ODD POISSON SUPERMANIFOLDS, COURANT ALGEBROIDS, HOMOTOPY STRUCTURES, AND DIFFERENTIAL OPERATORS

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Science and Engineering

2017

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Contents

Word count 32672

The University of Manchester

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Doctor of Philosophy
Odd Poisson Supermanifolds, Courant Algebroids, Homotopy Structures, and Differential Operators
June 19, 2017

In this thesis we investigate the role of odd Poisson brackets in related areas of supergeometry. In particular we study three different cases of their appearance: Courant algebroids and their homotopy analogues, weak Poisson structures and their relation to foliated manifolds, and the structure of odd Poisson manifolds and their modular class.

In chapter 2 we introduce the notion of a homotopy Courant algebroid, a subclass of which is suggested to stand as the double objects to L_∞-bialgebroids. We provide explicit formula for the higher homotopy Dorfman brackets introduced in this case, and the higher relations between these and the anchor maps. The homotopy Loday structure is investigated, and we begin a discussion of what other constructions in the theory of Courant algebroids can be carried out in this homotopy setting.

Chapter 3 is devoted to lifting a weak Poisson structure corresponding to a local foliation of a submanifold to a weak Koszul bracket, and interpreting the results in terms of the cohomology of an associated differential. This bracket is shown to produce a bracket on co-exact differential forms.

In chapter 5 studies classes of second order differential operators acting on semidensities on an arbitrary supermanifold. In particular, when the supermanifold is odd Poisson, we given an explicit description of the modular class of the odd Poisson manifold, and provide the first non-trivial examples of such a class. We also introduce the potential field of a general odd Laplacian, and discuss its relation to the geometry of the odd Poisson manifold and its status as a connection-like object.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

Acknowledgements

First and foremost I would like to thank my supervisors, Theodore Voronov and Hovhannes Khudaverdian, for the time they have given to me over the past years. They have taught me many things both inside and outside of mathematics, and have shown me what is fundamental when learning mathematics. To them I am most grateful.

It is also my pleasure to thank Kirill Mackenzie who provided many useful suggestions to improve parts of this text, and Simon Lyakhovich and Alexey Sharapov for their invitation to Tomsk State University where part of this work was done.

Also to my friends, especially those in Manchester, my officemates, and my housemates, who have made these past years some of the most enjoyable and stimulating. Always there for distraction, and without whom this most certainly would have been finished earlier.

Finally I would like to thank Emily and my family for their constant support, not only through the Ph.D. years, but in everything else also.

Introduction

Odd Poisson brackets first came into prominence with the publication of the seminal paper [BV81], in which the authors first introduced the *anti-bracket* together with its defining properties. The anti-bracket is in fact, the canonical odd Poisson bracket arising from the structure of an odd symplectic supermanifold, which had been known for several decades under a different guise. The Schouten bracket, first introduced in the work [Sch40], extends the even commutator of vector fields to the algebra of multivector fields, and only the observation that this algebra is isomorphic to the algebra of functions of an associated odd symplectic supermanifold is required in order to identify the Schouten bracket with the anti-bracket. The Schouten bracket however predates the language of supermathematics, and so was not recognised as an odd Poisson bracket until the first real studies of these structures began.

It should be noted however that the concept of odd brackets was already well established prior to the anti-bracket. Nijenhuis, who proved the main properties of the Schouten bracket [Nij55], together with Richardson introduced a pair $(G, \mathfrak{g})$ consisting of a Lie group G and what is now a Lie superalgebra $\mathfrak{g}$, whose even component stands as the Lie algebra of G [NR64]. The pair $(G, \mathfrak{g})$, known as a Harish-Chandra pair in representation theory, acts closely to that of a Lie supergroup. Odd brackets also appeared in the work of Gerstenhaber [Ger63] who introduced the structure of a graded Lie ring on the cohomology of an associative ring. The motivation for the introduction of this odd graded bracket lies in deformation theory, since such a graded cohomology ring equipped with this odd Gerstenhaber bracket naturally controls the associative deformations of the base ring. Despite the prior existence of these odd brackets, it was the anti-bracket's role in physics that served as the catalyst for the collective study of these objects, uniting these isolated constructions under a single theory, and giving rise to odd Poisson geometry.

At a first glance odd Poisson geometry seems deceptively similar to its even cousin, and one might initially presume that it is nothing but a trivial extension into the super category. For instance, both odd and even Poisson geometries admit Darboux type splitting theorems inducing symplectic foliations of the manifold, and the algebra of symmetries preserving each structure through canonical transformations is infinite-dimensional. However scratching at the surface begins to reveal the rather different behaviour that odd Poisson geometry actually exhibits. An odd symplectic manifold for example, admits no volume form which is preserved under canonical transformations, contrary to the even symplectic case which carries the distinguished Liouville form. As a result of this, a volume form on an odd symplectic manifold will always contain additional information to the odd symplectic structure, which is not the case for any even tensor geometry which always grants a distinguished volume.

The root of many of these peculiarities of odd Poisson geometry lies in the fact that the defining tensor field is symmetric, as opposed to the anti-symmetric even tensor field we are accustomed to in even Poisson geometry. This symmetry accounts for the unusual qualities that odd Poisson geometry actually shares with even Riemannian geometry! The most notable of these is that an odd Poisson supermanifold defines a distinguished class of odd Laplace-type operators, analogous to the Laplace-Beltrami operator of a Riemannian manifold. This parallel with the Riemannian case runs deeper still, where both second order operators are known to have intimate relations with the scalar curvature of the manifold when equipped with a compatible connection (see [BB09]).

This feature of odd Poisson geometry was exploited by the authors of [BV81] in order to develop a canonical quantisation procedure for an arbitrary Lagrangian gauge field theory. Their classical theory is developed over an odd symplectic supermanifold, the space of fields and anti-fields, where the classical master action $\mathcal{S}$ is required to satisfy the classical master equation

$$\{\mathcal{S}, \mathcal{S}\} = 0.$$

The bracket appearing in the classical master equation is the non-degenerate odd Poisson bracket arising from the odd symplectic structure. The structure of the odd symplectic supermanifold was not solely sufficient to develop a quantum formalism however, where an additional piece of information was required. An independent choice of volume form $\boldsymbol{\rho}$ must be made in order to introduce the volume-dependent quantum effective action $\mathcal{W}_{\boldsymbol{\rho}}$, together with an odd second order operator $\Delta_{\boldsymbol{\rho}}$ known as the BV-operator. These can then be used to formulate the quantum master equation

$$\Delta_{\boldsymbol{\rho}} \exp\left(\frac{i}{\hbar}\mathcal{W}_{\boldsymbol{\rho}}\right) = 0,$$

which the quantum action must satisfy to guarantee gauge invariance. This artificially introduced volume form must be chosen under certain restrictions, and is not unique in the sense that any volume satisfying these conditions may be chosen. The question then arises as to whether the BV-formalism can be developed independently of this artificial choice of volume.

The answer to this question is positive, and begins with Khudaverdian who provided an invariant definition of the BV-operator [Khu91] which was first defined in local Darboux coordinates. Khudaverdian describes the BV-operator in terms of the divergence of the associated Hamiltonian vector field with respect to the volume form:

$$\Delta_{\boldsymbol{\rho}} f = \mathrm{div}_{\boldsymbol{\rho}} \mathrm{grad} f,$$

for a smooth function f. Such an interpretation identifies the BV-operator as an odd Laplace type operator. At this point we should also mention the work of Kosmann-Schwarzbach and Monterde [KSM02], who independently investigated such divergence operators with regards to their properties as generating operators of odd Poisson brackets.

It was through further investigation of these operators that Khudaverdian initiated the important advancement to cast the BV-formalism in a new light. In the work [Khu04], Khudaverdian defined an odd Laplacian Δ_0 on an odd symplectic supermanifold, acting not on functions, but on semidensities. What is crucial about Khudaverdian's operator is that it requires no additional information to define, and arises as a consequence of the geometry of the odd symplectic manifold. It was further shown that with a choice of suitable volume form, Khudaverdian's canonical odd operator can be identified with the BV-operator on functions.

Around the same time as Khudaverdian's work, the article of Schwarz [Sch93] showed that the BV-formalism naturally leads to the consideration of integrals of $\exp\left(\frac{i}{\hbar}\mathcal{W}_{\boldsymbol{\rho}}\right)$ over Lagrangian submanifolds of the odd symplectic superspace, the integrands of which correspond precisely to semidensities on the entire odd symplectic space. With this observation and the introduction of Khudaverdian's operator, the BV-formalism could be rephrased in the language of densities, essentially bypassing the restrictions on the artificially introduced volume form. The importance of this canonical odd Laplacian provoked much further study, in particular, the operator was shown by Severa to arise naturally from the spectral sequence of the canonical associated bicomplex of the odd symplectic supermanifold [Sev06].

The use of densities of fixed weight is well-suited to the study of differential operators, having been particularly well exploited in low dimensional cases, see [OT05] and references therein for instance. The idea of Khudaverdian and Voronov to utilise the entire algebra of densities however provided much new insight into the geometry of these odd second order operators. See the works [KV12, KV02, KV04]. The approach via the whole algebra yielded interesting results entwining geometric constructions on the algebra of densities with the geometry of the base manifold, culminating with the definition of the KV-master groupoid determining how strongly a certain class of these operators depends on an associated connection [KV12].

As well as their contribution in physical theories, odd Poisson structures are also present in many areas of mathematics; the aforementioned Gerstenhaber bracket in deformation theory for example. They play a particularly useful role in the theory of Lie algebroids, especially when considering the Lie bialgebroids introduced by Mackenzie and Xu in [MX94] as linearisations of Poisson groupoids. Classically, Lie bialgebroids were described as a pair of Lie algebroids (A, A^*) satisfying the compatibility condition that the differential of one must act as a derivation over the bracket of the other. The reformulation by Kosmann-Schwarzbach places this into a more general setting [KS95]; since the differential of A acts on the space of sections $\Gamma(\wedge A^*)$, the definition should be modified slightly since the bracket of A^* acts only on $\Gamma(A^*)$. Kosmann-Schwarzbach showed that a Lie algebroid structure on A^* is equivalent to an odd Poisson bracket in the space $\Gamma(\wedge A^*)$, equivalent to Vaintrob's manifestation of a Lie algebroid A^* as an odd Poisson bracket in the algebra $C^\infty(\Pi A)$ under the natural identification [Vai97]. This improvement on the definition also provided a conceptually superior proof to that given by Mackenzie and Xu for the symmetry of a Lie bialgebroid, which was not immediately obvious.

With the introduction of Lie bialgebroids followed the question as to what should stand as the corresponding double object, analogous to the double introduced by Drinfeld in [Dri83] for Lie bialgebras. For a Lie bialgebra, Drinfeld's double construction produces a unique quasi-triangular Lie algebra suitable for quantisation, which in turn provides a valuable source of quantum groups. This same question for Lie bialgebroids turned out to be highly non-trivial, with complications arising which were not present in the Lie algebra setting. To answer this a variety of solutions have been proposed, the first of which was suggested by Liu, Weinstein and Xu [LWX97], who introduced the notion of a Courant algebroid to act as the analogue of the Drinfeld double.

For a Lie bialgebroid (A, A^*), the associated Courant algebroid is the direct sum $A \oplus A^*$ equipped with an anchor map, together with a skew-symmetric operation on sections which satisfies a Jacobi identity up to some measurable defect. This skew-symmetric operation is called the Courant bracket, originating from Courant who first introduced this as an integrability condition in his work [Cou90]. However, since this Courant bracket in not a Lie bracket, a Courant algebroid cannot be a Lie algebroid unlike the Drinfeld double of a Lie algebra which is again a Lie algebra. Such a discrepancy is unsatisfying, and shortly after the introduction of Courant algebroids two further solutions were offered.

The second utilised the super-language and was suggested by Roytenberg in his Ph.D. thesis [Roy99]. Roytenberg showed that a Lie bialgebroid (A, A^*) can be compactly described in terms of two commuting homological vector fields on the cotangent bundle $T^*\Pi A$. The sum of these vector fields was then defined to be the double, where the nilpotency of the odd vector field is equivalent to the Lie bialgebroid structure. Independently Mackenzie proposed a classical solution [Mac98, Mac11], defining the double of a Lie bialgebroid as the double Lie algebroid T^*A; a double vector bundle equipped with compatible Lie algebroid structures along each edge of the double bundle. A desirable property of these double Lie algebroids is that they can be thought as Lie algebroids in the category of Lie algebroids, so fulfilling the double role in Ehresmann's sense. Both Roytenberg's and Mackenzie's constructions reproduced Drinfeld's double when the Lie bialgebroid reduces to a Lie bialgebra, and further, the Courant algebroid picture sits in both these constructions due to the natural projection maps $T^*\Pi A \to \Pi(A \oplus A^*)$ and $T^*A \to A \oplus A^*$. In fact these two seemingly independent approaches are equivalent, as was shown by Voronov in [Vor12]. Paralleling the Q-manifold manifestation of a Lie algebroid [Vai97], Voronov proved that the double Lie algebroid structure of T^*A is equivalent to a pair of commuting homological vector fields on the space of "double reversed parity" $\Pi^2 T^*\Pi A \cong T^*\Pi A$. This pair of commuting vector fields coincides precisely with the two defined by Roytenberg, thus completely aligning the two constructions.

The detailed technicalities of the double Lie algebroid definition make the super language very adequate to describe such double structures. This is the case for other concepts, for example, the strongly homotopy Lie algebras of Lada and Stasheff [LS93]. A homotopy Lie algebra consists of a sequence of bracket operations of alternating parity, which satisfy complicated higher Jacobi identity type expressions. The super language encodes this entire structure into a single homological vector field on the vector space of reversed parity, providing a simple and compact description. With the introduction of homotopy Lie algebras, it was not long before this was advanced to Lie algebroids and Lie bialgebroids homotopy analogues. In particular, an L_∞-bialgebroid (A, A^*) consists of a pair of L_∞-algebroids in duality together with an extensive list of compatibility conditions. The classical definition is extremely bulky, yet the super description of Roytenberg for a Lie bialgebroid admits a straightforward generalisation as a pair of commuting (formal) homological vector fields on the cotangent bundle $T^*\Pi A$. These homotopy Lie bialgebroids should also admit suitable double objects, which will be the beginning of the main work of this thesis.

Content Outline

The thesis is organised in the following way:

Chapter 1 The first chapter introduces the essential background material and notation for the remainder of the thesis. In particular, it deals with Lie algebroids and their equivalent manifestations, together with the homotopy analogues of these structures. The chapter ends with a first result regarding the induced P_∞-structure from an L_∞-bialgebroid.

Chapter 2 The main focus of this chapter is to describe what geometrical object should stand as the double of an L_∞-bialgebroid. Double objects are introduced and their role as the double discussed, together with necessary conditions for their existence. This leads to the introduction of homotopy Courant algebroids, of which these double objects are examples. The structure of these homotopy Courant algebroids is discussed and examples provided. We conclude this chapter with a short discussion of what classical constructions appear in this new homotopy setting.

Chapter 3 Chapter 3 is devoted to the extension of a geometrical construction called a weak Poisson system. Such a system is introduced to describe arbitrary gauge field theories, which we develop to include covariant tensor fields. We show that all the information is encoded within a single homological vector field via its cohomology, and that the obtained weak Koszul bracket extends the even Poisson bracket introduced on co-exact differential forms for a generalised Hamiltonian mechanics.

Chapter 4 This is included primarily for interest and contains no new material. We discuss the symmetry of tensors in the super setting and the fundamental differences between even and odd tensor geometries on a supermanifold, a description of which is provided in terms of the Cartan prolongation of the associated Lie algebras of symmetries.

Chapter 5 The final chapter is devoted to the study of Laplace type operators acting on semidensities. We introduce connection-like objects associated to classes of these operators, which appear to possess a particularly deep connection to the geometry of the manifold when equipped with an odd Poisson structure. We also show for odd Poisson manifolds that the modular class can be described via these operators, and using this description provide the first examples of odd Poisson manifolds with a non-trivial modular class.

Notational Remarks

The language of supermanifolds will be consistently used throughout this work, and as a consequence we will always drop the prefix "super" when it is not necessary to distinguish from the "usual" non-super case. For example, a vector space will always be understood to be a super vector space, or when we specify a manifold we will always mean a supermanifold. A key feature of the super category is that all objects and morphisms carry a $\mathbb{Z}_2$-grading called parity. Homogeneous objects will be called even if they have parity 0, or odd if they carry parity 1. The term odd Poisson bracket refers then to a Poisson type bracket with Grassmann parity 1. We will denote the parity of an object a by $\tilde{a}$ when we wish to be explicit, though in working the parity will simply appear as the object itself to reduce notation. We also make the assumption that all the signs appear as a result of the Grassmann grading, and not as a consequence of any other gradings should they be present. For example, signs usually arising from the degree of a differential form will not contribute to formula in the usual sense. This will assist in keeping any additional gradings independent from the parity of the objects which is more natural in the super setting.

Since this work deals mostly with multi-linear multi-derivations, or Poisson brackets, there are a variety of different notations which will arise for these. It will be beneficial to fix certain brackets for canonical structures to avoid confusion. These are presented in the table:

Notation	Bracket
$[-,-]$	Commutator of vector fields
$(-,-)$	Canonical even Poisson bracket on T^*M
$[\![-,-]\!]$	Canonical odd Poisson bracket on ΠT^*M
$\{-,-\}$	Arbitrary Odd Poisson bracket
$[-,-]$	Arbitrary Even Poisson/ Loday bracket
$\{-,-\}_S$	Odd bracket derived from Hamiltonian S
$[-,-]_P$	Even bracket derived from Hamiltonian P
$\langle -,-\rangle$	Non-degenerate bilinear form

Accompanying the array of different brackets will be the associated manifolds where they are set. We will always attempt to keep a consistent choice of local coordinates on these manifolds, which again are listed here to avoid repetition:

Manifold	Notation	Local Coordinates
Smooth Manifold	M	x^a
Cotangent Bundle	T^*M	x^a, p_a
Odd Cotangent Bundle	ΠT^*M	x^a, x^*_a
Odd Tangent Bundle	ΠTM	x^a, η^a
Vector Bundle	A, E	x^a, u^i
"Odd" Vector Bundle	ΠA	x^a, ξ^i
Cotangent Double of ΠA	$T^*\Pi A$	x^a, ξ^i, p_a, π_i
Cotangent Double of ΠA^*	$T^*\Pi A^*$	$x^a, \theta_i, p_a, \pi^i$

Chapter 1

Poisson Manifolds and Lie Algebroid Structures

The purpose of this chapter is to introduce the main geometrical objects which feature throughout this work and to fix our notation for such structures. In particular, both even and odd Poisson manifolds are defined via derived brackets [KS04, Vor02], and those derived brackets which are linear in a precise sense are shown to be equivalent to Lie algebroid structures. This equivalence is discussed, and the homotopy analogues deriving from the L_∞-algebras of Lada and Stasheff [LS93] are introduced. We end by providing a first result in parallel with that found in [MX94] regarding the Poisson bracket induced by a Lie bialgebroid.

1.1 Poisson Manifolds

Definition 1.1.1. An *even Poisson algebra* is an associative algebra A equipped with an even bilinear operation $[-,-] : A \times A \to A$, called the even Poisson bracket, which satisfies:

1. $[a,b] = -(-1)^{ab}[b,a]$;
2. $[a,[b,c]] = [[a,b],c] + (-1)^{ab}[b,[a,c]]$;
3. $[a,bc] = [a,b]c + (-1)^{ab}b[a,c]$;

for homogeneous elements $a,b,c \in A$.

This definition corresponds to the usual definition of a Poisson algebra in the non-super setting, though we will call this an even Poisson algebra to distinguish from the odd version which will be of interest to us.

Definition 1.1.2. An *odd Poisson algebra*, or *Schouten algebra*, is an associative algebra A together with an odd bilinear operation $\{-,-\} : A \times A \to A$, called the odd Poisson bracket or the Schouten bracket, which satisfies:

1. $\{a,b\} = (-1)^{ab}\{b,a\}$;
2. $\{a,\{b,c\}\} = (-1)^{a+1}\{\{a,b\},c\} + (-1)^{(a+1)(b+1)}\{b,\{a,c\}\}$;
3. $\{a,bc\} = \{a,b\}c + (-1)^{(a+1)b}b\{a,c\}$;

for homogeneous elements $a, b, c \in A$.

Remark 1.1.1. Our convention will be to sit the parity of the bracket at the front, departing from the usual convention to sit the parity at the comma since this becomes inappropriate when considering higher bracket operations. In particular, our definition of an odd Poisson algebra differs in sign from the more classical definition, for example the Gerstenhaber algebras of [KS96]. The introduction of the sign becomes clear when considering the odd operation ad_a as a derivation of the odd bracket.

Remark 1.1.2. Even and odd Poisson algebras are even and odd Lie algebras where in addition, the bracket is a derivation of the associative multiplication in each argument. There are two equivalent manifestations of an even Lie algebra $\mathfrak{g}$: the algebra may be viewed with an anti-symmetric even bracket on $\mathfrak{g}$, or an odd symmetric bracket on $\Pi\mathfrak{g}$. These equivalent definitions are related by a shift in parity. If however the bracket is required to be Poisson, only the anti-symmetric description of definition 1.1.1 is compatible with the multiplication. Similarly, the symmetric definition 1.1.2 must be used for a Schouten algebra.

A manifold M for which its algebra of smooth functions $C^\infty(M)$ is an even or odd Poisson algebra will be called an even or odd Poisson manifold respectively. We will refer to both even and odd Poisson structures simply as Poisson structures when the parity is understood.

Canonical examples of even and odd Poisson structures arise naturally in physics. The prototypical example of an even Poisson manifold is the cotangent bundle of a smooth manifold; phase space in the Hamiltonian formalism where the bracket arises naturally from the bundle structure. A canonical odd Poisson bracket is obtained from the cotangent bundle's super analogue, the odd cotangent bundle, where the bracket again arises naturally from the structure of the manifold.

Example 1.1.1. Let M be a manifold and T^*M its cotangent bundle. Assign to local coordinates x^a on M, the corresponding momenta coordinates p_a which transform according to the Jacobian matrix J: for a change of coordinates $x = x(x')$,

$$p_a = J_a^{a'} p_{a'}, \qquad J_a^{a'} = \frac{\partial x^{a'}}{\partial x^a}, \qquad \tilde{p}_a = \tilde{x}^a. \tag{1.1}$$

The cotangent bundle carries a canonical even symplectic form, written locally as

$$\omega_0 = dp_a dx^a = d\left(p_a dx^a\right) = (-1)^{a+1} d\left(x^a dp_a\right), \tag{1.2}$$

where x^a, p_a are Darboux coordinates for this symplectic structure. This gives rise to a non-degenerate even Poisson bracket on $C^\infty(T^*M)$ with the local expression

$$(f, g) = (-1)^{a(f+1)} \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial x^a} - (-1)^{af} \frac{\partial f}{\partial x^a} \frac{\partial g}{\partial p_a}, \tag{1.3}$$

for functions $f, g \in C^\infty(T^*M)$.

Example 1.1.2. The odd cotangent bundle ΠT^*M is the cotangent bundle to M with reversed parity in the fibres. For local coordinates x^a on M, introduce the odd conjugate momenta x^*_a in the fibres. These conjugate momenta transform as in eq. (1.1), except now carry the parity $\tilde{x}^*_a = \tilde{x}^a + 1$. There exists a canonical odd symplectic form on ΠT^*M,

$$\omega_1 = dx^*_a dx^a = d\left(x^*_a dx^a\right) = d\left(x^a dx^*_a\right), \tag{1.4}$$

which yields a non-degenerate odd Poisson bracket

$$[\![f, g]\!] = (-1)^{a(f+1)} \frac{\partial f}{\partial x^*_a} \frac{\partial g}{\partial x^a} - (-1)^{(a+1)(f+1)} \frac{\partial f}{\partial x^a} \frac{\partial g}{\partial x^*_a}. \tag{1.5}$$

We will reserve the round brackets $(-, -)$ for the canonical even bracket (1.3), and the square bold brackets $[\![-, -]\!]$ for the canonical odd bracket (1.5).

Remark 1.1.3. The algebra of functions $C^\infty(\Pi T^*M)$ can be naturally identified with the algebra of multivector fields $\mathfrak{X}(M)$ on M via the odd $C^\infty(M)$-linear isomorphism

$$\pi : \mathfrak{X}(M) \to C^\infty(\Pi T^*M), \qquad \partial_a \mapsto x_a^*. \tag{1.6}$$

This takes the Schouten-Nijenhuis bracket of multivector fields [Nij55, Sch40] to the canonical odd Poisson bracket (1.5): for multivector fields $X, Y \in \mathfrak{X}(M)$ and their Schouten-Nijenhuis bracket $[X, Y]_{SN}$,

$$[\![\pi X, \pi Y]\!] = (-1)^X \pi \, [X, Y]_{SN} \, .$$

As a consequence, the canonical odd bracket is often referred to as the Schouten bracket, and we will often not distinguish between the two.

1.1.1 Derived Brackets

The derived bracket construction provides a description of bracket structures in terms of a differential graded Lie algebra together with a suitable Maurer-Cartan element. The construction, first formalised in [KS96] where the main properties were proved, has become a crucial tool in working with bracket operations. In the super-setting the construction was given by Voronov [Vor02], which was further extended to include higher homotopy brackets [Vor05b, Vor05a]. See the article [KS04] for an excellent survey on derived brackets.

In the case of Poisson manifolds the derived bracket method allows any Poisson bracket to be expressed in terms of the canonical even and odd brackets of eqs. 1.3 and 1.5.

Let M be an even Poisson manifold, and let $P \in \mathfrak{X}(M)$ be the even Poisson bivector field defining the bracket. Under the isomorphism (1.6), P is identified with an even fibre-wise quadratic function $P \in C^\infty(\Pi T^*M)$, which defines the Poisson bracket by the nested pair

$$[f, g]_P := [\![[\![P, f]\!], g]\!], \tag{1.7}$$

for functions $f, g \in C^\infty(M)$. The skew-symmetry and bi-derivation properties are inherited from the canonical odd bracket (1.5), whilst the Jacobi identity is equivalent to the condition

$$[\![P, P]\!] = 0. \tag{1.8}$$

In local coordinates, the Poisson Hamiltonian function $P = P(x, x^*)$ is given by

$$P = \frac{1}{2}P^{ab}(x)x_b^* x_a^*, \qquad P^{ab}(x) = (-1)^{(a+1)(b+1)}P^{ba}(x), \tag{1.9}$$

and describes the even Poisson bracket,

$$[f, g]_P = -(-1)^{af}\partial_a f P^{ab}(x)\partial_b g, \qquad [x^a, x^b]_P = -(-1)^a P^{ab}(x), \tag{1.10}$$

for $f, g \in C^\infty(M)$.

Remark 1.1.4. Equation (1.8) is at the heart of many geometrical and algebraic constructions, and in the physics literature bears the name "the master equation". We shall adopt this name, and refer to P as the master Hamiltonian of the bracket $[-, -]_P$.

The case is analogous for an odd Poisson manifold, however the corresponding Hamiltonian function does not correspond to an odd Poisson bivector as might be assumed. An odd fibre-wise quadratic Hamiltonian function $S \in C^\infty(T^*M)$ is required so that the commutation relation $(S, S) = 0$ remains non-trivial under the even bracket (1.3). The odd master function S defines an odd Poisson bracket on M by

$$\{f, g\}_S := ((S, f), g), \tag{1.11}$$

for which the master equation $(S, S) = 0$ is equivalent to the Jacobi identity for the derived odd bracket eq. (1.11). The Hamiltonian $S = S(x, p)$ is locally expressed as

$$S = \frac{1}{2}S^{ab}(x)p_b p_a, \qquad S^{ab}(x) = (-1)^{ab}S^{ba}(x), \tag{1.12}$$

and for functions $f, g \in C^\infty(M)$, the odd bracket is given by

$$\{f, g\}_S = (-1)^{f(a+1)}\partial_a f S^{ab}(x)\partial_b g, \qquad \{x^a, x^b\}_S = S^{ab}(x). \tag{1.13}$$

Notice the difference in signs between the even and odd derived brackets which are required for compatibility with the associative multiplication and linearity over odd constants.

1.2 Lie Algebroids

Lie algebroids are the infinitesimal counterparts to Lie groupoids, and unify the tangent bundle of a manifold and Lie algebras under one theory. These geometric structures admit useful interpretations in the super language as even and odd Poisson brackets on corresponding "neighbouring" manifolds which we will recall. For a classical introduction to the theory of Lie algebroids we point to the book by Mackenzie [Mac05].

Definition 1.2.1. A *Lie algebroid* is a vector bundle $A \to M$ together with a vector bundle map $\rho : A \to TM$ called the anchor, and an $\mathbb{R}$-linear Lie bracket $[-,-] = [-,-]_A$ on the $C^\infty(M)$-module of sections $\Gamma(A)$ satisfying the Leibniz rule,

$$[u, fv] = \rho(u|f)v + (-1)^{uf} f[u, v], \tag{1.14}$$

for sections $u, v \in \Gamma(A)$ and a function $f \in C^\infty(M)$.

The notation $\rho(u|f)$ is used for $\rho(u) \in \mathrm{Vect}(M)$ applied as a derivation to a function $f \in C^\infty(M)$. The anchor is naturally a morphism of Lie algebras $\rho : \Gamma(A) \to \mathrm{Vect}(M)$, taking the Lie bracket of sections to the commutator of vector fields on M,

$$\rho([u, v]_A) = [\rho(u), \rho(v)].$$

Example 1.2.1. The tangent bundle $A = TM$ is a trivial example of a Lie algebroid. The anchor is the identity map and the bracket is the commutator of vector fields.

Example 1.2.2. If $M = \{pt\}$ is a point, then a Lie algebra $\mathfrak{g}$ is a Lie algebroid when considered as a vector bundle over M. The anchor map is trivial whilst the bracket of sections is the Lie bracket of the Lie algebra $\mathfrak{g}$.

Example 1.2.3. Let M be a manifold with an integrable distribution $D \subset TM$. Then $D \to M$ is a Lie algebroid, and specifically is a Lie subalgebroid of the tangent Lie algebroid of example 1.2.1.

Notice that for any Lie algebroid $A \to M$, the image of the anchor $\mathrm{im}(\rho)$ is always an integrable distribution on M.

Example 1.2.4. Let G be a Lie group and M a G-manifold; a manifold with a smooth G-action $G \times M \to M$. The infinitesimal action $\mathfrak{g} \times M \to TM$ is used to define a Lie algebroid $\mathfrak{g} \times M \to M$ called the action Lie algebroid. The anchor is the action of $\mathfrak{g}$ on M, $(X, x) \mapsto X_x \in T_xM$ for $x \in M$, and the bracket of sections is defined by extending the bracket of constant sections of $\mathfrak{g}$ via the Leibniz rule.

Example 1.2.5. Suppose that M is an even Poisson manifold with bracket specified by a Poisson bivector P. Such a bivector generates a map $\Phi_P : T^*M \to TM$, raising indices with the Poisson tensor. This map forms the anchor of the cotangent Lie algebroid T^*M, whose Koszul bracket of 1-forms is

$$[\sigma, \tau]_{K_P} = \mathcal{L}_{\Phi_P(\sigma)}\tau - \mathcal{L}_{\Phi_P(\tau)}\sigma - dP(\sigma, \tau), \tag{1.15}$$

for 1-forms $\sigma, \tau \in \Gamma(T^*M)$, and where $\mathcal{L}_{\Phi_P(\sigma)}\tau$ is the Lie derivative of the form τ over the vector field $\Phi_P(\sigma)$.

Example 1.2.6. Let ω be a closed 2-form on a manifold M, and set $A_\omega = TM \oplus L$ where $L \to M$ is a trivial line bundle over M. A Lie algebroid structure can be defined on A_ω by taking the anchor of A_ω as the projection onto the first component, and the bracket as

$$[(X, f), (Y, g)] = \left([X, Y], \mathcal{L}_X g - \mathcal{L}_Y f + \omega(X, Y)\right),$$

where the space of sections $\Gamma(A_\omega)$ is identified with $\mathrm{Vect}(M) \times C^\infty(M)$. The Jacobi identity of the bracket is equivalent to the closed condition on ω. (If L is non-trivial simply replace the Lie derivative by some flat connection.)

Associated to any Lie algebroid A is the Lie algebroid differential d_A on the space of sections of the exterior product of the dual bundle $\Gamma(\wedge A^*)$. Considering examples 1.2.1 and 1.2.2, this differential is seen to unify the de Rham differential and the Chevalley-Eilenberg differential of a manifold and a Lie algebra respectively. The cohomology of the complex $(\Gamma(\wedge A^*), d_A)$ computes the cohomology of the Lie algebroid A.

Definition 1.2.1 is just one of four equivalent definitions of a Lie algebroid structure on A, with the other three described on the so called neighbours of A. It was first shown in [Vai97] that a Lie algebroid structure on A is equivalent to a certain odd vector field Q_A on the manifold ΠA, satisfying a non-trivial integrability condition

$$2Q_A^2 = [Q_A, Q_A] = 0.$$

Such an odd vector field is called a homological vector field, and a manifold equipped with a homological vector field will be called a Q-manifold. Let us make this precise. Any vector bundle $A \to M$ is naturally a graded manifold [Vor02]. This grading is called the weight, where for base coordinates x^a and fibre coordinates u^i, the weights are assigned as

$$w(x^a) = 0, \qquad w(u^i) = +1. \tag{1.16}$$

Coordinates u_i of the dual bundle A^* are assigned weight so that the natural pairing $u^i u_i$ maintains zero weight: given coordinates x^a, u_i of A^*, let $w(x^a) = 0$ and $w(u_i) = -1$. Since the coordinate changes of a vector bundle are linear in fibre variables, all coordinate changes are automatically weight preserving.

Let $u = u^i(x)e_i$ be a section of A for a local frame $\{e_i\}$. Define a canonical odd isomorphism $\imath : \Gamma(A) \to \mathrm{Vect}_{-1}(\Pi A)$ from sections of A to weight -1 vector fields on ΠA by

$$\imath(u) \mapsto \imath_u = (-1)^u u^i \frac{\partial}{\partial \xi^i}. \tag{1.17}$$

Let $Q_A \in \mathrm{Vect}_{+1}(\Pi A)$ be an odd, weight $+1$ vector field on ΠA. An arbitrary such vector field has the form

$$Q_A = \xi^i Q_i^a(x) \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{ji}^k(x) \frac{\partial}{\partial \xi^k}. \tag{1.18}$$

Then for sections $u, v \in \Gamma(A)$ and a function $f \in C^\infty(M)$, the anchor and bracket of a Lie algebroid structure on A can be expressed as

$$\rho(u|f) = [[Q_A, \imath_u], f], \qquad \rho(u|f) = u^i Q_i^a \frac{\partial f}{\partial x^a}, \tag{1.19}$$

and

$$\imath([u,v]_A) = (-1)^u [[Q_A, \imath_u], \imath_v], \tag{1.20}$$
$$[u,v]_A = \left(u^i Q_i^a \partial_a v^k - (-1)^{uv} v^i Q_i^a \partial_a u^k - (-1)^{i(v+1)} u^i v^j Q_{ji}^k\right) e_k.$$

The formula for the local frame $\{e_i\}$ are

$$\rho(e_i) = Q_i^a \frac{\partial}{\partial x^a}, \qquad [e_i, e_j] = (-1)^i Q_{ij}^k e_k.$$

The Jacobi identity for the bracket, together with the Leibniz condition (1.14) for the Lie algebroid A are compactly expressed in the integrability condition

$$[Q_A, Q_A] = 0. \tag{1.21}$$

As mentioned previously, this is a non-trivial condition for an odd vector field, since $[Q_A, Q_A] = 2Q_A^2$ is not necessarily zero. We will call the vector field Q_A the vector field associated to the Lie algebroid A. Notice that under the identification $\Gamma(\wedge A^*) \cong C^\infty(\Pi A)$, Q_A coincides with the Lie algebroid differential d_A, and the cohomology of the Lie algebroid is calculated via the complex $Q_A : C^\infty_k(\Pi A) \to C^\infty_{k+1}(\Pi A)$, now graded by weight.

Remark 1.2.1. More precisely, the space of sections $\Gamma(\wedge A^*)$ identifies with the subalgebra of functions on ΠA which are polynomial in the fibre variables. This distinction is often overlooked however, and the full identification allows to speak of pseudo-sections of A and so on. When M is a bosonic (purely even) manifold, the identification $\Gamma(\wedge A^*) \cong C^\infty(\Pi A)$ is a genuine isomorphism.

Example 1.2.7. Consider the tangent Lie algebroid $TM \to M$ as in example 1.2.1. The associated homological vector field on ΠTM equipped with canonical coordinates x^a, η^a is the weight $+1$ vector field

$$d = \eta^a \frac{\partial}{\partial x^a},$$

which coincides with the de Rham differential when functions on ΠTM are identified with (pseudo-)differential forms on M.

Example 1.2.8. Let $\mathfrak{g}$ be a Lie algebra viewed as a Lie algebroid over a point. The associated homological vector field on $\Pi\mathfrak{g}$ is

$$Q_{\mathfrak{g}} = \frac{1}{2}\xi^i\xi^j c^k_{ji}\frac{\partial}{\partial \xi^k}, \tag{1.22}$$

where the c^k_{ji} are structure constants of the Lie algebra. The condition $Q^2_{\mathfrak{g}} = 0$ is non-trivial and highlights the Jacobi identity for the bracket in terms of the structure constants.

