Ring constructions on spectral spaces

A thesis submitted to the University of Manchester for the degree of Doctor of Philosophy in the Faculty of Science and Engineering

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Contents

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Abstract

Ring constructions on spectral spaces; a thesis submitted to the University of Manchester for the degree of Doctor of Philosophy by Christopher Francis Tedd, September 27, 2016.

In the paper [14] Hochster gave a topological characterisation of those spaces X which arise as the prime spectrum of a commutative ring: they are the spectral spaces, defined as those topological spaces which are T_0, quasi-compact and sober, whose quasi-compact and open subsets form a basis for the topology and are closed under finite intersections. It is well known that the prime spectrum of a ring is always spectral; Hochster proved the converse by describing a construction which, starting from such a space X, builds a ring having the desired prime spectrum; however the construction given is (in Hochster's own words) very intricate, and has not been further exploited in the literature (a passing exception perhaps being the use of [14] Theorem 4 in the example on page 272 of [24]). In the finite setting, alternative constructions of a ring having a given spectrum are provided by Lewis [17] and Ershov [7], which, particularly in light of the work of Fontana in [8], appear to be more tractable, and at least more readily understood. This insight into the methods of constructing rings with a given spectrum is used to prove a result about which spaces may arise as the prime spectrum of a Noetherian ring: it is shown that every 1-dimensional Noetherian spectral space may be realised as the prime spectrum of a Noetherian ring. A close analysis of the two finite constructions considered here reveals considerable similarities between their underlying operation, despite their radically different presentations. Furthermore, we generalise the framework of Ershov's construction beyond the finite setting, finding that the ring we thus define on a space X contains the ring defined by Hochster's construction on X as a subring. We find that in certain examples these rings coincide, but that in general the containment is proper, and that the spectrum of the ring provided by our generalised construction is not necessarily homeomorphic to our original space. We then offer an additional condition on our ring which may (—and indeed in certain examples does) serve to repair this disparity. It is hoped that the analysis of the constructions presented herein, and the demonstration of the heretofore unrecognised connections between the disparate ring constructions proposed by various authors, will facilitate further investigation into the prime ideal structure of commutative rings.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Dedicated to the memory of

Gerard McCavana

01/10/1958—31/12/2013

&

Robert Ernest Charles Tedd

29/7/1927—01/04/2014

"craving peace its slow so slow
drop into our laps but as far
from it and myself as ever"

— Tom Paulin, *The Road to Inver* (after Pessoa)

Introduction

A topological space X is spectral if it satisfies the following properties:

(1) X is T_0;

(2) X is quasi-compact;

(3) X has a basis of sets which are quasi-compact and open;

(4) The quasi-compact open sets of X are closed under finite intersections; and

(5) X is sober — every non-empty irreducible closed set of X is the closure of a unique point.

A map $f : X \to Y$ of spectral spaces is a spectral map if the preimage under f of all quasi-compact open sets of Y are quasi-compact and open in X.

In the paper [14], Hochster showed that the spectral spaces are exactly the spaces that arise as the prime spectrum of some (commutative, unital) ring (— throughout the thesis a ring will always be commutative with 1), and that every spectral map of spectral spaces arises from a ring homomorphism under the prime spectrum functor, which takes a ring homomorphism $\varphi : A \to B$ to the map which sends a prime ideal of B to its preimage under φ. Indeed, it is easy to show that the prime spectrum of a ring is spectral, and that homomorphisms of rings induce spectral maps; Hochster demonstrated the converse by describing a construction which, starting from such a space X, builds a ring having the desired prime spectrum, in such a way that a spectral map between any two spectral spaces may be realised by a homomorphism of the constructed rings. This construction is the central object of study of this thesis, which, due to its intricate nature, is relatively unexplored and unexploited in the subsequent literature. It is hoped that the elucidation of this construction, and of the heretofore unrecognised connections between this and other ring constructions proposed by various authors ([17], [7]) will facilitate further investigation into the prime ideal structure of commutative rings.

The first chapter compiles relevant standard results from the literature for easy reference. In particular some detail is gone into regarding the theory of spectral spaces, due to the fairly idiosyncratic behaviour of such spaces and the relative unfamiliarity of their properties in the context of general topology. Of particular relevance to the remainder of this introduction is the specialisation order defined on a spectral space X (—indeed, on any T_0 space). The specialisation order on a space X, denoted $\rightsquigarrow$, is defined by

$$x \rightsquigarrow y \iff y \in \overline{\{x\}}.$$

The dimension of a spectral space X is then the dimension of $(X, \rightsquigarrow)$ as a poset. This ordering is trivial on any T_1 space, in particular any Hausdorff space, and on the space $\mathsf{Spec}\, A$ for a ring A this ordering is exactly the subset inclusion ordering on prime ideals considered as subsets of A. The references for this chapter are [1], [4] and [3], with the paper [14] also containing many of the stated results.

Chapter 2 gives an exposition of Hochster's construction of a ring having a given spectral space as its spectrum, as in the paper [14]. In particular notation is fixed which it is hoped will facilitate comparison of aspects of this construction with those presented elsewhere in the thesis. To summarise the construction: one may obtain a mapping from any topological space X into the prime spectrum of a ring of functions A on X in the case that each function f of A takes values in a domain at each $x \in X$: a point $x \in X$ is mapped to the prime ideal of A consisting of those functions $f \in A$ that are zero at x. When X is a spectral space we obtain four criteria which are necessary and sufficient for

$\mathsf{Spec}\, A$ to be homeomorphic to X: 1) we require that the range of the functions of A under evaluation at each point $x \in X$ is a domain; 2) if the preimage of the basic open sets $D(a)$ of $\mathsf{Spec}\, A$ are an open basis for X then X homeomorphically embeds into $\mathsf{Spec}\, A$; 3) if furthermore the preimage of these sets are quasi-compact, the inclusion map is a spectral map and X is called a spectral subspace of $\mathsf{Spec}\, A$; the properties of spectral subspaces are then used to produce a rather technical fourth and final criterion, see proposition 2.13, which if satisfied along with conditions (1)–(3) guarantees that $X \cong \mathsf{Spec}\, A$. We find it not too difficult to construct such a ring of functions A_X on a spectral space X which satisfies the first three criteria: we form a ring A_X consisting of polynomial expressions of characteristic functions χ_U for a collection of quasi-compact open sets $\{U\}_{U \in \mathcal{B}}$ which form a subbasis for the topology on X. However, the ring so constructed may yet be a long way from satisfying the fourth condition: our ring A_X may have many more prime ideals than there are points in the space X. Our procedure is to extend the ring A_X, or, to extend the domains which are the range of the functions in A_X evaluated on X towards their field of fractions, thereby in some sense 'inverting' certain functions in A_X and thus killing off unwanted prime ideals; however care must be taken that the three properties satisfied by A_X are not lost under this process of extension. Thus arises a procedure for determining in exactly what way the ring may be extended without the required properties being lost; it is hoped to give as clear an exposition of this procedure, which Hochster himself describes as "very intricate" in [14], as possible. Finally, we obtain a ring H_X satisfying all four properties, so that $\mathsf{Spec}\, H_X \cong X$; however it is by this stage not especially easy to determine exactly what ring this H_X actually is, being the union of an inductive process of ring extensions defined via the 'intricate procedure' mentioned above. On the other hand, it is comparatively simple to show that every spectral map between spectral spaces arises as the map induced from a homomorphism of rings having the appropriate spectra. This observation closes the chapter.

Chapter 3 gives details of the constructions of Lewis [17] and Ershov [7] of a ring having prime spectrum homeomorphic to a given *finite* spectral space. Again complete details are given in order to facilitate analysis and comparison of the various procedures used. Lewis's construction employs an inductive argument to iteratively build up a ring realising a given space via two techniques: the first technique adds a new layer of maximal ideals to a ring that is assumed to exist by induction, and the second technique "joins" points in the spectrum of the ring obtained which may have been inadvertently multiplicated as a consequence of the first technique. Fontana, in [8], observed that the two techniques used in Lewis's construction are both instances of taking the amalgamated or fibre sum of topological spaces, via its connection under certain hypotheses to the fibre product of rings having spectra homeomorphic to the spaces in question. That is, given spectral spaces and maps as in the following diagram:

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z, \end{array}$$

one may form the topological fibre sum $W = X \sqcup_Z Y$; this is a "gluing" construction that in some precise way amalgamates the topological structure of the spaces X and Y, see page 75. Then, under certain conditions, if we already have rings and ring homomorphisms $\varphi : A \to C$, $\psi : B \to C$ realising $f : X \to Z$ and $g : Y \to Z$, one may construct a ring realising W by taking the fibre product $A \times_C B$ over those homomorphisms. Phrasing the construction in the language of topological fibre sums not only simplifies the presentation of Lewis's method, but also allows us to extend the methods employed to demonstrate that every 1-dimensional Noetherian spectral space arises as the spectrum of a Noetherian ring; whilst it is expected that this fact is known, the author could not find an explicit reference to it in the literature, and it provides an example of how the fibre sum methodology may be used to control properties of rings having a certain given spectrum. It is worth noting that a finite space may arise as the prime spectrum of a Noetherian ring only in the case that the dimension of X is at most 1, see 1.41.

The construction of Ershov proceeds initially along similar lines to that of Hochster. For a finite spectral space X, we may find a subbasis for the topology on X that is in bijection with the points of X. The value at a point $z \in X$ of the characteristic function of a basis set corresponding to a point $x \in X$ is then determined by whether z lies below x in the specialisation order on X. Ershov then corresponds to each point $x \in X$ a ring $R_{x\uparrow}$, being the intersection of certain discrete valuation rings defined from the specialisation relations involving x in X; these are the domains in which our functions on X take their values at the point x. A ring R_X is then defined as those functions on X satisfying a compatibility condition: the values taken by f at points x and y in X must agree under the quotient maps associated to the valuations rings involved. The ring R_X then has the property that $\mathsf{Spec}\, R_X \cong X$.

Chapter 4 then proceeds to compare the constructions we have set out. Similarities are immediately found between our two finite constructions: we find that the elementary building blocks of Lewis's construction can also be viewed as intersections of discrete valuation rings arising in exactly the manner of Ershov's construction, except under the choice of an alternative subbasis for the topology of a finite spectral space. Furthermore, a strong similarity is found between the compatibility conditions by which Ershov picks out a ring R_X

having spectrum homeomorphic to a finite spectral space X, and the conditions arising in the definition of the fibre product over the rings defined on the parts of the space X that would be "amalgamated" in Lewis's methodology. However, whilst the topological consequences of taking either the ring R_X, or taking the fibre product as described, are identical (—we still obtain a ring whose spectrum is homeomorphic to X), the Ershov ring R_X is strictly smaller than the ring arising from the fibre product.

The second half of the chapter then proceeds from the similarity noted between the ring R_X defined by Ershov's construction and the ring A_X defined at the initial stage of Hochster's construction. In the finite case, our ring R_X was defined from a subbasis for X that was in bijection with the points of X; we find it is not at all difficult to generalise the definitions of our finite construction to start from any subbasis for X, thereby obtaining a ring R_X corresponding to any spectral space X. We then find a remarkable correspondence between the objects and conditions present in Hochster's construction and those present in this generalised Ershov construction. Firstly, Hochster defines a discrete valuation $\mathsf{v}_{x \rightsquigarrow y}$ to each relation $x \rightsquigarrow y$ that holds in the specialisation order on X; the generalised Ershov construction defines a discrete valuation ring $V_{x \rightsquigarrow y}$ corresponding to each such relation. We find 4.2 that $V_{x \rightsquigarrow y}$ is the valuation ring of the discrete valuation $\mathsf{v}_{x \rightsquigarrow y}$. Hochster then uses these valuations to impose conditions on which ring extensions are permitted in his construction; we find 4.3 that elements satisfying the condition of Hochster to be admitted to a ring extension always satisfy the condition of Ershov to be admitted to the general ring R_X; that is, where H_X is the union of all ring extensions permitted by Hochster's construction, we have that $H_X \subseteq R_X$. Indeed we find in a simple example that the rings H_X and R_X in fact coincide, but we also provide an example demonstrating that this does not hold in general, and furthermore (whilst we always have that $\mathsf{Spec}\, H_X \cong X$) we may not have that $\mathsf{Spec}\, R_X \cong X$ for R_X the general Ershov ring defined on a general spectral space X. This prompts us to investigate in two directions: on one hand, to consider subclasses of spectral spaces for which indeed we have $\mathsf{Spec}\, R_X \cong X$, and on the other hand to consider conditions which may be applied to R_X to guarantee isomorphism with the ring H_X (and thus ensuring that for the ensuing ring $\widehat{R}_X$ we indeed have $\mathsf{Spec}\, \widehat{R}_X \cong X$ for all spectral spaces X). Conjectural hypotheses are discussed; however due to the short time between the discovery of these lines of inquiry and the necessary completion date of this thesis, this concludes the extent of the concrete progress that has been able to be made.

Summary of notation

The symbol $\subseteq$ is used for subset inclusion; we use the symbol $\subsetneq$ to denote strict inclusion. Where $\leqslant$ denotes an order relation the symbol $<$ is reserved for strict ordering, i.e. $x < y \iff x \leqslant y \;\&\; x \neq y$.

If $\leqslant$ is an order relation on a set X, then given $Y \subseteq X$ we use $Y^{\uparrow}$ to denote the set

$$Y^{\uparrow} = \{x \in X \mid y \leqslant x \text{ for some } y \in Y\}$$

and $Y^{\downarrow}$ to denote the set

$$Y^{\downarrow} = \{x \in X \mid x \leqslant y \text{ for some } y \in Y\}.$$

In particular given some element $y \in X$ we use $y^{\uparrow}$ for

$$y^{\uparrow} = \{x \in X \mid y \leqslant x\}$$

and $y^{\upmapsto}$ for

$$y^{\upmapsto} = \{x \in X \mid y < x\} = y^{\uparrow} \setminus \{y\};$$

likewise we have

$$y^{\downarrow} = \{x \in X \mid x \leqslant y\}$$

and

$$y^{\rotatebox[origin=c]{90}{$\leftarrowtail$}} = \{x \in X \mid x < y\} = y^{\downarrow} \setminus \{y\}.$$

Given a poset X, the height of an element $x \in X$ is the maximum n such that there is a chain $x_0 < x_1 < \cdots < x_{n-1} < x_n = x$ in X, or ∞ where no such n exists (—note that minimal elements in X have height 0); we denote the height of $x \in X$ by $\mathsf{ht}\, x$. The dimension of X, denoted $\mathsf{dim}\, X$, is the maximum height of an element of X, or equivalently is the maximum length of a chain of distinct elements in X (under the same convention that a chain consisting of $n + 1$ distinct elements has length n), again with $\mathsf{dim}\, X = \infty$ if no such maximum exists.

Given sets X and Y then $X \sqcup Y$ denotes the disjoint union of X and Y. Given a subset $Y \subseteq X$ then $Y^{\mathsf{c}} = X \setminus Y$ denotes the complement of Y in X.

Given a ring A, we use the notation $A^{\times}$ for the set of units of A (in particular if A is a field then $A^{\times} = A \setminus \{0\}$). Given a subset $S \subseteq A$, then $\langle S \rangle$ denotes the ideal generated by S; in particular, for an element a of A then $\langle a \rangle$ denotes the ideal generated by a. We use the notation $\mathcal{Q}(A)$ to denote the total ring of quotients of A, the localisation of A at the set of non-zero-divisors of A; when A is an integral domain then $\mathcal{Q}(A)$ is the field of fractions or quotient field of A.

Throughout a ring is commutative with 1 and ring homomorphisms preserve the identity.

Given a topological space X and a subset $Y \subseteq X$, then $\overline{Y}$ denotes the closure of Y in X.

All notation not specified herein is either defined in the body of the text, or is hoped to be sufficiently standard as to be intelligible without ambiguity.

Summary of references

In addition to the textbooks [1], [3], [4] as mentioned above, the author relied on the excellent textbooks [15] and [10] as standard references on commutative algebra, with [19] providing alternative reference, and used [20] as a reference for basic general topology, with [11] providing a reference on the basic properties of Noetherian spaces.

Literature on the Hochster construction

Whilst frequent reference to Hochster's result is found throughout the subsequent literature, very few authors have engaged directly with the techniques of its construction, and the insights such an analysis may provide; an exception, notable for its rarity, being the use of [14] Theorem 4 in the example on page 272 of [24]; most often, the result of Hochster is used to prove a result of the form "there exists a ring having spectrum with certain properties", proved by producing a spectral space with such properties and then invoking Hochster's Theorem 6 to assert the existence of such a ring, without addressing any questions of what sort of a ring that provided by the construction which proves Theorem 6 might be.

Of examples where the "inner workings" of Hochster's method are actually referred to explicitly, the only examples this author is aware of are the appendix to the textbook [16], and the paper [2]. [16] is a re-presentation (in French) of the method of Hochster furnished with complete proofs and some small examples; it is in most regards identical to the presentation given in [14]. [2] uses the techniques of Hochster to prove constructive results concerning the class of lattices which may arise as the lattice of radical ideals of a ring; the nuances of constructivity aside (—which are of course the main concern of the paper in question), the paper again essentially reproduces the method of [14] unaltered, and thereby doesn't really furnish the reader with any greater insight as to why the construction works as it does. It is hoped the detailed analysis presented in this thesis may expose overlooked aspects of Hochster's results that could find some use in future work.

1. Preliminaries

The prime spectrum of a ring

The set of prime ideals of a ring A can be considered as the points of a topological space, the prime spectrum of A, denoted

$$\mathsf{Spec}\, A = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal of } A\},$$

with topology (named the Zariski topology) having closed sets defined by subsets $S \subseteq A$ as follows:

$$\mathcal{V}(S) = \{\mathfrak{p} \in \mathsf{Spec}\, A \mid S \subseteq \mathfrak{p}\},$$

by analogy (— as will become evident in due course) with the "algebraic sets" defined as the common zero-locus of sets of polynomials in classical algebraic geometry. Given an element $a \in A$, we write $\mathcal{V}(a)$ for $\mathcal{V}(\{a\}) = \{\mathfrak{p} \mid \{a\} \subseteq \mathfrak{p}\} = \{\mathfrak{p} \mid a \in \mathfrak{p}\}$.

It is common (— and equivalent) to see the topology defined as having closed sets $\mathcal{V}(I) = \{\mathfrak{p} \in \mathsf{Spec}\, A \mid I \subseteq \mathfrak{p}\}$ for ideals $I \subseteq A$, a convention we will adopt from now on. We begin by reviewing (without proof) relevant facts about this collection of closed sets, starting with the verification that it does in fact define a topology. A standard reference for facts concerning the prime spectrum of a ring is the textbook [1].

1.1. *For the subsets $\mathcal{V}(I) \subseteq \mathsf{Spec}\, A$ of the prime spectrum of a ring A as defined above, we have*

(1) $\mathcal{V}(A) = \emptyset$;

(2) $\mathcal{V}(\langle 0\rangle) = \mathsf{Spec}\, A$;

(3) $\bigcap_{\lambda\in\Lambda} \mathcal{V}(I_\lambda) = \mathcal{V}\big(\sum_{\lambda\in\Lambda} I_\lambda\big)$;

(4) $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(IJ)$.

1.2. *Given ideals I, J of A, we have $I \subseteq J \implies I \subseteq \mathsf{rad}\, J \iff \mathcal{V}(J) \subseteq \mathcal{V}(I)$. In particular, if I and J are radical ideals, then $I \subseteq J \iff \mathcal{V}(J) \subseteq \mathcal{V}(I)$.*

Then any closed set $C \subseteq \mathsf{Spec}\, A$ in the Zariski topology corresponds to a unique radical ideal I of A so that $C = \mathcal{V}(I)$; we use the notation $I = \mathcal{I}(C)$ to specify such a radical ideal.

We may in a very natural way define Spec as a (contravariant) functor from the category of rings and ring homomorphisms to that of topological spaces and continuous functions.

1.3. *The function $\mathsf{Spec}\,\varphi : \mathsf{Spec}\, B \to \mathsf{Spec}\, A$ defined from a ring homomorphism $\varphi : A \to B$ by $(\mathsf{Spec}\,\varphi)(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$ is a continuous function with respect to the Zariski topology on $\mathsf{Spec}\, B$, $\mathsf{Spec}\, A$.*

1.4. *Given homomorphisms of rings $A \xrightarrow{\ \varphi\ } B \xrightarrow{\ \psi\ } C$ then $\mathsf{Spec}(\psi\circ\varphi) = \mathsf{Spec}\,\varphi \circ \mathsf{Spec}\,\psi$.*

We observe an important property of the continuous functions between spectra which arise from surjective ring homomorphisms.

1.5. *Given a surjective ring homomorphism $\varphi : A \twoheadrightarrow B$ then the induced map $\mathsf{Spec}\,\varphi : \mathsf{Spec}\, B \hookrightarrow \mathsf{Spec}\, A$ is a closed embedding of topological spaces.*

In fact $\mathsf{Spec}\,\varphi\big(\mathsf{Spec}\, B\big) = \mathcal{V}(\ker\varphi)$.

We will return to examine further topological properties of the prime spectrum in due course. First, however, we introduce an important shift in perspective (— providing the promised analogy with classical algebraic geometry) that will recur throughout this investigation: that is, to consider our original ring A as a ring of functions on its spectrum. We define the value $f_\mathfrak{p}$ of an element $f \in A$ at a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ as the residue class of f in the ring $A/\mathfrak{p}$; that is, $f_\mathfrak{p} = [f]_{A/\mathfrak{p}} \in A/\mathfrak{p}$.

Then, since $[f]_{A/\mathfrak{p}} = 0 \iff f \in \mathfrak{p}$, we see, for a subset $S \subseteq A$,

$$\mathcal{V}(S) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f_\mathfrak{p} = 0 \text{ for all } f \in S\}.$$

We may always write a function on $\operatorname{Spec} A$ in terms of its graph, $f = \prod_{\mathfrak{p} \in \operatorname{Spec} A} f_\mathfrak{p}$. In such a way, we obtain a mapping of elements of A into the ring of all such functions taking values in the domain $A/\mathfrak{p}$ at $\mathfrak{p} \in \operatorname{Spec} A$, that is, we obtain a mapping

$$\varphi : A \to \prod_{\mathfrak{p} \in \operatorname{Spec} A} A/\mathfrak{p},$$

defined by $f \mapsto \prod_{\mathfrak{p} \in \operatorname{Spec} A} f_\mathfrak{p} = \prod_{\mathfrak{p} \in \operatorname{Spec} A} [f]_{A/\mathfrak{p}}$. It is natural to ask when do distinct elements of A actually define distinct functions on $\operatorname{Spec} A$, that is, when does this "ring of functions" viewpoint faithfully represent the original ring A. In fact,

1.6. *The mapping* $\varphi : A \to \prod_{\mathfrak{p} \in \operatorname{Spec} A} A/\mathfrak{p}$ *given by* $f \mapsto \prod_{\mathfrak{p} \in \operatorname{Spec} A} f_\mathfrak{p}$ *is injective if and only if A is a reduced ring, that is,* $\operatorname{nil} A = \{0\}$.

We see that given a space $X = \operatorname{Spec} A$ the prime spectrum of some ring A, there is in fact a reduced ring A' such that $\operatorname{Spec} A' \cong X$.

1.7. *Given any ring A, then* $\operatorname{Spec}\big(A/\operatorname{nil} A\big) \cong \operatorname{Spec} A$.

We now proceed to investigate the prime spectrum of a ring as a topological space. Our topology was defined by stating which sets were to be considered closed; we now turn our attention to the open sets of the Zariski topology.

Given a ring A, then the open sets of $\mathsf{Spec}\, A$ are sets of the form

$$\begin{aligned} D(I) = \mathsf{Spec}\, A \setminus \mathcal{V}(I) &= \mathsf{Spec}\, A \setminus \{\mathfrak{p} \mid I \subseteq \mathfrak{p}\} \\ &= \{\mathfrak{p} \mid I \not\subseteq \mathfrak{p}\} \end{aligned}$$

for $I \subseteq A$.

We define a principal open set given by an element $a \in A$ to be the set

$$D(a) = \{\mathfrak{p} \mid \langle a \rangle \not\subseteq \mathfrak{p}\} = \{\mathfrak{p} \mid a \notin \mathfrak{p}\}.$$

These sets turn out to be of central importance to the topological structure of $\mathsf{Spec}\, A$.

1.8. *The collection of principal open sets $\{D(a)\}_{a \in A}$ forms an open basis for the Zariski topology on* $\mathsf{Spec}\, A$.

1.9. *The principal open sets $D(a)$ are quasi-compact.*

1.10. $\mathsf{Spec}\, A$ *is quasi-compact,*

since $D(1) = \mathsf{Spec}\, A$.

1.11. *Given any $a, b \in A$, then $D(a) \cap D(b) = D(ab)$.*

In summary, we have demonstrated that the prime spectrum of a ring is a quasi-compact space with a basis of quasi-compact open sets which is closed under finite intersections. We continue, considering a class of topological spaces possessing similar properties.

Spectral spaces

We define that a topological space X is spectral if it satisfies the following properties:

(1) X is T_0;

(2) X is quasi-compact;

(3) X has a basis of sets which are quasi-compact and open;

(4) The quasi-compact open sets of X are closed under finite intersections; and

(5) X is sober — every non-empty irreducible closed set of X is the closure of a unique point.

Our standard reference for facts about spectral spaces is the forthcoming book [4]; however many results described herein are contained in e.g. [14] and elsewhere.

Of the four properties defining a spectral space, the least familiar is the latter; it in a certain sense guarantees that the topology exactly knows what set of points it is defined on. We remark that the requirement of uniqueness in the statement of property (5) actually implies that X is T_0: since point-closures are always already irreducible, then the condition of uniqueness requires that distinct points have distinct closures, which is exactly equivalent to T_0.

Then in a sober space distinct points are distinguished by the topology. The remainder of the condition — that every irreducible closed set has a generic point — in some sense stipulates that there are sufficient points to distinguish properties manifest in the topology.

We note some simple examples of classes of spectral spaces.

Proposition 1.12. *A Noetherian topological space X is spectral $\iff$ X is (T_0 and) sober*

as every subset of a Noetherian space X is quasi-compact, and so conditions (2)–(4) are automatically satisfied. $\square$

Proposition 1.13. *A finite T_0 space X is spectral.*

Again every subset of a finite space is quasi-compact (—indeed finite spaces are Noetherian) and so conditions (1)–(4) are satisfied. We observe that an irreducible closed set C of a finite space X is the closure of a point. Since C is finite, then $\bigcup_{c \in C} \overline{\{c\}}$ is a closed set, and so $C = \bigcup_{c \in C} \overline{\{c\}}$. Then as C is irreducible we must have that $C = \overline{\{c_0\}}$ for some c_0. Then if X is T_0 this point-closure is necessarily unique, hence X is sober and so hence spectral as required. □

The properties (2)–(4) of spectral spaces relate to the quasi-compact open sets of X. In fact, in the presence of the first four conditions, the final property is also dependent on the behaviour of the quasi-compact open sets of X, as we will see.

We introduce the notation

$$\overset{\circ}{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ is quasi-compact open}\}.$$

Observe that the collection of quasi-compact open sets of a topological space is closed under finite unions, since a union of open sets is open and a finite union of quasi-compact sets is quasi-compact. When X is a spectral space, we have that the set $\overset{\circ}{\mathcal{K}}(X)$ is also closed under finite intersections; that is, it is a sublattice of the lattice of open sets of X.

