l} \to \mathcal{L}_{2e}^{m}$ *be a bounded causal operator. Assume that:*

1. $\forall\tau \in [0,1]$*, the interconnection of* G *and* $\tau\Delta$ *is well posed;*
2. $\forall\tau \in [0,1]$*,* $\tau\Delta$ *satisfies the IQC defined by* $\Pi(j\omega)$*;*
3. *there exists* $\varepsilon_1 > 0$ *such that*
$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon_1 I \quad \forall\omega \in \mathbb{R}$$

Then, the feedback interconnection of G *and* Δ *is* $\mathcal{L}_2[0,\infty)$ *stable.*

Proof. See Theorem 1 in [35]. □

2.3.1 Stability Theorems based on loop transformations

The foundation of this analysis depends on the following definitions.

Definition 2.3.4. *[42, p. 2305-2306]* $\Pi \in \mathcal{RL}_\infty^{(l+m)\times(l+m)}$ *is a Hermitian function with the following structure:*

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^{\sim} & \Pi_{22} \end{pmatrix},$$

where $\Pi_{11} \in \mathcal{RL}_\infty^{l\times l}$, $\Pi_{12} \in \mathcal{RL}_\infty^{l\times m}$ *and* $\Pi_{22} \in \mathcal{RL}_\infty^{m\times m}$, Π *is called a strict Positive Negative IQC (PN-IQC) multiplier if positive scalars* $q, r > 0$ *exist, such that*

- $\Pi_{11}(j\omega) \geq qI_l \quad \forall\omega \in \mathbb{R}$,
- $\Pi_{22}(j\omega) \leq -rI_m \quad \forall\omega \in \mathbb{R}$.

Definition 2.3.5. *[42] A rational Function* $\Pi : j\mathbb{R} \to \mathbb{C}^{(l+m)\times(l+m)}$ *admits a hard IQC factorization if* $\hat{\Phi} \in \mathcal{RH}_\infty^{(l+m)\times(l+m)}$ *and* $J \in \mathbb{C}^{(l+m)\times(l+m)}$ *exist, such that* $\Pi = \hat{\Phi}^{\sim} J \hat{\Phi}$ *and any bounded causal operator* Δ *which satisfies the IQC defined by* Π *also satisfies*

$$\int_0^T \left(\Phi \begin{bmatrix} u \\ \Delta(u) \end{bmatrix}\right)^T J\Phi \begin{bmatrix} u \\ \Delta(u) \end{bmatrix} dt \geq 0$$

$\forall T \geq 0, u \in \mathcal{L}_2^l[0,\infty)$.

The following theorem shows that if an operator Δ holds a strict PN-IQC, it will also hold the same inner product for all truncated times.

Theorem 2.3.6. *[42] Let* $\Pi \in \mathcal{RL}_\infty^{(l+m)\times(l+m)}$ *be a strict PN-IQC multiplier. There then exists a hard IQC factorization* $\Pi = \Phi^{\sim} M \Phi$ *such that* $M := \begin{pmatrix} I_l & 0 \\ 0 & -I_m \end{pmatrix}$ *and* $\Phi := \begin{pmatrix} Y & 0 \\ P & X \end{pmatrix}$ *where* $Y, Y^{-1} \in \mathcal{RH}_\infty^{l\times l}$, $X \in \mathcal{RH}_\infty^{m\times m}$ *and* $P \in \mathcal{RH}_\infty^{m\times l}$.

Proof. See [42]. □

The results introduced in this section extend the use of dissipative stability [24] to non constant matrices. The resulting lemmas use IQC multipliers [14, 35] to describe the causal bounded operators Δ_1 and Δ_2. The main analytical tool used in these results are the strict PN-IQC multipliers, which allows to infer time domain conditions from the frequency dependent IQCs.

2.3.1.1 Stability analysis using the Small Gain Theorem

The first attempt to produce an extension of Dissipativity using IQC multipliers will be based on the Small Gain Theorem. Note that this theorem will use the strict PN-IQC multiplier to describe the properties of the non-linearities and then uses IQC hard factorization to formulate the stability conditions in the time domain. The latest step is inspired by the proof of Dissipativity in [24].

Lemma 2.3.7. *Let two causal bounded operators* $\Delta_1 : \mathcal{L}_{2e}^l \to \mathcal{L}_{2e}^m$, $\Delta_2 : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^l$ *be interconnected as in Figure 2.9. Suppose that:*

1. *the feedback loop* $[\Delta_1, \Delta_2]$ *is well posed,*
2. *there exists self-adjoint* $\Pi_1(j\omega), \Pi_2(j\omega) \in \mathcal{RL}_\infty^{(l+m)\times(l+m)}$ *with* $\Pi_{1,22}(j\omega) > 0$ *and* $\Pi_{2,11}(j\omega) > 0\ \forall\omega \in \mathbb{R}\cup\{\infty\}$ *such that*

$$\left\langle \begin{pmatrix} V_1(j\omega) \\ U_1(j\omega) \end{pmatrix}, \Pi_1(j\omega) \begin{pmatrix} V_1(j\omega) \\ U_1(j\omega) \end{pmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$

$$\forall U_1 \in \mathcal{H}_2^l, V_1 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{U_1\})\} \quad (2.19)$$

and

$$\left\langle \begin{pmatrix} U_2(j\omega) \\ V_2(j\omega) \end{pmatrix}, \Pi_2(j\omega) \begin{pmatrix} U_2(j\omega) \\ V_2(j\omega) \end{pmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$

$$\forall U_2 \in \mathcal{H}_2^m, V_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{U_2\})\}. \quad (2.20)$$

3. *there exists positive scalars* $\varepsilon, \lambda > 0$ *such that*

$$\Pi_1(j\omega) + \lambda\Pi_2(j\omega) \leq -\varepsilon I_{l+m} \quad \forall\omega \in \mathbb{R}, \quad (2.21)$$

4. *let* $H = \begin{bmatrix} 0 & I_m \\ I_l & 0 \end{bmatrix}$, *then noting that*

$$\begin{pmatrix} \hat{Y}_1^\sim(j\omega)\hat{Y}_1(j\omega) - \hat{P}_1^\sim(j\omega)\hat{P}_1(j\omega) & -\hat{P}_1^\sim(j\omega)\hat{X}_1(j\omega) \\ -\hat{X}_1^\sim(j\omega)\hat{P}_1(j\omega) & -\hat{X}_1^\sim(j\omega)\hat{X}_1(j\omega) \end{pmatrix} = H^T\Pi_1(j\omega)H,$$

$$\begin{pmatrix} \hat{Y}_2^\sim(j\omega)\hat{Y}_2(j\omega) - \hat{P}_2^\sim(j\omega)(j\omega)\hat{P}_2(j\omega) & -\hat{P}_2^\sim(j\omega)\hat{X}_2(j\omega) \\ -\hat{X}_2^\sim(j\omega)\hat{P}_2(j\omega) & -\hat{X}_2^\sim(j\omega)\hat{X}_2(j\omega) \end{pmatrix} = \Pi_2(j\omega),$$

assume that:

$$\begin{pmatrix} X_1 & P_1 \\ \sqrt{\lambda}P_2 & \sqrt{\lambda}X_2 \end{pmatrix}^{-1} \in \mathcal{RH}_\infty \tag{2.22}$$

Then the feedback interconnection $[\Delta_1, \Delta_2]$ *is* $\mathcal{L}_2[0,\infty)$ *stable.*

Proof. See Appendix A.1. □

The main source for conservatism in Lemma 2.3.7 is the need to find the PN-IQC multipliers that describe the non-linearities such that they hold the causal inverse in equation (2.22). The source of the problem is the lack of time domain information available from equation (2.21). This kind of restriction is also present in recent works such as [56], where time domain conditions are used in the proof, even when frequency domain conditions seem to be the most appropriate tool.

Note that by making Π_1, Π_2 constant matrices, Dissipativity Theorem 7.3 in [24], and Dissipativity Theorem from Proposition 5.8 in [10] can be recovered for the negative feedback interconnection of $\mathcal{L}_2[0,\infty)$ stable operators (output passive systems).

2.3.1.2 Stability analysis using the Passivity Theorem

The second attempt to produce an extension of Dissipativity using IQC multipliers will be based on the Passivity Theorem. This attempt is inspired by the mixed small gain and passivity properties that some systems can hold[56]. However, a new problem arises: the non-linearity is not represented using a standard IQC in the frequency domain, but it is represented using time domain integrals instead.

Lemma 2.3.8. *Let two causal bounded operators* $\Delta_1 : \mathcal{L}_{2e}^l \to \mathcal{L}_{2e}^m$, $\Delta_2 : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^l$ *be interconnected as in Figure 2.11.*

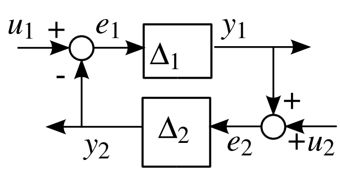


Figure 2.11: Closed loop proposition

Suppose that:

1. *the feedback interconnection* $[\Delta_1, \Delta_2]$ *is well posed,*

2. *there exists PN-IQC* $\tilde{\Pi}_1(j\omega), \tilde{\Pi}_2(j\omega) \in \mathcal{RL}_\infty^{(l+m)\times(l+m)}$

$$\Pi_1(j\omega) = \begin{bmatrix} \tilde{Y}_1(j\omega) & 0 \\ \tilde{P}_1(j\omega) & \tilde{X}_1(j\omega) \end{bmatrix}^\sim \begin{bmatrix} I_l & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} \tilde{Y}_1(j\omega) & 0 \\ \tilde{P}_1(j\omega) & \tilde{X}_1(j\omega) \end{bmatrix}, \tag{2.23}$$

$$\tilde{\Pi}_2(j\omega) = \begin{bmatrix} \tilde{Y}_2(j\omega) & 0 \\ \tilde{P}_2(j\omega) & \tilde{X}_2(j\omega) \end{bmatrix}^\sim \begin{bmatrix} I_m & 0 \\ 0 & -I_l \end{bmatrix} \begin{bmatrix} \tilde{Y}_2(j\omega) & 0 \\ \tilde{P}_2(j\omega) & \tilde{X}_2(j\omega) \end{bmatrix}, \tag{2.24}$$

such that

$$\left\langle \begin{bmatrix} Y_1 & 0 \\ P_1 & X_1 \end{bmatrix} \begin{pmatrix} (u_1(t))_T \\ (\Delta_1 u_1(t))_T \end{pmatrix}, \begin{bmatrix} I_l & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} Y_1 & 0 \\ P_1 & X_1 \end{bmatrix} \begin{pmatrix} (u_1(t))_T \\ (\Delta_1 u_1(t))_T \end{pmatrix} \right\rangle_{\mathcal{L}_2} \geq 0 \quad \forall u_1 \in \mathcal{L}_2^l[0,\infty), T \geq 0 \tag{2.25}$$

and

$$\left\langle \begin{bmatrix} Y_2 & 0 \\ P_2 & X_2 \end{bmatrix} \begin{pmatrix} (u_2(t))_T \\ (\Delta_2 u_2(t))_T \end{pmatrix}, \begin{bmatrix} I_m & 0 \\ 0 & -I_l \end{bmatrix} \begin{bmatrix} Y_2 & 0 \\ P_2 & X_2 \end{bmatrix} \begin{pmatrix} (u_2(t))_T \\ (\Delta_2 u_2(t))_T \end{pmatrix} \right\rangle_{\mathcal{L}_2} \geq 0 \quad \forall u_2 \in \mathcal{L}_2^m[0,\infty), T \geq 0 \tag{2.26}$$

where $\tilde{\Pi}_1$ *and* $\tilde{\Pi}_2$ *are the Fourier transforms of* Π_1 *and* Π_2.

3. *there exists positive scalars* $\varepsilon, \lambda > 0$ *such that*

$$\begin{pmatrix} 0 & I_m \\ I_l & 0 \end{pmatrix} \Pi_1(j\omega) \begin{pmatrix} 0 & I_l \\ I_m & 0 \end{pmatrix} + \lambda \Pi_2(j\omega) \leq -\varepsilon I_{l+m} \quad \forall \omega \in \mathbb{R}. \tag{2.27}$$

4. $-\lambda X_2^\sim P_2 = P_1^\sim X_1$.

Then the feedback interconnection $[\Delta_1, \Delta_2]$ *is* $\mathcal{L}_2[0,\infty)$ *stable.*

Proof. See Appendix A.2. □

An important question arises from this theorem. Why not use IQC to describe the non-linear system properties?. The information contained in the hard IQC needs to be reformulated such that it allows to use this information in the time domain. Take IQC from equation (2.18),

$$\left\langle \begin{bmatrix} U_i(j\omega) \\ V_i(j\omega) \end{bmatrix}, \Pi_i(j\omega) \begin{bmatrix} U_i(j\omega) \\ V_i(j\omega) \end{bmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0 \quad \forall U_i \in \mathcal{H}_2^l, V_i = \mathcal{F}\{\Delta_i(\mathcal{F}^{-1}\{U_i\})\},$$

then use Theorem 2.3.6 and the Parseval's Theorem to obtain the following equivalent time domain description of the non-linearity

$$\begin{aligned}
&\int_0^\infty (Y_i u(t))^T Y_i u(t) - (P_i u(t) + X_i \Delta_i u(t))^T (P_i u(t) + X_i \Delta_i u(t)) dt \geq 0 \quad \forall u \in \mathcal{L}_2^l[0,\infty) \\
&\iff \int_0^\infty (Y_i u_T(t))^T Y_i u_T(t) - (P_i u_T(t) + X_i \Delta_i u_T(t))^T (P_i u_T(t) + X_i \Delta_i u_T(t)) dt \geq 0 \\
&\qquad \forall u_T \in \mathcal{L}_{2e}^l. \quad (2.28)
\end{aligned}$$

In order to transform the non-linear terms from infinite time response $X_i \Delta_i u(t)_T$ to a truncated time response $X_i(\Delta_i u(t))_T$, add and subtract $\int_0^\infty (P_i u_T(t) + (X_i \Delta_i u(t))_T)^T (P_i u_T(t) + (X_i \Delta_i u(t))_T) dt$ to equation (2.28), i.e.:

$$\begin{aligned}
&\int_0^\infty (Y_i u_T(t))^T Y_i u_T(t) dt - \int_0^\infty u_T(t)^T P_i^T P_i u_T(t) dt - 2\int_0^\infty u_T(t)^T P_i^T X_i (\Delta_i u_T(t))_T dt \\
&\qquad - \int_0^\infty (\Delta_i u_T(t))_T^T X_i^T X_i (\Delta_i u_T(t))_T dt \\
&\qquad + 2\int_T^\infty u_T(t)^T P_i^T X_i ((\Delta_i u_T(t))_T - \Delta_i u_T(t)) dt \\
&+ \int_T^\infty (\Delta_i u_T(t))_T^T X_i^T X_i (\Delta_i u_T(t))_T dt - \int_T^\infty (\Delta_i u_T(t))^T X_i^T X_i \Delta_i u_T(t) dt \geq 0 \quad \forall u_T \in \mathcal{L}_{2e}^l. \\
&\qquad (2.29)
\end{aligned}$$

then, in order to show conditions (2.25) and (2.26) from IQC, i.e.

$$\begin{aligned}
&\int_0^\infty (Y_i u_T(t))^T Y_i u_T(t) - (P_i u_T(t) + X_i(\Delta_i u_T(t))_T)^T (P_i u_T(t) + X_i(\Delta_i u_T(t))_T) dt \geq 0 \\
&\qquad \forall u_T \in \mathcal{L}_{2e}^l \quad (2.30)
\end{aligned}$$

it is necessary to show that

$$\begin{aligned}
&2\int_T^\infty (P_i u_T(t))^T X_i ((\Delta_i u_T(t))_T - \Delta_i u_T(t)) dt \\
&\qquad + \int_T^\infty (X_i(\Delta_i u_T(t))_T)^T X_i (\Delta_i u_T(t))_T dt - \int_T^\infty (X_i \Delta_i u_T(t))^T X_i \Delta_i u_T(t) dt \leq 0 \\
&\qquad \forall u_T \in \mathcal{L}_{2e}^l \quad (2.31)
\end{aligned}$$

so this term can be safely removed without affecting the positivity of equation (2.29).

It is clear that for memoryless Δ_i, the solution is trivial, because $(\Delta_i u_T(t))_T = \Delta_i u_T(t)$, i.e. equation (2.32) is equal to zero.

It is also clear that for constant P_i, X_i the inequality (2.32) is true, because for $u_T =$

$0 \quad \forall t > T$ and $(\Delta_i u_T(t))_T = 0 \quad \forall t > T$:

$$\begin{aligned}
&2\int_T^\infty (P_i u_T(t))^T X_i((\Delta_i u_T(t))_T - \Delta_i u_T(t))dt \\
&\qquad + \int_T^\infty (X_i(\Delta_i u_T(t))_T)^T X_i(\Delta_i u_T(t))_T dt - \int_T^\infty (X_i \Delta_i u_T(t))^T X_i \Delta_i u_T(t) dt \\
&= 2\int_T^\infty 0 P_i^T X_i(0 - \Delta_i u_T(t))dt + \int_T^\infty 0 X_i^T X_i 0 dt - \int_T^\infty \Delta_i u_T(t)^T X_i^T X_i \Delta_i u_T(t) dt \\
&= -\int_T^\infty \Delta_i u_T(t)^T X_i^T X_i \Delta_i u_T(t) dt \leq 0 \quad \forall u_T \in \mathcal{L}_{2e}^l
\end{aligned} \tag{2.32}$$

It is worthwhile noting that this information is enough to obtain the Passivity Theorem with anti-causal multipliers as a corollary, using the conditions of Theorem VI.9.20 in [14].

Corollary 2.3.9. *Consider the feedback system of Figure 2.9, where $\Delta_1 : \mathcal{L}_2(-\infty,\infty) \rightarrow \mathcal{L}_2(-\infty,\infty)$, $\Delta_2 : \mathcal{L}_2(-\infty,\infty) \rightarrow \mathcal{L}_2(-\infty,\infty)$ are two causal bounded operator dynamical systems with gain $||\Delta_1||_{\mathcal{L}_2\rightarrow\mathcal{L}_2}, ||\Delta_2||_{\mathcal{L}_2\rightarrow\mathcal{L}_2}$. Suppose the negative feedback interconnection $[\Delta_1, -\Delta_2]$ is well posed. Let there be a non-causal multiplier $\hat{M}, \hat{M}^{-1} \in \mathcal{RL}_\infty$ with canonical factorization $M = M_- M_+$, where $M_-^\sim, M_+ \in \mathcal{RH}_\infty$, such that*

1. *for some $\delta > 0$*

$$< U_1, \hat{M}\Delta_1(U_1) >_{\mathcal{L}_2(-\infty,\infty)} \geq \frac{\delta}{2} < U_1, U_1 >_{\mathcal{L}_2(-\infty,\infty)} \quad \forall U_1 \in \mathcal{L}_2(-\infty,\infty) \tag{2.33}$$

2. *and it holds that:*

$$< U_2, \Delta_2(\hat{M}^\sim U_2) >_{\mathcal{L}_2(-\infty,\infty)} \geq 0 \quad \forall U_2 \in \mathcal{L}_2(-\infty,\infty) \tag{2.34}$$

then, the system is $\mathcal{L}_2[0,\infty)$ stable.

Proof. See Appendix A.3. □

For the sake of argument, a detailed proof of Corollary 2.3.9 is presented. The main objective of this exercise is to show that appropriate $\Pi_1(j\omega)$ and $\Pi_2(j\omega)$ exists such that Lemma 2.3.8 can be used.

However, due to the complexity of equation (2.32), dynamic multipliers for Dissipativity remain an open question using this approach.

This chapter has introduced the two classical approaches to studying non-linear systems and it has shown that using the apparently advantageous factorization from [42] results in a loss of information in the IQCs representing the non-linear systems. In the literature two alternative solutions are given for the problem; the first used in this work consists of the interconnection of circular graphs [43]. The second approach was presented in [57] and uses non standard loop transformations. However, these options are not discussed in this thesis.

Chapter 3

Search for SISO Zames-Falb multipliers

Acknowledgements

The material on Sections 2.2, 3.2.3, 3.3 and 3.4 was published in Systems & Control Letters, co-authored with Joaquin Carrasco, Alexander Lanzon and William P. Heath, entitled ”LMI searches for anti-causal and non-causal rational Zames-Falb multipliers”[58].

3.1 Introduction

This Chapter studies the stability of the feedback interconnection between a linear system and a memoryless non-linearity, typically described by a conic condition. Theorems such as the Passivity Theorem and the Small Gain Theorem [8, 9] test an open loop frequency-domain condition on the linear part of the system interconnection, and then conclude stability of the original non-linear problem. The success of these techniques lies in their simplicity, because they use analysis of open–loop properties to show the stability of closed loop systems.

The class of Zames–Falb multipliers is defined in Theorem 2.2.6 as follows: $M(s)$ is a Zames–Falb multiplier if its unit impulse response $m(t)$, is given by

$$m(t) = \delta(t) + \sum_{i=-\infty}^{\infty} z_i \delta(t - t_i) + z_a(t), \tag{3.1}$$

where $t_i \in \mathbb{R}$, $z_i \in \mathbb{R}$, δ is the Dirac delta, $z_a(t)$ is an absolutely integrable function, and

$$\sum_{i=-\infty}^{\infty} |z_i| + \int_{-\infty}^{\infty} |z_a(t)| dt < 1. \tag{3.2}$$

Given a linear plant $G(s)$, an appropriate multiplier within the class of Zames-Falb should be found such that $M(s)G(s) > 0$ is strictly positive real. A complete solution of the above problem remains open.

To date, only partial solutions have been presented in the literature. In this work only three structures are studied. Although they use slightly different non-linearities, multipliers for odd monotone non-linearities can be generated by approximating $\mathcal{L}_1(-\infty,\infty)$ norm using the triangle inequality [31]. However, no significant improvement is achieved for this comparison. Instead, this work uses the fact that the odd monotone non-linearities are a subset of the monotone non-linearities, and compares the performance assuming all non-linearities are odd.

We can summarize the three solutions as follows: In [28, 26, 29], the multiplier is construed by finite summation of Dirac delta distributions, with $z_a(t) = 0$ for all $t \in \mathbb{R}$; in [31], the multiplier has only one Dirac delta at $t = 0$, i.e. $z_i = 0 \quad \forall i \in \mathbb{Z}$, and the Laplace transform of $z_a(t)$ has a specific choice of its poles, in other words, the multiplier is formed by a summation of exponential functions. Finally, in [1], as in [31], the multiplier only has a Dirac delta at $t = 0$, and the Laplace transform of $z_a(t)$ is limited to be a causal transfer function of the same order as the linear system $G(s)$.

The objective of this Section is to summarize the properties of each method, study the conditions to compute a multiplier, and carry out a comparison to test their performance in several examples.

3.2 The multiplier synthesis

The solution given by Zames and Falb guarantees the absolute stability if a multiplier can be found. The following statement formalizes the problem:

Problem 3.2.1. *Given a linear plant $G(s) \in \mathcal{RH}_\infty$, find the largest $k > 0$ such that the feedback interconnection of G and a slope-restricted (and odd) monotone non-linearity in the sector $[0,k]$ is absolutely stable, or alternatively, find the largest $k > 0$ such that there exists a Zames-Falb multiplier $M(s) \in \mathcal{M}$ ($M(s) \in \mathcal{M}_{odd}$ respectively) such that:*

$$\mathrm{Re}\{M(j\omega)(1+kG(j\omega))\} \geq \varepsilon \quad \forall \omega \in \mathbb{R} \tag{3.3}$$

for some constant $\varepsilon > 0$.

In this section, basic descriptions of the three methods for solving Problem 3.2.1 are given. All of them require a linear search in k. Each one of the following methods focuses only in one part of the general multiplier in equation 3.1, either the summation of delta distributions, rational causal transfer functions and rational non-causal transfer functions. None of the methods successfully find the biggest sector boundary for all plants. This issue is discussed in Section 3.4.

3.2.1 Summation of $\delta(.)$ distributions

Safonov [28] proposes a parametrization of the Zames–Falb multiplier using only the $\delta(t-t_i)$ distributions in equation (3.1). Making $z_a(t) = 0$ and taking the restrictions of a Zames—Falb multiplier presented in equation (3.2), the method considers a finite N subset of the $\delta(t-t_i)$ summation, i.e.

$$m_N(t) = \delta(t) - h(t) = \delta(t) - \sum_{i=0}^{N} z_i \delta(t-t_i), \tag{3.4}$$

with a Laplace transform written as follows

$$M_N(s) = 1 - \sum_{k=1}^{N} z_i e^{t_i s}, \tag{3.5}$$

where $\sum_{k=1}^{N} z_i < 1$. It is worthwhile noting that the method is developed only for a search in the set $\mathcal{M}$, i.e. $z_i > 0 \quad \forall i \in \mathbb{Z}_+$. Then, Problem 3.2.1 is redefined as follows:

Problem 3.2.2 ([28, 26]). *Given a linear plant $G(s) \in \mathcal{RH}_\infty$, find the largest $k > 0$ such that the feedback interconnection of $G(s)$ and a slope-restricted non-linearity $[0,k]$ is absolutely stable, or alternatively, find the largest $k > 0$ such that there exists an integer N and two sequences $z = \{z_1, ..., z_N\}$ and $T_N = \{t_1, ..., t_N\}$ such that:*

1. $\sum_{k=0}^{N} z_i < 1$
2. *for some* $\varepsilon > 0$

$$\mathrm{Re}\{(1 - \sum_{k=1}^{N} z_i e^{j t_i \omega})\tilde{G}(j\omega)\} \geq \varepsilon \quad \forall \omega \in \mathbb{R} \tag{3.6}$$

where $\tilde{G}(s) = (1 + kG(s))$, and

3. $z_i \geq 0 \quad \forall i.$

For a fixed k, the problem is to find a Zames–Falb multiplier that maximizes the following function:

$$\psi_N(z) = \min_{\omega \in \mathbb{R}} \mathrm{Re}\{(1 - \sum_{k=1}^{N} z_i e^{jt_i\omega})\tilde{G}(j\omega)\}. \tag{3.7}$$

Algorithm 1 summarizes the search for the multiplier [26]. Step 3) in Algorithm 1 is developed as the optimization Algorithm 2. These instructions will generate a linear program that can be solved using any commercial solver.

Algorithm 1 Summation of $\delta(t)$ distributions

Step 1)
INITIALIZE PROGRAM
  $\omega_1 \leftarrow 0$
  $[\omega_2, ..., \omega_{N\omega}] \leftarrow$frequencies where $|\angle\tilde{G}(j\omega)| = 1^\circ$
  Set $N_T \leftarrow$ length of T
  Set $T_N = [t_1, ..., t_{N_t}] \leftarrow [-\pi/\omega_2, ..., -\pi/\omega_{N_\omega}]$ {initialization taken from [28]}
  Set $\varepsilon \leftarrow$ small number > 0 {complementary initialization taken from [26]}
Step 2)
  SOLVE $\lambda_0^\top = \min_{\omega \in \mathbb{R}} Re\{\tilde{G}(j\omega)\}$
Step 3)
  SOLVE $\lambda_N^\top = \max_{z \in Z_N} \psi_N(z)$ (Algorithm 2)
if $|\lambda_N^\top - \lambda_{N-1}^\top| < \varepsilon$ **then**
  go to Step 4
else
  Set $T_{N+1} \leftarrow T_N \bigcup t_{N+1}$ (optimize equation (15) in [26])
  Set $N \leftarrow N + 1$ and go to Step 3
end if
Step 4)
if $\lambda_N > 0$ **then**
  The closed loop system is absolutely stable.
else
  Inconclusive.
end if

Gapski [26] developed a method to find the new time t_{N+1} iteratively by optimizing of the right directional derivative of $\psi_{N+1}(z^l)$ in the direction of t_{n+1} using only the optimal solution z^l obtained with the vector T_N. This solution is implemented in Step 3 of Algorithm 1. Chang [29] proposes a method to increase the performance of Algorithm 2 in order to optimize the search of the time t_{N+1}, however, in his work an initial t_1 is

Algorithm 2 SOLVE $\lambda_N = \max_{z \in Z_N} \psi_N(z)$ (step 3 in Algorithm 1, taken from Step b) in [26])

Step a)
For N fixed, set $\varepsilon > 0$ sufficiently small, $L \leftarrow 1$
$z_1 = 0 \in \mathbb{R}^N$ and calculate $\psi_N(z^1) \in \delta\psi_N(z^1)$,
and $\mu^1 \in \delta\psi_N(z^1)$ from equation (12) in [26]
Step b)
Solve $\lambda^{L+1} = \max_{\lambda \leq 1, z \in Z_N} \{\lambda : \psi_N(z_l)$
$+ < \mu^l, z - z^l > \geq \lambda, l = 1, ..., L\}$
and let z_{L+1} be the optimal solution
Step c)
if $|\lambda^l - \psi_N(z^l)| < \varepsilon$ **then**
stop
else
$\lambda_N \leftarrow \lambda^{L+1}$, $L \leftarrow L+1$
calculate $\psi_N(z^{L+1}) \in \delta\psi_N(z^{L+1})$ and
calculate $\mu^{L+1} \in \delta\psi_N(z^{L+1})$
go back to step b)
end if

chosen in each case so that the program starts close to the optimal values; this choice minimizes the effect of the latest improvement. Algorithms 1 and 2 are reproduced here for the sake of completeness.

The main disadvantage of this method is the numerical search over frequency, where there is no analytical tool to confirm stability. This problem is made evident by Examples 5 and 6 in Section 3.4.