Example 1.2.9. The action Lie algebroid $\mathfrak{g} \times M \to M$ of example 1.2.4 has the associated homological vector field,

$$Q_{\mathfrak{g}\times M} = \xi^i X^a_i(x)\frac{\partial}{\partial x^a} + \frac{1}{2}\xi^i\xi^j c^k_{ji}\frac{\partial}{\partial \xi^k}$$

on $\Pi\mathfrak{g} \times M$, where the first term encodes the action $\mathfrak{g} \times M \to M$ and the second term corresponds to the Lie bracket of $\mathfrak{g}$ eq. (1.22).

Together with homological vector fields, Lie algebroids also manifest themselves as even and odd Poisson structures. In order to describe these however, we must first recall some facts concerning the double vector bundle structure of $T^*\Pi A$, together with some associated canonical isomorphisms.

1.2.1 The Double Vector Bundle $T^*\Pi A$

Given a vector bundle ΠA, the manifold $T^*\Pi A$ carries the structure of a double vector bundle,

$$\begin{array}{ccc} T^*\Pi A & \longrightarrow & \Pi A^* \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M, \end{array}$$

where both the horizontal and vertical arrows possess the structure of a vector bundle. Introduce coordinates x^a, ξ^i, p_a, π_i on $T^*\Pi A$. Together with the weight grading (1.16) carried by ΠA and ΠA^*, the bundle $T^*\Pi A$ carries a bi-grading: assign the grades $(0,1,2,1)$ to the local coordinates respectively, and define the (ϵ_1, ϵ_2)-bi-grading by

$$\epsilon_1 = \#p + \#\xi, \qquad \epsilon_2 = \#p + \#\pi. \tag{1.23}$$

From this bi-grading, a total grading ϵ may be defined as

$$\epsilon = \epsilon_1 + \epsilon_2, \tag{1.24}$$

corresponding to the Euler vector field

$$\epsilon = 2p_a \frac{\partial}{\partial p_a} + \xi^i \frac{\partial}{\partial \xi^i} + \pi_i \frac{\partial}{\partial \pi_i},$$

on $T^*\Pi A$.

The definition of $T^*\Pi A$ as a double vector bundle is naturally symmetric, and is equivalent to that of $T^*\Pi A^*$. Introducing local coordinates $y^a, \theta_i, q_a, \pi^i$ on $T^*\Pi A^*$, there is a canonical isomorphism

$$\kappa : T^*\Pi A \to T^*\Pi A^*, \tag{1.25}$$

$$(x^a, \xi^i, p_a, \pi_i) \mapsto (y^a, \theta_i, q_a, \pi^i) = (x^a, \pi_i, -p_a, (-1)^{i+1}\xi^i).$$

Mackenxie and Xu gave a global, coordinate independent proof of this for arbitrary even vector bundles in [MX94] using methods of Tulczyjew [Tul77] who proved the isomorphism earlier in the case of tangent bundles. For a summary of double vector bundle isomorphisms we refer to the book [Mac05]. The Mackenzie-Xu isomorphism was extended to the super case by Voronov [Vor02], who also proved the existence for odd cotangent bundles.

Our conventions take κ as an anti-symplectomorphism, introducing the minus sign on T^*M, the core of the double vector bundle. Other conventions take κ as a symplectomorphism where the minus is introduced on one side of the diagram. For the canonical symplectic form ω_0 (1.2),

$$\left(\kappa^{-1}\right)^* (d\pi_i d\xi^i + dp_a dx^a) = -d\pi^i d\theta_i - dq_a dy^a.$$

It is very illuminating to view this isomorphism acting locally. Consider a change of coordinates $x = x(x')$ on M. The coordinate transformations induced in $T^*\Pi A$ are

$$\xi^i = \xi^{i'} T^i_{i'}, \quad p_a = J^{a'}_a p_{a'} + (-1)^{a(j+1)} \xi^{j'} T^j_{j'} \partial_a T^{i'}_j \pi_{i'}, \quad \pi_i = T^{i'}_i \pi_{i'}.$$

For the coordinates on $T^*\Pi A^*$,

$$\theta_i = T^{i'}_i \theta_{i'}, \quad q_a = J^{a'}_a q_{a'} - (-1)^{a(j+1)+i'} \pi^{i'} T^j_{i'} \partial_a T^{j'}_j \theta_{j'}, \quad \pi^i = (-1)^{i+i'} \pi^{i'} T^i_{i'}.$$

The isomorphism can be readily written down by comparing these coordinate changes.

The odd super analogue described in [Vor02] is

$$\begin{aligned} \kappa_\Pi : \Pi T^* A &\to \Pi T^* \Pi A^*, \\ (x^a, u^i, x^*_a, u^*_i) &\mapsto (y^a, \theta_i, y^*_a, \theta^i_*) = (x^a, u^*_i, -x^*_a, u^i), \end{aligned} \tag{1.26}$$

for an arbitrary vector bundle A with local coordinates x^a, u^i. Notice the sign difference between the even and odd versions.

Of course, since both κ and κ_Π are (anti-)symplectomorphisms, they both have interpretations via Lagrangian submanifolds, and are sometimes referred to as Legendre transformations. See [Roy99] for example.

1.2.2 Lie Algebroids as Poisson Manifolds

The two other neighbouring structures to a Lie algebroid structure on A present themselves as even and odd Poisson structures on the algebras $C^\infty(A^*)$ and $C^\infty(\Pi A^*)$ respectively. That a Lie algebroid is equivalent to an even Poisson structure on A^* is classical. For example, for a Lie algebra $\mathfrak{g}$ with coordinates u^i, the even linear Lie-Poisson bracket on $C^\infty(\mathfrak{g}^*)$ is given locally by

$$[u_i, u_j] = (-1)^i c^k_{ij} u_k,$$

where c_{ij}^k are the structure constants of the Lie algebra, and u_i are dual coordinates to u^i. (Notice that this can be defined via the homological vector field $Q_{\mathfrak{g}}$ (1.22) together with the odd Mackenzie-Xu isomorphism (1.26).) Due to the nature of the super language however, it is the odd Poisson structure on $C^\infty(\Pi A^*)$ also introduced by Vaintrob [Vai97] that will prove to be more versatile. Let Q_A be the homological vector field associated to A (eq. (1.18)), and let $H_A \in C^\infty(T^*\Pi A)$ be the odd Hamiltonian function such that $(H_A, -)\big|_{\Pi A} = Q_A$ under the canonical bracket (1.3). In particular, H_A must satisfy the master equation $(H_A, H_A) = 0$, which ensures the integrability condition (1.21) of Q_A. Locally, H_A has the form

$$H_A = \xi^i Q_i^a p_a + \frac{1}{2}\xi^i \xi^j Q_{ji}^k \pi_k. \tag{1.27}$$

Let $S_A = (\kappa^{-1})^* H_A \in C^\infty(T^*\Pi A^*)$, where κ is the Mackenzie-Xu isomorphism (1.25), which satisfies its own master equation $(S_A, S_A) = 0$ since κ takes brackets to brackets. The odd Poisson bracket on $C^\infty(\Pi A^*)$ is then expressed via a derived bracket (1.11): for functions $f, g \in C^\infty(\Pi A^*)$,

$$\begin{aligned}\{f, g\}_A = (-1)^{i(f+1)} \frac{\partial f}{\partial \theta_i} S_i^a \frac{\partial g}{\partial x^a} \\ +(-1)^{fg} \frac{\partial g}{\partial \theta_i} S_i^a \frac{\partial f}{\partial x^a} + (-1)^{i(f+1)+j} \frac{\partial f}{\partial \theta_i} S_{ij}^k \theta_k \frac{\partial g}{\partial \theta_i},\end{aligned}$$

where x^a, θ_i are local coordinates on ΠA^*. In terms of these coordinates,

$$\{\theta_i, x^a\}_A = (-1)^i S_i^a, \qquad \{\theta_i, \theta_j\}_A = (-1)^{i+j} S_{ij}^k \theta_k, \tag{1.28}$$

which is the odd fibre-wise linear bracket associated to a Lie algebroid A. A consequence of the four equivalent definitions of a Lie algebroid is that each definition may be used when most appropriate, which becomes particularly useful when dealing with their morphisms. For simplicity, we will only describe morphisms of Lie algebroids over the same base manifold. A description of those over different base manifolds is more involved and we refer [Mac05].

Let A_1 and A_2 be Lie algebroids over the same manifold M. A Lie algebroid morphism Φ is a morphism of vector bundles $\Phi : A_1 \to A_2$ such that:

1. $[\Phi(u), \Phi(v)]_{A_2} = \Phi\left([u, v]_{A_1}\right)$;

2. $\rho_{A_2}\left(\Phi(u)\right) = \rho_{A_1}(u)$;

for sections $u, v \in \Gamma(A_1)$. The morphism Φ is equivalent to three other neighbouring morphisms for each equivalent manifestation:

1. For the associated homological vector fields Q_{A_1} and Q_{A_2} there is a map $\Phi_\Pi : \Pi A_1 \to \Pi A_2$ of Q-manifolds. A map of Q-manifolds, or a Q-map, is such that the vector fields Q_{A_1} and Q_{A_2} are Φ_Π-related: for $f \in C^\infty(\Pi A_2)$,

$$Q_{A_1}(\Phi_\Pi^* f) = \Phi_\Pi^* Q_{A_2}(f). \tag{1.29}$$

2. For the even Poisson structure on the manifolds A_1^* and A_2^*, the map $\Phi^* : C^\infty(A_2^*) \to C^\infty(A_1^*)$ is a Poisson map: for $f, g \in C^\infty(A_2^*)$,

$$[\Phi^* f, \Phi^* g]_{A_1} = \Phi^*[f, g]_{A_2}. \tag{1.30}$$

3. As in the even case, for the odd Poisson structure on the manifolds ΠA_1^* and ΠA_2^*, the map $\Phi_\Pi^* : C^\infty(\Pi A_2^*) \to C^\infty(\Pi A_1^*)$ is a Poisson map of the odd linear Poisson structures,

$$\{\Phi_\Pi^* f, \Phi_\Pi^* g\}_{A_1} = \Phi_\Pi^*\{f, g\}_{A_2}, \tag{1.31}$$

for functions $f, g \in C^\infty(\Pi A_2^*)$.

Corollary 1.2.1. *For a Lie algebroid A with associated vector field Q_A, there is a map $\rho_\Pi : \Pi A \to \Pi TM$ such that Q_A and the de Rham differential d* (1.2.7) *are ρ_Π-related. Equivalently, there is a map $\rho_\Pi^* : C^\infty(\Pi T^*M) \to C^\infty(\Pi A^*)$ taking the canonical odd Poisson bracket of multivector fields* (1.5) *to the odd linear Schouten bracket* (1.28) *on ΠA^*.*

1.2.3 Lie Bialgebroids

A Lie bialgebroid generalises the concept of a Lie bialgebra, and consists of a pair of Lie algebroids in duality. They are understood as the infinitesimal versions of Poisson Lie groupoids and were introduced in the article [MX94].

Let A be a vector bundle and A^* its dual, and suppose that both A and A^* carry the structure of a Lie algebroid. Let $H_A \in C^\infty(T^*\Pi A)$ be the Hamiltonian function of the homological vector field Q_A associated to A (eq. (1.27)), and let $S_{A^*} \in C^\infty(T^*\Pi A)$ be the master Hamiltonian function generating the odd Poisson bracket on $C^\infty(\Pi A)$, equivalent to the Lie algebroid structure on A^*.

Definition 1.2.2. The pair $\mathcal{A} = (A, A^*)$ is a *Lie bialgebroid* if, for $\Theta = H_A + S_{A^*}$,

$$(\Theta, \Theta) = 0, \tag{1.32}$$

under the canonical Poisson bracket (1.3) on $T^*\Pi A$.

This compact form contains all the information about the structure on A and A^*, together with their compatibility. This is a consequence of the double vector bundle structure on $T^*\Pi A$ and the induced bi-grading of eq. (1.23). Observe that $\epsilon(\Theta) = 3$, and the gradings of the Hamiltonian functions H_A and S_{A^*} are

$$\epsilon_1(H_A) = 2, \quad \epsilon_2(H_A) = 1, \quad \epsilon(H_A) = 3,$$
$$\epsilon_1(S_{A^*}) = 1, \quad \epsilon_2(S_{A^*}) = 2, \quad \epsilon(S_{A^*}) = 3.$$

Condition (1.32) decomposes by the bi-grading into the three separate conditions

$$(H_A, H_A) = 0, \qquad (H_A, S_{A^*}) = 0, \qquad (S_{A^*}, S_{A^*}) = 0.$$

In particular, the first and third equations carry the Lie algebroid relations for the Lie algebroids A and A^* respectively, whilst the second condition translates that ΠA be a QS-manifold in the sense of [Vor02], i.e. an odd Poisson manifold equipped with a homological vector field such that the vector field is a derivation over the odd bracket. In this case, ΠA is equipped with Q_A and the odd Poisson bracket induced from the structure of A^*; that Q_A is a derivation follows from the fact that Q_A preserves the Hamiltonian, $\mathcal{L}_{Q_A} S_{A^*} = 0$.

Remark 1.2.2. In the works [ASZK97, Sch93] the term QS-manifold is used to describe a Q-manifold equipped with an invariant volume element, whilst QP-manifold is used to describe a manifold with a homological vector field and a compatible, odd (non-degenerate) Poisson structure. We prefer to follow [Vor02], and reserve QP-manifold for a manifold equipped with a homological vector field and compatible even Poisson bracket.

Remark 1.2.3. Lie bialgebroids were classically described by the condition that $d_{A^*}[X, Y]_A = [d_{A^*}X, Y]_A + [X, d_{A^*}Y]_A$, where $[-, -]_A$ is the brackets of sections X, Y of A and d_{A^*} is the differential of A^* [MX94]. Immediately afterwards [KS95] showed that this implies that each differential must be a derivation of the opposing bracket.

That this definition is equivalent to definition 1.2.2 was shown in [Roy99], and indeed, can be seen by identifying the differential d_A with the vector field Q_A. The definition of a Lie bialgebroid is symmetric, which was first shown in [MX94] using the non-super description. The proof of this was rather involved, and because of this the super language is very natural for such structures since this result is evident from the double vector bundle structure of $T^*\Pi A$.

Example 1.2.10. Let $\mathfrak{g}$ and $\mathfrak{g}^*$ be Lie algebras. Then $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra if $\Pi\mathfrak{g}$ is a QS-manifold. Equivalently

$$Q_{\mathfrak{g}}\{u, v\}_{\mathfrak{g}^*} = -\{Q_{\mathfrak{g}}u, v\}_{\mathfrak{g}^*} + (-1)^{u+1}\{u, Q_{\mathfrak{g}}v\}_{\mathfrak{g}^*},$$

for functions $u, v \in C^\infty(\Pi\mathfrak{g})$, the homological vector field $Q_{\mathfrak{g}}$ associated to $\mathfrak{g}$, and the Schouten bracket induced by $\mathfrak{g}^*$.

Example 1.2.11. The tangent bundle and the cotangent bundle of a Poisson manifold provide a natural example of a Lie bialgebroid. Given a Poisson bivector $P \in C^\infty(\Pi T^*M)$, an odd binary bracket can be introduced in the algebra of differential forms $C^\infty(\Pi TM)$ (see example 1.2.5) as follows. Let $\Upsilon \in C^\infty(T^*\Pi T^*M)$ be the canonical odd Hamiltonian function generating the odd Schouten bracket on ΠT^*M,

$$\Upsilon : \qquad ((\Upsilon, -), -)\big|_{\Pi T^*M} = [\![-, -]\!], \qquad \Upsilon = (-1)^a p_*^a p_a. \tag{1.33}$$

Using Υ, the Poisson Hamiltonian P can be lifted to a fibre-wise linear function on $T^*\Pi T^*M$. Define $\Upsilon_P \in C^\infty(T^*\Pi T^*M)$ as

$$\Upsilon_P = (\Upsilon, P), \tag{1.34}$$

which Poisson commutes as a consequence of the Jacobi identity for P and since $(\Upsilon, \Upsilon) = 0$ is trivially satisfied. Under the canonical isomorphism (1.25), write

$$K_P = \kappa^*\Upsilon_P = -(\kappa^*\Upsilon, \kappa^*P), \tag{1.35}$$

where the last equality follows since κ is an anti-symplectomorphism. The Hamiltonian function K_P is an odd fibre-wise quadratic function on $T^*\Pi TM$ such that $(K_P, K_P) = 0$, and hence defines an odd derived bracket on ΠTM by formula (1.11). In canonical local coordinates x^a, η^a on ΠTM, the bracket has the expression

$$\{x^a, \eta^b\}_{K_P} = -P^{ab}, \qquad \{\eta^a, \eta^b\}_{K_P} = -\eta^c \partial_c P^{ab}. \tag{1.36}$$

When functions on ΠTM are identified with differential forms, $\eta^a \leftrightarrow dx^a$, this bracket coincides with the classical Koszul bracket (compare with example 1.2.5).

Notice that the homological vector field d (eq. (1.2.7)) defining the structure of the Lie algebroid TM is generated by the function $\kappa^*\Upsilon$,

$$d = (\kappa^*\Upsilon, -)\big|_{\Pi TM}.$$

It is easily checked that d is a derivation of the Koszul bracket by applications of the Jacobi identity for the canonical bracket $(-,-)$, and by observing that $(\kappa^*\Upsilon, \Upsilon_P) = 0$ by the Poisson-nilpotency of Υ. Hence the pair (TM, T^*M) forms a Lie bialgebroid called the cotangent Lie bialgebroid, where the cotangent bundle is equipped with the Koszul bracket of 1-forms.

The cotangent Lie bialgebroid is an example of a triangulated Lie bialgebroid. For a Lie algebroid A, let $\Lambda \in C^\infty(\Pi A^*)$ be an even fibre-wise quadratic function, an even A-bivector. If Λ satisfies the master equation $\{\Lambda, \Lambda\}_A = 0$, then $\Gamma(\wedge A^*) \cong C^\infty(\Pi A)$ inherits an odd binary bracket which is compatible with the structure on A, precisely in the sense that $\mathcal{A} = (A, A^*)$ is a Lie bialgebroid. In the case $A = TM$ and M is an even Poisson manifold, Λ is taken to be the Poisson bivector P and the odd binary bracket is the Koszul bracket of differential forms.

Example 1.2.12. A well-known example of a triangulated Lie bialgebroid is a triangulated Lie algebra. Let $\mathfrak{g}$ be a Lie algebra and choose an even quadratic function $R \in C^\infty(\Pi\mathfrak{g}^*)$ which can be identified with an even element $r \in \wedge^2\mathfrak{g}$. Classically, this is known as a triangulated r-matrix, and is required to satisfy the classical Yang-Baxter equation (see [KS95, KS97] for example). The element r satisfies the Yang-Baxter equation precisely when R satisfies the master equation $\{R, R\}_\mathfrak{g} = 0$ in $C^\infty(\Pi\mathfrak{g}^*)$.

A Lie bialgebroid $\mathcal{A} = (A, A^*)$ induces a Poisson structure on the base manifold M. Let d_A and d_{A^*} be the differentials associated to A and A^* respectively. Then the Poisson bracket on M is classically described by

$$[f, g]_\mathcal{A} = (-1)^f \langle d_A f, d_{A^*} g\rangle,$$

for the canonical pairing $\langle -,-\rangle$ of A and A^* and functions $f, g \in C^\infty(M)$. In terms of the structure of the QS-manifold ΠA, the bracket is expressed as

$$[f, g]_\mathcal{A} = \{Q_A f, g\}_{A^*}. \tag{1.37}$$

That these coincide can be seen by translating the first into the language of homological vector fields. The Jacobi identity and skew-symmetry properties follow from those of the canonical bracket $(-,-)$.

Remark 1.2.4. A choice of Poisson structure on M breaks the symmetry of the Lie bialgebroid discussed in remark 1.2.3. One could equivalently define the Poisson bracket (1.37) using the QS-manifold structure on ΠA^*. This differs by a minus sign however, since the identification (1.25) is an anti-symplectomorphism.

1.3 Homotopy Structures

Homotopy analogues of the structures discussed so far appear naturally when considering deformation theory or when dealing with gauge theories for example. These higher structures are the L_∞-algebra structures of Lada and Stasheff [LS93], their Poisson counterparts the P_∞ and S_∞-structures, and the higher Lie algebroid structures associated to these. This section extends the previous sections to include these definitions, with almost all carrying over mutatis mutandis.

Definition 1.3.1. An L_∞-*algebra* is a vector space V equipped with a sequence of $\mathbb{R}$-multilinear anti-symmetric operations $[\cdots] : V^{\times k} \to V$ called the k-brackets, such that these operations have parity $k \bmod 2$ and satisfy the higher k-Jacobi identities

$$\sum_{i+j=k+1} \sum_{\sigma \in Sh(i,k-i)} \chi(\sigma)(-1)^{i(j-1)}[[v_{\sigma(1)}, \ldots, v_{\sigma(i)}], v_{\sigma(i+1)}, \ldots, v_{\sigma(k)}] = 0, \tag{1.38}$$

where $Sh(i, k-i)$ is the set of $(i, k-i)$-unshuffles, permutations preserving the order of the first i elements and the last $k-i$ elements, and $\chi(\sigma)$ is the sign given by the anti-symmetric permutation by σ.

We use the notation $V^{\times k}$ to mean $V \times \ldots \times V$ k times. Interesting examples of L_∞-algebra structures may be found in [BL09] or [RW98] for example.

Remark 1.3.1. Analogous to remark 1.1.2, there are two conventions that can be taken in defining an L_∞-algebra. The first, and the one we shall adopt, is to follow definition 1.3.1 as given in [LS93], where the brackets are of alternating parity and anti-symmetric. The second is to require symmetric brackets $\{\cdots\}$, all of which now carry parity +1. This equivalent notion of an L_∞-algebra is known as an L_∞-anti-algebra, or a $L_\infty[1]$-algebra, since it is defined on the space of reversed parity $\Pi V \equiv V[1]$. These conventions differ by a shift in parity given by the isomorphism

$$\{\Pi v_1, \ldots, \Pi v_k\} = (-1)^{(k-1)v_1+(k-2)v_2+\cdots+v_{k-1}}\Pi[v_1, \ldots, v_k]. \tag{1.39}$$

However, if compatibility with an associative multiplication is imposed then this choice of convention is lost. Anti-symmetric brackets must be adopted for an even homotopy Poisson algebra, whilst symmetric operations must be used for a homotopy Schouten algebra.

Definition 1.3.2. An *even homotopy Poisson algebra*, or a *P_∞-algebra*, is an algebra A equipped with an anti-symmetric L_∞-algebra structure in the sense of definition 1.3.1 such that each bracket is a derivation of the associative multiplication in each argument.

Definition 1.3.3. An *odd homotopy Poisson algebra*, a *homotopy Schouten algebra*, or an *S_∞-algebra*, is an algebra equipped with a symmetric L_∞-algebra structure, or an $L_\infty[1]$-structure of remark 1.3.1, such that each bracket is a derivation of the associative multiplication in each argument.

The complicated Jacobi identities (1.38) can be compactly rephrased in terms of formal homological vector fields. Such a description is due to Voronov [Vor05b, Vor05a], where a higher derived bracket construction was introduced in order to describe such homotopy structures.

Let V be an L_∞-algebra. Considering V as a vector bundle over a point, the odd isomorphism (1.17) identifies an element $v \in V$ with an odd weight -1 vector field $\imath_v$ on ΠV. Introduce a formal homological vector field $Q_V \in \mathrm{Vect}(\Pi V)$, with the local description

$$Q_V = \xi^i Q_i^k \frac{\partial}{\partial \xi^k} + \frac{1}{2}\xi^i\xi^j Q_{ji}^k \frac{\partial}{\partial \xi^k} + \frac{1}{3!}\xi^i\xi^j\xi^l Q_{lji}^k \frac{\partial}{\partial \xi^k} + \cdots, \tag{1.40}$$

(compare with eq. (1.18)), which decomposes by weight, $Q_V = Q_0 + Q_1 + Q_2 + \cdots$. The brackets on V may then be defined in terms of the nested commutators of vector fields,

$$\imath_{[v_1,\ldots,v_k]} := (-1)^{(k-1)v_1+(k-2)v_2+\cdots+v_{k-1}}[\cdots[Q_V, \imath_{v_1}], \cdots, \imath_{v_k}](0), \tag{1.41}$$

where the commutators are evaluated at 0 in ΠV after the kth bracket. For a basis $\{e_i\}$ of V, the first three brackets on V are calculated to be

$$[e_i] = Q_i^k e_k, \qquad [e_i, e_j] = (-1)^i Q_{ij}^k e_k,$$
$$[e_i, e_j, e_k] = (-1)^j Q_{ijk}^l e_l.$$

Remark 1.3.2. When speaking about L_∞-algebras, we will always assume that the L_∞-algebras are strict, or without background. This means that there is no term of weight -1 appearing in Q_V which would give rise to a 0-bracket or a distinguished non-zero element of the vector space V.

Definition 1.3.4. An L_∞-algebra will be called *minimal* if the homological vector field (1.40) has no term of weight 0. That is, $Q_0 \equiv 0$ and there is no 1-bracket present.

Notice that if an L_∞-algebra V is minimal, then the binary bracket is a genuine Lie bracket on V since $Q_1^2 = 0$.

Definition 1.3.5. A *morphism* $\Phi : V \rightsquigarrow W$ *of homotopy Lie algebras* is a collection of maps $\Phi^k : V^{\times k} \to W$ which assemble as the Taylor coefficients of a map $\Phi_\Pi : \Pi V \to \Pi W$, where Φ_Π is a Q-map (see eq. (1.29)) relating the homological vector fields Q_V and Q_W generating the two L_∞-structures.

We must make an important observation: *there is no single map $V \to W$ which represents the L_∞-morphism*, hence the notation $V \rightsquigarrow W$. An L_∞-morphism is a collection of maps, which assemble into a single map of vector spaces of reversed parity. See [Kon03] for details.

Recall that for a Lie algebroid A, a weight $+1$ homological vector field on ΠA encodes the algebroid structure. The Lie bracket is recovered from the formula (1.20). Compare then with the higher derived brackets (1.41); we can define a sequence of brackets on the sections of A by lifting the weight restriction on the homological vector field, obtaining an L_∞-algebroid.

Definition 1.3.6. Let $A \to M$ be a vector bundle. Then A is an L_∞-*algebroid* if there is a sequence of maps $\rho_k : A^{\times(k-1)} \to TM$ called the k-anchors, and a sequence of $\mathbb{R}$-linear brackets $[\cdots]_k : \Gamma(A)^{\times k} \to \Gamma(A)$ called the k-brackets, such that the $C^\infty(M)$-module $\Gamma(A)$ has the structure of an L_∞-algebra, and

$$[u_1, \ldots, u_{k-1}, f u_k]_k = \rho_k(u_1, \ldots, u_{k-1}|f) u_k + (-1)^{f(u_1+\cdots+u_{k-1}+k)} f [u_1, \ldots, u_k]_k,$$

for sections $u_1, \dots, u_k \in \Gamma(A)$ and a function $f \in C^\infty(M)$.

Remark 1.3.3. Algebraically, these are described as homotopy Lie-Rinehart algebras, homotopy analogues of classical Lie-Rinehart algebras which stand as the algebraic counterpart to Lie algebroids. Homotopy Lie-Rinehart algebras are L_∞-algebras possessing a $C^\infty(M)$-module structure such that $C^\infty(M)$ is an L_∞-module over the algebra. See [Kje01, Vit15] for details and references therein.

The homological vector field on ΠA describing an L_∞-algebroid A admits a more general expression similar to that in eq. (1.40),

$$Q_A = Q_0 + Q_1 + Q_2 + \cdots,$$

decomposing by weight. The higher brackets and anchors can then be expressed by the nested commutators,

$$\rho_k(u_1, \dots, u_{k-1}|f) = (-1)^{(k-1)u_1 + \cdots + u_{k-2}} \big[[\cdots[Q_A, \imath_{u_1}], \dots, \imath_{u_{k-1}}], f\big](0),$$

$$\imath\big([u_1, \dots, u_k]\big) = (-1)^{(k-1)u_1 + \cdots + u_{k-1}} \big[\cdots[Q_A, \imath_{u_1}], \dots, \imath_{u_k}\big](0),$$

for sections $u_1, \dots, u_k \in \Gamma(A)$ and a function $f \in C^\infty(M)$. The entire structure of the L_∞-algebroid is compactly contained in the equation $Q_A^2 = 0$. Paralleling the Lie algebroid case, any L_∞-algebroid gives rise to even and odd homotopy Poisson algebras by analogous higher derived bracket constructions.

Definition 1.3.7. An L_∞-algebroid A is *minimal* if there is no weight 0 term present in the homological vector field associated to A, $Q_A = Q_0 + Q_1 + \cdots$ and $Q_0 = 0$.

In particular, there is no unary bracket or distinguished anchor associated to a minimal L_∞-algebroid, and the term Q_1 generates a genuine Lie algebroid structure on A since $Q_1^2 = 0$.

Proposition 1.3.1 ([Vor14])**.** *The higher k-anchors assemble into a morphism of L_∞-algebroids $\rho : A \rightsquigarrow TM$, where TM has the canonical Lie algebroid structure of example 1.2.1.*

Proof. For an arbitrary Q-manifold N with local coordinates x^a, the homological vector field defines the map $Q : N \to \Pi TN$ by $Q^*(\eta^a) = Q^a(x)$. Such a map is automatically a map of Q-manifolds, and from the L_∞-algebroid A with associated homological vector field Q_A, we define the map $Q_A : \Pi A \to \Pi T(\Pi A)$ as a Q-morphism. Next observe that the differential of any map of manifolds is a Q-morphism of odd tangent bundles relating the de Rham differentials. Define the projection $\beta_{\Pi A} : \Pi A \to M$; the odd tangent map $\Pi T\beta_{\Pi A} : \Pi T(\Pi A) \to \Pi TM$ takes the de Rham differential on $\Pi T(\Pi A)$ to that on ΠTM. Finally, ρ_Π is defined as the composition $\rho_\Pi = \Pi T\beta_{\Pi A} \circ Q_A$ which is a morphism of Q-manifolds since it is a composition of Q-morphisms. □

Definition 1.3.8. An L_∞-bialgebroid $\mathcal{A}$ is a pair of L_∞-algebroids (A, A^*) such that the two associated Hamiltonian functions $H_A, S_{A^*} \in C^\infty(T^*\Pi A)$ generate the structure of a homotopy QS-manifold on ΠA.

Notice that the strictness of the L_∞-algebroid structure on A (resp. A^*) is equivalent to the vanishing of the Hamiltonian function H_A (resp. S_{A^*}) on the zero section ΠA of $T^*\Pi A$ (resp. ΠA^* of $T^*\Pi A^*$).

Remark 1.3.4. There are alternative understandings of what an L_∞-bialgebroid should be. Bashkirov and A.Voronov, [BV16], define an L_∞-bialgebroid to be the double vector bundle $T^*\Pi E$ for some vector bundle $E \to M$, equipped only with a Poisson commuting function which is required to be linear in the base momenta p_a. This definition allows for operations with mixed inputs and outputs, and indeed such a case should not be discounted. We will discuss later the reason for our choice of definition 1.3.8 when speaking about the doubles of such structures, but note that definition 1.3.8, though more restrictive than that in [BV16], still incorporates the important class of examples of triangulated L_∞-bialgebroids.

We also draw attention to the work of Kravchenko [Kra07], and Bashkirov and A.Voronov [BV15], on L_∞-bialgebras which allow for mixed operations, and also the work of Mehta [Meh11], who defines a more restrictive L_∞-bialgebra as a Lie algebra equipped with a compatible homotopy Schouten structure arising from linearising a Lie group equipped with homotopy Poisson structure.

Example 1.3.1. An L_∞-bialgebra is a pair of L_∞-algebras $(\mathfrak{g}, \mathfrak{g}^*)$ whose associated Hamiltonian functions $H_\mathfrak{g}$ and $S_{\mathfrak{g}^*}$ satisfy $(H_\mathfrak{g}, S_{\mathfrak{g}^*}) = 0$ on $T^*\Pi\mathfrak{g} \cong \Pi(\mathfrak{g} \oplus \mathfrak{g}^*)$. With local coordinates ξ^i, π_i on $\Pi(\mathfrak{g} \oplus \mathfrak{g}^*)$, the Hamiltonians are of the form

$$H_\mathfrak{g} = Q^k(\xi)\pi_k, \qquad S_{\mathfrak{g}^*} = \xi^k S_k(\pi).$$

Example 1.3.2. Let A be a Lie algebroid, and $\Lambda \in C^\infty(\Pi A^*)$ be an even A-multivector such that $\{\Lambda, \Lambda\}_A = 0$. Then $\mathcal{A}_\Lambda = (A, A^*, \Lambda)$ is a triangulated L_∞-bialgebroid. Let S_A be the odd fibrewise-linear Hamiltonian function in $C^\infty(T^*\Pi A^*)$ generating the Schouten structure corresponding to the Lie algebroid A. We define the odd function $\Upsilon_\Lambda \in C^\infty(T^*\Pi A)$ as

$$\Upsilon_\Lambda = (S_A, \Lambda), \tag{1.42}$$

which satisfies $(\Upsilon_\Lambda, \Upsilon_\Lambda) = 0$ since Λ satisfies the master equation. Then $K_\Lambda = \kappa^*\Upsilon_\Lambda = -(H_A, \kappa^*\Lambda)$ is a Poisson-nilpotent function on $C^\infty(T^*\Pi A)$, which defines the sequence of odd Poisson brackets on $C^\infty(\Pi A)$ by the derived bracket formula

$$\{f_1, \ldots, f_k\}_{K_\Lambda} = (\cdots(K_\Lambda, f_1), \ldots, f_k)\big|_{\Pi A}, \tag{1.43}$$

for functions $f_1, \ldots, f_k \in C^\infty(\Pi A)$. These brackets provide A^* with an L_∞-algebroid structure, and since $\mathcal{L}_{Q_A} K_\Lambda = 0$ by the nilpotency of H_A, (A, A^*) is an L_∞-bialgebroid. Compare with example 1.2.11 in the case when $A = TM$ and Λ is a generalised Poisson structure. In this case we obtain the sequence of higher Koszul brackets on $C^\infty(\Pi TM)$ constructed in the work [KV08].

Example 1.3.3. The triangulated L_∞-bialgebra produced in [BV15] provides an example of an L_∞-bialgebroid. For an L_∞-algebra $\mathfrak{g}$, an even element $R \in C^\infty(\Pi\mathfrak{g}^*)$ satisfies the generalised Maurer-Cartan equation if

$$\sum_{k\geq 1} \frac{1}{k!}\{R, \ldots, R\}_{\mathfrak{g}} = 0,$$

for the induced homotopy Schouten structure on $C^\infty(\Pi\mathfrak{g}^*)$. Notice in the case $k = 2$, i.e. $\mathfrak{g}$ is a Lie algebra, the generalised Maurer-Cartan equation degenerates to the master equation $\{R, R\}_{\mathfrak{g}} = 0$. The even function R is identified with an "r_∞-matrix" in $\mathfrak{g}$, and defines a compatible L_∞-bialgebra structure on $\mathfrak{g}^*$ via the same construction as in example 1.3.2.

Though this is will be known to experts, the author does not know of any literature where this is stated, so we will detail the following proposition.

Proposition 1.3.2. *Let $\mathcal{A}$ be an L_∞-bialgebroid. Then M is a homotopy Poisson manifold, or a P_∞-manifold, with the brackets defined as*

$$[f_1, \ldots, f_k]_{\mathcal{A}} = (-1)^{(k-2)f_1+\cdots+f_{k-2}}\{Q_A f_1, \ldots, Q_A f_{k-1}, f_k\}_{A^*},$$

for functions $f_1, \ldots, f_k \in C^\infty(M)$, and where Q_A and $\{\cdots\}_{A^*}$ generate the QS-manifold structure of ΠA.