We note at this stage,

Proposition 1.14. *If $\mathcal{U}$ is a collection of quasi-compact open sets of a topological space X containing a basis for X, then if $\mathcal{U}$ is closed under finite unions then $\mathcal{U} = \overset{\circ}{\mathcal{K}}(X)$,*

since given a quasi-compact open set $U \in \overset{\circ}{\mathcal{K}}(X)$ then as $\mathcal{U}$ contains a basis then U can be written as a union of sets from $\mathcal{U}$; but as U is quasi-compact then it is a finite such union, and so is contained in $\mathcal{U}$. □

Given the central role of quasi-compact open sets in the behaviour of spectral spaces, we introduce the notion of a spectral map: A function of spectral spaces $f : X \to Y$ is spectral if for every $U \in \overset{\circ}{\mathcal{K}}(Y)$ then we have $f^{-1}(U) \in \overset{\circ}{\mathcal{K}}(X)$. Note

Proposition 1.15. *A spectral map $f : X \to Y$ of spectral spaces is continuous,*

since $\overset{\circ}{\mathcal{K}}(Y)$, $\overset{\circ}{\mathcal{K}}(X)$ are an open basis for Y, X respectively. □

We will generally restrict our attention to the subcategory of the category Top of topological spaces and continuous maps which consists of spectral spaces and spectral maps. It is important to note that this is not a full subcategory, in the sense that there are continuous maps in Top between spectral spaces which are not spectral maps.

We then define a spectral subspace of a spectral space X: A subset $Y \subseteq X$ is a spectral subspace of X if

(1) Y is a spectral space in the subspace topology inherited from X; and

(2) The inclusion map $Y \hookrightarrow X$ is a spectral map.

Then spectral subspaces are those such that both the subspace Y and the inclusion map $Y \hookrightarrow X$ are present in our subcategory consisting of spectral spaces and spectral maps.

The behaviour of spectral maps and spectral subspaces, along with the fifth property of the definition of a spectral space, that of being sober, can be stated in terms of a finer topology defined from the quasi-compact open sets of a spectral space X. We first introduce the notation

$$\overline{\mathcal{K}}(X) = \{X \setminus U \mid U \in \overset{\circ}{\mathcal{K}}(X)\}$$

for the collection of complements of quasi-compact open sets of X. We then define the patch or constructible topology on X to be the topology having

$\overset{\circ}{\mathcal{K}}(X) \cup \overline{\mathcal{K}}(X)$ as an open subbasis; that is, the topology generated by declaring the quasi-compact open sets of X to be closed and open. We denote X with the constructible topology by X_{con}. We see

Proposition 1.16. *The constructible topology on a spectral space X refines the original topology,*

since it contains the basis $\overset{\circ}{\mathcal{K}}(X)$. □

Proposition 1.17. *The constructible topology on a spectral space X is Hausdorff.*

The original topology on X is T_0, and the sets $\overset{\circ}{\mathcal{K}}(X)$ are a basis for this topology. Then, given distinct points $x \neq y$, there is a basis set $U \in \overset{\circ}{\mathcal{K}}(X)$ with either $x \in U$ *&* $y \notin U$, or vice-versa. But then $X \setminus U \in \overline{\mathcal{K}}(X)$, and so U and $X \setminus U$ are disjoint open sets in X_{con} with $x \in U$ and $y \in X \setminus U$ (or vice-versa), so X_{con} is Hausdorff. □

Proposition 1.18. *The sets of the form $U \cap V$ for $U \in \overset{\circ}{\mathcal{K}}(X)$ and $V \in \overline{\mathcal{K}}(X)$ form an open basis for the constructible topology on a spectral space X.*

Each set $U \cap V$ is certainly open in X_{con}, being a finite intersection of subbasic open sets. Further as $X \in \overset{\circ}{\mathcal{K}}(X)$ and (since $\emptyset \in \overset{\circ}{\mathcal{K}}(X)$,) $X \in \overline{\mathcal{K}}(X)$, then this collection contains every set $U \in \overset{\circ}{\mathcal{K}}(X)$ and $V \in \overline{\mathcal{K}}(X)$ in our subbasis in the form $U \cap X$ or $X \cap V$. Then as a collection of open sets containing the subbasis, it will suffice to show it is closed under finite intersections. As X is spectral we know $\overset{\circ}{\mathcal{K}}(X)$ is closed under finite intersections; furthermore as $\overset{\circ}{\mathcal{K}}(X)$ is closed under finite unions then $\overline{\mathcal{K}}(X)$ is closed under finite intersections. Then, given a finite intersection $\bigcap_i^n \left(U_i \cap V_i\right)$ with $U_i \in \overset{\circ}{\mathcal{K}}(X)$ and $V_i \in \overline{\mathcal{K}}(X)$, we have

$$\bigcap_i^n (U_i \cap V_i) = \bigcap_i^n U_i \cap \bigcap_i^n V_i,$$

with $\bigcap_i^n U_i \in \overset{\circ}{\mathcal{K}}(X)$ and $\bigcap_i^n V_i \in \overline{\mathcal{K}}(X)$ as required. □

Corollary 1.19. *The sets of the form $U \cup V$ for $U \in \overset{\circ}{\mathcal{K}}(X)$ and $V \in \overline{\mathcal{K}}(X)$ form a closed basis for the constructible topology on a spectral space X,*

since given an open basic set $U' \cap V'$ with $U' \in \overset{\circ}{\mathcal{K}}(X)$ and $V' \in \overline{\mathcal{K}}(X)$, then $X \setminus (U' \cap V') = (X \setminus U') \cup (X \setminus V') = V \cup U$ where $V = (X \setminus U') \in \overline{\mathcal{K}}(X)$ and $U = (X \setminus V') \in \overset{\circ}{\mathcal{K}}(X)$. □

The most significant property of the constructible topology on a spectral space X is that X_{con} is itself quasi-compact. In fact,

Proposition 1.20. *If X is a topological space satisfying properties (1)–(4) in the definition of a spectral space, then X is sober (that is, X is in fact a spectral space) $\iff$ X_{con} is quasi-compact.*

Let X be a topological space satisfying properties (1)–(4) in the definition of a spectral space; that is, X is T_0, quasi-compact, and the collection $\overset{\circ}{\mathcal{K}}(X)$ contains a basis and is closed under finite intersections. Observe that the conclusions of the preceding three propositions and the corollary still hold for such a space X, as their proofs (at most) use only the fact that X is T_0 rather than the full sobriety property. We relate sets that are closed and irreducible in the original topology on X to collections of subsets closed in the constructible topology on X having the finite intersection property.

Note that a closed subset $C \subseteq X$ is irreducible iff any collection of open sets which intersect C has non-empty finite intersections within C. Indeed, it suffices to consider basic open sets; then, C is irreducible if, given any collection $\mathcal{U}_\Lambda = \{U_\lambda \mid U_\lambda \in \overset{\circ}{\mathcal{K}}(X) \text{ and } U_\lambda \cap C \neq \emptyset\}$, we have that for any finite subcollection $U_{\lambda_1}, \ldots, U_{\lambda_n}$ we have $\left(\bigcap_i^n U_{\lambda_i}\right) \cap C \neq \emptyset$.

Suppose X_{con} is quasi-compact, and take C a set closed and irreducible in the original topology on X. We consider the collection $\mathcal{U}_C = \{U \in \overset{\circ}{\mathcal{K}}(X) \mid U \cap C \neq \emptyset\}$. As C is irreducible, this has the finite intersection property. Furthermore, it is a collection of basic closed sets for the constructible topology. Then, if X_{con} is quasi-compact, we must have $\bigcap \mathcal{U}_C \neq \emptyset$. Take $x \in \bigcap \mathcal{U}_C$. Then x is contained in every basic open set (for the original topology) that intersects C, that is, $C = \overline{\{x\}}$, and so (—since X is already T_0, and so x is uniquely specified,) X is sober.

Alternatively let $\mathcal{C}$ be a collection of closed basic subsets of X_{con} (—that is, sets of the form $U \cup V$ for $U \in \mathring{\mathcal{K}}(X)$ and $V \in \overline{\mathcal{K}}(X)$) having the finite intersection property. We see that chains of such collections (that is, of closed basic subsets of X_{con} having the finite intersection property) satisfy the condition of Zorn's lemma, since the union of such a chain retains the finite intersection property, as any finite collection of sets from the union of the chain must be already present in some set in the chain. Let $\mathcal{M}$ be a maximal such collection containing $\mathcal{C}$. Then define $\mathcal{V} = \{V \in \mathcal{M} \mid V \in \overline{\mathcal{K}}(X)\}$ and let $C = \bigcap \mathcal{V}$. We see that

$$\{U \in \mathring{\mathcal{K}}(X) \mid U \in \mathcal{M}\} = \{U \in \mathring{\mathcal{K}}(X) \mid U \cap C \neq \emptyset\}.$$

Since, if $U \notin \mathcal{M}$ then $\mathcal{M} \cup \{U\}$ must fail the finite intersection property, that is, we have $M_1, \ldots, M_n \in \mathcal{M}$ with $M_1 \cap \cdots \cap M_n \cap U = \emptyset$. Then $U^{\mathsf{c}} \in \mathcal{M}$, since $M_1 \cap \cdots \cap M_n \cap U^{\mathsf{c}} = M_1 \cap \cdots \cap M_n$, so for any $A_1, \ldots, A_m \in \mathcal{M} \setminus \{U^{\mathsf{c}}\}$ we see that

$$A_1 \cap \cdots \cap A_m \cap U^{\mathsf{c}} \supseteq \Big(\bigcap^m A_i\Big) \cap M_1 \cap \cdots \cap M_n \cap U^{\mathsf{c}} = \Big(\bigcap^m A_i\Big) \cap \Big(\bigcap^n M_j\Big) \neq \emptyset,$$

and so adding U^{c} preserves the finite intersection property. Furthermore $U^{\mathsf{c}} \in \overline{\mathcal{K}}(X)$, so $U^{\mathsf{c}} \in \mathcal{V}$; then $C = \bigcap \mathcal{V} \subseteq U^{\mathsf{c}}$ and so $U \cap C = \emptyset$.

Conversely, if $U \in \mathcal{M}$ then we consider the collection $\{U \cap V \mid V \in \mathcal{V}\}$. This is a collection of relatively-closed (—per the original topology) subsets in the quasi-compact subset U; furthermore it has the finite intersection property, since U and each V are in $\mathcal{M}$, and so a finite intersection from this collection is a finite intersection of sets from $\mathcal{M}$. Then, by quasi-compactness of U this collection must have non-empty total intersection, that is, $U \cap \bigcap V \neq \emptyset$, i.e. $U \cap C \neq \emptyset$ as required.

We show C is irreducible. We have seen that the set of basic open sets having non-empty intersection with C is exactly the set of basic open sets present in our collection $\mathcal{M}$; then given a finite collection $U_1, \ldots, U_n$ of such basic open sets, we have that $U_1 \cap \cdots \cap U_n \in \mathcal{M}$, since adding this set preserves the finite intersection property of $\mathcal{M}$ since each constituent set is already present in $\mathcal{M}$. Then as $U_1 \cap \cdots \cap U_n$ is in $\mathring{\mathcal{K}}(X)$, we have $U_1 \cap \cdots \cap U_n \in \mathcal{M} \implies (U_1 \cap \cdots \cap U_n) \cap C \neq \emptyset$; so C is irreducible as required. Then if X is sober, we have that $C = \overline{\{x\}}$ for some point $x \in X$. We show that $x \in \bigcap \mathcal{M}$. $\mathcal{M}$ is a collection of basic closed sets for the constructible topology on X, that is, it is a collection of sets of the form $U \cup V$ for $U \in \mathring{\mathcal{K}}(X)$ and $V \in \overline{\mathcal{K}}(X)$. In fact we see, given such a set $U \cup V \in \mathcal{M}$, then we have either $U \in \mathcal{M}$ or $V \in \mathcal{M}$. Since, if neither were in $\mathcal{M}$, by maximality we must have finite intersections $\left(\bigcap^m A_i\right) \cap U = \emptyset$ and $\left(\bigcap^n B_j\right) \cap V = \emptyset$ for $A_i, B_j \in \mathcal{M}$. But then $\left(\bigcap^m A_i\right) \cap \left(\bigcap^n B_j\right) \cap (U \cup V) = \emptyset$, contradicting $U \cup V \in \mathcal{M}$. Then $x \in U \cup V$, since, if $U \in \mathcal{M}$ then $U \cap C \neq \emptyset$, and so (since U is open) $x \in U$, and if $V \in \mathcal{M}$ then $C \subseteq V$ by definition. Then $x \in \bigcap \mathcal{M}$ and so certainly $x \in \bigcap \mathcal{C}$, and so we have shown that collections of closed basic subsets of X_{con} having the finite intersection property have non-empty total intersection, that is, X_{con} is quasi-compact. □

We now see how the behaviour of spectral maps relates to the constructible topology on a spectral space.

Proposition 1.21. *A map $f : X \to Y$ of spectral spaces X and Y is a spectral map $\iff$ f is continuous and is a continuous map with respect to the constructible topologies on X and Y.*

We know a spectral map $f : X \to Y$ is continuous; furthermore since for a spectral map we have $f^{-1}(\overset{\circ}{\mathcal{K}}(Y)) \subseteq \overset{\circ}{\mathcal{K}}(X)$ then likewise $f^{-1}(\overline{\mathcal{K}}(Y)) \subseteq \overline{\mathcal{K}}(X)$ and so the preimage of a subbasic set for Y_{con} is in the subbasis for X_{con}, so f is continuous as a map $X_{\mathsf{con}} \to Y_{\mathsf{con}}$.

Conversely, to show f is a spectral map we need to show that the preimage of a quasi-compact open set U of Y is quasi-compact open in X. If f is continuous then the preimage of U is open. If furthermore f is a continuous map between the constructible topologies on X and Y, then as U is a closed set in Y_{con} then the preimage of U is closed in X_{con}. Then, as X_{con} is quasi-compact, $f^{-1}(U)$ is quasi-compact in X_{con}, and as the original topology on X is a coarsening of the constructible topology then $f^{-1}(U)$ is quasi-compact in the original topology on X as required. □

Subsets of a spectral space X that are closed in the constructible topology have some remarkable properties. Firstly,

Proposition 1.22. *If $Y \subseteq X$ is closed in the constructible topology of X, then we have $x \in \overline{Y} \iff x \in \overline{\{y\}}$ for some $y \in Y$.*

Given some $x \in \overline{Y}$ we consider the collection of basic open neighbourhoods of x, $\mathcal{U}_x = \{U \in \overset{\circ}{\mathcal{K}}(X) \mid x \in U\}$. As x is in the closure of Y, then every open neighbourhood of x intersects Y; so the collection $\mathcal{U}_x \cup \{Y\}$ has the finite intersection property (since a finite intersection of sets from $\mathcal{U}_x$ is again an open neighbourhood of x, and so intersects Y). Furthermore, each set in this collection is closed in the constructible topology. Then, as X_{con} is quasi-compact, this collection has non-empty total intersection, that is there is an element $y \in Y$ contained in every basic open neighbourhood of x; then $x \in \overline{\{y\}}$ as required. □

In fact the spectral subspaces of a spectral space X are exactly the subsets closed in the constructible topology.

Proposition 1.23. *A subset Y of a spectral space X is a spectral subspace of $X \iff Y$ is a closed set in the constructible topology on X.*

Suppose Y is a spectral subspace of X. Then Y is a spectral space in the inherited topology, and inclusion $Y \hookrightarrow X$ is a spectral map. As Y is a spectral space, we have that Y_{con} is quasi-compact. As inclusion is a spectral map, then it is continuous between $Y_{\mathsf{con}} \to X_{\mathsf{con}}$; then, being the image of a quasi-compact set under a continuous map, Y is quasi-compact in X_{con}. Then finally as X_{con} is Hausdorff, Y is closed in X_{con} as required.

Conversely suppose Y is closed in X_{con}. We find $\overset{\circ}{\mathcal{K}}(Y) = \{U \cap Y \mid U \in \overset{\circ}{\mathcal{K}}(X)\}$. As Y and each $U \in \overset{\circ}{\mathcal{K}}(X)$ are closed in X_{con} then $U \cap Y$ is closed, and hence quasi-compact, in X_{con} (—since X_{con} is quasi-compact), and so also in X (since the original topology on X is a coarsening of the constructible topology), and then also in Y (since quasi-compactness is inherited by subsets of Y in the subspace topology). Furthermore this collection is a basis for the inherited topology on Y and is closed under finite unions, so it accounts for all quasi-compact open sets in Y. Clearly $\overset{\circ}{\mathcal{K}}(Y)$ is also closed under finite intersections; then Y will be a spectral space in the subspace topology iff it is sober. Since $\overset{\circ}{\mathcal{K}}(Y) = \{U \cap Y \mid U \in \overset{\circ}{\mathcal{K}}(X)\}$, and so also $\overline{\mathcal{K}}(Y) = \{V \cap Y \mid V \in \overline{\mathcal{K}}(X)\}$, it is clear that the constructible topology defined on Y as a spectral space (i.e. that generated by $\overset{\circ}{\mathcal{K}}(Y) \cup \overline{\mathcal{K}}(Y)$) is exactly the same as the subspace topology inherited by Y as a subset of X_{con}. As Y is closed in X_{con} then it is quasi-compact in X_{con}, so Y_{con} is quasi-compact (as quasi-compactness is inherited) and hence Y is sober as required. The fact that $\overset{\circ}{\mathcal{K}}(Y) = \{U \cap Y \mid U \in \overset{\circ}{\mathcal{K}}(X)\}$ shows that preimages of quasi-compact open sets of X are quasi-compact open in Y, hence inclusion is a spectral map, and so Y is a spectral subspace. □

We take this opportunity to investigate some further topological notions which will become relevant in later discussions.

The specialisation order

We have seen in various places in the preceding section properties of spectral spaces that relate to point-closures, that is, to closures of singleton sets in the spectral topology. We introduce an order relation on a topological space that encapsulates this notion.

Given a topological space X, we define the specialisation order on X, denoted $\rightsquigarrow$, by

$$x \rightsquigarrow y \iff y \in \overline{\{x\}}.$$

Where we have $x \rightsquigarrow y$ we say y is a specialisation of x and x is a generisation of y; x specialises to y and y generalises to x.

We fix notation as follows:

$$x^{\uparrow} = \{y \mid x \rightsquigarrow y\}, \text{ the set of all specialisations of } x;$$

$$x^{\downarrow} = \{z \mid z \rightsquigarrow x\}, \text{ the set of all generisations of } x.$$

Clearly by the definition we have $x^{\uparrow} = \overline{\{x\}}$.

In general, $\rightsquigarrow$ is a pre-order, that is, it is reflexive and transitive, since closure in a topological space is reflexive and transitive, i.e.

- $\forall x \in X$, $x \in \overline{\{x\}}$, so $\forall x \in X$, $x \rightsquigarrow x$, and
- $y \in \overline{\{x\}}$ *&* $z \in \overline{\{y\}} \implies z \in \overline{\{x\}}$, so $x \rightsquigarrow y$ *&* $y \rightsquigarrow z \implies x \rightsquigarrow z$.

We see

Proposition 1.24. *The specialisation order $\rightsquigarrow$ on a topological space X is a partial order (—i.e., it is anti-symmetric) $\iff$ X satisfies the T_0 separation property.*

Anti-symmetry of $\rightsquigarrow$ is the statement that $x \rightsquigarrow y$ *&* $y \rightsquigarrow x \implies x = y$, which translates via the definition to $y \in \overline{\{x\}}$ *&* $x \in \overline{\{y\}} \implies x = y$. Taking the contrapositive, we obtain $x \neq y \implies y \notin \overline{\{x\}}$ or $x \notin \overline{\{y\}}$, i.e. distinct points are distinguished by some closed set; equivalently by some open set, which is the T_0 property. □

Relating to the specialisation order on a topological space is the notion of an order-compatible topology on a poset. Given a poset $(X, \leqslant)$, a topology τ on X is order-compatible if the specialisation order on (X, τ) is $\leqslant$. Note that by stipulating that $\leqslant$ is a partial order we have that any order-compatible topology will satisfy the T_0 property, by the preceding observation.

A comprehensive overview of order-compatible topology as it pertains to spectral spaces is given in the paper [22]; see also [23], [18], [5], [21].

It is easy to see that

Proposition 1.25. *Given a set X, and a partial order $\leqslant$ and a topology τ on X, we have τ is order-compatible (with $\leqslant$) exactly when:*

(1) For each $x \in X$, the set $x^{\uparrow} = \{y \mid x \leqslant y\}$ is a closed set in τ, and

(2) For each closed set C in τ, $x^{\uparrow} \subseteq C$ for each $x \in C$.

We note as an immediate consequence of this observation that if we are given a poset $(X, \leqslant)$, then the order-compatible topologies on X form an interval in the collection of all topologies on X: the coarsest possible order-compatible topology is that given by taking the collection $\{x^{\uparrow} \mid x \in X\}$ as a closed subbasis; we call this topology the coarse lower topology or sometimes the closures-of-points or COP topology, and the finest topology is that given by allowing any set C containing $x^{\uparrow}$ for every $x \in C$ (i.e. all upper sets $C = C^{\uparrow}$) to be closed; in which case we obtain exactly the lower sets $U = U^{\downarrow}$ as the open sets, hence we can think of this as the topology having the collection $\{x^{\downarrow} \mid x \in X\}$ as an open basis; we call this topology the fine lower topology, or in some references the left topology. Given a poset $(X, \leqslant)$, we write X^l for the space given by X with the coarse lower topology, and X^L for X with the fine lower topology.

Note that when X is finite then we have that all up-closed sets $C \subseteq X$ (i.e. all closed sets of X^L) are of the form $S^\uparrow$ for some finite set $S \subseteq X$ (we can take the finite set $S = C$), that is, C is closed in X^l, hence $X^L = X^l$ and we may refer to this unique order-compatible topology as the lower topology on a finite poset X. We note that by 1.13, as an order-compatible topology on a poset is T_0, then every finite poset $(X, \leqslant)$ arises as the specialisation order of a spectral space, namely the finite T_0 space consisting of X equipped with the lower topology.

We now see that the specialisation order exactly characterises the Noetherian spectral spaces; per 1.12 these are the sober Noetherian spaces. Our first observation in this direction is that Noetherian spectral spaces are uniquely determined (up to homeomorphism) by their specialisation order; that is,

Proposition 1.26. *Given Noetherian spectral spaces X, Y where $(X, \rightsquigarrow_X) \cong (Y, \rightsquigarrow_Y)$ as posets, then $X \cong Y$ as topological spaces.*

It is a well known property of Noetherian spaces that every closed set of a Noetherian space X is a finite union of irreducible closed sets. In the case that X is spectral, and so sober, then every irreducible closed set is the closure of a point, and so every closed set C of X is a finite union $C = \overline{\{x_1\}} \cup \cdots \cup \overline{\{x_n\}} = x_1^\uparrow \cup \cdots \cup x_n^\uparrow$ (and of course every such finite union gives a closed set of X). That is, the topology on X is entirely specified by the partial order relation $\rightsquigarrow_X$, and so two isomorphic specialisation posets necessarily define the same topology. □

Note as a consequence of this observation that the topology on a Noetherian spectral space X is always equal to the coarse lower topology on $(X, \rightsquigarrow_X)$.

In fact there is a purely order-theoretic characterisation of exactly those posets which arise as the specialisation poset of a Noetherian spectral space, first given in [23]:

Theorem 1.27 ([23], discussion following Proposition 1). *A poset* $(X, \leqslant)$ *is isomorphic to* $(X', \rightsquigarrow_{X'})$ *for* X' *a Noetherian spectral space if and only if:*

(1) X *has the ascending chain condition;*

(2) X *has finitely many minimal elements* $\{x_1, \dots, x_n\}$ *and* $\{x_1, \dots, x_n\}^{\uparrow} = X$*; and*

(3) Every pair of elements $x, y \in X$ *has finitely many minimal upper bounds in* X*.*

We remark only on the necessity of these conditions, that is, that if X is a Noetherian spectral space then the poset $(X, \rightsquigarrow)$ satisfies these properties. An ascending chain $\dots \rightsquigarrow x_i \rightsquigarrow x_{i+1} \rightsquigarrow x_{i+2} \rightsquigarrow \dots$ corresponds to a *descending* chain of closed sets $\dots \supseteq \overline{\{x_i\}} \supseteq \overline{\{x_{i+1}\}} \supseteq \overline{\{x_{i+2}\}} \supseteq \dots$; thus if X is Noetherian the chain must stabilise. As in 1.26, the closed set X has finitely many irreducible components $X = \overline{\{x_1\}} \cup \dots \cup \overline{\{x_n\}} = x_1^{\uparrow} \cup \dots \cup x_n^{\uparrow}$ which correspond to points minimal in the specialisation order; likewise the closed set $x^{\uparrow} \cap y^{\uparrow}$ has finitely many irreducible components $x^{\uparrow} \cap y^{\uparrow} = z_1^{\uparrow} \cup \dots \cup z_m^{\uparrow}$, corresponding to the minimal common upper bounds of x and y in $(X, \rightsquigarrow)$. □

The inverse space of a spectral space

Given a spectral space X we use the collection $\overline{\mathcal{K}}(X)$ to define an alternative topology on the set X: We define the inverse space X_{inv} of X to be the set X with topology (—the inverse or opposite-order topology) defined by taking the set $\mathcal{B} = \overline{\mathcal{K}}(X) = \{X \setminus U \mid U \in \mathring{\mathcal{K}}(X)\}$ as an open subbasis for X. In fact as we have previously observed, $\mathcal{B} = \overline{\mathcal{K}}(X)$ is closed under finite intersections, so is in fact a basis for X_{inv}.

Remarkably, the space so defined is again a spectral space. We see

Proposition 1.28. X_{inv} *is quasi-compact,*

since the topology of X_{inv} is a coarsening of the topology of X_{con}, which is quasi-compact. □

Furthermore,

Proposition 1.29. X_{inv} *has a basis of quasi-compact open sets, closed under finite intersections,*

since firstly $\mathcal{B} = \overline{\mathcal{K}}(X)$ is closed under finite intersections; furthermore each set in $\overline{\mathcal{K}}(X)$ is closed in X_{con}, so is quasi-compact in X_{con} and hence is quasi-compact in X_{inv} (again since the topology of X_{inv} is a coarsening of the topology of X_{con}). □

In fact

Proposition 1.30. $\mathring{\mathcal{K}}(X_{\mathsf{inv}}) = \mathcal{B} = \overline{\mathcal{K}}(X)$,

that is our basis $\mathcal{B}$ accounts for all quasi-compact open sets in X_{inv}: since, as $\mathring{\mathcal{K}}(X)$ is closed under finite intersections then $\mathcal{B} = \overline{\mathcal{K}}(X)$ is closed under finite unions. But then $\mathcal{B}$ is a basis of quasi-compact open sets which is closed under finite unions, therefore by 1.14 it accounts for all quasi-compact open sets in X_{inv}, that is, $\mathring{\mathcal{K}}(X_{\mathsf{inv}}) = \mathcal{B}$. □

Corollary 1.31. $\overline{\mathcal{K}}(X_{\mathsf{inv}}) = \mathring{\mathcal{K}}(X)$,

since by definition, $\overline{\mathcal{K}}(X_{\mathsf{inv}})$ is the set of complements $\{X \setminus U \mid U \in \mathring{\mathcal{K}}(X_{\mathsf{inv}})\}$ of quasi-compact open sets of X_{inv}. But then, by the preceding proposition we have

$$\begin{aligned}\overline{\mathcal{K}}(X_{\mathsf{inv}}) &= \{X \setminus U \mid U \in \mathring{\mathcal{K}}(X_{\mathsf{inv}})\} \\ &= \{X \setminus U \mid U \in \overline{\mathcal{K}}(X)\} \\ &= \mathring{\mathcal{K}}(X).\end{aligned}$$

□

This allows us to observe,

Proposition 1.32. $(X_{\mathsf{inv}})_{\mathsf{inv}} = X$*,*

since $(X_{\mathsf{inv}})_{\mathsf{inv}}$ is the space defined as having $\overline{\mathcal{K}}(X_{\mathsf{inv}})$ as an open subbasis. But $\overline{\mathcal{K}}(X_{\mathsf{inv}}) = \mathring{\mathcal{K}}(X)$ is a basis for the original topology on X, so $(X_{\mathsf{inv}})_{\mathsf{inv}} = X$. □

Furthermore,

Proposition 1.33. $(X_{\mathsf{inv}})_{\mathsf{con}} = X_{\mathsf{con}}$*,*

since $\mathring{\mathcal{K}}(X_{\mathsf{inv}}) \cup \overline{\mathcal{K}}(X_{\mathsf{inv}}) = \overline{\mathcal{K}}(X) \cup \mathring{\mathcal{K}}(X)$. □

And then finally, we have

Proposition 1.34. X_{inv} *is a spectral space.*

We have already noted that X_{inv} is a quasi-compact space with a basis of quasi-compact open sets closed under finite intersections. It is clearly T_0, as we know distinct points in the set X are separated by some $U \in \mathring{\mathcal{K}}(X)$ (since the original topology on X is T_0); and so they are likewise separated by $X \setminus U \in \overline{\mathcal{K}}(X)$. Furthermore $(X_{\mathsf{inv}})_{\mathsf{con}} = X_{\mathsf{con}}$ is quasi-compact; so X_{inv} is sober, and so is hence a spectral space, as required. □

We finally note how the inverse topology and the specialisation order on a spectral space X relate.