3.2.2 Summation of exponential functions

Impulses are impractical to implement and require high frequency analysis. In order to approximate the effect of a multiplier constructed by impulses, but based on a summation of exponential functions, the following lemma is used [31]:

Lemma 3.2.3 ([59])**.** *If $f(t) \in \mathcal{L}_1[0,\infty)$, then for every $\varepsilon > 0$, there exists a vector $(a_0, a_1, \cdots, a_N) \in \mathbb{R}^{N+1}$ such that*

$$\int_0^\infty |f(t) - \sum_{i=0}^{N} a_i e^{-t} t^{i+\alpha}| dt < \varepsilon \tag{3.8}$$

where $\alpha > -1$

The construction of the multiplier is reproduced from [31]. Replacing t by $-t$, the same result can also be determined for $\mathcal{L}_1(-\infty,0]$. By choosing $\alpha=0$, an orthogonal basis for the approximation can be obtained: $e_i^+(t)=e^{-t}t^i, t\geq 0$ ($e_i^+(t)=0$ for $t\leq 0$), $e_i^-(t)=e^t t^i, t\leq 0$ ($e_i^-(t)=0$ for $t\geq 0$), then the N_{th} can be approximated by a function $m_N(t)\in\mathcal{L}_1(-\infty,\infty)$ for monotone odd non-linearity. However, in order to construct a convex search, the function $m_N(t)$ needs to be composed of two positive functions $x_N(t), y_N(t)\geq 0$, $m_N(t)=x_N(t)-y_N(t)$. Then using the absolute value of a sum, the $\mathcal{L}_1(-\infty,\infty)$ norm of $m_N(t)$ is bounded by a function that does not contain absolute values,

$$\int_{-\infty}^{\infty}|m_N(t)|dt=\int_{-\infty}^{\infty}|x_N(t)-y_N(t)|dt\leq\int_{-\infty}^{\infty}(x_N(t)+y_N(t))dt\leq 1.$$

Now, using Lemma 3.2.3, define $x_N(t)=\sum_{i=1}^N(a_ie_i^+(t)+b_ie_i^-(t))$ and $y_N(t)=\delta(t)+\sum_{i=1}^N(c_ie_i^+(t)+d_ie_i^-(t))$, the resulting multiplier is

$$m_N(t)=\delta(t)-h(t)=\delta(t)-\sum_{i=1}^N(a_ie_i^+(t)+b_ie_i^-(t))+\sum_{i=1}^N(c_ie_i^+(t)+d_ie_i^-(t)). \quad (3.9)$$

By obtaining the Fourier transform of equation (3.9)

$$M_N(j\omega)=1-\sum_{i=0}^N\left(\frac{a_i}{(j\omega+1)^{i+1}}-\frac{b_i}{(j\omega-1)^{i+1}}\right)i!$$
$$+\sum_{i=0}^N\left(\frac{c_i}{(j\omega+1)^{i+1}}-\frac{d_i}{(j\omega-1)^{i+1}}\right)i!, \quad (3.10)$$

then Problem 3.2.1 can be reformulated as follows:

Problem 3.2.4 ([31])**.** *Given a stable LTI plant $G(s)\in\mathcal{RH}_\infty$, find the largest $k>0$ such that the feedback interconnection of $G(s)$ and a slope-restricted non-linearity $[0,k]$ is absolutely stable, or alternatively, find the largest $k>0$ such that there exists an integer N and four diagonal matrices $A_N=diag\{a_1,...,a_N\}$, $B_N=diag\{b_1,...,b_N\}$, $C_N=diag\{c_1,...,c_N\}$ and $D_N=diag\{d_1,...,d_N\}$ satisfying:*

1. $\sum_{i=0}^N(a_i+(-1)^ib_i+c_i+(-1)^id_i)i!<1$;
2. *for some* $\varepsilon>0$
$$Re\{M_N(j\omega)\tilde{G}(j\omega)\}\geq\varepsilon,\quad\forall\omega\in\mathbb{R} \quad (3.11)$$
where $\tilde{G}(s)=(1+kG(s))$*; and*

3. *the following inequalities hold*

$$\frac{\sum_{i=0}^{N} a_i(-1)^i(j\omega)^{2i}}{(1-j\omega)^N(1+j\omega)^N} \geq 0 \;\forall\omega\in\mathbb{R}, \tag{3.12}$$

$$\frac{\sum_{i=0}^{N} b_i(j\omega)^{2i}}{(1-j\omega)^N(1+j\omega)^N} \geq 0 \;\forall\omega\in\mathbb{R}. \tag{3.13}$$

$$\frac{\sum_{i=0}^{N} c_i(-1)^i(j\omega)^{2i}}{(1-j\omega)^N(1+j\omega)^N} \geq 0 \;\forall\omega\in\mathbb{R}, \tag{3.14}$$

$$\frac{\sum_{i=0}^{N} d_i(j\omega)^{2i}}{(1-j\omega)^N(1+j\omega)^N} \geq 0 \;\forall\omega\in\mathbb{R}. \tag{3.15}$$

4. $a_i, b_i, c_i, d_i \geq 0 \quad \forall i = 1, 2, ..., N.$

In [31], a search method is not proposed. However [29] proposes a solution based on the use of a linear program using a frequency swap, but this method can be time consuming, and just as inaccurate as the Summation of $\delta(.)$ distributions due to the frequency search. In a later version of [31], published as [32], a Zames-Falb multiplier search is presented using only LMI solutions for Problem 3.2.4. However, the details of the technique using only LMI are inaccurate [32, p. 643-647]. This section corrects the state space representations.

The first step of Problem 3.2.4 is a summation, that can be written as a LMI parametrized by the affine variables A_N, B_N, C_N, D_N. Then, the first part of the problem can be rewritten as:

Find $A_N \geq 0$, $B_N \geq 0$, $C_N \geq 0$, $D_N \geq 0$ such that

$$\sum_{i=0}^{N}(a_i + (-1)^i b_i + c_i + (-1)^i d_i)i! < 1$$

$$\iff \begin{bmatrix} 0! \\ 1! \\ \vdots \\ (N-1)! \\ N! \end{bmatrix}^T (A_N + C_N) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} (-1)^0 0! \\ (-1)^1 1! \\ \vdots \\ (-1)^{N-1}(N-1)! \\ (-1)^N N! \end{bmatrix}^T (B_N + D_N) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} - 1 < 0 \tag{3.16}$$

Then, from the second step on Problem 3.2.4, consider equation (3.10) and write it in explicit form:

$$M_N(j\omega) = 1 - \sum_{i=0}^{N}\left(\frac{a_i}{(j\omega+1)^{i+1}} - \frac{b_i}{(j\omega-1)^{i+1}}\right)i!, \\ + \sum_{i=0}^{N}\left(\frac{c_i}{(j\omega+1)^{i+1}} - \frac{d_i}{(j\omega-1)^{i+1}}\right)i!$$

$$= 1 - \frac{\sum_{i=0}^{N} a_i i! \sum_{k=0}^{N-i}\binom{N-i}{k}(j\omega)^k}{\sum_{k=0}^{N+1}\binom{N+1}{k}(j\omega)^k} + \frac{\sum_{i=0}^{N} b_i i! \sum_{k=0}^{N-i}\binom{N-i}{k}(j\omega)^k(-1)^{N-i-k}}{\sum_{k=0}^{N+1}\binom{N+1}{k}(j\omega)^k(-1)^{N+1-k}}$$
$$+ \frac{\sum_{i=0}^{N} c_i i! \sum_{k=0}^{N-i}\binom{N-i}{k}(j\omega)^k}{\sum_{k=0}^{N+1}\binom{N+1}{k}(j\omega)^k} - \frac{\sum_{i=0}^{N} d_i i! \sum_{k=0}^{N-i}\binom{N-i}{k}(j\omega)^k(-1)^{N-i-k}}{\sum_{k=0}^{N+1}\binom{N+1}{k}(j\omega)^k(-1)^{N+1-k}} \tag{3.17}$$

where $\binom{x}{y} = \frac{x!}{y!(x-y)!}$. Using the Controllable Canonical Form [60, p. 75], the following state space representation is proposed.

$$M_N(j\omega) = 1 + \left[\begin{array}{c|c} A_a & B_a \\ \hline -C_a & -D_a \end{array}\right] + \left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right] + \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] + \left[\begin{array}{c|c} A_d & B_d \\ \hline -C_d & -D_d \end{array}\right], \tag{3.18}$$

Note that the denominator for a_i and c_i is the same, therefore

$$A_a = A_c = A_{ac} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & -(N+1) & -\frac{(N+1)N}{2} & \cdots & -\frac{(N+1)N}{2} & -(N+1) \end{bmatrix}. \tag{3.19}$$

Using the Controllable Canonical form, the matrix for B_a is equal to B_c

$$B_a = B_c = B_{ac} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.20}$$

Moreover, numerators for a_i and c_i differ only in the constant coefficients, therefore $C_a - C_c$ is

$$C_c - C_a = \begin{bmatrix} 0! \\ 1! \\ \cdots \\ (N-2)! \\ (N-1)! \\ N! \end{bmatrix}^T (-A_N + C_N) \begin{bmatrix} 1 & N & \frac{N(N-1)}{2} & \cdots & N & 1 \\ 1 & (N-1) & \frac{(N-1)(N-2)}{2} & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{3.21}$$

Given that all the transfer functions are strictly proper,

$$D_a = D_c = 0. \tag{3.22}$$

Note that the denominator for b_i and d_i is the same, therefore

$$A_b = A_d = A_{bd} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ (-1)^N & (-1)^{N-1}(N+1) & (-1)^{N-2}\frac{(N+1)N}{2} & \cdots & -\frac{(N+1)N}{2} & (N+1) \end{bmatrix} \tag{3.23}$$

Using the Controllable Canonical form, the matrix for B_b is equal to B_d

$$B_b = B_d = B_{bd} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.24}$$

The numerators for b_i and d_i only differ in the constant coefficients, therefore $C_b - C_d$ is expressed as

$$C_b - C_d = \begin{bmatrix} 0! \\ 1! \\ \vdots \\ (N-2)! \\ (N-1)! \\ N! \end{bmatrix}^T (B_N - D_N)$$
$$\times \begin{bmatrix} (-1)^N & (-1)^{N-1}N & (-1)^{N-2}\frac{N(N-1)}{2} & \cdots & -N & 1 \\ (-1)^{N-1} & (-1)^{N-2}(N-1) & (-1)^{N-3}\frac{(N-1)(N-2)}{2} & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{3.25}$$

Given that all the transfer functions are strictly proper,

$$D_b = D_d = 0. \tag{3.26}$$

Finally, equation (3.18) can be rewritten as

$$
\begin{aligned}
M_N(j\omega) &= 1 + \left[\begin{array}{c|c} A_{ac} & B_{ac} \\ \hline C_c - C_a & D_c - D_a \end{array}\right] + \left[\begin{array}{c|c} A_{bd} & B_{bd} \\ \hline C_b - C_d & D_b - D_d \end{array}\right] \\
&= \left[\begin{array}{cc|c} A_{ac} & 0 & B_{ac} \\ 0 & A_{bd} & B_{bd} \\ \hline C_c - C_a & C_b - C_d & 1 + D_c - D_a + D_b - D_d \end{array}\right]. \qquad (3.27)
\end{aligned}
$$

For the second step of Problem 3.2.4, make state space representation of $\tilde{G}$:

$$
\tilde{G} = \left[\begin{array}{c|c} A_G & B_G \\ \hline kC_G & kD_G + 1 \end{array}\right]. \qquad (3.28)
$$

Then, equation (3.11) can be rewritten as

$$
Re\left\{\left[\begin{array}{c|c} A_{MG} & B_{MG} \\ \hline C_{MG} & D_{MG} \end{array}\right]\right\} = Re\left\{\left[\begin{array}{ccc|c} A_{ac} & 0 & kB_{ac}C_G & B_{ac}(kD_G + 1) \\ 0 & A_{bd} & kB_{bd}C_G & B_{bd}(kD_G + 1) \\ 0 & 0 & A_G & B_G \\ \hline C_c - C_a & C_b - C_d & kC_G & kD_G + 1 \end{array}\right]\right\} > \varepsilon
$$

$$
\forall \omega \in \mathbb{R}. \qquad (3.29)
$$

Note that A_N, B_N, C_N, D_N are affine variables, in accordance to [32]. In other words, by letting A_{MG}, B_{MG} be constant matrices, the solution of the LMI depends linearly on the variables X_{MG}, A_N, B_N, C_N, D_N. Using the Kalman-Yakubovich-Popov Lemma [41], the inequality (3.29) is equivalent to the following problem:

Find $X_{MG} = X_{MG}^T$ and $A_N \geq 0$, $B_N \geq 0$, $C_N \geq 0$, $D_N \geq 0$ such that

$$
\begin{bmatrix} X_{MG}A_{MG} + A_{MG}^T X_{MG} & X_{MG}B_{MG} - C_{MG}^T \\ B_{MG}^T X_{MG} - C_{MG} & -D_{MG} - D_{MG}^T + \varepsilon \end{bmatrix} < 0 \qquad (3.30)
$$

For the third step of Problem 3.2.4, check the positivity of the multipliers using equations (3.12), (3.13), (3.14) and (3.15). First, rewrite the inequalities as a summation of a constant element with a strictly proper transfer function:

$$0 \leq \frac{\sum_{i=0}^{N} a_i(-1)^i(j\omega)^{2i}}{(1-j\omega)^N(1+j\omega)^N} = a_N + \frac{\sum_{i=0}^{N-1}\left(a_i - a_N\begin{pmatrix}N\\i\end{pmatrix}\right)(j\omega)^{2i}(-1)^{N-i}}{\sum_{i=0}^{N}\begin{pmatrix}N\\i\end{pmatrix}(j\omega)^{2i}(-1)^{N-i}} \forall\omega\in\mathbb{R}, \tag{3.31}$$

$$0 \leq \frac{\sum_{i=0}^{N} b_i(j\omega)^{2i}}{(-1+j\omega)^N(1+j\omega)^N} = b_N(-1)^N + \frac{\sum_{i=0}^{N-1}\left(b_i - b_N\begin{pmatrix}N\\i\end{pmatrix}(-1)^{N-i}\right)(j\omega)^{2i}(-1)^{N}}{\sum_{i=0}^{N}\begin{pmatrix}N\\i\end{pmatrix}(j\omega)^{2i}(-1)^{N-i}} \forall\omega\in\mathbb{R}, \tag{3.32}$$

$$0 \leq \frac{\sum_{i=0}^{N} c_i(-1)^i(j\omega)^{2i}}{(1-j\omega)^N(1+j\omega)^N} = c_N + \frac{\sum_{i=0}^{N-1}\left(c_i - c_N\begin{pmatrix}N\\i\end{pmatrix}\right)(j\omega)^{2i}(-1)^{N-i}}{\sum_{i=0}^{N}\begin{pmatrix}N\\i\end{pmatrix}(j\omega)^{2i}(-1)^{N-i}} \forall\omega\in\mathbb{R}, \tag{3.33}$$

$$0 \leq \frac{\sum_{i=0}^{N} d_i(j\omega)^{2i}}{(-1+j\omega)^N(1+j\omega)^N} = d_N(-1)^N + \frac{\sum_{i=0}^{N-1}\left(d_i - d_N\begin{pmatrix}N\\i\end{pmatrix}(-1)^{N-i}\right)(j\omega)^{2i}(-1)^{N}}{\sum_{i=0}^{N}\begin{pmatrix}N\\i\end{pmatrix}(j\omega)^{2i}(-1)^{N-i}} \forall\omega\in\mathbb{R}. \tag{3.34}$$

Using the Controllable Canonical Form [60, p. 75], the following state space representations are obtained from equations (3.31), (3.32), (3.33), (3.34)

$$Z_a(j\omega) = \left[\begin{array}{c|c} A_{ap} & B_{ap} \\ \hline C_{ap} & D_{ap} \end{array}\right] \geq 0 \quad \forall\omega\in\mathbb{R}, \tag{3.35}$$

$$Z_b(j\omega) = \left[\begin{array}{c|c} A_{bp} & B_{bp} \\ \hline C_{bp} & D_{bp} \end{array}\right] \geq 0 \quad \forall\omega\in\mathbb{R}, \tag{3.36}$$

$$Z_c(j\omega) = \left[\begin{array}{c|c} A_{cp} & B_{cp} \\ \hline C_{cp} & D_{cp} \end{array}\right] \geq 0 \quad \forall\omega\in\mathbb{R}, \tag{3.37}$$

$$Z_d(j\omega) = \left[\begin{array}{c|c} A_{dp} & B_{dp} \\ \hline C_{dp} & D_{dp} \end{array}\right] \geq 0 \quad \forall \omega \in \mathbb{R}, \tag{3.38}$$

where the state spaces matrices are the following. First notice that all denominators are equal, then

$$A_{ap} = A_{bp} = A_{cp} = A_{dp} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ (-1)^{N-1} & 0 & (-1)^{N-2}N & \cdots & N & 0 \end{bmatrix}. \tag{3.39}$$

Using the controllable canonical form in all equations results in

$$B_{ap} = B_{bp} = B_{cp} = B_{dp} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.40}$$

Noting that only the numerators are different, the following matrices are defined:

$$\begin{aligned} C_{ap} &= \begin{bmatrix} (-1)^N(a_0 - a_N) & 0 & (-1)^{N-1}(a_1 - a_N N) & \cdots & -(a_{N-1} - a_N N) & 0 \end{bmatrix} \\ &= \begin{bmatrix} (-1)^N \\ (-1)^{N-1} \\ \vdots \\ (-1)^1 \\ (-1)^0 \end{bmatrix}^T A_N \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -(-1)^N & 0 & -N(-1)^{N-1} & \cdots & -N(-1)^1 & 0 \end{bmatrix} \end{aligned} \tag{3.41}$$

$$D_{ap} = a_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T A_N \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.42}$$

$$C_{bp} = \begin{bmatrix} (b_0 - b_N(-1)^N)(-1)^N & 0 & (b_1 - b_N N(-1)^{N-1})(-1)^N & \cdots & (b_{N-1} + b_N N)(-1)^N & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)^N \\ (-1)^N \\ \vdots \\ (-1)^N \\ (-1)^N \end{bmatrix}^T B_N \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -(-1)^N & 0 & -N(-1)^{N-1} & \cdots & -N(-1)^1 & 0 \end{bmatrix} \tag{3.43}$$

$$D_{bp} = b_N(-1)^N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (-1)^N \end{bmatrix}^T B_N \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.44}$$

$$C_{cp} = \begin{bmatrix} (-1)^N(c_0 - c_N) & 0 & (-1)^{N-1}(c_1 - c_N N) & \cdots & -(c_{N-1} - c_N N) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)^N \\ (-1)^{N-1} \\ \vdots \\ (-1)^1 \\ (-1)^0 \end{bmatrix}^T C_N \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -(-1)^N & 0 & -N(-1)^{N-1} & \cdots & -N(-1)^1 & 0 \end{bmatrix} \tag{3.45}$$

$$D_{cp} = c_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T C_N \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.46}$$

$$C_{dp} = \begin{bmatrix}(d_0 - d_N(-1)^N)(-1)^N & 0 & (d_1 - d_N N(-1)^{N-1})(-1)^N & \cdots & (d_{N-1} + d_N N)(-1)^N & 0\end{bmatrix}$$

$$= \begin{bmatrix}(-1)^N \\ (-1)^N \\ \vdots \\ (-1)^N \\ (-1)^N\end{bmatrix}^T D_N \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -(-1)^N & 0 & -N(-1)^{N-1} & \cdots & -N(-1)^1 & 0 \end{bmatrix} \tag{3.47}$$

$$D_{dp} = d_N(-1)^N = \begin{bmatrix}0 \\ 0 \\ \vdots \\ 0 \\ (-1)^N\end{bmatrix}^T D_N \begin{bmatrix}0 \\ 0 \\ \vdots \\ 0 \\ 1\end{bmatrix} \tag{3.48}$$

Noting that the positivity of a transfer functions implies the positivity of the real part of that transfer function, to check the positivity of equations (3.12), (3.13), (3.14), (3.15), it is necessary to show

$$Re\{Z_a(j\omega)\} \geq 0 \quad \forall \omega \in \mathbb{R}, \tag{3.49}$$

$$Re\{Z_b(j\omega)\} \geq 0 \quad \forall \omega \in \mathbb{R}, \tag{3.50}$$

$$Re\{Z_c(j\omega)\} \geq 0 \quad \forall \omega \in \mathbb{R}, \tag{3.51}$$

$$Re\{Z_d(j\omega)\} \geq 0 \quad \forall \omega \in \mathbb{R}. \tag{3.52}$$

Once again, note that A_N, B_N, C_N, D_N are affine variables in accordance to[32]. In other words, by letting A_{ap}, A_{bp}, A_{cp}, A_{dp}, B_{ap}, B_{bp}, B_{cp}, B_{dp} be constant matrices, the resulting LMIs depend linearly on the variables X_{AZ}, X_{BZ}, X_{CZ}, X_{DZ}, A_N, B_N, C_N, D_N. Then, using the Kalman-Yakubovich-Popov Lemma [41], the inequalities (3.49), (3.50), (3.51) and (3.52) are equivalent to the following problems:

Find $X_{AZ} = X_{AZ}^T$ and $A_N \geq 0$ such that

$$\begin{bmatrix} X_{AZ}A_{ap} + A_{ap}^T X_{AZ} & X_{AZ}B_{ap} - C_{ap}^T \\ B_{ap}^T X_{AZ} - C_{ap} & -D_{ap} - D_{ap}^T \end{bmatrix} \leq 0, \tag{3.53}$$

find $X_{BZ} = X_{BZ}^T$ and $B_N \geq 0$ such that

$$\begin{bmatrix} X_{BZ}A_{bp} + A_{bp}^T X_{BZ} & X_{BZ}B_{bp} - C_{bp}^T \\ B_{bp}^T X_{BZ} - C_{bp} & -D_{bp} - D_{bp}^T \end{bmatrix} \leq 0, \tag{3.54}$$

find $X_{CZ} = X_{CZ}^T$ and $C_N \geq 0$ such that

$$\begin{bmatrix} X_{CZ}A_{cp} + A_{cp}^T X_{CZ} & X_{CZ}B_{cp} - C_{cp}^T \\ B_{cp}^T X_{CZ} - C_{cp} & -D_{cp} - D_{cp}^T \end{bmatrix} \leq 0, \tag{3.55}$$

find $X_{DZ} = X_{DZ}^T$ and $D_N \geq 0$ such that

$$\begin{bmatrix} X_{DZ}A_{dp} + A_{dp}^T X_{DZ} & X_{DZ}B_{dp} - C_{dp}^T \\ B_{dp}^T X_{DZ} - C_{dp} & -D_{dp} - D_{dp}^T \end{bmatrix} \leq 0. \tag{3.56}$$

Finally, Problem 3.2.4 can be formalized as:

Problem 3.2.5 ([31]). *Given a stable LTI plant $G(s) \in \mathcal{RH}_\infty$, find the largest $k > 0$ such that the feedback interconnection of $G(s)$ and a slope-restricted non-linearity $[0,k]$ is absolutely stable, or alternatively, find the largest $k > 0$ such that there exists an integer N and four diagonal matrices $A_N = diag\{a_1,...,a_N\} \geq 0$, $B_N = diag\{b_1,...,b_N\} \geq 0$, $C_N = diag\{c_1,...,c_N\} \geq 0$ and $D_N = diag\{d_1,...,d_N\} \geq 0$,satisfying:*

1. *There exists A_N, B_N, C_N, $D_N \geq 0$ such that*

$$\begin{bmatrix} 0! \\ 1! \\ \vdots \\ (N-1)! \\ N! \end{bmatrix}^T (A_N + C_N) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} (-1)^0 0! \\ (-1)^1 1! \\ \vdots \\ (-1)^{N-1}(N-1)! \\ (-1)^N N! \end{bmatrix}^T (B_N + D_N) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} - 1 < 0,$$

2. *there exists $X_{MG} = X_{MG}^T$ and A_N, B_N, C_N, $D_N \geq 0$ such that*

$$\begin{bmatrix} X_{MG}A_{MG} + A_{MG}^T X_{MG} & X_{MG}B_{MG} - C_{MG}^T \\ B_{MG}^T X_{MG} - C_{MG} & -D_{MG} - D_{MG}^T + \varepsilon \end{bmatrix} < 0,$$

3. • *there exists* $X_{AZ} = X_{AZ}^T$ *and* $A_N \geq 0$ *such that*

$$\begin{bmatrix} X_{AZ}A_{ap} + A_{ap}^T X_{AZ} & X_{AZ}B_{ap} - C_{ap}^T \\ B_{ap}^T X_{AZ} - C_{ap} & -D_{ap} - D_{ap}^T \end{bmatrix} \leq 0,$$

• *there exists* $X_{BZ} = X_{BZ}^T$ *and* $B_N \geq 0$ *such that*

$$\begin{bmatrix} X_{BZ}A_{bp} + A_{bp}^T X_{BZ} & X_{BZ}B_{bp} - C_{bp}^T \\ B_{bp}^T X_{BZ} - C_{bp} & -D_{bp} - D_{bp}^T \end{bmatrix} \leq 0,$$

• *there exists* $X_{CZ} = X_{CZ}^T$ *and* $C_N \geq 0$ *such that*

$$\begin{bmatrix} X_{CZ}A_{cp} + A_{cp}^T X_{CZ} & X_{CZ}B_{cp} - C_{cp}^T \\ B_{cp}^T X_{CZ} - C_{cp} & -D_{cp} - D_{cp}^T \end{bmatrix} \leq 0,$$

• *there exists* $X_{DZ} = X_{DZ}^T$ *and* $D_N \geq 0$ *such that*

$$\begin{bmatrix} X_{DZ}A_{dp} + A_{dp}^T X_{DZ} & X_{DZ}B_{dp} - C_{dp}^T \\ B_{dp}^T X_{DZ} - C_{dp} & -D_{dp} - D_{dp}^T \end{bmatrix} \leq 0.$$

3.2.3 Causal transfer functions

In [1], a LMI search has been proposed for multipliers of fixed order. The multiplier is restricted to the set of causal rational transfer functions of the same order as the plant, and restrict all coefficients $z_i = 0$ from equation (3.1), removing the delta distributions from the multiplier:

$$M(j\omega) = 1 - C_h(j\omega I - A_h)^{-1}B_h, \tag{3.57}$$

where the dimensions of the matrices A_h, B_h and C_h are the same as the state–space minimal representation of the plant. Moreover, the original search presented in [1] uses the IQC Theorem, and as a consequence, the analysis was done using positive feedback. In this section it is used negative feedback in order to compare the result with previous methods. However, the results remain equivalent [61].

Following [50], D_h is fixed to zero. Furthermore, imposing the LMI for bounded Peak-to-Peak Gain, described in [62], on matrices A_h, B_h and C_h to ensure that $M \in \mathcal{M}_{\text{odd}}$, Problem 3.2.1 can be reformulated as follows:

Problem 3.2.6 ([1])**.** *Given a stable LTI plant* $G(s) \in \mathcal{RH}_\infty$*, find the largest* $k > 0$ *such that the feedback interconnection of* $G(s)$ *and an odd slope-restricted non-linearity*

$[0,k]$ *is absolutely stable, or alternatively, find the largest* $k > 0$ *such that there exist a multiplier* $M(s)$ *satisfying the conditions:*

1. *there exists a positive matrix* $Y > 0$ *and positive constants* $\mu > 0$, $\lambda > 0$ *such that*

$$\begin{bmatrix} A_h^\top Y + YA_h + \lambda Y & YB_h \\ B_h^\top Y & -\mu I \end{bmatrix} < 0, \tag{3.58}$$

$$\begin{bmatrix} \lambda Y & 0 & C_h^\top \\ 0 & (1-\mu)I & 0 \\ C_h & 0 & 1 \end{bmatrix} \geq 0; \tag{3.59}$$

2. *there is an* $\varepsilon > 0$ *such that*

$$Re\{M(j\omega)\tilde{G}(j\omega)\} \geq \varepsilon I \quad \forall \omega \in \mathbb{R}, \tag{3.60}$$

where $\tilde{G}(s) = (1 + kG(s))$.

As opposed to the previous two methods, the search can only be carried out within $\mathcal{M}_{\text{odd}}$. In the spirit of a multi-objective synthesis [62], the existence of the multiplier is ensured with the following Proposition.