Proof. Skew-symmetry and the Leibniz rule are inherited from the Schouten brackets and the derivation Q_A. The only condition to check are the Jacobi identities (1.38),

$$\sum_{i+j=k+1} \sum_{\sigma \in Sh(i,k-i)} \chi(\sigma)(-1)^{i(j-1)}[[f_{\sigma(1)}, \ldots, f_{\sigma(i)}]_{\mathcal{A}}, f_{\sigma(i+1)}, \ldots, f_{\sigma(k)}]_{\mathcal{A}} = 0.$$

Consider the Jacobi identity for the homotopy Schouten structure on ΠA, where $\chi'(\sigma)$ is the (symmetric) sign acquired from the shuffle σ, and we split about the final argument,

$$\sum_{\sigma \in Sh(i,k-1-i)} \chi'(\sigma)\{\{Q_A f_{\sigma(1)}, \ldots, Q_A f_{\sigma(i)}\}_{A^*}, Q_A f_{\sigma(i+1)}, \ldots, Q_A f_{\sigma(k-1)}, f_k\}_{A^*}$$
$$+ \sum_{\sigma \in Sh(i-1,k-1-i)} \chi'(\sigma)\{\{Q_A f_{\sigma(1)}, \ldots, Q_A f_{\sigma(i-1)}, f_k\}_{A^*}, Q_A f_{\sigma(i)}, \ldots, Q_A f_{\sigma(k)}\}_{A^*} = 0.$$

Bringing Q_A out of the first term, switching the position of the nested bracket in the second, and inserting the correct signs brings us to

$$\sum_{\sigma \in Sh(i,k-1-i)} \chi'(\sigma)(-1)^{\varepsilon}[[f_{\sigma(1)}, \ldots, f_{\sigma(i)}]_{\mathcal{A}}, f_{\sigma(i+1)}, \ldots, f_{\sigma(k-1)}, f_k]_{\mathcal{A}}$$
$$+ \sum_{\sigma \in Sh(i,k-1-i)} \chi'(\sigma)(-1)^{\delta}[[f_{\sigma(1)}, \ldots, f_{\sigma(i-1)}, f_k]_{\mathcal{A}}, f_{\sigma(i)}, \ldots, f_{\sigma(k)}]_{\mathcal{A}} = 0,$$

where ε and δ are the signs

$$\varepsilon = i(j-1) + (k-2)f_{\sigma(1)} + \cdots + (k-i+1)f_{\sigma(i-2)}$$
$$+(k-i)f_{\sigma(i-1)} + (k-i-1)f_{\sigma(i)} + \cdots + f_{\sigma(k-2)},$$
$$\delta = \varepsilon + (j-1)f_k.$$

Observe that $\chi'(\sigma)$, the sign from the shuffle, is symmetric with respect to a shift in parity. The sign we must obtain is anti-symmetric with respect to the original parity. These differ precisely by a "parity shift" of the first $k-2$ arguments analogous to the isomorphism (1.39), and produces the sign $\varepsilon - i(j-1)$. It remains to observe that the additional sign of $(j-1)f_k$ in δ must be also incorporated into this shift, to remove those terms created by the presence of the odd derivation Q_A. From this, we obtain the Jacobi identity of the homotopy Poisson structure,

$$\sum_{i+j=k+1} \sum_{\sigma \in Sh(i,k-i)} \chi(\sigma)(-1)^{i(j-1)}[[f_{\sigma(1)}, \ldots, f_{\sigma(i)}]_{\mathcal{A}}, f_{\sigma(i+1)}, \ldots, f_{\sigma(k)}]_{\mathcal{A}} = 0.$$

□

Chapter 2

Homotopy Courant Algebroids, and Doubles of L_∞-Bialgebroids

Courant algebroids were introduced in the work [LWX97] in the search for the double objects of Lie bialgebroids. Their interpretation in terms of a homological vector field on a supermanifold was developed in [Roy99, Roy02], which is adapted in this chapter in the investigation as to what should stand as the double of an L_∞-bialgebroid. From this investigation we are lead to the notion of a homotopy Courant algebroid.

2.1 Courant Algebroids

Definition 2.1.1. A *Courant algebroid* is a vector bundle $E \to M$ together with a non-degenerate bilinear form $\langle -, - \rangle$ on sections, a map $\rho : E \to TM$ called the anchor, and a bracket $[-, -]$ on sections of E such that:

1. $[u, [v, w]] = [[u, v], w] + (-1)^{uv}[v, [u, w]]$;
2. $[u, fv] = \rho(u|f)v + (-1)^{uf} f[u, v]$;
3. $\rho([u, v]) = [\rho(u), \rho(v)]$;
4. $\rho(u|\langle v, w\rangle) = \langle [u, v], w\rangle + (-1)^{uv}\langle v, [u, w]\rangle$;
5. $\rho(u|\langle v, w\rangle) = \langle u, [v, w] + (-1)^{vw}[w, v]\rangle$,

for sections $u, v, w \in \Gamma(E)$ and a function $f \in C^\infty(M)$.

Properties (1), (2) and (4) can be rephrased by observing that a section $u \in \Gamma(E)$ generates a derivation of the $C^\infty(M)$-module $\Gamma(E)$ by $u \mapsto \mathrm{ad}_u = [u, -]$. The three properties (1), (2) and (4) are then equivalent to saying that the structure of the Courant algebroid is preserved under the action of ad_u. It was noticed in [Uch02] that the number of axioms for a Courant algebroid may be reduced, and in [KS05] it was shown that it could in fact be reduced to properties (1), (4), and (5), with (2) and (3) consequences of the others. A detailed history of the Courant algebroid can be found in the survey [KS13].

Our choice of bracket is the Dorfman bracket, a bracket originally introduced in [Dor93] for the study of integrable evolution equations. The bracket first defined by Courant in [Cou90] is a particular case of the more general Courant bracket which also bears Courant's name, and which is a skew-symmetrisation of the Dorfman bracket. Courant algebroids were originally defined using the Courant bracket [LWX97], however such a definition introduces defects in terms such as the Jacobi identity and the Leibniz rules. It happens that it is more practical to work with the Dorfman bracket for our purposes which removes the defects at the sacrifice of symmetry.

Remark 2.1.1. The Dorfman bracket is an example of a Loday bracket, a bilinear operation satisfying a Jacobi identity of the form (1) in definition 2.1.1, which provides the module $\Gamma(E)$ with the structure of a Loday algebra. As such, a Courant algebroid can be considered as a Loday algebroid equipped with a non-degenerate bilinear form [GKP13]. Loday algebras are also referred to as Leibniz algebras [KW01] and appear as derived brackets [KS04].

2.1.1 Graded Symplectic 2-Manifolds

An alternate description of Courant algebroids was developed by Roytenberg in terms of non-negatively graded manifolds [Roy02]. Described originally for those Courant algebroids arising as the suggested doubles of Lie bialgebroids [Roy99], Roytenberg showed that any Courant algebroid can be realised as a degree 2 symplectic manifold equipped with an appropriate degree 3 master function. The structure of the Courant algebroid was then shown to be expressible via this master function and the non-degenerate Poisson bracket by a derived bracket type formula.

A symplectic 2-manifold is a non-negatively graded manifold $\mathcal{M}$ (see [Vor02]), equipped with a symplectic form of degree 2. The degree of the symplectic form constrains $\mathcal{M}$ to have degree at most 2, which therefore exhibits the fibred structure

$$\mathcal{M} \to \Pi E \to M, \tag{2.1}$$

where M is an arbitrary smooth manifold and ΠE is a vector bundle. Roytenberg showed that symplectic 2-manifolds admit a one-to-one correspondence with pseudo-Euclidean vector bundles E, where the total space of the bundle of reversed parity ΠE inherits the structure of a Poisson manifold by the extension of the bilinear form as a derivation in each argument. The additional structure of a Courant algebroid is provided through a degree 3 function $\Theta \in C^\infty(\mathcal{M})$ via the non-degenerate Poisson bracket on $\mathcal{M}$. Let us recall this in detail.

Let E be a pseudo-Euclidean vector bundle. The cotangent bundle $T^*\Pi E$ carries the structure of a symplectic 2-manifold by considering the total ϵ-grading given in eq. (1.24) for the double vector bundle. The canonical even 2-form on $T^*\Pi E$ is of ϵ-degree 2, giving rise to the canonical degree -2 Poisson bracket (eq. (1.3)) on $C^\infty(T^*\Pi E)$. Using the canonical double vector bundle isomorphism eq. (1.25), the two projection maps

$$\beta : T^*\Pi E \to \Pi E, \qquad \beta'_* = \beta_* \circ \kappa : T^*\Pi E \to \Pi E^*,$$

can be combined into a single canonical map

$$\mathcal{X}_E : T^*\Pi E \to \Pi(E \oplus E^*). \tag{2.2}$$

The pull-back of $\mathcal{X}_E$ is a Poisson map $\mathcal{X}_E^* : C^\infty(\Pi(E \oplus E^*)) \to C^\infty(T^*\Pi E)$, taking the Poisson structure on $C^\infty(\Pi(E \oplus E^*))$ obtained from the extension of the natural pairing to the canonical non-degenerate bracket.

The isometric embedding $E \hookrightarrow E \oplus E^*$ given by $u \mapsto u + \frac{1}{2}\langle u, - \rangle$ then defines a map $\Pi E \to \Pi(E \oplus E^*)$, over which the bundle $T^*\Pi E$ may be pulled back producing an affine bundle $\mathcal{M} \to \Pi E$,

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[d] & T^*\Pi E \arrow[d] \\ \Pi E \arrow[r] & \Pi(E \oplus E^*). \end{tikzcd}$$

The pull-back of the canonical symplectic form on $T^*\Pi E$ provides a symplectic form on $\mathcal{M}$, whence $\mathcal{M}$ is called the minimal symplectic realisation of the Poisson manifold ΠE. The map $\mathcal{M} \to \Pi E$ is an affine fibration and induces the Poisson map $C^\infty(\Pi E) \to C^\infty(\mathcal{M})$; the affine structure of this fibration stems from the non-linear transformation of the degree 2 local coordinates in the fibres. Consider the set of canonical local coordinates x^a, ξ^i, p_a, π_i on $T^*\Pi E$, where under a change of coordinates $x = x(x')$,

$$\xi^i = \xi^{i'} T^i_{i'}, \quad p_a = J^{a'}_a p_{a'} + (-1)^{a(j+1)} \xi^{j'} T^j_{j'} \partial_a T^{i'}_j \pi_{i'}, \quad \pi_i = T^{i'}_i \pi_{i'}. \tag{2.3}$$

Due to the affine transformation of the base momenta p_a, it is beneficial to introduce a connection ∇ in the vector bundle E, compatible with the metric g, in order to redefine the coordinates. Such a connection is called a metric connection and satisfies the condition $\nabla g = 0$. Note that the existence of metric connections can always be assumed, see [Rot91] for instance. Let ∇ be defined by the connection coefficients A^i_{aj}, and prescribe new base momenta coordinates

$$q_a = p_a - (-1)^{a(j+1)} \xi^j A^i_{aj} \pi_i, \tag{2.4}$$

which obey the transformation law $q_a = J^{a'}_a q_{a'}$ with respect to the Jacobian matrix J. Equipped with this connection, the cotangent bundle splits into the direct sum $\mathcal{E}_\nabla = T^*M \oplus \Pi(E \oplus E^*)$ via the symplectomorphism

$$\varphi_{\mathcal{E}_\nabla} : \mathcal{E}_\nabla \to T^*\Pi E, \tag{2.5}$$

twisting the canonical symplectic form on $T^*\Pi E$ by the connection coefficients. Since $\varphi_{\mathcal{E}_\nabla}$ is a symplectomorphism, the composition $\varphi^*_{\mathcal{E}_\nabla} \circ \mathcal{X}^*_E$ defines the Poisson map

$$\mathcal{X} := \varphi^*_{\mathcal{E}_\nabla} \circ \mathcal{X}^*_E : C^\infty(\Pi(E \oplus E^*)) \to C^\infty(\mathcal{E}_\nabla), \tag{2.6}$$

which can be viewed as the inclusion of the Poisson subalgebra $C^\infty(\Pi(E \oplus E^*))$ into the algebra $C^\infty(\mathcal{E}_\nabla)$.

The degree 2 coordinates on the symplectic realisation $\mathcal{M}$ obey a similar affine transformation law to those on $T^*\Pi E$, and it is again beneficial to introduce a metric connection in E in order to redefine these. Given local coordinates x^a on M, natural coordinates ξ^i on ΠE can be introduced such that, for a local frame $\{e_i\}$ of E, the metric on E is described by constant functions

$$\langle e_i, e_j \rangle = (-1)^i g_{ij}.$$

The symplectic realisation $\mathcal{M}$ can then be prescribed local coordinates x^a, ξ^i, q_a, where the degree 2 coordinates q_a depend on the metric connection coefficients A^i_{aj}, but satisfy the transformation law $q_a = J^{a'}_a q_{a'}$ for a change of base coordinates $x = x(x')$. Again, the choice of connection splits the symplectic manifold $\mathcal{M}$ via the symplectomorphism

$$\varphi_{\mathcal{M}_\nabla} : \mathcal{M}_\nabla := T^*M \oplus \Pi E \to \mathcal{M},$$

which gives the corresponding Poisson map

$$\mathcal{X} : C^\infty(\Pi E) \to C^\infty(\mathcal{M}_\nabla). \tag{2.7}$$

(We denote both Poisson maps in eqs. (2.6) and (2.7) by the symbol $\mathcal{X}$, which should not cause confusion in context.) The even symplectic form on $\mathcal{M}_\nabla$ now depends on the connection coefficients, and coincides with the symplectic structure defined in [Rot91] defined from a metric connection. The corresponding non-degenerate Poisson bracket on $\mathcal{M}_\nabla$ is described by the local expressions

$$\begin{aligned} [q_a, x^b]_\nabla &= \delta^b_a, & [\xi^i, \xi^j]_\nabla &= (-1)^i g^{ij}, \\ [q_a, \xi^i]_\nabla &= \xi^j A^i_{aj}, & [q_a, q_b]_\nabla &= (-1)^i \xi^i \xi^j R_{abji}, \end{aligned} \tag{2.8}$$

where the coefficients g^{ij} are inverse to the metric coefficients g_{ij}, and R_{abji} are the coefficients of the curvature tensor of the connection ∇ in E.

We must mention that although the split manifold $\mathcal{M}_\nabla$ depends on the choice of connection, two different metric connections ∇ and ∇' produce symplectomorphic split manifolds $\mathcal{M}_\nabla \cong \mathcal{M}_{\nabla'}$. It therefore makes sense to fix a single metric connection ∇ on E throughout the remainder of the chapter.

Definition 2.1.2. Let E be a pseudo-Euclidean vector bundle equipped with a metric connection ∇ on E. A *Courant algebroid* is the pair $(\mathcal{M}_\nabla, \Theta)$, where $\Theta \in C^\infty(\mathcal{M}_\nabla)$ is a degree 3 function on the minimal symplectic realisation of ΠE such that

$$[\Theta, \Theta]_\nabla = 0. \tag{2.9}$$

It was shown in the work [Roy02] that the master equation (2.9) for the function Θ contains all the structural information of a Courant algebroid. The Poisson map in eq. (2.7) can then be used to recover the anchor and bracket of the Courant algebroid via derived brackets. Define the odd isomorphism

$$\jmath : \Gamma(E) \to C^\infty_{+1}(\Pi E), \tag{2.10}$$

identifying sections of E with degree 1 (fibre-linear) functions on ΠE via the non-degenerate metric g; for a section $u \in \Gamma(E)$, write $\jmath(u) = \jmath_u \in C^\infty(\Pi E)$. Then for sections $u, v \in \Gamma(E)$, the symmetric bilinear form on $\Gamma(E)$ may be expressed as

$$\langle u, v\rangle = (-1)^u [\jmath_u, \jmath_v]_\nabla,$$

where the non-degeneracy follows from that of the bracket. The anchor and bracket of E are given analogously to formula (1.19) and (1.20) for a Lie algebroid,

$$\begin{aligned} \rho(u|f) &= [[\Theta, \mathcal{X}(\jmath_u)]_\nabla, f]_\nabla, \\ \mathcal{X}(\jmath_{[u,v]_\Theta}) &= (-1)^u [[\Theta, \mathcal{X}(\jmath_u)]_\nabla, \mathcal{X}(\jmath_v)]_\nabla. \end{aligned} \tag{2.11}$$

We will introduce the notation $\bar{u} = \mathcal{X}(\jmath_u)$ for the section $u \in \Gamma(E)$ viewed as a degree 1 function on $C^\infty(\mathcal{M}_\nabla)$ with shifted parity.

2.2 The Double of an L_∞-Bialgebroid

In the work [Dri83], Drinfeld defined what is now known as the Drinfeld double of a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, producing a quasi-triangular Lie bialgebra which provides a source of quantum groups. Classically a Lie bialgebra is described as a Lie algebra $\mathfrak{g}$ whose dual space $\mathfrak{g}^*$ is also a Lie algebra, such that the comultiplication $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ satisfies the cocycle condition

$$\delta([u, v]) = (\mathrm{ad}_u \otimes 1 + 1 \otimes \mathrm{ad}_u)\delta(v) - (\mathrm{ad}_v \otimes 1 + 1 \otimes \mathrm{ad}_v)\delta(u),$$

for elements $u, v \in \mathfrak{g}$ and the Lie bracket on $\mathfrak{g}$.

The Drinfeld double of a Lie bialgebra is defined as the unique Lie bialgebra structure on the direct sum $\mathfrak{g} \oplus \mathfrak{g}^*$, such that both $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras, the natural inner product is ad-invariant, and the cobracket of $\mathfrak{g} \oplus \mathfrak{g}^*$ is given by the coboundary of an element $r \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g}^*)$. This last fact means that the double is actually a quasi-triangular Lie bialgebra since r satisfies (a modified version of) the Yang-Baxter equation, see example 1.2.12.

This entire construction is conveniently described in the super language. For a pair of Lie algebras $(\mathfrak{g}, \mathfrak{g}^*)$, the corresponding Hamiltonian functions $H_\mathfrak{g}$ and $S_{\mathfrak{g}^*}$ defining the Q-structure and the odd Poisson structure of $\Pi\mathfrak{g}$ commute under the canonical

Poisson bracket on $T^*\Pi\mathfrak{g}$ if and only if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra. That is, $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra if and only if $\Pi\mathfrak{g}$ is a QS-manifold.

It is this approach which is most adequate when defining the double of a Lie bialgebroid. To answer this several solutions were offered, the most encompassing of which were those given by Roytenberg [Roy99] and Mackenzie [Mac98, Mac11]. (These two solutions were shown to be equivalent by Voronov [Vor12]). The advantage of Roytenberg's approach was that it utilised the super language, which simplifies the complicated structures which were necessary in Mackenzie's classical approach via double Lie algebroids. (As is usually the case, for example in number theory, the elementary proofs are often the hardest.)

Given a Lie bialgebroid $\mathcal{A}$, we take as the double the pair of commuting homological vector fields on the cotangent bundle $T^*\Pi A$ defined by the pair of Hamiltonians H_A and S_{A^*} associated to $\mathcal{A}$. The sum of these homological vector fields then provides the manifold $T^*\Pi A$ with Q-structure. In comparison with the construction of the Drinfeld double, this sum provides the Lie algebra structure to the double $\mathfrak{g} \oplus \mathfrak{g}^*$. The cobracket structure obtained from the S-structure however remains undefined in the Courant case, but which is expected to exist. It is in fact the Q-manifold $T^*\Pi A$ together with this unknown structure that should be the correct Drinfeld double of a Lie bialgebroid.

Now let $\mathcal{A}$ be an L_∞-bialgebroid (as defined in definition 1.3.8). It makes good sense to begin to define a possible double in line with Roytenberg's approach for a Lie bialgebroid. Recall that an L_∞-bialgebroid $\mathcal{A}$ defines the QS-manifold ΠA generated by the odd Hamiltonians $H_A, S_{A^*} \in C^\infty(T^*\Pi A)$.

Definition 2.2.1. An L_∞-bialgebroid is *minimal* if both A and A^* are minimal L_∞-algebroids in the sense of definition 1.3.7.

Definition 2.2.2. The *double of a minimal L_∞-bialgebroid* $\mathcal{A}$ is the cotangent bundle $D = T^*\Pi A$ equipped with the commuting homological vector fields

$$Q = (H_A, -), \qquad K = (S_{A^*}, -),$$

defined from the Hamiltonians generating the L_∞-bialgebroid structure, together with some additional unknown structure.

Despite not defining this additional structure, the Q-structure on $T^*\Pi A$ is itself of interest, and must satisfy a non-trivial existence condition. This is the reason for the requirement of minimality. The Drinfeld double of a Lie bialgebra has the property that both the Lie algebra and its dual are Lie subalgebras of the structure. Translating into the language of homological vector fields, we require that when the two homological vector fields defining the structure of the Drinfeld double on $T^*\Pi\mathfrak{g}$ are restricted to the zero sections $\Pi\mathfrak{g}$ and $\Pi\mathfrak{g}^*$ in turn, they are both tangent to the zero sections, and the structure of both Lie algebras is recoverable via derived brackets. For an L_∞-bialgebroid we will require the same condition, that the zero sections ΠA and ΠA^* (viewed as the zero section of $T^*\Pi A^*$ under the canonical double vector bundle isomorphism,) should inherit the homological vector fields defining the L_∞-structures of both A and A^* upon restriction.

Proposition 2.2.1. *For an L_∞-bialgebroid $\mathcal{A}$, the homological vector field $Q_D = Q + K$ on the double D restricts to the vector fields Q_A and Q_{A^*} associated to the L_∞-algebroids A and A^* if and only if both L_∞-algebroids are minimal.*

Proof. Let $H_A, S_{A^*} \in C^\infty(D)$ be the Hamiltonian functions defining the structure of $\mathcal{A}$. In natural coordinates x^a, ξ^i, p_a, π_i,

$$H_A = Q^a(x,\xi)p_a + Q^k(x,\xi)\pi_k, \qquad S_{A^*} = S^a(x,\pi)p_a + \xi^k S_k(x,\pi). \tag{2.12}$$

Then calculating:

$$\begin{aligned} Q_D = &(Q^a + S^a)\frac{\partial}{\partial x^a} + \left(Q^i + \frac{\partial S^a}{\partial \pi_i}p_a + \frac{\partial S_k}{\partial \pi_i}\xi^k\right)\frac{\partial}{\partial \xi^i} \\ &- (-1)^a\left(\frac{\partial Q^b}{\partial x^a}p_b + \frac{\partial Q^i}{\partial x^a}\pi_i + \frac{\partial S^b}{\partial x^a}p_b + \frac{\partial S_k}{\partial x^a}\xi^k\right)\frac{\partial}{\partial p_a} \\ &\qquad + (-1)^i\left(\frac{\partial Q^a}{\partial \xi^i}p_a + \frac{\partial Q^k}{\partial \xi^i}\pi_k + S_i\right)\frac{\partial}{\partial \pi_i}. \end{aligned}$$

On the restriction of Q_D to the zero section ΠA we obtain the tangent vector field

$$Q_D\big|_{\Pi A} = Q^a(x,\xi)\frac{\partial}{\partial x^a} + Q^i(x,\xi)\frac{\partial}{\partial \xi^i} + S^a(x)\frac{\partial}{\partial x^a} + \xi^k S^i_k(x)\frac{\partial}{\partial \xi^i},$$

where we make clear the dependence upon the variables x, ξ. Hence we obtain the vector field Q_A associated to the L_∞-algebroid A if and only if the lowest degree components of S are zero. That is, A^* is a minimal L_∞-bialgebroid. The argument is then symmetrical for A. ◻

Definition 2.2.2 is incomplete for an arbitrary L_∞-bialgebroid, with the unknown structure still to be defined. When the base manifold M is a single point however, i.e. the L_∞-bialgebroid reduces to an L_∞-bialgebra, this structure can be described and presents itself as a classical binary Schouten bracket on the manifold $T^*\Pi A$[1].

Let $(\mathfrak{g}, \mathfrak{g}^*)$ be an L_∞-bialgebra, equivalent to the QS-manifold structure of $\Pi\mathfrak{g}$. The Poisson-commuting Hamiltonian functions $H_\mathfrak{g}, S_{\mathfrak{g}^*} \in C^\infty(T^*\Pi\mathfrak{g})$ define a single homological vector field

$$Q_D \in \mathrm{Vect}(\Pi(\mathfrak{g} \oplus \mathfrak{g})),$$

via the natural identification $T^*\Pi\mathfrak{g} \cong \Pi(\mathfrak{g} \oplus \mathfrak{g}) =: D$. Let ξ^i, ξ_i be natural coordinates on D, in which Q_D takes the form

$$Q_D = \left(Q^i + \frac{\partial S_k}{\partial \xi_i} \xi^k \right) \frac{\partial}{\partial \xi^i} + (-1)^i \left(\frac{\partial Q^k}{\partial \xi^i} \xi_k + S_i \right) \frac{\partial}{\partial \xi_i}. \tag{2.13}$$

Now introduce the iterated cotangent bundle T^*D with conjugate momenta coordinates π_i, π^i. The odd Hamiltonian function $H_D \in C^\infty(T^*D)$ corresponding to Q_D is given locally by

$$H_D = Q^i \pi_i + \frac{\partial S_k}{\partial \xi_i} \xi^k \pi_i + (-1)^i \pi^i S_i + (-1)^i \pi^i \frac{\partial Q^k}{\partial \xi^i} \xi_k. \tag{2.14}$$

The fact that $(\!(H_D, H_D)\!) = 0$, where we use the thickened double bracket $(\!(-,-)\!)$ to denote the canonical even Poisson bracket on T^*D, is a consequence of the nilpotency of Q_D.

There is a naturally defined element $r = (-1)^i \xi^i \xi_i$ in $C^\infty(D)$ (more correctly, it is defined in $C^\infty(D^*)$ where we then map back to $C^\infty(D)$ via the canonical identification), which defines a canonical element $R \in C^\infty(T^*D)$, where $R = (-1)^i \pi^i \pi_i$. Define the odd Hamiltonian function $S_D \in C^\infty(T^*D)$ by

$$S_D := \frac{1}{2} (\!(R, H_D)\!), \quad S_D = (-1)^{i+j} \frac{1}{2} \pi^i \pi^j \frac{\partial^2 Q^k}{\partial \xi^j \partial \xi^i} \xi_k - \frac{1}{2} \pi_i \pi_j \frac{\partial^2 S_k}{\partial \xi_j \partial \xi_i} \xi^k. \tag{2.15}$$

In fact S_D is the image under the canonical isomorphism of double vector bundles of the element r lifted by the Hamiltonian H_D viewed as generating the S-structure corresponding to the L_∞-algebra structure on $\mathfrak{g} \oplus \mathfrak{g}$. (Compare with the Hamiltonian K_Λ defined from the function (1.42) in example 1.3.2. In our setting $R = \kappa^* \Lambda$ and $S_D = K_\Lambda$.)

[1]The existence of this bracket was already known to A. Bruce, who considered the double construction for L_∞-bialgebras, unpublished.

Theorem 2.2.1. *The function S_D defines a binary Schouten bracket on D which is locally given by*

$$\{\xi^i,\xi^j\}_{S_D} = -\frac{\partial^2 S_k}{\partial\xi_i\partial\xi_j}\xi^k \qquad \{\xi^i,\xi_j\}_{S_D}=0 \qquad \{\xi_i,\xi_j\}_{S_D} = \frac{\partial^2 Q^k}{\partial\xi^i\partial\xi^j}\xi_k.$$

This structure is further preserved by the vector field Q_D, i.e., Q_D acts as a derivation over this bracket.

Proof. The local formula follow by direct calculation. What is remarkable is that the cross terms vanish completely from the Hamiltonian S_D. To check that the bracket satisfies the Jacobi identity is to check the master equation $(\!(S_D,S_D)\!) = 0$ for the Hamiltonian. Consider

$$\begin{aligned}(\!(S_D,S_D)\!) &= (\!((\!(R,H_D)\!),(\!(R,H_D)\!))\!)\\ &= (\!((\!((\!(R,H_D)\!),R)\!),H_D)\!) + (\!(R,(\!((\!(R,H_D)\!),H_D)\!))\!).\end{aligned}$$

The last term vanishes by the nilpotency of H_D, and is equivalent to the condition that Q_D be a derivation over the Schouten bracket. Hence we are left with the first term $(\!(H_D,(\!((\!(H_D,R)\!),R)\!))\!)$. The inner term $(\!((\!(H_D,R)\!),R)\!)$ corresponds to the right hand side of the generalised Maurer-Cartan equation $GMC(r) = \sum\{r,\ldots,r\}_{\mathfrak{g}\oplus\mathfrak{g}^*} = 0$ in example 1.3.3, which can be seen by local consideration, expanding both in terms of fibre variables. This is a direct generalisation of the classical situation. In fact r does not satisfy the generalised Maurer-Cartan equation, $GMC(r)\neq 0$, however it is invariant under the flow of Q_D, and so $(\!(H_D,(\!((\!(H_D,R)\!),R)\!))\!) = 0$, and hence $(\!(S_D,S_D)\!) = 0$.

The fact that it is invariant follows from the Jacobi identities for $H_\mathfrak{g}$ and $S_{\mathfrak{g}^*}$. Consider expression (2.15) for S_D, and write $S^{(1)} + S^{(2)} = S_D$ where

$$S^{(1)} = (-1)^{i+j}\frac{1}{2}\pi^i\pi^j\frac{\partial^2 Q^k}{\partial\xi^j\partial\xi^i}\xi_k,$$

and $S^{(2)}$ is defined as the remainder. Then $(\!(S_D,S_D)\!)$ decomposes by $S^{(1)}$ and $S^{(2)}$. Consider

$$(\!(S^{(1)},S^{(1)})\!) = (-1)^{n+m+j}\pi^m\pi^n\frac{\partial^2 Q^i}{\partial\xi^n\partial\xi^m}\pi^j\frac{\partial^2 Q^k}{\partial\xi^j\partial\xi^i}\xi_k.$$

Then an expansion of Q coincides precisely with the Jacobi identity $(H_\mathfrak{g},H_\mathfrak{g})$ with certain variables ξ^i substituted with $(-1)^i\pi^i$, and hence $(\!(S^{(1)},S^{(1)})\!) = 0$ if and only if $(H_\mathfrak{g},H_\mathfrak{g}) = 0$. The result is the same with $(\!(S^{(1)},S^{(2)})\!)$ depending on $(H_\mathfrak{g},S_{\mathfrak{g}^*})$ and $(\!(S^{(2)},S^{(2)})\!)$ depending on $(S_{\mathfrak{g}^*},S_{\mathfrak{g}^*})$. Hence $(\!(S_D,S_D)\!) = 0$ is a result of the QS-manifold structure of $\Pi\mathfrak{g}$. $\square$

Definition 2.2.3. The *double of a minimal L_∞-bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$ is the QS-manifold $D = T^*\Pi\mathfrak{g}$, where the QS-structure is generated by the two Hamiltonian functions $H_D, S_D \in C^\infty(T^*D)$ defined in equations (2.14) and (2.15).

It would be ideal to obtain a classical L_∞-algebra structure from D on the space $\Pi D \cong \mathfrak{g} \oplus \mathfrak{g}^*$ in complete analogy with Drinfeld. However, notice that the Hamiltonian S_D only generates a binary Schouten bracket on D which is in general non-linear in the variables ξ^i, ξ_j. This non-linear Schouten structure cannot hope to correspond to any linear Lie algebra structure as the bracket of eq. (1.28) describes. Hence the double of an L_∞-bialgebra is not an L_∞-bialgebra unless all brackets except the binary brackets vanish, in which case this reduces to the Drinfeld double of a Lie bialgebra.

2.3 Homotopy Courant Algebroids

The double of a Lie bialgebroid $\mathcal{A}$ produces a Courant algebroid structure due to the natural projection $T^*\Pi A \to \Pi(A \oplus A^*)$ and the derived bracket type formula (2.11). This remains true for the homotopy case, and by direct analogy we can endow the direct sum $A \oplus A^*$ with a sequence of higher brackets and anchors, producing an example of what will be a homotopy Courant algebroid.

2.3.1 The Higher Dorfman Brackets

Given an L_∞-algebroid $\mathcal{A}$ with associated Hamiltonian functions $H_A, S_{A^*} \in C^\infty(T^*\Pi A)$, the symplectomorphism (2.5) allows us to define the pull-backs

$$h := \varphi^*_{\mathcal{E}_\nabla} H_A, \qquad s := \varphi^*_{\mathcal{E}_\nabla} S_{A^*}, \tag{2.16}$$

to the manifold $\mathcal{E}_\nabla$, which satisfy $[h+s, h+s]_\nabla = 0$ since $\varphi_{\mathcal{E}_\nabla}$ preserves the Poisson brackets. Define the sum of these functions as $\vartheta = h + s$.

The graded algebra of functions $C^\infty(\mathcal{E}_\nabla)$ can be equipped with the structure of a homotopy Loday algebra as introduced in the work [Uch11], which endows $\mathcal{E}_\nabla$ with a sequence of higher operations satisfying a Loday-Jacobi identity. Define the kth higher Loday bracket Φ^k_ϑ as

$$\Phi^k_\vartheta(f_1, \dots, f_k) := (-1)^{(k-1)f_1 + \dots + f_{k-1}} \big[\cdots [\vartheta, f_1]_\nabla, \dots, f_k \big]_\nabla, \tag{2.17}$$

for functions $f_1, \dots, f_k \in C^\infty(\mathcal{E}_\nabla)$. These operations satisfy higher homotopy Jacobi identities written in a non-skew-symmetric form if and only if $[\vartheta, \vartheta]_\nabla = 0$; automatically a consequence of the L_∞-bialgebroid structure. We refer to [Uch11] for the details of this construction. An important observation to make is that the Loday brackets $\Phi^k_\vartheta, k \geq 1$, are built from graded operations, and so each bracket inherits the natural ϵ-grading from $\mathcal{E}_\nabla$. As such, the higher Jacobi identities satisfied by the brackets also decompose by the ϵ-grading of the manifold $\mathcal{E}_\nabla$.

This homotopy Loday structure allows the definition of a sequence of higher Dorfman brackets and anchors on the space of sections of the vector bundle $E = A \oplus A^*$. Recall that for a section $u \in \Gamma(E)$, the function $\bar{u} \in C^\infty(\mathcal{E}_\nabla)$ is the corresponding odd degree 1 function on $\mathcal{E}_\nabla$. Then the kth Dorfman bracket on $\Gamma(E)$ is defined by the nested sequence

$$\overline{[u_1, \dots, u_k]_\vartheta} = \Phi^k_\vartheta(\bar{u}_1, \dots, \bar{u}_k)\big|_{\epsilon=1}, \tag{2.18}$$

and similarly, the kth higher anchor as

$$\rho_k(u_1, \dots, u_{k-1}|f) = \Phi^k_\vartheta(\bar{u}_1, \dots, \bar{u}_{k-1}, f)\big|_{\epsilon=0}, \tag{2.19}$$

for sections $u_1, \dots, u_k \in \Gamma(A \oplus A^*)$, and a function $f \in C^\infty(M)$ viewed as a degree zero function on $\mathcal{E}_\nabla$. The notation $\big|_{\epsilon=1}$ means to restrict to the ϵ-degree 1 terms and so on. That these restrictions to the individual graded components actually specify higher bracket operations is a consequence of the observation that the Jacobi identities also decompose with respect to the ϵ-grading, and hence each graded piece of each operation satisfies a Jacobi identity.

The structure of an L_∞-bialgebroid $\mathcal{A}$ allows for a description of these higher Dorfman brackets by adapting the operator representation of a Courant algebroid introduced in [KS05]. For the functions $h, s \in C^\infty(\mathcal{E}_\nabla)$, introduce the two operators

$$\hat{h} = [h, -]_\nabla, \qquad \hat{s} = [s, -]_\nabla,$$

in $\mathrm{End}(C^\infty(\mathcal{E}_\nabla))$, together with the insertion and multiplication operators

$$\hat{\imath_X} = [\bar{X}, -]_\nabla, \quad m_X = \bar{X}, \quad \hat{\imath_\eta} = [\bar{\eta}, -]_\nabla, \quad m_\eta = \bar{\eta},$$

for sections $X \in \Gamma(A)$ and $\eta \in \Gamma(A^*)$. Cartan's formula then defines the Lie derivative operators by the commutators

$$\hat{\mathcal{L}}_X = [\hat{h}, \hat{\imath_X}], \qquad \hat{\mathcal{L}}^*_\eta = [\hat{s}, \hat{\imath_\eta}].$$

Notice that in general these Lie derivative operators are of inhomogeneous non-negative degree, due to the inhomogeneity of the functions h and s. It is easy to see that when the functions h and s define a Lie bialgebroid then these coincide with the usual Lie derivatives defined in [KS05]. Manipulating the Jacobi identity for the commutator of endomorphisms shows that these operators satisfy the following relations:

$$[\hat{\imath_X}, m_\eta] = m_{\imath_X \eta}, \qquad [\hat{h}, m_\eta] = m_{d\eta}, \qquad [\hat{\mathcal{L}}_X, m_\eta] = m_{\mathcal{L}_X \eta},$$
$$[\hat{s}, m_X] = m_{d_* X}, \qquad [\hat{\mathcal{L}}^*_\eta, m_X] = m_{\mathcal{L}^*_\eta X}.$$

We use hat-operators $\hat{h}, \hat{\imath_X}, \hat{\mathcal{L}}_X$ to denote endomorphisms of $C^\infty(\mathcal{E}_\nabla)$ and non-hat operators $d, \imath_X, \mathcal{L}_X$ to denote operations on sections of the L_∞-algebroids. The algebroid differentials d and d_* are also inhomogeneous of arbitrary positive degree, for which we will write d^k for the component of d with degree k. In particular, $d = d^0 + d^1 + \cdots$, where d^1 corresponds to the differential of a Lie algebroid.

Proposition 2.3.1. *Given an L_∞-bialgebroid $\mathcal{A}$, the following formula specify a sequence of higher Dorfman brackets $[\cdots]_\vartheta$ on the vector bundle $E = A \oplus A^*$: for sections $X_i, Y_j \in \Gamma(A)$ and $\eta_i, \tau_j \in \Gamma(A^*)$,*

$$[X]_\vartheta = [X]_A + d^0_* X, \qquad [\eta]_\vartheta = [\eta]_{A^*} + d^0 \eta,$$
$$[X, \eta]_\vartheta = \mathcal{L}_X \eta - \imath_\eta d^1_* X, \quad [\eta, X]_\vartheta = \mathcal{L}^*_\eta X - \imath_X d^1 \eta,$$

and for brackets of 3 or more arguments, where $l = i + j$,

$$[X_1, \ldots, X_i]_\vartheta = [X_1, \ldots, X_i]_A, \qquad [\eta_1, \ldots, \eta_j]_\vartheta = [\eta_1, \ldots, \eta_j]_{A^*},$$
$$\begin{aligned}[X_1, \ldots, X_i, \eta, Y_1, \ldots, Y_j]_\vartheta &= (-1)^{j + \frac{l(l-1)}{2}} \imath_{Y_j} \cdots \imath_{Y_1} \imath_{X_i} \cdots \imath_{X_1} d^l \eta \\ &\quad - (-1)^{j + \frac{l(l-1)}{2}} \sum_{m=1}^{i} (-1)^m \imath_{Y_j} \cdots \imath_{Y_1} \imath_{X_i} \cdots \widehat{\imath_{X_m}} \cdots \imath_{X_1} d^l \langle X_m, \eta \rangle, \\ [\eta_1, \ldots, \eta_i, X, \tau_1, \ldots, \tau_j]_\vartheta &= (-1)^{j + \frac{l(l-1)}{2}} \imath_{\tau_j} \cdots \imath_{\tau_1} \imath_{\eta_i} \cdots \imath_{\eta_1} d^l_* X \\ &\quad - (-1)^{j + \frac{l(l-1)}{2}} \sum_{m=1}^{i} (-1)^m \imath_{\tau_j} \cdots \imath_{\tau_1} \imath_{\eta_i} \cdots \widehat{\imath_{\eta_m}} \cdots \imath_{\eta_1} d^l_* \langle \eta_m, X \rangle,\end{aligned}$$

where $\hat{}$ denotes omission, and all other brackets vanish.