Given an order relation $\leqslant$, we may consider the opposite or dual order $\leqslant_{\mathsf{opp}}$ defined by

$$x \leqslant_{\mathsf{opp}} y \iff y \leqslant x.$$

Of course, $\leqslant_{\mathsf{opp}}$ is just the familiar order relation $\geqslant$.

Now, given a spectral space X, let $\rightsquigarrow_X$ denote the specialisation order on X, and consider the opposite order $(\rightsquigarrow_X)_{\mathsf{opp}}$.

Then, considering the inverse space X_{inv} of X, and writing the specialisation order on X_{inv} as $\rightsquigarrow_{X_{\mathsf{inv}}}$, we find

Proposition 1.35. $\rightsquigarrow_{X_{\mathsf{inv}}} = (\rightsquigarrow_X)_{\mathsf{opp}}$,
that is, $x \rightsquigarrow_{X_{\mathsf{inv}}} y \iff y \rightsquigarrow_X x$.

Recall we have $y \rightsquigarrow_X x \iff x \in \overline{\{y\}} \iff$ every basic closed set of X containing y contains x. Then $y \rightsquigarrow_X x \iff x \in \bigcap\{V \in \overline{\mathcal{K}}(X) \mid y \in V\}$. But considering this statement in X_{inv}, then as the sets $V \in \overline{\mathcal{K}}(X)$ are an open basis for the inverse topology on X, this says that every basic open neighbourhood of y (in X_{inv}) contains x, which is exactly equivalent to $y \in \overline{\{x\}}$ in X_{inv}, i.e. $x \rightsquigarrow_{X_{\mathsf{inv}}} y$. □

We now return to considering how the topological properties investigated here relate to the prime spectrum of a ring.

$\mathsf{Spec}\,A$ as a spectral space

As might be anticipated, we have

Proposition 1.36. *The prime spectrum of a ring A is a spectral space.*

We saw at the end of the first section that $\mathsf{Spec}\,A$ satisfies the second and third properties of a spectral space via the basis $\mathbb{D}(A)$ of quasi-compact 'principal open' sets, that is that $\mathsf{Spec}\, A$ is quasi-compact, and has a basis of quasi-compact open sets. We also observed that the principal open sets were closed under finite intersection; note the definition of a spectral space requires that the collection of *all* quasi-compact open sets be closed under finite intersections. We can see that, as $\mathbb{D}(A)$ is a basis, then a general quasi-compact open set U in $\mathsf{Spec}\, A$ is a finite union of sets from $\mathbb{D}(A)$,

$$U = D(a_1) \cup \cdots \cup D(a_n)$$

for some $a_1, \ldots, a_n \in A$. Then given two such quasi-compact open sets U, V, we see that

$$\begin{aligned} U \cap V &= \big(D(a_1) \cup \cdots \cup D(a_n)\big) \cap \big(D(b_1) \cup \cdots \cup D(b_m)\big) \\ &= \bigcup_{i,j} \big(D(a_i) \cap D(b_j)\big) \\ &= \bigcup_{i,j} \big(D(a_i b_j)\big), \end{aligned}$$

i.e. it is a finite union of principal quasi-compact open sets, and so is quasi-compact open. To show $\mathsf{Spec}\, A$ is a spectral space, then, it remains to show that every non-empty irreducible closed set of $\mathsf{Spec}\, A$ is the closure of a unique point (—recall that uniqueness will then imply that $\mathsf{Spec}\, A$ is T_0).

Let $\emptyset \neq C \subseteq \mathsf{Spec}\, A$ be an irreducible closed set. We consider $I = \mathcal{I}(C)$ the radical ideal of A such that $C = \mathcal{V}(I)$. We show that C being irreducible implies that I is in fact a prime ideal. Since, suppose we have ideals J, K of A with $JK \subseteq I$. Then (by 1.2 and 1.14) we have $\mathcal{V}(I) \subseteq \mathcal{V}(JK) = \mathcal{V}(J) \cup \mathcal{V}(K)$. As $\mathcal{V}(I)$ is irreducible we must have one of $\mathcal{V}(I) \subseteq \mathcal{V}(J)$ or $\mathcal{V}(I) \subseteq \mathcal{V}(K)$; then, as I is a radical ideal we have either $J \subseteq I$ or $K \subseteq I$; this shows that I is a prime ideal of A. Then the set $C = \mathcal{V}(I)$ is the closure in $\mathsf{Spec}\, A$ of the point

I, since for any closed set $\mathcal{V}(S)$ defined by some subset $S \subseteq A$, we have

$$I \in \mathcal{V}(S) \implies S \subseteq I \implies \mathcal{V}(I) \subseteq \mathcal{V}(S),$$

that is, $\mathcal{V}(I)$ is the smallest closed set of $\operatorname{Spec} A$ containing I. Then we have shown that every irreducible closed set of $\operatorname{Spec} A$ is the closure of a point; the required uniqueness is equivalent to distinct points having distinct closures, but given distinct prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ in $\operatorname{Spec} A$ then (— since prime ideals are radical ideals) the sets $\overline{\{\mathfrak{p}\}} = \mathcal{V}(\mathfrak{p})$ and $\overline{\{\mathfrak{q}\}} = \mathcal{V}(\mathfrak{q})$ are distinct. □

It is much harder to show—though no less true—that every spectral space arises as the prime spectrum of some ring, which shall be the topic of the following chapter.

In the meantime, we investigate some further properties of the prime spectrum of a ring, from the point of view of spectral spaces. Again the textbooks [4],[1] and in some instances the paper [14] provide standard references for such facts.

1.37. *Every ring homomorphism $\varphi : A \to B$ induces a spectral map $\operatorname{Spec} \varphi : \operatorname{Spec} B \to \operatorname{Spec} A$.*

We have already that $\operatorname{Spec} \varphi$ is continuous, from 1.3; we must show the preimage of a quasi-compact open set in $\operatorname{Spec} A$ is quasi-compact open in $\operatorname{Spec} B$. But this is true for the basis sets

$$\begin{aligned}\big(\operatorname{Spec} \varphi\big)^{-1}(D(a)) = \{\mathfrak{q} \in \operatorname{Spec} B \mid \big(\operatorname{Spec} \varphi\big)(\mathfrak{q}) \in D(a)\} &= \{\mathfrak{q} \mid a \notin \varphi^{-1}(\mathfrak{q})\} \\ &= \{\mathfrak{q} \mid \varphi(a) \notin \mathfrak{q}\} \\ &= D(\varphi(a))\end{aligned}$$

and so is true for finite unions of such, which account for all quasi-compact open sets in $\operatorname{Spec} A$. □

1.38. *The specialisation order on* Spec A *is the subset-inclusion relation between prime ideals as subsets of* A*, that is,* $\mathfrak{p} \rightsquigarrow \mathfrak{q} \iff \mathfrak{p} \subseteq \mathfrak{q}$ *as subsets of* A*.*

We have

$$\mathfrak{p} \rightsquigarrow \mathfrak{q} \iff \mathfrak{q} \in \overline{\{\mathfrak{p}\}} = \mathcal{V}(\mathfrak{p}) \iff \mathfrak{q} \in \{\mathfrak{q}' \in \mathsf{Spec}\, A \mid \mathfrak{p} \subseteq \mathfrak{q}'\} \iff \mathfrak{p} \subseteq \mathfrak{q}.$$

□

We have observed that the quasi-compact open sets of Spec A are the finite unions of basic open sets $D(a)$. We use this to note the following:

1.39. *Given* $X = \mathsf{Spec}\, A$*, sets of the form* $D(a) \cap \big(\bigcap_i^n \mathcal{V}(b_i)\big)$ *for* $a, b_1, \ldots, b_n \in A$ *form a basis for the space* $X_{\mathsf{con}} = \big(\mathsf{Spec}\, A\big)_{\mathsf{con}}$*.*

From 1.18 we have that sets of the form $U \cap V$ for $U \in \mathring{\mathcal{K}}(X)$ and $V \in \overline{\mathcal{K}}(X)$ form a basis for the constructible topology on a spectral space X. Given $X =$ Spec A, a general set in $\mathring{\mathcal{K}}(X)$ is of the form $\bigcup_j^m D(a_j)$ for some finite subset $a_1, \ldots, a_m \in A$, and a general set in $\overline{\mathcal{K}}(X)$ is the complement of such a set, i.e. is of the form $\big(\bigcup_i^n D(b_i)\big)^{\mathsf{c}} = \bigcap_i^n \big(D(b_i)\big)^{\mathsf{c}} = \bigcap_i^n \mathcal{V}(b_i)$ for some finite subset $b_1, \ldots, b_n \in A$. Then we have that sets of the form

$$\bigcup_j^m D(a_j) \cap \bigcap_i^n \mathcal{V}(b_i)$$

for finite subsets $a_1, \ldots, a_m$, $b_1, \ldots, b_n \in A$ give a basis for the constructible topology on Spec A. But, as any set Y open in X_{con} is a union of such sets, then Y is also a union of the sets $D(a_i) \cap \bigcap_{b \in B} \mathcal{V}(b)$ for each a_i; thus the collection of sets of this form also forms a basis for X_{con}. □

1.40. *Given* $X = \mathsf{Spec}\, A$*, sets of the form* $\mathcal{V}(a) \cup \bigcup_i^n D(b_i)$ *for* $a, b_1, \ldots, b_n \in A$ *form a closed basis for the space* $X_{\mathsf{con}} = \big(\mathsf{Spec}\, A\big)_{\mathsf{con}}$

by taking complements in the preceding result. □

We remarked in the preceding section on the characterisation of Noetherian spectral spaces by their specialisation posets. Whilst we have that if A is a Noetherian ring, then $\mathsf{Spec}\,A$ is indeed a Noetherian spectral space, unfortunately it is not the case that every specialisation poset of a Noetherian spectral space can arise as the inclusion ordering of prime ideals of a Noetherian ring; that is, there are Noetherian spectral spaces X for which there is no Noetherian ring A such that $\mathsf{Spec}\,A \cong X$. Since, observe that any finite chain, for example $X = \{x_0 < x_1 < x_2\}$, satisfies the conditions of 1.27, such that the coarse lower topology on X is a Noetherian spectral space. However it is not hard to see

1.41. *If A is a Noetherian ring of dimension at least 2, then $\mathsf{Spec}\,A$ is infinite.*

Let $\mathfrak{p} \subseteq A$ be a prime ideal of height 2. We write

$$\mathfrak{p} = \bigcup_{a \in \mathfrak{p}} \langle a \rangle.$$

The Krull principal ideal theorem (see, for example, [15]) states that a prime ideal $\mathfrak{q}_a$ in a Noetherian ring that is minimal with respect to the inclusion of a principal ideal $\langle a \rangle \subseteq \mathfrak{q}_a$ has height at most 1. For each $a \in \mathfrak{p}$ let $\mathfrak{q}_a$ be a prime ideal minimal over a; then

$$\mathfrak{p} \subseteq \bigcup_{a \in \mathfrak{p}} \mathfrak{q}_a.$$

If $\mathsf{Spec}\,A$ were finite then $\mathfrak{p}$ would necessarily be contained in a finite union of height-1 primes; then by prime avoidance it must be contained in one of them, giving $\mathfrak{p} \subseteq \mathfrak{q}_{a'}$ for some $\mathfrak{q}_{a'}$ of height less than or equal to 1. But this contradicts that the height of $\mathfrak{p}$ is at least 2; and so A must have infinitely many prime ideals. □

Thus the poset X defined above cannot arise as the inclusion ordering of any Noetherian ring.

Valuations and valuation rings

The theory of (discrete) valuations on a field is centrally involved in many of the discussions in this thesis; we summarise without proof the basic properties and results of this theory. The standard reference for this section is [3].

A valuation on a field k is a map $v : k \to \Gamma \cup \{\infty\}$ for Γ an ordered group and some symbol $\infty \notin \Gamma$ defined as greater than every element of Γ (with the obvious extension of the group operation: $\infty + g = g + \infty = \infty$ for any $g \in \Gamma$), satisfying the following properties:

(1) $v(\alpha) = \infty \iff \alpha = 0$;

(2) $v(\alpha\beta) = v(\alpha) + v(\beta)$ for all $\alpha, \beta \in k$;

(3) $v(\alpha + \beta) \geqslant \mathsf{min}(v(a), v(b))$ for all $\alpha, \beta \in k$.

A valuation $v : k \to \Gamma \cup \{\infty\}$ on a field k is improper if $v(\alpha) = 0$ for all $\alpha \in k^{\times}$, v being proper otherwise; some authors exclude improper valuations from their definition.

A valuation $v : k \to \Gamma \cup \{\infty\}$ is a discrete valuation on k if $\Gamma \cong \mathbb{Z}$; in practice all valuations considered in the remainder of this thesis will be discrete, however unless otherwise specified the results summarised in this section are valid for all valuations.

We obtain the following identities:

1.42. *Given a field k and $v : k \to \Gamma \cup \{\infty\}$ a valuation on k, then:*

(1) $v(1) = 0$;

(2) $v(\alpha) = v(-\alpha)$ for all $\alpha \in k$;

(3) If $v(\alpha) \neq v(\beta)$ then $v(\alpha + \beta) = \mathsf{min}(v(\alpha), v(\beta))$;

Given a valuation $v : k \to \Gamma \cup \{\infty\}$ on a field k the valuation ring of v is the ring $V = \{\alpha \in k \mid v(\alpha) \geqslant 0\}$. Generally, a ring R (—necessarily a domain) is a valuation ring if it is the valuation ring of some valuation v on $\mathcal{Q}(R)$.

We note

1.43. *A domain R is a valuation ring if for every $\alpha \in \mathcal{Q}(R)^\times$, we have either $\alpha \in R$ or $\alpha^{-1} \in R$.*

1.44. *Given a valuation $v : k \to \Gamma \cup \{\infty\}$ on a field k, the valuation ring V is a local ring, having unique maximal ideal $\mathfrak{m} = \{\alpha \in k \mid v(\alpha) > 0\}$.*

In fact,

1.45. *The set of ideals of a valuation ring are totally-ordered by subset inclusion.*

Then every valuation ring comes with a canonical quotient map $\pi : R \twoheadrightarrow R/\mathfrak{m}$, and we refer to the field $K = R/\mathfrak{m}$ as the quotient field of R.

A valuation ring V of a valuation v is a discrete valuation ring if the valuation v is discrete.

1.46. *A ring R is a discrete valuation ring if and only if R is a local Noetherian domain whose unique maximal ideal $\mathfrak{m}$ is a principal ideal; equivalently* iff *R is a local principal ideal domain. In particular (at least in the case of a proper discrete valuation ring, so that $\mathfrak{m} \neq 0$), $\mathfrak{m}$ is generated by any element r of R for which $v(r) = 1$.*

2. Hochster's construction

Representing a spectral space X within the spectrum of a ring of functions on X

We have seen that given a reduced ring A, we may represent the ring A as a ring of functions on $\mathsf{Spec}\, A$ taking values in the domain $A/\mathfrak{p}$ at each $\mathfrak{p} \in \mathsf{Spec}\, A$. We aim to replicate that situation starting from a general spectral space X.

Given X a topological space and A a ring of functions $X \to R$ for some ring R, then considering the "evaluation at x" map $\mathsf{ev}_x : A \to R$ for each $x \in X$ that takes a function $f \in A$ to the value $f(x) \in R$, this map defines

- a subring of R, $R_x = \mathsf{ev}_x(A) = \{f(x) \mid f \in A\}$; and
- an ideal of A, $I_x = \mathsf{ker}(\mathsf{ev}_x) = \{f \mid f(x) = 0_R\}$.

We may then, as we did earlier, write elements of A 'graph-wise' as elements of $\prod_{x\in X} R_x$, with $f \mapsto \prod_{x\in X} \mathsf{ev}_x(f) = \prod_{x\in X} f(x)$.

Proposition 2.1. *If R_x is a domain (for every $x \in X$) then I_x is a prime ideal (for every $x \in X$),*

since I_x is the kernel of the surjective homomorphism $\mathsf{ev}_x : A \twoheadrightarrow R_x$. □

Thus in the case that R_x is a domain for every $x \in X$, we obtain a mapping of X into the prime spectrum of A, which we will write as $\Phi : X \to \mathsf{Spec}\, A$ given by $\Phi(x) = \mathsf{ker}(\mathsf{ev}_x) = I_x$.

We compare the (induced) topology on $\Phi(X)$ with the topology on X, particularly in the case that X is a spectral space, which will be assumed from now on. We have the collection $\mathbb{D}(A)$ as a basis for the topology on $\mathsf{Spec}\,A$; we look at the preimage of principal open sets $D(f)$ under Φ. To each $f \in A$ we define the subset $d(f) \subseteq X$ by

$$\begin{aligned} d(f) = \Phi^{-1}\big(D(f)\big) &= \{x \mid \Phi(x) \in D(f)\} \\ &= \{x \mid \mathsf{ker}(\mathsf{ev}_x) \in D(f)\} \\ &= \{x \mid f \notin \mathsf{ker}(\mathsf{ev}_x)\} \\ &= \{x \mid f(x) \neq 0\}.^{[1]} \end{aligned}$$

Then the following propositions, per Section 3 of [14], give successive conditions on the sets $d(f) \subseteq X$ which relate the topology of X to that of $\mathsf{Spec}\,A$.

Proposition 2.2. *If* $\{d(f)\}_{f \in A}$ *is a basis for* X *then* $X \overset{\cong}{\hookrightarrow} \mathsf{Spec}\,A$*.*

Since X is a spectral space, then it is T_0, and so if $\{d(f)\}_{f\in A}$ is a basis for X then Φ is injective, since distinct points $x \neq y$ must be distinguished by some basic set $d(f)$, in which case we cannot have $\Phi(x) = \Phi(y)$. Then identifying X with its image $\Phi(X)$, clearly we have $d(f) = D(f) \cap \Phi(X)$, and so, the topology defined by $\{d(f)\}_{f\in A}$ is exactly the subspace topology inherited from $\mathsf{Spec}\,A$. □

Proposition 2.3. *If furthermore each* $d(f)$ *is quasi-compact in* X *then* X *is a spectral subspace of* $\mathsf{Spec}\,A$*,*

since in that case given a quasi-compact open set of $\mathsf{Spec}\,A$, being a finite union $D(a_1)\cup\cdots\cup D(a_n)$, then its preimage under Φ, the set $d(a_1)\cup\cdots\cup d(a_n)$, is quasi-compact open in X, and so inclusion is a spectral map. □

[1] It is worth noting for posterity that convincing myself of a suspected typo at exactly this point in Hochster's exposition was the first non-trivial bit of maths I undertook in this project; his paper [14] gives the definition of $d(f)$ as the set of those x such that $f(x) = 0$.

We shall in fact not find it too difficult, given a spectral space X, to define rings A and R as above so that X is a spectral subspace of $\mathsf{Spec}\, A$. We will however find it significantly more difficult to obtain a ring for which $X \cong \mathsf{Spec}\, A$.

Given a spectral space X take $\mathcal{B}_X \subseteq \overset{\circ}{\mathcal{K}}(X)$ an open subbasis for X. We fix a field k. Take $T_X = \{t_U\}_{U \in \mathcal{B}_X}$ a collection of indeterminates indexed by $\mathcal{B}_X$ and define corresponding "characteristic" functions $\chi_U : X \to k[T_X]$ by

$$\chi_U(x) = \begin{cases} t_U & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

Let $\mathcal{X}_X = \{\chi_U\}_{U \in \mathcal{B}_X}$; we consider the ring $k[\mathcal{X}_X]$, that is, the ring of polynomial expressions in χ's with coefficients in k. We then consider this as a ring of functions $X \to k[T_X]$ in the obvious way, with the value of a polynomial expression $\sum \alpha_i \chi_{U_1}^{\beta_1} \cdot \ldots \cdot \chi_{U_m}^{\beta_m}$ in $k[\mathcal{X}_X]$ at a point $x \in X$ defined via $(fg)_x = f_x g_x$, $(f+g)_x = f_x + g_x$, and the definitions of the functions χ_U; we then define the ring A_X to be this ring of functions. An important caveat at this stage is that distinct expressions in A_X may not truly be distinct functions $X \to k[T_X]$. For example, suppose that our subbasis $\mathcal{B}_X$ contains non-empty sets U, V such that $U \cap V = \emptyset$ (—note in this case that the space X is reducible). Then the expression $\chi_U \chi_V \in k[\mathcal{X}_X]$ is identically zero on X.

We examine this set-up in light of our earlier discussion. Given $x \in X$, we see that the range of evaluation at x, that is, the ring R_x as given above, is the subring of $k[T_X]$ generated by those indeterminates t_U for which $x \in U$. That is, given $x \in X$, define $T_x = \{t_U \in T_X \mid x \in U\}$. Then $R_x = k[T_x]$. Evidently, this is a domain for each $x \in X$. Then, we consider A_X as a subring of the product of these domains, that is, we identify an element $f \in A_X$ with its graph $\prod_{x \in X} f_x$. Note, as per the preceding paragraph, the action of identifying expressions in $k[\mathcal{X}_X]$ with the function they induce in A_X may not be injective; however this identification is (of course) onto, and so we will frequently consider elements of

A_X as expressions in $k[\mathcal{X}_X]$, when strictly we mean the result of evaluating this expression at each $x \in X$, that is, the function in A_X induced by the expression. Thus we consider A_X as the subring

$$A_X = \{\prod_{x\in X} \mathsf{ev}_x(f) \mid f \in k[\mathcal{X}_X]\} \subseteq \prod_{x\in X} \mathsf{ev}_x(k[\mathcal{X}_X]) = \prod_{x\in X} k[T_x],$$

and so obtain a mapping $\Phi : X \to \mathsf{Spec}\, A_X$ by $\Phi(x) = \mathsf{ker}(\mathsf{ev}_x)$. We find

Proposition 2.4. *X is a spectral subspace of $\mathsf{Spec}\, A_X$, that is, the sets $d(f) = \Phi^{-1}(D(f))$ for $f \in A_X$ are quasi-compact open and form a basis for X.*

A general $f \in A_X$ is of the form $f = \sum_i \lambda_i m_i$ for coefficients $\lambda_i \in k^\times$ and m_i distinct monomials from $k[\mathcal{X}_X]$ (that is, $m_i = \chi_{U_1}^{\alpha_1}\chi_{U_2}^{\alpha_2}\dots\chi_{U_N}^{\alpha_N}$ for some $U_j \in \mathcal{B}_X$ and $\alpha_j \in \mathbb{N}$, where by convention we include 1 as the 'empty' monomial). Then at a particular $x \in X$, we have that

$$f_x = \big(\lambda_1 m_1\big)_x + \cdots + \big(\lambda_n m_n\big)_x,$$

where

$$\big(\lambda_i m_i\big)_x = \big(\lambda_i \chi_{U_1}^{\alpha_1}\chi_{U_2}^{\alpha_2}\dots\chi_{U_N}^{\alpha_N}\big)_x = \begin{cases} \lambda_i t_{U_1}^{\alpha_1} t_{U_2}^{\alpha_2}\dots t_{U_N}^{\alpha_N} & \text{in the case } x \in U_j \text{ for all } j \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$d(m_i) = \{x \mid x \in U_j \text{ for all } j\} = \bigcap_j U_j,$$

and

$$d(f) = d(m_1) \cup d(m_2) \cup \cdots \cup d(m_n)$$

(note that as the distinct monomials m_i evaluate either to distinct monomials $(m_i)_x$ or to 0, we cannot have terms cancelling each other to zero, so if m_i is non-zero at x then f is indeed non-zero at x, that is $x \in d(m_i) \implies x \in d(f)$; the reverse inclusion is obvious).

That is, we have that $d(m_i)$ is a finite intersection of quasi-compact open sets of X, so is itself quasi-compact open; and then $d(f)$ is a finite union of quasi-compact open sets, and so is quasi-compact open. We need finally that our collection $\{d(f)\}_{f\in A_X}$ is a basis for X. But for $U \in \mathcal{B}_X$ we have $\chi_U \in A_X$, and $d(\chi_U) = U$, so $\mathcal{B}_X \subseteq \{d(f)\}_{f\in A_X}$; furthermore we observed that $d(\chi_{U_1}\chi_{U_2}\ldots\chi_{U_n}) = \bigcap_i U_i$, so $\{d(f)\}_{f\in A_X}$ contains our subbasis and all finite intersections thereof, and therefore is a basis for X as required. □

We may at this stage be a long way from having $\mathsf{Spec}\, A_X \cong X$ (which we will have exactly when $\Phi(X) = \mathsf{Spec}\, A_X$). We observe a necessary condition for $\mathsf{Spec}\, A_X \cong X$, which is in fact also sufficient in the case that X is a spectral subspace of $\mathsf{Spec}\, A_X$ (in which case we make the identification between X and $\Phi(X)$ as a subspace of $\mathsf{Spec}\, A_X$). We have $X \subsetneq \mathsf{Spec}\, A_X$ if and only if there is a prime ideal $\mathfrak{p} \in \mathsf{Spec}\, A_X$ such that $\mathfrak{p} \notin X$. One situation where we may be able to detect this is where such a $\mathfrak{p}$ lies in some $\mathcal{V}(I)$ thereby distinguishing it from some other set $\mathcal{V}(J)$ with $\mathfrak{p} \notin \mathcal{V}(J)$, so that $\mathcal{V}(I) \not\subseteq \mathcal{V}(J)$; but such that all primes in $\mathcal{V}(I)\backslash\mathcal{V}(J)$ fall outside the range of Φ (that is, outside the image of X), so that in the subspace X we do in fact have $(\mathcal{V}(I)\cap X) \subseteq (\mathcal{V}(J)\cap X)$ (we take this opportunity to define $z(I) = \mathcal{V}(I)\cap X = \{x \in X \mid f_x = 0 \text{ for all } f \in I\}$; equivalently $z(I) = \Phi^{-1}(\mathcal{V}(I))$, and note for $f \in A_X$ then $z(f) = X \setminus d(f)$). In fact for X a spectral subspace of $\mathsf{Spec}\, A_X$, it is sufficient to check such situations do not arise for I, J such that J is principal and I is finitely generated, in order to guarantee $X = \mathsf{Spec}\, A_X$. That is,

Proposition 2.5 ([14], Theorem 2). *Given a spectral space X and a ring of functions B from X into some ring R satisfying the conditions of propositions 2.1, 2.2 and 2.3, so that X is a spectral subspace of $\mathsf{Spec}\, B$ (and hence we identify X with $\Phi(X)$ as a subspace of $\mathsf{Spec}\, B$), then $X = \mathsf{Spec}\, B$ if and only if for all $f \in B$ and finite subsets $G \subseteq B$, we have that $z(G) \subseteq z(f) \implies f \in \mathsf{rad}\, G$.*

From the preceding discussion, since $\mathcal{V}(G) \subseteq \mathcal{V}(f) \iff f \in \operatorname{rad} G$ we clearly have necessity. Conversely, given that X is a spectral subspace of $\operatorname{Spec} B$, then from 1.23 we have that X is closed in the patch space $(\operatorname{Spec} B)_{\mathsf{con}}$. Then, if $X \subsetneq \operatorname{Spec} B$ then there is some closed basis subset C of $(\operatorname{Spec} B)_{\mathsf{con}}$ such that $X \subseteq C \subsetneq \operatorname{Spec} B$. From 1.40 we have that the sets of the form $\mathcal{V}(f) \cup \bigcup_i^n D(g_i)$ for $f, g_1, \ldots, g_n \in B$ form a closed basis for $(\operatorname{Spec} B)_{\mathsf{con}}$. Then if $X \subsetneq \operatorname{Spec} B$ there is some $f \in B$ and some finite $G = \{g_1, \ldots, g_n\} \subseteq B$ such that

$$X \subseteq \mathcal{V}(f) \cup \bigcup_i^n D(g_i) \subsetneq \operatorname{Spec} B,$$

or, taking complements,

$$X \subseteq \mathcal{V}(f) \cup \bigcup_i^n \big(\mathcal{V}(g_i)\big)^{\mathsf{c}} = \mathcal{V}(f) \cup \big(\bigcap_i^n \mathcal{V}(g_i)\big)^{\mathsf{c}} = \mathcal{V}(f) \cup \big(\mathcal{V}(\langle G \rangle)\big)^{\mathsf{c}} \subsetneq \operatorname{Spec} B$$

(the equality $\bigcap_i^n \mathcal{V}(g_i) = \mathcal{V}(\langle G \rangle) = \mathcal{V}(G)$) coming from property 3 in 1.1). Then

$$\begin{aligned} X \subseteq \mathcal{V}(f) \cup \big(\mathcal{V}(G)\big)^{\mathsf{c}} &\iff (\mathcal{V}(f) \cap X) \cup \big(\big(\mathcal{V}(G)\big)^{\mathsf{c}} \cap X\big) = X \\ &\iff \big(\mathcal{V}(G) \cap X\big) \subseteq (\mathcal{V}(f) \cap X) \\ &\iff z(G) \subseteq z(f). \end{aligned}$$

But if for all $f \in B$ and finite $G \subseteq B$ we have that $z(G) \subseteq z(f) \implies f \in \operatorname{rad} G$, then $z(G) \subseteq z(f) \implies f \in \operatorname{rad} G \implies \mathcal{V}(G) \subseteq \mathcal{V}(f) \implies \mathcal{V}(f) \cup \big(\mathcal{V}(G)\big)^{\mathsf{c}} = \operatorname{Spec} B$, i.e. $\mathcal{V}(f) \cup \bigcup_i^n D(g_i) = \operatorname{Spec} B$ for all basic closed subsets of $(\operatorname{Spec} B)_{\mathsf{con}}$ such that $X \subseteq \mathcal{V}(f) \cup \bigcup_i^n D(g_i)$; in which case we must have that $X = \operatorname{Spec} B$ as required. □

In fact in the example of our ring A_X as defined above (and for all subsequent rings considered in this chapter), this condition simplifies, due to the following sequence of observations:

Proposition 2.6 ([14], Theorem 4). *Given a ring of functions B on X as in proposition 2.5 (i.e. so that X is a spectral subspace of $\operatorname{Spec} B$), if for every function $f \in B$ we have that the image of f in R is finite (i.e. the set*

$\{\mathsf{ev}_x(f)\}_{x\in X} = \{f_x\}_{x\in X}$ is a finite subset of R), then for all finite collections $g_1,\dots,g_n \in B$ there is some $g \in B$ such that $z(g_1)\cap\cdots\cap z(g_n) = z(g)$ (equivalently, such that $d(g_1)\cup\cdots\cup d(g_n) = d(g)$).