Proposition 3.2.7 ([1]). *Given a LTI plant* $G(s) \in \mathcal{RH}_\infty \sim (A_p, B_p, C_p, D_p)$ *and a constant* $k > 0$. *There exists a multiplier* $M(s)$ *fulfilling conditions of Problem 3.2.6 if there exist positive definite symmetric matrices* $S_{11} > 0, P_{11} > 0$, *unstructured matrices* $\tilde{A}_u, \tilde{B}_u, \tilde{C}_u$, *and scalars* $\mu > 0$ *and* $\lambda > 0$ *such that following inequalities are satisfied.*

$$\begin{bmatrix} S_{11}A_p + A_p^T S_{11} & S_{11}A_p + A_p^T P_{11} + KC_p^T \bar{B}_u^T + \bar{A}_u^T & S_{11}B_p - KC_p^T + \bar{C}_u^T \\ \star & P_{11}A_p + K\bar{B}_u C_p + A_p^T P_{11} + KC_p^T \bar{B}_u^T & P_{11}B_p + \bar{B}_u(I + KD_p) - KC_p^T \\ \star & \star & -(I + KD_p) - (I + KD_p^T) \end{bmatrix} < 0 \tag{3.61}$$

$$\begin{bmatrix} -\bar{A}_u^T - \bar{A}_u + \lambda(P_{11} - S_{11}) & -\bar{B}_u \\ -\bar{B}_u^T & -\mu I \end{bmatrix} < 0, \tag{3.62}$$

$$\begin{bmatrix} \lambda(P_{11} - S_{11}) & 0 & \bar{C}_u^T \\ 0 & 1-\mu & 0 \\ \bar{C}_u & 0 & 1 \end{bmatrix} > 0; \tag{3.63}$$

Proof. See Proposition 2 in [1]. □

That is, consider the system in Figure 3.1 a) and perform the loop transformation of Passivity Theorem (Theorem 2.2.6) over the following representation:

$$\begin{aligned} u &= r - \phi y, \\ y &= Gu. \end{aligned} \quad (3.64)$$

The resulting loop (Figure 3.1 c)) is the following representation.

$$\begin{aligned} y &= \frac{1}{k}\phi y + \tilde{y} \\ \tilde{y} &= Gr - \frac{1}{k}(kG+I)\phi y \end{aligned}$$

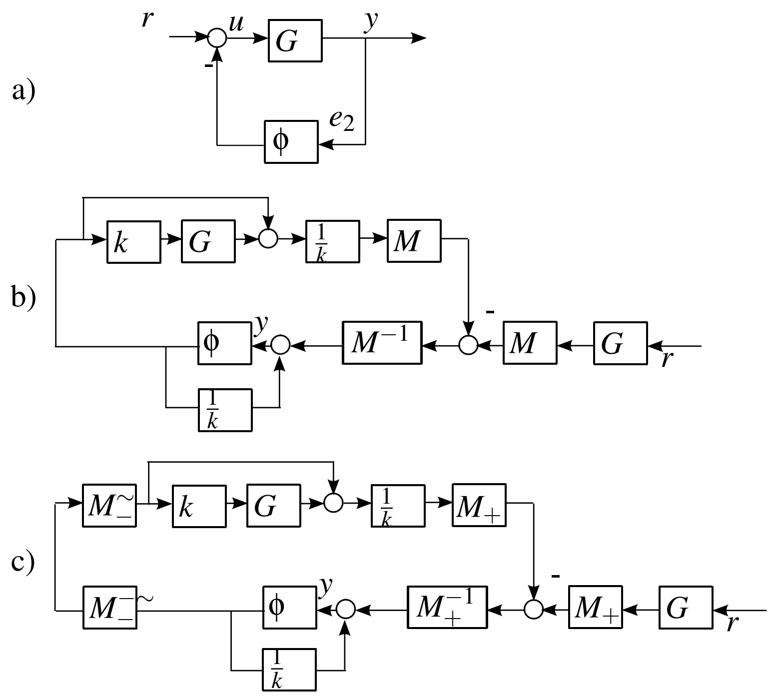


Figure 3.1: Turner loop transformation[1]

A relaxation for searching the boundary of the $\mathcal{L}_1(-\infty,\infty)$ norm is proposed in [1] was demonstrated as incorrect, because it allowed multipliers to obtained where the $\mathcal{L}_1(-\infty,\infty)$ norm was larger than 1. The results have been corrected in [50]. Although λ is suggested to be fixed to a small value in [1], a linear search over λ is needed to obtain a good result.

3.2.3.1 Addition of Popov multiplier

Recently, the addition of a Popov multiplier has been proposed in [63, 33] for strictly proper plants, i.e. $D_p = 0$. Two new parameters are added to the multiplier

$$M(s) = M_{ZF} + (\nu + \eta s) \tag{3.65}$$

where $\nu > 0$, $\eta \in \mathbb{R}$ and $M_{ZF} \in \mathcal{M}_{\text{odd}}$.

This addition can improve the result significantly since the equivalent Zames–Falb multiplier can be non-causal if $\eta < 0$. As a result, Proposition 3.2.7 can be rewritten

Proposition 3.2.8. *Given a LTI plant $G(s) \in \mathcal{RH}_\infty \sim (A_p, B_p, C_p, 0)$ and a constant $k > 0$ such that the feedback interconnection of $G(s)$, there exist a multiplier $M(s) = M_{ZF} + (\nu + \eta s)$, $M_{ZF} \in \mathcal{M}_{odd}$ fulfilling conditions of Theorem 2.2.6 if there exist positive definite symmetric matrices $S_{11} > 0, P_{11} > 0$, unstructured matrices $\tilde{A}_u, \tilde{B}_u, \tilde{C}_u$, and scalars $\nu > 0$, $\eta \in \mathbb{R}$, $\mu > 0$ and $\lambda > 0$ such that the following inequalities are satisfied.*

$$\begin{bmatrix} S_{11}A_p + A_p^T S_{11} & S_{11}A_p + A_p^T P_{11} + KC_p^T \bar{B}_u^T + \bar{A}_u^T & S_{11}B_p - K(I + vI + \eta A_p^T)C_p^T + \bar{C}_u^T \\ \star & P_{11}A_p + A_p^T P_{11} + K\bar{B}_u C_p + KC_p^T \bar{B}_u^T & P_{11}B_p + \bar{B}_u - K(I + vI + \eta A_p^T)C_p^T \\ \star & \star & -2I - 2vI - \eta K(C_p B_p + B_p^T C_p^T) \end{bmatrix} < 0 \tag{3.66}$$

$$\begin{bmatrix} -\bar{A}_u^T - \bar{A}_u + \lambda(P_{11} - S_{11}) & -\bar{B}_u \\ -\bar{B}_u^T & -\mu I \end{bmatrix} < 0, \tag{3.67}$$

$$\begin{bmatrix} \lambda(P_{11} - S_{11}) & 0 & \bar{C}_u^T \\ 0 & 1-\mu & 0 \\ \bar{C}_u & 0 & 1 \end{bmatrix} > 0; \tag{3.68}$$

Proof. See Proposition 1 in [33]. □

The maximum slope results from this method are presented in Section 3.4. The apparent poor performance of this method compared to the summation of $\delta(.)$ distributions and summation of exponential functions was explained in Section 2.2. The following section then shows a way to complement this powerful tool in order to exploit the benefits of anti-causal multipliers.

3.3 Anti-causal Transfer Functions

The last method described, introduced by [1], shows promising results, combining simplicity and performance. However, as was discussed in Section 2.2, this method is artificially limited by its applicability of only causal multipliers. This section describes a modification that allows the use of non-causal multipliers, and is considered as a complementary method. The general class of Zames-Falb multipliers can be factorized into causal and anti-causal parts using the canonical factorization of Lemma 3 in [25], i.e. $M(j\omega) = M_-(j\omega)M_+(j\omega)$, where $M_-^\sim, (M_-^\sim)^{-1}, M_+, (M_+)^{-1} \in \mathcal{RH}_\infty$. In other words, using the algorithm of [1], $M_-(j\omega)$ is assumed to be the identity, while this work will propose to make $M_+(j\omega)$ the identity. The results are described in two methods to introduce the non-causal multiplier: a Causal Search, and an Anti-causal Search.

3.3.1 Causal Search

This first method inverts the loop transformation. Take the condition from equation (2.9), and by writing the multiplier in the general factorizable form, the following statements are equivalent:

$$\begin{aligned} & Re\{M(j\omega)\tilde{G}(j\omega)\} > 0 \quad \forall\omega \in \mathbb{R}, \\ \iff & Re\{\tilde{G}(j\omega)M^{-\sim}(j\omega)\} > 0 \quad \forall\omega \in \mathbb{R}, \\ \iff & Re\{(M^\sim(j\omega)\tilde{G}^{-1}(j\omega))^{-1}\} > 0 \quad \forall\omega \in \mathbb{R}, \\ \iff & Re\{M^\sim(j\omega)\tilde{G}^{-1}(j\omega)\} > 0 \quad \forall\omega \in \mathbb{R}. \end{aligned} \tag{3.69}$$

The inverse $\tilde{G}^{-1}(j\omega) = (1 + kG(j\omega))^{-1}$ exists and is stable because without loss of generality, the feedback interconnection $[G,k]$ is stable. Using the canonical factorization from Lemma 3 in [25], $M(j\omega) = M_-(j\omega)M_+(j\omega)$, then making $M_+ = I$ and using equation (3.69) yields the following equivalence

$$\begin{aligned} & Re\{M_-(j\omega)\tilde{G}(j\omega)\} > 0 \quad \forall\omega \in \mathbb{R} \\ \iff & Re\{M_-^\sim(j\omega)\tilde{G}^{-1}(j\omega)\} > 0 \quad \forall\omega \in \mathbb{R}. \end{aligned} \tag{3.70}$$

Thus, the search for an anti-causal multiplier is equivalent to a search of the adjoint multiplier using the inverse of the plant $\tilde{G}^{-1}(j\omega) = (I + kG(j\omega))^{-1}$. The adjoint of any anti-causal multiplier is a causal multiplier, therefore the search is done over the equivalent causal multiplier. The method can be stated as follows:

Proposition 3.3.1. *Given a LTI plant $G(s) \in \mathcal{RH}_\infty \sim (A_p, B_p, C_p, D_p)$ and a constant $k > 0$, assume without loss of generality that the feedback interconnection $[G, k]$ is stable. Let us define the matrices:*

$$\hat{A}_p = A_p - B_p(I + kD_p)^{-1}C_p, \tag{3.71}$$

$$\hat{B}_p = -B_p(I + kD_p)^{-1}, \tag{3.72}$$

$$\hat{C}_p = k(I + kD_p)^{-1}C_p, \tag{3.73}$$

$$\hat{D}_P = (I + kD_p)^{-1}. \tag{3.74}$$

Then, there exists a multiplier $M(s)$ fulfilling conditions of Problem 3.2.6 if there exist positive definite symmetric matrices $S_{11} > 0, P_{11} > 0$, unstructured matrices $\tilde{A}_u, \tilde{B}_u, \tilde{C}_u$, and scalars $\mu > 0$ and $\lambda > 0$ such that the following inequalities are satisfied.

$$\begin{bmatrix} S_{11}\hat{A}_p + \hat{A}_p^T S_{11} & S_{11}\hat{A}_p + \hat{A}_p^T P_{11} + \hat{C}_p^T \bar{B}_u^T + \bar{A}_u^T & S_{11}\hat{B}_p - \hat{C}_p^T + \bar{C}_u^T \\ \star & P_{11}\hat{A}_p + \hat{A}_p^T P_{11} + \bar{B}_u\hat{C}_p + \hat{C}_p^T \bar{B}_u^T & P_{11}\hat{B}_p + \bar{B}_u\hat{D}_p - \hat{C}_p^T \\ \star & \star & -\hat{D}_p - \hat{D}_p^T \end{bmatrix} < 0 \tag{3.75}$$

$$\begin{bmatrix} -\bar{A}_u^T - \bar{A}_u + \lambda(P_{11} - S_{11}) & -\bar{B}_u \\ -\bar{B}_u^T & -\mu I \end{bmatrix} < 0, \tag{3.76}$$

$$\begin{bmatrix} \lambda(P_{11} - S_{11}) & 0 & \bar{C}_u^T \\ 0 & 1 - \mu & 0 \\ \bar{C}_u & 0 & 1 \end{bmatrix} > 0. \tag{3.77}$$

The reconstruction of the multiplier can be carried on as suggested in [1], solving the following equations:

$$\bar{A}_u = P_{12}A_h Q_{12}^T S_{11} \tag{3.78}$$

$$\bar{B}_u = P_{12}B_h \tag{3.79}$$

$$\bar{C}_u = C_h Q_{12}^T S_{11}. \tag{3.80}$$

This process starts using $P_{22} = I$ with the following equation:

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & I \end{bmatrix}^{-1} = \begin{bmatrix} (P_{11} - P_{12}P_{12}^T)^{-1} & -(P_{11} - P_{12}P_{12}^T)^{-1}P_{12} \\ -P_{12}^T(P_{11} - P_{12}P_{12}^T)^{-1} & I + P_{12}^T(P_{11} - P_{12}P_{12}^T)^{-1}P_{12} \end{bmatrix} \tag{3.81}$$

where it follows that

$$P_{12}P_{12}^T = P_{11} - S_{11} \tag{3.82}$$

$$Q_{12}^T S_{11} = -P_{12}^T, \tag{3.83}$$

Then, the solutions are

$$A_h = -P_{12}^{-1}\bar{A}_u P_{12}^{-T} \tag{3.84}$$

$$B_h = P_{12}^{-1}\bar{B}_u \tag{3.85}$$

$$C_h = -\bar{C}_u P_{12}^{-T}. \tag{3.86}$$

Finally, the multiplier $M(j\omega)$ that fulfills all conditions for Theorem 2.2.6 is $M(j\omega) = M_-(j\omega) = I - B_h^T(j\omega I + A_h^T)C_h^T$.

3.3.2 Anti-causal Search

The original method from Proposition 3.2.7 published in [1] is based in the multi-objective synthesis developed in [62]. The method proposed in this section is a complementary search, where $P > 0$ in Proposition 3.2.7 is replaced by $P < 0$, where the non-singularity of P allows the substitution. A prior lemma is needed to bound the $\mathcal{L}_1(-\infty,\infty)$ norm of the anti-causal transfer function.

Lemma 3.3.2. *Given a strictly proper transfer function $H(s) \in \mathcal{RH}_\infty^\perp$, parametrized by $H(s) = C(Is-A)^{-1}B$, where $-A$ is Hurwitz. Assume that there exist $Y < 0$, $\mu > 0$, and $\lambda > 0$ such that*

$$\begin{bmatrix} A^TY + YA - \lambda Y & -YB \\ -B^TY & -\mu \end{bmatrix} < 0, \tag{3.87}$$

$$\begin{bmatrix} -\lambda Y & 0 & C^T \\ 0 & \xi - \mu & 0 \\ C & 0 & \xi \end{bmatrix} > 0, \tag{3.88}$$

then $||H(s)||_1 < \xi$.

Proof. The result is straightforward, since $||H(s)||_1$ is the same as $||(H^\sim(s))^T||_1$, where $H^\sim(s)$ is given by

$$(H^\sim(s))^T = (-1)C(sI-(-A))^{-1}B \tag{3.89}$$

moreover, the factor (-1) can be ignored when taking norms. Therefore, taking $W =$

$-Y$ in equations (3.90) and (3.91), then there exists $W > 0$, $\mu > 0$ and $\lambda > 0$ such that

$$\begin{bmatrix} -A^T W + W(-A) + \lambda W & WB \\ B^T W & -\mu \end{bmatrix} < 0, \tag{3.90}$$

$$\begin{bmatrix} \lambda W & 0 & C^T \\ 0 & \xi - \mu & 0 \\ C & 0 & \xi \end{bmatrix} > 0, \tag{3.91}$$

As a result, using the bound on the Peak-to-Peak gain in [62], the $\mathcal{L}_1(-\infty,\infty)$ norm is equivalent to $||-(H^{\sim}(s))^T||_1 = ||H(s)||_1 < \xi$. □

Using this lemma, a new proposition is made to find an anti-causal Zames-Falb multiplier.

Proposition 3.3.3. *Given a LTI plant $G(s) \in \mathcal{RH}_\infty \sim (A_p, B_p, C_p, D_p)$ and a constant $k > 0$. There exists a multiplier $M(s)$ fulfilling conditions of Problem 3.2.6 if there exist positive definite symmetric matrices $S_{11} > 0, P_{11} > 0$, unstructured matrices $\tilde{A}_u, \tilde{B}_u, \tilde{C}_u$, and scalars $\mu > 0$ and $\lambda > 0$ such that following inequalities are satisfied.*

$$\begin{bmatrix} S_{11}A_p + A_p^T S_{11} & S_{11}A_p + A_p^T P_{11} + KC_p^T \bar{B}_u^T + \bar{A}_u^T & S_{11}B_p - KC_p^T + \bar{C}_u^T \\ \star & P_{11}A_p + K\bar{B}_u C_p + A_p^T P_{11} + KC_p^T \bar{B}_u^T & P_{11}B_p + \bar{B}_u(I + KD_p) - KC_p^T \\ \star & \star & -(I + KD_p) - (I + KD_p^T) \end{bmatrix} < 0, \tag{3.92}$$

$$\begin{bmatrix} -\bar{A}_u^T - \bar{A}_u - \lambda(P_{11} - S_{11}) & \bar{B}_u \\ \bar{B}_u^T & -\mu I \end{bmatrix} < 0, \tag{3.93}$$

$$\begin{bmatrix} -\lambda(P_{11} - S_{11}) & 0 & \bar{C}_u^T \\ 0 & 1 - \mu & 0 \\ \bar{C}_u & 0 & 1 \end{bmatrix} > 0. \tag{3.94}$$

Proof. Note that the following statements are equivalent, from equation (3.11)

$$Re\{M(j\omega)\tilde{G}(j\omega)\} \geq \varepsilon I \quad \forall \omega \in \mathbb{R},$$

$$\iff Re\left\{ \left[\begin{array}{cc|c} A_p & 0 & B_p \\ KB_h C_p & A_h & B_h(I + KD_p) \\ \hline KC_p & -C_h & I + KD_p \end{array} \right] \right\} \geq \varepsilon I \forall \omega \in \mathbb{R} \tag{3.95}$$

using the Kalman-Yakubovich-Popov Lemma [41], equation (3.95) is equivalent to

$\iff \exists P = P^T$ such that

$$\begin{bmatrix} P\begin{bmatrix} A_p & 0 \\ KB_hC_p & A_h \end{bmatrix} + \begin{bmatrix} A_p^T & KC_p^TB_h^T \\ 0 & A_h^T \end{bmatrix} P & P\begin{bmatrix} B_p \\ B_h(I+KD_p) \end{bmatrix} - \begin{bmatrix} KC_p^T \\ -C_h^T \end{bmatrix} \\ \begin{bmatrix} B_p^T & (I+KD_p^T)B_h^T \end{bmatrix} P - \begin{bmatrix} KC_p & -C_h \end{bmatrix} & -(I+KD_p)-(I+KD_p^T) \end{bmatrix} < 0, \quad (3.96)$$

where $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$. Note that positivity of P is not required. Assume that P and P_{12} are not singular, and $P_{22} < 0$. The inverse of $P^{-1} = Q$ can be expressed as follows

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = \begin{bmatrix} Q_{11}P_{11}+Q_{12}P_{12}^T & Q_{11}P_{12}+Q_{12}P_{22} \\ Q_{12}^TP_{11}+Q_{22}P_{12}^T & Q_{12}^TP_{12}+Q_{22}P_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (3.97)$$

then, the following relations are true:

$$\begin{bmatrix} Q_{11} & I \\ Q_{12}^T & 0 \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ I & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ P_{11} & P_{12} \end{bmatrix} \quad (3.98)$$

Applying a congruence transformation $diag(\begin{bmatrix} Q_{11} & Q_{12} \\ I & 0 \end{bmatrix}, I)$, and then a second congruence transformation, $diag(S_{11}, I, I) = diag(Q_{11}^{-1}, I, I)$, matrix (3.96) is equivalent to

$$\begin{bmatrix} S_{11}A_p + A_p^TS_{11} & S_{11}A_p + A_p^TP_{11} + KC_p^TB_h^TP_{12}^T + S_{11}Q_{12}A_h^TP_{12}^T & S_{11}B_p - KC_p^T + S_{11}Q_{12}C_h^T \\ \star & P_{11}A_p + KP_{12}B_hC_p + A_p^TP_{11} + KC_p^TB_h^TP_{12}^T & P_{11}B_p + P_{12}B_h(I+KD_p) - KC_p^T \\ \star & \star & -(I+KD_p)-(I+KD_p^T) \end{bmatrix} < 0 \quad (3.99)$$

Now, define the unstructured matrices:

$$\bar{A}_u = P_{12}A_hQ_{12}^TS_{11}, \quad (3.100)$$

$$\bar{B}_u = P_{12}B_h, \quad (3.101)$$

$$\bar{C}_u = C_hQ_{12}^TS_{11}. \quad (3.102)$$

Then, equation (3.99) is equivalent to

$$\begin{bmatrix} S_{11}A_p + A_p^TS_{11} & S_{11}A_p + A_p^TP_{11} + KC_p^T\bar{B}_u^T + \bar{A}_u^T & S_{11}B_p - KC_p^T + \bar{C}_u^T \\ \star & P_{11}A_p + K\bar{B}_uC_p + A_p^TP_{11} + KC_p^T\bar{B}_u^T & P_{11}B_p + \bar{B}_u(I+KD_p) - KC_p^T \\ \star & \star & -(I+KD_p)-(I+KD_p^T) \end{bmatrix} < 0, \quad (3.103)$$

which is condition (3.92).

Now the attention can shift to the description of the anti-causal multiplier. From Lemma 3.3.2 consider equation (3.90) and apply the congruence transformation $diag(S_{11}Q_{12}, I)$, then equation (3.90) is equivalent to

$$\begin{bmatrix} S_{11}Q_{12}A_h^TYQ_{12}^TS_{11} + S_{11}Q_{12}YA_hQ_{12}^TS_{11} - \lambda S_{11}Q_{12}YQ_{12}^TS_{11} & -S_{11}Q_{12}YB_h \\ -B_h^TYQ_{12}^TS_{11} & -\mu I \end{bmatrix} < 0. \quad (3.104)$$

Let $Y = P_{22}$, using the definition of the inverse of P from equation (3.97) and the shorthand notation from equations (3.100),(3.101),(3.102), equation (3.104) is equivalent to

$$\begin{bmatrix} -\bar{A}_u^T - \bar{A}_u - \lambda(P_{11} - S_{11}) & \bar{B}_u \\ \bar{B}_u^T & -\mu I \end{bmatrix} < 0,$$

which is equation (3.93).

Similarly, from Lemma 3.3.2 consider equation (3.91) and use the congruence transformation $diag(S_{11}Q_{12}, I, I)$. Assume again $Y = P_{22}$, use the definition of the inverse of P from equation (3.97) and the shorthand notation from equations (3.100), (3.101), (3.102), then equation (3.91) is equivalent to

$$\begin{bmatrix} -\lambda(P_{11} - S_{11}) & 0 & \bar{C}_u^T \\ 0 & (1-\mu)I & 0 \\ \bar{C}_u & 0 & 1 \end{bmatrix} > 0,$$

which is equation (3.94).

Finally, in order to remove the assumption about P, it is necessary to show that the feasibility of equations (3.92), (3.93) and (3.94) ensures the existence of a non-singular P_{12}, P and $P_{22} < 0$.

Using equation (3.92) and the Kalman-Yakubovich-Popov Lemma [41],

$$S_{11} = Q_{11}^{-1} > 0 \iff Q_{11} > 0,$$

and therefore non-singular. Using equation (3.94) it is known that $P_{11} - S_{11} < 0$. Without loss of generality, make $P_{22} = -I$, then:

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & -I \end{bmatrix} = \begin{bmatrix} Q_{11}P_{11} + Q_{12}P_{12}^T & Q_{11}P_{12} - Q_{12} \\ Q_{12}^TP_{11} + Q_{22}P_{12}^T & Q_{12}^TP_{12} - Q_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (3.105)$$

where it follows that

$$-(P_{11}-S_{11})>0 \iff -Q_{11}P_{11}+I>0,$$

then there exists $Q_{12}P_{12}^T>0$ such that $-Q_{11}P_{11}+I-Q_{12}P_{12}^T=0$. Using

$$Q_{11}P_{12}-Q_{12}=0 \iff Q_{11}P_{12}=Q_{12},$$

implies that P_{12} is non-singular, by noting that

$$Q_{12}P_{12}^T>0 \iff Q_{11}P_{12}P_{12}^T>0 \iff P_{12}P_{12}^T>0.$$

Finally, using the fact that

$$-Q_{11}P_{11}+I-Q_{12}P_{12}^T=0 \iff Q_{11}=(P_{11}+P_{12}P_{12}^T)^{-1},$$

the structure of the inverse of P is:

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & -I \end{bmatrix}^{-1} = \begin{bmatrix} (P_{11}+P_{12}P_{12}^T)^{-1} & (P_{11}+P_{12}P_{12}^T)^{-1}P_{12} \\ P_{12}^T(P_{11}+P_{12}P_{12}^T)^{-1} & -I+P_{12}^T(P_{11}+P_{12}P_{12}^T)^{-1}P_{12} \end{bmatrix},$$

consequently, P is not singular.

Therefore, conditions of Problem 3.2.6 are fulfilled, and the feedback interconnection is $\mathcal{L}_2[0,\infty)$ stable. □

3.3.2.1 Addition of Popov multiplier

Popov multiplier is a limiting case of the Zames-Falb multiplier [64], i.e.

$$1+qs=\lim_{\varepsilon\to 0}\frac{1+qs}{1+\varepsilon s}.$$

A detailed analysis can be found in [54]. Therefore, since the method proposed originally in [1] is restricted to causal multipliers and its anti-causal part has been developed in the previous section, the addition of a Popov multiplier as proposed in [33, 63] to these causal or anti-causal searches improves the parametrization of the Zames-Falb multiplier. The result in [33] generates a non-causal Zames-Falb multiplier with a limited anti-causal part whereas the following result will generate Zames-Falb multipliers with a limited causal part.

Proposition 3.3.4. *Given a LTI plant $G(s) \in \mathcal{RH}_\infty \sim (A_p, B_p, C_p, 0)$ and a constant $k > 0$. There exist a multiplier $M(s) = M_{ZF} + (\nu + \eta s)$, $M_{ZF} \in \mathcal{M}_{odd}$ fulfilling conditions of Problem 3.2.6 if there exist positive definite symmetric matrices $S_{11} > 0, P_{11} > 0$, unstructured matrices $\tilde{A}_u, \tilde{B}_u, \tilde{C}_u$, and scalars $\nu > 0$, $\eta \in \mathbb{R}$, $\mu > 0$ and $\lambda > 0$ such that the following inequalities are satisfied.*

$$\begin{bmatrix} S_{11}A_p + A_p^T S_{11} & S_{11}A_p + A_p^T P_{11} + KC_p^T \bar{B}_u^T + \bar{A}_u^T & S_{11}B_p - K(I + vI + \eta A_p^T)C_p^T + \bar{C}_u^T \\ \star & P_{11}A_p + A_p^T P_{11} + K\bar{B}_u C_p + KC_p^T \bar{B}_u^T & P_{11}B_p + \bar{B}_u - K(I + vI + \eta A_p^T)C_p^T \\ \star & \star & -2I - 2vI - \eta K(C_p B_p + B_p^T C_p^T) \end{bmatrix} < 0 \tag{3.106}$$

$$\begin{bmatrix} -\bar{A}_u^T - \bar{A}_u - \lambda(P_{11} - S_{11}) & \bar{B}_u \\ \bar{B}_u^T & -\mu I \end{bmatrix} < 0, \tag{3.107}$$

$$\begin{bmatrix} -\lambda(P_{11} - S_{11}) & 0 & \bar{C}_u^T \\ 0 & 1 - \mu & 0 \\ \bar{C}_u & 0 & 1 \end{bmatrix} > 0; \tag{3.108}$$

This result can be shown either using IQC machinery and following [1], [63], or using classical loop transformation techniques. Here, the latter is preferred for the sake of additional insight.

Proof. In this case, the class of Zames-Falb multiplier plus Popov is defined by the addition of a Popov multiplier and a Zames-Falb multiplier, i.e.

$$M_{PZF}(s) = M_{ZF}(s) + v + \eta s \tag{3.109}$$

where $M_{ZF} \in \mathcal{M}_{\text{odd}}$, $\nu > 0$, $\eta \in \mathbb{R}$. Then, the existence of a multiplier $M_{PZF}(s)$ such that

$$Re\{M_{PZF}(j\omega)(1 + kG(j\omega))\} > 0 \tag{3.110}$$

implies the $\mathcal{L}_2[0,\infty)$ stability of the feedback interconnection of Figure 2.4 via Theorem 2.2.6.