In particular, notice that when the functions h and s define a Lie bialgebroid we recover the well-known binary Dorfman bracket

$$[X + \eta, Y + \tau]_\vartheta = [X, Y]_A + \mathcal{L}^A_X \tau - \imath_Y d_A \eta + [\eta, \tau]_{A^*} + \mathcal{L}^{A^*}_\eta Y - \imath_\tau d_{A^*} X.$$

Proof. First notice that the bi-gradings carried by the hamiltonian functions h and s prohibit brackets of two or more mixed arguments: for example,

$$[[[[h_{(k,1)}, \bar{X}]_\nabla, \bar{Y}]_\nabla, \bar{\eta}]_\nabla, \bar{\tau}]_\nabla \equiv 0$$

since the term is of bi-degree $(k-2,-1)$. Hence only the expressions in the proposition remain. The unary brackets follow straightforwardly from the investigation of minimality, and the binary brackets coincide with the classical Dorfman bracket by comparison of degree.

Consider then the expression $[X_1, \ldots, X_i, \eta, Y_1, \ldots, Y_j]_\vartheta$ for $l = i + j \geq 1$. Manipulations of the Jacobi identity for the Loday brackets give the following expression

$$\begin{aligned}\Phi_\vartheta^{l+1}(\bar{X}_1, \ldots, \bar{X}_i, \bar{\eta}, \bar{Y}_1, \ldots, \bar{Y}_j) &= (-1)^i \big[\cdots [h, \bar{\eta}], \bar{X}_1], \ldots, \bar{Y}_j\big] \\ &\quad - \sum_{m=1}^{i} (-1)^{i-m} \big[\cdots [h, [\bar{X}_m, \bar{\eta}]], \bar{X}_1, \ldots, \bar{Y}_j\big] \\ = (-1)^{j+\frac{l(l-1)}{2}} \Big(\big[\bar{Y}_j, \ldots, [\bar{X}_1, [h, \bar{\eta}]] \cdots \big] &- \sum_{m=1}^{i} (-1)^m \big[\bar{Y}_j, \ldots, [\bar{X}_1, [h, [\bar{X}_m, \bar{\eta}]]] \cdots \big] \Big).\end{aligned}$$

We can then identify the terms in the bracket with the corresponding operators. The term involving multiple 1-forms and a single vector field is symmetric (as in the binary case). □

Example 2.3.1. The motivating example for considering such double constructions appears when M is a homotopy Poisson manifold, and the L_∞-bialgebroid in question is the cotangent L_∞-bialgebroid (TM, T^*M) of example 1.3.2.

Let $[\cdots]_{K_P}$ be the sequence of higher Koszul brackets providing T^*M with the structure of an L_∞-algebroid. Then the higher Dorfman brackets on the vector bundle $TM \oplus T^*M$ are described by proposition 2.3.1. For vector fields $X_1, \ldots, X_k \in \Gamma(TM)$ and differential 1-forms $\eta_1, \ldots, \eta_k \in \Gamma(T^*M)$,

$$[X + \eta]_\vartheta = [\eta]_{K_P},$$

$$[X_1 + \eta_1, X_2 + \eta_2]_\vartheta = [X_1, X_2] + \mathcal{L}_{X_1}\eta_2 - \imath_{X_2} d\eta_1 + [\eta_1, \eta_2]_{K_P} + \mathcal{L}^*_{\eta_1} X_2 - \imath_{\eta_2} d^1_P X_1,$$

and for $k \geq 3$,

$$[X_1 + \eta_1, \ldots, X_k + \eta_k]_\vartheta = [\eta_1, \ldots, \eta_k]_{K_P} + \sum_{i=1}^{k} [\eta_1, \ldots, \eta_{i-1}, X_i, \eta_{i+1}, \ldots, \eta_k]_\vartheta,$$

where, for the differential d_P of multivector fields,

$$[\eta_1, \ldots, \eta_{i-1}, X_i, \eta_{i+1}, \ldots, \eta_k]_\vartheta = (-1)^{i+\frac{(k-2)(k+1)}{2}} \imath_{\eta_k} \cdots \imath_{\eta_{i+1}} \imath_{\eta_{i-1}} \cdots \imath_{\eta_1} d_P^l X$$
$$-(-1)^{i+\frac{(k-2)(k+1)}{2}} \sum_{m=1}^{i-1} (-1)^m \imath_{\eta_k} \cdots \imath_{\eta_{i+1}} \imath_{\eta_{i-1}} \cdots \widehat{\imath_{\eta_m}} \cdots \imath_{\eta_1} d_P^l \langle \eta_m, X_i \rangle.$$

These can be seen as higher Dorfman operations on $TM \oplus T^*M$, the symmetrisation of which will give rise to higher homotopy Courant brackets.

Notice that to define the higher Dorfman brackets we do not require the homotopy Poisson structure to be minimal, since we do not in fact need to start from the double of the L_∞-bialgebroid. It is simply the manifold $\mathcal{E}_\nabla$ equipped with a Poisson-commuting function of arbitrary degree which is needed to describe the Loday structure on $\mathcal{E}_\nabla$ and the Dorfman brackets on the direct sum.

2.3.2 Homotopy Courant Algebroids

As the doubles of Lie bialgebroids define a subset of Courant algebroids, the doubles of L_∞-bialgebroids provide examples of a Courant algebroid's homotopy analogue. This homotopy Courant algebroid is a pseudo-Euclidean vector bundle E equipped with a sequence of higher Dorfman brackets and anchors, the structure of which is defined analogously to the structure of a Courant algebroid via the symplectic realisation $\mathcal{M}_\nabla$. Following [Uch11], we begin by defining a homotopy Loday structure on $C^\infty(\mathcal{M}_\nabla)$: let the kth Loday bracket be given by the nested sequence

$$\Phi_\Theta^k(f_1, \ldots, f_k) := (-1)^{(k-1)f_1+\cdots+f_{k-1}} \big[\cdots [\Theta, f_1]_\nabla, \ldots, f_k \big]_\nabla, \tag{2.20}$$

for functions $f_1, \ldots, f_k \in C^\infty(\mathcal{M}_\nabla)$, and $\Theta \in C^\infty(\mathcal{M}_\nabla)$ such that $[\Theta, \Theta]_\nabla = 0$. The kth anchor and kth Dorfman bracket may then be defined on the space of sections $\Gamma(E)$ by the derived brackets,

$$\rho(u_1, \ldots, u_{k-1}|f) = \Phi_\Theta^k(\bar{u}_1, \ldots, \bar{u}_{k-1}, f)\big|_{\epsilon=0}, \tag{2.21}$$

$$\overline{[u_1, \ldots, u_k]_\Theta} = \Phi_\Theta^k(\bar{u}_1, \ldots, \bar{u}_k)\big|_{\epsilon=1}, \tag{2.22}$$

where we recall that $\bar{u} \in C^\infty(\mathcal{M}_\nabla)$ is the degree 1 function corresponding to the section $u \in \Gamma(E)$.

A priori there are no additional restrictions on the Loday master function Θ, which is only required to satisfy the master equation $[\Theta, \Theta]_\nabla = 0$ to define the necessary structure. However, notice that an arbitrary function Θ will possess nonlinear terms in the fibre variables of the fibration $\mathcal{M} \to \Pi E$, corresponding to a nonlinear dependence on the base momenta q_a in the manifold $\mathcal{M}_\nabla = T^*M \oplus \Pi E$. Terms linear in q_a govern the anchor maps $\rho_k : \Gamma(E)^{k-1} \to TM$, whilst these nonlinear additions correspond to "higher multi-anchors", maps $\rho_k^l : \Gamma(E)^{\times k-1} \to S^l\Gamma(TM)$ into the lth symmetric power of the tangent bundle. In order to keep our construction "close enough" to the homotopy Courant algebroid structure acquired from an L_∞-bialgebroid, we will assume that Θ is at most linear in the fibration $\mathcal{M} \to \Pi E$, and hence takes the local form

$$\Theta = \Theta^a(x, \xi)q_a + \Theta_0(x, \xi).$$

The homotopy Loday structure constructed in [Uch11] however incorporates this option for a nonlinear Θ, and it would be interesting to see what kind of structure is present if we remove this assumption.

Remark 2.3.1. In general Θ should be allowed arbitrary positive degree, where the degree 3 component contributes to a genuine Courant algebroid structure. The double of an L_∞-bialgebroid has no terms in Θ of degree less than 3, since the minimality assumptions kills all terms of degree 2 and the assumption that our L_∞-algebroids are strict zeros terms of degree 1. Though a homotopy Courant algebroid may well possess a unary bracket and anchor deriving from a degree 2 term, it is unnatural to allow degree 1 components in Θ which would correspond to a distinguished section of E, hence we will always assume that $\epsilon(\Theta) \geq 2$.

Proposition 2.3.2. *Given a pseudo-Euclidean vector bundle E with metric $\langle -, - \rangle$, the higher Dorfman brackets and anchors defined from a suitable function $\Theta \in C^\infty(\mathcal{M}_\nabla)$ satisfy the following relations:*

$$\rho_k(u_1, \ldots, u_{k-1}|\langle v, w\rangle)$$
$$= \langle [u_1, \ldots, u_{k-1}, v]_k, w\rangle + (-1)^{v(u_1+\cdots u_{k-1}+k)}\langle v, [u_1, \ldots, u_{k-1}, w]_k\rangle;$$

$$(-1)^{(u_i+u_{i+1})(u_{i+2}+\cdots+u_{k+1})}\rho_k(u_1, \ldots, \hat{u}_i, \hat{u}_{i+1}, \ldots, u_{k+1}|\langle u_i, u_{i+1}\rangle)$$
$$= \langle [u_1, \ldots, u_i, u_{i+1}, \ldots, u_k] + (-1)^{u_iu_{i+1}}[u_1, \ldots, u_{i+1}, u_i, \ldots, u_k], u_{k+1}\rangle,$$

for sections $u_1, \ldots, u_k, v, w \in \Gamma(E)$, and where we denote the omission of a section u_i by $\hat{u}_i$.

Proof. Either side of the expressions may be represented in terms of the Loday brackets on $C^\infty(\mathcal{M}_\nabla)$, for example:

$$\begin{aligned} \rho(u|\langle v, w\rangle) &= (-1)^v \left[[\Theta, \bar{u}]_\nabla, [\bar{v}, \bar{w}]_\nabla\right]_\nabla \\ &= (-1)^v \left[[[\Theta, \bar{u}]_\nabla, \bar{v}]_\nabla, \bar{w}\right]_\nabla + (-1)^{v+u(v+1)} \left[\bar{v}, [[\Theta, \bar{u}]_\nabla, \bar{w}]_\nabla\right]_\nabla \\ &= \langle [u, v], w\rangle + (-1)^{uv}\langle v, [u, w]\rangle, \end{aligned} \tag{2.23}$$

for sections $u, v, w \in \Gamma(E)$. The proof then is directly analogous to this calculation, keeping track of signs and the nested commutators. $\square$

Proposition 2.3.2 then allows us to state the definition of a homotopy Courant algebroid.

Definition 2.3.1. A *homotopy Courant algebroid* is a pseudo-Euclidean vector bundle E equipped with a sequence of higher Dorfman brackets $[\cdots]$ forming a homotopy Loday algebra $\Gamma(E)$, together with higher anchors $\rho_k : E^k \to TM$ assembling into a morphism of Loday structures, all of which satisfy the relations contained in proposition 2.3.2.

Note that there do exist notions of higher Courant algebroids already in the literature, [Zam12] for example develops a Courant bracket on the vector bundle $TM \oplus \wedge T^*M$, yet such objects do not provide an adequate setting for homotopy structures. Higher Courant and Dorfman brackets have also been considered independently from Courant algebroids in a selection of works [KW15, Bor15, Ban15, Ber07], most of which concern the construction of non-abelian higher derived brackets. These works however offer no interpretation of these higher brackets in an algebroid setting.

Observe that by construction the definition of a homotopy Courant algebroid is equivalent to the existence of a suitable master function Θ on the minimal symplectic realisation of the Poisson manifold ΠE, in parallel with [Roy02]. When Θ is homogeneous of degree 3, it is clear that we recover the usual Courant algebroid structure.

Example 2.3.2. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be an L_∞-bialgebra. Then $\mathfrak{g} \oplus \mathfrak{g}^*$ inherits the structure of a homotopy Courant algebroid with trivial anchor maps which is defined from the

Q-structure on the double $\Pi(\mathfrak{g}\oplus\mathfrak{g}^*)$ eq. (2.13). In this case the Dorfman brackets are completely skew-symmetric, coinciding with the Lie brackets on $\mathfrak{g}\oplus\mathfrak{g}^*$.

Example 2.3.3. Similarly, let (A, A^*) be an L_∞-bialgebroid. Analogous to example 2.3.2, the bundle $E = A\oplus A^*$ is a homotopy Courant algebroid with the structure generated by $h+s$ in eq. (2.16). For an arbitrary L_∞-bialgebroid we will obtain non-trivial anchor maps in general, and the higher Dorfman brackets are given by proposition 2.3.1.

Example 2.3.4. Consider an L_∞-algebra $\mathfrak{g}$ acting on a manifold M as in the work [MZ12]. This infinitesimal action is described by the L_∞-morphism $\mathfrak{g}\times M \rightsquigarrow TM$ which stands as the anchor morphism of the L_∞-action algebroid $\mathfrak{g}\times M\to M$ (compare with example 1.2.4).

The L_∞-action algebroid is defined by a homological vector field on $\Pi\mathfrak{g}\times M$,

$$Q_{\mathfrak{g}\times M} = X + Q_{\mathfrak{g}}, \qquad Q_{\mathfrak{g}\times M} = X^a(x,\xi)\frac{\partial}{\partial x^a} + Q^i(\xi)\frac{\partial}{\partial \xi^i},$$

where $Q_{\mathfrak{g}}$ defines the L_∞-algebra structure on $\mathfrak{g}$, and X is an odd vector field on M taking formal values in $\Pi\mathfrak{g}$. The anchor of the algebroid is governed by X which specifies a curved L_∞-morphism $\mathfrak{g}\rightsquigarrow \mathrm{Vect}(M)$ via the maps $\phi^k_X : \wedge^k\mathfrak{g}\to\mathrm{Vect}(M)$, defined by

$$\phi^k_X(e_1\wedge\ldots\wedge e_k) = [\cdots[X,\imath_{e_1}],\ldots,\imath_{e_k}](0)$$

for a basis $\{e_i\}$ of $\mathfrak{g}$, analogous to the derived bracket formula in eq. (1.41). That X defines such a morphism is equivalent to the homological property of $Q_{\mathfrak{g}\times M}$. See [MZ12] for details and examples of these L_∞-algebra actions.

Let the bundle $\mathfrak{g}^*\times M$ be given the trivial L_∞-structure. Then $T^*(\Pi\mathfrak{g}\times M)\cong T^*M\times\Pi(\mathfrak{g}\times\mathfrak{g}^*)$ equipped with the Poisson-nilpotent function

$$\Theta = Q^i(\xi)\xi_i + X^a(x,\xi)p_a,$$

for fibre coordinates p_a, ξ_i on $T^*(\Pi\mathfrak{g}\times M)$, defines a homotopy Courant algebroid structure. The higher anchors are controlled by the vector field X, whilst the brackets are specified by the L_∞-algebra structure on $\mathfrak{g}$.

Example 2.3.5. Let $\mathcal{A}$ be a proto-bialgebroid in the sense of Kosmann-Schwarzbach [KS05]; a Lie bialgebroid $\mathcal{A}$ equipped with a trivector X and 3-form ϕ acting as background. The proto-bialgebroid structure is encoded in the master equation of the degree 3 master function $\Theta = \bar{X} + H_A + S_{A^*} + \bar{\phi} \in C^\infty(T^*\Pi A)$. It was shown in [KS05] that the operator method used in proposition 2.3.1 can extend the Dorfman bracket associated to a Lie bialgebroid to incorporate these background elements.

The concept of a proto-bialgebroid gives rise to a proto-L_∞-bialgebroid, which is described by an L_∞-bialgebroid $\mathcal{A}$ equipped with a sequence of multivector fields and differential forms such that the degree k component of Θ carries a degree k multivector field and k-form background. The brackets in proposition 2.3.1 can then be adjusted to accommodate these background fields, providing examples of homotopy Courant algebroids.

Homotopy Courant algebroids should in fact be examples of the more general homotopy Loday algebroids in parallel with Courant algebroids and Loday algebroids. There are varied definitions of a Loday algebroid found in the literature. The definition we shall employ simply relaxes the symmetry of the bracket in a Lie algebroid. In this case a Loday algebroid is to a Loday algebra as a Lie algebroid is to a Lie algebra.

Definition 2.3.2. A *Loday algebroid* is a vector bundle $E \to M$ together with a Loday bracket on the space of sections, an anchor map $\rho : E \to TM$, and a vector bundle morphism

$$\alpha : E \to TM \otimes \mathrm{End}(E), \tag{2.24}$$

such that the right and left Leibniz rules hold:

$$[u, fv] = (-1)^{fu} f[u, v] + \rho(u|f)v;$$
$$[fu, v] = f[u, v] - (-1)^{v(f+u)}\rho(v|f)u + \alpha(v)(df \otimes u),$$

for a function $f \in C^\infty(M)$ and sections $u, v \in \Gamma(E)$.

The obvious difference between a Loday algebroid and a Lie algebroid is the lack of skew-symmetry which forces us to consider $C^\infty(M)$-linearity in the left bracket argument as well as in the right. We could equally as well require no such linearity, though the structure of a Courant algebroid automatically generates the map (2.24) providing us with the left Leibniz rule, which therefore makes it a natural condition to require.

Remark 2.3.2. The definition of a Loday algebroid as given in [SX08] requires the vector bundle E to come equipped with a pseudo-metric which forces the map α to take a particular form. Though this more closely resembles a Courant algebroid, there seems no reason to ask for such specific conditions.

As mentioned, the structure of a Courant algebroid generates automatically the left Leibniz rule. Define a map $D : C^\infty(M) \to \Gamma(E)$ by the formula $\langle Df, u\rangle = (-1)^{uf}\rho(u|f)$, which satisfies the relations

$$D\langle u, u\rangle = 2[u, u], \qquad \langle D\langle u, u\rangle, v\rangle = 2\langle \mathrm{ad}_v u, u\rangle.$$

The map D will be referred to as the defect map, since these formula show that D measures the defect in the symmetry of the Dorfman bracket. The defect map was originally introduced in [LWX97] to measure the discrepancy in the Jacobi identity for the Courant bracket, the skew-symmetrisation of the Dorfman bracket. In terms of derived brackets the map D is given by

$$\overline{Df} = \Phi^1_\Theta(f)\big|_{\epsilon=1}.$$

All the above relations are immediately deduced from the Jacobi identity via the derived bracket formula. In particular, we see that the map (2.24) for a Courant algebroid can be defined by

$$\alpha(v)(df \otimes u) = Df\langle v, u\rangle, \tag{2.25}$$

for sections $u, v \in \Gamma(E)$, which provides us with the left Leibniz relation for the Dorfman bracket,

$$[fu, v] = f[u, v] - (-1)^{v(u+f)}\rho(v|f)u + Df\langle u, v\rangle.$$

The structure of a homotopy Courant algebroid allows the definition of an infinite sequence of defect maps D_k, together with "higher α-maps" defining the k-Leibniz rules for each of the k-brackets.

From proposition 2.3.2 the higher defect maps $D_k : \Gamma(E)^{\times k-2} \times C^\infty(M) \to \Gamma(E)$ may be defined by the homotopy Loday brackets

$$\overline{D_k(u_1, \ldots, u_{k-2}; f)} = (-1)^{(k-2)u_1+\cdots u_{k-3}}\Phi^k_\Theta(\bar{u}_1, \ldots, \bar{u}_{k-2}, f)\big|_{\epsilon=1}, \tag{2.26}$$

which satisfy the relation

$$\langle D_k(u_1,\dots,u_{k-2};f),v\rangle = (-1)^{fv}\rho_k(u_1,\dots,u_{k-2},v|f), \tag{2.27}$$

as an easy consequence of the Jacobi identity. Notice that these higher defect maps measure the symmetry of the higher Dorfman brackets by proposition 2.3.2.

Proposition 2.3.3. *The defect map $D_k : \Gamma(E)^{\times k-2}\times C^\infty(M)\to\Gamma(E)$ defined by formula* (2.26) *is anti-symmetric in the arguments $u_1,\dots,u_{k-2}$, and acts as a derivation of $C^\infty(M)$ in the final argument.*

Proof. This is evident from the definition in formula (2.26) or from formula (2.27). □

One can then observe how these maps appear when considering linearity in each of the arguments of the higher Dorfman brackets. Suppose that E is a homotopy Courant algebroid equipped with a 3-bracket $[-,-,-]_\Theta$. For sections $u,v,w\in\Gamma(E)$, the following formula hold:

$$\begin{aligned}
[u,v,fw]_\Theta &= (-1)^{f(u+v+1)}f[u,v,w]_\Theta - \rho_3(u,v|f)w;\\
[u,fv,w]_\Theta &= (-1)^{f(u+1)}f[u,v,w]_\Theta + (-1)^{w(f+v)}\rho_3(u,w|f)v + D_3(u;f)\langle v,w\rangle;\\
[fu,v,w]_\Theta &= (-1)^{f}f[u,v,w]_\Theta - (-1)^{(f+u)(v+w)}\rho_3(v,w|f)u\\
&\qquad + (-1)^{v(u+f)}D_3(v;f)\langle u,w\rangle - (-1)^{w(u+v+f)}D_3(w;f)\langle u,v\rangle.
\end{aligned}$$

These formula generalise immediately to any higher bracket.

Proposition 2.3.4. *Let E be a homotopy Courant algebroid. The kth Dorfman bracket satisfies the following formula considering linearity over $C^\infty(M)$ in the ith position:*

$$\begin{aligned}
[u_1,\dots,fu_i,\dots,u_k] &= (-1)^{f(u_1+\cdots+u_{i-1}+k)}f[u_1,\dots,u_i,\dots,u_k]\\
&\quad + (-1)^{i+(u_i+f)(u_{i+1}+\cdots+u_k)}\rho(u_1,\dots,\hat{u}_i,\dots,u_k|f)u_i\\
&\quad - \sum_{j=i+1}^{k}(-1)^{j-i+\delta}D_k(u_1,\dots,\hat{u}_i,\dots,\hat{u}_j,\dots,u_k;f)\langle u_i,u_j\rangle,
\end{aligned} \tag{2.28}$$

where $\delta=(f+u_i)(u_{i+1}+\cdots+\hat{u}_j+\cdots+u_k)+u_j(u_{j+1}+\cdots+u_k)$, and $\hat{u}_i$ denotes omission of the section u_i.

Proof. The formula follows from repeated applications of the Jacobi identity and the Leibniz rule for the even Poisson bracket $[-,-]_\nabla$ on $\mathcal{M}_\nabla$. A useful observation in manipulating these formula is that

$$\Phi^k_\Theta(\bar{u}_1, \ldots, f, \bar{u}_j, \ldots, \bar{u}_k) = (-1)^{f(u_j+\cdots+u_k)}\Phi^k_\Theta(\bar{u}_1, \ldots, \bar{u}_k, f), \tag{2.29}$$

since the bracket of a function $f \in C^\infty(M)$ with any section $\bar{u}$ is zero. □

Definition 2.3.3. For a homotopy Courant algebroid E over a bosonic manifold M, define for each kth Dorfman bracket, the finite sequence of maps

$$\alpha^k_i : \underbrace{E \times \ldots \times E}_{k-1\ times} \to TM \otimes \mathrm{End}(E), \qquad i = 1, \ldots, k, \tag{2.30}$$

such that

$$\begin{aligned}[u_1, \ldots, fu_i, \ldots, u_k]_\Theta &= f[u_1, \ldots, u_i, \ldots, u_k]_\Theta \\ &\quad + (-1)^i \rho_k(u_1, \ldots, \hat{u}_i, \ldots, u_k|f)u_i + \alpha^k_i(u_1, \ldots, \hat{u}_i, \ldots, u_k)[df \otimes u_i].\end{aligned}$$

The maps are defined for bosonic manifolds simply to reduce the signs present in the formula. These can be inserted easily as necessary.

Proposition 2.3.5. *For a bosonic manifold M, the map $\alpha^k_i : E^{\times k-1} \to TM \otimes \mathrm{End}(E)$ is given by the following:*

$$\alpha^k_i(u_1, \ldots, u_{k-1})(df \otimes v) = -\sum_{j=i+1}^{k} (-1)^{j-i} D_k(u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_k; f)\langle u_i, u_j\rangle,$$

as the anti-symmetrisation of the defect maps D_k around the final i arguments.

Remark 2.3.3. If we relax the condition that Θ be linear in the fibration $\mathcal{M} \to \Pi E$ then these expressions become considerably more complicated due to the presence of the higher multi-anchors $\rho^l_k : \Gamma(E^{k-1}) \to S^l\Gamma(TM)$. These multi-anchors appear in the expressions as a consequence of the modification of observation (2.29) by additional terms.

2.4 Discussion

Through investigating the double objects to L_∞-bialgebroids, we have defined the Drinfeld double for an L_∞-bialgebra, and provided an explicit expression for the coboundary structure on the double. The double of an L_∞-bialgebroid should be similarly defined, combining the approaches of L_∞-bialgebras and Lie bialgebroids to obtain the Q-structure already defined in definition 2.2.2, together with the still undiscovered compatible S-structure which is expected to exist.

These doubles also lead to the introduction of homotopy Courant algebroids. We defined a sequence of higher Dorfman brackets and anchors compromising a homotopy Loday algebra structure on a pseudo-Euclidean vector bundle, and have given explicit formula for these when the homotopy Courant algebroid defined arises from an L_∞-bialgebroid structure. It would be interesting to see what kind of classical structures transfer to this homotopy setting. Partial progress has been made here, for instance we can say that there is no notion of a co-anchor for a homotopy Courant algebroid, obscuring what exact homotopy Courant algebroids should be. The notion of a Dirac structure however carries over, and null Dirac structures are possible to define and examples to be found. The non-linearity present in the homotopy case however raises issues with the isotropy of Dirac structures defined from Hamiltonian operators, a problem which does not seem to be resolvable. It is this non-linearity which is the root of many of the problems when working with these homotopy Courant algebroids.

Chapter 3

Weak Poisson Systems and an Extension to Differential Forms

The authors of [LS05] introduced a geometrical construction which may be adapted as necessary in order to describe an arbitrary gauge field theory, possibly one which is not Hamiltonian or Lagrangian. This construction, called a weak Poisson system, defines on the leaf space of a possibly singular foliation "smooth" contravariant tensor fields together with an even Poisson bracket.

In this chapter we review this set-up and show that it admits an extension to covariant tensor fields. In so doing, the weak Poisson bracket is lifted to a weak Koszul-type bracket via the approach detailed in [KV08] and example 1.2.11, which extends the generalised Hamiltonian mechanics proposed in [Mic85] to any gauge field theory. The majority of the material in this chapter appears in the article [LPS17]. Throughout this chapter we will assume that M is a usual (non-super) manifold simply to reduce the signs in the formula, however all the results follow for any supermanifold by inserting the signs as necessary.

3.1 Weak Poisson Systems

Let M be a usual (bosonic) manifold with local coordinates x^a, and let $\Sigma \subset M$ be a smooth submanifold of codimension $k < \dim M$ specified by a system of equations $T^i(x) = 0$, for $i = 1, \ldots, k$. Further, suppose that R_α, for $\alpha = 1, \ldots, n$, are a set of vector fields on M tangent to Σ, satisfying the integrability condition

$$[R_\alpha, R_\beta] = f^\gamma_{\alpha\beta} R_\gamma + T^i X_{i\alpha\beta}, \tag{3.1}$$

for smooth functions $f^\gamma_{\alpha\beta}$ and vector fields $X_{i\alpha\beta}$. The vector fields R_α define an integrable distribution over the submanifold Σ, but which is not necessarily integrable over the whole of M. In order words, the vector fields form an open Lie algebra over Σ.

Remark 3.1.1. For simplicity we will assume that the vector fields R_α are global, linearly independent vector fields on M which are tangent to Σ. These assumptions are not necessary however, and the construction needs only to be modified to incorporate local vector fields which are functionally dependent.

The submanifold Σ is foliated by the integrable submanifolds of the distribution, where we use N to represent the space of leaves of this foliation. In general N is far from being a smooth manifold, though the independence of the vector fields forces the foliation to be regular, this does not guarantee that N should even be Hausdorff. Despite this, the concept of "smooth" tensor fields on N can be introduced as those tensors on M which are constant along the leaves of the foliation. These tensors can be seen to "descend" to the space of leaves and in fact hold physical importance.

A gauge field theory in the Hamiltonian formalism consists of a phase space with a possible system of equations defining a constraint surface, together with a set of gauge transformations; vector fields on the constraint surface defining an integrable distribution. Those functions which are invariant under gauge transformation, i.e. constant along the gauge orbits, define the physical observables of the system. The phase space may be identified with the manifold M, the constraint surface with Σ, and the gauge generators with the vector fields R_α. Notice that for a general gauge system, the gauge generators are usually only locally defined, dependent vector fields which are integrable only over the constraint surface [HT92]. We will sometimes use terminology from the physics literature, referring to the equations $T^i(x) = 0$ as the constraint equations, the vector fields R_α as the gauge generators, and the integral submanifolds as the gauge orbits.

Remark 3.1.2. When considering a gauge theory in the Lagrangian formalism, the manifold M is identified with the space of trajectories of the system, and the submanifold Σ corresponds to the equations of motion which may or may not arise as the stationary surface of an action functional. The vector fields R_α again correspond to the generators of the gauge transformations. The interest in this geometrical construction follows from the observation that not all gauge field theories are defined by a Hamiltonian function or action functional (see [LS05] and references therein). A weak Poisson system provides a method to consistently treat field theories possessing gauge symmetries without reference to such data.

Since smooth functions on N correspond to the physical observables of a gauge theory, it is of interest to introduce a Poisson structure in the algebra of functions $C^\infty(N)$ in order to quantise via deformation. To achieve this, the authors of [LS05] introduced a weak Poisson bracket on M, a Poisson bracket which satisfies a Jacobi identity up to terms proportional to the constraint equations and the gauge generators. This bracket becomes a true Poisson bracket however if it is considered as a bracket on the space $C^\infty(N)$, and hence it makes sense to ask about bivector fields, and more generally contravariant tensor fields, which can generate such structures.

A weak Poisson system is well described in terms of vector bundles, therefore we will translate the above set-up into this language.

Let $E \to M$ be a vector bundle of rank k over M. The submanifold Σ is specified by the zero locus of a section $T \in \Gamma(E)$, given locally by

$$T = T^i(x)e_i, \tag{3.2}$$

for a local frame $\{e_i\}$ over an open set $U \subset M$. To ensure that the zero locus of T does indeed describe a smooth submanifold, it is sufficient to fix a connection ∇_E in E. Using this, we require that the bundle homomorphism defined by the covariant derivative of T, $\nabla_E T : TM \to E$, be of constant rank in a tubular neighbourhood of Σ. This is an equivalent notion to the constant rank condition imposed on the Jacobian matrix of the system of equations $T^i(x) = 0$, since locally

$$\nabla_E T = dx^a \nabla_{E,a} T^i e_i = dx^a \left(\partial_a T^i + T^j A^i_{aj}\right) e_i,$$

for connection coefficients A^i_{aj}, which is of constant rank over Σ if and only if $||\partial_a T^i||$ is. Note that if the functions T^i are linearly independent then the section T intersects the base transversally, and the map $\nabla_E T$ is required to be of constant rank only on Σ.

The vector fields R_α also define a bundle homomorphism $R : F \to TM$, from an appropriate rank n vector bundle F into the tangent bundle of M. From the identification $\mathrm{Hom}(F, TM) \cong \Gamma(F^* \otimes TM)$, the homomorphism R defines and is defined by a section of $F^* \otimes TM$,

$$R = R^a_\alpha f^\alpha \otimes \frac{\partial}{\partial x^a} \in \Gamma(F^* \otimes TM), \tag{3.3}$$

for a local frame $\{f^\alpha\}$ of F^* dual to a frame $\{f_\alpha\}$ of F. As with the map $\nabla_E T$, R is also required to be of constant rank in a tubular neighbourhood of Σ. These constant rank requirements on the morphisms $\nabla_E T$ and R are referred to as the regularity conditions in [KLS05]. When restricted to Σ the two maps define the exact sequence of vector bundles

$$0 \to F \xrightarrow{R} TM \xrightarrow{\nabla_E T} E \to 0, \tag{3.4}$$

but which in general, do not form an exact sequence away from Σ. The assumption that the vector fields are linearly independent is equivalent to the injectivity of the vector bundle morphism R, the image of which is always an integrable distribution over Σ.

Remark 3.1.3. With the assumption of linear independence of the functions T^i and the vector fields R_α, the short sequence terminates in eq. (3.4). However if there is linear dependence present in the system, i.e. there is reducibility in the gauge algebra or amongst the constraint equations, then the sequence must be extended appropriately on either side to accommodate this dependence [KLS05].

Those functions and multivector fields which are constant along the integral submanifolds defined by the foliation over Σ can intuitively be described as those tensor fields $\mathcal{T}$ on M such that

$$\mathcal{L}_{R_\alpha} \mathcal{T} = \mathcal{T}^\alpha R_\alpha + T^i \mathcal{T}_i,$$

for the Lie derivative $\mathcal{L}_{R_\alpha}$ and smooth tensor fields $\mathcal{T}^\alpha$ and $\mathcal{T}_i$ on M. This is naturally phrased in terms of the algebra $C^\infty(\Pi T^* M)$. Define an ideal $\mathcal{I} \subset C^\infty(\Pi T^* M)$ generated by the elements T^i and R_α,

$$\mathcal{I} = \langle T^i, R_\alpha \rangle, \tag{3.5}$$

where we identity the odd function $R^a_\alpha x^*_a \in C^\infty(\Pi T^*M)$ with the even vector field $R_\alpha \in \mathfrak{X}(M)$. That the ideal is closed under the Schouten bracket follows from the assumption that the vector fields R_α are tangent to Σ, and the integrability condition of eq. (3.1). In the language of sections, $\mathcal{I}$ is obtained from the image of the morphism R defined from eq. (3.3), together with the image of the map $\Gamma(E^*) \to C^\infty(M)$ obtained by fixing the section T and using the natural pairing of sections of E with those on the dual.

Definition 3.1.1. A multivector field on M is said to be *trivial* if it is a linear combination of the constraint functions T^i and the vector fields R_α. Trivial multivector fields denoted by $\mathfrak{X}_\mathcal{I}$ are identified with functions in $\mathcal{I}$ under the isomorphism (1.6).

Definition 3.1.2. A multivector field U on M is *projectible* if

$$[\![U, \mathcal{I}]\!] \subset \mathcal{I}, \tag{3.6}$$

where $U \in C^\infty(\Pi T^*M)$ is the corresponding function on the odd cotangent bundle and the bracket is the Schouten bracket of multivector fields (1.5).

In particular, trivial multivector fields can be considered to be zero on the space of leaves N, and projectible multivector fields are those multivector fields on M which are tangent to Σ and constant over the integrable submanifolds defined by the foliation over Σ. Projectible multivector fields form an equivalence relation in the algebra of functions $C^\infty(\Pi T^*M)$ given by

$$U \sim V \qquad \Leftrightarrow \qquad U - V \in \mathcal{I}. \tag{3.7}$$

This is interpreted as two projectible multivector fields are equivalent if they differ by a “zero” vector field on the leaf space.

Projectible multivector fields form a closed subalgebra $\mathfrak{X}_P(M) \subset \mathfrak{X}(M)$ defining the Poisson normaliser of the ideal $\mathfrak{X}_\mathcal{I}$ of trivial multivector fields under the Schouten bracket. The space of smooth multivector fields on N is then expressed as the quotient

$$\mathfrak{X}(N) = \mathfrak{X}_P(M)/\mathfrak{X}_\mathcal{I}(M), \tag{3.8}$$

of the projectible multivector fields modulo the trivial ones. Some subspaces of interest are $\mathfrak{X}^0(N)$, $\mathfrak{X}^1(N)$ and $\mathfrak{X}^2(N)$. The first coincides with the algebra of smooth functions on N, and when the construction is identified with a gauge system, corresponds to the algebra of physical observables of the theory. The second is the space of vector fields on N forming a Lie algebra under the Schouten bracket. The algebra of physical observables $\mathfrak{X}^0(N) \cong C^\infty(N)$ is a module over $\mathfrak{X}^1(N)$, where

$$\dot{f} = [\![U, f]\!], \tag{3.9}$$

defines a function, the Lie derivative of $f \in C^\infty(N)$ over $U \in \mathfrak{X}^1(N)$. The function $\dot{f}$ will remain projectible over the flow of U so long as U remains projectible.