By an elementary induction it suffices to consider the situation for two elements $f, g \in B$. Let $f_{x_1} \neq f_{x_2} \neq \cdots \neq f_{x_m}$, $g_{z_1} \neq g_{z_2} \neq \cdots \neq g_{z_n}$ enumerate all possible non-zero values for f and g (Note that if either f or g takes no non-zero values, i.e. if $z(f) = X$ or $z(g) = X$, then either $z(f)\cap z(g) = z(f)$ or $z(f)\cap z(g) = z(g)$ and so we are done). Furthermore consider the sets $W_{ij} = \{y \in X \mid f_y = f_{x_i} \,\&\, g_y = g_{z_j}\}$ for some x_i and z_j, and choose an element $y_{ij} \in W_{ij}$ whenever W_{ij} is non-empty. Let

$$Y = \{x_1,\dots,x_m,z_1,\dots,z_n\}\cup\{y_{ij} \text{ where defined}\}.$$

We consider the set $S = \{\sum f^k g^l\}$ of all combinations of f and g (i.e. the set of all sums $f^{k_1}g^{l_1}+\cdots+f^{k_n}g^{l_n}$ for all possible $k_1,l_1,\dots,k_n,l_n,n \in \mathbb{N}$). Since for each $y \in Y$ we have that one of f_y or g_y is non-zero, then $S \not\subseteq \Phi(y) = \mathsf{ker}(\mathsf{ev}_y) = \{h \in B \mid h_y = 0\}$ for each $y \in Y$. Then as each $\Phi(y)$ is a prime ideal of B and S is an additively and multiplicatively closed subset of B, by prime avoidance we have that $S \not\subseteq \bigcup_{y\in Y}\Phi(y) = \{h \in B \mid h_y = 0 \text{ for some } y \in Y\}$. Then, there is some $r \in S$ non-zero on the whole of Y, in which case r will be non-zero at all points $x \in X$ where f or g take non-zero values, since the values of f_x and g_x at such a point (and hence the value of r_x) must be the same as the value taken at some $y \in Y$. Then, $d(r) = d(f)\cup d(g)$ as required. □

Proposition 2.7. *Given a ring of functions B on X as in proposition 2.5, such that for all finite collections $g_1,\dots,g_n \in B$ there is some $g \in B$ such that $z(g_1)\cap\cdots\cap z(g_n) = z(g)$, then we will have $X = \mathsf{Spec}\,B$ if and only if for all $f, g \in B$ we have that $z(g)\subseteq z(f) \implies f \in \mathsf{rad}\,g$,* as is obvious by replacing the set $z(G) = \bigcap_i^n z(g_i)$ in 2.5 by the set $z(g)$ guaranteed by our hypothesis. $\square$

Corollary 2.8. *Given a ring of functions B on X as in proposition 2.5, if for every function $f \in B$ we have that the image of f in R is finite, we have $X = \mathsf{Spec}\, B$ if and only if for all $f, g \in B$ we have that $z(g) \subseteq z(f) \implies f \in \mathsf{rad}\, g$.*

Then we note

Proposition 2.9. *The ring A_X as defined on page 54 has the property that for each $f \in A_X$ the image of f in $R = k[T_X]$ is finite.*

Given that for a general $f = \sum_i \lambda_i m_i \in A_X$ the monomials $m_i = \chi_{U_1}^{\alpha_1} \dots \chi_{U_N}^{\alpha_N}$ take only the values $\lambda_i t_{U_1}^{\alpha_1} \dots t_{U_N}^{\alpha_N}$ or 0 according as whether $x \in \bigcap_j U_j$ or not, then clearly such an f only takes finitely many distinct values in $k[T_X]$ under evaluation at points $x \in X$. $\square$

Indeed, looking forward to our eventual goal, the ring H_X we ultimately construct such that $\mathsf{Spec}\, H_X \cong X$ also shares this property, so that for every $g_1, \dots, g_n \in H_X$ there is some $g \in H_X$ such that $z(g_1) \cap \dots \cap z(g_n) = z(g)$. But as we will have $\mathsf{Spec}\, H_X \cong X$ then we have $z(g_i) = \mathcal{V}(g_i)$ and $d(g_i) = D(g_i)$, so that, as noted in [24], the ring H_X has the property that for every $g_1, \dots, g_n \in H_X$ there is some $g \in H_X$ such that

$$\mathcal{V}(g_1) \cap \dots \cap \mathcal{V}(g_n) = \mathcal{V}(g)$$

and

$$D(g_1) \cup \dots \cup D(g_n) = D(g);$$

that is, the collections $\mathbb{V}(H_X) = \{\mathcal{V}(a)\}_{a \in H_X}$ and $\mathbb{D}(H_X) = \{D(a)\}_{a \in H_X}$ are lattices under the operations $\cap, \cup$ (note that as $\mathcal{V}(f) \cup \mathcal{V}(g) = \mathcal{V}(fg)$; $D(f) \cap D(g) = D(fg)$ they are in general semi-lattices under $\cup$, $\cap$ respectively); a property that is shared, for example, by all unique factorisation domains, as well as by all Bézout rings, but which is not satisfied in general in an arbitrary ring B.

The semi-lattice $\mathbb{D}(A)$ is strongly linked to divisibility in the ring A: we have $a \mid b \iff b \in \langle a \rangle \iff \langle b \rangle \subseteq \langle a \rangle \implies D(b) \subseteq D(a)$ (—the reverse implication proceeds $D(b) \subseteq D(a) \implies b \in \mathsf{rad}\, a \implies a \mid b^n$ for some $n \in \mathbb{N}$); it would be interesting to see what further insights may follow from the fact that for any ring A, there is a ring having the same prime spectrum as A (namely, the ring H_X for $X = \mathsf{Spec}\, A$) with the property that $\mathbb{D}(H_X)$ is a lattice.

We now give an illustration of the first step of our construction, and of how we may fall short of having $X = \mathsf{Spec}\, A_X$ at this first stage, by calculating an example.

The example of $X = \mathsf{Spec}\, \mathbb{Z}$

Let $X = \mathsf{Spec}\, \mathbb{Z}$. The first step of our construction of a ring A_X from X is to pick a subbasis for X; we may choose the sets $D(p)$ for (positive) prime numbers p. We then take a set of corresponding indeterminates $T_X = \{t_p \mid p \in \mathbb{N} \text{ is a prime}\}$ and a set $\mathcal{X}_X$ of characteristic functions $\chi_p : X \to k[T_X]$ defined by

$$\chi_p(\mathfrak{q}) = \begin{cases} t_p & \text{if } \mathfrak{q} \in D(p) \\ 0 & \text{if } \mathfrak{q} \notin D(p) \end{cases}$$

for points of X, i.e. prime ideals $\mathfrak{q} \in \mathsf{Spec}\, \mathbb{Z}$. Then the range of evaluation at $x = \mathfrak{q}$, that is, the ring R_x as given above, is the subring of $k[T_X]$ generated by those indeterminates t_p for which $\mathfrak{q} \in D(p)$. But we have $\mathfrak{q} \in D(p) \iff p \notin \mathfrak{q} \iff \mathfrak{q} \neq \langle p \rangle$. Then for a prime ideal $\mathfrak{q} = \langle q \rangle$ we have that $T_{\mathfrak{q}} = T_X \setminus \{t_q\}$, and where $\mathfrak{q} = \langle 0 \rangle$ we have $T_{\langle 0 \rangle} = T_X$. We then define $A_X = \{\prod_{\mathfrak{q} \in \mathsf{Spec}\, \mathbb{Z}} \mathsf{ev}_{\mathfrak{q}}(f) \mid f \in k[\mathcal{X}]\} \subseteq \prod_{\mathfrak{q} \in \mathsf{Spec}\, \mathbb{Z}} k[T_{\mathfrak{q}}]$.

In fact clearly expressions $f \in k[\mathcal{X}_X]$ evaluate "injectively" at the point $\langle 0 \rangle$

under the mapping $\chi_p \mapsto t_p$, and so A_X is isomorphic to the ring $k[T_X] = k[t_2, t_3, t_5, t_7, \dots] = R_{\langle 0\rangle}$ together with projections $R_{\langle 0\rangle} \to R_{\mathfrak{p}}$ given by $f_{\langle 0\rangle} \mapsto f_{\langle 0\rangle} \operatorname{\mathsf{mod}} t_p$ for p the prime number such that $\mathfrak{p} = \langle p\rangle$.

We observe that for $\mathfrak{q} \neq \langle 0\rangle$ we have

$$\Phi(\mathfrak{q}) = \mathsf{ker}(\mathsf{ev}_{\mathfrak{q}}) = \{f \in A_X \mid f_{\mathfrak{q}} = 0\} \cong \langle t_q\rangle,$$

and

$$\Phi(\langle 0\rangle) = \{f \in A_X \mid f_{\langle 0\rangle} = 0\} \cong \{0\},$$

thus seeing how X embeds into $\mathsf{Spec}\, A_X$. But we can without difficulty find suitable $f, g \in A_X$ so that $z(g) \subseteq z(f)$ but $f \notin \mathsf{rad}\, g$: Let $f = \chi_2$ and $g = \chi_2 + \chi_3$, that is, f is the function taking value $f_{\mathfrak{p}} = 0$ at the point $\mathfrak{p} = \langle 2\rangle$ and $f_{\mathfrak{q}} = t_2$ elsewhere, and g takes values $g_{\langle 2\rangle} = t_3$; $g_{\langle 3\rangle} = t_2$ and $g_{\mathfrak{p}} = t_2 + t_3$ elsewhere. Then $z(g) = \emptyset$ and $z(f) = \{\langle 2\rangle\}$ so that $z(g) \subseteq z(f)$ as required, but $f \in \mathsf{rad}\, g \iff \exists r \in A, n \in \mathbb{N}$ such that $f^n = r \cdot g$, in which case we must have, for example, that $r_{\langle 0\rangle} \cdot g_{\langle 0\rangle} = f^n_{\langle 0\rangle}$ for some $r_{\langle 0\rangle} \in R_{\langle 0\rangle} = k[T_X]$, that is that $r_{\langle 0\rangle} \cdot (t_2 + t_3) = t_2^n$. But clearly there is no $r_{\langle 0\rangle} \in k[T_X]$ for which this is possible, hence there is no $r \in A_X$ such that $f^n = r \cdot g$ and so $f \notin \mathsf{rad}\, g$ and so hence $X \neq \mathsf{Spec}\, A_X$. Indeed the ideal $\langle t_2 + t_3\rangle$ is prime in $R_{\langle 0\rangle} = k[t_2, t_3, t_5, t_7, \dots]$, and so the ideal of A_X consisting of those functions induced by some expression in the ideal $\langle \chi_2 + \chi_3\rangle$ in $k[\mathcal{X}_X]$ is a prime ideal of A_X outside the range of Φ.

Extensions and indices

The construction continues by defining a discrete valuation $\mathsf{v}_{x \rightsquigarrow y}$ on $\mathcal{Q}(R_x) = k(T_x)$ for every specialisation $x \rightsquigarrow y$ in X (this collection of valuations Hochster refers to as an index). We note that where we have $x \rightsquigarrow y$, then $T_y \subseteq T_x$. Since, if $x \rightsquigarrow y$ then $y \in \overline{\{x\}}$, so that every open neighbourhood of y contains x. Then given $U \in \mathcal{B}$, if $y \in U$ (so that $t_U \in T_y$) then $x \in U$, so $t_U \in T_x$. We define the valuation $\mathsf{v}_{x\rightsquigarrow y}$ as follows: given $t_U \in T_x$ (—that is, where $x \in U$), then $\mathsf{v}_{x\rightsquigarrow y}(t_U)$ is either 1 or 0 according as whether t_U vanishes at y or not. That is,

$$\mathsf{v}_{x\rightsquigarrow y}(t_U) = \begin{cases} 1 & y \notin U, \\ 0 & y \in U. \end{cases}$$

We then define $\mathsf{v}_{x\rightsquigarrow y}$ on $R_x = k[T_x]$ by

- $\mathsf{v}_{x\rightsquigarrow y}(t_1^{\alpha_1} \dots t_n^{\alpha_n}) = \sum \alpha_i$ for those i such that $\mathsf{v}_{x\rightsquigarrow y}(t_i) = 1$; and
- $\mathsf{v}_{x\rightsquigarrow y}(\sum \lambda_i m_i) = \mathsf{min}(\{\mathsf{v}_{x\rightsquigarrow y}(m_i)\})$ for m_i distinct monomials in T_x and $\lambda_i \in k^\times$.

It then has a uniquely defined extension to a discrete valuation $\mathsf{v}_{x\rightsquigarrow y} : k(T_x) \to \mathbb{Z} \cup \{\infty\}$. Furthermore it has the following properties, which will be important not only in the ensuing discussions but also in light of later chapters:

Proposition 2.10. *For all $f \in A_X$, then at any R_x, and for all specialisations $x \rightsquigarrow y$, we have that $\mathsf{v}_{x\rightsquigarrow y}(f_x) \geqslant 0$.*

Evidently, if $f \in A_X$ then $f_x \in k[T_x]$ and so $\mathsf{v}_{x\rightsquigarrow y}(f_x) \geqslant 0$. □

Proposition 2.11. *For all $f \in A_X$, then at any R_x, and for all specialisations $x \rightsquigarrow y$, we have that $\mathsf{v}_{x\rightsquigarrow y}(f_x) > 0 \iff f_y = 0$.*

Since, given $f \in A_X$ then $f_x \in k[T_x]$. Then by the definition of our valuation, we have $\mathsf{v}_{x\rightsquigarrow y}(f_x) = 0 \iff$ there is some term λm of f for some $\lambda \in k^\times$ and some monomial m such that $\mathsf{v}_{x\rightsquigarrow y}(\lambda m_x) = 0$, which occurs if and only if m consists solely of χ_U's such that $y \in U$ (which we consider to be vacuously satisfied in the case that the term in question is a constant). But in such a case then $\lambda m_y \neq 0$ and so $f_y \neq 0$; and conversely if $f_y \neq 0$ then there must be some term λm of f which does not vanish at y, in which case we must have that m consists wholly of χ_U's such that $y \in U$, and so then $\mathsf{v}_{x\rightsquigarrow y}(f_x) = 0$ as required. □

Furthermore the next property will be crucial.

Proposition 2.12. *For all $f \in A_X$ there is an integer N_f such that, at any R_x, and for all specialisations $x \rightsquigarrow y$, if $f_x \neq 0$ then $\mathsf{v}_{x \rightsquigarrow y}(f_x) \leqslant N_f$.*

As we have observed several times before, monomials in A_X evaluate either to the corresponding monomial in $k[T_x]$ or to zero; therefore the degree of some $f_x \in k[T_x]$, and so hence any possible value $\mathsf{v}_{x \rightsquigarrow y}(f_x)$ (— provided $f_x \neq 0$), is bounded by the degree of f as a polynomial in $k[\mathcal{X}_X]$. □

Our construction then proceeds by considering $f, g \in A_X$ where at all R_x, for all $x \rightsquigarrow y$, that firstly $\mathsf{v}_{x \rightsquigarrow y}(f_x) \geqslant \mathsf{v}_{x \rightsquigarrow y}(g_x)$; and furthermore that if $\mathsf{v}_{x \rightsquigarrow y}(f_x) = \mathsf{v}_{x \rightsquigarrow y}(g_x)$ then either $\mathsf{v}_{x \rightsquigarrow y}(g_x) = \infty$ or $\mathsf{v}_{x \rightsquigarrow y}(g_x) = 0$. Once again these conditions will be not only significant for the progress of the current construction (—in some sense they ensure that the two properties discussed in the preceding propositions are maintained when we extend our ring), but also for when we reconsider this set-up in light of later chapters.

The naïve idea is then that we extend A_X to the subring of $\prod_{x \in X} \mathcal{Q}(R_x)$ generated by A_X and the element $\prod_{x \in X} \frac{f_x}{g_x}$. Of course we may have that at a certain $x \in X$ we have that both f_x and g_x are zero (— the condition on our valuations guarantees that whenever $g_x = 0$ we have already that $f_x = 0$); then, technically speaking, we extend to the subring of $\prod_{x \in X} \mathcal{Q}(R_x)$ generated by A_X and the element $\frac{f}{=}{g} \in \prod_{x \in X} \mathcal{Q}(R_x)$ defined as

$$\left(\frac{f}{=}{g}\right)_x = \begin{cases} \frac{f_x}{g_x} & g_x \neq 0; \\ 0 & g_x = 0. \end{cases}$$

Hochster observes that, when the valuation conditions are satisfied for a certain f and g, then we maintain X as a spectral subspace of the ring $A' = A_X[\frac{f}{=}{g}]$. As we have seen, this will be the case if

- The range $R'_x = \mathsf{ev}_x(A') = \{f'(x) \mid f' \in A'\}$ is a domain for all $x \in X$;
- The collection $\{d(g')\}_{g' \in A'}$ is an open basis for X, and
- For every $h' \in A'$ we have that $d(h')$ is quasi-compact open in X.

We have the first condition since R'_x is a subring of $\mathcal{Q}(R_x)$; and so we have a map $\Phi' : X \to \mathsf{Spec}\, A'$ given by $x \mapsto \mathsf{ker}(\mathsf{ev}_x)$. Furthermore clearly the last condition implies the second condition, as the collection $\{d(h')\}_{h' \in A'}$ contains the basis $\{d(f)\}_{f \in A_X}$ for X —note that for $f \in A_X$ the set $d(f) = \{x \mid \mathsf{ker}(\mathsf{ev}_x) \in D(f)\} = \{x \mid f_x \neq 0\}$ still defines the same subset of X when we extend A_X to A'. Then furthermore we have that Φ' is injective (as the sets $d(f)$ for $f \in A_X$ still separate points in X) and so identifying X with its image $\Phi'(X)$ we have for each $h' \in A'$ that $d(h') = D(h') \cap \Phi'(X)$. We need to check that each $d(h')$ is quasi-compact open in X.

Proposition 2.13 ([14], Theorem 3). *Given $f, g \in A_X$ such that for every specialisation $x \rightsquigarrow y$ in X we have that $\mathsf{v}_{x \rightsquigarrow y}(f_x) \geqslant \mathsf{v}_{x \rightsquigarrow y}(g_x)$, with $\mathsf{v}_{x \rightsquigarrow y}(f_x) = \mathsf{v}_{x \rightsquigarrow y}(g_x)$ only if either $\mathsf{v}_{x \rightsquigarrow y}(g_x) = \infty$ or $\mathsf{v}_{x \rightsquigarrow y}(g_x) = 0$, and defining $A' = A_X[\frac{f}{g}]$ for $\frac{f}{g}$ as given above, then for every $h' \in A'$ we have that $d(h')$ is quasi-compact open in X.*

We consider a general element h' in $A' = A_X[\frac{f}{g}]$,

$$h' = f_n\left(\frac{f}{g}\right)^n + \cdots + f_1\left(\frac{f}{g}\right) + f_0$$

for $f_0, \ldots, f_n \in A_X$. It will be not too difficult to show that $d(h')$ is quasi-compact in X. We will then show it is open by showing its complement, the set $z(h') = \{x \in X \mid h'_x = 0\}$, is closed in X.

The behaviour of h' at a point $x \in X$ falls into two possible cases according as whether $\left(\frac{f}{g}\right)_x = 0$ or not. We can see that if $\left(\frac{f}{g}\right)_x = 0$ then $h'_x = 0 \iff (f_0)_x = 0$, and we know that we will have $\left(\frac{f}{g}\right)_x = 0$ exactly when $f_x = 0$.

In the alternative case, that is, where $\left(\frac{f}{g}\right)_x \neq 0$, we consider the function $g^n h'$ in A'. We see in fact (— regardless of which case in the definition of $\left(\frac{f}{g}\right)_x$ applies at x) that

$$g_x^n h_x' = (f_n)_x f_x^n + \cdots + (f_1)_x f_x g_x^{n-1} + (f_0)_x g_x^n$$

at all $x \in X$, that is $g^n h' = f_n f^n + \cdots + f_1 f g^{n-1} + f_0 g^n$ and so $g^n h'$ is an element of A. Then in the case when $\left(\frac{f}{g}\right)_x \neq 0$ then certainly $g_x \neq 0$, and so $h_x' = 0 \iff g_x^n h_x' = 0$ (—since R_x is a domain). Then, to sum up, if $f_x = 0$ then $h_x' = 0 \iff (f_0)_x = 0$ and if $f_x \neq 0$ then $h_x' = 0 \iff g_x^n h_x' = 0$; that is,

$$d(h') = (z(f) \cap d(f_0)) \cup (d(f) \cap d(g^n h')).$$

Observe (as for every $a \in A_X$ we have $d(a)$ is quasi-compact open in X, and we have that $f, f_0, g^n h'$ are all in A_X) that each set in the above expression for $d(h')$ is a closed set in X_{con}, and so $d(h')$ is closed in X_{con}; then $d(h')$ is quasi-compact in X_{con}, and so is quasi-compact in X.

We now consider the complement of $d(h')$. From the above discussion, we can see that

$$z(h') = (z(f) \cap z(f_0)) \cup (d(f) \cap z(g^n h')),$$

and so we can likewise conclude that $z(h')$ is closed in X_{con}. Then by proposition 1.22 it is closed in X if it contains the closure of each of its points; that is, if we have for each $x \in z(h')$ and $x \rightsquigarrow y$, that $y \in z(h')$. Then, given $x \in z(h')$ we have either $x \in z(f) \cap z(f_0)$ or $x \in d(f) \cap z(g^n h')$. If $x \in z(f) \cap z(f_0)$ then as $f, f_0 \in A_X$ we have $z(f) \cap z(f_0)$ is closed in X (since their complements $d(f)$, $d(f_0)$ are open), so certainly for any $x \rightsquigarrow y$ we have $y \in z(f) \cap z(f_0)$ and so $y \in z(h')$. Then, suppose $x \notin z(f)$, that is, $f_x \neq 0$. Then $x \in d(f) \cap z(g^n h')$ and so given $x \rightsquigarrow y$ we have (as $z(g^n h')$ is closed in X) that $y \in z(g^n h')$. If $y \in d(f)$ then $y \in d(f) \cap z(g^n h')$ and so we are done. Suppose not, that is, suppose $f_y = 0$. Then we need to show that $y \in z(f) \cap z(f_0)$; we have $y \in z(f)$ and so we will have $y \in z(h')$ if and only if $y \in z(f_0)$.

We have $f_x \neq 0$ so that $\left(\frac{f}{g}\right)_x = \frac{f_x}{g_x}$, and we have $h'_x = 0$. Then

$$h'_x = (f_n)_x\big(\tfrac{f_x}{g_x}\big)^n + \cdots + (f_1)_x\tfrac{f_x}{g_x} + (f_0)_x = 0, \text{ so}$$
$$(f_n)_x\big(\tfrac{f_x}{g_x}\big)^n + \cdots + (f_1)_x\tfrac{f_x}{g_x} = -(f_0)_x \text{ and so}$$
$$\mathsf{v}_{x\rightsquigarrow y}\Big(\sum_{i=1}^{n}(f_i)_x\big(\tfrac{f_x}{g_x}\big)^i\Big) = \mathsf{v}_{x\rightsquigarrow y}(-(f_0)_x) = \mathsf{v}_{x\rightsquigarrow y}((f_0)_x).$$

By proposition 2.11 then as $f_y = 0$ we have $\mathsf{v}_{x\rightsquigarrow y}(f_x) \neq 0$, and so by assumption $\mathsf{v}_{x\rightsquigarrow y}(f_x) > \mathsf{v}_{x\rightsquigarrow y}(g_x)$; then $\mathsf{v}_{x\rightsquigarrow y}(\frac{f_x}{g_x}) > 0$ and so

$$\mathsf{v}_{x\rightsquigarrow y}((f_0)_x) = \mathsf{v}_{x\rightsquigarrow y}\Big(\sum_{i=1}^{n}(f_i)_x\big(\tfrac{f_x}{g_x}\big)^i\Big) \geqslant \min_{i\geqslant 1}\Big(\Big\{\mathsf{v}_{x\rightsquigarrow y}\Big((f_i)_x\big(\tfrac{f_x}{g_x}\big)^i\Big)\Big\}\Big)$$
$$= \min_{i\geqslant 1}\Big(\Big\{\mathsf{v}_{x\rightsquigarrow y}\big((f_i)_x\big) + i\mathsf{v}_{x\rightsquigarrow y}\big(\tfrac{f_x}{g_x}\big)\Big\}\Big)$$

by the standard properties of valuations; then as $f_i \in A_X$ then $\mathsf{v}_{x\rightsquigarrow y}\big((f_i)_x\big) \geqslant 0$ for each i, and so $\mathsf{v}_{x\rightsquigarrow y}\big((f_i)_x\big) + i\mathsf{v}_{x\rightsquigarrow y}(\frac{f_x}{g_x}) > 0$ for all $i \geqslant 1$ and hence $\mathsf{v}_{x\rightsquigarrow y}((f_0)_x) > 0$ likewise. But again by proposition 2.11 we have $\mathsf{v}_{x\rightsquigarrow y}((f_0)_x) > 0 \implies (f_0)_y = 0$, that is, $y \in z(f_0)$ as required. □

We observe

Proposition 2.14. *For all $h' \in A'$ we have that h' has finite image as a function $X \to k(T_X)$,*

since this is the case for f, g and all possible $f_0, \ldots, f_n \in A_X$. □

As mentioned above, for the continuation of our construction we need the following properties for the ring A':

Proposition 2.15. *For all $h' \in A'$, then for all specialisations $x \rightsquigarrow y$ we have that $\mathsf{v}_{x\rightsquigarrow y}(h'_x) \geqslant 0$.*

Proposition 2.16. *For all $h' \in A'$, then for all specialisations $x \rightsquigarrow y$ we have that $\mathsf{v}_{x\rightsquigarrow y}(h'_x) = 0 \iff h'_y \neq 0$.*

Proposition 2.17. *For all $h' \in A'$, there is an integer $N_{h'}$ such that for all specialisations $x \rightsquigarrow y$ where $h'_x \neq 0$ we have that $\mathsf{v}_{x\rightsquigarrow y}(h'_x) \leqslant N_{h'}$.*

The first result follows since $\mathsf{v}_{x\rightsquigarrow y}(\big(\frac{f}{g}\big)_x) \geqslant 0$ for all specialisations $x \rightsquigarrow y$ by our condition on f and g, and so for

$$h' = f_n\big(\frac{f}{g}\big)^n + \cdots + f_1\big(\frac{f}{g}\big) + f_0$$

we have that the value $\mathsf{v}_{x\rightsquigarrow y}(h'_x) \geqslant \mathsf{min}(\mathsf{v}_{x\rightsquigarrow y}\big((f_i)_x(\big(\frac{f}{g}\big)_x)^i\big)) \geqslant 0$.