First, note that the structure of $M_{PZF}(j\omega)(1 + kG(j\omega))$ is non-proper unless the constant term in $G(s)$ is zero. Then, the following expressions are equivalent to equation (3.110):

$$Re\{(1 - C_h(j\omega I - A_h)^{-1}B_h + vI + \eta j\omega I)\tilde{G}(j\omega)\} \geq \varepsilon I \forall \omega \in \mathbb{R}$$

$$\Longleftrightarrow Re\left\{\left[\begin{array}{ccc|c} A_p & 0 & 0 & B_p \\ KB_hC_p & A_h & 0 & B_h \\ 0 & 0 & A_p & B_p \\ \hline (I+vI)KC_p & -C_h & \eta KC_pA_p & I+vI+\eta KC_pB_p \end{array}\right]\right\} \geq \varepsilon I \forall\omega\in\mathbb{R}. \tag{3.111}$$

Using the transformation $T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -I & 0 & I \end{bmatrix}$, and removing uncontrollable modes, equation (3.111) results in

$$\Longleftrightarrow Re\left\{\left[\begin{array}{cc|c} A_p & 0 & B_p \\ KB_hC_p & A_h & B_h \\ \hline KC_p(I+vI+\eta A_p) & -C_h & I+vI+\eta KC_pB_p \end{array}\right]\right\} \geq \varepsilon I \forall\omega\in\mathbb{R}. \tag{3.112}$$

Now, using the proof for Proposition 3.3.1, replace equation (3.95) with equation (3.112), the rest of the proof follows in similar fashion. Finally, equation (3.103) is replaced by equation (3.106), which is ensured in the conditions for this proposition. □

Remark 3.3.5. *As commented in [50] and [62], a search over λ is required for obtaining competitive results. In the causal Zames-Falb search [1, 50] as well as in the anti-causal search presented in this section, the maximum slope k appears to have a quasi-convex dependence with respect to λ. However, the addition of the Popov multiplier, in [33] and in this section changes this behaviour, and several local maxima can appear.*

Now, the reconstruction of the multiplier can be carried on as suggested in [1], solving the following equations:

$$\bar{A}_u = P_{12}A_hQ_{12}^TS_{11}$$
$$\bar{B}_u = P_{12}B_h$$
$$\bar{C}_u = C_hQ_{12}^TS_{11}.$$

This process starts using $P_{22} = -I$ with the following equation:

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & -I \end{bmatrix} = \begin{bmatrix} Q_{11}P_{11}+Q_{12}P_{12}^T & Q_{11}P_{12}-Q_{12} \\ Q_{12}^TP_{11}+Q_{22}P_{12}^T & Q_{12}^TP_{12}-Q_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

where it follows that

$$
\begin{aligned}
P_{12}P_{12}^T &= S_{11} - P_{11}, \\
Q_{12}^T S_{11} &= P_{12}^T.
\end{aligned}
$$

Then, the solutions are

$$
\begin{aligned}
A_h &= P_{12}^{-1}\bar{A}_u P_{12}^{-T}, \\
B_h &= P_{12}^{-1}\bar{B}_u, \\
C_h &= \bar{C}_u P_{12}^{-T}.
\end{aligned}
$$

Finally, the multiplier $M(j\omega)$ that fulfils all conditions for Theorem 2.2.6 is $M_{PZF}(j\omega) = 1 - C_h(j\omega I - A_h)^{-1}B_h + v(I + \frac{\eta}{v}j\omega I)$.

3.4 Numerical Results

The multiplier set in Theorem 2.2.6 is divided according to their structure in Figure 3.2. For monotone non-linearities there exists two multiplier synthesis: summation of deltas [28], and the specialization of [31]. In contrast, for the class of odd monotone non-linearities, the list of multipliers are the following: causal rational transfer function [1] and the extension to non-causal multipliers using Popov multiplier [63, 33]; a complement of the causal method, namely, the anti-causal Proposition 3.3.1 and the extension to non-causal Proposition 3.3.4; finally summation of exponentials [31] and Park's method [30], which are originally non-causal. It can be noted that none of the structures are able to cover the complete multipliers set. At the same time it can be seen that there is no method that provides the best results for all of the examples.

Nine examples (see Table 3.1) are revisited in order to analyse the performance of each method. Examples 1-6 were proposed in [1]. Example 7 was given in [35] and example 8 was proposed in [29]. Example 9 is new. In each example, the non-linearity is allowed within sector and slope $[0,k]$, and then the bisection method is used to find the maximum k for which the closed loop is stable. For examples 1, 2 and 9, results for the anti-causal methods are obtained using $1/k + G(j\omega)$ rather than $1 + kG(j\omega)$ as the numerical results sometimes differ.

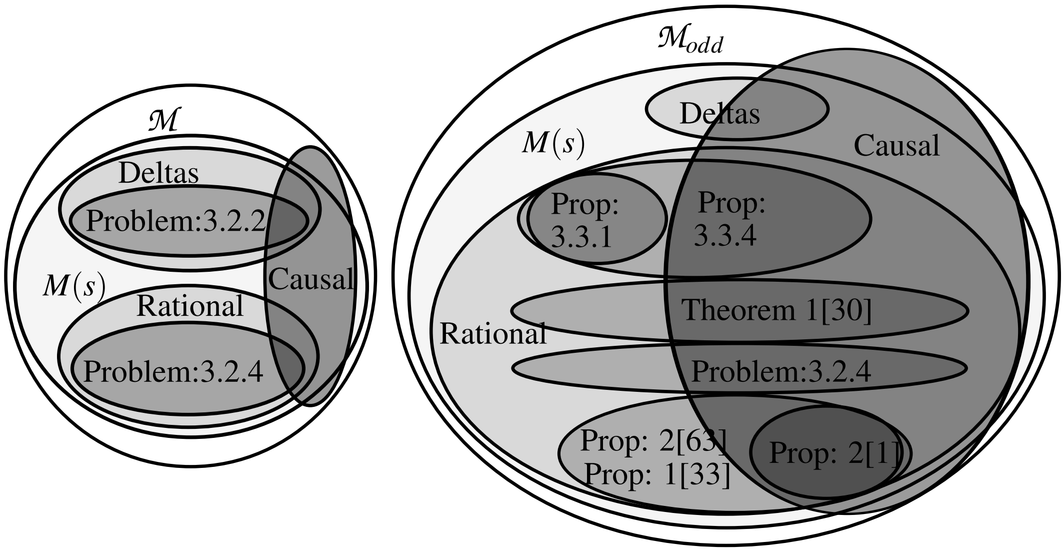


Figure 3.2: Schematic representation of multiplier sets

Table 3.1: Example list

Example	Transfer Function
1 [1]	$P_1(s) = \frac{s^2-0.2s-0.1}{s^3+2s^2+s+1}$
2 [1]	$P_2(s) = -P_1(s)$
3 [1]	$P_3(s) = \frac{s^2}{s^4+0.2s^3+6s^2+0.1s+1}$
4 [1]	$P_4(s) = -P_3(s)$
5 [1]	$P_5(s) = \frac{s^2}{s^4+0.0003s^3+10s^2+0.0021s+9}$
6 [1]	$P_6(s) = -P_5(s)$
7 [35]	$P_7(s) = \frac{s^2}{s^3+2s^2+2s+1}$
8 [29]	$P_8(s) = 9.432\frac{(s^2+15.6s+147.8)(s^2+2.356s+56.21)(s^2-0.332s+26.15)}{(s^2+2.588s+90.9)(s^2+11.79s+113.7)(s^2+14.84s+84.05)(s+8.83)}$
9 (new)	$P_9(s) = \frac{s^2}{s^4+5.001s^3+7.005s^2+5.006s+6}$

Table 3.2: Sector/slope bound obtainable using various multiplier synthesis methods

Example	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6	Ex. 7	Ex. 8	Ex. 9
Park [30]	**4.5894**	**1.0894**	0.7083	0.7883	0.00183	0.00183	**10,000+**	62.5691	26.0097
Turner [1, 63, 33] (Proposition 3.2.7)	2.2428	**1.0894**	0.7049	0.8526	0.00181	0.00095	17.605	**87.3854**	5.2643
Anti-causal Turner (Proposition 3.3.1)	**4.5894**	1.0745	0.9846	0.6135	0.00095	0.00182	**10,000+**	21.6190	38.5982
Turner+Popov [63, 33] (Proposition 3.2.8)	3.2897	**1.0894**	0.7760	1.0792	**0.00333**	0.00318	17.724	**87.3854**	13.7834
Anti-causal Turner+Popov (Proposition 3.3.4)	**4.5894**	1.0745	1.4513	0.7222	0.00319	**0.00333**	**10,000+**	22.4304	**91.0858**
Deltas [28, 26, 29] (Algorithm 1)	**4.5894**	**1.0894**	**1.6122**	**1.2652**	Unreliable	Unreliable	95.406	83.1430	80.2735
Exponentials [32]	**4.5894**	1.0803	1.1192	0.9107	0.000576	0.0012	**10,000+**	13.4375	49.5643
Nyquist value	4.5894	1.0894	∞	3.5	∞	1.7142	∞	87.3854	∞

Table 3.2 shows the results for the plants in Table 3.1. The Nyquist value is shown as an upper limit for the maximum sector-slope, and it represents the maximum k such that KG satisfies the Nyquist criterion for all $0 < K < k$. Given that examples 1 and 2 are third order plants, the Kalman Conjecture is valid for these plants [53], therefore the Nyquist value becomes the supremum for the maximum stabilizing sector-slope.

The summation of $\delta(.)$ distributions, using Algorithm 1, can achieve better results on Example 4, but it is necessary to perform an evaluation of equation (3.6) in Problem 3.2.2 for a large number of frequencies and times T_N in order to be accurate, because the transfer function is not rational. Examples 5 and 6 show that this lack of accuracy leads to bad performance with lightly damped plants. Additionally, the initial time t_1 is crucial for the convergence of the problem, because the algorithm can easily converge to a local minimum. This method contains no conservatism when computing the $\mathcal{L}_1(-\infty,\infty)$ norm for the function $M_N(s)$, and therefore the main identified source of conservatism lies in the correct choice of the vector T_N.

The summation of exponentials is more accurate in the computation of equation (2.9) when using Theorem 2.2.6, because the test of Problem 3.2.4 can be executed using LMIs. The $\mathcal{L}_1(-\infty,\infty)$ norm (equation (2.8)) for the generated multiplier $M_N(s)$ in [32] is approximated, and seems to be the only source of conservatism. This method reaches good performance for Examples 1 and 7 with low order multipliers, however, in order to improve the results for the rest of the examples, a multiplier of greater order was required. This method presents an important advantage over Safonov's original approach, because it requires no initial selection of time t_1. This automatic search explains why this method is able to find the maximum sector boundaries proposed by the Kalman conjecture in Examples 1 and 2 without problem. In [1] the method proposed by Chen [31] is assumed to be more conservative than Park's method [30]. Nevertheless, using the synthesis described in Section 3.2.2, the above statement is not general as shown in examples 3 and 4.

Park's method was included as reference for minimal expected performance, since it can be seen as a convex search within the first order Zames-Falb multipliers [65] with no conservatism in the $\mathcal{L}_1(-\infty,\infty)$ norm, even with lightly damped plants.

Despite this, these two methods can synthesis multipliers within $\mathcal{M}$ as close as desired to any multiplier in this set by increasing N. Park's results show that there exist multipliers within $\mathcal{M}$ which cannot be achieved by these two methods. Therefore, numerical issues are the main source of conservatism in these methods for lightly damped plants, see examples 5 and 6.

Remarkably, results for the anti-causal Turner+Popov method improves the maximum slope where the Park's method is better than the causal method [1] (examples 1, 3 and 6). Nevertheless, the addition of the Popov multiplier in [33, 63] and its implementation for the anti-causal method provides a reliable and competitive method if they are combined.

Example 9 has been designed to show under what circumstances the methods proposed in this work are expected to provide better results than alternative methods in the literature. Anti-causal multipliers are expected to be more appropriate than causal multipliers for achieving negative values of the phase. In addition, temporal searches such as summation of $\delta(.)$ in Section 3.2.1 can be very inaccurate for lightly damped plants and summation of exponentials from Section 3.2.2 can be conservative within the set of Zames-Falb multipliers, due to the use of a triangle inequality for bounding the $\mathcal{L}_1(-\infty,\infty)$ norm (see equation (18) in [32]); one more time, in plants with slightly damped poles it can be a drawback. Therefore, the proposed example has two resonant poles at $-0.0005 \pm 1j$, two zeros at 0 to ensure the Nyquist value at infinity, and two other poles at -2 ant -3 so the order is more than 3.

The numerical results show that the causal multipliers are more appropriate when the Nyquist plot of the plant reaches the minimum value of its real part in the third quadrant (Examples 4 and 6), whereas anti-causal multipliers are more appropriate when this minimum is reached in the second quadrant (Examples 1,3,5 and 7). This empirical rule agrees with the analysis in Section 2.2.2.

3.5 Conclusions

The set of Zames–Falb multipliers is not fully explored by the current methods for multiplier synthesis. The elusive description of the $\mathcal{L}_1(-\infty,\infty)$ norm of a function as a frequency condition has introduced fundamental conservatism into most methods that synthesize multipliers based in rational transfer functions. However, this problem is well-known in the literature and has been the focus of the multiplier synthesis problem.

The summation of exponentials deals with the problem by allowing only signals with positive impulse in order to find the value of the norm. The summation of delta distributions appears as the simplest approach to find the exact value of the norm of the function. However, the simplicity of the norm calculation trades off with the computational complexity required to verify any other conditions expressed in the frequency domain. This complexity leads to the need for expert experience in using the method in order to achieve its best performance. In contrast, the rational causal multiplier introduces only an approximate upper bound for the $\mathcal{L}_1(-\infty,\infty)$ norm, but simplifies the test for any other frequency condition, because they can be reduced an LMI. The main advantages of these methods is their multiplier flexibility, both summation of exponentials and summation of delta distributions require little information from the Linear Time Invariant plant in order to define their structure. This flexibility is a potential advantage when Zames-Falb multipliers are used in problems outside the domain of Absolute Stability. Problems like Absolute Stability with delay [66, 67, 68, 69, 70, 71, 72, 73, 74, 75] and stability of systems with hysteresis [76] should find in this summary an ideal starting point when choosing a way to synthesize Zames-Falb multipliers.

In order to extend and simplify the use of Zames-Falb multipliers for more general problems, this thesis presented an algebraic correction to the LMI search proposed in [32] to perform a multiplier search.

In contrast, the search for causal transfer functions seems to be an optimal solution to find Zames-Falb multipliers, given that it can define the position of poles and zeros in one step, but fails to achieve the maximum performance for all the selected examples. For the first order Zames-Falb multipliers, theoretical results have shown that causal Zames-Falb multipliers have a corresponding constraint on their phase lead. An example given in the literature has been used to show that a non-causal multiplier obtained by inspection beats all the convex searches if they are restricted artificially to causal Zames-Falb multipliers. Therefore, the causality of the transfer function has been identified via examples in this paper to be a significant source of conservatism.

Using the method developed in [1], a search of anti-causal multipliers has been proposed, which is a complementary solution to the search of causal multipliers. The new search has been tested and it improves the results given by Turner's method [1] in the examples where it is not competitive. A similar extension to that of [63] is proposed to avoid the anti-causal limitation. The anti-causal search developed in this paper confirms that a major source of conservatism for some examples in [1] is the restriction to causal multipliers. The combination of causal and anti-causal methods with the addition of the Popov multiplier generates results at least competitive with the best in the literature. However, the delta method can provide better results in some cases due to its advantages measuring the $\mathcal{L}_1(-\infty,\infty)$ norm of the multiplier. Finding an efficient search over the entire class of Zames-Falb multipliers remains an open problem.

Chapter 4

IQC and Dissipativity

4.1 Introduction

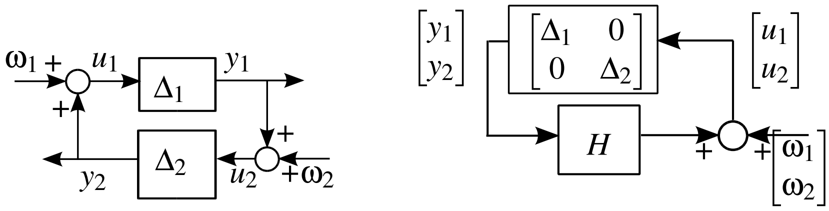
The Introduction of this thesis presented the two stability results, obtained using the factorization proposed by [42]. The first result, Lemma 2.3.7, attempts to introduce non-causal multipliers for dissipative systems. This result makes use of IQC framework [35] as the way to analyse the non-linearities because this descriptions allows to transforms without difficulty a test for stability into a Linear Matrix Inequality (LMI) problem. Moreover, Lemma 2.3.7 is a result that does not depend on any homotopy argument. However, the way loop transformations are used imposed a restrictive condition on the multiplier structure for the equivalent system, namely its causal invertibility.

The second result, Lemma 2.3.8, attempts to use conditions similar to that of the IQC Theorem to show stability via Passivity Theorem. However, this lemma is unable to completely capture the structure of the IQC to describe non-linear systems and therefore falls short in achieving a generalization of Passivity with arbitrary dynamic multipliers.

The main results of this chapter are Theorem 4.2.2, Corollary 4.2.4 and Lemma 4.2.5. Theorem 4.2.2 makes successful use of the IQC framework to show the stability of the feedback interconnection of two non-linear systems. The trade off made by Theorem 4.2.2 is that it requires that every operator to be open loop $\mathcal{L}_2[0,\infty)$ stable before doing the feedback interconnection. Subsequently, Corollary 4.2.4 presents a link between the standard representation of IQCs from [35] and the standard representation of Dissipativity, but with the added freedom of dynamic multipliers. Finally, Lemma 4.2.5 shows the equivalence of Theorem 4.2.2 and the IQC Theorem [35].

4.2 Stability Using the IQC Theorem

This section develops an explicit stability theorem using exclusively the IQC Theorem for the system in Figure 4.1a.



(a) Original closed loop (b) Modified closed loop

Figure 4.1: Equivalent System Closed loop

For the feedback interconnection in Figure 4.1a, the following remarks summarize the conditions for well posedness and stability. Consider $H = \begin{bmatrix} 0 & I_n \\ I_m & 0 \end{bmatrix}$.

Lemma 4.2.1. *The well posedness of the non-linear operator*

$$\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \left[I - H \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \right]^{-1} \tag{4.1}$$

is equivalent to the well posedness of the system in Figure 4.1a.

Furthermore, the $\mathcal{L}_2[0,\infty)$ *stability of the causal non-linear operator* (4.1) *is equivalent to the* $\mathcal{L}_2[0,\infty)$ *stability of the system in Figure 4.1a.*

Proof. For two causal bounded non-linear operators $\Delta_1 : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^n$ and $\Delta_2 : \mathcal{L}_{2e}^n \to \mathcal{L}_{2e}^m$, the feedback interconnection $[\Delta_1, \Delta_2]$ in Figure 4.1a can also be represented by the following equation:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \left[I - H \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \right]^{-1} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}. \tag{4.2}$$

Then, the well posedness of the mapping $[\omega_1^T, \omega_2^T]^T$ to outputs $[y_1^T, y_2^T]^T$ is equivalent to the well posedness of the operator $\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \left[I - H \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \right]^{-1}$ shown in Figure 4.1b. Additionally, if the system in Figure 4.1a is well posed, the $\mathcal{L}_2[0,\infty)$ stability of the mapping from $[\omega_1^T, \omega_2^T]^T$ to outputs $[y_1^T, y_2^T]^T$ (Figure 4.1a) is equivalent to the $\mathcal{L}_2[0,\infty)$ stability of the non-linear causal operator $\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \left[I - H \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \right]^{-1}$ (Figure 4.1b).

□

Note that the feedback interconnection in Figure 4.1a can be represented by the equivalent loop in Figure 4.1b. Hence, to show the stability of the system in Figure 4.1a, it could be used the equivalent representation in Figure 4.1b via Remark 4.2.1.

The following Theorem presents an improved version of Lemma 2.3.7 and Lemma 2.3.8, but deduced from the IQC Theorem. Note that this result will still depend on the continuous well posedness of the interconnection $[\tau\Delta_1, \tau\Delta_2]$ for all $\tau \in [0,1]$. However, this is an equivalent representation of the IQC Theorem, thus successfully removes condition in equation (2.22) from Lemma 2.3.7, and unlike Lemma 2.3.8, it makes full use of the IQC library.

Theorem 4.2.2. *Given two Hermitian measurable functions* $\Pi_1, \Pi_2 : \mathbb{R} \to \mathbb{C}^{m+n\times m+n}$ *and given two causal bounded operators* $\Delta_1 : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^n$, $\Delta_2 : \mathcal{L}_{2e}^n \to \mathcal{L}_{2e}^m$. *Suppose that:*

1. $\forall\tau \in [0,1]$, *the feedback interconnection* $[\tau\Delta_1, \tau\Delta_2]$ *is well posed,*
2. Δ_1 *and* Δ_2 *satisfy the following IQCs:*

$$\left\langle \begin{pmatrix} \tilde{y}_1 \\ \tilde{x}_1 \end{pmatrix}, \begin{pmatrix} \Pi_{1,11}(j\omega) & \Pi_{1,12}(j\omega) \\ \Pi_{1,12}^{\sim}(j\omega) & \Pi_{1,22}(j\omega) \end{pmatrix} \begin{pmatrix} \tilde{y}_1 \\ \tilde{x}_1 \end{pmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$
$$\forall \tilde{x}_1 \in \mathcal{H}_2^m, \tilde{y}_1 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{\tilde{x}_1\})\}, \quad (4.3)$$

$$\left\langle \begin{pmatrix} \tilde{x}_2 \\ \tilde{y}_2 \end{pmatrix}, \begin{pmatrix} \Pi_{2,11}(j\omega) & \Pi_{2,12}(j\omega) \\ \Pi_{2,12}^{\sim}(j\omega) & \Pi_{2,22}(j\omega) \end{pmatrix} \begin{pmatrix} \tilde{x}_2 \\ \tilde{y}_2 \end{pmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$
$$\forall \tilde{x}_2 \in \mathcal{H}_2^n, \tilde{y}_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{\tilde{x}_2\})\}. \quad (4.4)$$

with $\Pi_{1,22}(j\omega), \Pi_{2,11}(j\omega) \geq 0 \quad \forall\omega \in \mathbb{R}$

Then, the feedback interconnection $[\Delta_1, \Delta_2]$ *is* $\mathcal{L}_2[0,\infty)$ *stable if there exist some constants* $\lambda, \varepsilon > 0$ *such that*

$$\Pi_1(j\omega) + \lambda\Pi_2(j\omega) \leq -\varepsilon I \quad \forall\omega \in \mathbb{R}. \quad (4.5)$$

Proof. Firstly, note equation (4.5) implies that $\Pi_{1,11}(j\omega), \Pi_{2,22}(j\omega)$ are uniformly strictly negative over all frequencies since

$$\Pi_{1,11}(j\omega) + \lambda\Pi_{2,11}(j\omega) + \varepsilon I \leq 0 \implies \Pi_{1,11}(j\omega) \leq -\lambda\Pi_{2,11}(j\omega) - \varepsilon I < 0 \quad \forall\omega \in \mathbb{R},$$

and

$$\Pi_{1,22}(j\omega)+\lambda\Pi_{2,22}(j\omega)+\varepsilon I \leq 0 \implies \lambda\Pi_{2,22}(j\omega) \leq -\Pi_{1,22}(j\omega)-\varepsilon I < 0 \quad \forall\omega\in\mathbb{R}.$$

For simplicity of the proof, define: $\tilde{\Pi}_1(j\omega) = H^T\Pi_1(j\omega)H$. Therefore, the following IQCs hold for all $\tau \in [0,1]$ if, and only if, the products (4.3) and (4.4) hold, as shown in Remark 2 in [35].

$$\left\langle \begin{pmatrix} \tilde{x}_1 \\ \tau\tilde{y}_1 \end{pmatrix}, \tilde{\Pi}_1(j\omega)\begin{pmatrix} \tilde{x}_1 \\ \tau\tilde{y}_1 \end{pmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0 \quad \forall \tilde{x}_1 \in \mathcal{H}_2^m, \tilde{y}_1 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{\tilde{x}_1\})\}, \tag{4.6}$$

$$\left\langle \begin{pmatrix} \tilde{x}_2 \\ \tau\tilde{y}_2 \end{pmatrix}, \Pi_2(j\omega)\begin{pmatrix} \tilde{x}_2 \\ \tau\tilde{y}_2 \end{pmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0 \quad \forall \tilde{x}_2 \in \mathcal{H}_2^n, \tilde{y}_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{\tilde{x}_2\})\}. \tag{4.7}$$

The feedback loop can be seen as the bounded causal non-linear operator $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$ interconnected in positive feedback with the linear static system $H = \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix} \in \mathcal{RH}_\infty^{m+n\times m+n}$.
Then, the following conditions hold:

1. $\forall\tau \in [0,1]$, the interconnection $[H, \tau\Delta]$ is well posed

2. Adding equation (4.6) and equation (4.7), multiplied by some constants $\alpha_1, \alpha_2 > 0$, will remain positive $\forall\tau \in [0,1]$, i.e.

$$\alpha_1\left\langle \begin{pmatrix} \tilde{x}_1 \\ \tau\tilde{y}_1 \end{pmatrix}, \tilde{\Pi}_1(j\omega)\begin{pmatrix} \tilde{x}_1 \\ \tau\tilde{y}_1 \end{pmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} + \alpha_2\left\langle \begin{pmatrix} \tilde{x}_2 \\ \tau\tilde{y}_2 \end{pmatrix}, \Pi_2(j\omega)\begin{pmatrix} \tilde{x}_2 \\ \tau\tilde{y}_2 \end{pmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$
$$\forall \tilde{x}_1 \in \mathcal{H}_2^m, \tilde{x}_2 \in \mathcal{H}_2^n, \tilde{y}_1 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{\tilde{x}_1\})\}, \tilde{y}_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{\tilde{x}_2\})\}.$$

The following statements are equivalent:

$\forall\tau \in [0,1]$ and $\forall\alpha_1, \alpha_2 > 0$,

$$\left\langle \begin{pmatrix} \tilde{x}_1 \\ \tau\tilde{y}_1 \\ \hdashline \tilde{x}_2 \\ \tau\tilde{y}_2 \end{pmatrix}, \left(\begin{array}{c:c} \alpha_1\tilde{\Pi}_1(j\omega) & 0 \\ \hdashline 0 & \alpha_2\Pi_2(j\omega) \end{array} \right) \begin{pmatrix} \tilde{x}_1 \\ \tau\tilde{y}_1 \\ \hdashline \tilde{x}_2 \\ \tau\tilde{y}_2 \end{pmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$

$$\forall \tilde{x}_1 \in \mathcal{H}_2^m, \tilde{x}_2 \in \mathcal{H}_2^n, \tilde{y}_1 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{\tilde{x}_1\})\}, \tilde{y}_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{\tilde{x}_2\})\}.$$

$\iff$ $\forall\tau \in [0,1]$ and $\forall\alpha_1, \alpha_2 > 0$,

$$\left\langle \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \hdashline \tau\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} \end{pmatrix}, \Gamma \left(\begin{array}{c:c} \alpha_1\tilde{\Pi}_1(j\omega) & 0 \\ \hdashline 0 & \alpha_2\Pi_2(j\omega) \end{array} \right) \Gamma \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \hdashline \tau\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} \end{pmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$

$$\forall \tilde{x}_1 \in \mathcal{H}_2^m, \tilde{x}_2 \in \mathcal{H}_2^n, \tilde{y}_1 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{\tilde{x}_1\})\}, \tilde{y}_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{\tilde{x}_2\})\}, \quad (4.8)$$

$$\text{where } \Gamma = \left(\begin{array}{cc:cc} I_m & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ \hdashline 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_m \end{array} \right).$$

Given that $\tau\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}$ is the Fourier transform of $\tau\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, with $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathcal{F}^{-1}\left\{\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}\right\}$ then the inner product (4.8) satisfies the Integral Quadratic Constraint $\forall\tau \in \varepsilon[0,1]$ with

$$\Pi(j\omega) = \Gamma \left[\begin{array}{c:c} \alpha_1\tilde{\Pi}_1(j\omega) & 0 \\ \hdashline 0 & \alpha_2\Pi_2(j\omega) \end{array} \right] \Gamma$$

parametrized by α_1, α_2.

3. The following statements are equivalent:

   $\exists\varepsilon_1 > 0$, $\alpha_1, \alpha_2 \geq 0$ such that

$$\begin{bmatrix} H \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} H \\ I \end{bmatrix} \leq -\varepsilon_1 I \quad \forall\omega \in \mathbb{R},$$

$\iff$ $\exists\varepsilon_1 > 0$, $\alpha_1, \alpha_2 \geq 0$ such that

$$\begin{bmatrix} H \\ I \end{bmatrix}^* \left(\begin{array}{c|c} \alpha_1 \tilde{\Pi}_1(j\omega) & 0 \\ \hline 0 & \alpha_2 \Pi_2(j\omega) \end{array} \right) \begin{bmatrix} H \\ I \end{bmatrix} \leq -\varepsilon_1 I \quad \forall \omega \in \mathbb{R},$$

since $\begin{bmatrix} H \\ I \end{bmatrix}^* \Gamma = \begin{bmatrix} H \\ I \end{bmatrix}^*$

$\iff$ $\exists\varepsilon > 0$, $\lambda \geq 0$ such that

$$\Pi_1(j\omega) + \lambda \Pi_2(j\omega) \leq -\varepsilon I \quad \forall \omega \in \mathbb{R}. \tag{4.9}$$

[Multiply all the inequality by $\frac{1}{\alpha_1}$, and let $\lambda = \frac{\alpha_2}{\alpha_1}$ and $\varepsilon = \frac{\varepsilon_1}{\alpha_1}$. Note also that $\tilde{\Pi}_1(j\omega) = H^T \Pi_1(j\omega) H$]

Note that the last equation is true from equation (4.5).

Using the IQC Theorem and Conditions 1,2,3, then the feedback interconnection $\left[H, \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \right]$ is $\mathcal{L}_2[0,\infty)$ stable. By Lemma 4.2.1, the equivalent feedback interconnection $[\Delta_1, \Delta_2]$ is $\mathcal{L}_2[0,\infty)$ stable.

□

Although Jönsson [43] first outlines these kind of interconnections, it does not explore the limitations imposed by the IQC Theorem over the non-linear systems, namely equation 4.9.

Regarding Lemma 2.3.7 and Lemma 2.3.8, first note that the resulting corollary has no link with Passivity or Dissipativity, because this result depends on an homotopy argument to show $\mathcal{L}_2[0,\infty)$ stability. Second, note that the operators $\Pi_1(j\omega)$ and $\Pi_2(j\omega)$ are required to hold $\Pi_{1,22}(j\omega) \geq 0$, $\Pi_{2,11}(j\omega) \geq 0$, and from equation (4.5), $\Pi_{1,11}(j\omega) < 0$, $\Pi_{2,22}(j\omega) < 0$. However, these operators are not restricted to the class of PN-IQC multipliers, because $\Pi_1(j\omega)$ and $\Pi_2(j\omega)$ are not required to belong to $\mathcal{RL}_\infty$ (i.e. includes irrational operators).