Finally, $\mathfrak{X}^2(N)$ consists of all bivector fields on N. A Poisson structure on $C^\infty(N)$ is introduced in the article [LS05] by a choice of projectible bivector field $P \in \mathfrak{X}^2_P(M)$, where P satisfies a weak Jacobi identity,

$$[\![P, P]\!] \in \mathcal{I}. \tag{3.10}$$

Such a bivector field produces a weak Poisson bracket on $C^\infty(M)$ by the derived bracket construction,

$$[f, g]_P := [\![[\![P, f]\!], g]\!],$$

a skew-symmetric bilinear operation satisfying a Leibniz rule in each argument, but whose Jacobi identity is satisfied up to trivial terms. The bivector field P however induces a genuine Poisson bracket on $C^\infty(N)$ since the defects in the Jacobi identity vanish on passage to the quotient. A projectible vector field U on M will be called weakly Poisson if it preserves the Poisson structure up to trivial terms,

$$[\![U, P]\!] \in \mathcal{I}.$$

A weakly Poisson vector field on M, together with a weak Poisson bivector field P, defines a Hamiltonian structure on the space of functions $C^\infty(N)$.

Example 3.1.1. Let $M = \mathbb{R}^4$ have global coordinates x, y, z, t, and consider the Lorentz Lie algebra L generated by the following set of vector fields:

$$R_x = -z\partial_y + y\partial_z, \quad R_y = -x\partial_z + z\partial_x, \quad R_z = -y\partial_x + x\partial_y,$$
$$B_x = x\partial_t + t\partial_x, \quad B_y = y\partial_t + t\partial_y, \quad B_z = z\partial_t + t\partial_z,$$

subject to the relations

$$[R_a, R_b] = \pm R_c, \qquad [R_a, B_b] = \pm B_c,$$
$$[B_a, B_b] = \pm R_c,$$

for $a, b, c \in \{x, y, z, t\}$. The exact signs can be deduced from the expressions above. The important observation is that the space spanned by the rotational vector fields $R = \{R_x, R_y, R_z\}$ is a Lie subalgebra of L, whilst the space spanned by the so called boosts $B = \{B_x, B_y, B_z\}$ is equipped with an R-action.

Define R to be a set of gauge generators on $\mathbb{R}^4$ (we specify no constraint equations), and define a bivector field P on $\mathbb{R}^4$ by

$$P = zB_x \wedge B_y + xB_y \wedge B_z + yB_z \wedge B_x.$$

Then P is gauge invariant and satisfies a weak Jacobi identity. Indeed, the action of the vector field R_x gives

$$\begin{aligned} [\![R_x, P]\!] &= R_x(z)B_x \wedge B_y + zB_z \wedge B_x + R_x(y)B_z \wedge B_x + yB_y \wedge B_x \\ &= y\,(B_x \wedge B_y + B_y \wedge B_x) = 0, \end{aligned}$$

and similarly for R_y and R_z. The bivector P is constant under “spatial rotations”. From calculations, the weak Jacobi identity is proportional to the trivector

$$[\![P, P]\!] \propto R \wedge B \wedge B.$$

Hence P defines a Poisson structure on the space of R-invariant functions (identified with those functions with are invariant under “space-like” rotations in $(1, 3)$-Minkowski space).

One could extend this example by adjoining to L the spatial translations $\partial_x, \partial_y, \partial_z, \partial_t$ in order to consider the full Poincaré Lie algebra.

Example 3.1.2. Consider the 3-dimensional Heisenberg group $H_3(\mathbb{R})$,

$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}.$$

The corresponding Lie algebra $\mathfrak{h}_3$ is generated by the matrices

$$p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which satisfy the commutation relation

$$[p, q] = r.$$

Identify $H_3(\mathbb{R})$ with $\mathbb{R}^3$ equipped with local coordinates x, y, z. Under this identification the generators p, q, r of $\mathfrak{h}_3$ coincide with the vector fields

$$X = \partial_x - \frac{1}{2} y \partial_z, \quad Y = \partial_y + \frac{1}{2} x \partial_z, \quad Z = \partial_z.$$

Consider Z as a gauge generator on $\mathbb{R}^3$. The bivector

$$P = X \wedge Y,$$

defines a weak Poisson bivector on $\mathbb{R}^3$ since it satisfies the weak Jacobi identity

$$[\![P, P]\!] = 2X \wedge Y \wedge Z.$$

We can associate to this the weak Poisson vector field

$$V = y\partial_x - x\partial_y.$$

Therefore $\mathbb{R}^2$ as the quotient of $\mathbb{R}^3$ by the z-axial gauge orbits is equipped with the Poisson structure $P = \partial_x \wedge \partial_y$, together with a subgroup of Poisson automorphisms generated by V. Under the flow of V functions of the form $f = f(x^2+y^2)$ are invariant, which is clear since V generates rotations about the origin.

Example 3.1.3. A Jacobi manifold (M, P, R) is a manifold M equipped with a bivector field P and a distinguished vector field R such that

$$[\![P, R]\!] = 0, \qquad [\![P, P]\!] = 2P \wedge R.$$

Considering R as a single gauge generator, a Jacobi manifold is an example of a weak Poisson system where the weak Poisson bivector field is given by P. In the case when M is a contact manifold of dimension $2n + 1$ there exists a natural Jacobi structure on M, and so a natural weak Poisson structure. Let σ be the 1-form on M defining the contact structure: σ is such that $\sigma \wedge (d\sigma)^n$ is no-where vanishing, and there exist coordinates $(q^1, \ldots, q^n, p^1, \ldots, p^n, t)$ where

$$\sigma = dt - \sum_{i=1}^{n} p^i dq^i.$$

In these coordinates the multivector fields R and P have the expressions

$$R = \frac{\partial}{\partial t}, \qquad P = \sum_{i=1}^{n} \left(\frac{\partial}{\partial q^i} + p^i \frac{\partial}{\partial t} \right) \wedge \frac{\partial}{\partial p^i},$$

defining the weak Poisson system with gauge generator R and weak Poisson bivector P. The space of leaves of the foliation defined by R is a $2n$-dimensional symplectic manifold, where the coordinates q^i, p^i are Darboux coordinates for the induced symplectic structure.

3.2 A BRST Embedding

In the work [LS05], it was shown that any weak Poisson system admitted, in the physical language, an embedding into the BRST framework. This embedding involves the construction of an acyclic complex and the subsequent perturbation of the associated differential, which plays the role of a Koszul-Tate type differential. For a weak Poisson system this complex is the graded algebra of multivector fields on some extended manifold, from which a master function can be described, encoding all the information about the constraint equations, the gauge generators, and their relations.

Geometrically this embedding involves extending the original manifold M by adjoining additional coordinates, and introducing a homological vector field on this extended manifold. This homological vector field is often referred to as the corresponding BRST operator of the theory.

3.2.1 The Master Hamiltonian

To begin, define the extended manifold $\mathcal{M}$ as the total space of the vector bundle $\Pi E \oplus \Pi F \to M$, the direct sum of the vector bundles E and F of the sequence (3.4) with shifted parity in the fibres. This supplements the original bosonic coordinates x^a with additional Grassmann odd variables η^i, c^α called *ghosts*. The conjugate odd momenta are adjoined by extending to the odd cotangent bundle $\Pi T^*\mathcal{M}$, providing $\Pi T^*\mathcal{M}$ with the entire set of local coordinates $x^a, \eta^i, c^\alpha, y_a^*, \eta_i^*, c_\alpha^*$. Notice that the variables y_a^* transform according to the expression

$$y_a^* = J_a^{a'} y_{a'}^* + (-1)^{aB} z^{B'} T_{B'}^{B} \partial_a \left(T_B^{A'} \right) z_{A'}^*,$$

where we write

$$z^A = (\eta^i, c^\alpha), \quad z^*_A = (\eta^*_i, c^*_\alpha), \quad J^{a'}_a(x) = \frac{\partial x^{a'}}{\partial x^a}(x), \quad \text{and} \quad z^A = z^{A'} T^A_{A'}(x).$$

We wish to interpret the odd momenta y^*_a in terms of multivector fields on $\mathcal{M}$, and as the set-up currently stands the coordinates do not transform correctly. To remedy this, we split $\Pi T^*\mathcal{M}$ into a direct sum by introducing a linear connection $\nabla = \nabla_{\Pi E} \oplus \nabla_{\Pi F}$ on $\mathcal{M}$; the sum of connections on ΠE and ΠF respectively. The connection coefficients A^B_{aA} obey the transformation law,

$$A^B_{aA} = \partial_a \left(T^{A'}_A\right) T^B_{A'} + (-1)^{a(A+A')} T^{A'}_A J^{a'}_a A^{B'}_{a'A'} T^B_{B'}.$$

Using ∇, define new coordinates $x^*_a = y^*_a - (-1)^{aB} z^B A^A_{aB} z^*_A$ referred to as *long momenta* in [Vor02] (also introduced similarly in eq. (2.4)). The new coordinates satisfy $x^*_a = J^{a'}_a x^*_{a'}$, transforming with respect to the Jacobian matrix as multivector fields on $\mathcal{M}$. We will fix the connection ∇ such that the odd cotangent bundle decomposes into a direct sum,

$$\mathcal{N} = \Pi E \oplus \Pi F \oplus E^* \oplus F^* \oplus \Pi T^* M.$$

The manifold $\mathcal{N}$ carries several gradings in addition to the Grassmann parity, some of which artificially arise from the general theory of the BRST formalism. The first is the natural $\mathbb{N}$-grading called the *momentum degree*, which is the degree of a polynomial function on $\mathcal{N}$ in the momenta coordinates. The second is a $\mathbb{Z}$-grading called the *ghost grading*, a name inherited from the physics literature. The assumption that the base manifold M is bosonic couples the ghost grading to the Grassmann parity where the parity equals the ghost degree modulo 2, though this does not hold for a general supermanifold. The final grading is called the *resolution degree*. The following table displays all the variables together with their respective gradings:

	x^a	η^i	c^α	x^*_a	η^*_i	c^*_α
Parity $(\tilde{\cdot})$	0	1	1	1	0	0
Ghost (gh)	0	-1	1	1	2	0
Momentum (deg)	0	0	0	1	1	1
Resolution (res)	0	1	0	0	0	1

(3.11)

It will be convenient to introduce the collective notation $\phi^A = (x^a, \eta^i, c^\alpha)$ and $\phi^*_A = (x^*_a, \eta^*_i, c^*_\alpha)$ from which it can be observed that

$$\mathrm{gh}(\phi^*) = 1 - \mathrm{gh}(\phi).$$

The manifold $\Pi T^*\mathcal{M}$ carries a canonical odd symplectic form defined by eq. (1.4), which pulls back to $\mathcal{N}$, twisting the form by the connection coefficients. The odd symplectic form on $\mathcal{N}$ is exact with primitive 1-form

$$\sigma = dx^a x_a^* + \nabla_{\Pi E}\eta^i \eta_i^* + \nabla_{\Pi F} c^\alpha c_\alpha^*,$$

where

$$\nabla_{\Pi E}\eta^i = d\eta^i - \eta^j dx^a A^i_{aj}, \qquad \nabla_{\Pi F} c^\alpha = dc^\alpha - c^\beta dx^a A^\alpha_{a\beta}.$$

The symplectic form then gives rise to the odd non-degenerate Poisson bracket of ghost degree -1 which has the following local appearance:

$$\begin{array}{ccc} \{x_a^*, c^\alpha\}_\nabla = c^\beta A^\alpha_{a\beta}, & \{x_a^*, c_\alpha^*\}_\nabla = A^\beta_{a\alpha} c_\beta^*, & \{\eta_i^*, \eta^j\}_\nabla = \delta_i^j, \\ \{x_a^*, \eta^i\}_\nabla = \eta^j A^i_{aj}, & \{x_a^*, \eta_i^*\}_\nabla = A^j_{ai}\eta_j^*, & \{c_\alpha^*, c^\beta\}_\nabla = \delta_\alpha^\beta, \\ \{x_a^*, x^b\}_\nabla = \delta_a^b, & \{x_a^*, x_b^*\}_\nabla = c^\beta \mathcal{R}^\alpha_{\beta ab} c_\alpha^* + \eta^i \mathcal{R}^j_{iab}\eta_j^*, & \end{array} \tag{3.12}$$

where all other brackets vanish identically, and the terms $\mathcal{R}^j_{iab}$ and $\mathcal{R}^\alpha_{\beta ab}$ are the components of the curvatures of the connections $\nabla_{\Pi E}$ and $\nabla_{\Pi F}$ respectively. Notice that when the bundles ΠE and ΠF are trivial, the connection and curvature components vanish and we recover the usual canonical odd Poisson structure.

Using this odd Poisson bracket we can define an odd differential on the algebra $C^\infty(\mathcal{N})$ by specifying a master Hamiltonian $\mathcal{S} \in C^\infty(\mathcal{N})$ with the gradings

$$\tilde{\mathcal{S}} = 0, \qquad \mathrm{gh}(\mathcal{S}) = 2, \qquad \deg(\mathcal{S}) > 0,$$

together with the assumption that $\mathcal{S}$ satisfies the master equation

$$\{\mathcal{S}, \mathcal{S}\}_\nabla = 0. \tag{3.13}$$

The function $\mathcal{S}$ is called the master function, and contains all the information about the section T of the vector bundle E, the homomorphism $R : F \to TM$, and the compatibility conditions between them. It may also encode the structure of a weak Poisson bracket, and hence $\mathcal{S}$ can completely describe a weak Poisson system. Grading $\mathcal{S}$ by the resolution degree,

$$\mathcal{S} = \sum_{r \geq 0} \mathcal{S}_r,$$

local expressions for $\mathcal{S}_0$ and $\mathcal{S}_1$ can be obtained:

$$\mathcal{S}_0 = T^i \eta_i^* + c^\alpha R_\alpha^a x_a^* + P^{ab} x_b^* x_a^*; \tag{3.14}$$

$$\mathcal{S}_1 = \left(c^\beta c^\alpha U_{\alpha\beta}^\gamma + V^{\gamma ab} x_b^* x_a^* + c^\alpha W_\alpha^{\gamma a} x_a^* + Y^{\gamma i} \eta_i^*\right) c_\gamma^*$$
$$+\eta^i \left(c^\beta c^\alpha A_{\alpha\beta i}^a x_a^* + c^\alpha B_{\alpha i}^{ab} x_b^* x_a^* + D_i^{abc} x_c^* x_b^* x_a^* + c^\alpha E_{\alpha i}^j \eta_j^* + F_i^{aj} \eta_j^* x_a^*\right),$$

where the coefficients T, R and P correspond to the section T, the homomorphism R and a weak Poisson bivector P. The other coefficient functions are higher structure functions, encoding higher relations between the ghost variables. (The term $\mathcal{S}_0$ is referred to as the boundary condition for the master action $\mathcal{S}$, see [HT92].)

To ensure the regularity conditions on T and R of eq. (3.4), we further require that $\mathcal{S}$ satisfies the non-degeneracy condition

$$\operatorname{rank} \left(\frac{\partial^2 \mathcal{S}}{\partial \phi^A \phi_B^*} \right)\Bigg|_{d\mathcal{S}=0} = (n, k),$$

where k and n are the ranks of the bundles E and F respectively.

The master equation (3.13) ensures that

$$Q := \{\mathcal{S}, -\}_\nabla\big|_{\phi^*=0}, \tag{3.15}$$

defines a homological vector field on the Lagrangian submanifold $\mathcal{M} \subset \mathcal{N}$. The vector field Q is of ghost degree $+1$, and has the local expression

$$Q = T^i \frac{\partial}{\partial \eta^i} + c^\alpha R_\alpha^a \frac{\partial}{\partial x^a} + \cdots .$$

In physics literature, Q is often called the BRST operator associated to the gauge system [HT92]. Notice that it contains all the information about the constraints and the gauge generators, but does not know about the artificially introduced Poisson structure which is present in $\mathcal{S}$ in momentum degree $+2$.

Since $\mathcal{S}$ satisfies the master equation it defines a homotopy Poisson structure, or a P_∞-structure, on the base manifold $\mathcal{M}$ by a sequence of derived brackets. The k-bracket is given by the nested odd Poisson brackets

$$[F_1, \ldots, F_k]_{\mathcal{S}} = \big\{ \cdots \{\mathcal{S}, F_1\}_\nabla, \ldots, F_k \big\}_\nabla\big|_{\mathcal{M}}, \tag{3.16}$$

for functions $F_1, \ldots, F_k \in C^\infty(\mathcal{M})$ on the extended manifold. In particular, the binary bracket satisfies the homotopy Jacobi identity

$$[F, [G, H]_{\mathcal{S}}]_{\mathcal{S}} - [[F, G]_{\mathcal{S}}, H]_{\mathcal{S}} - (-1)^{FG} [G, [F, H]_{\mathcal{S}}]_{\mathcal{S}} \tag{3.17}$$
$$= [[F]_{\mathcal{S}}, G, H]_{\mathcal{S}} - (-1)^F [F, [G]_{\mathcal{S}}, H]_{\mathcal{S}} + (-1)^{F+G} [F, G, [H]_{\mathcal{S}}]_{\mathcal{S}} + [[F, G, H]_{\mathcal{S}}]_{\mathcal{S}}.$$

3.2.2 The Existence and Uniqueness of $\mathcal{S}$

The existence of $\mathcal{S}$ is not guaranteed a priori, but follows from the construction of an appropriate differential in an acyclic complex. Here we will briefly describe this and provide a sketch of the existence of $\mathcal{S}$. Details for the construction of a "BRST charge" in the Hamiltonian formalism for both irreducible and reducible cases can be found in [HT92] or [FHST89] for example.

To the algebra $C^\infty(\Pi T^*M)$ of multivector fields on M, we introduce the odd ghost variables η^i, c^*_α for each constraint coefficient T^i and gauge generator R_α. The variables are assigned resolution degree $+1$, after which the algebra is further extended to the whole of $C^\infty(\mathcal{N})$ by adjoining the resolution degree 0 variables η^*_i, c^α. A differential δ may be introduced which is required to act solely on the variables in positive resolution degree,

$$\delta = T^i\frac{\partial}{\partial\eta^i} + R^a_\alpha x^*_a\frac{\partial}{\partial c^*_\alpha}, \qquad \mathrm{gh}(\delta) = +1, \quad \mathrm{res}(\delta) = -1. \tag{3.18}$$

The differential is such that the image $\mathrm{im}(\delta)$ corresponds precisely to the (extended) ideal of trivial multivector fields $\mathcal{I}$. It is well known that such a differential is acyclic. Indeed, either see the result in [HT92], or introduce a contracting homotopy as follows. Let g be a metric in the fibres of the vector bundle E whose space of sections contain the section T. Define an odd operator

$$\zeta_T = \eta^i g_{ik} T^k, \qquad \mathrm{gh}(\zeta_T) = -1, \qquad \mathrm{res}(\zeta_T) = +1. \tag{3.19}$$

Then

$$[\delta, \zeta_T] = T^i g_{ik} T^k \quad \Rightarrow \quad [\delta, \frac{1}{\langle T, T\rangle_g}\zeta_T] = 1,$$

providing a contracting homotopy for δ. Hence the differential δ is acyclic in positive resolution degree,

$$H_k(\delta) = 0, \qquad k > 0. \tag{3.20}$$

Using δ we can construct a secondary derivation Δ_0 with the properties and gradings:

$$[\delta, \Delta_0] = \delta\Delta_0 + \Delta_0\delta = 0, \qquad \Delta_0^2 = -[\delta, \Delta_1], \tag{3.21}$$
$$\deg(\Delta_0) = 0, \quad \mathrm{gh}(\Delta_0) = 1, \quad \mathrm{res}(\Delta_0) = 0,$$

where Δ_1 is some derivation of resolution degree $+1$. This is constructed explicitly using the projectibility conditions contained in (3.6).

We can now sketch the existence of $\mathcal{S}$. Grading $\mathcal{S} = \sum_{r\geq 0} \mathcal{S}_r$ by resolution degree, where $\mathcal{S}_0$ is given by the expression (3.14), the master equation (3.13) generates the chain of equations

$$\delta \mathcal{S}_{r+1} = \lambda_r(\mathcal{S}_0, \dots, \mathcal{S}_r),$$

for a function λ_r of resolution degree r.

For $r > 0$, the acyclicity of δ ensures the existence of $\mathcal{S}_{r+1}$, since by the Jacobi identity $\{\mathcal{S}, \{\mathcal{S}, \mathcal{S}\}_\nabla\}_\nabla = 0$, the term $\lambda_r(\mathcal{S}_0, \dots, \mathcal{S}_r)$ is δ-closed and hence δ-exact. Therefore $\mathcal{S}$ exists if and only if the initial equation in resolution degree 0 has a solution,

$$\delta \mathcal{S}_1 = \lambda_0(\mathcal{S}_0) = \{\mathcal{S}, \mathcal{S}\}_\nabla\big|_{\text{res}=0}.$$

An analysis of the bracket (3.12) gives that this equation encodes precisely the compatibility conditions between the constraints T, the gauge generators R and the weak Poisson bivector field P, and so is also automatically satisfied.

Notice that in each term $\mathcal{S}_r$ there is always the possibility to add any δ-exact term. This ambiguity in $\mathcal{S}$ presents no issues however, since this can always be absorbed by a canonical transformation of the odd Poisson manifold $\mathcal{N}$ [LS05].

3.2.3 The Cohomology of Q

The BRST operator Q of eq. (3.15) turns the algebra $C^\infty(\mathcal{N})$ into a bi-graded differential complex $\mathcal{C}(\mathcal{N})$ with ghost degree l and the momentum degree k,

$$Q : \mathcal{C}^k_l(\mathcal{N}) \to \mathcal{C}^k_{l+1}(\mathcal{N}).$$

It is evident that the cohomology of Q also decomposes with respect to the ghost and momentum degrees,

$$\mathcal{H}(Q) = \bigoplus_{k,l} \mathcal{H}^k_l(Q).$$

Lemma 3.2.1. *For $k > l$, the cohomology groups $\mathcal{H}^k_l(Q)$ are trivial.*

Proof. The differential Q can be expanded according to the resolution degree

$$Q = \delta + \Delta_0 + \sum_{i\geq 1} \Delta_i,$$

where δ and Δ_0 are the differentials in eqs. (3.18) and (3.21) respectively. From table (3.11) we make the following observation: for any variable ϕ or ϕ^*,

$$\text{res}(\phi^{(*)}) \geq \deg(\phi^{(*)}) - \text{gh}(\phi^{(*)}). \tag{3.22}$$

So for any function $F \in \mathcal{C}_l^k(\mathcal{N})$ where $k > l$, $\text{res}(F) > 0$. Grading F in terms of the resolution degree

$$F = \sum_{r>0} F_r,$$

we wish to show that there exists $G \in \mathcal{C}_{l-1}^k(\mathcal{N})$ such that $QG = F$ when $QF = 0$.

Let r_0 be the lowest resolution degree appearing in F. Then the resolution degree $r_0 - 1$ term of the cocycle condition $QF = 0$ is

$$\delta F_{r_0} = 0,$$

which, by the acyclicity of δ and since $\text{res}(F_{r_0}) > 0$, tells us that $F_{r_0} = \delta G_{r_0+1}$ for some resolution degree $r_0 + 1$ function G_{r_0+1} of ghost degree $l - 1$.

The resolution degree r_0 component of $QF = 0$ reads,

$$\delta F_{r_0+1} + \Delta_0 F_{r_0} = 0 \quad \Rightarrow \quad \delta F_{r_0+1} + \Delta_0 \delta G_{r_0+1} = 0.$$

The identity $Q^2 F \equiv 0$ in the lowest resolution degree gives the relation

$$\delta\Delta_0 + \Delta_0\delta = 0,$$

which is also provided by the construction of Δ_0 in eq. (3.21). Hence

$$\delta F_{r_0+1} = -\Delta_0 \delta G_{r_0+1} = \delta \Delta_0 G_{r_0+1},$$

and again using the acyclicity of δ, we can write

$$\delta(F_{r_0+1} - \Delta_0 G_{r_0+1}) = 0 \qquad \Rightarrow \qquad F_{r_0+1} - \Delta_0 G_{r_0+1} = \delta G_{r_0+2},$$

for some function G_{r_0+2}. Hence we find F_{r_0+1} to be the resolution degree $r_0 + 1$ term

$$F_{r_0+1} = \Delta_0 G_{r_0+1} + \delta G_{r_0+2}.$$

Continuously solving with those equations in higher resolution degree allows a function G to be constructed such that $F = QG$ when $F \in \mathcal{H}_l^k(Q)$. Therefore the cohomology groups $\mathcal{H}_l^k(Q)$ are trivial for $k > l$. $\square$

The use of the BRST framework for a weak Poisson system is contained within the next proposition.

Proposition 3.2.1 ([LS05]). *Any projectible multivector field U on M can be lifted to a Q-cocycle F_U in $C^\infty(\mathcal{N})$, and further, the cohomology group $\mathcal{H}^k_k(Q)$ is isomorphic to the algebra $\mathfrak{X}^k(N)$, the projectible multivector fields of degree k on M modulo the trivial ones.*

Proof. Let U be a degree k multivector field on M considered as a function on $\mathcal{N}$ with the properties:

$$U = U^{i_1 \cdots i_k}(x) x^*_{i_k} \dots x^*_{i_1}, \qquad \mathrm{gh}(U) = k = \deg(U), \quad \mathrm{res}(U) = 0.$$

Let $F = F_U$ denote a function extending U over $\mathcal{N}$ with the prescribed gradings

$$\mathrm{gh}(F) = k, \quad \deg(F) = k, \quad \mathrm{res}(F) \geq 0.$$

Expanding $F = \sum_{r \geq 0} F_r$ in terms of resolution degree, notice that $F_0 = U$ due to the restrictions on the ghost and momentum degrees. Suppose that F is a Q-cocycle, then decomposing the condition $QF = 0$ by resolution degree, we obtain the system of equations

$$\delta F_{r+1} = \lambda_r(F_0, \dots, F_r), \tag{3.23}$$

in resolution degree $r \geq 0$. For $r > 0$, the acyclicity of δ guarantees the existence of the term F_r, since the equation $\lambda_r(F_0, \dots, F_r)$ is δ closed by the identity $Q^2 F \equiv 0$. The cocycle condition admits solutions in positive resolution degree, and only the case $r = 0$ remains. Consider the equation

$$\delta F_1 + \Delta_0 F_0 = 0 \quad \Leftrightarrow \quad \{S, F\}_\nabla \Big|_{\substack{\deg = k \\ \mathrm{res} = 0}} = 0.$$

An analysis of the bracket eq. (3.12), together with the restrictions on the momentum and resolution degrees returns precisely the projectibility relations for the multivector field U. For example, let $U = U^a x^*_a$ be a vector field considered as a degree 1, ghost 1 function on $\mathcal{N}$. An arbitrary extension over $\mathcal{N}$ has the general expression

$$F_U = U^a x^*_a + \eta^i \left(c^\alpha U^a_{\alpha i} x^*_a + U^j_i \eta^*_j \right) + c^\beta U^\alpha_\beta c^*_\alpha + \cdots .$$

The resolution degree 0 term of the cocycle condition $\delta F_1 + \Delta_0 F_0 = 0$ is satisfied precisely when eq. (3.6) holds. Hence we can begin the induction on F_U if and only if U is projectible.

Suppose now that $F_U, F_V \in \mathcal{C}^k_k(\mathcal{N})$ are Q-cocycles representing the same cohomology class in $\mathcal{H}^k_k(Q)$, so there exists a function $G \in \mathcal{C}^k_{k-1}(\mathcal{N})$ such that $F_U = F_V + QG$. In resolution degree 0 we obtain

$$U = V + \delta G_1.$$

Notice that there is no term of resolution degree 0 in G, since observation (3.22) ensures that $\mathrm{res}(G) > 0$ since $\deg(G) = k > \mathrm{gh}(G) = k - 1$. Now the image of δ constructed in eq. (3.18) is precisely the trivial multivector fields, and so δG_1 represents a trivial multivector field and therefore $U \sim V$ under the equivalence relation in eq. (3.7). Hence the spaces $\mathcal{H}^k_k(Q)$ and $\mathfrak{X}^k(N)$ are isomorphic. □

Consider the binary bracket on $\mathcal{M}$ given by the derived brackets of eq. (3.16). Due to the homotopy Jacobi identity in eq. (3.17), we see that the bracket

$$[F, G]_{\mathcal{S}} = \big\{\{\mathcal{S}, F\}_\nabla, G\big\}_\nabla\big|_{\phi^*=0}, \tag{3.24}$$

induces a genuine even Poisson bracket on the cohomology group $\mathcal{H}^0(Q) = \bigoplus_l \mathcal{H}^0_l(Q)$. This bracket is of ghost degree 0, and hence the algebra of projectible functions $\mathcal{H}^0_0(Q) \cong C^\infty(N)$ forms a Poisson subalgebra of $\mathcal{H}^0(Q)$. This induced Poisson bracket on $\mathcal{H}^0_0(Q)$ coincides with the Poisson bracket on $C^\infty(N)$ obtained from the weak Poisson structure on M. As such, a BRST embedding of a weak Poisson system gives a complete cohomological description of the system in terms of the BRST operator Q.

3.3 Lifting the Weak Poisson Structure

Detailed in [KV08], any homotopy Poisson structure on a manifold M has a canonical lift to a homotopy Schouten structure on the odd tangent bundle ΠTM – a higher Koszul bracket hierarchy. Using this idea, the Poisson bracket (3.24) on the $\mathcal{Q}$-cohomology $C^\infty(N) \cong \mathcal{H}^0_0(Q)$ can be lifted to a Koszul bracket on the space $C^\infty(\Pi TN)$, corresponding to differential forms on the leaf space N. This is achieved by lifting the extended Poisson structure of eq. (3.16) on $\mathcal{M}$ to the space of ghost extended differential forms $C^\infty(\Pi T\mathcal{M})$, and using the cohomology of a corresponding differential to describe those which descend to the space N.

Introduce the odd tangent bundle $\Pi T\mathcal{N}$ to the manifold $\mathcal{N}$, equipped with the odd velocities as expressed in the table:

	x^a	η^i	c^α	x^*_a	η^*_i	c^*_α	ξ^a	u^i	ν^α	ξ^*_a	u^*_i	ν^*_α
Parity $(\tilde{\cdot})$	0	1	1	1	0	0	1	0	0	0	1	1
Ghost (gh)	0	-1	1	1	2	0	-1	-2	0	0	1	-1
Momentum (deg)	0	0	0	1	1	1	0	0	0	0	0	0
Res (res)	0	1	0	0	0	1	0	1	0	0	0	1
Form (Deg)	0	0	0	0	0	0	1	1	1	1	1	1

The form degree Deg will be the natural grading in the vector bundle $\Pi T\mathcal{N}$, where the fibre coordinates are assigned degree $+1$, and the base coordinates are assigned degree 0. Collectively, write $\psi^A = (\xi^a, u^i, \nu^\alpha)$ and $\psi^*_A = (\xi^*_a, u^*_i, \nu^*_\alpha)$. We also draw attention to the resolution degree held by the coordinates u^i.

This manifold carries a natural even symplectic form; since $\Pi T\mathcal{N}$ may be canonically identified with $T^*\mathcal{N}^*$ by the composition of the two identifications

$$\Pi T\mathcal{N} \cong T^*\mathcal{N} \quad \text{and} \quad T^*\mathcal{N} \cong T^*\mathcal{N}^*,$$

the canonical 2-form on $T^*\mathcal{N}^*$ can be pulled back under the symplectomorphism

$$\begin{gathered} \chi : \Pi T\mathcal{N} \to T^*\mathcal{N}^*, \\ (\phi^A, \phi^*_A, \psi^A, \psi^*_A) \mapsto (\phi^A, (-1)^A\psi^A, -\psi^*_A, \phi^*_A). \end{gathered} \tag{3.25}$$

The pull back of the canonical form equips $\Pi T\mathcal{N}$ with an even non-degenerate Poisson bracket, written as $[-,-]$, which is of ghost degree 0. The manifold $\mathcal{N}^* \cong \Pi T\mathcal{M}$ defines a Lagrangian surface in the cotangent bundle $T^*\mathcal{N}^*$, which in turn defines a Lagrangian surface $\mathcal{L} \subset \Pi T\mathcal{N}$ under the (anti-)symplectomorphism χ. Locally this is described by setting the "conjugate momenta" equal to zero,

$$\mathcal{L} = \left\{ (\phi, \phi^*, \psi, \psi^*) \in \Pi T\mathcal{N} \,\middle|\, \phi^* = 0 = \psi^* \right\}. \tag{3.26}$$

Since the functions on $\mathcal{N}^*$ are identified with differential forms on $\mathcal{M}$, under the map (3.25) a function $\Gamma \in C^\infty(\mathcal{L})$ is identified with a differential form on $\mathcal{M}$.

Let $\Psi = \psi^A\psi^*_A$ be the canonical ghost -1 odd function of the velocity variables. Simply by construction we have that Ψ is Poisson-nilpotent,

$$[\Psi, \Psi] = 0. \tag{3.27}$$

Now consider $\mathcal{S}$ as a function in $C^\infty(\Pi T\mathcal{N})$, and define the odd function

$$\Psi_{\mathcal{S}} = [\Psi, \mathcal{S}], \qquad \mathrm{gh}(\Psi_{\mathcal{S}}) = +1. \tag{3.28}$$

In particular, this function is of form degree +1 and carries the information of the weak Poisson system. (Notice the similarity to the function (1.34) defined in example 1.2.11.)

Lemma 3.3.1. *The function $\Psi_{\mathcal{S}}$ satisfies $[\Psi_{\mathcal{S}}, \Psi_{\mathcal{S}}] = 0$ if $\mathcal{S}$ satisfies the master equation $\{\mathcal{S}, \mathcal{S}\}_\nabla = 0$.*

Proof. By applications of the Jacobi identity,

$$[\Psi_{\mathcal{S}}, \Psi_{\mathcal{S}}] = [[\Psi, \mathcal{S}], [\Psi, \mathcal{S}]] = -[[\Psi, [\Psi, \mathcal{S}]], \mathcal{S}] + [\Psi, [[\Psi, \mathcal{S}], \mathcal{S}]].$$

Observe then that Ψ generates the bracket in eq. (3.12) on $\mathcal{N}$ by

$$[[\Psi, \mathcal{S}], \mathcal{S}] = \{\mathcal{S}, \mathcal{S}\}_\nabla, \tag{3.29}$$

and hence $[\Psi_{\mathcal{S}}, \Psi_{\mathcal{S}}] = 0$ if $\{\mathcal{S}, \mathcal{S}\}_\nabla = 0$. (Notice in fact that $[\Psi_{\mathcal{S}}, \Psi_{\mathcal{S}}] = 0$ if and only if $\{\mathcal{S}, \mathcal{S}\}_\nabla$ is a de Rham cocycle on $\mathcal{N}$.) □

Since $\mathcal{S}$ satisfies the master equation (3.13), lemma 3.3.1 ensures that $\Psi_{\mathcal{S}}$ defines a homological vector field

$$\mathcal{Q} = [\Psi_{\mathcal{S}}, -]\big|_{\mathcal{L}}, \tag{3.30}$$

on the Lagrangian submanifold $\mathcal{L}$. Some explicit terms in $\mathcal{Q}$ are

$$\mathcal{Q} = T^i \frac{\partial}{\partial \eta^i} + \xi^a \partial_a T^i \frac{\partial}{\partial u^i} + c^\alpha R^a_\alpha \frac{\partial}{\partial x^a} + \left(c^\alpha \xi^b \partial_b R^a_\alpha + \nu^\alpha R^a_\alpha\right) \frac{\partial}{\partial \xi^a} + \cdots,$$

where we see terms involving the constraints, the gauge generators and their differentials appearing.

Further, $\Psi_{\mathcal{S}}$ generates a sequence of higher derived Schouten brackets on $\mathcal{L}$ by the formula

$$\{\Gamma_1, \ldots, \Gamma_k\}_{\Psi_{\mathcal{S}}} = \left[\cdots [\Psi_{\mathcal{S}}, \Gamma_1], \ldots, \Gamma_k\right]\big|_{\mathcal{L}}, \tag{3.31}$$

for functions $\Gamma_1, \ldots, \Gamma_k \in C^\infty(\mathcal{L})$. This sequence of odd Poisson brackets is precisely the lift of the homotopy Poisson structure on $\mathcal{M}$ given in eq. (3.16), and coincides exactly with the higher Koszul brackets introduced in [KV08] under the image of the map $\chi^* : C^\infty(\Pi T\mathcal{M}) \to C^\infty(\mathcal{L})$.