Furthermore if $f_x = 0$ then $h'_x = (f_0)_x$ and the second result will hold since $f_0 \in A_X$; furthermore those values $\mathsf{v}_{x\rightsquigarrow y}(h'_x)$ for which $f_x = 0$ but $h'_x \neq 0$ will be bounded by N_{f_0}.

Then to complete the result we need only look at those x for which $f_x \neq 0$, in which case

$$h'_x = (f_n)_x\big(\tfrac{f_x}{g_x}\big)^n + \cdots + (f_1)_x\tfrac{f_x}{g_x} + (f_0)_x.$$

Note then that

$$h'_x = \frac{(f_n)_x f_x^n + g_x(f_{n-1})_x f_x^{n-1} \cdots + g_x^{n-1}(f_1)_x f_x + g_x^n (f_0)_x}{g_x^n},$$

where the numerator is the image at x of the element $f' = f_n f^n + \cdots + g^n f_0$ of A_X, therefore at those x where $f_x \neq 0$ and $h'_x \neq 0$ the value of any valuation $\mathsf{v}_{x\rightsquigarrow y}(h'_x) = \mathsf{v}_{x\rightsquigarrow y}(f'_x) - \mathsf{v}_{x\rightsquigarrow y}(g_x^n)$ is bounded by $N_{f'}$ (since $g \in A_X$ and so hence $\mathsf{v}_{x\rightsquigarrow y}(g_x) \geqslant 0$); therefore in general the value of $\mathsf{v}_{x\rightsquigarrow y}(h'_x)$ is bounded by $\mathsf{max}(N_{f_0}, N_{f'})$.

It remains therefore to address the second required property: that $\mathsf{v}_{x\rightsquigarrow y}(h'_x) = 0 \iff h'_y \neq 0$.

If $f_y = 0$ then $\mathsf{v}_{x \rightsquigarrow y}(f_x) > 0$ and so $\mathsf{v}_{x \rightsquigarrow y}(\frac{f_x}{g_x}) > 0$; and furthermore if $f_y = 0$ then $\left(\frac{f}{g}\right)_y = 0$ so that $h'_y = (f_0)_y$. We have that

$$\mathsf{v}_{x \rightsquigarrow y}(h'_x) \geqslant \min_{i \geqslant 0}(\{\mathsf{v}_{x \rightsquigarrow y}\big((f_i)_x\big) + i\mathsf{v}_{x \rightsquigarrow y}(\tfrac{f_x}{g_x})\}),$$

and for all $i \geqslant 1$ we have $\mathsf{v}_{x \rightsquigarrow y}\big((f_i)_x\big) + i\mathsf{v}_{x \rightsquigarrow y}(\frac{f_x}{g_x}) > 0$ so that $\mathsf{v}_{x \rightsquigarrow y}(h'_x) = 0 \implies \mathsf{v}_{x \rightsquigarrow y}\big((f_0)_x\big) = 0$; but furthermore if $\mathsf{v}_{x \rightsquigarrow y}\big((f_0)_x\big) = 0$ then by the property 1.423 of valuations (as every other term in the range on min is strictly bigger than zero) we in fact have equality, i.e. if $\mathsf{v}_{x \rightsquigarrow y}\big((f_0)_x\big) = 0$ then

$$\mathsf{v}_{x \rightsquigarrow y}(h'_x) = \min_{i \geqslant 0}(\{\mathsf{v}_{x \rightsquigarrow y}\big((f_i)_x\big) + i\mathsf{v}_{x \rightsquigarrow y}(\tfrac{f_x}{g_x})\}) = \mathsf{v}_{x \rightsquigarrow y}\big((f_0)_x\big) = 0.$$

Then we have $\mathsf{v}_{x \rightsquigarrow y}(h'_x) = 0 \iff \mathsf{v}_{x \rightsquigarrow y}\big((f_0)_x\big) = 0 \iff (f_0)_y \neq 0 \iff h'_y \neq 0$.

Otherwise if $f_y \neq 0$ then furthermore $g_y \neq 0$, and so firstly $\mathsf{v}_{x \rightsquigarrow y}(g_x) = 0$, so that

$$\mathsf{v}_{x \rightsquigarrow y}((g^n h')_x) = n\mathsf{v}_{x \rightsquigarrow y}(g_x) + \mathsf{v}_{x \rightsquigarrow y}(h'_x) = \mathsf{v}_{x \rightsquigarrow y}(h'_x);$$

and also as in the previous proposition we have that

$$h'_y = 0 \iff (g^n h')_y = 0;$$

and so $\mathsf{v}_{x \rightsquigarrow y}(h'_x) = 0 \iff \mathsf{v}_{x \rightsquigarrow y}((g^n h')_x) = 0 \iff (g^n h')_y \neq 0 \iff h'_y \neq 0$ as required (the second equivalence holding since $g^n h'$ is an element of A_X). □

This allows us to extend our original ring A_X in the following way: call a pair f, g **admissible** if they satisfy the condition of proposition 2.13. Note that if we have f, g and f', g' admissible in A_X, then the pair f', g' is still admissible in the ring $A' = A_X[\frac{f}{g}]$ (since admissibility depends only on the valuations $\mathsf{v}_{x \rightsquigarrow y}$, which are already defined on the whole of $\prod_{x \in X} \mathcal{Q}(R_x)$, and therefore unchanged by whatever extensions we may carry out within this ring), and furthermore, by propositions 2.15 and 2.16, every part of the proof of 2.13 is still valid when extending the ring A' by $\frac{f'}{g'}$ (Hochster phrases this by stating that the collection of valuations $\mathsf{v}_{x \rightsquigarrow y}$ is also an index for the ring A'). Then in fact the proofs of 2.15, 2.16 and 2.17 are also valid for the extension $A'[\frac{f'}{g'}]$ (that is, our valuations are also an index for the ring $A'' = A'[\frac{f'}{g'}]$ and each $h \in A''$ has a bound N_h on its values at these valuations), and each $h \in A''$ again has finite image as a function $X \to k(T_X)$, since it is a finite combination of functions with this property. Then we may extend in any order by finitely many admissible pairs and still have that the ensuing ring $\overline{A}$ is a ring of functions, each of finite image, with X as a spectral subspace of $\mathsf{Spec}\,\overline{A}$ (i.e. the conclusion of 2.13 holds for $\overline{A}$), that our collection of valuations are still 'an index' for $\overline{A}$ — that is that we may again extend by any admissible pair in $\overline{A}$, and that each $h \in \overline{A}$ has a bound N_h on its values. This allows the following inductive argument: Set $A_0 = A_X$ and let $B = B_0 = \{\frac{f}{g} \mid f, g \in A_X$ satisfy the condition of proposition 2.13$\}$. Then given any $h \in A_1 = A_X[B]$, h arises from some finite extension of A_X by admissible elements. Therefore for all $h \in A_1$ we have that the conclusions of propositions 2.13—2.17 hold for h, so that X is a spectral subspace of $\mathsf{Spec}\,A_1$, with A_1 a ring of functions of finite image in their codomain, and our collection of valuations $\mathsf{v}_{x \rightsquigarrow y}$ are an index for A_1 bounded on each $h \in A_1$, so that we may extend A_1 by any admissible pair in A_1. We then apply identical reasoning, inductively forming the set $B_i = \{\frac{f}{g} \mid f, g \in A_i$ satisfy the condition of proposition 2.13$\}$ and extending to the ring $A_{i+1} = A_i[B_i]$, where at all stages n we maintain X as a spectral subspace of the ring A_n. We then set $A_\omega = \bigcup_{n \in \mathbb{N}} A_n$. We find

Theorem 2.18 ([14], Theorem 6). *Defining $H_X = A_\omega$ as given above, we have $X \cong \mathsf{Spec}\,H_X$.*

We summarise some vital facts about the ring of functions H_X. These are

(1) X is a spectral subspace of $\mathsf{Spec}\,H_X$;

(2) Each element $f \in H_X$ has finite image in the ring $k(T_X)$;

(3) For each element $f \in H_X$ there is an integer N_f such that for all specialisations $x \rightsquigarrow y$ in X where $f_x \neq 0$, we have that $\mathsf{v}_{x \rightsquigarrow y}(f_x) \leqslant N_f$.

We show that it has the property that for all $f, g \in H_X$ we have that $z(g) \subseteq z(f) \implies f \in \mathsf{rad}\, g$.

Suppose we have $f, g \in H_X$ such that $z(g) \subseteq z(f)$. We need to show that there is some $r \in H_X$ and some $n \in \mathbb{N}$ such that $f^n = r \cdot g$, so that $f \in \mathsf{rad}\, g$. We consider a ring A_i in our inductive procedure for i sufficiently large that both f and g are contained in A_i. There is an integer N_g such that for all specialisations $x \rightsquigarrow y$ where $g_x \neq 0$ we have that $\mathsf{v}_{x \rightsquigarrow y}(g_x) \leqslant N_g$. Then at all specialisations $x \rightsquigarrow y$ we have $\mathsf{v}_{x \rightsquigarrow y}\big(f_x^{N_g+1}\big) = (N_g + 1)\mathsf{v}_{x \rightsquigarrow y}(f_x) > \mathsf{v}_{x \rightsquigarrow y}(g_x)$ except if $\mathsf{v}_{x \rightsquigarrow y}(g_x) = \infty$ or $\mathsf{v}_{x \rightsquigarrow y}(g_x) = 0$, where $\mathsf{v}_{x \rightsquigarrow y}\big(f_x^{N_g+1}\big) = \mathsf{v}_{x \rightsquigarrow y}(g_x)$. Since firstly $g_x = 0 \implies f_x = 0$ since $z(g) \subseteq z(f)$; and secondly $\mathsf{v}_{x \rightsquigarrow y}(f_x) = 0$ only if $f_y \neq 0$, in which case by the same reasoning necessarily $g_y \neq 0$ and so $\mathsf{v}_{x \rightsquigarrow y}(g_x) = 0$. Then $\frac{f^{N_g+1}}{g}$ is admissible in A_i, and so $\frac{f^{N_g+1}}{g} \in A_{i+1}$ and hence $\frac{f^{N_g+1}}{g} \in H_X$; finally for $r = \frac{f^{N_g+1}}{g} \in H_X$ we have that $f^{N_g+1} = r \cdot g$, as either $f_x = 0$ in which case $r_x = 0$ and $f_x^{N_g+1} = 0 = 0 \cdot g_x$, or $r_x = \frac{f_x^{N_g+1}}{g_x}$ and so $r_x g_x = \frac{f_x^{N_g+1}}{g_x} g_x = f_x^{N_g+1}$; thus $f \in \mathsf{rad}\, g$ as required.

Then by 2.8 we have that $X = \mathsf{Spec}\, H_X$. □

Representing spectral maps via ring homomorphisms

As we saw in the preliminaries, every ring homomorphism gives rise to a spectral map between the corresponding spectral spaces. Our construction given above allows us, without too much difficulty, to show that every spectral map $f : X \to Y$ of spectral spaces arises from a ring homomorphism $f^* : H_Y \to H_X$ such that $\mathsf{Spec}\, H_X \cong X$, $\mathsf{Spec}\, H_Y \cong Y$ and $\mathsf{Spec}(f^*) \cong f$.

A spectral map $f : X \to Y$ induces a mapping $\mathring{\mathcal{K}}(f) : \mathring{\mathcal{K}}(Y) \to \mathring{\mathcal{K}}(X)$ of the quasi-compact open sets of X and Y by $U \mapsto f^{-1}(U)$; when we consider our construction of a ring from the spaces X and Y we see that this map restricts to a map $r : \mathcal{B}_Y \to \mathring{\mathcal{K}}(X)$. We then require that we choose $\mathcal{B}_X$ so that $r(\mathcal{B}_Y) \subseteq \mathcal{B}_X$; furthermore we wish that the map r be injective, as this will simplify the situation greatly. Obviously in general this may not be possible, when the map $\mathring{\mathcal{K}}(f)$ is not itself injective; thus we extend the definition of $\mathcal{B}_X$, and so the corresponding definitions of the sets T_X and $\mathcal{X}_X$, to allow us to specify an injective mapping $r : \mathcal{B}_Y \hookrightarrow \mathcal{B}_X$, as follows: we let $\mathcal{B}_X$ be any set, equipped with an identification function $i : \mathcal{B}_X \to \mathring{\mathcal{K}}(X)$ so that the range $i(\mathcal{B}_X)$ is a subbasis for X; thus we think of the characteristic function χ_u corresponding to some element $u \in \mathcal{B}_X$ as being defined as taking the value t_u on the set $i(u) \in \mathring{\mathcal{K}}(X)$ and 0 elsewhere. We can then define an injective function $r : \mathcal{B}_Y \hookrightarrow \mathcal{B}_X$ simply by taking $\mathcal{B}_X$ to be the set $\mathcal{B}_Y \times \mathring{\mathcal{K}}(X)$ with r defined as $r(U) = (U, f^{-1}(U))$ and the identification function $i : \mathcal{B}_X \to \mathring{\mathcal{K}}(X)$ to be projection onto the second co-ordinate (of course instead of $\mathring{\mathcal{K}}(X)$ in the above definition we could take any subbasis $\mathcal{B} \subseteq \mathring{\mathcal{K}}(X)$ such that $f^{-1}(\mathcal{B}_Y) \subseteq \mathcal{B}$, with maps defined in an identical way). We can see by examination of the argument in 2.4 that this definition does not materially alter the behaviour of our ring of functions A_X defined as on page 54 from $\mathcal{X}_X$, and so all the properties claimed for the ring A_X and all subsequent constructions thereon are maintained; hence $\mathsf{Spec}\, H_X \cong X$ as required.

Furthermore, injectivity of r allows us to guarantee, when we define a ring homomorphism $f^* : H_Y \to H_X$ from the map r, that distinct monomials in H_Y evaluate to distinct monomials in H_X, greatly simplifying the argument.

Recall the observation that not every distinct expression in $k[\mathcal{X}_X]$ corresponds to a unique function. In the case of a single characteristic function χ_u, this is only a problem if $i(u) = \emptyset$ so that χ_u is identically zero on X; as otherwise it will take the value t_u somewhere on X, which is taken by no other characteristic function $\chi_{u'}$ for $u \neq u'$. Then the homomorphism $\varphi : A_Y \to A_X$ defined by $\varphi(\chi_U) = \chi_{r(U)}$ is injective on individual characteristic functions χ_U except where $i(r(U)) = f^{-1}(U) = \emptyset$, where $\varphi(\chi_U) = 0$. Then for distinct monomials m, m' in A_Y, we have $\varphi(m) \neq \varphi(m')$ except where $\varphi(m) = 0 = \varphi(m')$. This allows us to use the argument as in 2.4 that given $h = \sum_j \lambda_j m_j \in A_Y$ we have that $d(\varphi(h)) = d(\varphi(m_1)) \cup \dots \cup d(\varphi(m_n))$.

Then for all $h \in A_Y$ we have that $f^{-1}(d(h)) = d(\varphi(h))$: since, per the preceding discussion, this reduces to checking $f^{-1}(d(m)) = d(\varphi(m))$ on monomials $m = \chi_{U_1}^{\alpha_1} \dots \chi_{U_N}^{\alpha_N}$, in which case we have $d(m) = \bigcap_j U_j$. But

$$\varphi(m) = \varphi(\chi_{U_1}^{\alpha_1}) \cdot \dots \cdot \varphi(\chi_{U_N}^{\alpha_N}) = \chi_{r(U_1)}^{\alpha_1} \cdot \dots \cdot \chi_{r(U_N)}^{\alpha_N}$$

so that

$$d(\varphi(m)) = d\big(\chi_{r(U_1)}^{\alpha_1} \cdot \dots \cdot \chi_{r(U_N)}^{\alpha_N}\big) = \bigcap_j i(r(U_j)) = \bigcap_j f^{-1}(U_j) = f^{-1}(d(m))$$

as required.

The homomorphism φ is defined (via the identification of A_Y with the ring $\{\prod_{y \in Y} \mathsf{ev}_y(g) \mid g \in k[\mathcal{X}_Y]\}$) on all of R_y for each $y \in Y$ and so obviously has a uniquely defined extension to the ring $\prod_{y \in Y} \mathcal{Q}(R_y)$; and so remains defined throughout our extension process.

Then if we verify that $\varphi^{-1}(\Phi_X(x)) = \Phi_Y(f(x))$ for all $x \in X$, that is, that $\varphi^{-1}(\mathsf{ker}(\mathsf{ev}_x)) = \mathsf{ker}(\mathsf{ev}_{f(x)})$ for all x, we will have that the natural extension of φ to $H_Y \subseteq \prod_{y \in Y} \mathcal{Q}(R_y)$ is the ring homomorphism f^* such that $f^* = \mathsf{Spec}\,\varphi$.

We see

$$\begin{aligned} g \in \varphi^{-1}(\mathsf{ker}(\mathsf{ev}_x)) \iff \varphi(g)_x = 0 \iff x \notin d(\varphi(g)) \iff f(x) \notin d(g) \\ \iff g_{f(x)} = 0 \iff g \in \mathsf{ker}(\mathsf{ev}_{f(x)}) \end{aligned}$$

as required.

3. Ring constructions on finite posets

We now present two constructions, per Lewis [17], and Ershov [7], of a ring having prime spectrum homeomorphic to any *finite* spectral space. Note that in the finite case, from 1.26, that two spectral spaces are homeomorphic if and only if their specialisation posets are isomorphic; note furthermore that every finite poset $(X, \leqslant)$ arises as the specialisation poset of some finite spectral space, namely the spectral space consisting of X in the lower topology; therefore we may consider finite spectral spaces and finite posets interchangeably, and to achieve the desired result it suffices to find for every finite poset $(X, \leqslant)$ a ring R such that $(\mathsf{Spec}\, R, \subseteq) \cong (X, \leqslant)$. As a consequence, the topological details of a finite spectral space X are reduced to the details of a partial order relation on X, which discards many of the intricacies present in the general topological situation, thereby greatly simplifying the presentation and discussion of the results.

Summary of techniques

We introduce two techniques that are used repeatedly in both the constructions under consideration, to obtain rings with certain (finite) prime ideal structure.

Intersecting (discrete) valuation rings

We have per 1.45 that the ideals, in particular the prime ideals, of a valuation ring V are totally-ordered by subset inclusion; that is, the prime spectrum of

V is a total order in the specialisation order. In particular, a (proper) discrete valuation ring is a local principal ideal domain, so that it has exactly two prime ideals $\langle 0 \rangle \subsetneq \mathfrak{m}$. We see how this may be used, via the following propositions, to obtain certain finite posets as the prime spectrum of a ring.

Proposition 3.1 ([3], §7, Proposition 2). *If $A_1, \ldots, A_n$ are inclusion-incomparable valuation rings of a field k, then for $R = \bigcap_i A_i$ we have that the maximal ideals of R are in bijection with the maximal ideals $\mathfrak{m}_i$ of the valuation rings A_i, with each maximal ideal $\mathfrak{n}_i$ of R being of the form $\mathfrak{n}_i = (\mathfrak{m}_i \cap R)$.*

Proposition 3.2 (op. cit, §7, Proposition 1). *Given $R, A_1, \ldots, A_n$ as above, the prime ideals of A_i are in order-preserving bijection with the prime ideals of R contained in $\mathfrak{n}_i$,*

since we have that $R_{\mathfrak{n}_i} = A_i$. □

Corollary 3.3 (loc. cit.). $R/\mathfrak{n}_i \cong A_i/\mathfrak{m}_i$,

since $R/\mathfrak{n}_i \cong R_{\mathfrak{n}_i}/\mathfrak{n}_i R_{\mathfrak{n}_i} \cong A_i/\mathfrak{m}_i$. □

In particular if $A_1, \ldots, A_n$ are inclusion-incomparable discrete valuation rings of a field k, then $R = \bigcap_i A_i$ has exactly n non-zero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of height 1 corresponding to the maximal ideals $\mathfrak{m}_i$ of each A_i; furthermore the quotient map $R \twoheadrightarrow R/\mathfrak{p}_i$ is isomorphic to the restriction of the canonical quotient map $\pi_i : A_i \twoheadrightarrow A_i/\mathfrak{m}_i$ to R.

Fibre sums of spaces

The fibre sum of topological spaces is a construction that gives the pushout of a diagram in the category of topological spaces and continuous maps. That is, given a diagram

$$\begin{array}{ccc} & & X \\ & & \big\uparrow f \\ Y & \xleftarrow{\quad g \quad} & Z \end{array}$$

of topological spaces and continuous maps, the fibre sum provides a space $X \sqcup_Z Y$ and maps f', g' having the universal property in the diagram

$$\begin{array}{ccc} X \sqcup_Z Y & \xleftarrow{\quad g' \quad} & X \\ f' \big\uparrow & & \big\uparrow f \\ Y & \xleftarrow{\quad g \quad} & Z. \end{array}$$

Most details in the following discussion may be found in [8] and the references therein; a detailed exploration of fibre sum constructions in relation to spectral spaces may be found in the textbook [4].

The fibre sum $X \sqcup_Z Y$ as a set is the disjoint union $X \sqcup Y$ modulo the equivalence relation $\sim$ given by $f(z) \sim g(z)$ for each $z \in Z$, topologised with the quotient topology defined from the surjection $X \sqcup Y \twoheadrightarrow X \sqcup Y/\sim$. We will in fact only need to consider the circumstance where one of f or g is a closed embedding, which simplifies the description of the fibre sum. In the case that g is a closed embedding (and therefore an injective function) we identify Z with its image under g as a subset of Y, and the underlying set of $X \sqcup_Z Y$ is then the disjoint union $(Y \setminus Z) \sqcup X$, where the maps g', f' are the identity on $X, (Y \setminus Z)$ respectively, and where f' maps $z \in Z$ to $f(z) \in X$. Then the topology of $X \sqcup_Z Y$ has as closed sets $C \subseteq (Y \setminus Z) \sqcup X$ such that $C \cap X$ is closed in X and $(C \cap (Y \setminus Z)) \cup f^{-1}(C \cap f(Z))$ is closed in Y. We note that X homeomorphically embeds into $X \sqcup_Z Y$ as a closed set and $(Y \setminus Z)$ (considered as a subspace of Y) homeomorphically embeds into $X \sqcup_Z Y$ as an open set. A particularly important observation for the purposes of our construction is that we have a specialisation $x \rightsquigarrow y$ in $X \sqcup_Z Y$ if and only if one of the following hold:

(1) We have a specialisation $x \rightsquigarrow y$ in $(Y \setminus Z)$;

(2) We have a specialisation $x \rightsquigarrow y$ in X, or

(3) There is some $z \in Z$ and specialisations $x \rightsquigarrow z$ in Y and $f(z) \rightsquigarrow y$ in X.

It is worth noting (per [4]) that in general the topological fibre sum $X \sqcup_Z Y$ of two spectral spaces X, Y over a third spectral space Z may not itself be a spectral space; however in the case that one of the maps f or g is a closed embedding then the space $X \sqcup_Z Y$ is indeed spectral.

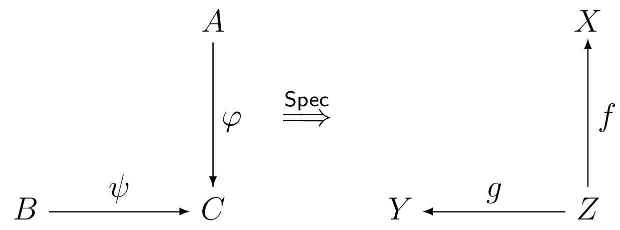
On the other side of our considerations, we have that Spec is a contravariant functor from the category of rings and ring homomorphisms to the category of topological spaces and continuous maps, so that the relevant corresponding notion in the category of rings is the pullback; it is well known that this is given by the fibre product of rings. That is, given rings and ring homomorphisms

$$\begin{array}{ccc} & & A \\ & & \Big\downarrow{\scriptstyle \varphi} \\ B & \xrightarrow{\psi} & C \end{array}$$

then the fibre product $A \times_C B = \{(a, b) \in A \times B \mid \varphi(a) = \psi(b)\}$ has the universal property in the diagram

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\psi'} & A \\ {\scriptstyle \varphi'}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi} \\ B & \xrightarrow{\psi} & C \end{array}$$

(with the maps φ', ψ' being the obvious projections). We consider the situation where we have spectral spaces $X = \mathsf{Spec}\, A$, $Y = \mathsf{Spec}\, B$, $Z = \mathsf{Spec}\, C$, and corresponding maps $f = \mathsf{Spec}\, \varphi$, $g = \mathsf{Spec}\, \psi$, as in the following diagram.



As noted above, it may be the case that the fibre sum $X \sqcup_Z Y$ is not a spectral space; in which case we cannot hope to associate the fibre product of rings with the corresponding fibre sum of spaces. However we know from 1.5 that in the case the ring homomorphism ψ (for example) is surjective, we have that the corresponding spectral map $g = \mathsf{Spec}\,\psi$ is a closed embedding, in which case the fibre sum is a spectral space; in fact in this case we have

Theorem 3.4 ([8], 1.4). *Given rings A, B, C and corresponding spectra $X = \mathsf{Spec}\,A$, $Y = \mathsf{Spec}\,B$, $Z = \mathsf{Spec}\,C$ as in the above diagram, then if $\psi : B \twoheadrightarrow C$ is a surjective ring homomorphism, then $\mathsf{Spec}(A \times_C B) \cong X \sqcup_Z Y$.*

This then gives us a general method to use the properties of the fibre sum discussed above to construct rings with particular spectra.

An important addition to theorem 3.4 concerns the properties of the fibre product in relation to Noetherian rings. Unfortunately, it is not sufficient that the rings A, B, C be Noetherian to guarantee their fibre product $D = A \times_C B$ is Noetherian. Recalling that a ring homomorphism $\varphi : A \to B$ is said to be finite in the case that B can be finitely generated as a module by the image of A under φ, we find

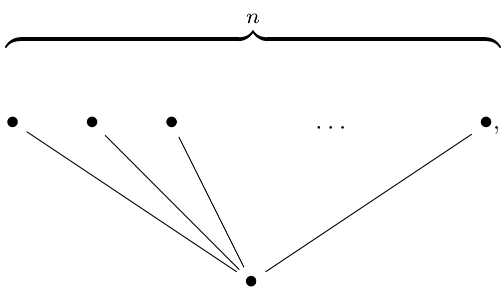
Proposition 3.5 ([8], 1.8). *Given A, B, C and $\varphi : A \to C$, $\psi : B \twoheadrightarrow C$ as above, then the fibre product $D = A \times_C B$ is Noetherian if and only if A and B are Noetherian and φ is a finite homomorphism.*

This observation will be critical when considering which spectral spaces can arise as the prime spectrum of a Noetherian ring.