Moreover, the representation in Theorem 4.2.2 also makes it easy to connect IQCs and other areas of non-linear control. The corollaries in Section 4.3 will explore some of the classical Passivity and Dissipativity Theorems and their connection to the IQC structure.

In order to make the connection between Theorem 4.2.2 and the IQC Theorem, first is necessary to define a class of multipliers that hold conditions of Theorem 4.2.2

directly.

Definition 4.2.3. *An Hermitian measurable function* $\Pi : \mathbb{R} \to \mathbb{C}^{m+n \times m+n}$ *with structure*

$$\Pi(j\omega) = \begin{bmatrix} \Pi_{11}(j\omega) & \Pi_{12}(j\omega) \\ \Pi_{12}^{\sim}(j\omega) & \Pi_{22}(j\omega) \end{bmatrix}.$$

is said to be a positive-negative multiplier if $\Pi_{11}(j\omega) \geq 0, \Pi_{22}(j\omega) \leq 0 \quad \forall \omega \in \mathbb{R}$

Do not confuse PN-IQCs with positive-negative multipliers, as the later only refers to non-rational operators, but does not implies the existence of any time domain condition such that of hard IQC multipliers[35].

The direct implementation of Theorem 4.2.2 initially implies a restriction to use positive-negative multipliers. This limitation will be removed later.

Corollary 4.2.4. *Let* $G \in \mathcal{RH}_\infty^{l\times m}$ *and* $\Delta : \mathcal{L}_2[0,\infty)^l \to \mathcal{L}_2[0,\infty)^m$ *be a causal bounded operator. Assume that there exists an Hermitian measurable function* $\Pi : \mathbb{R} \to \mathbb{C}^{m+n \times m+n}$ *with* $\Pi_{11}(j\omega) \geq 0, \Pi_{22}(j\omega) \leq 0 \quad \forall \omega \in \mathbb{R}$*, and*

1. *the feedback interconnection of* $[G, \tau\Delta]$ *is well posed* $\forall \tau[0,1]$*;*

2. Δ *satisfies the IQC defined by* $\Pi(j\omega)$*, i.e.*

$$\left\langle \begin{pmatrix} U(j\omega) \\ V(j\omega) \end{pmatrix}, \Pi(j\omega) \begin{pmatrix} U(j\omega) \\ V(j\omega) \end{pmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$
$$\forall U(j\omega) \in \mathcal{H}_2^l, V = \mathcal{F}\{\Delta(\mathcal{F}^{-1}\{U\})\}; \quad (4.10)$$

3. *there exists* $\varepsilon_1 > 0$ *such that*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^{\sim} \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon_1 I \quad \forall \omega \in \mathbb{R}. \quad (4.11)$$

Then, the feedback interconnection $[\Delta, G]$ *is* $\mathcal{L}_2[0,\infty)$ *stable.*

Proof. For this proof, make $\Delta_1 = G$ and $\Delta_2 = \Delta$. Because of the linearity of G, the well posedness of $[G, \tau\Delta]$ implies the well posedness of $[\sqrt{\tau}G, \sqrt{\tau}\Delta]$ $\forall \tau \in [0,1]$.

- Case 1: $||\Delta||_{\mathcal{L}_2}^2 ||G||_\infty^2 - 1 < 0$. Consider $\Delta_1 = G$, $\Delta_2 = \Delta$. Choose λ as in (4.23), ε as in (4.24), Π_1 as in (4.25) and Π_2 as in (4.26).

- Case 2: $||\Delta||^2_{\mathcal{L}_2}||G||^2_\infty - 1 = 0$. Choose

$$\lambda = 1,$$

$$\varepsilon = \frac{\varepsilon_1||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2}}{1+||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2}},$$

$$\Pi_1(j\omega) = -\Pi(j\omega) - \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_1 I \end{pmatrix} - \varepsilon_1||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2}\frac{2+||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2}}{1+||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2}}\begin{pmatrix} I & 0 \\ 0 & -||G||^2_\infty I \end{pmatrix},$$

$$\Pi_2(j\omega) = \Pi(j\omega) + \varepsilon_1\begin{pmatrix} ||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2} I & 0 \\ 0 & -I \end{pmatrix}.$$

- Case 3: $||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2}||G||^2_\infty - 1 > 0$. Choose

$$\lambda = 1,$$

$$\varepsilon = \frac{\varepsilon_1}{(1+||G||^2_\infty)(1+||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2})},$$

$$\Pi_1(j\omega) = -\Pi(j\omega) - \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_1 I \end{pmatrix} - \frac{\varepsilon_1||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2} + \varepsilon_1(||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2}||G||^2_\infty - 1)}{(||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2}||G||^2_\infty - 1)(1+||G||^2_\infty)}\begin{pmatrix} I & 0 \\ 0 & -||G||^2_\infty I \end{pmatrix},$$

$$\Pi_2(j\omega) = \Pi(j\omega) + \frac{\varepsilon_1||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2}}{(1+||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2})(||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2}||G||^2_\infty - 1)}\begin{pmatrix} ||\Delta||^2_{\mathcal{L}_2\to\mathcal{L}_2} I & 0 \\ 0 & -I \end{pmatrix},$$

Then G,Δ satisfy the IQCs (4.3) and (4.4). $\Pi_1(j\omega)$ and $\Pi_2(j\omega)$ fulfill $\Pi_{1,22}(j\omega)$, $\Pi_{2,11}(j\omega) > 0 \quad \forall\omega \in \mathbb{R}\cup\{\infty\}$. Meanwhile, $\Pi_1(j\omega), \Pi_2(j\omega), \lambda, \varepsilon$ satisfy (4.5). Then the system is $\mathcal{L}_2[0,\infty)$ stable via Theorem 4.2.2. □

The following lemma shows that a feedback interconnection $[G,\Delta]$ holding conditions 1-3 of Theorem 2.3.3 will also hold the conditions of Theorem 4.2.2. In other words, the original closed loop can be decomposed into a succession of equivalent systems via loop transformations, where each one holds positive-negative multipliers. This procedure is necessary when $\Pi_{22}(j\omega)$ is not sign definite.

Lemma 4.2.5. *Let $G \in \mathcal{RH}^{l\times m}_\infty$ and $\Delta : \mathcal{L}^l_2[0,\infty) \to \mathcal{L}^m_2[0,\infty)$ be a bounded causal operator. Assume that:*

1. *for every* $\tau \in [0,1]$*, the feedback interconnection of* $[G, \tau\Delta]$ *is well posed;*

2. *for every* $\tau \in [0,1]$*, the IQC defined by a hermitian* $\Pi(j\omega)$ *is satisfied by* $\tau\Delta$*, i.e.*

$$\left\langle \begin{pmatrix} U \\ \tau V \end{pmatrix}, \Pi(j\omega) \begin{pmatrix} U \\ \tau V \end{pmatrix} \right\rangle_{\mathcal{H}_2} \geq 0 \quad \forall U \in \mathcal{H}_2, V = \mathcal{F}\{\Delta(\mathcal{F}^{-1}\{U\})\}. \tag{4.12}$$

3. *there exists* $\varepsilon > 0$ *such that*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^{\sim} \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon_1 I \quad \forall \omega \in \mathbb{R} \tag{4.13}$$

Then, the feedback interconnection of G and Δ *is* $\mathcal{L}_2[0,\infty)$ *stable.*

Proof. Because of the linearity of G, the well posedness of $[G, \tau\Delta]$ implies the well posedness of $[\sqrt{\tau}G, \sqrt{\tau}\Delta]$ $\forall \tau \in [0,1]$. The rest of the proof will build from Corollary 4.3.1 and Corollary 4.2.4.

- Assume $\Pi_{22}(j\omega) \leq 0$. choose $\tau = 1$ and use Corollary 4.2.4.

- Assume $\Pi_{22}(j\omega)$ is not sign definite.

This proof is divided in two steps. The first step is to find a $\tau_0 \in [0,1]$ such that the feedback interconnection $[G, \tau_0\Delta]$ is stable. A trivial solution for this step is to choose $\tau_0 = \frac{1}{2\|G\|_\infty \|\Delta\|_{\mathcal{L}_2 \to \mathcal{L}_2}}$ and use Small Gain Theorem, or Corollary 4.3.1. Then, the positivity of $\Pi_{22}(j\omega)$ is not an obstacle to find two positive-negative multipliers.

The second step is to show the stability for $\tau = 1$. This step is made by an iterative process of successive small modifications of the stable system. Figure 4.2 represents a sketch of the proof.

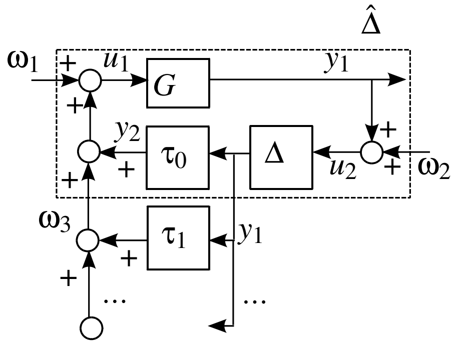


Figure 4.2: Closed loop iterative process

The first problem is to find an IQC parametrized by simple positive-negative multiplier $\Pi_2(j\omega)$ that $\hat{\Delta}$ fulfils, with input ω_3 and output y_1.

$$\left\langle \begin{pmatrix} W_3 \\ Y_1 \end{pmatrix}, \Pi_2(j\omega) \begin{pmatrix} W_3 \\ Y_1 \end{pmatrix} \right\rangle_{\mathcal{H}_2} \geq 0 \quad \forall W_3 \in \mathcal{H}_2^m, Y_1 = \mathcal{F}\{\hat{\Delta}(\mathcal{F}^{-1}\{W_3\})\}.$$

Using equations (4.12) and (4.13), the closed loop in Figure 4.2 holds the following IQC, for any $\gamma, \kappa > 0$:

$$\begin{aligned} \int_{-\infty}^{\infty} \gamma \begin{pmatrix} U_2 \\ \tau_0 Y_1 \end{pmatrix}^{\sim} \Pi(j\omega) \begin{pmatrix} U_2 \\ \tau_0 Y_1 \end{pmatrix} & \\ + \kappa \begin{pmatrix} G(j\omega)U_1 \\ U_1 \end{pmatrix}^{\sim} \left(-\Pi(j\omega) - \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_1 I \end{pmatrix} \right) \begin{pmatrix} G(j\omega)U_1 \\ U_1 \end{pmatrix} d\omega \geq 0 & \\ \forall U_1 \in \mathcal{H}_2^m, U_2 \in \mathcal{H}_2^l, Y_1 = \mathcal{F}\{\Delta(\mathcal{F}^{-1}\{U_2\})\}, & \quad (4.14) \end{aligned}$$

where U_1, U_2 are the Fourier transforms of u_1, u_2. Using the stable closed loop of Figure 4.2, signals u_1 and u_2 are

$$\begin{aligned} u_1 &= \omega_1 + \omega_3 + \tau_0 \Delta(u_2) \\ u_2 &= \omega_2 + G u_1. \end{aligned}$$

Without loss of generality, make $\omega_2 = \omega_1 = 0$. Then

$$\begin{aligned} u_2 = G u_1 &= G\omega_3 + \tau_0 G \Delta(u_2), \\ u_1 &= \omega_3 + \tau_0 \Delta(u_2). \end{aligned}$$

Make the appropriate substitutions on equation (4.14). Consequently the system $\hat{\Delta}$ is well posed using Condition 1 and $\mathcal{L}_2[0,\infty)$ stable for τ_0, therefore it holds the following IQC

$$\begin{aligned} \int_{-\infty}^{\infty} \begin{pmatrix} W_3 \\ Y_1 \end{pmatrix}^{\sim} \Bigg[\gamma \begin{pmatrix} G(j\omega)^{\sim} & 0 \\ \tau_0 G(j\omega)^{\sim} & \tau_0 I \end{pmatrix} \Pi(j\omega) \begin{pmatrix} G(j\omega) & \tau_0 G(j\omega) \\ 0 & \tau_0 I \end{pmatrix} & \\ + \kappa \begin{pmatrix} G(j\omega)^{\sim} & I \\ \tau_0 G(j\omega)^{\sim} & \tau_0 I \end{pmatrix} \left(-\Pi(j\omega) - \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_1 I \end{pmatrix} \right) \begin{pmatrix} G(j\omega) & \tau_0 G(j\omega) \\ I & \tau_0 I \end{pmatrix} \Bigg] \begin{pmatrix} W_3 \\ Y_1 \end{pmatrix} d\omega \geq 0 & \\ \forall W_3 \in \mathcal{H}_2^m, Y_1 = \mathcal{F}\{\hat{\Delta}(\mathcal{F}^{-1}\{W_3\})\}, & \end{aligned}$$

where ω_3 is the desired input signal and $y_1 = \Delta(u_2)$ is the desired output. Choose $\gamma = 1, \kappa = 1$, then

$$\int_{-\infty}^{\infty} \begin{pmatrix} W_3 \\ Y_1 \end{pmatrix}^{\sim} \Pi_2(j\omega) \begin{pmatrix} W_3 \\ Y_1 \end{pmatrix} d\omega \geq 0 \quad \forall W_3 \in \mathcal{H}_2^m, Y_1 = \mathcal{F}\{\hat{\Delta}(\mathcal{F}^{-1}\{W_3\})\},$$

with

$$\Pi_2(j\omega) = \begin{bmatrix} -\Pi_{12}(j\omega)^{\sim}G(j\omega) - G(j\omega)^{\sim}\Pi_{12}(j\omega) - \Pi_{22}(j\omega) & -\tau_0\Pi_{12}(j\omega)^{\sim}G(j\omega) - \tau_0\Pi_{22}(j\omega) \\ -\tau_0 G(j\omega)^{\sim}\Pi_{12}(j\omega) - \tau_0\Pi_{22}(j\omega) & 0 \end{bmatrix} + \begin{bmatrix} -\varepsilon_1 I & -\tau_0\varepsilon_1 I \\ -\tau_0\varepsilon_1 I & -\tau_0^2\varepsilon_1 I \end{bmatrix}. \quad (4.15)$$

Note that $\Pi_{2,11}(j\omega) \geq 0 \quad \forall\omega \in \mathbb{R}$ using equation (4.13), i.e.

$$-\Pi_{12}^{\sim}(j\omega)G(j\omega) - G(j\omega)^{\sim}\Pi_{12}(j\omega) - \Pi_{22}(j\omega) - \varepsilon_1 I \geq G(j\omega)^{\sim}\Pi_{11}(j\omega)^{\sim}G(j\omega) \geq 0 \quad \forall\omega \in \mathbb{R}.$$

Then, $\Pi_2(j\omega)$ is a positive-negative multiplier.

In addition, note that the positive constant operator $\tau_1 I$, where $\tau_1 \in (0,1]$ holds the following IQC with a positive-negative multiplier:

$$\int_{-\infty}^{\infty} \begin{bmatrix} \tau_1 Y_1 \\ Y_1 \end{bmatrix}^{\sim} \Pi_1(j\omega) \begin{bmatrix} \tau_1 Y_1 \\ Y_1 \end{bmatrix} d\omega \geq 0 \quad \forall Y_1 \in \mathcal{H}_2^m,$$

with

$$\Pi_1(j\omega) = \begin{bmatrix} -\frac{1}{\tau_1^2}X(j\omega) & -\frac{1}{\tau_1}S(j\omega)^{\sim} \\ -\frac{1}{\tau_1}S(j\omega) & X(j\omega) + S(j\omega) + S(j\omega)^{\sim} + Z(j\omega) \end{bmatrix} \quad (4.16)$$

for any $X(j\omega) = X(j\omega)^{\sim} \geq 0$, $Z(j\omega) = Z(j\omega)^{\sim} \geq 0$ and $S(j\omega)$ such that $X(j\omega) + S(j\omega) + S(j\omega)^{\sim} + Z(j\omega) \geq 0$.

Choose

$$X(j\omega) = -\tau_1^2(\Pi_{22}(j\omega) + G(j\omega)^{\sim}\Pi_{12}(j\omega) + \Pi_{12}(j\omega)^{\sim}G(j\omega)),$$
$$S(j\omega) = -\tau_1\tau_0(\varepsilon_1 I + \Pi_{22}(j\omega) + G(j\omega)^{\sim}\Pi_{12}(j\omega)),$$
$$Z(j\omega) = \frac{\varepsilon_1\tau_0^2}{2}I.$$

Make

$$\lambda = 1,$$

then, adding equation (4.16) and (4.15) yields:

$$\Pi_1(j\omega) + \lambda\Pi_2(j\omega) = \begin{bmatrix} -\varepsilon_1 I & 0 \\ 0 & \begin{matrix} -\tau_1^2(\Pi_{22}(j\omega) + G(j\omega)^{\sim}\Pi_{12}(j\omega) + \Pi_{12}(j\omega)^{\sim}G(j\omega)) \\ -\tau_1\tau_0(2\varepsilon_1 I + 2\Pi_{22}(j\omega) + G(j\omega)^{\sim}\Pi_{12}(j\omega) + \Pi_{12}(j\omega)^{\sim}G(j\omega)) \\ -\tau_0^2\varepsilon_1\frac{1}{2}I \end{matrix} \end{bmatrix}.$$

As a result, considering the fact that $\Pi(j\omega)$ is bounded, and choosing τ_1 such that $\tau_1 \leq \tau_0$, the following inequality can be determined:

$$\Pi_1(j\omega) + \Pi_2(j\omega) \leq \begin{bmatrix} -\varepsilon_1 I & 0 \\ 0 & \tau_1\tau_0(3||\Pi_{22}||_\infty + 4||G||_\infty||\Pi_{12}||_\infty) - \tau_0^2\varepsilon_1\frac{1}{2}I \end{bmatrix} \quad \forall\omega \in \mathbb{R}. \quad (4.17)$$

Choose

$$\tau_1 = \frac{\tau_0\varepsilon_1\frac{1}{2}}{2\varepsilon_1 + 3||\Pi_{22}||_\infty + 4||G||_\infty||\Pi_{12}||_\infty} \quad (4.18)$$

Then, noting that $\tau_0 \in [0,1]$ implies

$$\Pi_1(j\omega) + \Pi_2(j\omega) \leq \begin{bmatrix} -\varepsilon_1 I & 0 \\ 0 & -\frac{\tau_0^2\varepsilon_1^2}{2\varepsilon_1+3||\Pi_{22}||_\infty+4||G||_\infty||\Pi_{12}||_\infty}I \end{bmatrix} \leq -\varepsilon I \quad \forall\omega \in \mathbb{R}.$$

where

$$\varepsilon = \frac{\tau_0^2\varepsilon_1^2}{2\varepsilon_1 + 3||\Pi_{22}||_\infty + 4||G||_\infty||\Pi_{12}||_\infty}.$$

On the other hand, using equation (4.13) it can be shown that

$$X(j\omega) = -\tau_1^2(\Pi_{22}(j\omega) + G(j\omega)^{\sim}\Pi_{12}(j\omega) + \Pi_{12}(j\omega)^{\sim}G(j\omega)) > \tau_1^2\varepsilon_1 I \quad \forall\omega \in \mathbb{R},$$

furthermore, considering the fact that $\Pi(j\omega)$ is bounded it also holds that

$$X(j\omega)+S(j\omega)+S(j\omega)^{\sim}+Z(j\omega)$$
$$> \tau_1^2\varepsilon_1 I+\frac{1}{2}\tau_0^2\varepsilon_1 I\frac{\varepsilon_1+2||\Pi_{22}||_\infty+4||G||_\infty||\Pi_{12}||_\infty}{2\varepsilon_1+3||\Pi_{22}||_\infty+4||G||_\infty||\Pi_{12}||_\infty} > 0,$$

therefore, $\Pi_1(j\omega)$ is a positive-negative multiplier.

Finally, since the system is bounded for $\tau=\tau_0$, where $1\geq\tau_0>0$, there exists τ_1 such that the closed loop is also stable for

$$1\geq\tau=\tau_0+\tau_1=\tau_0+\frac{\tau_0\varepsilon_1\frac{1}{2}}{2\varepsilon_1+3||\Pi_{22}||_\infty+4||G||_\infty||\Pi_{12}||_\infty}.$$

Since the system is bounded for $\tau=\tau_0+\tau_1$, where $1\geq\tau_0+\tau_1>0$, there exists τ_2 such that the closed loop is also stable for

$$1\geq\tau=\tau_0+\tau_1+\tau_2=\tau_0+\frac{(2\tau_0+\tau_1)\varepsilon_1\frac{1}{2}}{2\varepsilon_1+3||\Pi_{22}||_\infty+4||G||_\infty||\Pi_{12}||_\infty}.$$

Since the system is bounded for $\tau=\tau_0+\tau_1+\tau_2$, where $1\geq\tau_0+\tau_1+\tau_2>0$, there exists τ_3 such that the closed loop is also stable for

$$1\geq\tau=\tau_0+\tau_1+\tau_2+\tau_3=\tau_0+\frac{(3\tau_0+2\tau_1+\tau_2)\varepsilon_1\frac{1}{2}}{2\varepsilon_1+3||\Pi_{22}||_\infty+4||G||_\infty||\Pi_{12}||_\infty}....etc$$

This sequence describes a divergent series on $\sum_i\tau_i$. By induction, G and $\sum_i\tau_i\Delta$ satisfy the IQCs (4.3) and (4.4) for all i. Moreover, $\Pi_1(j\omega)$ and $\Pi_2(j\omega)$ fulfil the positive-negative conditions $\Pi_{1,22}(j\omega),\Pi_{2,11}(j\omega)>0\quad\forall\omega\in\mathbb{R}\cup\{\infty\}$. Meanwhile, $\Pi_1(j\omega),\Pi_2(j\omega),\lambda,\varepsilon$ satisfy (4.5). Then the system is $\mathcal{L}_2[0,\infty)$ stable for successive τ_i, $i=0,1,...$ until $\tau=1$ via Theorem 4.2.2. □

This corollary shows that Theorem 4.2.2 is a complete and equivalent reinterpretation of the IQC Theorem [35]. Moreover, it demonstrates the applicability of Theorem 4.2.2, where by choosing one system to be Linear Time Invariant, IQC Theorem can be recovered. The demonstration method used in this section have the additional advantage of being applicable to extend multipliers of Dissipative based proofs for IQC, such as Theorem 1 in [57], from strict PN-IQCs to multipliers $\Pi(j\omega)$ where $\Pi_{22}(j\omega)$ is indefinite.

The apparent disadvantage of Theorem 4.2.2 is the need for well posedness on a sector $\tau \in [0,1]$. This problem will be discussed in the following section.

4.2.1 Comments on the well posedness of the closed loop

Condition 1 on Theorem 4.2.2 asks for the well posedness of the closed loop for all $\tau \in [0,1]$. There exists some cases where this condition can be determined from when the systems are well posed for $\tau = 1$.

Take operators that delay all inputs, according to Definition 2.1.15:

Lemma 4.2.6. *Let $\Delta_1 : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ and $\Delta_2 : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ be $\mathcal{L}_2[0,\infty)$, if the operator $\Delta_1\Delta_2$ delays all inputs, then, the closed loop $[\tau\Delta_1, \tau\Delta_2]$ is well posed for all $\tau \in \mathbb{R}^+$.*

Proof. Scaling Δ_1, Δ_2 by some real constant produces operators $\tau\Delta_1, \tau\Delta_2$ that delay all inputs. The result follows Corollary 2.1.16. □

From this lemma, it is evident a feedback interconnection with a pure delay will always be well posed for all $\tau \in [0,1]$ if it is well posed for $\tau = 1$.

Similarly, assume Δ_1 to be strongly causal, uniform with respect to past inputs according to Definition 2.1.6, and Δ_2 to be locally Lipschitz continuous according to Definition 2.1.5, then:

Lemma 4.2.7. *Let $\Delta_1 : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ be strongly causal, uniform with respect to past inputs and $\Delta_2 : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ be $\mathcal{L}_2[0,\infty)$ be locally Lipschitz continuous. Then, the closed loop $[\tau\Delta_1, \tau\Delta_2]$ is well posed for all $\tau \in \mathbb{R}^+$.*

Proof. Multiplying Δ_1 by any τ results in a strongly causal operator $\tau\Delta_1$, uniform with respect to past inputs, and multiplying Δ_2 by any τ results in a locally Lipschitz continuous operator $\tau\Delta_2$. The result follows from Corollary 2.1.17. □

An important remark is made in [13] regarding this lemma. It is reproduced here for the sake of argument.

Remark 4.2.8. *Consider the case when the output y is related to the input u through the ordinary differential equation:*

$$\dot{x} = f(x(t), u(t), t)$$
$$y = g(x(t), t)$$

where $t > 0$, $x(0) \in \mathbb{R}^m$, $x \in \mathcal{L}_2^m[0,\infty)$, $u \in \mathcal{L}_2^l[0,\infty)$ and $y \in \mathcal{L}_2^n[0,\infty)$, f is continuous on $t > 0$ and Lipschitz continuous on $\mathcal{L}_2^m[0,\infty) \times \mathcal{L}_2^l[0,\infty)$, and g is continuous on $t > 0$

and Lipschitz continuous on $\mathcal{L}_2^m[0,\infty)$*. It is easy to show the mapping from* $u \in \mathcal{L}_2^l[0,\infty)$ *into* $y \in \mathcal{L}_2^n[0,\infty)$ *is well defined, strongly causal operator, uniformly with respect to past inputs.*

Then, Lemma 4.2.7 shows the well posedness for all $\tau \in [0,1]$ of the feedback interconnection of $[G, \tau\Delta]$, where $G \in \mathcal{RH}_\infty$ is strictly proper, and therefore strongly causal, uniform with respect to past inputs, and Δ is:

- Linear Time Invariant operator with bounded gain.
- Constant real scalar.
- Multiplication by harmonic oscillation.
- Slowly time varying scalar.
- Continuous "Popov" non-linearity.
- Monotonic and monotonic odd non-linearities.

For systems with feed-through, the following lemma enumerates the conditions for the well posedness of the feedback loop for all $\tau \in [0,1]$.

Let the attenuated feedback interconnection $[\tau\Delta_1, \tau\Delta_2]$ be represented by the following equations:

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & -I_m \\ I_n & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = u + Hy, \tag{4.19}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \tau \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \tau G e. \tag{4.20}$$

Lemma 4.2.9. *Let the uniform instantaneous gains of* G_{ij} *and* H_{ji} *at* $T \in [0,\infty)$ *be* $a_{ij}(T)$ *and* $b_{ji}(T)$*, respectively. Define the gain-product matrix* $\Theta(T) = \{\theta_{jj'}(T)\}$ *by*

$$\theta_{jj'}(T) = \sum_{i=1}^{m+n} b_{ji}(T) a_{ij'}(T), \quad j, j' = 1, \ldots, m+n. \tag{4.21}$$

Then, if the matrix $I - \Theta(T)$ *is an* $M-matrix$ *for all T, the closed loop system defined by equations* (4.19) *and* (4.20) *is well posed in the sense of Definition 2.1.7 for all* $\tau \in [0,1]$.

Proof. Note that the gain-product matrix $\hat{\Theta}(T) = \{\hat{\theta}_{jj'}(T)\}$ for the attenuated system in equations (4.19) and (4.20) is the following expression:

$$\hat{\theta}_{jj'}(T) = \tau \sum_{i=1}^{m+n} b_{ji}(T) a_{ij'}(T), \quad j, j' = 1, \ldots, m+n, \tag{4.22}$$

then the following inequality can be determined for all $i, j = 1, \ldots, m+n$:

$$0 < \hat{\theta}_{ij}(T) \leq \theta_{ij}(T).$$

From [46], it is known that if $I - \Theta(T)$ is an M-matrix, then the matrix $I - \hat{\Theta}(T)$ is also an M-matrix. The result follows from Theorem 2.1.13. □

Therefore, for those cases when the conditions for well posedness hold for $\tau = 1$, well posedness also will also hold for $\tau \in [0, 1]$. This Lemma can be used when Lemma 4.2.7 can no be applied: for example, when the Linear Time Invariant system is biproper $G \in \mathcal{RH}_\infty$.

The need for well posedness of the feedback loop for all $\tau \in [0, 1]$ can be a disadvantage when the IQC Theorem is applied [57]. However, there are many non-linear interconnections that are well posed under trivial conditions. This section illustrates that showing the well posedness of closed loops is not a complicated task for an important group of non-linearities. Furthermore, works like [76] and [77] show that conditions for the existence of unique solutions are also available for some non-continuous operators. When the operators are continuous, showing the well posedness can be a trivial task, as mentioned in the Appendix A in [3].

Well posedness is not the main focus of this work, and consequently, it will be assumed that the feedback interconnection is well posedness in the corollaries.

4.3 Corollaries

The following list of corollaries is a direct implementation of Theorem 4.2.2. This section will deduce some well known results, such as Small Gain Theorem, Dissipativity and Passivity for the interconnection of $\mathcal{L}_2[0, \infty)$ bounded operators. The latest requirement is necessary because the Theorem 4.2.2 requires the open loop operators to be $\mathcal{L}_2[0, \infty)$ bounded in order to show stability.

The objective of this exercise is to illustrate the utility of Theorem 4.2.2 to express stability results applicable to the feedback interconnections of operators that are

$\mathcal{L}_2[0,\infty)$ stable in open loop.

4.3.1 Small Gain Theorem

The Small Gain Theorem can be deduced directly from the boundedness of two non-linear systems. Note that the feedback interconnection is positive feedback in this version of the theorem, but that has no bearing on the result.