3.3.1 The Cohomology of $\mathcal{Q}$

Consider the algebra of functions $C^\infty(\mathcal{L})$. The operator $\mathcal{Q}$ turns this algebra into a complex which is graded by the ghost degree l and the form degree m,

$$\mathcal{Q} : \mathcal{C}_l^m(\mathcal{L}) \to \mathcal{C}_{l+1}^m(\mathcal{L}). \tag{3.32}$$

Let σ be a differential k-form on M expressed as a function on the Lagrangian submanifold $\mathcal{L}$,

$$\sigma = \xi^{a_1} \cdots \xi^{a_k} \sigma_{a_k \cdots a_1}(x), \qquad \mathrm{gh}(\sigma) = -k, \quad \mathrm{Deg}(\sigma) = k.$$

As in the case of multivector fields on M, we can extend σ to a homogeneous function Γ_σ over the entire of $\mathcal{L}$, subject to the degree restrictions

$$\mathrm{gh}(\Gamma_\sigma) = -k, \qquad \mathrm{Deg}(\Gamma_\sigma) = k, \qquad \mathrm{res}(\Gamma_\sigma) \geq 0.$$

For example, a 1-form $\sigma = \xi^a \sigma_a$ extends to a function Γ_σ such that in resolution degree 0 and 1,

$$\begin{aligned} \Gamma_\sigma\big|_{\mathrm{res}=0} &= \xi^a \sigma_a, \\ \Gamma_\sigma\big|_{\mathrm{res}=1} &= \xi^a \eta^i c^\alpha \sigma_{\alpha i a} + u^i c^\alpha \sigma_{\alpha i} + \nu^\alpha \eta^i \sigma_{i\alpha}. \end{aligned}$$

In light of proposition 3.2.1, we make the following definition.

Definition 3.3.1. Let σ be a differential k-form on M, and Γ_σ the homogeneous extension of σ viewed as a function on $\mathcal{L}$. Then σ is a *projectible differential form* if Γ_σ is a $\mathcal{Q}$-cocycle,

$$\mathcal{Q}\Gamma_\sigma = 0.$$

Intuitively, a projectible differential form should be one that acts trivially on the gauge generators, and whose Lie derivative over the vector fields R_α is also trivial:

$$\imath_{R_\alpha} \sigma \propto T, \qquad \mathcal{L}_{R_\alpha} \sigma \propto T + dT.$$

That this is a correct definition follows as a result of the next proposition.

Proposition 3.3.1. *The cohomology group $\mathcal{H}_{-k}^k(\mathcal{Q})$ is isomorphic to the algebra of projectible differential k-forms over M.*

Proof. For the simplicity of local expressions we will assume that both E and F admit flat connections. In particular, $\mathcal{N} = \Pi T^*\mathcal{M}$ and the canonical odd bracket (1.5) coincides with the bracket of eq. (3.12). This then implies that the bracket on $\Pi T\mathcal{N}$ is given locally by

$$[\psi^*_A, \phi^B] = \delta^B_A, \qquad [\psi^A, \phi^*_B] = \delta^A_B.$$

By the Cartan formula for the Lie derivative we see that the vector field $\mathcal{Q}$ coincides with the Lie derivative on $\mathcal{L}$ over the vector field Q,

$$\mathcal{Q} = [[\Psi, \mathcal{S}], -]\big|_{\mathcal{L}} := \mathcal{L}_Q, \tag{3.33}$$

from which it is now evident that $\mathcal{Q}^2 = \mathcal{L}_{[Q,Q]} = 0$. The expansion of Q by the resolution degree provides an expansion of $\mathcal{Q}$ also, and in resolution degree -1 $\mathcal{Q}$ takes the form

$$\mathcal{Q}_{-1} = \mathcal{L}_\delta = T^i \frac{\partial}{\partial \eta^i} + \xi^a \partial_a T^i \frac{\partial}{\partial u^i}. \tag{3.34}$$

In resolution degree 0 we find a more complicated expression,

$$\begin{aligned}\mathcal{Q}_0 = c^\alpha R^a_\alpha \frac{\partial}{\partial x^a} + c^\beta c^\alpha U^\gamma_{\alpha\beta} \frac{\partial}{\partial c^\gamma} + \eta^i c^\alpha E^j_{\alpha i} \frac{\partial}{\partial \eta^j} + \nu^\alpha \left(R^a_\alpha \frac{\partial}{\partial \xi^a} + c^\beta U^\gamma_{\beta\alpha} \frac{\partial}{\partial \nu^\gamma} + \eta^i E^j_{i\alpha} \frac{\partial}{\partial u^j} \right) \\ + \xi^b \partial_b \left(c^\alpha R^a_\alpha \frac{\partial}{\partial \xi^a} + c^\beta c^\alpha U^\gamma_{\alpha\beta} \frac{\partial}{\partial \nu^\gamma} + \eta^i c^\alpha E^j_{\alpha i} \frac{\partial}{\partial u^j} \right) + u^i c^\alpha E^j_{\alpha i} \frac{\partial}{\partial u^j}.\end{aligned}$$

Now let σ be a differential k-form on M viewed as a function on $\mathcal{L}$; σ is such that

$$\sigma = \xi^{a_1} \dots \xi^{a_k} \sigma_{a_k \cdots a_1}(x), \qquad \mathrm{gh}(\sigma) = -k, \quad \mathrm{Deg}(\sigma) = k, \quad \mathrm{res}(\sigma) = 0.$$

Let $\Gamma = \Gamma_\sigma$ be a function extending σ over $\mathcal{L}$. Decomposing Γ by resolution degree, the restriction on the other gradings shows that

$$\begin{aligned}\Gamma_0 &= \xi^{a_1} \dots \xi^{a_k} \sigma_{a_k \cdots a_1}, \\ \Gamma_1 &= \xi^{a_1} \dots \xi^{a_{k-1}} \big(u^i c^\alpha I_{\alpha i a_1 \cdots a_{k-1}} + \nu^\alpha \eta^i J_{i\alpha a_1 \cdots a_{k-1}} \big) + \nu^\alpha u^i \xi^{a_1} \dots \xi^{a_{k-2}} K_{a_1 \cdots a_{k-2} i\alpha}.\end{aligned}$$

The cocycle condition $\mathcal{Q}\Gamma = 0$ can also be decomposed by resolution degree in order to obtain a system of equations as in eq. (3.23). In resolution degree 0 we find

$$\mathcal{Q}_{-1}\Gamma_1 + \mathcal{Q}_0 \Gamma_0 = 0,$$

which recovers precisely the intuitive projectibility conditions

$$\imath_{R_\alpha} \sigma \propto T, \qquad \mathcal{L}_{R_\alpha} \sigma \propto T + dT,$$

when comparing the local expressions. The higher degree equations

$$\mathcal{Q}_{-1}\Gamma_{r+1} = \lambda_r(\Gamma_0, \ldots, \Gamma_r),$$

are satisfied by the acyclicity of the differential $\mathcal{Q}_{-1} = \mathcal{L}_\delta$, which follows from the acyclicity of the differential δ in eq. (3.18). The operator

$$\zeta'_T = \eta^i g_{ik} T^k + u^i g_{ik} \xi^a \partial_a T^i,$$

allows the definition of a contracting homotopy for the differential $\mathcal{L}_\delta$, analogous to the operator in eq. (3.19).

Suppose now that $\Gamma \in \mathcal{H}^k_{-k}$ is $\mathcal{Q}$-cocycle. Then $\Gamma = \mathcal{Q}\Xi$ for some function $\Xi \in C^\infty(\mathcal{L})$, such that

$$\mathrm{res}(\Xi) \geq 0, \qquad \mathrm{Deg}(\Xi) = k, \qquad \mathrm{gh}(\Xi) = -(k+1).$$

With these degree restrictions we observe that there must be no resolution degree 0 term present in Ξ, and in resolution degree +1,

$$\Xi_1 = \eta^i \xi^{a_1} \ldots \xi^{a_k} \Xi_{a_k \cdots a_1 i} + u^i \xi^{a_1} \ldots \xi^{a_{k-1}} \Xi_{a_{k-1} \cdots a_1 i}.$$

The function $\mathcal{Q}\Xi$ does contain resolution degree 0 terms however, where

$$\Gamma_0 = \mathcal{L}_\delta \Xi_1 = T^i \xi^{a_1} \ldots \xi^{a_k} \Xi_{a_k \cdots a_1 i} + \xi^a \partial_a T^i \xi^{a_1} \ldots \xi^{a_{k-1}} \Xi_{a_{k-1} \cdots a_1 i}.$$

On the surface Σ, this is a differential k-form proportional to the differentials of the constraint equations. Therefore, if $\Gamma_\sigma = \Gamma_\tau$ in the cohomology group $\mathcal{H}^k_{-k}$, the corresponding differential forms σ and τ on M differ by a trivial differential form. □

Analogous to the bracket described in eq. (3.24), the odd binary bracket given by eq. (3.31) satisfies the Jacobi identity up to $\mathcal{Q}$-coboundaries. Therefore when passing to the cohomology of the complex (3.32), the binary bracket realises a true odd Poisson bracket on the $\mathcal{Q}$-cohomology $\mathcal{H}(\mathcal{Q})$. Notice that unlike the even Poisson bracket which was introduced only on the momentum degree 0 functions $\mathcal{H}^0(Q)$, this odd bracket is defined on the entire cohomology $\mathcal{H}(\mathcal{Q})$ with arbitrary form degree. It can be seen that this is the odd, ghost +1 lift of the even Poisson bracket on projectible functions. Indeed, since the de Rham differential $d = [\Psi, -]\big|_{\mathcal{L}}$ commutes with $\mathcal{Q}$ as a consequence of the Jacobi identity, d acts on the cohomology $d : \mathcal{H}^m_l(\mathcal{Q}) \to \mathcal{H}^{m+1}_{l-1}(\mathcal{Q})$.

For functions $F \in \mathcal{H}^0_l(\mathcal{Q})$ and $G \in \mathcal{H}^0_k(\mathcal{Q})$, the odd bracket $\{dF, G\}_{\Psi_\mathcal{S}} \in \mathcal{H}^0_{k+l}(\mathcal{Q})$, where

$$\{dF, G\}_{\Psi_\mathcal{S}} = \big[[\Psi_\mathcal{S}, [\Psi, F]], G\big]\big|_\mathcal{L} = -\big[[\Psi, [\Psi_\mathcal{S}, F]], G\big]\big|_\mathcal{L},$$

by the Jacobi identity. Using observation (3.29) it can be seen that

$$-\big[[\Psi, [\Psi_\mathcal{S}, F]], G\big]\big|_\mathcal{L} = -\big\{\{\mathcal{S}, F\}_\nabla, G\big\}_\nabla = -[F, G]_\mathcal{S}.$$

Similarly,

$$d[F, G]_\mathcal{S} = -(-1)^F \{dF, dG\}_{\Psi_\mathcal{S}},$$

and we obtain the Koszul bracket of formula (1.36).

Proposition 3.3.2. *The cohomology group $\mathcal{H}(\mathcal{Q})$ is an odd Poisson algebra equipped with the odd binary bracket, which is closed under the exterior derivative.*

Proof. For $\Gamma, \Xi \in \mathcal{H}(\mathcal{Q})$, the fact that $\mathcal{Q}\{\Gamma, \Xi\}_{\Psi_\mathcal{S}} = 0$ follows from an application of the Jacobi identity, where

$$\big[\Psi_\mathcal{S}, [[\Psi_\mathcal{S}, \Gamma], \Xi]\big] = 0,$$

under the conditions that $\mathcal{Q}\Gamma = 0$, $\mathcal{Q}\Xi = 0$ and $[\Psi_\mathcal{S}, \Psi_\mathcal{S}] = 0$. That the algebra is closed under the de Rham differential follows from its commutativity with $\mathcal{Q}$. $\square$

Proposition 3.3.3. *The algebra $\mathcal{H}(\mathcal{Q})$ is a $\mathcal{H}_1(Q)$-module under the interior product.*

Proof. Given $F \in \mathcal{H}_1(Q)$, the interior product of F with a function Γ on $\mathcal{L}$ is given by

$$\imath_F \Gamma = (-1)^F [F, \Gamma]\big|_\mathcal{L}. \tag{3.35}$$

Then for an element $\Xi \in \mathcal{H}(\mathcal{Q})$,

$$\begin{aligned}\mathcal{Q}(\imath_F \Xi) &= (-1)^F \big[\Psi_\mathcal{S}, [F, \Xi]\big]\big|_\mathcal{L} \\ &= (-1)^F \big[[\Psi_\mathcal{S}, F], \Xi\big]\big|_\mathcal{L} + \big[F, [\Psi_\mathcal{S}, \Xi]\big]\big|_\mathcal{L}.\end{aligned}$$

The term $[\Psi_\mathcal{S}, F]$ vanishes due to observation (3.29) which corresponds to the projectibility of F. The second term vanishes since Ξ is a $\mathcal{Q}$-cocycle. Hence $\imath_F \Xi$ is also $\mathcal{Q}$-closed. $\square$

Notice that when $F = F_U$ and $\Gamma = \Gamma_\sigma$ for a vector field U and a differential form σ, we recover the interior product $\imath_U \sigma$ on the space of leaves N in the lowest resolution degree.

From the Cartan formula $\mathcal{L}_F = [d, \imath_F]$, the Lie derivative of a $\mathcal{Q}$-cocycle Γ over a Q-cocycle F is given by

$$\mathcal{L}_F\Gamma = (-1)^F \big[[\Psi, F], \Gamma\big]\big|_{\mathcal{L}}. \tag{3.36}$$

Defining the function $\Psi_F = (-1)^F[\Psi, F]$ with $\mathrm{gh}(\Psi_F) = \mathrm{gh}(F) + 1$, the Lie derivative can be written as the bracket $\mathcal{L}_F\Gamma = [\Psi_F, \Gamma]$. Notice that the operations have the following degrees:

$$d : \mathcal{H}_l^m(\mathcal{Q}) \to \mathcal{H}_{l-1}^{m+1}(\mathcal{Q}), \qquad \imath_F : \mathcal{H}_l^m(\mathcal{Q}) \to \mathcal{H}_{l+1}^{m-1}(\mathcal{Q}),$$
$$\mathcal{L}_F : \mathcal{H}_l^m(\mathcal{Q}) \to \mathcal{H}_l^m(\mathcal{Q}),$$

if F is homogeneous of degree $\deg(F) = 1$. It is clear that if F and Γ are both appropriate cocycles, then so is the Lie derivative.

Proposition 3.3.4. *For functions $F, G \in \mathcal{H}_1(Q)$, the following formula hold:*

$$[\mathcal{L}_F, \mathcal{L}_G] = \mathcal{L}_{\{F,G\}_\nabla}, \qquad [\imath_F, \imath_G] = 0,$$
$$[\mathcal{L}_F, \imath_G] = \imath_{\{F,G\}_\nabla}.$$

Proof. All follow from applications of the Jacobi identity. ☐

Proposition 3.3.5. *Let $\Gamma_\sigma, \Gamma_\tau \in \mathcal{H}_{-k}^k(\mathcal{Q})$ correspond to projectible differential k-forms on M, and let $F_U \in \mathcal{H}_1^1(Q)$ correspond to a projectible vector field which preserves these differential forms, $\mathcal{L}_{F_U}\Gamma_\sigma = \mathcal{L}_{F_U}\Gamma_\tau = 0$. Then the Koszul bracket is also preserved,*

$$\mathcal{L}_{F_U}\{\Gamma_\sigma, \Gamma_\tau\}_{\Psi_\mathcal{S}} = 0. \tag{3.37}$$

Proof. By the Jacobi identity,

$$[\Psi_{F_U}, [[\Psi_\mathcal{S}, \Gamma_\sigma], \Gamma_\tau]] = [[\Psi_{F_U}, [\Psi_\mathcal{S}, \Gamma_\sigma]], \Gamma_\tau] + [[\Psi_\mathcal{S}, \Gamma_\sigma], [\Psi_{F_U}, \Gamma_\tau]]$$
$$= [[[\Psi_{F_U}, \Psi_\mathcal{S}], \Gamma_\sigma]], \Gamma_\tau] + [[\Psi_\mathcal{S}, [\Psi_{F_U}, \Gamma_\sigma]], \Gamma_\tau] + [[\Psi_\mathcal{S}, \Gamma_\sigma], [\Psi_{F_U}, \Gamma_\tau]].$$

The first term vanishes due to the projectibility of U and the nilpotency of Ψ; the second follows from proposition 3.3.3, and the third vanishes from the projectibility of σ. ☐

Corollary 3.3.1. *Let $F \in \mathcal{H}_0^0(\mathcal{Q})$ be a $\mathcal{Q}$-cocycle on $\mathcal{M}$ and U a projectible vector field such that $\mathcal{L}_{F_U}F = 0$. Then for a projectible 1-form σ, $\{\Gamma_\sigma, F\}_{\Psi_\mathcal{S}}$ corresponds to a projectible function invariant over the flow of U on N. That is, if U describes dynamics on the leaf space N, then an integral of this motion can be used to produce further integrals from projectible 1-forms.*

3.3.2 A Generalised Poisson Bracket

For an even Poisson manifold M, the article [Mic85] introduced a generalised even Poisson bracket on the space of differential forms modulo exact forms – the coexact differential forms. Unlike the Koszul bracket which is an odd lift of the even Poisson bracket, this generalised Poisson bracket is seen to be a direct even lift of the base structure. A consequence of the existence of this even lift is that the space of differential forms must be restricted to ensure all the conditions of the Poisson bracket are fulfilled. It was shown in [CV92] that the de Rham differential d provides a homomorphism of Lie algebras in such a way that we may define the generalised even Poisson bracket on coexact forms as the bracket $[-,-]_{co}$ such that

$$d[\sigma,\tau]_{co} = (-1)^{\sigma}\{d\sigma, d\tau\}_{K_P},$$

for coexact differential forms σ and τ and the Koszul bracket (1.36).

The higher Koszul bracket hierarchy defined in [KV08] allows a generalised homotopy Poisson structure to be introduced on the algebra of coexact differential forms in precisely the same way. Define the kth bracket $[\cdots]_{co}$ by

$$d[\sigma_1,\ldots,\sigma_k]_{co} = (-1)^{(k-1)\sigma_1+\cdots+\sigma_{k-1}}\{d\sigma_1,\ldots,d\sigma_k\}_{K_P},$$

for coexact differential forms on M and the kth Koszul bracket. These brackets form a homotopy Poisson algebra which can be checked by the manipulation of signs to obtain the Jacobi identity as in the proof of proposition 1.3.2.

The space of projectible coexact forms

$$\mathcal{H}(\mathcal{Q})/d\mathcal{H}(\mathcal{Q}),$$

can thus be equipped with an even generalised Poisson bracket, corresponding to the binary Koszul bracket introduced on $\mathcal{H}(\mathcal{Q})$. Therefore, the generalised Hamiltonian mechanics introduced in [Mic85] can be naturally extended to include systems with gauge symmetries using the idea of projectible differential forms.

Chapter 4

Odd Tensor Geometries

On a supermanifold there are four types of tensor controlled geometries that arise, each corresponding to one of the four different types of rank 2 tensor fields; those tensors that are even or odd, and symmetric or anti-symmetric. When the tensors in question are even, the super Riemannian or super symplectic geometries that are obtained can be seen as trivial extensions of those on a non-super manifold by accommodating the odd variables. Such even structures have been extensively studied, and results transfer mutatis mutandis to the super category. The case of odd tensors however generates two geometries which have no classical analogue: the odd symplectic geometry of the BV-formalism [BV81], and the mysterious odd Riemannian geometry.

In the usual case, even Riemannian and even symplectic geometries possess a sharp contrast in the symmetry of the defining tensor fields which properly distinguishes these geometries. The objective of this chapter is to discuss the odd case where this dichotomy of symmetry no longer exists, and we suggest that the Lie algebra of infinitesimal isometries of the structures should be the correct objects to consider in order to properly distinguish between these.

The majority of the material in this chapter is text-book knowledge, and is solely included for discussion and interest, though the discussion between the differences of the odd geometries appears in the short article [KP16].

4.1 Symmetry of Tensors

Even Riemannian geometry is defined by a Riemannian metric, a non-degenerate rank 2 symmetric covariant tensor field G. This metric determines a very rigid structure with the space of infinitesimal isometries, the Killing vector fields, possessing finite dimension. On the other hand, a Riemannian structure offers a very rich set of invariants, for example the Levi-Civita connection allows the definition of the curvature tensor together with its covariant derivatives which completely define the Riemannian structure.

Even symplectic geometry offers a polar contrast to the Riemannian case, where the symplectic structure is now defined by an anti-symmetric, non-degenerate rank 2 covariant tensor field ω. The space of infinitesimal isometries of this geometry is infinite-dimensional, since each isometry is given by a Hamiltonian vector field generated by a function on the manifold. The flexibility of the 2-form is now offset by the lack of local invariants; Darboux's theorem implies that all symplectic manifolds are locally equivalent, and only the dimension of the manifold can offer any kind of distinguishing local feature.

In both of these cases the symmetry of the defining tensor fields can be used to properly distinguish between the two structures.

Consider then what can be obtained for the odd analogues. Odd Riemannian geometry still remains obscured, since there lacks a detailed study of even its basic properties. Much more is known however in the case of odd symplectic geometry, which has gained popularity since the advent of the BV-formalism for Lagrangian gauge field theories [BV81]. There is for example, a Darboux theorem for an odd symplectic structure which again tells us that the dimension of the manifold is the only local invariant. What is perhaps peculiar is that odd symplectic geometry shares many features with even Riemannian geometry. For instance, both admit a canonical Laplace operator which is deeply related to the scalar curvature of the manifold [BB09]. It is this unique blend of features from both symmetric and anti-symmetric backgrounds which suggests that the symmetry of an odd tensor may not be such of a defining feature of the corresponding geometry.

Below is a table listing the 4 types of tensor geometries on a supermanifold together with their inverse tensors and transformation laws:

Even Riemannian	Odd Riemannian	Even Symplectic	Odd Symplectic
$g_{ab} = \underset{(a+1)(b+1)}{-g_{ba}}$	$\chi_{ab} = \underset{ab}{\chi_{ba}}$	$\omega_{ab} = \underset{ab}{-\omega_{ba}}$	$E_{ab} = \underset{(a+1)(b+1)}{E_{ba}}$
$g^{ab} = \underset{ab}{g^{ba}}$	$\chi^{ab} = \underset{(a+1)(b+1)}{\chi^{ba}}$	$P^{ab} = \underset{(a+1)(b+1)}{P^{ba}}$	$S^{ab} = \underset{ab}{S^{ba}}$
$g_{ab} = \underset{b(k+1)}{x_a^i g_{ik} x_b^k}$	$\chi_{ab} = \underset{b(k+1)}{x_a^i \chi_{ik} x_b^k}$	$\omega_{ab} = \underset{k(b+1)}{x_a^i \omega_{ik} x_b^k}$	$E_{ab} = \underset{k(b+1)}{x_a^i E_{ik} x_b^k}$

Shorthand notation is used here such that $\underset{(a+1)(b+1)}{-g_{ba}} = -(-1)^{(a+1)(b+1)} g_{ba}$ and so on. In the super case there is a convention of which sign rules to adopt, and we will require the symmetry of the components g_{ab} and E_{ab} as above so as to write their inverse tensors as symbols of second order differential operators without introducing extra signs.

Let V be a vector space. From the table we see that for a rank 2 contravariant tensor T with inverse t such that $T^{ab} t_{bc} = \delta_c^a$, it obeys the symmetry rule

$$T^{ab} = \pm(-1)^{ab} T^{ba} \quad \Rightarrow \quad t_{ab} = \mp(-1)^{(a+1)(b+1)+t} t_{ba}.$$

If T (and hence t,) is even, then a symmetric tensor on V has an anti-symmetric inverse on the space with reversed parity ΠV, and similarly, an anti-symmetric tensor on V has a symmetric inverse on ΠV. This apparent change in symmetry is purely cosmetic however. The parity reversal functor $\Pi : V \mapsto \Pi V$ defines a canonical isomorphism $V \otimes V \cong \Pi V \otimes \Pi V$, which locally maps tensors

$$T^{ab} \mapsto \hat{T}^{ab} = (-1)^a T^{ab}.$$

This isomorphism induces another canonical isomorphism $S^2(V) \cong \bigwedge^2(\Pi V)$ from the symmetric square of V to the wedge square of ΠV. Hence in the even case, the inverse of a symmetric/anti-symmetric tensor is again symmetric/anti-symmetric.

The contrast appears in the odd case where the inverse of a symmetric odd tensor on V is a symmetric odd tensor on ΠV, isomorphic to an anti-symmetric tensor on V. There is a shift in symmetry, since the inverse of a symmetric tensor must be anti-symmetric to negate the sign introduced by the parity of the odd elements. Similarly, the inverse of an anti-symmetric odd tensor must now be symmetric. Because of this, both symmetric and anti-symmetric tensor fields lead to odd Riemannian and odd symplectic geometries, which suggests that the symmetry of the defining tensors should not be such a strongly distinguishing feature as in the even cases. For further details on the symmetry properties of tensors, we refer to the book [Man97], or the appendix of [Vor16].

4.2 Cartan Prolongation

Let V be a vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra of the Lie algebra $\mathfrak{gl}(V)$. One defines the Lie algebra of formal vector fields on V as

$$\mathrm{Vect}(V) := \bigoplus_{k \geq 0} S^k V^* \otimes V,$$

where $S^k V^*$ is the space of rank k symmetric covariant tensors on V. The natural Lie bracket on $\mathrm{Vect}(V)$ lowers the polynomial degree by 1, and hence

$$\mathfrak{gl}(V) \cong V^* \otimes V \hookrightarrow \mathrm{Vect}(V),$$

is a Lie subalgebra of $\mathrm{Vect}(V)$. The space V acts naturally on $\mathrm{Vect}(V)$ by insertion; for $u \in V$ and $g \in \mathrm{Vect}(V)$, define the map $\imath_u : S^k V^* \otimes V \to S^{k-1} V^* \otimes V$ by

$$\imath_u g = [g, u].$$

Definition 4.2.1. [RU12] The *Cartan prolongation* of the Lie algebra $\mathfrak{g}$ is the space $\mathfrak{g}^{(\infty)} = V \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \cdots$, where $\mathfrak{g}^{(0)} = \mathfrak{g}$, and the kth Cartan prolongation $\mathfrak{g}^{(k)}$ is defined inductively as

$$\mathfrak{g}^{(k)} = \left\{ g \in S^k V^* \otimes V \,\middle|\, \forall u \in V,\ \imath_u g \in \mathfrak{g}^{(k-1)} \right\}.$$

In particular, elements of the kth Cartan prolongation of a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ are tensors $T^i_{jm_1 \cdots m_k}$, which are symmetric over all lower indices, and for fixed values of $m_1, \cdots, m_k$, the tensor $T^i_{jm_1 \cdots m_k}$ belongs to $\mathfrak{g}$.

Example 4.2.1. Consider the Lie algebra consisting of all rotations and dilations of $\mathbb{R}^n$,

$$\mathfrak{co}(n) = \left\{ X \in \mathrm{Vect}(\mathbb{R}^n) \,\middle|\, \mathcal{L}_X G = \lambda G \right\},$$

dilating the standard Euclidean metric G by $\lambda = \lambda(x) \neq 0$. From the condition $\mathcal{L}_X G = \lambda G$, we obtain the equation

$$\frac{\partial X^k}{\partial x^i} I_{kj} + I_{ik} \frac{\partial X^k}{\partial x^j} = \lambda \delta_{ij},$$

and writing $T_i^j = \partial_i X^j$ we come to

$$T_j^i + T_i^j = \lambda \delta_{ij}, \tag{4.1}$$

in the individual matrix entries. Differentiating (4.1) by any fixed coordinate x^k we obtain

$$T_{kj}^i + T_{ki}^j = \partial_k \lambda \delta_{ij} = \lambda_k \delta_{ij}. \tag{4.2}$$

The tensors T_{kj}^i are certainly symmetric with respect to the lower indices, and belong to the first Cartan prolongation $\mathfrak{co}^{(1)}(n)$ of $\mathfrak{co}(n)$. Rearranging (4.2) as $T_{kj}^i = \lambda_k \delta_{ij} - T_{ki}^j$ and observing the symmetry of the lower indices gives

$$T_{kj}^i = \frac{1}{2} \left(\lambda_k \delta_{ij} - \lambda_i \delta_{kj} + \lambda_j \delta_{ik} \right).$$

The first Cartan prolongation of $\mathfrak{co}(n)$ is seen to be spanned by tensors of this form, and so $\mathfrak{co}(n)$ admits an n-dimensional first Cartan prolongation $\mathfrak{co}^{(1)}(n)$. Consider repeating this for the second prolongation:

$$T_{lkj}^i + T_{lki}^j = \lambda_{lk} \delta_{ij}, \qquad T_{lkj}^i = \frac{1}{2} \left(\lambda_{lk} \delta_{ij} - \lambda_{li} \delta_{kj} + \lambda_{lj} \delta_{ik} \right). \tag{4.3}$$

Comparing the symmetry of the lower indices we come to the conclusion that $\lambda_{lk} = 0$. Therefore the second Cartan prolongation of $\mathfrak{co}(n)$ is trivial, and hence

$$\mathfrak{co}^{(\infty)}(n) = \mathbb{R}^n \oplus \mathfrak{co}(n) \oplus \mathfrak{co}^{(1)}(n),$$

which is of dimension $n + n(n-1)/2 + 1 + n = (n+1)(n+2)/2$. The Cartan prolongation of the conformal Lie algebra on $\mathbb{R}^n$ therefore consists of n generators of translations, $n(n-1)/2$ generators of rotations, 1 dilation, and n special conformal transformation generators. We mention that during the calculation of (4.3) we find that $n \neq 2$. In dimension 2 there is an infinite-dimensional group of local conformal transformations, and indeed, the calculations blow up in this case reflecting this.

Now consider the case of the Lie algebras preserving even Riemannian and even symplectic structure on a $2n$-dimensional space $\mathbb{R}^{2n}$.

Let G be the standard Euclidean metric, and let K be a Killing vector field; $\mathcal{L}_K G = 0$. Then locally,

$$\frac{\partial K^i}{\partial x^j} + \frac{\partial K^j}{\partial x^i} = 0.$$

Write $T^i_j = \partial_j K^i$, and differentiate this with respect to any coordinate x^k,

$$T^i_{kj} = \frac{\partial K^i}{\partial x^k \partial x^j}.$$

Since the tensor is symmetric in indices k and j, and anti-symmetric with respect to indices i and j, it must vanish identically:

$$T^i_{kj} = -T^k_{ij} = -T^k_{ji} = T^j_{ki} = T^j_{ik} = -T^i_{jk} = -T^i_{kj}.$$

Since $T^i_{jk} \equiv 0$, we see that $K^i = x^j A^i_j + B^i$, showing that all infinitesimal isometries of a Euclidean metric are infinitesimal translations and rotations. Since each T^i_{jk} belongs to the first Cartan prolongation of the Lie algebra $\mathfrak{so}(2n)$, the first Cartan prolongation of $\mathfrak{so}(2n)$ vanishes identically, and so the rigidity of a Riemannian metric is encoded in the vanishing of the first prolongation of its Lie algebra of infinitesimal symmetries.

One can consider a symplectic structure on $\mathbb{R}^{2n}$ given in standard Darboux coordinates. An infinitesimal canonical transformation L gives the equation

$$\frac{\partial L^i}{\partial x^j} - \frac{\partial L^j}{\partial x^i} = 0. \tag{4.4}$$

Then analogous to the Riemannian case, one can calculate the first Cartan prolongation of the symplectic Lie algebra $\mathfrak{sp}(n)$. Since the tensor arising from equation (4.4) is symmetric in all indices, there is an infinite dimensional space of solutions for this. Indeed, let J be the matrix defining the standard even symplectic structure in Darboux coordinates. Then we have

$$L^i = J^{ij}\partial_j f, \qquad f \in C^\infty(\mathbb{R}^{2n}),$$

for all functions f. This coincides with the well-known fact that all Hamiltonian vector fields preserve the symplectic structure. Observe that all higher Cartan prolongations are also non-trivial. Any arbitrary rank $k+2$ symmetric tensor $L_{kjm_1\cdots m_k}$ defines a tensor $L^i_{jm_1\cdots m_k} = J^{ik} L_{kjm_1\cdots m_k}$ belonging to the kth Cartan prolongation.

The conclusions are analogous in the super case for the odd structures, only the signs must be inserted in the calculations. This suggests that the distinguishing feature of odd Riemannian and odd symplectic geometry should not be in the symmetry of the defining tensors, but in the differences in the Cartan prolongation of the corresponding Lie algebras of infinitesimal symmetries.

Chapter 5

Odd Laplace Operators, their Potential Fields, and the Modular Class of an Odd Poisson Manifold

The goal of this chapter is to introduce and study a class of odd Laplace type differential operators associated to an odd Poisson manifold. In the odd symplectic case, this class will incorporate the famous BV-operator introduced in [BV81], or more precisely, Khudaverdian's extension of the BV-operator to semidensities [Khu04]. To any operator in this class we assign a geometrical object called the potential field, whose properties suggest a deep connection to the geometry of an odd Poisson manifold. We then show that such a class of operators may be used to describe the modular class of an odd Poisson manifold introduced in [KV02], and we use this operator description to provide the first examples of an odd Poisson manifold with a non-trivial modular class. The majority of the material in this chapter appears in the two articles [KP17] and [KP16].

5.1 The Algebra of Densities

To begin, we recall some necessary facts concerning the algebra of densities $\mathcal{F}(M)$; a commutative algebra associated to any manifold M which provides a natural setting to study differential operators (see for example [OT05, KV13] and references therein).

Definition 5.1.1. A *density of weight* $\lambda \in \mathbb{R}$ on a manifold M is a geometric object with the appearance $\mathbf{s} = s(x)|\mathcal{D}x|^{\lambda}$, such that, under a change of local coordinates $x = x(x')$, the density transforms as

$$\mathbf{s} = s(x(x'))\mathrm{Ber}\left(\frac{\partial x}{\partial x'}\right)^{\lambda} |\mathcal{D}x'|^{\lambda}. \tag{5.1}$$

Densities of weight λ transform according to the λth power of the Berezinian of the Jacobian matrix associated to the coordinate change. There exist densities on a manifold with certain distinguished weights: those of weight zero are identified with smooth functions on M, whilst those of weight 1 correspond to volume elements. Our interest will primarily be in densities of weight 1/2; half-densities or semidensities. Note that bold face will be used to distinguish densities and geometric objects acting on densities from those acting on the algebra of functions $C^{\infty}(M)$.

The vector space of all homogeneous densities of weight λ on M is written $\mathcal{F}^{\lambda} = \mathcal{F}^{\lambda}(M)$. Together these spaces form a natural $\mathbb{R}$-graded algebra $\mathcal{F}(M) = \oplus_{\lambda\in\mathbb{R}}\mathcal{F}^{\lambda}(M)$, where the multiplication of two homogeneous densities $\mathbf{s} \in \mathcal{F}^{\lambda}, \mathbf{t} \in \mathcal{F}^{\mu}$ is defined by

$$\mathbf{st} = s(x)t(x)|\mathcal{D}x|^{\lambda+\mu} \in \mathcal{F}^{\lambda+\mu}.$$

An arbitrary inhomogeneous density on M is a finite linear combination of homogeneous densities of arbitrary weights. Such a finite linear combination under an $\mathbb{R}$-grading is called a pseudo-polynomial [KV14].

Remark 5.1.1. Densities of weight λ on a manifold can equivalently be described as sections of the determinant line bundle $|\mathrm{Vol}(M)|^{\otimes\lambda} = |\mathrm{Ber}(T^*M)\backslash M|^{\otimes\lambda}$, for which a local frame is provided by the element $|\mathcal{D}x|^{\lambda}$. Multiplication of densities then corresponds to the natural identification $|\mathrm{Vol}(M)|^{\otimes\lambda}\otimes|\mathrm{Vol}(M)|^{\otimes\mu} \cong |\mathrm{Vol}(M)|^{\otimes(\lambda+\mu)}$ [KV14].

The algebra of densities can be interpreted as a subalgebra of the function space of an extended manifold $\hat{M}$. Introducing a formal variable t to replace $|\mathcal{D}x|$, which is assumed to be positive, invertible and of even Grassmann parity, the variables x^a, t may be viewed as local coordinates on a line bundle $\hat{M} \to M$. This line bundle is the positive half of the bundle $\mathrm{Ber}(TM)$, where the positive section t of $\mathrm{Ber}(T^*M)$ is seen as a function on the dual bundle. Under a change of coordinates $x = x(x')$, the variable t transforms as

$$t = \mathrm{Ber}\left(\frac{\partial x}{\partial x'}\right) t'.$$

The algebra of densities $\mathcal{F}(M)$ can then be identified with a subalgebra of the algebra of functions $C^\infty(\hat{M})$, consisting of all pseudo-polynomial functions in t under the $\mathbb{R}$-grading.

Remark 5.1.2. The distinction of the subalgebra is important. In particular, the $\mathbb{R}$-grading does not extend to the whole algebra $C^\infty(\hat{M})$. As a consequence it is beneficial to assume that $\hat{M}$ is an $\mathbb{R}$-graded manifold, and so the algebra of densities becomes the entire algebra of functions on the now $\mathbb{R}$-graded $\hat{M}$. Assuming this, we will be able to interpret derivations of $\mathcal{F}(M)$ as vector fields on $\hat{M}$ and so on.

For densities $\mathbf{s}, \mathbf{t} \in \mathcal{F}(M)$, a derivation $\mathbf{X}$ of $\mathcal{F}(M)$ satisfies

$$\mathbf{X}(\mathbf{st}) = \mathbf{X}(\mathbf{s})\mathbf{t} + (-1)^{\mathbf{Xs}}\mathbf{s}\mathbf{X}(\mathbf{t}).$$

The $\mathbb{R}$-grading extends to derivations, and it can be seen that a homogeneous vector field $\mathbf{X}$ of weight λ on $\hat{M}$ has the local expression

$$\mathbf{X}(x,t) = t^\lambda X^a(x)\partial_a + t^{\lambda+1}X^0(x)\partial_t.$$

The algebra of densities also comes equipped with a natural scalar product. For two compactly supported densities $\mathbf{s} \in \mathcal{F}^\lambda$ and $\mathbf{t} \in \mathcal{F}^\mu$, define

$$(\mathbf{s},\mathbf{t}) = \begin{cases} \int_M s(x)t(x)|\mathcal{D}x| & \text{if } \lambda+\mu = 1, \\ 0 & \text{otherwise }. \end{cases} \tag{5.2}$$

This canonical scalar product is non-degenerate due to the non-degeneracy of the Berezin integral.