The method of Lewis

The construction in [17] uses iteratively two techniques to build up a given finite poset as the spectrum of a ring: the first being “gluing” (the spectrum of) a domain onto a maximal ideal in (the spectrum of) some other ring; the second being “joining” maximal ideals of some ring so that they form a single point in the spectrum. In fact it was observed by Fontana [8] that both techniques are instances of forming the fibre sum of spectral spaces, via its connection with the fibre product of the corresponding rings. This insight greatly simplifies the analysis of Lewis's construction, and it is within this framework that it will be presented here.

The basic unit of Lewis's construction is the spectrum of a polynomial ring in finitely many indeterminates, localised at the height-1 prime ideals corresponding to the principal ideals generated by each indeterminate. This gives a ring having n (—where n is the number of indeterminates) non-zero prime ideals each of height 1. In detail: we fix a field k, and consider the polynomial ring $k[t_1, \ldots, t_n]$ for some n. For each t_i, we have that the ideal $\langle t_i \rangle$ is a prime ideal of height 1. Forming the set $S = \big(\langle t_1 \rangle \cup \cdots \cup \langle t_n \rangle\big)^{\mathsf{c}}$ we take $A = k[t_1, \ldots, t_n]_S$. By the well-known properties of localisation, we have that the prime ideals of A are in order-preserving bijection with the prime ideals of $k[t_1, \ldots, t_n]$ disjoint from S; these are exactly the height-1 prime ideals $\langle t_i \rangle$ for $i = 1, \ldots, n$, and the zero ideal $\langle 0 \rangle$. Then the space $\mathsf{Spec}\, A$ has specialisation structure as in the following diagram:



where specialisation relations proceed "upwards", that is, a diagram $\begin{array}{c} \bullet^{y} \\ \big| \\ \bullet^{x} \end{array}$ denotes the existence of a specialisation $x \rightsquigarrow y$.

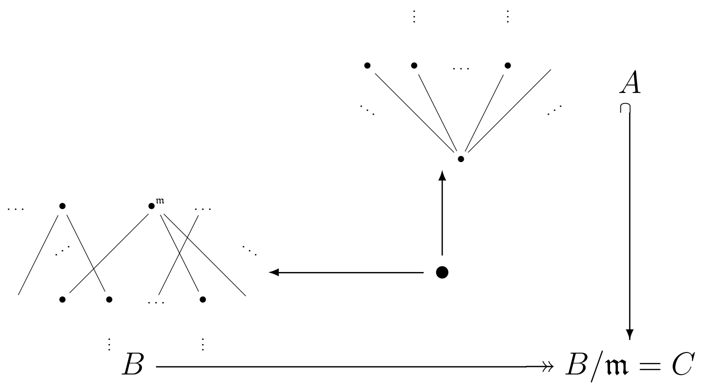
We summarise the two instances of the fibre sum of spectral spaces utilised by Lewis and describe the iterative process by which we may obtain a ring with spectrum homeomorphic to a given finite poset.

'Gluing' of domains over maximal ideals

Working over some field K, let A be some domain contained in K (—in practice A will be the ring $k[t_1, \dots, t_n]_S$ as described above, and we will take $K = k\big(t_1, \dots, t_n)\big)$, and let B be a K-algebra having some maximal ideal $\mathfrak{m}$. Then setting $C = B/\mathfrak{m}$ we have an inclusion $\varphi : A \hookrightarrow C$, and a surjective quotient map $\psi : B \twoheadrightarrow C$, as in the following diagram:

$$\begin{array}{ccc} & & A \\ & & \big\downarrow{\scriptstyle \varphi} \\ B & \xrightarrow{\ \psi\ } & C \end{array}$$

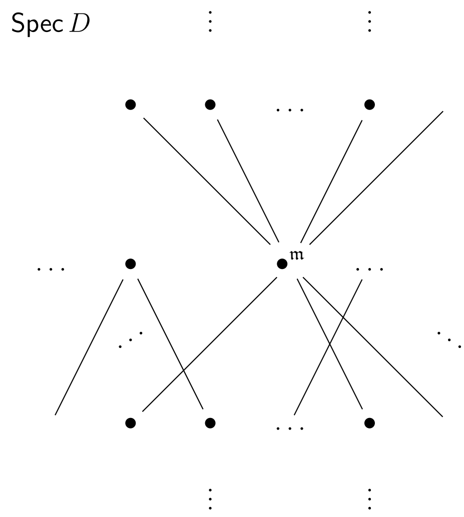
Considering the image of this diagram under the functor Spec, we have that $\mathsf{Spec}\,\varphi : \mathsf{Spec}\,C \to \mathsf{Spec}\,A$ maps the single point $\langle 0 \rangle$ of the spectrum of the field C to $\varphi^{-1}(\{0\}) = \mathsf{ker}\,\varphi = \{0\}$ in $\mathsf{Spec}\,A$, and the map $\mathsf{Spec}\,\psi : \mathsf{Spec}\,C \to \mathsf{Spec}\,B$ is a closed embedding taking the point $\langle 0 \rangle$ of $\mathsf{Spec}\,C$ to $\psi^{-1}(\{0\}) = \mathsf{ker}\,\psi = \mathfrak{m}$ in $\mathsf{Spec}\,B$ (hence the image of $\mathsf{Spec}\,C$ is the closed subset $\{\mathfrak{m}\}$). The following picture describes the situation:



Then, as the map ψ is surjective, forming the fibre product $D = A \times_C B$, we will have that $\mathsf{Spec}\, D$ is homeomorphic to the fibre sum of $X = \mathsf{Spec}\, A$ and $Y = \mathsf{Spec}\, B$ over $Z = \mathsf{Spec}\, C$; that is, we have the situation represented by the following diagram:

$$\begin{array}{ccc} D = A \times_C B & \xrightarrow{\psi'} & A \\ {\scriptstyle \varphi'} \downarrow & & \downarrow {\scriptstyle \varphi} \\ B & \xrightarrow{\psi} & C \end{array} \quad \overset{\mathsf{Spec}}{\Longrightarrow} \quad \begin{array}{ccc} W = X \sqcup_Z Y & \xleftarrow{g'} & X \\ {\scriptstyle f'} \uparrow & & \uparrow {\scriptstyle f} \\ Y & \xleftarrow{g} & Z \end{array}$$

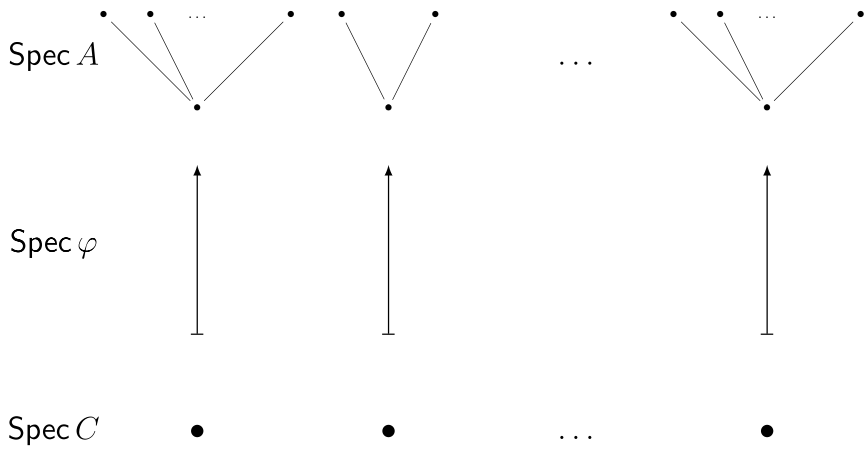
Then the spectrum of D, being the space $W = X \sqcup_Z Y$, has the topological structure of $Y = \mathsf{Spec}\, B$ with the closed set $\{\mathfrak{m}\}$ replaced by the closed set $X = \mathsf{Spec}\, A$; that is, it is the result of gluing the space $\mathsf{Spec}\, A$ onto the maximal point $\mathfrak{m}$ of $\mathsf{Spec}\, B$, as in the following picture:



Examining $D = A \times_C B$ as a ring, we have

$$
\begin{aligned}
D = \{(a,b) \in A \times B \mid \varphi(a) = \psi(b)\} &= \{(a,b) \in A \times B \mid b \operatorname{\mathsf{mod}} \mathfrak{m} = a\} \\
&\cong \{b \in B \mid b \operatorname{\mathsf{mod}} \mathfrak{m} \in A\} \\
&= A + \mathfrak{m} \subseteq B.
\end{aligned}
$$

We may perform this operation on finitely many maximal points of $\mathsf{Spec}\, B$ simultaneously. That is, suppose $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are among the maximal ideals of a K-algebra B, and we have finitely many domains $A_1, \ldots, A_n$ each contained in K. Then the product $A = A_1 \times \cdots \times A_n$ is contained in the product $C = B/\mathfrak{m}_1 \times \cdots \times B/\mathfrak{m}_n$, giving an inclusion homomorphism $\varphi : A \hookrightarrow C$, and by the Chinese remainder theorem we again obtain a surjective quotient map $\psi : B \twoheadrightarrow C$. We examine the image of these maps under the functor Spec. Write C_i for the field $B/\mathfrak{m}_i$, so that $C = C_1 \times \cdots \times C_n$. Then C, being a finite product of fields, has n prime ideals each of the form $\mathfrak{p}_i = (C_1, \ldots, C_{i-1}, 0, C_{i+1}, \ldots, C_n)$ for $i = 1, \ldots, n$. Then the image of the point $\mathfrak{p}_i \in \mathsf{Spec}\, C$ under the induced map $\mathsf{Spec}\, \varphi$, that is, the preimage of $\mathfrak{p}_i$ under φ, is the prime ideal $(A_1, \ldots, A_{i-1}, 0, A_{i+1}, \ldots, A_n)$ of A.



The image of such a point $\mathfrak{p}_i$ under $\mathsf{Spec}\,\psi$ is the kernel of the quotient map $B \twoheadrightarrow B/\mathfrak{m}_i$; that is, it is the maximal ideal $\mathfrak{m}_i$ of B. Then the image of $Z = \mathsf{Spec}\,C$ under the closed embedding $g = \mathsf{Spec}\,\psi$ is the closed set of maximal points $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\} \subseteq \mathsf{Spec}\,B$

Then taking the fibre product $D = A \times_C B$, again $\mathsf{Spec}\,D$ is homeomorphic to the fibre sum $W = X \sqcup_Z Y$ of $X = \mathsf{Spec}\,A$ and $Y = \mathsf{Spec}\,B$ over $Z = \mathsf{Spec}\,C$. We retain the labelling of the diagram on page 80. Then $\mathsf{Spec}\,D$ has the structure of $\mathsf{Spec}\,B$ with the closed set $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ replaced by the closed set $\mathsf{Spec}\,A$. In particular, recall that we have a specialisation $x \rightsquigarrow y$ in $W \cong \mathsf{Spec}\,D$ if and only if either there is already such a specialisation in $X = \mathsf{Spec}\,A$ or in $Y \setminus Z = \mathsf{Spec}\,B \setminus \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$, or else that for $x \in Y$ and $y \in X$ there is a point $\mathfrak{m}_i \in \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$, such that there is a specialisation $x \rightsquigarrow \mathfrak{m}_i$ in Y and a specialisation $f(\mathfrak{m}_i) \rightsquigarrow y$ in X. As can be seen from the above description of $f = \mathsf{Spec}\,\varphi$, this has the result of attaching the spectrum of each A_i directly on to the maximal point $\mathfrak{m}_i$ of $\mathsf{Spec}\,B$, with no further relations other than those guaranteed by transitivity ensuing.

Then examining $D = A \times_C B$ as a ring, we have

$$\begin{aligned} D &= \{(a,b) \in A \times B \mid \varphi(a) = \psi(b)\} \\ &= \{((a_1, a_2, \ldots, a_n), b) \in (A_1 \times \cdots \times A_n) \times B \mid \\ &\qquad (b \bmod \mathfrak{m}_1, \ldots, b \bmod \mathfrak{m}_n) = (a_1, \ldots, a_n)\} \\ &\cong \{b \in B \mid b \bmod \mathfrak{m}_i \in A_i \text{ for all } i\} \\ &= \bigcap_i (A_i + \mathfrak{m}_i) \subseteq B. \end{aligned}$$

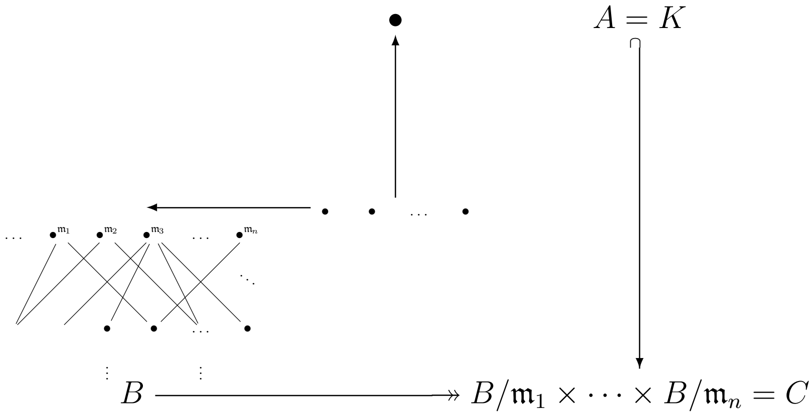
‘Joining’ of maximal ideals

Suppose that $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are among the maximal ideals of a K-algebra B and again let $C = B/\mathfrak{m}_1 \times \cdots \times B/\mathfrak{m}_n = C_1 \times \cdots \times C_n$, with $\psi : B \twoheadrightarrow C$ the surjective quotient homomorphism guaranteed by the Chinese remainder theorem. Then let $A = K$, and let $\varphi : A \hookrightarrow C = C_1 \times \cdots \times C_n$ be the diagonal map $K \hookrightarrow K \times \cdots \times K$, i.e the map that sends $a \mapsto (a, \ldots, a)$ for $a \in K$.

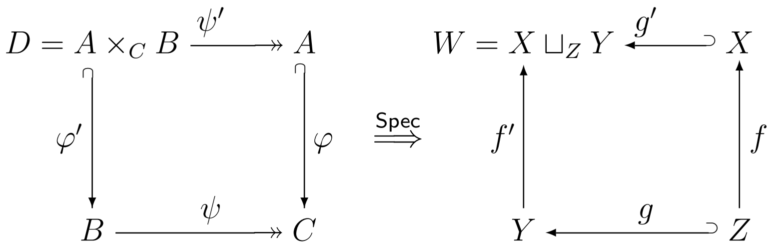
$$\begin{array}{ccc} & & A = K \\ & & \Big\downarrow{\scriptstyle \varphi} \\ B & \xrightarrow{\ \psi\ } & C = C_1 \times \cdots \times C_n \end{array}$$

Again, applying the functor Spec, the image of the point $\mathfrak{p}_i = (C_1, \ldots, 0, \ldots, C_n) \in \mathsf{Spec}\, C$ under the induced map $\mathsf{Spec}\, \psi$ is the kernel of the quotient map $B \twoheadrightarrow B/\mathfrak{m}_i$; that is, it is the maximal ideal $\mathfrak{m}_i$ of B. Then the image of $Z = \mathsf{Spec}\, C$ under the closed embedding $g = \mathsf{Spec}\, \psi$ is the closed set of maximal points $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\} \subseteq Y = \mathsf{Spec}\, B$.

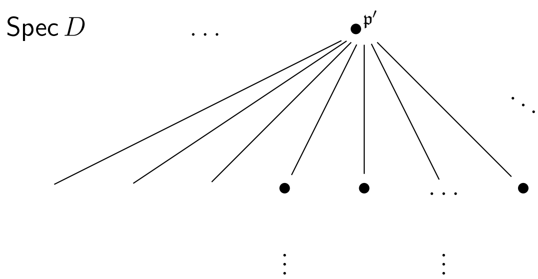
The preimage of $\mathfrak{p}_i = (C_1, \ldots, 0, \ldots, C_n)$ under φ is $\{0\}$, so that for every $\mathfrak{p}_i \in Z = \mathsf{Spec}\, C$ we have that $f(\mathfrak{p}_i) = \varphi^{-1}(\mathfrak{p}_i) = \langle 0 \rangle$, the sole point in $X = \mathsf{Spec}\, K$.



Then once again, as ψ is surjective, taking $D = A \times_C B$ we have that $\mathsf{Spec}\, D$ is homeomorphic to the fibre sum $W = X \sqcup_Z Y$.



Then $\mathsf{Spec}\, D$ has the structure of $\mathsf{Spec}\, B$ with the closed set $Z = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ replaced by the single point $X = \mathsf{Spec}\, A = \mathsf{Spec}\, K$; that is, it is the result of amalgamating the points $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ into a single point $\mathfrak{p}'$:



(note we have a specialisation $x \rightsquigarrow \mathfrak{p}'$ in $W \cong \mathsf{Spec}\, D$ if and only if we have a specialisation $x \rightsquigarrow \mathfrak{m}_i$ in $Y = \mathsf{Spec}\, B$ for some i).

We examine the ensuing ring:

$$\begin{aligned}
D &= \{(a,b) \in A \times B \mid \varphi(a) = \psi(b)\} \\
&= \{(a,b) \mid (a,\dots,a) = (b \bmod \mathfrak{m}_1, \dots, b \bmod \mathfrak{m}_n)\} \\
&= \{(a,b) \mid a = b \bmod \mathfrak{m}_1 = \dots = b \bmod \mathfrak{m}_n\} \\
&\cong \{b \in B \mid \exists m_1 \in \mathfrak{m}_1, m_2 \in \mathfrak{m}_2, \dots, m_n \in \mathfrak{m}_n, a \in K\colon \\
&\qquad b = a + m_1 = a + m_2 = \dots = a + m_n\} \\
&= \{b \mid \exists m \in \bigcap_i \mathfrak{m}_i, a \in K\colon\ b = a + m\} \\
&= K + \bigcap_i \mathfrak{m}_i = K + (\mathfrak{m}_1\mathfrak{m}_2 \dots \mathfrak{m}_n) \subseteq B.
\end{aligned}$$

Once again, we may apply this procedure on finitely many finite sets of maximal ideals at once, joining the maximal ideals of each set to a (distinct) single point in the ensuing spectrum of the fibre product: We take A to be a product of copies of K, one for each set of maximal ideals, we take C to again be the product of the quotient rings of B modulo each maximal ideal, and we map A diagonally component-wise into C, that is, the component of A corresponding to a particular set of maximal ideals maps diagonally into all quotient rings which are the quotient of B by a member of this set of maximal ideals. Explicitly, let $M_j = \{\mathfrak{m}_{i_j}\}$, $j = 1, \dots, m$ be disjoint finite subsets of maximal ideals of B. Let $A = K^m$ and $C = \prod_j \prod_{i_j} B/\mathfrak{m}_{i_j}$. Then we take $\varphi : A \hookrightarrow B$ to be the map sending

$$(a_1, a_2, \dots, a_m) \mapsto ((a_1, a_1, \dots, a_1), (a_2, a_2, \dots, a_2), \dots, (a_m, a_m, \dots, a_m)),$$

that is, where each component a_j maps diagonally into $\prod_{i_j} B/\mathfrak{m}_{i_j}$. Then a point $\mathfrak{m}_{i_j} \in M_j$ maps under $\operatorname{Spec}\varphi$ to the point $(K, \dots, K_{j-1}, 0, K_{j+1}, \dots, K)$ in $\operatorname{Spec} A$; and so, as in the comparable situation for 'gluing' finitely many spaces at once, we have that each point $\mathfrak{m}_{i_j}$ of M_j is joined to a single point

$\mathfrak{m}_j$ in $\mathsf{Spec}\, D$, and no other relations are created between points from $\mathsf{Spec}\, B$ in the ensuing space $\mathsf{Spec}\, D$.

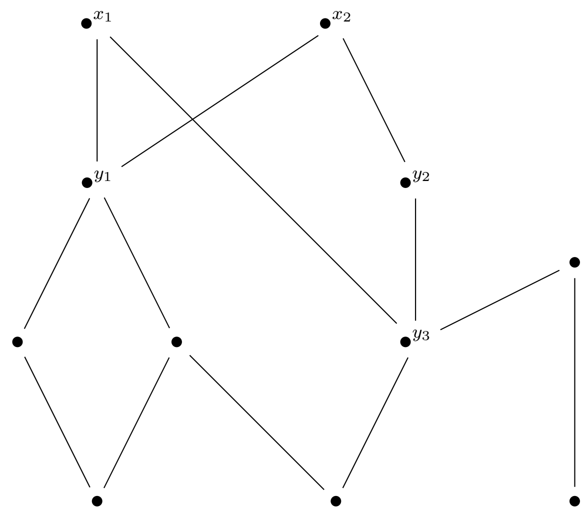
Finally,

$$\begin{aligned}
D &= \{(a,b) \in A \times B \,|\varphi(a) = \psi(b)\} \\
&= \{((a_1,\dots,a_m),b) \,|(a_1,\dots,a_1) = (b \bmod \mathfrak{m}_{1_1},\dots,b \bmod \mathfrak{m}_{n_1}) \;\&\; \\
&\qquad \vdots \\
&\qquad (a_m,\dots,a_m) = (b \bmod \mathfrak{m}_{1_m},\dots,b \bmod \mathfrak{m}_{n'_m})\} \\
&\cong \{b \in B \,|\big[\exists d_1 \in \bigcap_i \mathfrak{m}_{i_1}, a_1 \in K\colon\; b = a_1 + d_1\big] \;\&\; \\
&\qquad \vdots \\
&\qquad \big[\exists d_m \in \bigcap_i \mathfrak{m}_{i_m}, a_m \in K\colon\; b = a_m + d_m\big]\} \\
&= \{b \mid b \in K + \bigcap_i \mathfrak{m}_{i_1} \;\&\; \dots \;\&\; b \in K + \bigcap_i \mathfrak{m}_{i_m}\} \\
&= \bigcap_j \big(K + \bigcap_i \mathfrak{m}_{i_j}\big) \subseteq B.
\end{aligned}$$

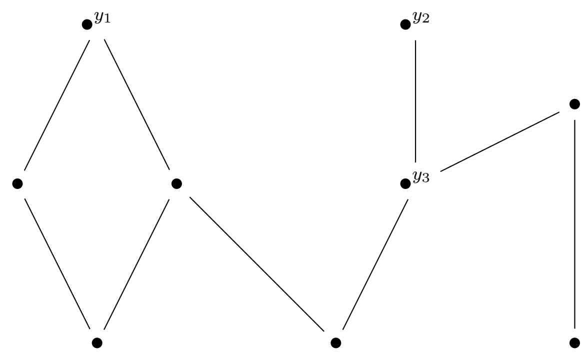
The iterative procedure

We now describe how we build a ring L_X having spectrum isomorphic to a given finite poset X. We proceed by induction on the dimension of X; that is, we build up X ‘layer-by-layer’, increasing the dimension of the poset obtained by 1 at each stage.

The idea is that by removing the points of maximum height of X, we obtain a poset X' of dimension $n-1$, to which an inductive hypothesis can be applied to give a ring $L_{X'}$ with $\mathsf{Spec}\, L_{X'} \cong X'$. We then “reconstruct” the poset X first by gluing a layer of (spectra of) 1-dimensional domains above appropriate maximal ideals of $L_{X'}$ using the ‘gluing of domains’ technique described from page 79 onwards, and then, where we have in X that some point of maximum height lies above several different predecessors, we join together maximal points in the ring ensuing from our gluing of domains to obtain a ring having the structure of X, using the 'joining of maximal ideals' technique as described from page 83 onwards. More precisely, let $X_{\mathsf{ht}\,n} = \{x_1, \ldots, x_m\}$ be the points of maximum height of X. Let $X_{<n} = X \setminus X_{\mathsf{ht}\,n}$ and let $Y = \{y \in X \mid y$ is an immediate predecessor of some $x_i \in X_{\mathsf{ht}\,n}\}$: these are the points that we would wish to 'glue' a dimension-1 spectrum onto. Note however that in general we need not have that every element $y \in Y$ is indeed maximal in $X_{<n}$: consider for example the case where X is a poset as given by the following diagram:



Then we can see that the point y_3 is an immediate predecessor of the point x_1 of maximum height in X, but that y_3 is not maximal in the ensuing poset $X_{<n} = X \setminus X_{\mathsf{ht}\,n}$.



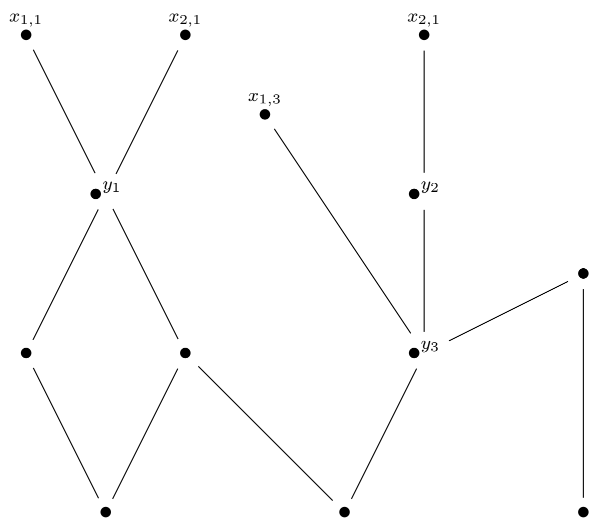
We would therefore not be able to create the relation between y_3 and x_1 by gluing a point above y_3, as our technique only applies to maximal points.

However, we know that, given a ring having a certain prime ideal structure, we may use the "joining of maximal ideals" technique to combine several maximal points into one. We are then able to avoid the difficulties of non-maximal immediate predecessors of points of maximum height by considering a poset where we have "split" the points of maximum height into multiple copies, so that each copy has a unique immediate predecessor, as shall be seen. We know then know that, given a ring representing this poset, we may obtain a ring representing our original poset by joining the multiplicated maximal points.

Then, labelling elements of Y as $y_1, \ldots, y_M$, we define a new set of points

$$X^+ = \{x_{i,j} \mid x_i \in X_{\mathsf{ht}\, n} \text{ and } y_j \in Y \text{ is an immediate predecessor of } x_i \text{ in } X\}.$$

We then let $X_1 = X_{<n} \cup X^+$ with partial order defined by $x_{i,j} > z \iff y_j \geqslant z$ (with relations in $X_{<n}$ inherited from X).



The poset X_1 has two key properties. Firstly, every point of maximum height in X_1 has a unique immediate predecessor: note that the height n points in X_1 will be the points $x_{i,j}$ such that $\mathsf{ht}\, y_j = n - 1$ (—in general, $\mathsf{ht}\, x_{i,j} = \mathsf{ht}\, y_j + 1$); guaranteeing that in X_1 every immediate predecessor of a point of maximum height is maximal in the poset $X' = X_1 \setminus (X_1)_{\mathsf{ht}\, n}$. Secondly, we can obtain our original poset X from X_1 by joining maximal points of X_1: for each original point of maximum height $x_i \in X_{\mathsf{ht}\,n}$ we wish to join all duplicates $x_{i,j} \in X^+$ to a single point.

We now describe how we set up an induction to provide rings with the required spectra.

Theorem 3.6 ([17], Theorem (2.10)). *Given X a finite poset, equivalently a finite spectral space, there exists a ring L_X such that $\mathsf{Spec}\, L_X \cong X$.*

We build a ring L_X having spectrum isomorphic to a given finite poset X by induction on the dimension of X.

More precisely, our induction hypothesis will be that for any field k and for any finite poset X' with $\mathsf{dim}\, X' < n$, there is a k-algebra $L_{X'}$ such that $\mathsf{Spec}\, L_{X'} \cong X'$; an important point is that although at the induction step we will be trying to show that a k-algebra L_X such that $\mathsf{Spec}\, L_X \cong X$ exists for a given field k, our hypothesis allows us to assume the existence of a k'-algebra such that $L_{X'}$ such that $\mathsf{Spec}\, L_{X'} \cong X'$ for *any* field k'. In the base case where $\mathsf{dim}\, X = 0$ we have that X consists of finitely many (m, say) points, each of height 0. Then setting $L_X = k^m$ clearly we have that L_X is a k-algebra with $\mathsf{Spec}\, L_X = \mathsf{Spec}\, k^m \cong X$.