Corollary 4.3.1. *Let $\Delta_1 : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$, $\Delta_2 : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ be two bounded causal operators with gains $||\Delta_1||_{\mathcal{L}_2\to\mathcal{L}_2}$ and $||\Delta_2||_{\mathcal{L}_2\to\mathcal{L}_2}$, such that $||\Delta_1||_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||_{\mathcal{L}_2\to\mathcal{L}_2} < 1$ and the feedback interconnection $[\tau\Delta_1, \tau\Delta_2]$ is well posed $\forall\tau \in [0,1]$, then the feedback interconnection in Figure 4.1a is $\mathcal{L}_2[0,\infty)$ stable.*

Proof. Choose

$$\lambda = \frac{1+||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}}{1+||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}}, \tag{4.23}$$

$$\varepsilon = \frac{1-||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}}{1+||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}}, \tag{4.24}$$

$$\Pi_1 = \begin{bmatrix} -I & 0 \\ 0 & ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2} I \end{bmatrix}, \tag{4.25}$$

$$\Pi_2 = \begin{bmatrix} ||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} I & 0 \\ 0 & -I \end{bmatrix}. \tag{4.26}$$

Then, Δ_1, Δ_2 satisfy the IQCs (4.3) and (4.4), parametrized by the strict positive-negative multipliers Π_1 and Π_2. Meanwhile, $\Pi_1, \Pi_2, \lambda, \varepsilon$ satisfy (4.5). Then the system is $\mathcal{L}_2[0,\infty)$ stable via Theorem 4.2.2. □

4.3.2 Dissipativity

This corollary takes the conditions from Theorem 7.4 in [24], which is a version of Dissipativity multi-input, multi-output operators. However, this corollary is restricted to be used only for non-linear systems that are bounded and connected in negative feedback, i.e. $R_1, R_2 \geq 0$. Note that Theorem 2 in [20] shows that the requirement of finite gain stability over the non-linearities ensures that there exists at least one (P,Q,S) triplet with $Q_1, Q_2 < 0$.

Corollary 4.3.2. *Let $\Delta_1 : \mathcal{L}_2^l[0,\infty) \to \mathcal{L}_2^m[0,\infty), \Delta_2 : \mathcal{L}_2^m[0,\infty) \to \mathcal{L}_2^l[0,\infty)$ be bounded causal operator with gains $||\Delta_1||_{\mathcal{L}_2\to\mathcal{L}_2}$ and $||\Delta_2||_{\mathcal{L}_2\to\mathcal{L}_2}$, such that the negative feedback* *interconnection* $[\tau\Delta_1, -\tau\Delta_2]$ *is well posed* $\forall\tau \in [0,1]$. *Furthermore, let* $R_1, Q_2 \in \mathbb{R}^{m\times m}$ *and* $R_2, Q_1 \in \mathbb{R}^{l\times l}$ *be any symmetric matrices and let* $S_1 \in \mathbb{R}^{m\times l}$, $S_2 \in \mathbb{R}^{l\times m}$. *It holds that* Δ_1 *and* Δ_2 *are* (Q,R,S)*-operator dissipative, i.e.:*

$$\langle U_1, R_1 U_1\rangle_T + 2\langle U_1, S_1 V_1\rangle_T + \langle V_1, Q_1 V_1\rangle_T \geq 0 \quad \forall T > 0, \forall U_1 \in \mathcal{L}_{2e}^l, V_1 = \Delta_1(U_1) \tag{4.27}$$

and

$$\langle U_2, R_2 U_2\rangle_T + 2\langle U_2, S_2 V_2\rangle_T + \langle V_2, Q_2 V_2\rangle_T \geq 0 \quad \forall T > 0, U_2 \in \mathcal{L}_{2e}^m, V_2 = \Delta_2(U_2) \tag{4.28}$$

where
$R_1 \geq 0.$
$R_2 \geq 0.$
and the following inequality holds for some positive scalars $\lambda, \varepsilon_1 > 0$

$$\begin{pmatrix} Q_1 + \lambda R_2 & (-1)S_1^T + \lambda S_2 \\ (-1)S_1 + \lambda S_2^T & R_1 + \lambda Q_2 \end{pmatrix} \leq -\varepsilon_1 I, \tag{4.29}$$

then the feedback system is $\mathcal{L}_2[0,\infty)$ *stable.*

Proof. Note that via Theorem 2.3.6, equations (4.27) and (4.28) imply

$$\left\langle \begin{bmatrix} V_1 \\ U_1 \end{bmatrix}, \begin{bmatrix} Q_1 & S_1^T \\ S_1 & R_1 \end{bmatrix} \begin{bmatrix} V_1 \\ U_1 \end{bmatrix} \right\rangle_{\mathcal{H}_2} \geq 0 \quad \forall U_1 \in \mathcal{H}_2^l, V_1 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{U_1\})\} \tag{4.30}$$

and

$$\left\langle \begin{bmatrix} U_2 \\ V_2 \end{bmatrix}, \begin{bmatrix} R_2 & S_2 \\ S_2^T & Q_2 \end{bmatrix} \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} \right\rangle_{\mathcal{H}_2} \geq 0 \quad \forall U_2 \in \mathcal{H}_2^m, V_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{U_2\})\} \tag{4.31}$$

Now, choose

$$\lambda > 0,$$
$$\varepsilon = \frac{\varepsilon_1}{2},$$
$$\Pi_1 = \begin{bmatrix} Q_1 & -S_1^T \\ -S_1 & R_1 + \frac{\varepsilon_1}{2} I \end{bmatrix}, \tag{4.32}$$

$$\Pi_2 = \begin{bmatrix} R_2 + \frac{\varepsilon_1}{\lambda 2} I & S_2 \\ S_2^T & Q_2 \end{bmatrix}. \tag{4.33}$$

Adding (4.32) to λ (4.33) yields

$$\Pi_1 + \lambda \Pi_2 = \begin{bmatrix} Q_1 + \lambda R_2 + \frac{\varepsilon_1}{2} I & S_1^T + \lambda S_2 \\ S_1 + \lambda S_2^T & R_1 + \frac{\varepsilon_1}{2} I + \lambda Q_2 \end{bmatrix} \leq -\frac{\varepsilon_1}{2} I.$$

Then Δ_1, Δ_2 satisfy the IQCs (4.3) and (4.4), parametrized by the strict positive-negative multipliers Π_1 and Π_2. Meanwhile, $\Pi_1, \Pi_2, \lambda, \varepsilon$ satisfy (4.5). Then the system is $\mathcal{L}_2[0,\infty)$ stable via Theorem 4.2.2. □

4.3.3 Passivity

For this corollary, the conditions used are taken from Theorem 5.4 in [10, p. 257-259]. However, the non-linearity studied in this Corollary is restricted to be open loop $\mathcal{L}_2[0,\infty)$ stable. Note that Theorem 2 in [20] shows that the finite gain stability over the non-linearities imposes an implicit restriction over the structure of the multiplier Π_1 and Π_2, where there always will be a set of multipliers such that $\delta_1, \delta_2 > 0$.

Corollary 4.3.3. *Consider the feedback system of Figure 4.1a with two causal bounded operator* $\Delta_1 : \mathcal{L}_{2e}^l[0,\infty) \to \mathcal{L}_{2e}^m[0,\infty)$, $\Delta_2 : \mathcal{L}_{2e}^m[0,\infty) \to \mathcal{L}_{2e}^l[0,\infty)$ *with gains* $||\Delta_1||_{\mathcal{L}_2 \to \mathcal{L}_2}$, $||\Delta_2||_{\mathcal{L}_2 \to \mathcal{L}_2}$. *Suppose the negative feedback interconnection* $[-\tau\Delta_1, \tau\Delta_2]$ *is well posed* $\forall \tau \in [0,1]$.

1. *Let the systems* Δ_1 *and* Δ_2 *satisfy*

$$< u_1, \Delta_1(u_1) >_T \geq \delta_1 ||\Delta_1(u_1)||_T^2 + \varepsilon_1 ||u_1||_T^2 \quad \forall u_1 \in \mathcal{L}_{2e}^l, T > 0 \tag{4.34}$$

and

$$< u_2, \Delta_2(u_2) >_T \geq \delta_2 ||\Delta_2(u_2)||_T^2 + \varepsilon_2 ||u_2||_T^2 \quad \forall u_1 \in \mathcal{L}_{2e}^m, T > 0, \tag{4.35}$$

2. *where the scalars* $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2 \in \mathbb{R}$ *hold:*

$$\varepsilon_1 + \delta_2 > 0, \quad \varepsilon_2 + \delta_1 > 0. \tag{4.36}$$

Then, the system is $\mathcal{L}_2[0,\infty)$ stable.

Proof. Note that using the boundedness of Δ_1, Δ_2, the negative feedback interconnection, and using the Parseval's Theorem, equations (4.34) and (4.35) imply the following IQC:

$$\left\langle \begin{bmatrix} V_1 \\ U_1 \end{bmatrix}, \begin{bmatrix} -\delta_1 I & -\frac{1}{2}I \\ -\frac{1}{2}I & -\varepsilon_1 I \end{bmatrix} \begin{bmatrix} V_1 \\ U_1 \end{bmatrix} \right\rangle_{\mathcal{H}_2} \geq 0 \quad \forall U_1 \in \mathcal{H}_2^l, V_1 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{U_1\})\}$$

and

$$\left\langle \begin{bmatrix} U_2 \\ V_2 \end{bmatrix}, \begin{bmatrix} -\varepsilon_2 I & \frac{1}{2}I \\ \frac{1}{2}I & -\delta_2 I \end{bmatrix} \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} \right\rangle_{\mathcal{H}_2} \geq 0 \quad \forall U_2 \in \mathcal{H}_2^m, V_2 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{U_2\})\}.$$

Now, lets consider the possible combinations of the scalar parameters $\delta_1, \varepsilon_1, \delta_2, \varepsilon_2$:

- Case 1: $1 - ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2} ||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} > 0$. Consider λ from equation (4.23), ε from equation (4.24), Π_1 from equation (4.25) and Π_2 from equation (4.26).

- Case 2: $\varepsilon_1 > 0, \varepsilon_2 > 0, \delta_1 > -\frac{\varepsilon_1}{||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}}, \delta_2 > -\frac{\varepsilon_2}{||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}}$.

  Choose

$$\varepsilon = \min\left\{\frac{\frac{\varepsilon_1}{||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}} + \delta_1}{2}, \frac{\frac{\varepsilon_2}{||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}} + \delta_2}{2}\right\},$$

$$\lambda = 1,$$

$$\Pi_1 = \begin{bmatrix} \left(-\frac{\varepsilon_1}{||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}} - \delta_1\right)I & -\frac{1}{2}I \\ -\frac{1}{2}I & \frac{\frac{\varepsilon_2}{||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}} + \delta_2}{2} \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} \frac{\frac{\varepsilon_1}{||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}} + \delta_1}{2} & \frac{1}{2}I \\ \frac{1}{2}I & \left(-\frac{\varepsilon_2}{||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}} - \delta_2\right)I \end{bmatrix}.$$

- Case 3: $1 - ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2} ||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} < 0$.

  Now, two cases can happen:

- Case 3.1: $\varepsilon_1 \geq 0, \varepsilon_2 < 0, \delta_1 \geq 0, \delta_2 \geq 0$.

  Choose

$$\varepsilon = \frac{||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}(\delta_1 + \varepsilon_2) + \delta_2 + \varepsilon_1}{(1 + ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2})(1 + ||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2})},$$

$$\lambda = 1,$$

$$\Pi_1 = \begin{bmatrix} -\delta_1 I & -\frac{1}{2}I \\ -\frac{1}{2}I & -\varepsilon_1 I \end{bmatrix} + \frac{(||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} + ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} - 1)(\delta_2+\varepsilon_1) + (\delta_1+\varepsilon_2)}{(1+||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2})(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} - 1)} \begin{bmatrix} -I & 0 \\ 0 & ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2} I \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} -\varepsilon_2 I & \frac{1}{2}I \\ \frac{1}{2}I & -\delta_2 I \end{bmatrix} + \frac{||\Delta_2||^4_{\mathcal{L}_2\to\mathcal{L}_2}(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}(\delta_1+\varepsilon_2) + (\delta_2+\varepsilon_1))}{(1+||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2})(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} - 1)} \begin{bmatrix} I & 0 \\ 0 & -\frac{1}{||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}} I \end{bmatrix},$$

- Case 3.2: $\varepsilon_1, \varepsilon_2 \leq 0$.

  Choose

$$\varepsilon = \frac{(\delta_1+\varepsilon_2) + (\delta_2+\varepsilon_1)||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}}{(1+||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2})(1+||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2})} > 0,$$

$$\lambda = 1,$$

$$\Pi_1 = \begin{bmatrix} -\delta_1 I & -\frac{1}{2}I \\ -\frac{1}{2}I & -\varepsilon_1 I \end{bmatrix} + \frac{||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}(\delta_1+\varepsilon_2) + ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}(\varepsilon_1+\delta_2)}{(1+||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2})(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} - 1)} \begin{bmatrix} -I & 0 \\ 0 & ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2} I \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} -\varepsilon_2 I & \frac{1}{2}I \\ \frac{1}{2}I & -\delta_2 I \end{bmatrix} + \frac{(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} - 1 + ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2})(\delta_1+\varepsilon_2) + (\varepsilon_1+\delta_2)}{(1+||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2})(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} - 1)} \begin{bmatrix} ||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} I & 0 \\ 0 & -I \end{bmatrix}.$$

- Case 4: $1 = ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}$. Then, two cases happen;

- Case 4.1: $\varepsilon_1 \geq 0, \varepsilon_2 < 0, \delta_1 \geq 0, \delta_2 \geq 0$. Choose

$$\varepsilon = \min\{\frac{\delta_1+\varepsilon_2}{2}, \frac{\delta_2+\varepsilon_1}{2}\},$$

$$\lambda = 1,$$

$$\Pi_1 = \begin{bmatrix} -\delta_1 I & -\frac{1}{2}I \\ -\frac{1}{2}I & -\varepsilon_1 I \end{bmatrix} + \begin{bmatrix} -\frac{\varepsilon_1}{||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}}I & 0 \\ 0 & (\varepsilon_1 + \frac{\delta_2+\varepsilon_1}{2})I \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} -\varepsilon_2 I & \frac{1}{2}I \\ \frac{1}{2}I & -\delta_2 I \end{bmatrix} + \varepsilon_1 \begin{bmatrix} \frac{1}{||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}}I & 0 \\ 0 & -I \end{bmatrix}.$$

- Case 4.2: $\varepsilon_1, \varepsilon_2 \leq 0$.

  Choose

$$\varepsilon = \min\{\frac{\delta_1+\varepsilon_2}{2}, \frac{\delta_2+\varepsilon_1}{2}\},$$

$$\lambda = 1,$$

$$\Pi_1 = \begin{bmatrix} -\delta_1 I & -\frac{1}{2}I \\ -\frac{1}{2}I & (-\varepsilon_1 + \frac{\delta_2+\varepsilon_1}{2})I \end{bmatrix}$$

$$\Pi_2 = \begin{bmatrix} (-\varepsilon_2 + \frac{\delta_1+\varepsilon_2}{2})I & \frac{1}{2}I \\ \frac{1}{2}I & -\delta_2 I \end{bmatrix}$$

  Then Δ_1 and Δ_2 satisfy the IQCs (4.3) and (4.4), parametrized by the strict positive-negative multipliers Π_1 and Π_2. Meanwhile, $\Pi_1, \Pi_2, \lambda, \varepsilon$ satisfy (4.5). Then the system is $\mathcal{L}_2[0,\infty)$ stable via Theorem 4.2.2.

□

This Corollary implies that the negative feedback interconnection of systems that at least hold Outputs Strict Passivity, i.e. ($\varepsilon_1 \geq 0, \varepsilon_2 \geq 0, \delta_1 > 0, \delta_2 > 0$) will be $\mathcal{L}_2[0,\infty)$ stable. This limitation happens because bounded operators will always hold Output Passivity [20]. This corollary hence does not contain completely Passivity Theorem.

4.3.4 Passivity with non-causal multipliers

For this corollary the conditions are taken from Passivity Theorem 9.20, in chapter VI of [14] with the following condition in the multipliers: $\hat{M}, \hat{M}^{-1} \in \mathcal{RL}_\infty$.

Corollary 4.3.4. *Consider the feedback system of Figure 4.1a, where* $\Delta_1 : \mathcal{L}_2(-\infty,\infty) \to \mathcal{L}_2(-\infty,\infty)$, $\Delta_2 : \mathcal{L}_2(-\infty,\infty) \to \mathcal{L}_2(-\infty,\infty)$ *are two causal bounded operator dynamical systems with gain* $||\Delta_1||_{\mathcal{L}_2\to\mathcal{L}_2}, ||\Delta_2||_{\mathcal{L}_2\to\mathcal{L}_2}$. *Suppose the negative feedback interconnection* $[\tau\Delta_1, -\tau\Delta_2]$ *is well posed* $\forall\tau \in [0,1]$. *Let there be a multiplier* $\hat{M}, \hat{M}^{-1} \in \mathcal{RL}_\infty$ *such that*

1. *for some* $\delta > 0$

$$< U_1, \hat{M}\Delta_1(U_1) >_{\mathcal{L}_2(-\infty,\infty)} \geq \frac{\delta}{2} < U_1, U_1 >_{\mathcal{L}_2(-\infty,\infty)} \quad \forall U_1 \in \mathcal{L}_2(-\infty,\infty) \quad (4.37)$$

2. *and it holds that:*

$$< U_2, \Delta_2(\hat{M}^{-1}U_2) >_{\mathcal{L}_2(-\infty,\infty)} \geq 0 \quad \forall U_2 \in \mathcal{L}_2(-\infty,\infty) \qquad (4.38)$$

then, the system is $\mathcal{L}_2[0,\infty)$ *stable.*

Proof. Note that equation (4.37) and negative feedback implies the following IQC using Parseval's Theorem

$$\left\langle \begin{bmatrix} \hat{V}_1 \\ \hat{U}_1 \end{bmatrix}, \begin{bmatrix} 0 & \hat{M}^{\sim}(j\omega) \\ \hat{M}(j\omega) & -\delta I \end{bmatrix} \begin{bmatrix} \hat{V}_1 \\ \hat{U}_1 \end{bmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$
$$\forall \hat{U}_1 \in \mathcal{H}_2, \hat{V}_1 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{\hat{U}_1\})\}. \quad (4.39)$$

Given that Δ_1 is finite gain, the previous matrix can be perturbed by $\gamma_1 > 0$ and remain positive:

$$\left\langle \begin{bmatrix} \hat{V}_1 \\ \hat{U}_1 \end{bmatrix}, \begin{bmatrix} -\frac{\gamma_1}{||\Delta_1||^2_{\mathcal{L}_2 \to \mathcal{L}_2}} I & \hat{M}^{\sim}(j\omega) \\ \hat{M}(j\omega) & (\gamma_1 - \delta) I \end{bmatrix} \begin{bmatrix} \hat{V}_1 \\ \hat{U}_1 \end{bmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$
$$\forall \hat{U}_1 \in \mathcal{H}_2, \hat{V}_1 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{\hat{U}_1\})\}.$$

Likewise, (4.38) implies the following inequality using Parseval's Theorem.

$$\left\langle \begin{bmatrix} \hat{U}_2 \\ -\hat{V}_2 \end{bmatrix}, \begin{bmatrix} 0 & -\hat{M}^{\sim}(j\omega) \\ -\hat{M}(j\omega) & 0 \end{bmatrix} \begin{bmatrix} \hat{U}_2 \\ -\hat{V}_2 \end{bmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$
$$\forall \hat{U}_2 \in \mathcal{H}_2, \hat{V}_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{\hat{U}_2\})\}.$$

Given the finite gain of Δ_2, the previous product can be perturbed by $\gamma_2 > 0$ and remain positive

$$\left\langle \begin{bmatrix} \hat{U}_2 \\ -\hat{V}_2 \end{bmatrix}, \begin{bmatrix} \gamma_2 I & -\hat{M}^{\sim}(j\omega) \\ -\hat{M}(j\omega) & -\frac{\gamma_2}{||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}} I \end{bmatrix} \begin{bmatrix} \hat{U}_2 \\ -\hat{V}_2 \end{bmatrix} \right\rangle_{\mathcal{L}_2(j\mathbb{R})} \geq 0$$

$$\forall \hat{U}_2 \in \mathcal{H}_2, \hat{V}_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{\hat{U}_2\})\} \quad (4.40)$$

Then, there are 3 cases for which stability can be verified:

- Case 1: $||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} - 1 < 0$. Consider λ from equation (4.23), ε from equation (4.24), Π_1 from equation (4.25) and Π_2 from equation (4.26).

- Case 2: $1 = ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}$. Choose

$$\varepsilon = \frac{\delta}{2 + ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}},$$

$$\lambda = 1,$$

$$\Pi_1(j\omega) = \begin{bmatrix} 0 & \hat{M}^{\sim}(j\omega) \\ \hat{M}(j\omega) & -\delta I \end{bmatrix} + 2\delta \begin{bmatrix} -\frac{1}{||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}} I & 0 \\ 0 & I \end{bmatrix},$$

$$\Pi_2(j\omega) = \begin{bmatrix} 0 & -\hat{M}^{\sim}(j\omega) \\ -\hat{M}(j\omega) & 0 \end{bmatrix} + \frac{\delta(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2} + 4)}{||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2} + 2)} \begin{bmatrix} I & 0 \\ 0 & -\frac{1}{||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}} I \end{bmatrix}.$$

- Case 3: $1 - ||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} < 0$. Choose

$$\varepsilon = \frac{\delta}{(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2} + 1)(||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} + 1)},$$

$$\lambda = 1,$$

$$\Pi_1(j\omega) = \begin{bmatrix} 0 & \hat{M}^{\sim}(j\omega) \\ \hat{M}(j\omega) & -\delta I \end{bmatrix} + \left(\delta + \frac{\delta}{(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} - 1)(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2} + 1)}\right) \begin{bmatrix} -\frac{1}{||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}} I & 0 \\ 0 & I \end{bmatrix},$$

$$\Pi_2(j\omega) = \begin{bmatrix} 0 & -\hat{M}^{\sim}(j\omega) \\ -\hat{M}(j\omega) & 0 \end{bmatrix} + \frac{\delta ||\Delta_2||^4_{\mathcal{L}_2\to\mathcal{L}_2}}{(||\Delta_1||^2_{\mathcal{L}_2\to\mathcal{L}_2}||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} - 1)(||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2} + 1)} \begin{bmatrix} I & 0 \\ 0 & -\frac{1}{||\Delta_2||^2_{\mathcal{L}_2\to\mathcal{L}_2}} I \end{bmatrix}.$$

Then Δ_1, Δ_2 satisfy the IQCs (4.3) and (4.4), parametrized by the strict positive-negative multipliers $\Pi_1(j\omega)$ and $\Pi_2(j\omega)$. Meanwhile, $\Pi_1(j\omega), \Pi_2(j\omega), \lambda, \varepsilon$ satisfy (4.5). Then the system is $\mathcal{L}_2[0,\infty)$ stable via Theorem 4.2.2. □

This result does not need or imply the existence of a canonical factorization, a fact that was originally stated in [35]. This is because stability is not verified by a passive equivalent feedback interconnection. Instead, a homotopy argument starts from a stable feedback interconnection $[\tau\Delta_1, -\tau\Delta_2]$ and then proceeds to modify it until it transforms into the complete feedback interconnection $[\Delta_1, -\Delta_2]$. Because the use of homotopy technique, the well posedness of $[\tau\Delta_1, -\tau\Delta_2]$ is required for all $\tau \in [0,1]$ instead of the canonical factorization. Note that as stated in [35], this is trivial because Passivity IQC is strict PN-IQC.

However, this result can only be used for Δ_1 and Δ_2 that are $\mathcal{L}_2[0,\infty)$ bounded, and therefore it is not equivalent to Passivity Theorem.

4.4 Example

Recently, the interconnection of a delay and a so-called *output strictly equilibrium-independent passive operator with equilibrium-independent roll-off* was outlined in [78]. This interconnection is an important example where the need for description of non-linear systems in closed loop needs to be explicitly stated.

First the properties of the non-linearity are defined for the example. These properties are taken from [78].

Definition 4.4.1. *Δ_{OP} is a dynamical system described as*

$$\dot{x} = f(x,u), \tag{4.41}$$

$$y = h(x). \tag{4.42}$$

Δ_{OP} is said to be output strictly equilibrium-independent passive (OSEIP) with gain $\gamma_1 > 0$ if for every $\tilde{u} \in \mathbb{R}^m$, there exists a once differentiable storage function $S_{\tilde{u}} : \mathbb{R}^n \to \mathbb{R}$ such that $S_{\tilde{u}}(x) > 0 \quad \forall x \neq \tilde{x}$, $S_{\tilde{u}}(\tilde{x}) = 0$, and

$$\nabla_x S_{\tilde{u}} \cdot f(x,u) \leq (u-\tilde{u})^T(y-\tilde{y}) - \frac{1}{\gamma_1}(y-\tilde{y})^T(y-\tilde{y})$$

for all $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, where $\tilde{y} = h(\tilde{x})$.

Then, in order to use IQCs, define the corner frequency as follows:

Definition 4.4.2. *Δ_{OP} in equations* (4.41) *and* (4.42) *is said to have equilibrium-independent roll-off with corner frequency $\omega_c > 0$ and gain γ_2 if for every $\tilde{u} \in \mathbb{R}^m$, there exists a once differentiable storage function $V_{\tilde{u}}(x) : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\tilde{u}}(x) > 0 \quad \forall x \neq \tilde{x}$, $V_{\tilde{u}}(\tilde{x}) = 0$, and*

$$0 \leq -\nabla V_{\tilde{u}} \cdot f(x,u) + \gamma_2^2 (u-\tilde{u})^T(u-\tilde{u}) - \bar{z}^T\bar{z},$$

$\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m$, where

$$\bar{z} = \frac{\partial h}{\partial x} f(x,u)\frac{1}{\omega_c} + y - \tilde{y},$$

where $\tilde{y} = h(\tilde{x})$.

We will call this non-linearity Δ_{EIRO}. In [78], it is shown that Δ_{EIRO} holds the IQC with the following multiplier:

$$\Pi_{2,\xi,\gamma_2}(j\omega) = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1+(\frac{\omega}{\omega_c})^2}{\gamma_2^2(1+\xi(\frac{\omega}{\omega_c})^2)} \end{bmatrix}, \tag{4.43}$$

where $\gamma_2 > 0$, $\omega_c > 0$ is the corner frequency and $0 < \xi \ll 1$ is introduced to make Π_{2,ξ,γ_2} proper.

It also holds an IQC with the passivity multiplier

$$\Pi_{1,\gamma_1}(j\omega) = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\gamma_1} \end{bmatrix}, \tag{4.44}$$

where $\gamma_1 > 0$.

The feedback interconnection is represented in Figure 4.3, with $\Delta_2 = \Delta_{EIRO}$ and $\Delta_1 = \gamma e^{-sT}$:

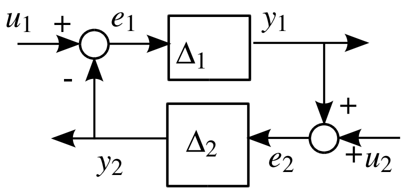


Figure 4.3: Negative feedback interconnection

The following equations describe the feedback interconnection:

$$e_1 = -u_1 + \Delta_2 e_2,$$
$$e_2 = u_2 - \Delta_1 e_1.$$

In order to be able to apply Theorem 4.2.2, it is necessary to show that the feedback interconnection is well posed. However, this is a trivial task because the series $\Delta_1\Delta_2(u(t),t) = \Delta_{EIRO}(u(t-\theta),t-\theta)$ delays all inputs. This can be shown by placing the series $\Delta_1\Delta_2$ in series with a predictor of θ prediction time, the resulting operator is equal to $\Delta_{EIRO}(u(t),t)$, which is causal. Consequently, using Corollary 4.2.6 the feedback interconnection $[\tau\Delta_1, \tau\Delta_2]$ is well posed for all $\tau \in [0,1]$. Furthermore, note that the feedback interconnection can also be equivalently represented by the following equations:

$$e_1 = -u_1 + \Delta_2 e_2, \tag{4.45}$$
$$e_2 = u_2 - \gamma\Delta_2 e_2 + e_3, \tag{4.46}$$
$$e_3 = \gamma\Delta_2 e_2 - \Delta_1 e_1. \tag{4.47}$$

The equivalent representation is presented in the Figure 4.4.