Remark 5.1.3. We will always assume that the integral exists, in particular, that any orientation conditions are satisfied if necessary. For details on orientation conditions on supermanifolds we refer to the book [Vor91].

It is known that any unital algebra possessing a non-degenerate scalar product admits a canonical divergence operator [KV04]. Recall that for such an algebra A, a divergence operator is an even linear map $\mathrm{div} : \mathrm{Der}A \to A$ such that, for $D \in \mathrm{Der}A$ and $a \in A$,

$$\mathrm{div}(aD) = a\mathrm{div}D + (-1)^{aD}D(a).$$

For details on divergence operators see [KSM02]. Therefore when equipped with the canonical scalar product (5.2), the algebra of densities admits a canonical divergence operator defined as follows.

Definition 5.1.2. The *divergence* of a graded vector field $\mathbf{X}$ on $\hat{M}$ is given by

$$\mathrm{div}\mathbf{X} = -\left(\mathbf{X} + \mathbf{X}^{+}\right), \tag{5.3}$$

where $\mathbf{X}^{+}$ is the adjoint of $\mathbf{X}$ regarded as a differential operator on $\mathcal{F}(M)$, defined by $(\mathbf{Xs}, \mathbf{t}) = (-1)^{\mathbf{Xs}}(\mathbf{s}, \mathbf{X}^{+}\mathbf{t})$ for densities $\mathbf{s}, \mathbf{t} \in \mathcal{F}(M)$.

Corollary 5.1.1. *A vector field $\boldsymbol{X}$ on $\hat{M}$ is divergenceless if and only if $\boldsymbol{X}$ is anti-self-adjoint.*

Now let $\mathbf{X}$ be a divergenceless homogeneous derivation of $\mathcal{F}(M)$ of weight zero,

$$\mathbf{X}(x,t) = X^{a}(x)\partial_{a} + tX^{0}(x)\partial_{t}.$$

Then

$$\mathrm{div}\mathbf{X} = 0 \quad \Leftrightarrow \quad \mathbf{X} = -\mathbf{X}^{+} \quad \Rightarrow \quad \partial_{a}X^{a} - X^{0} = 0.$$

Therefore any divergenceless derivation of weight zero takes the local form

$$\mathbf{X}(x,t) = X^{a}(x)\partial_{a} + t\partial_{a}X^{a}(x)\partial_{t}.$$

Proposition 5.1.1. *Let $\boldsymbol{X}$ be a homogeneous derivation of weight zero. If $\boldsymbol{X}$ is anti-self-adjoint, that is, $\boldsymbol{X}$ has zero divergence, then $\boldsymbol{X}$ acts on densities by the Lie derivative of the associated vector field $X = X^{a}(x)\partial_{a}$ on the manifold M: for a homogeneous density $\boldsymbol{s}$ of weight λ,*

$$\boldsymbol{Xs} = X^{a}\partial_{a}\boldsymbol{s} + \lambda\partial_{a}X^{a}\boldsymbol{s} = \mathcal{L}_{X}\boldsymbol{s}.$$

Remark 5.1.4. The relationship between divergence operators, connections and volume forms on a manifold is well-studied in the article [KSM02]. Consider a volume form $\boldsymbol{\rho} = \rho(x)|\mathcal{D}x|$ on M. This defines a non-degenerate scaler product in the algebra of functions $C^{\infty}(M)$ by $(f,g)_{\boldsymbol{\rho}} = \int_{M} fg\boldsymbol{\rho}$, and as such, an associated canonical divergence operator: for $X \in \mathrm{Vect}(M)$,

$$\left(\mathrm{div}_{\boldsymbol{\rho}}X\right)\boldsymbol{\rho} = \mathcal{L}_{X}\boldsymbol{\rho}, \qquad \mathrm{div}_{\boldsymbol{\rho}}X = (-1)^{a(X+1)}\frac{1}{\rho}\partial_{a}\left(\rho X^{a}\right). \tag{5.4}$$

A choice of volume form on M further allows the identification of semidensities with functions. For a semidensity $\mathbf{s} = s(x)\sqrt{|\mathcal{D}x|}$ on M, define the function $f_{\mathbf{s}}$ by

$$f_{\mathbf{s}} = \mathbf{s}\boldsymbol{\rho}^{-\frac{1}{2}} = \frac{s(x)}{\sqrt{\rho(x)}} \tag{5.5}$$

since $\rho(x)$ is assumed to be a Grassmann even, invertible function. This one to one correspondence identifies the canonical divergence operator (5.3) with that depending on the volume form (5.4).

The notion of a divergence operator is equivalent to the existence of a connection in the bundle Vol(M). This induces a covariant derivative of volume forms over vector fields on M, locally given by

$$\nabla_X \boldsymbol{\rho} = (X^a \partial_a \rho + X^a \gamma_a \rho)\, |\mathcal{D}x|,$$

where γ_a are connection coefficients that transform according to

$$\gamma_a = x_a^{a'} \left(\gamma_{a'} + \partial_{a'} \log \mathrm{Ber} \left(\frac{\partial x}{\partial x'} \right) \right). \tag{5.6}$$

(Recall the notation $x_a^{a'} = \partial x^{a'} / \partial x^a$.) Conversely, to any connection in the line bundle Vol(M), one can associate a divergence operator on M by

$$\mathrm{div}_\nabla X = (-1)^{a(X+1)} \left(\partial_a - \gamma_a \right) X^a. \tag{5.7}$$

A choice of volume form specifies a canonical flat connection in Vol(M) by defining

$$\gamma_a = -\partial_a \log \rho. \tag{5.8}$$

With this choice, the two induced divergence operators (5.4) and (5.7) coincide. Another important example comes from an affine connection ∇ on M whose connection coefficients are given by the Christoffel symbols Γ^c_{ab}. A connection in Vol(M) may then be defined by setting $\gamma_a = -\mathrm{sTr}\Gamma^c_{ab} = -\Gamma^b_{ab}(-1)^b$.

5.2 Laplace Operators and their Potential Fields

An arbitrary odd second order differential operator Δ of weight zero acting on densities on a manifold M has the local expression

$$\Delta = \frac{1}{2} \left(S^{ab}(x) \partial_b \partial_a + t T^a(x) \partial_a \partial_t + t^2 R(x) \partial_t^2 + A^a(x) \partial_a + t B(x) \partial_t + C(x) \right).$$

Since the algebra of densities comes equipped with a canonical scalar product eq. (5.2), one can speak naturally about adjoint operators without the need to introduce further data such as a volume form. For an arbitrary weight zero operator Δ acting on densities, the adjoint operator Δ^+ is an odd operator defined by

$$(\Delta\mathbf{s}, \mathbf{t}) = (-1)^{\mathbf{s}} \left(\mathbf{s}, \Delta^+\mathbf{t}\right), \tag{5.9}$$

for densities $\mathbf{s}, \mathbf{t} \in \mathcal{F}(M)$. It will be of interest to restrict these operators to the subspace of semidensities, where an arbitrary operator Δ takes the local form

$$\Delta = \frac{1}{2}\left(S^{ab}(x)\partial_b\partial_a + V^a(x)\partial_a + U(x)\right).$$

A special class of operators acting on semidensities are those which are self-adjoint, $\Delta = \Delta^+$. In this case, the local form of a self-adjoint operator has a particularly simple expression:

$$\Delta = \frac{1}{2}\left(\partial_b\left(S^{ba}\partial_a\right) + U\right), \tag{5.10}$$

where U is some odd term of order zero.

Under a change of coordinates $x = x(x')$, the expression for a semidensity $\mathbf{s} = \mathbf{s}(x)$ in the new coordinates x' is

$$\mathbf{s}(x') = \mathbf{s}(x(x')), \qquad \mathbf{s}(x') = s(x(x'))\sqrt{\mathcal{J}}\sqrt{|\mathcal{D}x'|},$$

where $\mathcal{J}$ is the Berezinian of the Jacobian matrix of the coordinate transformation. The second order terms of a self-adjoint operator Δ acting on semidensities are then seen to satisfy the transformation law

$$S^{ab} = (-1)^{a(a'+1)} x^a_{a'} S^{a'b'} x^b_{b'},$$

coinciding with the transformation law of an odd rank 2 symmetric contravariant tensor field S.

Recall that there is a natural even identification between symmetric contravariant tensor fields on M and functions on the cotangent bundle T^*M which are polynomial in the fibre variables. Under this identification, an odd rank 2 tensor field S corresponds to an odd fibre-wise quadratic function $S \in C^\infty(T^*M)$,

$$S = \frac{1}{2}S^{ab}\partial_b \otimes \partial_a \quad \leftrightarrow \quad S = \frac{1}{2}S^{ab}p_b p_a.$$

This identification between symmetric polynomial functions on T^*M and symmetric contravariant tensor fields on M will be implicit throughout the rest of this chapter.

We will refer to the function S as the principal symbol of the self-adjoint operator Δ with the local form (5.10), though this is an abuse of language, since the principal symbol of such an operator is actually a function in the extended space of functions $C^\infty(T^*\hat{M})$.

Definition 5.2.1. Fix an odd rank 2 symmetric contravariant tensor field S on M. Define *the class $\mathcal{D}_S$ of odd second order differential operators* acting on semidensities on M such that:

1. An operator $\Delta \in \mathcal{D}_S$ is self-adjoint;

2. The principal symbol of any operator in $\mathcal{D}_S$ is given by the odd fibre-wise quadratic function $S \in C^\infty(T^*M)$ corresponding to the odd tensor field S.

In other words, $\Delta \in \mathcal{D}_S$ has the local appearance (5.10), where S^{ab} are components of the tensor field S.

Proposition 5.2.1. *The class $\mathcal{D}_S$ is non-empty, and any two operators $\Delta, \Delta' \in \mathcal{D}_S$ differ by a scalar function,*

$$\Delta' - \Delta = F(x). \tag{5.11}$$

One can say that $\mathcal{D}_S$ is an affine space of second order operators associated to the space of functions on M.

Proof. First suppose that $\Delta, \Delta' \in \mathcal{D}_S$. Since both Δ and Δ' have the same principal symbol, their difference $\Delta' - \Delta$ is an operator of order at most 1. Suppose that $\Delta' - \Delta = L^a\partial_a + F$ is a first order operator. Since the difference is self-adjoint,

$$L^a\partial_a + F = -\partial_a\left(L^a\right) + F = -L^a\partial_a - \partial_a L^a + F,$$

and so $L^a \equiv 0$. Therefore their difference is an odd scalar function F.

To show that the class is non-empty we shall construct such an operator (using a partition of unity argument). Let $\boldsymbol{\rho} = \rho(x)|\mathcal{D}x|$ be a volume form on M and $S = \frac{1}{2}S^{ab}\partial_b \otimes \partial_a$ be an odd rank 2 symmetric contravariant tensor field.

Given a semidensity $\mathbf{s} \in \mathcal{F}^{1/2}$ we can construct a function $f_{\mathbf{s}} = \mathbf{s}\boldsymbol{\rho}^{-1/2}$ by eq. (5.5), and via the tensor field S, a gradient Hamiltonian vector field $X_{f_{\mathbf{s}}} = S(df_{\mathbf{s}})$. Now consider the divergence of $X_{f_{\mathbf{s}}}$ with respect to the volume form $\boldsymbol{\rho}$ as an odd second order operator on semidensities. Define $\Delta^{\boldsymbol{\rho}}$ as

$$\Delta^{\boldsymbol{\rho}}\mathbf{s} = \frac{1}{2}\sqrt{\boldsymbol{\rho}}\,\mathrm{div}_{\boldsymbol{\rho}} X_{f_{\mathbf{s}}}. \tag{5.12}$$

In local coordinates,

$$\Delta^{\boldsymbol{\rho}}\mathbf{s} = \frac{1}{2}\left(\partial_b\left(S^{ba}\partial_a\right)\mathbf{s} - \frac{1}{2}\partial_a\left(S^{ab}\partial_b\log\rho\right)\mathbf{s} - \frac{1}{4}\partial_a\log\rho S^{ab}\partial_b\log\rho\mathbf{s}\right)\sqrt{|\mathcal{D}x|}.$$

That the operator is self-adjoint follows from that of the corresponding operator $\Delta_{\boldsymbol{\rho}}$ on functions. Define the operator $\Delta_{\boldsymbol{\rho}}$ by

$$\Delta_{\boldsymbol{\rho}} f = \boldsymbol{\rho}^{-1/2}\Delta^{\boldsymbol{\rho}}(f\boldsymbol{\rho}^{1/2}), \tag{5.13}$$

for a function $f \in C^\infty(M)$. The operator $\Delta_{\boldsymbol{\rho}}$ is self-adjoint due to the Green's integral identity

$$\int_M \Delta_{\boldsymbol{\rho}}(f)g\boldsymbol{\rho} = \int_M X_f g\boldsymbol{\rho} = (-1)^{fg}\int_M X_g f\boldsymbol{\rho} = (-1)^f\int_M f\Delta_{\boldsymbol{\rho}}(g)\boldsymbol{\rho},$$

(see [KV02]), and the self-adjoint property of $\Delta^{\boldsymbol{\rho}}$ follows. □

Example 5.2.1. Consider a pair of vector fields X and Y on M of different Grassmann parity. As the Lie derivative $\mathcal{L}_X$ of semidensities on M is a first order anti-self-adjoint differential operator of parity $\tilde{X}$ by proposition 5.1.1, we can define an odd self-adjoint operator Δ on semidensities by

$$\Delta = \frac{1}{2}\left(\mathcal{L}_X\mathcal{L}_Y + \mathcal{L}_Y\mathcal{L}_X\right), \tag{5.14}$$

where the odd rank 2 contravariant tensor field S is defined as the symmetric product $S = X \odot Y$ of the vector fields X and Y.

Notice that any symmetric rank 2 tensor field defines such an operator, since there exists a (non-unique) decomposition $S = \sum_\lambda X_\lambda \odot Y_\lambda$ of S into the symmetric product of vector fields. In general, each different decomposition will correspond to a different operator in the same class $\mathcal{D}_S$.

Proposition 5.2.2. *Let $\Delta \in \mathcal{D}_S$ be an odd self-adjoint second order differential operator on semidensities. Given a function $f \in C^\infty(M)$,*

$$\Delta(f\boldsymbol{s}) = (-1)^f f\Delta\boldsymbol{s} + \mathcal{L}_{X_f}\boldsymbol{s}, \tag{5.15}$$

where X_f is the associated gradient vector field $X_f = S(df)$.

Proof. Write $\Delta(f\mathbf{s}) = (-1)^f f\Delta\mathbf{s} + [\Delta, f]\mathbf{s}$. Then $[\Delta, f]$ is a first order operator on semidensities, and is shown to be anti-self-adjoint directly from the definition (5.9). Hence, with principal symbol $X_f = S(df)$, the operator coincides with the Lie derivative over the vector field X_f by proposition 5.1.1. $\square$

5.2.1 The Potential Field

Together with the odd symmetric tensor field S, an odd object of order zero is required to full describe an operator $\Delta \in \mathcal{D}_S$. This is clear from expression (5.10). In fact this object is geometrical, and later we see that some of its properties suggest close connections with the geometry of the manifold when equipped with an odd Poisson structure.

Definition 5.2.2. Given a manifold M and an odd rank 2 symmetric contravariant tensor field S, the geometrical object U in expression (5.10) is called *the potential field* of the operator $\Delta \in \mathcal{D}_S$, or *a second order compensating field* of the operator Δ.

The name "second order compensating field" is a consequence of its action under diffeomorphisms on M. In the same way that connection coefficients compensate local coordinate changes for a covariant derivative in the first derivatives, the potential field compensates those in the operator Δ in second derivatives. We will see that the potential field U has many similarities with that of a connection.

Under a change of coordinates $x = x(x')$, the compensating field transforms as

$$U = U' + \frac{1}{2}\partial_{a'}\left(S^{a'b'}\partial'_b \log \mathcal{J}\right) + \frac{1}{4}\left(\partial_{a'} \log \mathcal{J} S^{a'b'}\partial_{b'} \log \mathcal{J}\right), \tag{5.16}$$

where $\mathcal{J}$ is the Berezinian of the Jacobian matrix of the coordinate transformation.

Proposition 5.2.3. *An operator $\Delta \in \mathcal{D}_S$ is well defined by two geometrical objects:*

1. *The odd rank 2 symmetric contravariant tensor field S defining the principal symbol of the operator;*

2. *A potential field U which transforms according to equation (5.16).*

The space of compensating fields is an affine space associated to the space of functions over M.

Consider some examples of potential fields. Throughout these we will assume that we fix an odd tensor field $S = \frac{1}{2}S^{ab}\partial_b \otimes \partial_a$ on a manifold M.

Example 5.2.2. Let M be equipped with a volume form $\boldsymbol{\rho} = \rho(x)|\mathcal{D}x|$. Then we can consider the operator $\Delta^{\boldsymbol{\rho}}$ on semidensities defined in eq. (5.12). The expression of this operator in coordinates shows that the potential field is locally given by

$$U_{\boldsymbol{\rho}} = -\frac{1}{2}\partial_a\left(S^{ab}\partial_b \log\rho\right) - \frac{1}{4}\partial_a \log\rho S^{ab}\partial_b \log\rho. \tag{5.17}$$

Example 5.2.3. A volume form $\boldsymbol{\rho}$ defines the divergence operator (5.12) with potential $U_{\boldsymbol{\rho}}$ as above. It also induces a connection in the bundle of volume forms defined by the symbols

$$\gamma_a = -\partial_a \log\rho.$$

Expressing the potential $U_{\boldsymbol{\rho}}$ in terms of γ_a,

$$U_{\boldsymbol{\rho}} = U_\gamma = \frac{1}{2}\partial_a\gamma^a - \frac{1}{4}\gamma^a\gamma_a, \tag{5.18}$$

where $\gamma^a = S^{ab}\gamma_b$ defines an upper connection, or a contravariant connection, in the bundle $\mathrm{Vol}(M)$. Formula (5.18) for the potential field holds for any connection in the space of volume forms, for example, when M is equipped with an affine connection. In this case the corresponding second order operator is defined by the divergence with respect to the connection as in eq. (5.7) as opposed to a volume form.

Both of the previous examples rely on the introduction of extra information, whether it be a volume form or a connection in the bundle $\mathrm{Vol}(M)$. The potential field may arise as a primary object however, as the next example describes.

Example 5.2.4. The odd tangent bundle ΠTM is equipped with a natural volume form. For canonical local coordinates x^a, η^a, this is written as

$$\boldsymbol{\rho} = |\mathcal{D}(x,\eta)|. \tag{5.19}$$

This form remains invariant under a change of coordinates: for $x = x(x')$, $\eta^a = \eta^{a'}J^a_{a'}$, we see that

$$|\mathcal{D}(x,\eta)| = \mathrm{Ber}\begin{pmatrix} \frac{\partial x}{\partial x'} & 0 \\ \eta'\frac{\partial^2 x}{\partial x'\partial x'} & \frac{\partial x}{\partial x'} \end{pmatrix} |\mathcal{D}(x',\eta')| = |\mathcal{D}(x',\eta')|.$$

Hence, considering the potential field (5.17), we obtain a canonical potential field given by $U = U_{\boldsymbol{\rho}} \equiv 0$ in coordinates adapted to the structure of the bundle ΠTM.

Remark 5.2.1. An important example of a potential field occurs when one considers operators of non-zero weights. Consider the real line $\mathbb{R}$ and a symmetric tensor density $\partial_x \otimes \partial_x |\mathcal{D}x|^2$. Such a density gives rise to an operator

$$\Delta = \left(\partial_x^2 + U\right) |\mathcal{D}x|^2,$$

where $U = U(x)|\mathcal{D}x|^2$ is the potential density called a projective connection [KV12, OT05]. Under a change of coordinates $y = y(x)$, U transforms by the Schwartzian derivative

$$U(y)|\mathcal{D}y|^2 = U(x)|\mathcal{D}x|^2 - \frac{1}{2}\left(\frac{y_{xxx}}{y_x} - \frac{3}{2}\frac{y_{xx}^2}{y_x^2}\right)|\mathcal{D}x|^2.$$

In particular, it was shown that the existence of a projective connection on the base manifold M gives rise to a canonical connection in the bundle of volume forms, and hence to a canonical second order operator [Geo08].

5.3 The Modular Class of an Odd Poisson Manifold

In the previous section we considered the class of operators $\mathcal{D}_S$ depending on an arbitrary odd rank 2 symmetric tensor field S. A desirable property of such Laplace type operators is nilpotency; for example, the BV-operator introduced in [BV81] is required to be nilpotent.

5.3.1 Nilpotency Conditions

Proposition 5.3.1. *Let S be an odd rank 2 symmetric contravariant tensor field on M and let $\mathcal{D}_S$ be the associated class of differential operators. For any operator $\Delta \in \mathcal{D}_S$, the operator Δ^2 is an even anti-self-adjoint operator of order at most 3. Further:*

1. *The order of the operator Δ^2 is less than 3 if and only if the principal symbol S defines an odd Poisson structure on the manifold M;*

2. *If the order of Δ^2 is less than 3, then it is at most 1. In this case,*

$$\Delta^2 = \mathcal{L}_X, \tag{5.20}$$

for some even vector field X on M;

3. *The operator* $\Delta^2 = 0$ *if and only if the principal symbol* S *defines an odd Poisson structure and the vector field* X *in equation* (5.20) *vanishes.*

Proof. Given semidensities $\mathbf{s}, \mathbf{t} \in \mathcal{F}^{1/2}$,

$$\left(\Delta^2 \mathbf{s}, \mathbf{t}\right) = (-1)^{\Delta(\Delta+\mathbf{s})} \left(\Delta \mathbf{s}, \Delta \mathbf{t}\right) = (-1)^{\Delta(\Delta+\mathbf{s})+\Delta\mathbf{s}} \left(\mathbf{s}, \Delta^2 \mathbf{t}\right),$$

and so $(\Delta^2 \mathbf{s}, \mathbf{t}) = -(\mathbf{s}, \Delta^2 \mathbf{t})$. Hence the square of Δ is an anti-self-adjoint even operator. Since Δ is odd, the square may be expressed as a commutator

$$\Delta^2 = \frac{1}{2}[\Delta, \Delta],$$

and so it must have order at most 3.

Considering the operator Δ^2 in local coordinates,

$$\Delta^2 = \frac{1}{2}(-1)^a S^{ai} \partial_i S^{bc} \partial_c \partial_b \partial_a + \text{ lower order terms.}$$

The third order terms coincide precisely with the local expression of the Jacobi identity for the odd Hamiltonian function $S \in C^\infty(T^*M)$. It can be seen that the symbol of Δ^2 defines the cubic Hamiltonian function $(S, S) \in C^\infty(T^*M)$, the vanishing of which is precisely the master equation for the master Hamiltonian S of an odd Poisson structure.

If $S \in C^\infty(T^*M)$ defines an odd Poisson structure on M then the third order terms vanish and the operator Δ^2 is at most second order. However, since Δ^2 is anti-self-adjoint, the second order terms must also vanish. Therefore, if S defines an odd Poisson bracket, then Δ^2 is a first order anti-self-adjoint operator and so by proposition 5.1.1 must be the Lie derivative of some vector field X on M. It is clear then that $\Delta^2 = 0$ if and only if the vector field X vanishes. $\square$

From now we will assume that M is an odd Poisson manifold with fixed master Hamiltonian function S, to which we associate the canonical class of second order differential operators $\mathcal{D}_S$ acting on semidensities on M.

Definition 5.3.1. For an odd Poisson manifold M with master Hamiltonian $S \in C^\infty(T^*M)$ and an operator $\Delta \in \mathcal{D}_S$, the vector field $X = X_\Delta$ such that

$$\Delta^2 = \mathcal{L}_{X_\Delta}$$

is called *the modular vector field associated to the operator* Δ.

Proposition 5.3.2. *For an odd Poisson manifold M and an operator $\Delta \in \mathcal{D}_S$, the modular vector field X_Δ associated to Δ is a Poisson vector field. Further, if $\Delta' = \Delta + F$ is another operator in $\mathcal{D}_S$, the associated modular vector field X_Δ changes by the Hamiltonian vector field X_F,*

$$X_{\Delta'} = X_\Delta + X_F.$$

Proof. The vector field X_Δ preserves the operator since, for any semidensity $\mathbf{s} \in \mathcal{F}^{1/2}$,

$$\mathcal{L}_{X_\Delta} \Delta \mathbf{s} = \Delta^2 \circ \Delta \mathbf{s} = \Delta \circ \Delta^2 \mathbf{s} = \Delta \circ \mathcal{L}_{X_\Delta} \mathbf{s}.$$

It therefore preserves the odd Poisson bracket since it preserves the principal symbol of the operator. Considering a second operator $\Delta' = \Delta + F$,

$$(\Delta')^2 = (\Delta + F)^2 = \Delta^2 + [\Delta, F],$$

since F is an odd function. The operator $[\Delta, F]$ is a first order operator with principal symbol that of the Hamiltonian vector field $X_F = \{F, -\}_S$. It is anti-self-adjoint and hence is given by the Lie derivative over X_F. Therefore

$$\mathcal{L}_{X_{\Delta'}} = \mathcal{L}_{X_\Delta} + \mathcal{L}_{X_F}.$$

□

Recall that the Poisson-Lichnerowicz cohomology of an odd Poisson manifold is obtained from the complex $d_S : C^\infty(T^*M) \to C^\infty(T^*M)$, where $d_S = (S, -)$ is the odd super-analogue of the Lichnerowicz differential. Intuitively, the first Poisson-Lichnerowicz cohomology group $\mathcal{H}^1_{d_S}(M)$ is interpreted as the space of Poisson vector fields on M modulo those which are Hamiltonian. Therefore the modular vector field X_Δ associated to any operator $\Delta \in \mathcal{D}_S$ is a representative of a cohomology class in the first Poisson-Lichnerowicz cohomology, which moreover is independent of the choice of Δ since the modular vector field $X_{\Delta'}$ associated to any other operator $\Delta' \in \mathcal{D}_S$ differs only by a Hamiltonian vector field. We come to the following definition.

Definition 5.3.2. For an odd Poisson manifold M with master Hamiltonian S and associated class of operators $\mathcal{D}_S$, *the modular class of the odd Poisson manifold* is the cohomological class in the first Poisson-Lichnerowicz cohomology represented by the modular vector field associated to any operator $\Delta \in \mathcal{D}_S$.

This class was first introduced in [KV02] in consideration of the operator $\Delta_{\boldsymbol{\rho}}$ acting on functions (eq. (5.13)). In the same paper many properties of the odd operator $\Delta_{\boldsymbol{\rho}}$ were proved, some of which will be beneficial to recall.

Proposition 5.3.3. *For functions $f, g \in C^\infty(M)$, the operator $\Delta_{\boldsymbol{\rho}}$ defined by*

$$\Delta_{\boldsymbol{\rho}} f = \frac{1}{2}\mathrm{div}_{\boldsymbol{\rho}} X_f,$$

satisfies the following together with the odd Poisson bracket $\{-,-\}$ on M:

$$\Delta_{\boldsymbol{\rho}}(fg) = \Delta_{\boldsymbol{\rho}}(f)g + (-1)^f f\Delta_{\boldsymbol{\rho}}(g) + 2\{f,g\}; \tag{5.21}$$

$$\Delta_{\boldsymbol{\rho}}\{f,g\} = -\{\Delta_{\boldsymbol{\rho}} f, g\} - (-1)^f\{f, \Delta_{\boldsymbol{\rho}} g\}; \tag{5.22}$$

$$\mathcal{L}_{\Delta^2_{\boldsymbol{\rho}}}\boldsymbol{\rho} = 0. \tag{5.23}$$

In particular, property (5.21) implies that, up to a factor of 2, the odd Laplacian generates the odd Poisson bracket on M. Odd Poisson algebras whose generating operators are of square zero are called Batalin-Vilkovisky algebras, or BV-algebras [KS00, Xu99]. In this case the first order operator $\Delta^2_{\boldsymbol{\rho}}$ is a vector field, and further is a Hamiltonian vector field if and only if the modular class of the odd Poisson manifold vanishes. It is then seen that the modular class is an obstruction to finding a square zero generator of the odd bracket: if the modular class vanishes, then $\Delta^2_{\boldsymbol{\rho}} = X_f$ for some function $f \in C^\infty(M)$. Define then a new volume form $\boldsymbol{\rho}'$ where

$$\boldsymbol{\rho}' = e^g\boldsymbol{\rho} \qquad \Rightarrow \qquad \Delta^2_{\boldsymbol{\rho}'} = \Delta^2_{\boldsymbol{\rho}} + X_{\Delta_{\boldsymbol{\rho}} g + \frac{1}{2}\{g,g\}},$$

for $g \in C^\infty(M)$. In particular, if $X_f = -X_{\Delta_{\boldsymbol{\rho}} + \frac{1}{2}\{g,g\}}$ then the new operator $\Delta_{\boldsymbol{\rho}'}$ is nilpotent. Notice that if we suppose that $\Delta^2_{\boldsymbol{\rho}} = 0$, then redefining $\boldsymbol{\rho}$ by $\boldsymbol{\rho}'$ again provides a nilpotent operator if g satisfies a Maurer-Cartan type equation

$$\Delta_{\boldsymbol{\rho}} g + \frac{1}{2}\{g,g\} = 0.$$

(Notice that this Maurer-Cartan equation is the quantum master equation when g describes a quantum effective action on an odd symplectic supermanifold.)

The derivation property (5.22) is equivalent to the condition that the operator be self-adjoint. The proof of this relies on the fact that the curvature of the divergence operator with respect to the volume form is precisely zero (see [KSM02]), which is equivalent to the flatness of the induced connection in $\mathrm{Vol}(M)$. The flatness of this connection is equivalent to the self-adjoint property of the operator $\Delta^{\boldsymbol{\rho}}$ acting on semidensities, which is self-adjoint if and only if $\Delta_{\boldsymbol{\rho}}$ is.

5.3.2 Even Poisson Manifolds

Recall that in [ELW99, Wei97], the modular class of an even Poisson manifold (which readily extends to even Poisson supermanifolds,) was constructed as follows. Let M be an even Poisson manifold with corresponding Poisson tensor P, and let $\boldsymbol{\rho}$ be a volume form on M. Then an operator directly analogous to the operator (5.13) may be defined on functions by

$$\Delta_{\boldsymbol{\rho}} f = \mathrm{div}_{\boldsymbol{\rho}} X_f = \frac{1}{\rho}(-1)^{a(X_f+1)}\partial_a \left(\rho P^{ab}\partial_b f\right). \tag{5.24}$$

In the even case, the anti-symmetry of the Poisson bivector P reduces the operator (5.24) to a first order differential operator on functions. The operator is therefore a vector field, and further, it is a Poisson vector field preserving the even Poisson bracket on M. Changing the volume form $\boldsymbol{\rho}' = e^{\sigma}\boldsymbol{\rho}$ alters the vector field $\Delta_{\boldsymbol{\rho}}$ by the Hamiltonian vector field X_σ. Therefore the Poisson vector field $\Delta_{\boldsymbol{\rho}}$ defines a cohomology class in the first Poisson-Lichnerowicz cohomology, the modular class of the even Poisson manifold.

By definition, the modular class of an even Poisson manifold has an intimate relation with volume forms on the manifold. Recall that the operator $\Delta_{\boldsymbol{\rho}}$ is equivalently defined by

$$\Delta_{\boldsymbol{\rho}} f \boldsymbol{\rho} = \mathcal{L}_{X_f}\boldsymbol{\rho},$$

as the coefficient of the volume form when acted upon by the Lie derivative of a Hamiltonian vector field. A volume form $\boldsymbol{\rho}$ is said to be invariant if it remains invariant over the flow of all Hamiltonian vector fields on M, in which case the Poisson manifold is said to be unimodular. This occurs if and only if the modular vector field $\Delta_{\boldsymbol{\rho}}$ vanishes.

For example, for any even symplectic manifold the Liouville form is a natural volume form invariant over the flow of every Hamiltonian vector field. The operator associated to this form vanishes, and so does the modular class of any even symplectic manifold. An example of an even Poisson manifold with non-vanishing modular class is given by $\mathfrak{g}^*$ for a Lie algebra $\mathfrak{g}$. The modular class of $\mathfrak{g}^*$ is represented by the infinitesimal modular character of the Lie algebra, a vector field with constant value $x \mapsto \mathrm{sTr}(\mathrm{ad}_x)$.

As well as for even Poisson manifolds, modular vector fields are well-defined for Lie algebroids, which are connected to generating operators of the corresponding Schouten structure [KS00, KS08, Xu99]. The definition of these however rely on the notion of a top exterior bundle of the corresponding Lie algebroid, employing connections and representations of the Lie algebroid. In the super case this should be replaced by the determinant line bundle, since there is no notion of a top exterior bundle. It would be interesting to interpret the modular class of a Lie algebroid equipped with an odd bracket in the space of sections in terms of the determinant line bundle, and whether any relation exists between the modular class of a Lie algebroid A and the modular class of the equivalent odd Poisson manifold ΠA^*.

5.3.3 A Local Description

It is interesting to provide a local description of the modular vector field associated to an operator Δ. For an arbitrary operator $\Delta \in \mathcal{D}_S$ with local description (5.10), an expression for Δ^2 is obtained, equal to the Lie derivative of the associated modular vector field,

$$X_\Delta = \frac{1}{4}\partial_c\left(S^{cd}\partial_d\partial_b S^{ba}\right)\partial_a + \frac{1}{2}(-1)^a S^{ab}\partial_b U\partial_a. \tag{5.25}$$

For ease, write $\chi^a\partial_a = \frac{1}{4}\partial_c\left(S^{cd}\partial_d\partial_b S^{ba}\right)\partial_a$. Then the Lie derivative

$$\begin{aligned}\mathcal{L}_{X_\Delta} &= \chi^a\partial_a + \frac{1}{2}(-1)^a S^{ab}\partial_b U\partial_a + \frac{1}{2}(-1)^a\partial_a\left(\chi^a + \frac{1}{2}(-1)^a S^{ab}\partial_b U\right)\\ &= X_\Delta + \frac{1}{4}\partial_a\left(S^{ab}\partial_b U\right),\end{aligned}$$

comparing with local calculations for Δ^2. Notice that the naive divergence of χ, $(-1)^a\partial_a\chi^a$, is identically zero. Since χ depends solely on the odd Poisson structure this fact must have a geometrical meaning. Secondly, observe that the term $\chi^a\partial_a$ is not a vector field alone, but depends on terms from the potential field U to transform as a vector field under an arbitrary change of coordinates.

Proposition 5.3.4. *The term χ in the modular vector field X_Δ associated to an operator Δ transforms in the following way for a change of coordinates $x = x(x')$,*

$$\begin{aligned}\chi^{a'} = \partial_c&\left(S^{cd}\partial_d\left(\partial_b S^{ba} + \partial_b\log\mathcal{J}S^{ba}\right)\right)x_a^{a'}\\ &+ \partial_c\log\mathcal{J}S^{cd}\partial_d\left(\partial_b S^{ba} + \partial_b\log\mathcal{J}S^{ba}\right)x_a^{a'}.\end{aligned}$$

Recall that for a connection in the bundle of volume forms, the symbols γ_a transform according to eq. (5.6), which is strikingly similar to components of the term χ. This local analysis suggests that a modular vector field, together with the potential field, should possess close relations with the geometry of the odd Poisson manifold.

We finally observe that for a unimodular change of coordinates (volume preserving coordinate changes such that $\mathcal{J} = \operatorname{Ber}\left(\frac{\partial x}{\partial x'}\right) \equiv 1$,), we obtain

$$\chi^{a'} = \partial_c \left(S^{cd}\partial_d\partial_b S^{ba}\right) x_a^{a'}, \qquad U = U' \tag{5.26}$$

transforming as a vector field and a function.

5.4 Examples

5.4.1 The Symplectic Potential Field

Odd symplectic manifolds are the motivating example for the study of odd Laplace type operators since their introduction in the BV-formalism [BV81] for gauge field theories. Let M be an odd symplectic manifold with master Hamiltonian $E \in C^\infty(T^*M)$, and let x^a, θ_a be Darboux coordinates on M. An odd second order operator Δ_0 acting on semidensities can be naively written as

$$\Delta_0 \mathbf{s} = \frac{\partial^2 s(x,\theta)}{\partial\theta_a \partial x^a}\sqrt{|\mathcal{D}(x,\theta)|}, \tag{5.27}$$

for a semidensity $\mathbf{s} \in \mathcal{F}^{1/2}(M)$. What is remarkable is that this operator remains invariant under canonical transformations, and hence is a well-defined operator on M. This is a consequence of the Batalin-Vilkovisky lemma [BV81, Khu04], which states that

$$\Delta_0 \operatorname{Ber}\left(\frac{\partial(x,\theta)}{\partial(x',\theta')}\right)^{1/2} = 0,$$

for a coordinate change from one set of Darboux coordinates x, θ to another x', θ'. The canonical operator Δ_0 acting on semidensities can be found under the name of Khudaverdian's Δ-operator or the canonical odd Laplacian [Khu04]. In order to relate Δ_0 with the BV-operator on functions, we require a choice of volume form $\boldsymbol{\rho}$ on M such that $\boldsymbol{\rho}$ can be expressed as $|\mathcal{D}(x,\theta)|$ in some set of Darboux coordinates. Such a volume form is called normal, and specifically has constant coefficient 1 in this

Darboux coordinate chart. In this case, the BV-operator on functions $\Delta_{\boldsymbol{\rho}}$ is defined by the conjugation

$$\Delta_{\boldsymbol{\rho}} f = \frac{1}{\sqrt{\boldsymbol{\rho}}} \Delta_0 f \sqrt{\boldsymbol{\rho}} = \frac{\partial^2 f}{\partial \theta_a \partial x^a}, \tag{5.28}$$

for a function $f \in C^\infty(M)$. The benefit of the operator Δ_0 acting on semidensities is that no volume form is needed, lifting the restriction on the existence of a normal volume form.