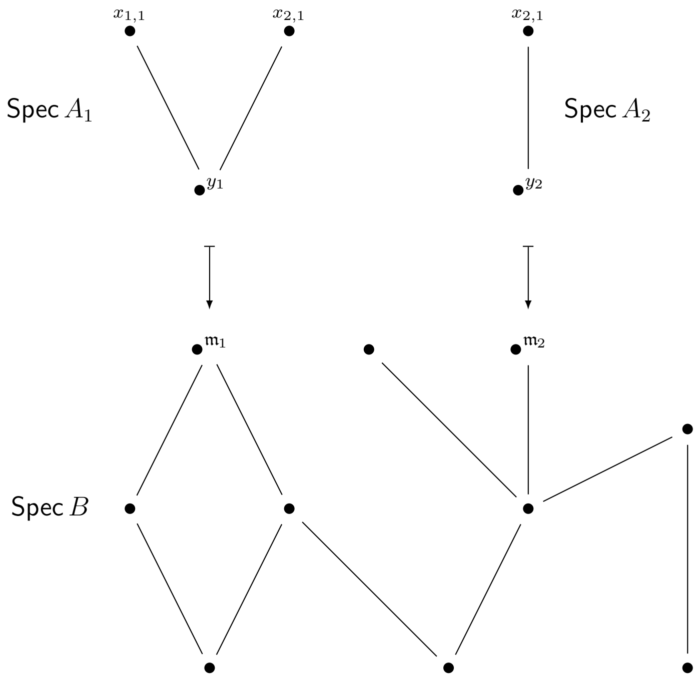
Now suppose X is a finite poset with $\mathsf{dim}\, X = n$. We assume that for any finite X' with $\mathsf{dim}\, X' < n$ and any field k' there is a k'-algebra $L_{X'}$ such that $\mathsf{Spec}\, L_{X'} \cong X'$. We must prove that for a given field k, there is a k-algebra L_X such that $\mathsf{Spec}\, L_X \cong X$. We fix k.

We first show

Lemma 3.7. *Under the present inductive assumptions, if X_1 is the poset obtained from X by the above-described procedure of "splitting" the points of maximum height of X into multiple copies, then there is a k-algebra L_{X_1} such that $\mathsf{Spec}\, L_{X_1} \cong X_1$.*

Let $y_1, \dots, y_{M_0}$ be the immediate predecessors of the points of maximum height of X_1; then $X_1 = (X_1)_{<n} \cup \bigsqcup_j y_j^{\uparrow}$ for $j = 1, \dots, M_0$. We describe domains $A_1, \dots, A_{M_0}$ having spectrum isomorphic to $y_j^{\uparrow}$ (—that is, the set $y_j^{\uparrow}$ plus a minimum point corresponding to the point y_j itself) for each $j = 1, \dots, M_0$. Take $T = \{t_{x_i} \mid x_i \in X_{\mathsf{ht}\, n}\}$ a set of indeterminates corresponding to each point $x_i \in X_{\mathsf{ht}\, n}$, and let $T_{y_j^{\uparrow}} = \{t_{x_i} \mid x_i \in y_j^{\uparrow}\}$. Then for each $j = 1, \dots, M_0$, working over the ring $k[T_{y_j^{\uparrow}}]$, let $S_j = \left(\bigcup \langle t_{x_i} \rangle\right)^{\mathsf{c}}$ for $t_{x_i} \in T_{y_j^{\uparrow}}$. Then set $A_j = k[T_{y_j^{\uparrow}}]_{S_j}$. Per the discussion at the beginning of this section, this is a 1-dimensional domain having height-1 prime ideals corresponding to the indeterminates t_{x_i}; that is, as a poset, it has prime ideal structure isomorphic to the sub-poset $y_j^{\uparrow}$.

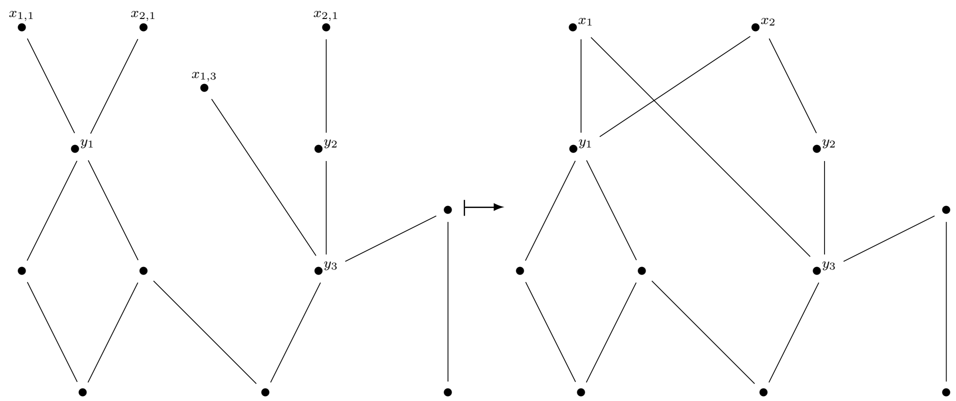
Let $K = k(T)$ so that $A_j \hookrightarrow K$ for all $j = 1, \dots, M_0$, let $A = A_1 \times \cdots \times A_{M_0}$ and let B be the K-algebra guaranteed by our induction hypothesis such that $\mathsf{Spec}\, B \cong X'$. Then for each $y_j \in X'$ there is a maximal ideal $\mathfrak{m}_j$ of B corresponding to y_j. Let $C = B/\mathfrak{m}_1 \times \cdots \times B/\mathfrak{m}_{M_0}$ and take the obvious inclusion map $\varphi : A \hookrightarrow C$ and the canonical surjective quotient map $\psi : B \twoheadrightarrow C$. We are in exactly the situation for applying the "gluing of domains over maximal ideals" construction as per page 79 onwards. Then for $D = A \times_C B = \bigcap_j \left(A_j + \mathfrak{m}_j\right)$ we have that $\mathsf{Spec}\, D$ has the structure of $\mathsf{Spec}\, B$, which is to say of X', with each point $\mathfrak{m}_j$ replaced by $\mathsf{Spec}\, A_j \cong y_j^{\uparrow}$; that is, $\mathsf{Spec}\, D$ has exactly the structure of the poset X_1. Furthermore, since $k \hookrightarrow A_j$ for every $j = 1, \dots, M_0$, we have that D is a k-algebra. We set $L_{X_1} = D$.



□

Lemma 3.8. *Given a k-algebra L_{X_1} such that* Spec $L_{X_1} \cong X_1$*, where X_1 is the poset obtained from X from the procedure described above, then there is a k-algebra L_X such that* Spec $L_X \cong X$*.*

We have that to each $x_{i,j} \in X_1$ there corresponds a unique maximal ideal $\mathfrak{n}_{i,j}$ of L_{X_1}. We wish to join those ideals $\mathfrak{n}_{i,j}$ as j ranges over the immediate predecessors y_j of x_i into a single point corresponding to x_i, for each $i = 1, \ldots, m$. Let $A' = k^m$ and let $C' = \prod_i^m \prod_j L_{X_1}/\mathfrak{n}_{i,j}$ over all such j as y_j is an immediate predecessor of x_i in X; then take $\varphi' : A' \hookrightarrow C'$ the "component-wise" diagonal map as described in our "joining of maximal ideals" section that takes the i-th component of A' diagonally into $\prod_j L_{X_1}/\mathfrak{n}_{i,j}$, and let $\psi' : L_{X_1} \twoheadrightarrow C$ be the canonical surjective quotient map. Then we are in exactly the situation for applying the "joining of maximal ideals" construction per page 83 onwards. Thus setting $L_X = L_{X_1} \times_{C'} A' = \bigcap_i \left(k + \bigcap_j \mathfrak{n}_{i,j}\right)$ we have that Spec L_X has the structure of Spec L_{X_1}, which is to say of X_1, with the exception that for each x_i, the points $x_{i,j}$ corresponding to the maximal ideals $\mathfrak{n}_{i,j}$ of L_{X_1} have been joined into a single point; thus $\mathsf{Spec}\, L_X \cong X$, and L_X is a k-algebra as required.



□

So, given our n-dimensional poset X, we form the poset X_1. From lemma 3.7, we are provided a k-algebra $L_{X'}$ such that $\mathsf{Spec}\, L_{X_1} \cong X_1$. Therefore from 3.8 we are provided a k-algebra L_X such that $\mathsf{Spec}\, L_X \cong X$, and the induction is complete. □

Application to 1-dimensional Noetherian spectral spaces

In this subsection we temporarily broaden our scope beyond finite spectral spaces, to investigate an aspect of the long-standing problem of characterising which posets can arise as the inclusion-ordering of prime ideals of a Noetherian ring; equivalently, as the specialisation order on the prime spectrum of a Noetherian ring. We noted in the first chapter that, whilst this problem has been solved completely in the case of Noetherian spectral *spaces*, it is also known that this characterisation is insufficient for spectra of Noetherian *rings*: whilst the spectrum of a Noetherian ring always gives rise to a specialisation poset satisfying the conditions of 1.27, we saw an example of such a poset which can never arise from the spectrum of a Noetherian ring. Note that in view of 1.26 the problem of finding a ring with spectrum homeomorphic to a given Noetherian spectral space X remains equivalent to finding one with isomorphic specialisation order; we wish to investigate to what extent can we represent such a space, or poset, by the spectrum of a Noetherian ring.

We use the tools we have just developed, to show,

Theorem 3.9. *If X is a Noetherian spectral space of dimension 1, then there is a Noetherian ring N_X such that $\mathsf{Spec}\, N_X \cong X$.*

We recall that if X is a Noetherian space, then under $\rightsquigarrow$, we have

(1) X has the ascending chain condition;

(2) X has finitely many minimal points $y_1, \ldots, y_n$ with $y_1^{\uparrow} \cup \cdots \cup y_n^{\uparrow} = X$; and

(3) Every pair $x, y \in X$ has finitely many minimal upper bounds in X.

We see that this means we can take a product of finitely many (Noetherian) domains whose spectra correspond to $y_j^{\uparrow}$ for each minimal point y_j, and that we need perform only finitely many 'joining' operations between height-1 points of these spaces to obtain the desired space X. However, the crucial concern will be to ensure that our operations preserve Noetherianeity for the rings constructed.

Let $X_{\mathsf{min}} = \{y_1, \ldots, y_n\}$ consist of the minimal elements of X and let $X' = \{x \in X \mid \mathsf{ht}\, x = 1\} = X \setminus X_{\mathsf{min}}$. If X is infinite take k an algebraically-closed field with $|k| = |X'|$ and identify elements of X' with elements of k under the implied bijection; otherwise take k to be any algebraically-closed field of characteristic 0 and identify each $x_1, x_2, x_3, \ldots, x_i, \ldots$ in X' with 1, 2, 3, ..., i, ... in k. We consider localisations of $k[T]$. Observe that as the maximal ideals of $k[T]$ are in bijection with k, to each point $x \in X'$ there corresponds a maximal ideal $\mathfrak{m}_x = \langle T - x \rangle$ of $k[T]$.

Then considering $y_j^{\uparrow}$ for $y_j \in X_{\mathsf{min}}$, let

$$S_j = k[T] \setminus \bigcup_{x \in y_j^{\uparrow}} \langle T - x \rangle$$

and set $B_j = k[T]_{S_j}$; this is a Noetherian ring such that $\mathsf{Spec}\, B_j \cong y_j^{\uparrow}$; that is, the points $x \in y_j^{\uparrow}$ correspond bijectively to the maximal ideals $\mathfrak{n}_x = \mathfrak{m}_x k[T]_{S_j}$ of $\mathsf{Spec}\, B_j$

Then we take

$$X_1' = \bigcup_{i \neq j} y_i^{\uparrow} \cap y_j^{\uparrow} = \{x \mid x \in y_i^{\uparrow} \cap y_j^{\uparrow} \text{ for some } i \neq j\}$$

to be the set of all elements of X lying above more than one minimal element. By property 3, this is finite; let $X_1' = \{x_1', \dots, x_m'\}$.

We set $B = B_1 \times \cdots \times B_n$; this is again a Noetherian ring and we have that $\mathsf{Spec}\, B$ has the structure of the disjoint union of the $y_j^{\uparrow}$; note that the $x_i' \in X_1'$ are exactly those points of X appearing more than once as maximal ideals in $\mathsf{Spec}\, B$; we label each instance of a x_i', equivalently, a $\mathfrak{n}_{x_i'}$, as $\mathfrak{n}_{x_{i,j}'}$ in the case that $x_i' \in y_j^{\uparrow}$.

Then we let $C = \prod_i^m \prod_j B/\mathfrak{n}_{x_{i,j}'}$ for such i, j as appear; note as X_1' is finite then C is a Noetherian ring and (—provided it is not empty) we get a surjective homomorphism $\psi : B \twoheadrightarrow C$. Let $C_{i,j}$ denote the component of C corresponding to $B/\mathfrak{n}_{x_{i,j}'}$ for each i, j. We see that the image under $\mathsf{Spec}\, \psi$ of the point corresponding to the zero ideal of $C_{i,j}$ (that is, the ideal consisting of all elements in $\prod_{i',j'} C_{i',j'}$ that are zero in the $C_{i,j}$ component) is the point $\mathfrak{n}_{x_{i,j}'}$ (—if X_1' is empty then $\mathsf{Spec}\, B \cong X$ and we are done already. We suppose not).

We now take $A_i = k[T]/\mathfrak{m}_{x_i'}$ for each $x_i' \in X_1'$, where $\mathfrak{m}_{x_i'}$ is the "original" maximal ideal of $k[T]$ corresponding to x_i'; we have $A_i \cong C_{i,j} = B/\mathfrak{n}_{x_{i,j}'}$ for every j such that $x_i' \in y_j^{\uparrow}$.

Then we take $A = \prod_i A_i$ so that again A is Noetherian, and we define φ :

$A \hookrightarrow B$ as in the "joining of maximal ideals" section to be the 'component-wise diagonal' map where each component a_i maps diagonally into $\prod_j B/\mathfrak{m}_{i,j}$. Then each point corresponding to the zero ideal of some $C_{i,j'}$ maps under $\mathsf{Spec}\,\varphi$ to the point $(A_1, \ldots, A_{i-1}, 0, A_{i+1}, \ldots, A_m)$ in $\mathsf{Spec}\,A$. Thus we have the following set-up.

$$\begin{array}{ccc} & & A \\ & & \big\downarrow \varphi \\ B & \xrightarrow{\ \psi\ } & C \end{array}$$

Importantly at this point we note that the map φ as so defined is a finite ring homomorphism; that is, we observe that the ring $C = \prod_i^m \prod_j C_{i,j}$ is finitely generated by the image of A and the standard basis $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 1)$.

Then, taking the fibre product $D = A \times_C B$ of A and B over C, we have that in $\mathsf{Spec}\,D$ each maximal ideal $\mathfrak{n}_{i,j}$ of $\mathsf{Spec}\,B$ corresponding to a multiplicated point x'_i has been joined into a single point, with exactly the relevant specialisations and no others being preserved; thus $\mathsf{Spec}\,D \cong X$; and furthermore, as A and B are Noetherian rings and φ is a finite homomorphism, we have from 3.5 that $D = A \times_C B$ is a Noetherian ring, and the result is proved. □

Note that we only used one of our two techniques: that of 'joining' maximal ideals. In fact, certainly in the set-up we have considered here, it is not possible to use the other technique, that of 'gluing' the spectrum of some domain above a maximal ideal, and still preserve Noetherianeity of the ensuing constructed ring. Observe, to set up the maps required for the application of our topological fibre sum construction, given a domain A we took $k = \mathcal{Q}(A)$ and a k-algebra B, so that we would obtain a surjective map $\psi : B \twoheadrightarrow B/\mathfrak{m}$ and an inclusion

$\varphi : A \hookrightarrow k \hookrightarrow B/\mathfrak{m}$ for $\mathfrak{m}$ the maximal ideal in question. But in such a case the map φ cannot be finite, since ([1](5.3)) a homomorphism is finite if and only if it is integral and of finite type; but ([1](5.7)) if k is a field and $A \hookrightarrow k$ is integral then A is a field. Thus any non-trivial gluing operation that utilises the inclusion of a domain into its field of fractions necessarily occasions a non-Noetherian ring; indeed, this method, christened the "$D + \mathfrak{m}$" construction (*viz.* the description of the ensuing ring, page 81), has been exploited [9] as a technique for providing examples of non-Noetherian rings!

In the context of Noetherian ring constructions, the paper [6] offers an interesting alternative approach to exploring what partial-order structures can arise within the spectrum of a Noetherian ring: there it is shown, by a similar methodology to the above, that any finite poset X can be order-preservingly embedded into the spectrum of a Noetherian ring A in a particularly strong way: the embedding $\varphi : X \hookrightarrow \mathsf{Spec}\, A$ is bijective on the sets of minimal points and on the sets of maximal points of the respective posets, and there exists a saturated chain of a given length d between $\varphi(x_1)$ and $\varphi(x_2)$ in $\mathsf{Spec}\, A$ if and only if a saturated chain of length d exists between x_1 and x_2 in X. The construction employed utilises exactly the "joining of maximal ideals" method as one of two elements of its procedure; the other being "adding a layer" to (the spectrum of) a ring B by extending to $B[X]$, then localising at those new maximal ideals that are to be preserved. It would be of great interest to explore further how much this combination of techniques allows one to control what possibilities arise as the prime spectrum of a Noetherian ring. In the literature addressing the question of which orderings can arise from the spectrum a Noetherian ring, the survey paper [23] provides a comprehensive overview of the state of the art in dimension 2; also relevant in this context are the papers [12], [24], [13], [26].

The method of Ershov

In the paper [7] Ershov gives his own construction of a ring R_X from a finite poset X such that the prime ideals of R_X have the structure of X under subset inclusion; equivalently, such that the (spectral) space given by X in the lower topology is homeomorphic to $\mathsf{Spec}\, R_X$.

We examine this construction here, in light of the considerations of the previous chapter.

We saw that the construction of Hochster begins with taking a subbasis of open sets $\mathcal{B}_X$, a collection of indeterminates $T_X = \{t_U\}_{U \in \mathcal{B}_X}$ indexed by $\mathcal{B}_X$, and rings R_x for $x \in X$ given by considering $T_x = \{t_U \in T_X \mid x \in U\}$, then setting $R_x = k[T_x]$ over our chosen field k.

Given a finite spectral space X, we know that the sets $x^{\downarrow}$ give an open subbasis for the (lower) topology on X. Then, following Hochster, we may take one indeterminate t_x for each point $x \in X$, and characteristic functions χ_x defined by

$$\chi_x(z) = \begin{cases} t_x & \text{if } z \in x^{\downarrow} \\ 0 & \text{if } z \notin x^{\downarrow}. \end{cases}$$

We then see that

$$\chi_x(z) = \begin{cases} t_x & \text{if } z \leqslant x \\ 0 & \text{if } z \nleqslant x, \end{cases}$$

and so the ring R_z corresponding to a point z is that generated by those indeterminates t_x for which $x \geqslant z$, i.e. $R_z = k[T_z]$ for $T_z = \{t_x \mid x \geqslant z\} = \{t_x \mid x \in z^{\uparrow}\}$. We will adopt the shorthand of writing $T_{z^{\uparrow}}$ for this collection of indeterminates. Note as before that if $x \rightsquigarrow y$ then $T_{y^{\uparrow}} \subseteq T_{x^{\uparrow}}$.

Thinking of the poset X as a spectral space, we define a discrete valuation ring corresponding to each specialisation $y \rightsquigarrow z$ in X, which we will write as

$V_{y \rightsquigarrow z}$.

We utilise the sets of indeterminates $T_{z^{\uparrow}}$ etc. as defined above. Given a specialisation $y \rightsquigarrow z$, firstly, in the case that $z = y$ we define the ring $V_{y \rightsquigarrow y}$ to be the improper valuation ring

$$V_{y \rightsquigarrow y} = k(T_{y^{\uparrow}}) = \mathcal{Q}(R_y).$$

In the case that $z \neq y$, so that $y \in y^{\uparrow} \setminus z^{\uparrow}$, then, writing in the obvious way $T_{y^{\mapsup}} = T_{y^{\uparrow}} \setminus \{t_y\}$, we define a set $\frac{T_{y^{\mapsup}} \setminus T_{z^{\uparrow}}}{t_y} = \{\frac{t_{x'}}{t_y} \mid t_{x'} \in T_{y^{\mapsup}} \setminus T_{z^{\uparrow}}\} = \{\frac{t_{x'}}{t_y} \mid x' \in y^{\mapsup} \setminus z^{\uparrow}\}$. Note that this set may well be empty. We then define the discrete valuation ring $V_{y \rightsquigarrow z}$ to be

$$V_{y \rightsquigarrow z} = k(T_{z^{\uparrow}}, \frac{T_{y^{\mapsup}} \setminus T_{z^{\uparrow}}}{t_y})[t_y]_{\langle t_y \rangle}.$$

That is, writing $K_{y \rightsquigarrow z}$ for the field $k(T_{z^{\uparrow}}, \frac{T_{y^{\mapsup}} \setminus T_{z^{\uparrow}}}{t_y})$, then $V_{y \rightsquigarrow z}$ is the localisation of the polynomial ring $K_{y \rightsquigarrow z}[t_y]$ at the ideal generated by t_y.

This ring has unique maximal ideal $\mathfrak{m}_{y \rightsquigarrow z} = t_y V_{y \rightsquigarrow z}$; thus we have verified (per 1.46) it is a discrete valuation ring. Corresponding to each $V_{y \rightsquigarrow z}$ we then have the canonical quotient map $\pi_{y \rightsquigarrow z} : V_{y \rightsquigarrow z} \to V_{y \rightsquigarrow z}/\mathfrak{m}_{y \rightsquigarrow z} = K_{y \rightsquigarrow z}$ (in the case of the improper valuation rings $V_{y \rightsquigarrow y}$ we take the map $\pi_{y \rightsquigarrow y}$ to be the identity).

We then define rings corresponding to $y^{\uparrow}$ for each $y \in X$, as follows:

$$R_{y^{\uparrow}} = \bigcap_{z \leftsquigarrow y} V_{y \rightsquigarrow z}$$

(it is worth noting in passing, though this will not become relevant until later in our discussions, that the ring $R_{y^{\uparrow}}$ is an intersection of finitely many discrete valuation rings).

We need to observe some important facts about how the rings of the form $R_{x^{\uparrow}}$ and those of the form $V_{x \rightsquigarrow y}$ relate to each other. Given some $x \in X$ we have that $R_{x^\uparrow} \subseteq V_{x \rightsquigarrow y}$ for each $y \geqslant x$, so that given $r \in R_{x^\uparrow}$, then $\pi_{x \rightsquigarrow y}(r)$ is defined for every $y \geqslant x$. But furthermore, fixing some $y' \geqslant x$, we find that $V_{y' \rightsquigarrow z} \subseteq K_{x \rightsquigarrow y'}$ for any $z \geqslant y'$ so that $R_{y'^\uparrow} \subseteq K_{x \rightsquigarrow y'}$: since

$$V_{y' \rightsquigarrow z} = k(T_{z^\uparrow}, \frac{T_{y'^\Uparrow} \setminus T_{z^\uparrow}}{t_{y'}})[t_{y'}]_{\langle t_{y'} \rangle} \subseteq k(T_{y'^\uparrow}) \subseteq k(T_{y'^\uparrow}, \frac{T_{x^\Uparrow} \setminus T_{y'^\uparrow}}{t_x}) = K_{x \rightsquigarrow y'}.$$

Then for $r \in R_{x^\uparrow}$ it makes sense to compare elements $s \in R_{y'^\uparrow}$ with the element $\pi_{x \rightsquigarrow y'}(r) \in K_{x \rightsquigarrow y'}$, as indeed we will do in the next definition.

Then finally we define the 'Ershov ring' $R_X \subseteq \prod_{x \in X} R_{x^\uparrow}$ to be the sub-ring of functions on X taking values at x in the ring $R_{x^\uparrow}$ defined by

$$R_X = \{f \in \prod_{x \in X} R_{x^\uparrow} \mid \forall x \in X, y \geqslant x \text{ then } f_y = \pi_{x \rightsquigarrow y}(f_x)\}.$$

Remark. In Ershov's paper, his construction is phrased very generally in terms of families of subsets of a finite poset X. In practice, however, the subsets considered are exactly those of the form $x^\uparrow$ for elements $x \in X$. For comparison purposes, we now relate the notions and notation of Ershov's paper with the objects we have defined above.

Let $\mathcal{W}$ be the collection $\{x^\uparrow\}_{x \in X}$ of subsets of X. Ershov's construction begins by defining discrete valuation rings R^Y_Z corresponding to subsets $Z \subseteq Y$ in $\mathcal{W}$; that is, to subsets $z^\uparrow \subseteq y^\uparrow$ for $z, y \in X$; equally, to elements y, z such that $y \rightsquigarrow z$ in X; Ershov's ring R^Y_Z is the ring we denote by $R_{y \rightsquigarrow z}$.

Given a set $Y \in \mathcal{W}$ (that is, where $Y = y^\uparrow$ for some $y \in X$), Ershov defines $\downarrow Y$ to be the subfamily $\{Z \in \mathcal{W} \mid Z \subseteq Y\}$; we see this is the collection $\{z^\uparrow \in \mathcal{W} \mid z^\uparrow \subseteq y^\uparrow\} = \{z^\uparrow \mid z \geqslant y\} = \{z^\uparrow \mid z \in y^\uparrow\}$. The ring Ershov denotes as R^Y, defined in reference to the notion $\downarrow Y$, is our ring $R_{y^\uparrow}$.

Given a subcollection $\mathcal{W}_0 \subseteq \mathcal{W}$, he defines $\downarrow \mathcal{W}_0$ to be $\bigcup_{Y \in \mathcal{W}_0} \downarrow Y$. We have a correspondence $Y_i = y_i^\uparrow$ for each $Y_i \in \mathcal{W}_0$; let us write $X_0 = \{y_i \mid y_i^\uparrow = Y_i \in \mathcal{W}_0\}$. Then $\downarrow \mathcal{W}_0$ is the collection of $z^\uparrow$ where $z^\uparrow$ is contained in some $\downarrow Y_i$, that is, where $z \geqslant y_i$, for some i. Then $\downarrow \mathcal{W}_0 = \{z^\uparrow \mid z \geqslant y_i \text{ for some } y_i \in X_0\} =$

$\{z^\uparrow \mid z \in X_0^\uparrow\}$. A subcollection $\mathcal{W}_0 \subseteq \mathcal{W}$ is called lower if $\downarrow \mathcal{W}_0 = \mathcal{W}_0$; we see that the collection $\mathcal{W}_0$ is lower exactly when $X_0 = X_0^\uparrow$. Then, on lower subfamilies $\mathcal{W}_0 \subseteq \mathcal{W}$, equivalently, on up-closed subsets $X_0 \subseteq X$, Ershov defines the ring $R_{\mathcal{W}_0}$; in the case that $\mathcal{W}_0 = \mathcal{W}$ then $R_{\mathcal{W}}$ is exactly our "Ershov ring" R_X. Given an up-closed subset $X_0 \subseteq X$, then the ring R_{X_0}, being the Ershov ring defined on the sub-poset X_0 with the inherited order, corresponds to the ring $R_{\mathcal{W}_0}$ in Ershov's notation.

As R_X is a ring of functions on X taking values in domains $R_{x^\uparrow} \subseteq \mathcal{Q}(R_x)$, we obtain a mapping $X \to \mathsf{Spec}\, R_X$ by $x \mapsto \mathsf{ker}(\mathsf{ev}_x)$ as before. In fact ([7], Theorem 1), when X is finite, we have $\mathsf{Spec}\, R_X \cong X$. We proceed towards a proof of this fact, noting other important properties of this construction along the way.