Showing the stability of $\hat{\Delta}_2$ is trivial. Δ_2 holds the IQC parametrized by equation (4.44), then it is evident that the passive system in feedback with negative gain is stable, i.e.

$$< y_2, \gamma y_2 >_T = \gamma ||y_2||_T^2 \quad \forall y_2 \in \mathcal{L}_{2e}, T > 0,$$
$$< e_2, \Delta_2 e_2 >_T \geq \frac{1}{\gamma_1} ||\Delta_2 e_2||_T^2 \quad \forall e_2 \in \mathcal{L}_{2e}, T > 0.$$

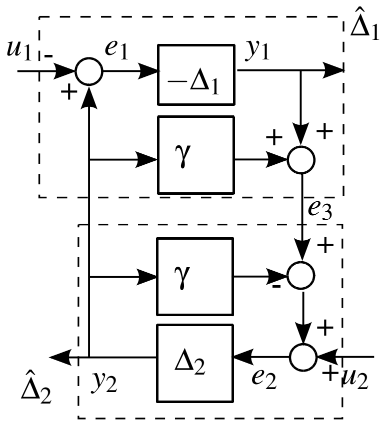


Figure 4.4: Delay encapsulation

By using Theorem 2.1.20 with $\Delta_1 = \gamma$ and $\Delta_2 = \Delta_{EIRO}$, feedback loop holds the following parametrization: $\delta_1 = \gamma$, $\delta_2 = \frac{1}{\gamma_1}$, $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$. It is easy to show that

$$\varepsilon_1 + \delta_2 = \frac{1}{\gamma_1} > 0,$$
$$\varepsilon_2 + \delta_1 = \gamma > 0.$$

Consequently, $\hat{\Delta}_2$ is $\mathcal{L}_2[0,\infty)$ stable. Therefore, the subsystem $\hat{\Delta}_2$ will hold the following IQC:

$$\left\langle \begin{bmatrix} E_2 \\ Y_2 \end{bmatrix}, \left[\alpha_1 \Pi_{1,\gamma_1}(j\omega) + \alpha_2 \Pi_{2,\xi,\gamma_2}(j\omega)\right] \begin{bmatrix} E_2 \\ Y_2 \end{bmatrix} \right\rangle_{\mathcal{H}_2} \geq 0$$
$$\forall E_2 \in \mathcal{H}_2, Y_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{E_2\})\}.$$

Making the appropriate substitutions from equations (4.45), (4.46) and making $u_2 = 0$, $\hat{\Delta}_2$ holds the following IQC

$$\left\langle \begin{bmatrix} E_3 \\ Y_2 \end{bmatrix}, \begin{bmatrix} I & 0 \\ -\gamma I & I \end{bmatrix} \left[\alpha_1 \Pi_{1,\gamma_1}(j\omega) + \alpha_2 \Pi_{2,\xi,\gamma_2}(j\omega)\right] \begin{bmatrix} I & -\gamma I \\ 0 & I \end{bmatrix} \begin{bmatrix} E_3 \\ Y_2 \end{bmatrix} \right\rangle_{\mathcal{H}_2} \geq 0$$
$$\forall E_2 \in \mathcal{H}_2, Y_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{E_2\})\}.$$

Choosing $\alpha_2 = \frac{\alpha_1}{2\gamma}$ yields

$$\left\langle \begin{bmatrix} E_3 \\ Y_2 \end{bmatrix}, \Pi_3(j\omega) \begin{bmatrix} E_3 \\ Y_2 \end{bmatrix} \right\rangle_{\mathcal{H}_2} \geq 0 \quad \forall E_2 \in \mathcal{H}_2, Y_2 = \mathcal{F}\{\Delta_2(\mathcal{F}^{-1}\{E_2\})\}, \tag{4.48}$$

with

$$\Pi_3(j\omega) = \begin{bmatrix} \frac{\alpha_1}{2\gamma} & 0 \\ 0 & -\frac{\alpha_1\gamma}{2} - \frac{\alpha_1}{\gamma_1} - \frac{\alpha_1}{2} \frac{1+(\frac{\omega}{\omega_c})^2}{\gamma\gamma_2^2(1+\xi(\frac{\omega}{\omega_c})^2)} \end{bmatrix}. \tag{4.49}$$

Meanwhile, the delay encapsulation holds the following absolute value:

$$|E_3| = |\gamma Y_2 - \gamma e^{-j\omega\tau} Y_2| = \gamma|1 - e^{-j\omega\tau}||Y_2| \leq |D(j\omega)||Y_2|, \tag{4.50}$$

Likewise, the norm of the encapsulated delay will hold

$$||E_3||^2_{\mathcal{H}_2} = ||\gamma Y_2 - \gamma e^{-j\omega\tau} Y_2||^2_{\mathcal{H}_2} = \gamma^2|1 - e^{-j\omega\tau}|^2||Y_2||^2_{\mathcal{H}_2} \leq D(j\omega)^\sim D(j\omega)||Y_2||^2_{\mathcal{H}_2}. \tag{4.51}$$

This function can be bounded by a causal linear filter $D(j\omega) \in \mathcal{RH}_\infty$. In this case, we are interested only in the filter already present in the IQC, held by equation (4.48). Now, considering the norms of the delay, the following IQC is fulfilled:

$$D(j\omega)^\sim D(j\omega)||Y_2||^2_{\mathcal{H}_2} - ||E_3||^2_{\mathcal{H}_2} \geq 0 \iff \begin{bmatrix} E_3 \\ Y_2 \end{bmatrix} \Pi_4(j\omega) \begin{bmatrix} E_3 \\ Y_2 \end{bmatrix} \geq 0 \quad \forall \omega \in \mathbb{R} \tag{4.52}$$

with

$$\Pi_4(j\omega) = \begin{bmatrix} -1 & 0 \\ 0 & D(j\omega)^\sim D(j\omega) \end{bmatrix}. \tag{4.53}$$

Adding equation (4.49) with $\lambda > 0$ times (4.53) yields

$$\Pi_3(j\omega) + \lambda\Pi_4(j\omega) = \begin{bmatrix} \frac{\alpha_1}{2\gamma} - \lambda & 0 \\ 0 & -\frac{\alpha_1}{2\gamma}(\gamma^2 + 2\frac{\gamma}{\gamma_1} + \frac{1+(\frac{\omega}{\omega_c})^2}{\gamma_2^2(1+\xi(\frac{\omega}{\omega_c})^2)}) + \lambda D(j\omega)^\sim D(j\omega) \end{bmatrix}.$$

Results in [78] assumes $\gamma_1 = \gamma_2 = 1$. For simplicity, choose $\varepsilon_h > 1$, $\varepsilon_l < 1$ such that

$\varepsilon_h\varepsilon_l > 1$, then choose

$$\lambda = \frac{\varepsilon_h \alpha_1}{2\gamma},$$
$$D(j\omega)^{\sim} D(j\omega) = \varepsilon_l(\gamma^2 + 2\gamma + \frac{1+(\frac{\omega}{\omega_c})^2}{1+\xi(\frac{\omega}{\omega_c})^2}).$$

Consequently, given that $\xi \ll 1$ results in

$$\Pi_3(j\omega) + \lambda\Pi_4(j\omega) = \begin{bmatrix} \frac{(1-\varepsilon_h)\alpha_1}{2\gamma} & 0 \\ 0 & \frac{(\varepsilon_h\varepsilon_l-1)\alpha_1}{2\gamma}(\gamma^2+2\gamma+\frac{1+(\frac{\omega}{\omega_c})^2}{1+\xi(\frac{\omega}{\omega_c})^2}) \end{bmatrix}$$

$$\leq \min\{\frac{(1-\varepsilon_h)\alpha_1}{2\gamma}, \frac{(\varepsilon_h\varepsilon_l - 1)\alpha_1}{2\gamma}(\gamma^2+2\gamma+\frac{1}{\xi})\}I \quad \forall\omega \in \mathbb{R}$$

Choose

$$\varepsilon = \min\{\frac{(1-\varepsilon_h)\alpha_1}{2\gamma}, \frac{(\varepsilon_h\varepsilon_l - 1)\alpha_1}{2\gamma}(\gamma^2+2\gamma+\frac{1}{\xi})\} \tag{4.54}$$

It is evident that ε_1 and ε_2 can then be made arbitrarily close to 1.

Then $\hat{\Delta}_1, \hat{\Delta}_2$ satisfy the IQCs (4.3) and (4.4), parametrized by the strict Positive Negative IQC multipliers $\Pi_3(j\omega)$ in equation (4.49) and $\Pi_4(j\omega)$ in equation (4.53). Meanwhile, $\Pi_3(j\omega), \Pi_4(j\omega), \lambda, \varepsilon$ satisfy (4.5). Then the system is $\mathcal{L}_2[0,\infty)$ stable via Theorem 4.2.2.

In order to compare this result, it is only left to find a relationship between the delay τ and the gain γ. This is evident from equation (4.51), i.e.

$$\gamma^2|1-e^{-j\omega\tau}|^2 < \gamma^2 + 2\gamma + \frac{1+(\frac{\omega}{\omega_c})^2}{1+\xi(\frac{\omega}{\omega_c})^2} \quad \forall\omega \in \mathbb{R}.$$

Making the substitution $\hat{\omega} = \tau\omega$, this equation can be written equivalently

$$2\gamma^2(1-cos(\hat{\omega})) < \gamma^2 + 2\gamma + \frac{1+(\frac{\hat{\omega}}{\omega_c\tau})^2}{1+\xi(\frac{\hat{\omega}}{\omega_c\tau})^2} \quad \forall\hat{\omega} \in \mathbb{R}. \tag{4.55}$$

An exact solution optimizing the relation between $\omega_c\tau$ and γ is not of the interest of this work. However, for the sake of argument, a numerical solution to equation (4.55) is presented in Figure 4.5. Notice that equation (4.55) and equation (23) in [78] are a very close match for $n = 1$. Furthermore, when $\tau \to 0$, the feedback interconnection of $\hat{\Delta}_2$ and a positive constant is recovered, and therefore γ can be unbounded. On the other hand, if $\tau \to \infty$, equation (4.55) is only true for $\gamma < 1$, therefore recovering Passivity Theorem and the Small Gain Theorem respectively.

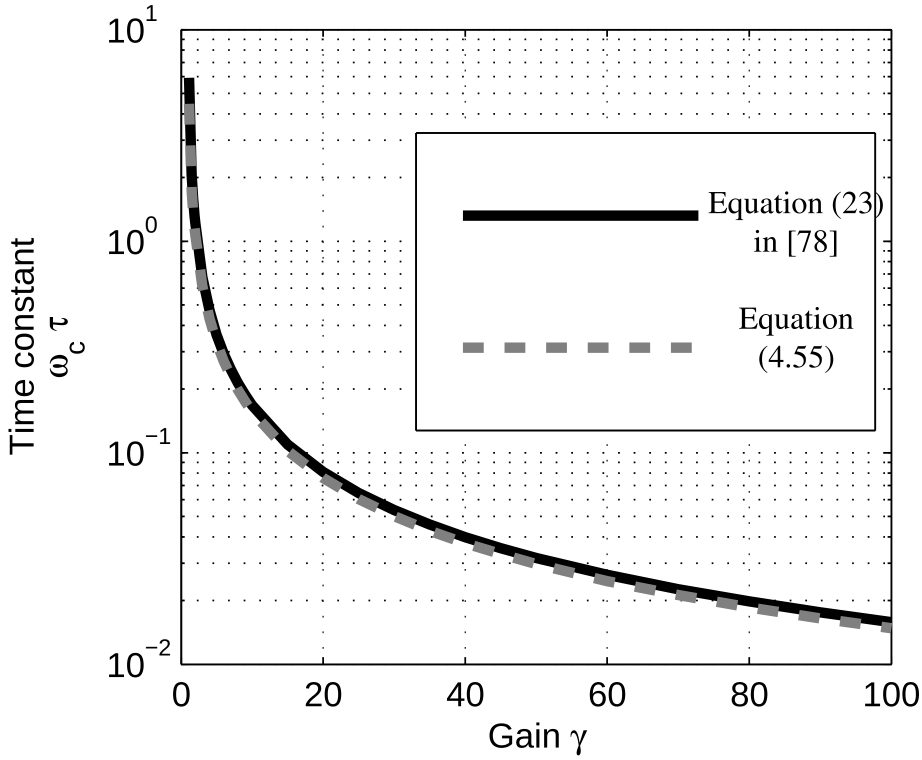


Figure 4.5: Gain Vs Time constant

This example depicts the novelty of Theorem 4.2.2, because it shows that generalizations such as Appendix 2.9 in [79] only partially contemplate the full potential of the IQC analysis, nonetheless, that is a step in the right direction. It is noteworthy that Theorem 4.2.2 has transformed the disturbance itself in the focus of attention.

Additionally, this work now suggest a way to show the stability of the feedback interconnections made by Δ_{EIRO} with the full library of IQCs.

4.5 Conclusions

This chapter shows the need for the reinterpretation of the IQC Theorem in the general form of Theorem 4.2.2, where two non-linear systems are interconnected in positive feedback. In this form, Passivity, Dissipativity and Small Gain Theorem are partially recovered when the systems interconnected are assumed to be bounded.

This work is an explicit and non trivial reinterpretation of the original IQC Theorem using loop transformations. However, this reinterpretation shows precise limitations on the conditions that these non-linearities have to hold in order to be compatible feedback interconnections, namely the requirement to hold PN-IQC multipliers.

In the introduction, there were developed two lemmas that can exploit IQCs description of non-linear uncertainties, but using Passivity or Small Gain Theorems to deduce stability. This effort is a focus in the literature with works like [57] that explicitly have found stability conditions using only PN-IQC multipliers. Although showing the well posedness only for $\tau = 1$ seems to save considerable effort, this section showed that many non-linear disturbances have no problem observing the conditions for the most basic definition of well posedness presented in [13] for all $\tau \in [0,1]$. Instead, the main focus of this work the reformulation of the IQC Theorem to an equivalent representation that explicitly studies feedback interconnections of two non-linear systems.

This work initially studies the Small Gain Theorem and how it can be recovered from Theorem 4.2.2 with the extra condition of well posedness for all $\tau \in [0,1]$. Then it studied how the IQC Theorem can be recovered first for strict PN-IQCs and then for the general class of IQCs. The resulting conclusion is that when a feedback interconnection holds the conditions of the IQC Theorem, it will have an equivalent representation that holds PN-IQC multipliers, although this would need to be done in small increments.

The study of the relation between IQC and Dissipative systems shows that the introduction of dynamic multipliers is still an open question. The corresponding corollary partially recovers the Dissipativity Theorem, but it is restricted to the feedback interconnection of Output Passive operators.

In the special case of Passivity, the need for canonical factorization was exchanged for the well posedness of the feedback loop for all $\tau \in [0,1]$, due to the proof strategy used in the IQC Theorem. This remark was originally made for the IQC Theorem in [35], albeit applicable originally for the interconnection of a Linear Time Invariant system with a bounded perturbation.

Finally, an example from the literature was selected that showcases the interconnection of Passivity and delay operators using IQC techniques and the potential utility brought out by the novel representation in Theorem 4.2.2.

Chapter 5

Conclusions

This chapter presents a summary of the contributions of this thesis and proposes future research based on the findings of this thesis.

5.1 Contributions to the Zames-Falb multipliers

This work was inspired by the need of anti-causal Zames-Falb multipliers to show the stability of a class of non-linear systems. In the introduction an example of such a system is developed and appropriate lemmas are collected that ensure the existence of anti-causal Zames-Falb multipliers for systems that fulfil the Kalman Conjecture. In order to illustrate this problem, methods available from the literature were modified such that only causal Zames-Falb multipliers are calculated. This effort showed indisputable evidence for the need of anti-causal multipliers search algorithms. Chapter 3 was devoted to solve this problem.

At the beginning of Chapter 3, this thesis studies the state of the art concerning Zames-Falb multiplier synthesis. A minor contribution presented in this chapter is the correct algebraic representation of the Zames-Falb synthesis presented in [32].

The main contribution made by Chapter 3 is an extension of the Zames-Falb multiplier search presented in [1] and [33] to anti-causal multipliers. The new proposed algorithms, anti-causal Zames-Falb syntheses and anti-causal Zames-Falb plus Popov multiplier synthesis, can then be used in tandem with [1] and [33] to find less conservative estimates for the stability margins in high order plants.

5.2 Contributions to the IQC theory

In the introduction to IQC analysis some possible loop transformations where explored that are traditionally implemented to use the Small Gain Theorem and the Passivity Theorem in order to adopt IQCs as the description of the non-linear systems. This work produced little results when used with the PN-IQC multiplier factorization presented in [42]. However, this analysis showed that PN-IQCs are an invaluable tool in order to construct a stable feedback interconnection of two non-linear systems.

The main contribution of this chapter is the solution to the stability problem of the feedback interconnection of two non-linear systems that hold IQCs is the reinterpretation of the IQC Theorem and the generalization of the PN-IQCs to positive-negative multipliers. The resulting theorem is then used to derive well known results, such as the Small Gain Theorem, Dissipativity and Passivity. Theorem 4.2.2 showed that the IQC Theorem can be sufficient to deal with an extended class of systems. Additionally, Corollary 4.2.4 and Lemma 4.2.5 demonstrated that Theorem 4.2.2 is an equivalent representation of the IQC Theorem with explicit conditions for the interconnection of non-linear systems. In order to show the need for the reinterpretation made in Theorem 4.2.2, a well-known example in the literature is reconstructed using only the new theorem, potentially opening the application of IQCs, where they no longer are restricted to describe uncertainties.

5.3 Directions for Future Research

In the area of Zames-Falb multipliers, there are two possible options in the continuation of the research of Chapter 3.

- Zames-Falb multipliers are used only for analysis. The pioneering work of [80] suggests that Zames-Falb multipliers can also be used to synthesize controllers.
- Zames-Falb multipliers are not exclusive to monotone non-linearities. The presented multiplier synthesis methods for the class $\mathcal{M}_{odd}$ can be used with the work in [76] to study hysteresis non-linearities.
- The problem of Absolute Stability with delay has been identified to be a candidate application of Zames-Falb multipliers combined with IQCs for delay. The state of the art of this area can be found in [10, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75].

In the area of the IQC Theorem, the following problems are put forward, which are a direct consequence of the feedback interconnection studied in Chapter 3.

- The last example opens a new question: Can Δ_{EIRO} be trivially interconnected with other non-linearities described in the IQCs rather than delay?.
- The work on Absolute Stability with delay has shown that more general structures than Δ_{EIRO} are already covered by classical Lyapunov analysis [74, 72, 70, 69, 68, 67, 10]. Theorem 4.2.2 suggests that IQCs can also be used to study this problem from the point of view of Input-Output analysis.
- The introduction left open the question of having a proof that shows the stability of the interconnection of two non-linear systems using IQCs and the Passivity Theorem. A generalization of a recent work in Dissipativity [57] may provide the correct solution to the problem.

Appendix A

Appendix

A.1 Proof to Lemma 2.3.7

This section presents the proof the stability problem solved by Lemma 2.3.7. The first part of the proof builds an equivalent loop transformation, the second part focuses in proving the stability of the closed loop using Small Gain Theorem. In this stage of the work, the inverse in equation (2.22) has shown to be very restrictive condition, only allowing multipliers with a very specific structure, and thus, it becomes the biggest restriction for the application of this result.

Proof. From equations (2.19) and (2.20), it is know that the plant $\Delta_i, i = 1,2$ holds the following inequalities, using Theorem 2.3.6 and Definition 2.3.5 :

$$\left\langle \begin{bmatrix} Y_i & 0 \\ P_i & X_i \end{bmatrix} \begin{bmatrix} u_i \\ \Delta_i(u_i) \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} Y_i & 0 \\ P_i & X_i \end{bmatrix} \begin{bmatrix} u_i \\ \Delta_i(u_i) \end{bmatrix} \right\rangle_T \geq 0 \quad \forall u_i \in \mathcal{L}_{2e} \tag{A.1}$$

The sum of (2.19) and (2.20) will remain positive, then the following expressions are equivalent:

$$\begin{aligned} \iff & \left\langle \begin{bmatrix} Y_1 & 0 \\ P_1 & X_1 \end{bmatrix} \begin{bmatrix} u_1 \\ \Delta_1(u_1) \end{bmatrix}, \begin{bmatrix} I_l & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} Y_1 & 0 \\ P_1 & X_1 \end{bmatrix} \begin{bmatrix} u_1 \\ \Delta_1(u_1) \end{bmatrix} \right\rangle_T \\ & +\lambda \left\langle \begin{bmatrix} Y_2 & 0 \\ P_2 & X_2 \end{bmatrix} \begin{bmatrix} u_2 \\ \Delta_2(u_2) \end{bmatrix}, \begin{bmatrix} I_m & 0 \\ 0 & -I_l \end{bmatrix} \begin{bmatrix} Y_2 & 0 \\ P_2 & X_2 \end{bmatrix} \begin{bmatrix} u_2 \\ \Delta_2(u_2) \end{bmatrix} \right\rangle_T \geq 0 \\ & \forall u_1 \in \mathcal{L}_{2e}^l, u_2 \in \mathcal{L}_{2e}^m. \end{aligned}$$

$$\iff \left\langle \begin{bmatrix} Y_1 & 0 & 0 & 0 \\ 0 & \sqrt{\lambda}Y_2 & 0 & 0 \\ P_1 & 0 & X_1 & 0 \\ 0 & \sqrt{\lambda}P_2 & 0 & \sqrt{\lambda}X_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \Delta_1(u_1) \\ \Delta_2(u_2) \end{pmatrix}, \right.$$

$$\left. \times \begin{bmatrix} I_l & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_l \end{bmatrix} \begin{bmatrix} Y_1 & 0 & 0 & 0 \\ 0 & \sqrt{\lambda}Y_2 & 0 & 0 \\ P_1 & 0 & X_1 & 0 \\ 0 & \sqrt{\lambda}P_2 & 0 & \sqrt{\lambda}X_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \Delta_1(u_1) \\ \Delta_2(u_2) \end{pmatrix} \right\rangle_T \geq 0$$

$\forall u_1 \in \mathcal{L}_{2e}^l, u_2 \in \mathcal{L}_{2e}^m, T \geq 0.$

$$\iff \left\langle \begin{bmatrix} \mathcal{Y} & 0 \\ \mathcal{P} & \mathcal{X} \end{bmatrix} \begin{pmatrix} \bar{u} \\ \Theta(\bar{u}) \end{pmatrix}, \begin{bmatrix} I_{l+m} & 0 \\ 0 & -I_{m+l} \end{bmatrix} \begin{bmatrix} \mathcal{Y} & 0 \\ \mathcal{P} & \mathcal{X} \end{bmatrix} \begin{pmatrix} \bar{u} \\ \Theta(\bar{u}) \end{pmatrix} \right\rangle_T \geq 0 \quad \forall \bar{u} \in \mathcal{L}_{2e}^{l+m}, T \geq 0$$

[Replace the following shorthand notations:]

$$\mathcal{Y} = \begin{bmatrix} Y_1 & 0 \\ 0 & \sqrt{\lambda}Y_2 \end{bmatrix},$$

$$\mathcal{P} = \begin{bmatrix} P_1 & 0 \\ 0 & \sqrt{\lambda}P_2 \end{bmatrix},$$

$$\mathcal{X} = \begin{bmatrix} X_1 & 0 \\ 0 & \sqrt{\lambda}X_2 \end{bmatrix},$$

$$\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$\Theta(\bar{u}) = \begin{pmatrix} \Delta_1(u_1) \\ \Delta_2(u_2) \end{pmatrix}.$$

$$\iff \left\langle \begin{pmatrix} \bar{z} \\ \mathcal{P}\mathcal{Y}^{-1}\hat{z} + \mathcal{X}\Theta(\mathcal{Y}^{-1}\hat{z}) \end{pmatrix}, \begin{bmatrix} I_{l+m} & 0 \\ 0 & -I_{m+l} \end{bmatrix} \begin{pmatrix} \bar{z} \\ \mathcal{P}\mathcal{Y}^{-1}\hat{z} + \mathcal{X}\Theta(\mathcal{Y}^{-1}\hat{z}) \end{pmatrix} \right\rangle_T \geq 0$$

$\forall \bar{z} \in \mathcal{L}_{2e}^{l+m}, T \geq 0.$

[Make the following change of variable: $\hat{z} = \mathcal{Y}\bar{u}$, given that $\mathcal{Y}, \mathcal{Y}^{-1} \in \mathcal{RH}_\infty$.]

$$\iff \ < \bar{z}, \bar{z} >_T$$
$$- < \mathcal{P}\mathcal{Y}^{-1}\hat{z} + \mathcal{X}\Theta(\mathcal{Y}^{-1}\hat{z}), \mathcal{P}\mathcal{Y}^{-1}\hat{z} + \mathcal{X}\Theta(\mathcal{Y}^{-1}\hat{z}) >_T \geq 0$$

$\forall \bar{z} \in \mathcal{L}_{2e}^{l+m}, T \geq 0$

$$\iff ||(\mathcal{P}\mathcal{Y}^{-1}+\mathcal{X}\Theta\mathcal{Y}^{-1})(\hat{z})||_T \leq ||\bar{z}||_T \quad \forall \bar{z} \in \mathcal{L}_{2e}^{l+m}, T \geq 0 \tag{A.2}$$

By executing the inner product, it was shown that the non-linear system on Figure A.3 holds finite gain stability less than 1.

$\hat{\Pi}_1(j\omega)$, $\hat{\Pi}_2(j\omega)$, $\hat{\mathcal{Y}}(j\omega)$, $\hat{\mathcal{P}}(j\omega)$, $\hat{\mathcal{X}}(j\omega)$ are the Fourier transforms of Π_1, Π_2, $\mathcal{Y}$, $\mathcal{P}$, $\mathcal{X}$ respectively. Note that equation (2.21) is equivalent to the following expressions:

$$\iff H^*\hat{\Pi}_1(j\omega)H + \lambda\hat{\Pi}_2(j\omega) < -\gamma I \quad \forall\omega \in \mathbb{R}$$

$$\iff H^*\hat{\mathcal{Y}}^*(j\omega)\hat{\mathcal{Y}}(j\omega)H - H^*\hat{\mathcal{P}}^*(j\omega)\hat{\mathcal{P}}(j\omega)H$$
$$- H^*\hat{\mathcal{P}}^*(j\omega)\hat{\mathcal{X}}(j\omega) - \hat{\mathcal{X}}^*(j\omega)\hat{\mathcal{P}}(j\omega)H - \hat{\mathcal{X}}^*(j\omega)\hat{\mathcal{X}}(j\omega) < -\gamma I \quad \forall\omega \in \mathbb{R}.$$

[Expanding the matrices, and replacing the shorthand notation.]

$$\implies \begin{bmatrix} \hat{\mathcal{Y}}(j\omega)H \\ \hat{\mathcal{P}}(j\omega)H + \hat{\mathcal{X}}(j\omega) \end{bmatrix}^* \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\mathcal{Y}}(j\omega)H \\ \hat{\mathcal{P}}(j\omega)H + \hat{\mathcal{X}}(j\omega) \end{bmatrix} > 0 \quad \forall\omega \in \mathbb{R}.$$

[Multiplying by -1 and removing the small variable γ.]

$$\implies \left\langle \begin{bmatrix} \mathcal{Y}H \\ \mathcal{P}H + \mathcal{X} \end{bmatrix} \bar{g}, \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{Y}H \\ \mathcal{P}H + \mathcal{X} \end{bmatrix} \bar{g} \right\rangle_T > 0 \quad \forall \bar{g} \in \mathcal{L}_{2e}, T \geq 0$$

[Using Theorem 2.3.6 and noting that $\hat{\mathcal{P}}H + \hat{\mathcal{X}}, (\hat{\mathcal{P}}H + \hat{\mathcal{X}})^{-1} \in \mathcal{RH}_\infty$.]

$$\iff \left\langle \begin{bmatrix} I \\ \mathcal{Y}H(\mathcal{X}+\mathcal{P}H)^{-1} \end{bmatrix} \bar{c}, \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I \\ \mathcal{Y}H(\mathcal{X}+\mathcal{P}H)^{-1} \end{bmatrix} \bar{c} \right\rangle_T$$
$$> 0 \quad \forall \bar{c} \in \mathcal{L}_{2e}, T \geq 0$$

[Make $\bar{g} = (\mathcal{X}+\mathcal{P}H)^{-1}\bar{c}$, noting that $\mathcal{P}H + \mathcal{X}, (\mathcal{P}H + \mathcal{X})^{-1} \in \mathcal{RH}_\infty$.]

$$\iff ||\mathcal{Y}(I + H\mathcal{X}^{-1}\mathcal{P})^{-1}H\mathcal{X}^{-1}\bar{c}||_T^2 < ||\bar{c}||_T^2 \quad \forall \bar{c} \in \mathcal{L}_{2e}, T \geq 0$$

$$||\mathcal{Y}(I + H\mathcal{X}^{-1}\mathcal{P})^{-1}H\mathcal{X}^{-1}\bar{c}||_T < ||\bar{c}||_T \quad \forall \bar{c} \in \mathcal{L}_{2e}, T \geq 0 \tag{A.3}$$

In order to use Small Gain Theorem, note that Figure A.1 represents the original closed loop.

The equations that represent the connection in Figure A.1 are the following expressions:

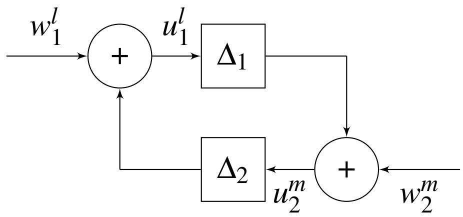


Figure A.1: Closed loop figure

$$u_1 = w_1 + \Delta_2(u_2)$$
$$u_2 = w_2 + \Delta_1(u_1).$$

$\iff$

$$\bar{v} = H^{-1}\bar{w} + \Theta(\bar{u})$$
$$\bar{u} = H\bar{v}$$

[Take $\Theta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$,$H = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.]