An interesting question as to what the canonical odd Laplacian looks like in arbitrary local coordinates was answered by Bering [Ber06]. In our notation the operator Δ_0 is such that $\Delta_0 = \frac{1}{2}\left(\partial_a \left(E^{ab}\partial_b\right) + U_0\right)$, where

$$U_0 = \frac{1}{4}\partial_b \partial_a E^{ab} - (-1)^{b(d+1)} \frac{1}{12} \partial_a E^{bc} E_{cd} \partial_b E^{da} \tag{5.29}$$

is the canonical symplectic potential field of the operator Δ_0, and the matrix $||E_{ab}||$ defines the odd symplectic form on M; the calculation of this formula is found at the end of this chapter. Notice that in Darboux coordinates the potential field U_0 vanishes, and the operator Δ_0 is clearly nilpotent. Hence the modular class of any odd symplectic manifold is zero paralleling the even case, despite the absence of a canonical volume form invariant under canonical transformations.

In the same article Bering introduced an odd scalar function associated to a volume form $\boldsymbol{\rho}$ on an odd symplectic manifold. Recall the operator $\Delta^{\boldsymbol{\rho}}$ on semidensities given by eq. (5.12), which has the associated potential field $U_{\boldsymbol{\rho}}$ of eq. (5.17). An odd scalar function $\nu_{\boldsymbol{\rho}}$, depending on $\boldsymbol{\rho}$, may be defined by

$$\Delta_0 \mathbf{s} = \Delta^{\boldsymbol{\rho}} \mathbf{s} + \nu_{\boldsymbol{\rho}} \mathbf{s}. \tag{5.30}$$

It can be seen that $\nu_{\boldsymbol{\rho}}$ is simply the odd function in eq. (5.11) obtained when considering the difference of the two Δ-operators. In particular, it was shown that if this scalar function is adjoined to the canonical BV-operator (5.28) on functions,

$$\Delta'_{\boldsymbol{\rho}} = \Delta_{\boldsymbol{\rho}} + \nu_{\boldsymbol{\rho}},$$

then the requirement that the volume form be normal can be removed completely in the BV-formalism [BB08b]. The addition of this odd scalar however breaks the nilpotency of the operator $\Delta_{\boldsymbol{\rho}}$, which is recovered if and only if the scalar function is an odd constant, i.e. a Casimir of the odd non-degenerate bracket. This construction has further implications in producing a modified quantum master equation of the BV-formalism.

As a consequence of the choice of volume $\boldsymbol{\rho}$, the determinant bundle $\mathrm{Vol}(M)$ comes equipped with a canonical flat connection as given in eq. (5.8). Then given an affine connection over M which is anti-symplectic, torsion free, and is such that the defining Christoffel symbols Γ^b_{ac} satisfy $\gamma_a = \Gamma^b_{ab}$, it was shown by Bering that the odd scalar function $\nu_{\boldsymbol{\rho}}$ is proportional to the scalar curvature of the manifold M. This remarkable connection to the scalar curvature, together with the peculiar transformation laws eqs. (5.16), (5.25) of the potential field and a modular vector field suggest that the potential fields associated to odd Laplace operators should stand as some connection-like objects. In particular, the uniqueness of the symplectic potential allows us to state a Levi-Civita type statement for odd symplectic manifolds.

Proposition 5.4.1. *For any odd symplectic manifold M, there exists a unique symplectic potential U_0 vanishing in Darboux coordinates and with local expression* (5.29) *defining the canonical Δ_0-operator* (5.27)*.*

The results of [Ber06] were extended to a more general setting when the odd Poisson structure is now degenerate but admits a compatible 2-form field in a precise sense [BB08a, Ber08]. In such a case, the odd scalar term $\nu_{\boldsymbol{\rho}}$ can still be defined and the connection with the scalar curvature remains true. Degenerate odd Poisson structures incorporate important examples such as Dirac brackets associated with second class constraints [HT92], or when considering the boundary of odd symplectic superspaces associated with physical theories.

5.4.2 Further Examples

Example 5.4.1. Let M be an even Poisson manifold with even Poisson tensor P, and consider ΠTM equipped with the Koszul bracket of differential forms. Since the manifold ΠTM is provided with a natural volume element $\boldsymbol{\rho} = |\mathcal{D}(x,\eta)|$ eq. (5.19), we may define the operator $\Delta^{\boldsymbol{\rho}}$ on semidensities by eq. (5.12) such that $U_{\boldsymbol{\rho}} = 0$.

The operator $\Delta^{\boldsymbol{\rho}}$ can then be identified with the operator $\Delta_{\boldsymbol{\rho}}$ on functions (5.13) by

$$\Delta_{\boldsymbol{\rho}} f = \boldsymbol{\rho}^{-1/2}\Delta^{\boldsymbol{\rho}}(f\boldsymbol{\rho}^{1/2}), \tag{5.31}$$

for a function $f \in C^\infty(M)$.

Recall that the Koszul-Brylinski operator $\partial_P = [d, \imath_P]$ is a generator of the Koszul bracket on ΠTM, and that with the choice of canonical volume, the operator $\Delta_{\boldsymbol{\rho}}$ defined on functions coincides with this operator (see [KSM02]). It follows from the Jacobi identity for the Poisson tensor P that the operator $\partial_P^2 = 0$, and hence $\Delta_{\boldsymbol{\rho}}^2 = 0$ under the identification. Equivalently, the operator $\Delta^{\boldsymbol{\rho}}$ is nilpotent on semidensities, and the associated modular vector field on the odd Poisson manifold ΠTM is seen to vanish.

This can also be seen from local considerations. The Poisson tensor P is locally of constant rank around any regular point x_0 in the even Poisson manifold M, which allows the introduction of Darboux-Lie coordinates p^a, q_a, y^i in the vicinity of x_0 as described in [Wei83], such that

$$[p^a, q_b]_P = \delta^a_b, \tag{5.32}$$

and all other even brackets vanish. The entries of the matrix of the Poisson tensor about x_0 are constant, and hence so are the entries of the matrix defining the odd Hamiltonian of the Koszul bracket in the corresponding local trivialisation of ΠTM. From the local expression of the associated modular vector field $X = \frac{1}{4}\partial_c \left(S^{cd}\partial_d \partial_b S^{ba}\right)\partial_a$, we see that this vanishes at all regular points of M, and since regular points are dense in M, the modular vector field must vanish at all points.

Example 5.4.2. Suppose that M is an $n|n$-dimensional supermanifold equipped with a regular odd Poisson structure. That is, the odd symplectic leaves of the Poisson manifold are of equal dimension $m|m$ for $m < n$ and admit no transverse Poisson structure. Then there exist Darboux-Lie coordinates of the form (5.32) from which a canonical operator Δ can be defined on semidensities as

$$\Delta \mathbf{s} = \frac{\partial^2 s(p,q,y)}{\partial p^a \partial q_a}\sqrt{|\mathcal{D}(p,q,y)|}, \tag{5.33}$$

analogous to eq. (5.27). The potential field of this operator is calculated in [BB08a] (since any regular Poisson structure admits a compatible 2-form field), which is shown to vanish in Darboux-Lie coordinates. The operator is of square zero, and again we see that the modular class of this odd Poisson manifold vanishes.

The examples so far show that a wide class of odd Poisson manifolds have a vanishing modular class. The following example however presents as the first known example of an odd Poisson manifold admitting a non-trivial modular class and is found in the article [KP17].

Example 5.4.3. Define for a manifold M the product $N = M \times \mathbb{R}^{0|1}$ equipped with local coordinates x^a on M supplemented by a single odd coordinate τ on $\mathbb{R}^{0|1}$. Let A be an even vector field on N such that in local coordinates,

$$A = A^a(x,\tau)\frac{\partial}{\partial x^a}.$$

Analogous to example 5.2.1, an odd second order self-adjoint operator may be defined on semidensities on N by

$$\Delta = \frac{1}{2}\left(\mathcal{L}_A\mathcal{L}_{\partial_\tau} + \mathcal{L}_{\partial_\tau}\mathcal{L}_A\right),$$

via the odd homological vector field ∂_τ. The operator Δ takes the local expression

$$\Delta = A^a\partial_a\partial_\tau + \frac{1}{2}\partial_\tau A^a\partial_a + (-1)^a\frac{1}{2}\partial_a A^a\partial_\tau + (-1)^a\frac{1}{4}\partial_\tau\partial_a A^a.$$

In order for the third order terms of Δ^2 to vanish, the vector field $A = A(x,\tau)$ must be chosen such that

$$\partial_\tau A^a A^b + (-1)^a A^a\partial_\tau A^b = 0. \tag{5.34}$$

Since $\tau^2 = 0$, $A^a(x,\tau) = K^a(x) + \tau\eta^a(x)$ for even and odd vector fields K and η. A trivial solution to eq. (5.34) is given by $\eta^a(x) = 0$, though this results in $\Delta^2 = 0$ also. A non-trivial solution however is provided by $A^a(x,\tau) = \tau\eta^a(x)$, from which arises the odd Poisson bracket on N given by

$$\{\tau, x^a\} = \tau\eta^a(x) = A^a(x,\tau).$$

A calculation of Δ^2 shows that

$$\Delta^2 = \frac{1}{4}\eta^b\partial_b\eta^a\partial_a + \frac{1}{8}\eta^b\partial_b\partial_a\eta^a = \mathcal{L}_X,$$

where X is the modular vector field associated to Δ, and is given by

$$X = \frac{1}{4}\eta^b\partial_b\eta^a\partial_a = \frac{1}{8}[\eta,\eta].$$

The claim is that the vector field X is not Hamiltonian. Indeed, if X is Hamiltonian then there exists $f \in C^\infty(N)$ such that $\{f, -\} = X$. But for any other function

$g \in C^\infty(N)$, the bracket $\{f, g\}$ belongs to the ideal generated by the odd variable τ, whereas the function $X(g)$ does not for $g = g(x)$. Thus the vector field cannot be Hamiltonian, and so X represents a non-trivial cohomology class in the first Poisson-Lichnerowicz cohomology of the odd Poisson manifold N. Such an example can be thought of as an odd time parameterisation of the manifold M, and an adjoint construction to the supermanifold ΠTM (see [KP17]).

Example 5.4.4. Example 5.4.3 fits into a larger class of examples. Let M and N be manifolds with coordinates x^a and y^i respectively, and suppose that N is equipped with a homological vector field $Q = Q^i(y)\partial_i$. Define on the product manifold $M \times N$, an even vector field $A = A^a(x, y)\partial_a$ depending on the variables y^i. Then an odd second order self-adjoint operator Δ may be defined on semidensities on $M \times N$ by

$$\Delta = \frac{1}{2}\left(\mathcal{L}_A \mathcal{L}_Q + \mathcal{L}_Q \mathcal{L}_A\right).$$

Calculating the square of the operator provides the necessary condition that the third order terms vanish:

$$Q^i \partial_i A^a A^b + (-1)^a A^a Q^i \partial_i A^b = 0. \tag{5.35}$$

A possible solution to eq. (5.35) is to choose A such that it separates variables, $A^a(x, y) = R(y)\eta^a(x)$, where $R = R(y)$ is an odd function on N and $\eta = \eta^a(x)\partial_a$ is an odd vector field on M. The principal symbol of Δ then defines an odd Poisson bracket on $M \times N$ by

$$\{y^i, x^a\} = Q^i(y)R(y)\eta^a(x). \tag{5.36}$$

Calculating Δ^2 brings us to the associated modular vector field

$$X = \frac{1}{4}\left(Q^i \partial_i R\right)^2 \eta^b \partial_b \eta^a \partial_a = \frac{1}{8}[Q(R)\eta, Q(R)\eta],$$

which is not Hamiltonian since any Hamiltonian vector field is proportional to the odd function R, whereas X is not in general.

A global construction can be provided in terms of Lie derivatives. For the operator Δ,

$$4\Delta^2 = \mathcal{L}_A \mathcal{L}_Q \mathcal{L}_A \mathcal{L}_Q + \mathcal{L}_Q \mathcal{L}_A \mathcal{L}_Q \mathcal{L}_A + \mathcal{L}_Q \mathcal{L}_A \mathcal{L}_A \mathcal{L}_Q,$$

since $Q^2 = 0$. Observing that the associated modular vector field is given by $X = \frac{1}{4}[Q, A]^2 = \frac{1}{8}[[Q, A], [Q, A]]$, we can write

$$4\mathcal{L}_X = \frac{1}{2}\mathcal{L}_{[[Q,A],[Q,A]]} = \mathcal{L}^2_{[Q,A]} = (\mathcal{L}_Q \mathcal{L}_A - \mathcal{L}_A \mathcal{L}_Q)^2.$$

Therefore,

$$\Delta^2 - \mathcal{L}_X = \frac{1}{2}\mathcal{L}_Q\mathcal{L}_A\mathcal{L}_A\mathcal{L}_Q,$$

where the terms $\mathcal{L}_Q\mathcal{L}_A\mathcal{L}_A\mathcal{L}_Q$ depend on the Jacobi identity. Relatively simple, albeit tedious manipulations of well-known formula confirm that if $A = R\eta$ then the expression $\mathcal{L}_Q\mathcal{L}_A\mathcal{L}_A\mathcal{L}_Q$ vanishes.

Proposition 5.4.2. *Let M and N be manifolds and suppose that N is equipped with a homological vector field Q. Then the product $M \times N$ can be endowed with an odd Poisson bracket defined by the principal symbol of the operator Δ acting on semidensities, where*

$$\Delta = \frac{1}{2}\left(\mathcal{L}_A\mathcal{L}_Q + \mathcal{L}_Q\mathcal{L}_A\right),$$

and A is an even vector field that separates variables $A = R(y)\eta(x)$ for an odd function R on N and an odd vector field η defined on M. The modular class of this odd Poisson structure is non-trivial in general, and is represented by the modular vector field

$$X_\Delta = \frac{1}{8}[Q(R)\eta, Q(R)\eta].$$

Example 5.4.5. Proposition 5.4.2 has a relation with the Nijenhuis bracket, the bracket defined on differential form-valued vector fields [FN56, Nij55]. Consider on the manifold ΠTM vector fields of the form

$$X = X^a(x,\eta)\frac{\partial}{\partial x^a}, \tag{5.37}$$

which can be identified with form-valued vector fields on the base M. Every vector field of the form (5.37) has a unique lift to a vector field $\hat{X}$ on the manifold ΠTM such that $\hat{X}$ commutes with the de Rham differential d, and the restriction of $\hat{X}$ to M is given by X. The Nijenhuis bracket $[X,Y]_N$ of two form-valued vector fields X and Y is defined by the relation

$$\widehat{[X,Y]_N} = [\hat{X},\hat{Y}].$$

In particular, if X is odd, the Nijenhuis bracket of $[X,X]_N$ is non-trivial in general.

Now let X be an arbitrary odd form-valued vector field on ΠTM. By proposition 5.4.2 in analogy with example 5.4.3, the pair $(\Pi TM, \hat{X})$ defines an odd Poisson supermanifold $\Pi TM \times \mathbb{R}^{0|1}$ such that the modular class of the manifold is represented by the vector field

$$\frac{1}{8}[\hat{X},\hat{X}] = \frac{1}{8}\widehat{[X,X]_N}.$$

5.5 Discussion

For an odd Poisson manifold M we have associated a class of second order differential operators acting on semidensities on M which may be used to describe the modular class of this odd Poisson manifold. Further, these operators allow for an easy description of examples of these, having provided both trivial and non-trivial cases. We have also introduced the potential field associated to any one of these differential operators and begun to analyse the properties it possesses. The peculiar transformation laws that both itself and any associated modular vector field share suggest that it should resemble some connection like object, which is further supported by the introduction of Bering's odd scalar function [Ber06] and its intimate relation to the scalar curvature of the odd Poisson manifold [BB08a]. It is hoped that the introduction of the potential field may have consequences in the BV-formalism, whose scalar counterpart has already been investigated in the works [BB08b, Ber08, BB09].

Another interesting advance would be to define a set of higher cohomology classes associated to an odd Poisson manifold; in the articles [LS04, LMS10], a series of characteristic classes were produced for any Q-manifold, a certain class of which coincided with the modular class of an even Poisson manifold. It is hoped that the modular class described here will fit into one of these series, and its higher counterparts obtained to parallel the even case. These higher classes could correspond to the case of homotopy Poisson structures, of which an operator description has not yet been obtained due to the complications of the transformation laws of densities.

5.6 Calculation of the Symplectic Potential

We give the details of the calculations involved in obtaining expression (5.29),

$$U(x) = \frac{1}{4}\partial_b\partial_a E^{ab}(x) - (-1)^{b(d+1)}\frac{1}{12}\partial_a E^{bc}(x)E_{cd}(x)\partial_b E^{da}(x),$$

the potential of the canonical Δ-operator on a symplectic manifold.

Let z^A be a system of Darboux coordinates on M and x^a be an arbitrary coordinate system. We will make use of the fact that infinitesimally,

$$\delta \log \mathcal{J} = \mathrm{sTr}\left(\mathcal{J}^{-1}\delta\mathcal{J}\right), \qquad \mathcal{J} = \mathrm{Ber}\left(\frac{\partial x}{\partial z}\right).$$

We see that

$$\begin{aligned}\left(\delta z^A\right)\partial_A \log \mathcal{J} &= \mathrm{sTr}\left(\mathcal{J}^{-1}\left(\delta z^A\right)\partial_A \mathcal{J}\right)\\ &= (-1)^a \frac{\partial z^B}{\partial x^a}\delta z^A \frac{\partial x^a}{\partial z^A z^B} = (-1)^{a(A+1)}\left(\delta z^A\right) z_a^B x_{BA}^a,\end{aligned}$$

from which we obtain

$$\partial_A \log \mathcal{J} = (-1)^{a(A+1)} z_a^B x_{BA}^a = \partial_A \mathcal{J}\mathcal{J}^{-1}. \tag{5.38}$$

It then follows that

$$\begin{aligned}\partial_B\partial_A \mathcal{J}\mathcal{J}^{-1} &= \partial_B\left(\partial_A \mathcal{J}\mathcal{J}^{-1}\right) - (-1)^{AB}\partial_A \mathcal{J}\partial_B \mathcal{J}^{-1}\\ &= \partial_B\partial_A \log \mathcal{J} + (-1)^{AB}\partial_A \mathcal{J}\mathcal{J}^{-1}\partial_B \mathcal{J}\mathcal{J}^{-1}\\ &= \partial_B\partial_A \log \mathcal{J} + \partial_B \log \mathcal{J}\partial_A \log \mathcal{J},\end{aligned}$$

where we use the matrix identity

$$\frac{\partial}{\partial z^A}\frac{\partial z^B}{\partial x^a} = -(-1)^{Aa}\frac{\partial z^C}{\partial x^a}\frac{\partial x^b}{\partial z^C \partial z^A}\frac{\partial z^B}{\partial x^b}.$$

Now let $\mathbf{s} = s(z)\sqrt{|\mathcal{D}z|}$ be a semidensity in Darboux coordinates z^A. If one changed from arbitrary coordinates x^a to coordinates z^A, then

$$s(z)\sqrt{|\mathcal{D}z|} = s(x(z))\mathrm{Ber}\left(\frac{\partial x}{\partial z}\right)^{1/2}\sqrt{|\mathcal{D}z|}$$

provides an expression of $\mathbf{s}$ in terms of the coordinates x^a.

In Darboux coordinates z^A the canonical symplectic potential (5.29) vanishes, and we can write the canonical operator Δ_0 as

$$\Delta_0 \mathbf{s} = I^{AB}\partial_B\partial_A s(z)\sqrt{|\mathcal{D}z|} = I^{AB}\partial_B\partial_A\left(s(x(z))\mathrm{Ber}\left(\frac{\partial x}{\partial z}\right)^{1/2}\right)\sqrt{|\mathcal{D}z|}.$$

Then under a change of coordinates $x = x(z)$,

$$\begin{aligned}&I^{AB}\partial_B\partial_A\left(s(x(z))\mathrm{Ber}\left(\frac{\partial x}{\partial z}\right)^{1/2}\right)\sqrt{|\mathcal{D}z|}\\ &= \left(I^{AB}\partial_B\partial_A s + 2I^{AB}\partial_B s\partial_A \mathcal{J}^{1/2} + I^{AB}\partial_B\partial_A \mathcal{J}^{1/2}s\right)\mathcal{J}^{-1/2}\sqrt{|\mathcal{D}x|}\\ &= I^{AB}\left(\partial_B\partial_A s + \partial_B s\underset{a(A+1)}{\left(z_a^C x_{CA}^a\right)} + \frac{1}{2}\partial_B\underset{a(A+1)}{\left(z_a^C x_{CA}^a\right)}s + \frac{1}{4}\underset{b(B+1)+a(A+1)}{z_b^D x_{DB}^b z_a^C x_{CA}^a s}\right)\sqrt{|\mathcal{D}x|},\end{aligned}$$

making use of formula (5.38) and its consequences. In the zero order terms we get

$$U(x(z)) = \frac{1}{2}\underset{a(A+1)}{I^{AB}\partial_B\left(z_a^C\right)x^a_{CA}} + \frac{1}{2}\underset{a(A+1)+aB}{I^{AB}z_a^C x^a_{CBA}} + \frac{1}{4}\underset{b(B+1)+a(A+1)}{I^{AB}z_b^D x^b_{DB} z_a^C x^a_{CA}}. \tag{5.39}$$

Now consider the expressions

$$\partial_b\partial_a E^{ab}, \qquad\qquad (-1)^{b(d+1)}\partial_a E^{bc}E_{cd}\partial_b E^{da}.$$

We shall change coordinates in each of these. Firstly,

$$\begin{aligned}
\partial_b\partial_a E^{ab} &= z_b^C\partial_C\left(\underset{a(A+1)}{z_a^D\partial_D\left(x^a_A I^{AB} x^b_B\right)}\right) = z_b^C\partial_C\left(\underset{a(A+1)}{z_a^D x^a_{DA}} I^{AB}x^b_B + I^{AB}x^b_{BA}\right)\\
&= -\underset{a(A+1)+Ca}{z_b^C z_a^E x^c_{EC} z_c^D x^a_{DA}} I^{AB} x^b_B + \underset{a(A+1)+Ca}{z_b^C z_a^D x^a_{DCA}} I^{AB} x^b_B + \underset{a(A+1)+C}{z_b^C z_a^D x^a_{DA}} I^{AB} x^b_{BC} + \underset{C}{z_b^C I^{AB} x^b_{BAC}}\\
&= -\underset{a(A+1)+Ba}{z_a^E x^c_{EB} z_c^D x^a_{DA}} I^{AB} + \underset{a(A+1)+Ba}{z_a^D x^a_{DBA}} I^{AB} + \underset{a(A+1)+C}{z_b^C z_a^D x^a_{DA}} I^{AB} x^b_{BC} + \underset{C}{z_b^C I^{AB} x^b_{BAC}}.
\end{aligned}$$

We come to

$$\partial_b\partial_a E^{ab} = 2\underset{a(A+B+1)}{I^{AB} z_a^D x^a_{DBA}} + \underset{a(A+1)+b(B+1)}{z_a^D x^a_{DA} I^{AB} z_b^C x^b_{CB}} - \underset{a(A+B+1)}{z_a^C x^b_{CB} z_b^D x^a_{DA} I^{AB}}. \tag{5.40}$$

The calculations for the second term are long but are performed in precisely the same way. We obtain

$$(-1)^{b(d+1)}\partial_a E^{bc}E_{cd}\partial_b E^{da} = 3\underset{a(A+B+1)}{z_a^C x^b_{CB} z_b^D x^a_{DA} I^{AB}} + \underset{S(b+1)+b(C+1)+D(A+1)+A(b+1)}{I^{AB} x^a_{BD} z_a^R I_{RS} z_b^S x^b_{AC} I^{CD}}.$$

By symmetry about the component I_{RS}, the last term vanishes, and so

$$(-1)^{b(d+1)}\partial_a E^{bc}E_{cd}\partial_b E^{da} = 3\underset{a(A+B+1)}{z_a^C x^b_{CB} z_b^D x^a_{DA} I^{AB}}. \tag{5.41}$$

Comparing eq. (5.39) with eqs. (5.40) and (5.41) we find that

$$U(x) = \frac{1}{4}\partial_b\partial_a E^{ab} - \frac{1}{12}(-1)^{b(d+1)}\partial_a E^{bc}E_{cd}\partial_b E^{da}.$$

Our calculations rely on the fact that the odd Poisson tensor E is non-degenerate. One can repeat these calculations for the case when E is degenerate but admits a compatible 2-form field as in the work of Bering [Ber08]. In this case Bering obtained potential fields for odd Laplace operators which could no doubt be obtained through similar calculations as above. These cases considered by Bering incorporate an important class of odd degenerate Poisson structures.

Appendix A

Elements of Supermathematics

We present here some of the basics of the theory of supermanifolds to aid in the understanding of the text. For a detailed introduction to the theory however, we point to the books [Man97, Rog07, Vor91] or more classically [Ber83]. There are also many articles containing a more limited, though often more "operational" exposition. See for example [CS11].

A.1 Super Algebra and Supermanifolds

The main concept of supermathematics is to extend existing structures by anti-commuting variables, corresponding to assigning a $\mathbb{Z}_2$-grading called parity. Most authors use $p(v)$, $\tilde{v}$, $|v|$ or $\epsilon(v)$ to denote the parity of an object, however we prefer to avoid the additional notation. The notation $\tilde{a}$ will be to used only when we wish to emphasise the parity of a.

Definition A.1.1. A *super vector space* is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. Elements of V_0 are called even, and elements of V_1 are called odd.

Definition A.1.2. A *super algebra* A is a super vector space equipped with an even associative product $A_i A_j \subset A_{i+j}$.

The parity reversal functor is a functor $\Pi : V \mapsto V$ on vector spaces such that

$$\Pi V = (\Pi V)_0 \oplus (\Pi V)_1 \,,$$

where

$$(\Pi V)_0 := V_1, \qquad (\Pi V)_1 := V_0.$$

The symbol Π should be treated as an odd symbol such that $\Pi^2 = 1$. Hence any super vector space can be viewed as a classical vector space together with a copy of a vector space with reversed parity.

Example A.1.1. The real superspace $\mathbb{R}^{n|m} = \mathbb{R}^n \oplus \Pi\mathbb{R}^m$ can be considered as a copy of $\mathbb{R}^n$ together with a copy of $\mathbb{R}^m$ with reversed parity.

The majority of constructions carry through in the super case by a simple extension of the $\mathbb{Z}_2$-grading. For example, a homogeneous derivation D of a super algebra A is a map $D : A \to A$ such that

$$D(ab) = D(a)b + (-1)^{Da} aD(b),$$

for $a, b \in A$. The symbol $(-1)^{Da}$ represents -1 raised to the parity of D multiplied by that of a. The derivation D is even if $\widetilde{Da} = \tilde{a}$, and odd if $\widetilde{Da} = \tilde{a} + 1$.

A basis $\{e_A\} = \{e_a, e_\alpha\}$, for $a = 1, \dots n$ and $\alpha = 1, \dots, m$, can be chosen for a super vector space V, where $\{e_a\}$ are basis vectors of the even space V_0 and $\{e_\alpha\}$ are basis vectors of the odd space V_1. The dimension of the super vector space V is then a pair, $\dim(V) = n|m$, called the super dimension.

Supermathematics offers a distinction between left and right coordinates; for a basis $\{e_A\}$ of a super vector space V, a vector $v \in V$ may be expressed either by a row such that $v = v^A e_A$, or by a column where $v = e_A v^A$. These differ by a sign, however we often choose to adopt left coordinates for a vector space and right coordinates for its dual. In this way, the canonical pairing $(v, \eta) = (v^A e_A, e^B \eta_B) = v^A \eta_A$ remains invariant for $\eta \in V^*$.

An even linear transformation between two bases $\{e_A\}$ and $\{e_{A'}\}$ is represented by a super matrix A, such that

$$\begin{pmatrix} e_{a'} \\ e_{\alpha'} \end{pmatrix} = A \begin{pmatrix} e_a \\ e_\alpha \end{pmatrix} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} e_a \\ e_\alpha \end{pmatrix},$$

where A_{00} is an $n \times n$ matrix of even elements, A_{01} is an $n \times m$ matrix of odd elements, A_{10} is an $m \times n$ matrix of odd elements, and A_{11} is an $m \times m$ matrix of even elements. This is the standard form for an even super matrix.

Such a matrix is an invertible element of the more general matrix group $\mathrm{Mat}(n|m)$ forming the space of all even super matrices. Matrices given by A form a linear subspace of invertible matrices $GL(n|m)$. Notice that one could also consider odd matrices given by a block decomposition similar to A, but with even and odd entries interchanged.

The Berezinian is a function Ber : $GL(n|m) \to \mathbb{R}^{1|0}$ such that

$$\mathrm{Ber}\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \frac{\det(A_{00} - A_{01}A_{11}^{-1}A_{10})}{\det A_{11}}, \tag{A.1}$$

and is the super analogue of the determinant. Notice the difference from the classical determinant in that the Berezinian is not necessarily a polynomial, nor is it defined on the whole space of matrices $\mathrm{Mat}(n|m)$ for which the fermi-fermi block A_{11} is not necessarily invertible.

As in the usual case for the determinant, there is an associated supertrace defined by

$$\mathrm{sTr}A = \mathrm{Tr}A_{00} - \mathrm{Tr}A_{11},$$

which obeys a Liouville formula of the form

$$\mathrm{Ber}(e^A) = e^{\mathrm{sTr}A}.$$

For details on super algebra and the Berezinian we refer specifically to the books [Man97] and [Vor91].

A.2 Supermanifolds and the Berezin Integral

A Grassmann algebra of dimension m is an algebra generated by m odd elements $\xi^1, \dots, \xi^m$, such that

$$\xi^\alpha \xi^\beta = -\xi^\beta \xi^\alpha.$$

A function of odd variables $f(\xi)$ is defined as an element of such a Grassmann algebra. Due to the nilpotency of the odd variables, the expansion of f always terminates,

$$f(\xi) = f_0 + \xi^\alpha f_\alpha + \frac{1}{2}\xi^\alpha \xi^\beta f_{\beta\alpha} + \cdots + \frac{1}{m!}\xi^{\alpha_1} \dots \xi^{\alpha_m} f_{\alpha_m \cdots \alpha_1}. \tag{A.2}$$

The odd variables ξ^α are called coordinates on the superspace $\mathbb{R}^{0|m}$. If the coefficients in expansion (A.2) depend on even variables x^a defined on $U^n \subset \mathbb{R}^n$, we obtain a function $f = f(x, \xi)$ on the open super domain $U^{n|m}$, a subset of real superspace $\mathbb{R}^{n|m}$.

If $U^{n|m}$ and $V^{n|m}$ are two open domains with non-empty intersection equipped with coordinates (x^a, ξ^α) and (y^a, θ^α) respectively, then the change of coordinates is given by

$$\begin{aligned} x^a &= x^a(y) + \theta^\alpha \theta^\beta x^a_{\beta\alpha}(y) + \cdots , \\ \xi^\alpha &= \theta^\beta \xi^\alpha_\beta(y) + \theta^\beta \theta^\gamma \theta^\delta \xi^\alpha_{\delta\gamma\beta}(y) + \cdots . \end{aligned} \tag{A.3}$$

A key feature in supermathematics is the ability to "mix" both even and odd variables as in eq. (A.3), which considerably enlarges the class of morphisms of super structures.

The algebra of smooth functions $C^\infty(U^{n|m})$ on $U^{n|m}$ defines a sheaf on $\mathbb{R}^n$ defined as

$$C^\infty(U^{n|m}) = C^\infty(U^n)[\xi^1, \ldots, \xi^m],$$

of smooth functions on U^n with values in the Grassmann algebra on m generators.

Definition A.2.1. A (real) *supermanifold* $M = M^{n|m}$ is a locally ringed space which is locally isomorphic to $(U^n, C^\infty(U^{n|m}))$.

Loosely speaking, a supermanifold is made from gluing copies of real superspace $\mathbb{R}^{n|m}$ using the coordinate transformations (A.3). The carrier manifold, or the support manifold, is the classical smooth manifold obtained by setting all the odd variables equal to zero. For example, the carrier of the supermanifold $\mathbb{R}^{n|m}$ is n-dimensional real space $\mathbb{R}^n$.

Since the odd variables are generators of a Grassmann algebra however, it makes no sense to assign numerical values other than zero, and so a function of odd variables cannot take values at any other point. Because of this, the concept of Λ-points is employed to obtain an intuitive description of what a "point" of a supermanifold is. For an arbitrary Grassmann algebra $\Lambda = \Lambda_0 \oplus \Lambda_1$, a Λ-point of a supermanifold $M^{n|m}$ is a homomorphism $C^\infty(M) \to \Lambda$, or equivalently a map $\mathbb{R}^{0|p} \to M$ where $\Lambda = C^\infty(\mathbb{R}^{0|p})$. The Λ-point has coordinates x^a, ξ^α where $x^a \in \Lambda_0$ and $\xi^\alpha \in \Lambda_1$. Then any function on M takes values in these Λ-points of M. Note that the set of all Λ-points forms a smooth manifold.

The topology of a supermanifold is equivalent to that of its carrier manifold which means that an "all or nothing" approach is needed when defining integration of odd variables. Given an odd superspace $\mathbb{R}^{0|m}$ with coordinates ξ^α, define the Berezin integral of a arbitrary function $f \in C^\infty(\mathbb{R}^{0|m})$ as

$$\int_{\mathbb{R}^{0|m}} \mathcal{D}\xi f(\xi) = f_{\alpha_1\cdots\alpha_m},$$

where $f_{\alpha_1\cdots\alpha_m}$ is the top coefficient of the expansion (A.2). The symbol $\mathcal{D}\xi$ corresponds to the integration element and will be explained later. When the coefficients of f now depend on coordinates x^a, the integral of a function $f \in C^\infty(\mathbb{R}^{n|m})$ which is rapidly decreasing or of compact support in x is defined by

$$\int_{\mathbb{R}^{n|m}} \mathcal{D}(x,\xi) f(x,\xi) = (-1)^{\frac{m(m-1)}{2}} \int_{\mathbb{R}^n} d^n x f_{\alpha_1\cdots\alpha_m}(x),$$

where the right hand side is the standard Lebesgue integral. The integral of a smooth function on an arbitrary supermanifold is obtained by a partition of unity. See [Rog07, Vor91].

The Berezin integral enjoys the following transformation under a change of coordinates $x = x(y,\theta), \xi = \xi(y,\theta)$: for a rapidly decreasing or compactly supported function $f \in C^\infty(\mathbb{R}^{n|m})$,

$$\int_{\mathbb{R}^{n|m}} \mathcal{D}(x,\xi) f(x,\xi) = \int_{\mathbb{R}^{n|m}} \mathcal{D}(y,\theta) \mathrm{Ber}\left(\frac{\partial(x,\xi)}{\partial(y,\theta)}\right) f(x(y,\theta),\xi(y,\theta)).$$

It also obeys the integration by parts formula

$$\int_{\mathbb{R}^{n|m}} \mathcal{D}x \frac{\partial f(x)}{\partial x^A} g(x) = -(-1)^{Af} \int_{\mathbb{R}^{n|m}} \mathcal{D}x f(x) \frac{\partial g(x)}{\partial x^A},$$

where the x^A are collective variables on $\mathbb{R}^{n|m}$.

Remark A.2.1. In order to properly define the integral over a supermanifold, some notion of orientation is needed. In the super case orientation is more involved, and relies on four manifestations of the Berezinian (A.1). We refer to [Vor91] for a detailed discussion about orientation, and simply assume that we have the integrals defined as required.

Finally, we return to the symbol $\mathcal{D}(x,\xi)$. This is shorthand notation, and refers to the symbol

$$\mathcal{D}(x,\xi) = [dx^1, \ldots, dx^n \,|\, d\xi^1, \ldots, d\xi^m].$$

These stand as the integration elements of a supermanifold, and are as practical to work with as with the common element $dx^1 \wedge \ldots \wedge dx^n$ for an n-dimensional manifold.

Under a change of coordinates $(x, \xi) = (x(y, \theta), \xi(y, \theta))$,

$$\mathcal{D}(x, \xi) = \mathrm{Ber}\left(\frac{\partial(x, \xi)}{\partial(y, \theta)}\right)\mathcal{D}(y, \theta).$$

Changing to a collective notation, the elements $\mathcal{D}x$ form a local frame for the line bundle $\mathrm{Vol}(M) = \mathrm{Ber}(T^*M)$, the determinant bundle for any supermanifold M. (Note that in Chapter 5 we use the notation $|\mathcal{D}x|$ to refer to the positive half of this bundle.) A generator of the space of sections of $\mathrm{Vol}(M)$ corresponds to a volume element $\boldsymbol{\rho}$ on M, where $\rho(x)$ is a Grassmann even, invertible function. See [KSM02] for a detailed construction of $\mathrm{Ber}(T^*M)$.

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