$\iff$ Figure A.2 shows the closed loop interconnection:

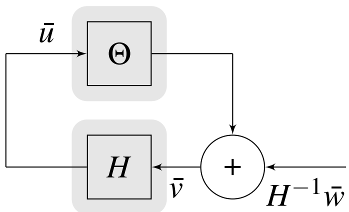


Figure A.2: Closed loop figure

$\iff$

$$\begin{cases} \mathcal{X}\bar{v} = \mathcal{X}H^{-1}\bar{w} + \mathcal{X}\Theta(\bar{u}) \\ \bar{u} = H\bar{v} \end{cases}$$

$\iff$

$$\begin{cases} \mathcal{X}\bar{v} + \mathcal{P}\mathcal{Y}^{-1}\bar{e} = \mathcal{X}H^{-1}\bar{w} + \mathcal{P}\mathcal{Y}^{-1}\bar{e} + \mathcal{X}\Theta(\mathcal{Y}^{-1}\bar{e}) \\ \mathcal{Y}^{-1}\bar{e} = H\bar{v} \end{cases}$$

[Given that $\mathcal{Y}^{-1} \in \mathcal{RH}_\infty$, then $\bar{u} = \mathcal{Y}^{-1}\bar{e}$.]

$\iff$

$$\begin{cases} \bar{c} = \mathcal{X}H^{-1}\bar{w} + \mathcal{P}\mathcal{Y}^{-1}\bar{e} + \mathcal{X}\Theta(\mathcal{Y}^{-1}\bar{e}) \\ \mathcal{Y}^{-1}\bar{e} = H\mathcal{X}^{-1}\bar{c} - H\mathcal{X}^{-1}\mathcal{P}\mathcal{Y}^{-1}\bar{e} \end{cases}$$

[Rewriting the equation in closed loop format requires the a further substitution $\bar{c} = \mathcal{X}\bar{v} + \mathcal{P}\mathcal{Y}^{-1}\bar{e}$. It is know that $\hat{\mathcal{X}}^{\sim}(j\omega)\hat{\mathcal{X}}(j\omega) > 0$, then the inverse $\mathcal{X}^{-1}$ exists. Make the following variable substitution $\bar{v} = \mathcal{X}^{-1}\bar{c} - \mathcal{X}^{-1}\mathcal{P}\mathcal{Y}^{-1}\bar{e}$.]

$\iff$

$$\begin{cases} \bar{c} = \mathcal{X}H^{-1}\bar{w} + (\mathcal{P} + \mathcal{X}\Theta)(\mathcal{Y}^{-1}\bar{e}) \\ \bar{e} = \mathcal{Y}(I + H\mathcal{X}^{-1}\mathcal{P})^{-1}H\mathcal{X}^{-1}\bar{c} \end{cases}$$

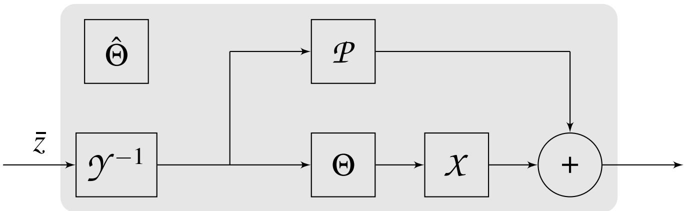


Figure A.3: Modified non-linearity

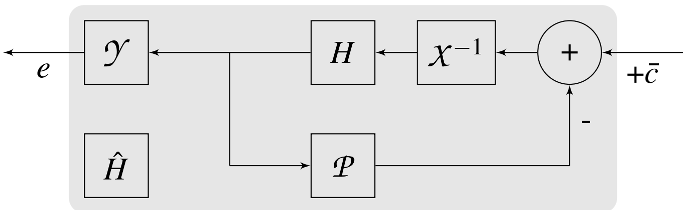


Figure A.4: Modified linearity

[The new non-linear plant is a linear combination of the original $\Theta()$ and the multipliers $\mathcal{P}, \mathcal{Y}^{-1}, \mathcal{X}$, can be represented in Figure A.3, while the linear part of the plant can be represented as Figure A.4.]

$\iff$ Figure A.5.

Therefore, for the equivalent system, using equations (A.2) and (A.3) and using the small gain Theorem 2.1.19, the closed loop is $\mathcal{L}_2[0, \infty)$.

□

A.2 Proof to Lemma 2.3.8

Proof. Let $\lambda > 0$. Adding equation (2.25) with to λ (2.26) will remain positive.

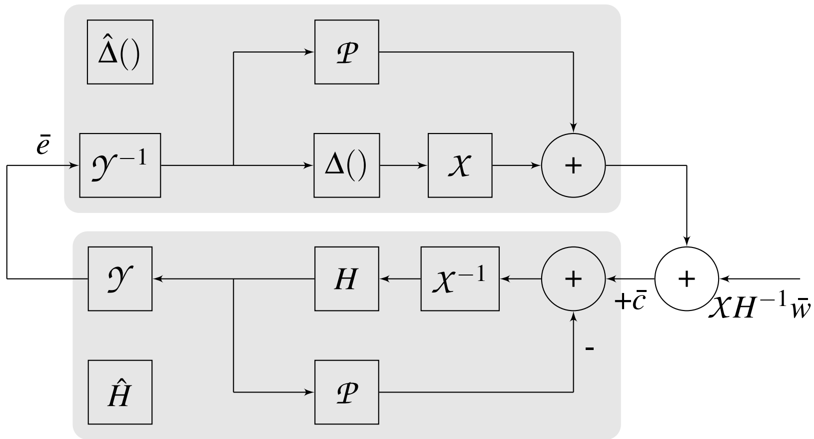


Figure A.5: Equivalent closed loop

$$\begin{aligned}&||Y_1(u_1)_T||^2_{\mathcal{L}_2} - ||P_1(u_1)_T||^2_{\mathcal{L}_2} - ||X_1(\Delta_1 u_1)_T||^2_{\mathcal{L}_2} - 2\,\langle X_1(\Delta_1 u_1)_T, P_1(u_1)_T\rangle_{\mathcal{L}_2}\\ &+\lambda||Y_2(u_2)_T||^2_{\mathcal{L}_2} - \lambda||P_2(u_2)_T||^2_{\mathcal{L}_2} - \lambda||X_2(\Delta_2 u_2)_T||^2_{\mathcal{L}_2} - 2\lambda\,\langle X_2(\Delta_2 u_2)_T, -P_2(u_2)_T\rangle_{\mathcal{L}_2}\\ &\geq 0 \quad \forall (u_1)_T \in \mathcal{L}_2^l[0,\infty), (u_2)_T \in \mathcal{L}_2^m[0,\infty), T \geq 0.\end{aligned}$$

now, taking the feedback interconnection:

$$\begin{aligned}u_1 &= w_1 + \Delta_2(u_2)\\ u_2 &= w_2 + \Delta_1(u_1)\end{aligned}$$

given the well posedness of the system, then, the truncated signals exists:

$$\begin{aligned}(u_1)_T &= (w_1)_T + (\Delta_2 u_2)_T\\ (u_2)_T &= (w_2) + (\Delta_1 u_1)_T\end{aligned}$$

In the other hand, because of causality, the inner product can be determined for all truncated signals. Therefore, making the substitution of $(\Delta_1 u_{1T})_T$ and $(\Delta_2 u_{2T})_T$

$$
\begin{aligned}
0 \leq ||Y_1(u_1)_T||^2_{\mathcal{L}_2} - ||P_1(u_1)_T||^2_{\mathcal{L}_2} - \lambda ||X_2(u_1)_T||^2_{\mathcal{L}_2} \\
+ \lambda ||Y_2(u_2)_T||^2_{\mathcal{L}_2} - \lambda ||P_2(u_2)_T||^2_{\mathcal{L}_2} - ||X_1(u_2)_T||^2_{\mathcal{L}_2} \\
- 2\langle P_1(u_1)_T, X_1(u_2)_T\rangle_{\mathcal{L}_2} - 2\lambda \langle X_2(u_1)_T, P_2(u_2)_T\rangle_{\mathcal{L}_2} \\
+ 2\langle P_1(u_1)_T, X_1(w_2)_T\rangle_{\mathcal{L}_2} + 2\lambda \langle X_2(u_1)_T, X_2(w_1)_T\rangle_{\mathcal{L}_2} \\
+ 2\langle X_1(u_2)_T, X_1(w_2)_T\rangle_{\mathcal{L}_2} + 2\lambda \langle P_2(u_2)_T, X_2(w_1)_T\rangle_{\mathcal{L}_2} \\
- \lambda \langle X_2(w_1)_T, X_2(w_1)_T\rangle_{\mathcal{L}_2} - \langle X_1(w_2)_T, X_1(w_2)_T\rangle_{\mathcal{L}_2} \\
\forall (u_1)_T \in \mathcal{L}_2^l[0,\infty), (u_2)_T \in \mathcal{L}_2^m[0,\infty), T \geq 0. \quad \text{(A.4)}
\end{aligned}
$$

now, from equation (2.27), the following inequality can be determined:

$$
\begin{bmatrix} -\tilde{X}_1^{\sim}(j\omega)\tilde{X}_1(j\omega) & -\tilde{X}_1^{\sim}(j\omega)\tilde{P}_1(j\omega) \\ -\tilde{P}_1^{\sim}(j\omega)\tilde{X}_1(j\omega) & \tilde{Y}_1^{\sim}(j\omega)\tilde{Y}_1(j\omega) - \tilde{P}_1^{\sim}(j\omega)\tilde{P}_1(j\omega) \end{bmatrix}
+ \begin{bmatrix} \lambda\tilde{Y}_2^{\sim}(j\omega)\tilde{Y}_2(j\omega) - \lambda\tilde{P}_2^{\sim}(j\omega)\tilde{P}_2(j\omega) & -\lambda\tilde{P}_2^{\sim}(j\omega)\tilde{X}_2(j\omega) \\ -\lambda\tilde{X}_2^{\sim}(j\omega)\tilde{P}_2(j\omega) & -\lambda\tilde{X}_2^{\sim}(j\omega)\tilde{X}_2(j\omega) \end{bmatrix} \leq -\varepsilon I_{l+m} \quad \forall \omega \in \mathbb{R}.
$$

therefore, the following relations hold:

$$
\begin{array}{lcll}
\lambda\tilde{Y}_2^{\sim}(j\omega)\tilde{Y}_2(j\omega) - \lambda\tilde{P}_2^{\sim}(j\omega)\tilde{P}_2(j\omega) - \tilde{X}_1^{\sim}(j\omega)\tilde{X}_1(j\omega) & \leq & -\varepsilon I_{l+m} & \forall \omega \in \mathbb{R} \\
\tilde{Y}_1^{\sim}(j\omega)\tilde{Y}_1(j\omega) - \tilde{P}_1^{\sim}(j\omega)\tilde{P}_1(j\omega) - \lambda\tilde{X}_2^{\sim}(j\omega)\tilde{X}_2(j\omega) & \leq & -\varepsilon I_{l+m} & \forall \omega \in \mathbb{R}
\end{array} \quad \text{(A.5)}
$$

Note that the integrals from equation (A.4) can be determined for infinite times with truncated inputs, then, using equation (A.5) the inner products are rewritten as follows, using equation (A.5) and assuming $-\lambda X_2^{\sim} P_2 = P_1^{\sim} X_1$

$$
\begin{aligned}
0 \leq ||Y_1(u_1)_T||^2_{\mathcal{L}_2} &- ||P_1(u_1)_T||^2_{\mathcal{L}_2} - \lambda ||X_2(u_1)_T||^2_{\mathcal{L}_2} \\
&+ \lambda ||Y_2(u_2)_T||^2_{\mathcal{L}_2} - \lambda ||P_2(u_2)_T||^2_{\mathcal{L}_2} - ||X_1(u_2)_T||^2_{\mathcal{L}_2} \\
&- 2\langle P_1(u_1)_T, X_1(u_2)_T\rangle_{\mathcal{L}_2} - 2\lambda \langle X_2(u_1)_T, P_2(u_2)_T\rangle_{\mathcal{L}_2} \\
&+ 2\langle P_1(u_1)_T, X_1(w_2)_T\rangle_{\mathcal{L}_2} + 2\lambda \langle X_2(u_1)_T, X_2 w_{1T}\rangle_{\mathcal{L}_2} \\
&+ 2\langle X_1(u_2)_T, X_1(w_2)_T\rangle_{\mathcal{L}_2} + 2\lambda \langle P_2(u_2)_T, X_2 w_{1T}\rangle_{\mathcal{L}_2} \\
&- \lambda ||X_2 w_{1T}||^2_{\mathcal{L}_2} - ||X_1(w_2)_T||^2_{\mathcal{L}_2} \\
&\leq -\varepsilon ||(u_1)_T||^2_{\mathcal{L}_2} - \varepsilon ||(u_2)_T||^2_{\mathcal{L}_2} \\
&+ 2\langle P_1(u_1)_T, X_1(w_2)_T\rangle_{\mathcal{L}_2} + 2\lambda \langle X_2(u_1)_T, X_2 w_{1T}\rangle_{\mathcal{L}_2} \\
&+ 2\langle X_1(u_2)_T, X_1(w_2)_T\rangle_{\mathcal{L}_2} + 2\lambda \langle P_2(u_2)_T, X_2 w_{1T}\rangle_{\mathcal{L}_2} \\
&- \lambda ||X_2 w_{1T}||^2_{\mathcal{L}_2} - ||X_1(w_2)_T||^2_{\mathcal{L}_2} \\
&\qquad \forall (u_1)_T \in \mathcal{L}_2^l[0,\infty), (u_2)_T \in \mathcal{L}_2^m[0,\infty), T \geq 0.
\end{aligned}
$$

Then:

$$
\begin{aligned}
\varepsilon ||(u_1)_T||^2_{\mathcal{L}_2} + \varepsilon ||(u_2)_T||^2_{\mathcal{L}_2} &\leq -\lambda ||X_2(w_1)_T||^2_{\mathcal{L}_2} - ||X_1(w_2)_T||^2_{\mathcal{L}_2} \\
&+ 2\langle P_1(u_1)_T, X_1(w_2)_T\rangle_{\mathcal{L}_2} + 2\lambda \langle X_2(u_1)_T, X_2(w_1)_T\rangle_{\mathcal{L}_2} \\
&+ 2\langle X_1(u_2)_T, X_1(w_2)_T\rangle_{\mathcal{L}_2} + 2\lambda \langle P_2(u_2)_T, X_2(w_1)_T\rangle_{\mathcal{L}_2} \\
&\qquad \forall (u_1)_T \in \mathcal{L}_2^l[0,\infty), (u_2)_T \in \mathcal{L}_2^m[0,\infty), T \geq 0.
\end{aligned}
$$

which is less or equal to the absolute values of the inner products

$$
\begin{aligned}
\varepsilon ||(u_1)_T||^2_{\mathcal{L}_2} + \varepsilon ||(u_2)_T||^2_{\mathcal{L}_2} &\leq +\lambda ||X_2||^2_\infty ||(w_1)_T||^2_{\mathcal{L}_2} + ||X_1||^2_\infty ||(w_2)_T||^2_{\mathcal{L}_2} \\
&+ 2||P_1^{\sim} X_1||_\infty ||(u_1)_T||_{\mathcal{L}_2} ||(w_2)_T||_{\mathcal{L}_2} + 2\lambda ||X_2||^2_\infty ||(u_1)_T||_{\mathcal{L}_2} ||(w_1)_T||_{\mathcal{L}_2} \\
&+ 2||X_1||^2_\infty ||(u_2)_T||_{\mathcal{L}_2} ||(w_2)_T||_{\mathcal{L}_2} + 2\lambda ||P_2^{\sim} X_2||_\infty ||(u_2)_T||_{\mathcal{L}_2} ||(w_1)_T||_{\mathcal{L}_2} \\
&\qquad \forall (u_1)_T \in \mathcal{L}_2^l[0,\infty), (u_2)_T \in \mathcal{L}_2^m[0,\infty), T \geq 0. \qquad \text{(A.6)}
\end{aligned}
$$

Using Cauchy-Schwartz and sub multiplicative inequalities. Now, eliminate $(u_1)_T$ from equation (A.6) using $(u_1)_T = (w_1)_T - (\Delta_2 u_2)_T$ and its norm: $||(u_1)_T||_{\mathcal{L}_2} = ||(w_1)_T - (\Delta_2 u_2)_T||_{\mathcal{L}_2} \leq ||(w_1)_T||_{\mathcal{L}_2} + ||\Delta_2||_{\mathcal{L}_2 \to \mathcal{L}_2} ||(u_2)_T||_{\mathcal{L}_2}$

$$||(u_2)_T||_{\mathcal{L}_2}^2 \leq \frac{3\lambda||X_2||_\infty^2||(w_1)_T||_{\mathcal{L}_2}^2 + ||X_1||_\infty^2||(w_2)_T||_{\mathcal{L}_2}^2 + 2||P_1^{\sim}X_1||_\infty||(w_1)_T||_{\mathcal{L}_2}||(w_2)_T||_{\mathcal{L}_2}}{\varepsilon}$$
$$+2\frac{(||X_1||_\infty^2 + ||P_1^{\sim}X_1||_\infty||\Delta_2||_{\mathcal{L}_2\to\mathcal{L}_2})||(w_2)_T||_{\mathcal{L}_2}}{\varepsilon}||(u_2)_T||_{\mathcal{L}_2}$$
$$+2\frac{\lambda(||X_2||_\infty^2||\Delta_2||_{\mathcal{L}_2\to\mathcal{L}_2} + ||P_2^{\sim}X_2||_\infty)||(w_1)_T||_{\mathcal{L}_2}}{\varepsilon}||(u_2)_T||_{\mathcal{L}_2}$$
$$\forall (u_2)_T \in \mathcal{L}_2^m[0,\infty), T \geq 0.$$

then:

$$||(u_2)_T||_{\mathcal{L}_2}^2 \leq \bar{c}(T) + 2\bar{b}(T)||(u_2)_T||_{\mathcal{L}_2} \quad \forall (u_2)_T \in \mathcal{L}_2^m[0,\infty), T \geq 0.$$

where $\bar{b}(T)$ and $\bar{c}(T)$ tend to finite values $\bar{b}$ and $\bar{c}$, respectively, as $T \to \infty$, since $w_1, w_2 \in \mathcal{L}_2[0,\infty)$.

therefore

$$||(u_2)_T||_{\mathcal{L}_2\to\mathcal{L}_2} \leq \bar{b}(T) + \sqrt{\bar{b}(T)^2 + \bar{c}(T)}$$

$\forall T \in [0,\infty)$, tending to a constant value as $T \to \infty$. Therefore $u_2 \in \mathcal{L}_2[0,\infty)$.

Likewise, eliminate $(u_2)_T$ from equation (A.6) using $(u_2)_T = (w_2)_T + (\Delta_1 u_1)_T$ and its norm: $||(u_2)_T||_{\mathcal{L}_2} = ||(w_2)_T + (\Delta_1 u_1)_T||_{\mathcal{L}_2} \leq ||(w_2)_T||_{\mathcal{L}_2} + ||\Delta_1||_{\mathcal{L}_2\to\mathcal{L}_2}||(u_1)_T||_{\mathcal{L}_2}$

$$||(u_1)_T||_{\mathcal{L}_2}^2 \leq \frac{\lambda||X_2||_\infty^2||(w_1)_T||_{\mathcal{L}_2}^2 + 3||X_1||_\infty^2||(w_2)_T||_{\mathcal{L}_2}^2 + 2\lambda||P_2^{\sim}X_2||_\infty||(w_2)_T||_{\mathcal{L}_2}||(w_1)_T||_{\mathcal{L}_2}}{\varepsilon}$$
$$+2\frac{(||P_1^{\sim}X_1||_\infty + ||X_1||_\infty^2||\Delta_1||_{\mathcal{L}_2\to\mathcal{L}_2})||(w_2)_T||_{\mathcal{L}_2}}{\varepsilon}||(u_1)_T||_{\mathcal{L}_2}$$
$$+2\frac{\lambda(||X_2||_\infty^2 + ||P_2^{\sim}X_2||_\infty||\Delta_1||_{\mathcal{L}_2\to\mathcal{L}_2})||(w_1)_T||_{\mathcal{L}_2}}{\varepsilon}||(u_1)_T||_{\mathcal{L}_2}$$
$$\forall (u_1)_T \in \mathcal{L}_2^l[0,\infty), T \geq 0.$$

then

$$||(u_1)_T||_{\mathcal{L}_2}^2 \leq \tilde{c}(T) + \tilde{b}(T)||(u_1)_T||_{\mathcal{L}_2} \quad \forall (u_1)_T \in \mathcal{L}_2^l[0,\infty), T \geq 0.$$

where $\tilde{b}(T)$ and $\tilde{c}(T)$ tend to finite values $\tilde{b}$ and $\tilde{c}$, respectively, as $T \to \infty$, since $w_1, w_2 \in \mathcal{L}_2[0,\infty)$.

therefore

$$||(u_1)_T||_{\mathcal{L}_2} \leq \tilde{b}(T) + \sqrt{\tilde{b}(T)^2 + \tilde{c}(T)}$$

$\forall T \in [0,\infty)$, tending to a constant value as $T \to \infty$. Therefore $u_1 \in \mathcal{L}_2[0,\infty)$. □

A.3 Proof to Corollary 2.3.9

Proof. Using equation (2.33), the following inner product can be determined:

$$< U_1, \hat{M}(j\omega)V_1 >_{\mathcal{H}_2} = < U_1, \hat{M}_-(j\omega)\hat{M}_+(j\omega)V_1 >_{\mathcal{H}_2}$$
$$= < \hat{M}_-^{\sim}(j\omega)U_1, \hat{M}_+(j\omega)V_1 >_{\mathcal{H}_2} \geq \frac{\delta}{2} < U_1, U_1 >_{\mathcal{H}_2}$$
$$U_1 \in \mathcal{H}_2, V_1 = \mathcal{F}\{\Delta_1(\mathcal{F}^{-1}\{U_1\})\}$$

Now, make the following substitution $W_1 = \hat{M}_-^{\sim}(j\omega)U_1 \iff \hat{M}_-^{-\sim}(j\omega)W_1 = U_1$ and $F_1 = \hat{M}_+(j\omega)V_1 \iff \hat{M}_+^{-1}(j\omega)F_1 = V_1$

$$< W_1, F_1 >_{\mathcal{H}_2} \geq \frac{\delta}{2} < \hat{M}_-^{-\sim}(j\omega)W_1, \hat{M}_-^{-\sim}(j\omega)W_1 >_{\mathcal{H}_2}$$
$$\forall \quad W_1 \in \mathcal{H}_2, F_1 = \mathcal{F}\{M_+\Delta_1 M_-^{-\sim}(\mathcal{F}^{-1}\{W_1\})\}$$

Now, given that $\hat{M}_-^{\sim}(j\omega)$ and $\hat{M}_-^{-\sim}(j\omega)$ are bounded, it is evident that: $||\hat{M}_-^{\sim}U||^2_{\mathcal{H}_2} \leq ||\hat{M}_-^{\sim}||^2_\infty ||U||^2_{\mathcal{H}_2}$. Replacing back $U = \hat{M}_-^{-\sim}(j\omega)V$, $||\hat{M}_-^{-\sim}V||^2_{\mathcal{H}_2} \geq \frac{1}{||\hat{M}_-^{\sim}||^2_\infty}||V||^2_{\mathcal{H}_2}$.

$$< W_1, F_1 >_{\mathcal{H}_2} \geq \frac{\delta}{2} < \hat{M}_-^{-\sim}(j\omega)W_1, \hat{M}_-^{-\sim}(j\omega)W_1 >_{\mathcal{H}_2} \geq \frac{\delta}{2||\hat{M}_-^{\sim}||^2_\infty} < W_1, W_1 >_{\mathcal{H}_2}$$
$$\forall \quad W_1 \in \mathcal{H}_2, F_1 = \mathcal{F}\{M_+\Delta_1 M_-^{-\sim}(\mathcal{F}^{-1}\{W_1\})\}$$

$$2 < W_1, F_1 >_{\mathcal{H}_2} - \frac{\delta}{||\hat{M}_-^{\sim}||^2_\infty} < W_1, W_1 >_{\mathcal{H}_2} \geq 0$$
$$\forall \quad W_1 \in \mathcal{H}_2, F_1 = \mathcal{F}\{M_+\Delta_1 M_-^{-\sim}(\mathcal{F}^{-1}\{W_1\})\} \quad \text{(A.7)}$$

For the Condition (2.34), the positivity of the following inner product holds:

$$< U_2, V_2 >_{\mathcal{H}_2} \geq 0 \quad \forall U_2 \in \mathcal{H}_2, V_2 = \mathcal{F}\{\Delta_2(M_+^{-1}\mathcal{F}^{-1}\{M_-^{-1}U_2\})\}$$

Then, make the following substitution: $W_2 = \hat{M}_-^{-1}(j\omega)U_2 \iff \hat{M}_-(j\omega)W_2 = U_2$

$$< \hat{M}_-(j\omega)W_2, V_2 >_{\mathcal{H}_2} = < W_2, \hat{M}_-^{\sim}(j\omega)V_2 >_{\mathcal{H}_2} \geq 0$$
$$\forall W_2 \in \mathcal{H}_2, V_2 = \mathcal{F}\{\Delta_2(M_+^{-1}\mathcal{F}^{-1}\{W_2\})\}$$

The stability test is now for the feedback interconnection $[M_+\Delta_1 M_-^{-\sim}, -M_-^{\sim}\Delta_2 M_+^{-1}]$. The feedback interconnection $[M_+\Delta_1 M_-^{-\sim}, -M_-^{\sim}\Delta_2 M_+^{-1}]$ is well posed because the feedback interconnection $[\Delta_1, -\Delta_2]$ is well posed. This follows from the equivalence of the feedback interconnections presented in Figure A.6.

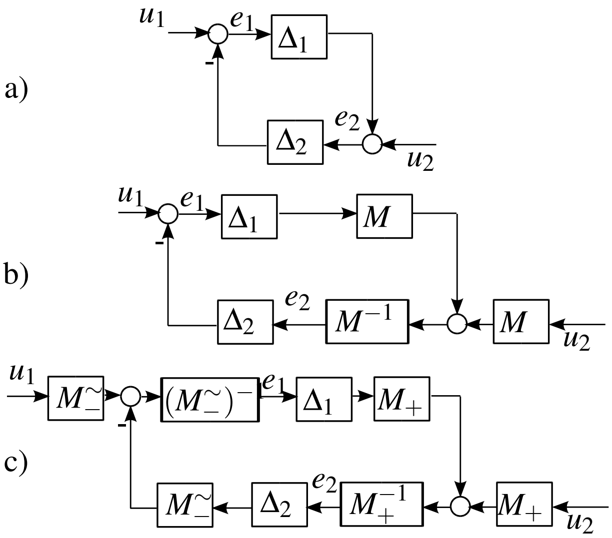


Figure A.6: Closed loop equivalent systems

In order to show stability, the following cases are taken into consideration:

- Case 1: $1 - ||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2} ||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2} > 0$ (Small Gain Theorem). Choose

$$\lambda = \frac{1 + ||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}}{1 + ||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2}},$$

$$\varepsilon = \frac{1 - ||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2} ||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2}}{1 + ||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2}},$$

$$\Pi_1 = \begin{bmatrix} -I & 0 \\ 0 & ||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2} I \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} ||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2} I & 0 \\ 0 & -I \end{bmatrix}.$$

- Case 2: $1 - ||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2} ||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2} < 0$.

  Choose

$$\lambda = 1,$$

$$\varepsilon = \frac{\frac{\delta}{||\hat{M}_-^{\sim}||_\infty^2}}{(||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}+1)(||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2}+1)},$$

$$\Pi_1 = \begin{bmatrix} 0 & I \\ I & -\frac{\delta}{||\hat{M}_-^{\sim}||_\infty^2} I \end{bmatrix} + \frac{\delta}{||\hat{M}_-^{\sim}||_\infty^2}\begin{bmatrix} -\frac{1}{||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}} I & 0 \\ 0 & I \end{bmatrix}$$
$$+ \frac{\delta \frac{1}{||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}+1}}{||\hat{M}_-^{\sim}||_\infty^2(||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2}-1)}\begin{bmatrix} -\frac{1}{||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Pi_2 = \frac{\frac{\frac{\delta}{||\hat{M}_-^{\sim}||_\infty^2}(||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2})^2}{||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2}+1}}{||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2}-1}\begin{bmatrix} I & 0 \\ 0 & -\frac{1}{||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2}} I \end{bmatrix} + \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$$

- Case 3: $1 = ||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2}$. Choose

$$\lambda = 1,$$

$$\varepsilon = \frac{\frac{\delta}{||\hat{M}_-^{\sim}||_\infty^2}}{||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}+2},$$

$$\Pi_1 = 2\frac{\delta}{||\hat{M}_-^{\sim}||_\infty^2}\begin{bmatrix} -\frac{1}{||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & I \\ I & -\frac{\delta}{||\hat{M}_-^{\sim}||_\infty^2} I \end{bmatrix}$$

$$\Pi_2 = \frac{\frac{\delta}{||\hat{M}_-^{\sim}||_\infty^2}(||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}+4)}{||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}(||M_+\Delta_1 M_-^{-\sim}||^2_{\mathcal{L}_2\to\mathcal{L}_2}+2)}\begin{bmatrix} I & 0 \\ 0 & -\frac{1}{||M_-^{\sim}\Delta_2 M_+^{-1}||^2_{\mathcal{L}_2\to\mathcal{L}_2}} I \end{bmatrix} + \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$$

Given that in all cases Π_1 and Π_2 are constant matrices, equation (2.32) holds. Then, Δ_1, Δ_2 satisfy equations (2.25) and (2.26), parametrized by the constant strict Positive Negative IQC multipliers Π_1 and Π_2. Meanwhile, $\Pi_1, \Pi_2, \lambda, \varepsilon$ satisfy (2.27). Then the system is $\mathcal{L}_2[0,\infty)$ stable via Lemma 2.3.8. □

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