We note,

3.10. *If the poset X has a unique minimal element x_0, then the ring R_X can be identified with a certain subring of the ring $R_{x_0^\uparrow}$:*

since we have for all $f \in R_X$ that $f_y = \pi_{x_0 \rightsquigarrow y}(f_{x_0})$ then $R_X \cong \{f_{x_0} \mid f \in R_X\} = \{r \in R_{x_0^\uparrow} \mid \pi_{x_0 \rightsquigarrow y}(r) \in R_{y^\uparrow}$ for all $y \in X$ \& $\pi_{y \rightsquigarrow z}(\pi_{x_0 \rightsquigarrow y}(r)) = \pi_{y' \rightsquigarrow z}(\pi_{x_0 \rightsquigarrow y'}(r))$ for all $y \rightsquigarrow z \leftsquigarrow y'$ in $X\}$. In general, for $x_1, \ldots, x_n$ the minimal elements of X then R_X can be identified with a subring of $R_{x_1^\uparrow} \times \cdots \times R_{x_n^\uparrow}$, namely $R_X \cong \{(f_{x_1}, \ldots, f_{x_n}) \mid f \in R_X\}$, which can be characterised by a similar condition as above, but further requiring that $\pi_{x_i \rightsquigarrow y}(r_i) = \pi_{x_j \rightsquigarrow y}(r_j)$ whenever we have $x_i \rightsquigarrow y \leftsquigarrow x_j$.

We now observe some relationships between the ring R_X we have constructed and certain rings R_Y corresponding to subsets Y of X.

Note that given any subset $Y \subseteq X$, we may define a ring of functions on Y

from R_X simply by projecting from $\prod_{x\in X} R_{x^\uparrow}$ to $\prod_{x\in Y} R_{x^\uparrow}$, that is, let

$$R_X{\restriction}_Y = \{\prod_{x\in Y} f_x \mid \text{ for } f \in R_X\}.$$

In fact

Theorem 3.11 ([7], Proposition 6). *If Y is up-closed in X then we have $R_X{\restriction}_Y = R_Y$ for R_Y the Ershov ring defined on the sub-poset Y with the inherited order.*

Let us assume for the purpose of an induction that $|X| = n$ and all posets X' with $|X'| < n$ have the property that for every upper-closed subset $Y' \subseteq X'$ then $R_{X'}{\restriction}_{Y'} = R_{Y'}$. In the base case where $|X| = 1$ we have the only upper-closed subset of X is X itself, and the restriction map $R_X \to R_X{\restriction}_X$ is identity. Continuing the induction, we consider two possible situations: Where X has a unique minimal element x_0, and where X has more than one minimal element.

Suppose X has a unique minimal element x_0, and let $X' = X\setminus\{x_0\}$. We note (—provided that there is some $y \neq x_0 \in X$) that $R_{x_0^\upharpoonright}$ is the intersection of the finitely many discrete valuation rings $V_{x_0\rightsquigarrow y}$ for $y \in x_0^\upharpoonright = X'$. We claim these are inclusion-incomparable; that is, for $y \neq z$ then we have that $V_{x_0\rightsquigarrow y} \not\subseteq V_{x_0\rightsquigarrow z}$ and vice-versa. Since, if $y \neq z$ suppose without loss of generality that $y \not\leq z$. Then $t_z \notin T_{y^\uparrow}$; and so $t_z \in T_{x_0^\upharpoonright} \setminus T_{y^\uparrow}$. We compare

$$V_{x_0\rightsquigarrow y} = k(T_{y^\uparrow}, \frac{T_{x_0^\upharpoonright} \setminus T_{y^\uparrow}}{t_{x_0}})[t_{x_0}]_{\langle t_{x_0}\rangle};$$
$$V_{x_0\rightsquigarrow z} = k(T_{z^\uparrow}, \frac{T_{x_0^\upharpoonright} \setminus T_{z^\uparrow}}{t_{x_0}})[t_{x_0}]_{\langle t_{x_0}\rangle},$$

seeing that $\frac{1}{t_z} \in V_{x_0\rightsquigarrow z}$, $\frac{1}{t_z} \notin V_{x_0\rightsquigarrow y}$; $\frac{t_z}{t_{x_0}} \in V_{x_0\rightsquigarrow y}$, $\frac{t_z}{t_{x_0}} \notin V_{x_0\rightsquigarrow z}$.

Then from the discussion at the start of this chapter, each maximal ideal $\mathfrak{m}_{x_0\rightsquigarrow y}$ of such valuation rings corresponds to a unique maximal ideal $\mathfrak{n}_y = (\mathfrak{m}_{x_0\rightsquigarrow y} \cap R_{x_0^\upharpoonright})$ of $R_{x_0^\upharpoonright}$, and furthermore the map $\pi_{x_0\rightsquigarrow y}{\restriction}_{R_{x_0^\upharpoonright}} : R_{x_0^\upharpoonright} \twoheadrightarrow K_{x_0\rightsquigarrow y}$

is surjective, since it is isomorphic to the quotient map $R_{x_0^\uparrow} \twoheadrightarrow R_{x_0^\uparrow}/\mathfrak{n}_y \cong V_{x_0\rightsquigarrow y}/\mathfrak{m}_{x_0\rightsquigarrow y} = K_{x_0\rightsquigarrow y}$. Then furthermore by the Chinese remainder theorem we get a surjection $\pi_0 : R_{x_0^\uparrow} \twoheadrightarrow \prod_{y\in X'} K_{x_0\rightsquigarrow y}$. Note that as we have $R_{y^\uparrow} \subseteq K_{x_0\rightsquigarrow y}$ for all $y \in X'$ then $R_{X'} \subseteq \prod_{y\in X'} K_{x_0\rightsquigarrow y}$. Let $R_0 = \{r \in R_{x_0^\uparrow} \mid \pi_0(r) \in R_{X'}\} = \pi_0^{-1}(R_{X'})$.

Then $R_X \cong R_0$, as given $f \in R_X$ then $f\restriction_{X'} \in R_{X'}$ and $\pi_0(f_{x_0}) = f\restriction_{X'}$ (as $\pi_0(f_{x_0})_y = \pi_{x_0\rightsquigarrow y}(f_{x_0}) = f_y$) so that $f_{x_0} \in R_0$. Conversely, given $r \in R_0$ then mapping r to the element f_r' of $\prod_{x\in X} R_{x^\uparrow}$ defined by

$$(f_r)_y = \begin{cases} r & y = x_0, \\ \pi_{x_0\rightsquigarrow y}(r) & y \in X' \end{cases}$$

we have that f_r is in R_X, since $\pi_{x_0\rightsquigarrow y}(r)$ is in $R_{X'}$ for all $y \neq x_0$; therefore the condition of R_X that $\pi_{y\rightsquigarrow z}((f_r)_y) = (f_r)_z$ is satisfied either trivially in the case $y = x_0$ or by the condition on X' in the case $y, z \in X'$; furthermore clearly $(f_r)_{x_0} = r$, so that $R_X \cong R_0$ as claimed. Then identifying R_X with R_0 we have that $R_X\restriction_{X'} = \pi_0(R_X) = R_{X'}$ as required.

Alternatively, suppose X has more than one minimal element; let x_1 be some minimal element of X and set $X_1 = x_1^\uparrow$, considered as a sub-poset of X. Since there is some minimal element $x_2 \neq x_1$ in X we have $X_1 \subsetneq X$ so that $|X_1| < n$. Now let $X_2 = (X \setminus x_1)^\uparrow$. As x_1 is minimal then $x_1 \notin X_2$, so that again $X_2 \subsetneq X$ and $|X_2| < n$; note that $X = X_1 \cup X_2$. Finally let $X^* = X_1 \cap X_2$. Let $R' = R_{X_1} \times_{R_{X^*}} R_{X_2}$ be the fibre product of R_{X_1} and R_{X_2} over R_{X^*} as in the following

$$\begin{array}{ccc} R' = R_{X_1} \times_{R_{X^*}} R_{X_2} & \twoheadrightarrow & R_{X_1} \\ \downarrow & & \downarrow \\ R_{X_2} & \twoheadrightarrow & R_{X^*}, \end{array}$$

where the maps $R_{X_1} \twoheadrightarrow R_{X^*}$, $R_{X_2} \twoheadrightarrow R_{X^*}$ are surjective by hypothesis (since

X^* is an up-closed subset of both X_1 and X_2 of size $< n$), thereby inducing surjectivity in the projections $R_{X_1} \times_{R_{X^*}} R_{X_2} \twoheadrightarrow R_{X_1}$, $R_{X_1} \times_{R_{X^*}} R_{X_2} \twoheadrightarrow R_{X_2}$,

We show that $R_X \cong R'$.

A general element of R' is of the form $r = (f_1, f_2)$ for $f_1 \in R_{X_1}$, $f_2 \in R_{X_2}$ such that $f_1{\restriction}_{X^*} = f_2{\restriction}_{X^*}$. As $X_1 \cup X_2 = X$, we may define an element $f = \varphi(r) \in \prod_{x \in X} R_{x\uparrow}$ by

$$f_x = \begin{cases} (f_1)_x & x \in X_1 \\ (f_2)_x & x \in X_2; \end{cases}$$

note this is well defined since f_1 and f_2 agree on their intersection. In fact we have $f \in R_X$, since every condition $\pi_{x \rightsquigarrow y}(f_x) = f_y$ appears in one of either X_1 or X_2 by their upwards-closedness, and so the condition $\pi_{x \rightsquigarrow y}(f_x) = f_y$ is satisfied for f_x as so defined as it is satisfied by whichever of $(f_1)_x \in R_{X_1}$ or $(f_2)_x \in R_{X_2}$ applies at x. Evidently the map φ is injective. Conversely given $f' \in R_X$ then we have $(f'{\restriction}_{X_1}, f'{\restriction}_{X_2}) \in R'$ since obviously these restrictions agree on their further restriction to X^*; and clearly $\varphi(f'{\restriction}_{X_1}, f'{\restriction}_{X_2}) = f'$, hence $R_X \cong R'$ as required; and so the restriction maps $R_X \twoheadrightarrow R_{X_1}$, $R_X \twoheadrightarrow R_{X_2}$ are surjective as in the above diagram. Finally given any (proper) up-closed subset Y of X clearly either $Y \subseteq X_1$ or $Y \subseteq X_2$ (according as whether $x_1 \in Y$ or not), and so we get a surjective restriction map $R_X \twoheadrightarrow R_Y$ via $R_X \twoheadrightarrow R_{X_i} \twoheadrightarrow R_Y$ by the result just proved and the induction hypothesis applied to the appropriate R_{X_i} for $i = 1$ or 2. □

Thus far we have examined relationships between rings defined on subsets of a finite poset X without any comment as to their prime ideal structure. We rectify this forthwith.

Theorem 3.12 ([7], Theorem 1). *Given X a finite poset, equivalently, a finite spectral space, then* $\mathsf{Spec}\, R_X \cong X$.

We note that the ring R_X is a ring of functions on a finite spectral space X taking values in a field $k(T_X)$; then the range $\mathsf{ev}_x(R_X)$ of a point $x \in X$, being just projection onto the x-coordinate of $R_X \subseteq \prod_{x\in X} R_{x^\uparrow}$ is a domain and so we have a mapping $x \mapsto \ker(\mathsf{ev}_x)$ of X into $\operatorname{Spec} R_X$; it will be seen that this gives the required homeomorphism.

We use an induction identical in structure to that of the preceding proposition.

In the base case where X consists of a single element $X = \{x_0\}$ we have the following rings

$$V_{x_0 \rightsquigarrow x_0} = k(t_{x_0});$$

$$R_{x_0^\uparrow} = \bigcap_{z \leftsquigarrow x_0} V_{x_0 \rightsquigarrow z} = V_{x_0 \rightsquigarrow x_0} = k(t_{x_0});$$

$$R_X = \{f \in \prod_{x \in X} R_{x^\uparrow} \mid \forall x \in X, y \geqslant x \text{ then } f_y = \pi_{x \rightsquigarrow y}(f_x)\} = \prod_{x\in X} R_{x^\uparrow} = R_{x_0^\uparrow} = k(t_{x_0})$$

as the condition on R_X is trivial, so that $\ker(\mathsf{ev}_{x_0}) = \{0\} = \operatorname{Spec} R_X$ and so $\operatorname{Spec} R_X \cong X$ as required.

In the inductive step we consider two possibilities: that X has a unique minimal element, or that X has more than one minimal element.

In the case that X has a unique minimal element x_0 let $X' = X \setminus \{x_0\}$ as before; by induction we have that $x \mapsto \ker(\mathsf{ev}_x)$ gives a homeomorphism $\operatorname{Spec} R_{X'} \cong X'$, and we have a surjective map $\pi_0 : R_X \twoheadrightarrow R_{X'}$.

We claim that for every non-zero prime $\mathfrak{p}$ of R_X we have $\ker \pi_0 \subseteq \mathfrak{p}$. Let $a \in \ker \pi_0$ and take a non-zero $b \in \mathfrak{p}$. Let $v_{x_0 \rightsquigarrow y}$ be the discrete valuation associated to the valuation ring $V_{x_0 \rightsquigarrow y}$, and let $N_b = \max(\{v_{x_0\rightsquigarrow y}(b)\}_{y \in X'})$;. We see that $a \in \ker \pi_0 \implies a \in \mathfrak{m}_{x_0 \rightsquigarrow y}$ for all y, so that $v_{x_0\rightsquigarrow y}(a) > 0$ for all y. Then $v_{x_0\rightsquigarrow y}(\frac{a^{N_b+1}}{b}) > 0$ for all y so that $\frac{a^{N_b+1}}{b} \in \ker \pi_0 \subseteq R_X$. But then $a^{N_b+1} = \left(\frac{a^{N_b+1}}{b}\right) \cdot b \in \mathfrak{p}$, so that $a \in \mathfrak{p}$ as required. Then the non-zero prime ideals of R_X are in order-preserving bijection with the prime ideals of $R_{X'}$, which by assumption are isomorphic to the sub-poset $X' = X \setminus \{x_0\}$ (and clearly for all $f \in R_X$ and $x' \in X'$ we have $f_{x'} = 0 \iff (\pi_0(f))_{x'} = 0$ so that $\mathsf{ker}(\mathsf{ev}_{x'})$ in R_X corresponds exactly with $\mathsf{ker}(\mathsf{ev}_{x'})$ in $R_{X'}$); thus $\mathsf{Spec}\, R_X$ has the structure of $\mathsf{Spec}\, R_{X'}$ plus the ideal $\langle 0 \rangle$. Verifying that $\mathsf{ker}(\mathsf{ev}_{x_0}) = \{0\}$, since $f_{x_0} = 0 \implies f_{x'} = 0$ for all $x' \in X$ (as $f_{x'} = \pi_{x_0 \rightsquigarrow x'}(f_{x_0})$), then we have that $x \mapsto \mathsf{ker}(\mathsf{ev}_x)$ gives a homeomorphism $\mathsf{Spec}\, R_X \cong X$ as required.

Alternatively suppose X has more than one minimal element. Again, let x_1 be some minimal element of X and take $X_1 = x_1^{\uparrow}$, $X_2 = (X \setminus x_1)^{\uparrow}$, $X^* = X_1 \cap X_2$ as before. We have $|X_1|, |X_2|, |X^*| < n$ and so an induction applies; we have that $\mathsf{Spec}\, R_{X_1} \cong X_1$, $\mathsf{Spec}\, R_{X_2} \cong X_2$ and $\mathsf{Spec}\, R_{X^*} \cong X^*$. Note that as X_1, X_2, X^* are up-closed, they are closed sets in the lower topology on X; furthermore X^* is a closed subspace of both X_1 and X_2. We may then easily observe, in light of the discussion at the start of this chapter, that $X \cong X_1 \sqcup_{X^*} X_2$ (perhaps this is most easily observed by considering that any specialisation $x \rightsquigarrow y$ in X must lie within X_1 or X_2, recalling that the topology on X is uniquely determined by its specialisations). But then we are in the set-up of theorem 3.4 as follows:

$$\begin{array}{ccc} & & R_{X_1} \\ & & \downarrow \\ R_{X_2} & \twoheadrightarrow & R_{X^*} \end{array} \quad \overset{\mathsf{Spec}}{\Longrightarrow} \quad \begin{array}{ccc} & & X_1 \\ & & \uparrow \\ X_2 & \hookleftarrow & X^*, \end{array}$$

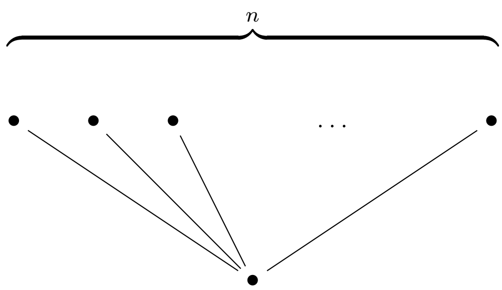
so that $\mathsf{Spec}(R_{X_1} \times_{R_{X^*}} R_{X_2}) \cong X_1 \sqcup_{X^*} X_2 \cong X$. But we have from the proof of proposition 3.11 that $R_X \cong R_{X_1} \times_{R_{X^*}} R_{X_2}$, so that $\mathsf{Spec}\, R_X \cong X$ as required. Again $f_{x'} = 0 \iff (f{\restriction}_{X_i})_{x'} = 0$ for the appropriate X_i containing x'; so that $x \mapsto \mathsf{ker}(\mathsf{ev}_x)$ again gives the homeomorphism. □

4. Comparison of constructions

Comparison of finite constructions

We now proceed to compare aspects of the constructions considered in the preceding chapter, and of the rings L_X and R_X we defined from a finite spectral space X.

The fundamental “building blocks” of the construction of L_X were the domains A_j defined as the localisation of a polynomial ring in finitely many indeterminates at the (union of the) principal ideals generated by each indeterminate, thus giving a ring with prime spectrum having the following structure



for n the number of indeterminates of our polynomial ring. That is, to obtain a ring A having prime ideal structure as in the above diagram for a certain n, we begin by working over the ring $k[t_1, \ldots, t_n]$. We then take the set $S = \big(\langle t_1 \rangle \cup \cdots \cup \langle t_n \rangle\big)^{\mathsf{c}}$ and define $A = k[t_1, \ldots, t_n]_S$.

On the other hand, the basic objects in the construction of R_X were the rings $R_{x^{\uparrow}}$, being an intersection of the finitely many discrete valuation rings $V_{x \rightsquigarrow y}$ where $y \in x^{\uparrow}$. We did not have cause to examine the prime ideal structure of these rings in our exposition, utilising only the properties of the associated quotient maps $\pi_{x \rightsquigarrow y}$. However, we know from the observation following 3.3 that (—at least in the case that there is some $y \neq x$ in $x^{\upfishtail}$) such a ring is a 1-dimensional domain with maximal ideals corresponding bijectively to the maximal ideals $\mathfrak{m}_{x\rightsquigarrow y}$ of each $V_{x\rightsquigarrow y}$ for $y \in x^{\uparrow}$.

Conversely, the rings of the form of A as above can be viewed as an intersection of discrete valuation rings. Observe if we wished to represent the poset

by a ring of such form, the ring obtained would $A_0 = k[t]_{\langle t\rangle}$. As A_0 is a local Noetherian domain with principal maximal ideal tA_0, we have that it is a discrete valuation ring of the field $\mathcal{Q}(A_0) = k(t)$. Suppose we have n such rings chosen so that they are each a valuation ring of the same field; that is, we choose

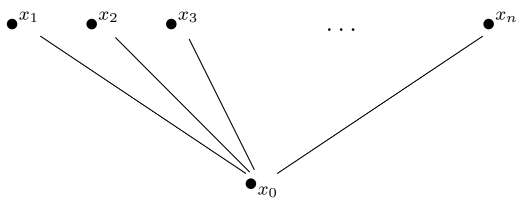
$$
\begin{aligned}
A_1 &= k(t_2, \dots, t_n)[t_1]_{\langle t_1\rangle}, \\
A_2 &= k(t_1, t_3, \dots, t_n)[t_2]_{\langle t_2\rangle}, \\
&\vdots \\
A_i &= k(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)[t_i]_{\langle t_i\rangle}, \\
&\vdots \\
A_n &= k(t_1, \dots, t_{n-1})[t_n]_{\langle t_n\rangle},
\end{aligned}
$$

so that $\mathcal{Q}(A_1) = \cdots = \mathcal{Q}(A_n) = k(t_1, \dots, t_n)$. Then for each i we have that $\dfrac{1}{t_i} \notin A_i$, $\dfrac{1}{t_i} \in A_j$ for all $j \neq i$, so that the rings are inclusion-incomparable, and it is easy to see that the intersection

$$\bigcap_i A_i = k(t_2, \dots, t_n)[t_1]_{\langle t_1\rangle} \cap \cdots \cap k(t_1, \dots, t_{n-1})[t_n]_{\langle t_n\rangle}$$

is the set of $\frac{f}{g} \in k(t_1, \dots, t_n)$ such that g is not in any of the ideals $\langle t_1\rangle, \dots, \langle t_n\rangle$; that is, $\bigcap_i A_i$ is the ring $k[t_1, \dots, t_n]_{(\langle t_1\rangle \cup \cdots \cup \langle t_n\rangle)^{\mathsf{c}}} = A$.

We find however that the ring associated by the construction of Ershov to a finite irreducible 1-dimensional spectral space X (that is, a space X having specialisation structure as in the diagram at the start of this section) appears significantly different to the ring just described. Let us label the points of X as follows:



Then for all $x_i \in X$ the ring $V_{x_i \rightsquigarrow x_i} = k(t_{x_i})$, and for $i = 1, \dots, n$, we have

$$V_{x_0 \rightsquigarrow x_i} = k(t_{x_i}, \frac{T_{j \neq i}}{t_{x_0}})[t_{x_0}]_{\langle t_{x_0} \rangle}$$

where $\dfrac{T_{j \neq i}}{t_{x_0}} = \{\frac{t_{x_j}}{t_{x_0}} \mid j \geqslant 1, \, j \neq i\}$.

Then for each $i \geqslant 1$ we have that $R_{x_i^{\uparrow}} = V_{x_i \rightsquigarrow x_i} = k(t_{x_i})$, and

$$R_{x_0^{\uparrow}} = \bigcap_i V_{x_0 \rightsquigarrow x_i} = \bigcap_i \big(k(t_{x_i}, \frac{T_{j \neq i}}{t_{x_0}})[t_{x_0}]_{\langle t_{x_0} \rangle} \big);$$

then

$$\begin{aligned} R_X &= \{ f \in \prod_{x \in X} R_{x^{\uparrow}} \mid \forall x \in X, y \geqslant x \text{ then } f_y = \pi_{x \rightsquigarrow y}(f_x) \} \\ &= \{ f \in \prod_{x \in X} R_{x^{\uparrow}} \mid f_{x_i} = \pi_{x_0 \rightsquigarrow x_i}(f_{x_0}) \text{ for all } i \geqslant 1 \} \\ &\cong \{ r \in R_{x_0^{\uparrow}} \mid \pi_{x_0 \rightsquigarrow x_i}(r) \in R_{x_i^{\uparrow}} = k(t_{x_i}) \text{ for all } i \geqslant 1 \}. \end{aligned}$$

Note that $R_{x_0^{\uparrow}}$ is the ring $\bigcap_i \big(k(t_{x_i}, \frac{T_{j \neq i}}{t_{x_0}})[t_{x_0}]_{\langle t_{x_0} \rangle} \big)$ and we have the quotient maps $\pi_{x_0 \rightsquigarrow x_i} : k(t_{x_i}, \frac{T_{j \neq i}}{t_{x_0}})[t_{x_0}]_{\langle t_{x_0} \rangle} \twoheadrightarrow K_{x_0 \rightsquigarrow x_i} = k(t_{x_i}, \frac{T_{j \neq i}}{t_{x_0}})$; it is not especially clear what restriction the condition on R_X imposes on members of $R_{x_0^{\uparrow}}$ but it seems plausible that it may be non-trivial. It is even not at all clear to this author what the set $R_{x_0^{\uparrow}}$ itself actually is.

However, recall that our initial step in the definition of our ring R_X was choosing a suitable subbasis $\mathcal{B}_X$ for the topology on X; as the topology is the lower topology, we chose the basis $\mathcal{B}_X = \{x^{\downarrow} \mid x \in X\}$, being, in general, a basis for the fine lower topology on a poset. Alternatively we could have chosen $\overline{\mathcal{B}}_X = \{(x^{\uparrow})^{\mathsf{c}} \mid x \in X\}$, a subbasis for the coarse lower topology, as the fine and coarse lower topologies on a finite poset coincide. Let $\overline{T}_X = \{\bar{t}_x \mid x \in X\}$ be the corresponding set of indeterminates. We would then obtain the characteristic functions

$$\overline{\chi}_x(z) = \begin{cases} \bar{t}_x & \text{if } z \in (x^{\uparrow})^{\mathsf{c}} \\ 0 & \text{if } z \in x^{\uparrow}. \end{cases}$$

We then see that

$$\overline{\chi}_x(z) = \begin{cases} \bar{t}_x & \text{if } x \nleqslant z \\ 0 & \text{if } x \leqslant z, \end{cases}$$

and so in this new set-up the ring $\overline{R}_z$ corresponding to a point z is that generated by those indeterminates $\bar{t}_x$ for which $x \nleqslant z$, i.e. $\overline{R}_z = k[\overline{T}_z]$ for $\overline{T}_z = \{\bar{t}_x \mid x \nleqslant z\} = \{\bar{t}_x \mid x \in (z^{\downarrow})^{\mathsf{c}}\}$.

Note that we still maintain the property that if $x \rightsquigarrow y$ then $\overline{T}_y \subseteq \overline{T}_x$.

Now let us follow the remainder of the Ershov construction on the poset X as given above, in this different setting. We first find that for each x_i, the set $\overline{T}_{x_i} = \{\bar{t}_{x_j} \mid x_j \nleqslant x_i\} = \{\bar{t}_{x_j} \mid \text{ for } j \text{ such that } i \neq j \neq 0\}$; we will label this set $\overline{T}_{x_i} = \overline{T}_{j\neq i}$.

Then, associated to each trivial specialisation $x_i \rightsquigarrow x_i$ is the ring $\overline{V}_{x_i \rightsquigarrow x_i} = k(\overline{T}_{j\neq i})$, and for $i = 1, \ldots, n$, considering the specialisation $x_0 \rightsquigarrow x_i$, we pick some uniformising element from the set $\overline{T}_{x_0} \setminus \overline{T}_{j\neq i}$ and divide the remaining elements by it. But we see that $\overline{T}_{x_0} \setminus \overline{T}_{j\neq i} = \{\bar{t}_{x_i}\}$, so there is nothing to do. Then, we have that

$$\overline{V}_{x_0 \rightsquigarrow x_i} = k(\overline{T}_{j\neq i})[\bar{t}_{x_i}]_{\langle \bar{t}_{x_i} \rangle}.$$

The situation thus begins to look a lot more like that of the ring A in Lewis's construction.

We first observe that the ring $\overline{V}_{x_0 \rightsquigarrow x_i}$ has unique maximal ideal

$$\mathfrak{m}_{x_0 \rightsquigarrow x_i} = \bar{t}_{x_i} \overline{V}_{x_0 \rightsquigarrow x_i},$$

with corresponding quotient map

$$\overline{\pi}_{x_0 \rightsquigarrow x_i} : \overline{V}_{x_0 \rightsquigarrow x_i} \twoheadrightarrow \overline{V}_{x_0 \rightsquigarrow x_i} / \mathfrak{m}_{x_0 \rightsquigarrow x_i} = k(\overline{T}_{j \neq i}).$$

We form

$$\begin{aligned} \overline{R}_{x_0^{\uparrow}} &= \bigcap_i k(\overline{T}_{j \neq i})[\bar{t}_{x_i}]_{\langle \bar{t}_{x_i} \rangle} \\ &= k[\bar{t}_{x_1}, \ldots, \bar{t}_{x_n}]_{(\langle \bar{t}_{x_1} \rangle \cup \cdots \cup \langle \bar{t}_{x_n} \rangle)^{\mathsf{c}}}; \end{aligned}$$

finally,

$$\begin{aligned} \overline{R}_X &= \{ f \in \prod_{x \in X} \overline{R}_{x^{\uparrow}} \mid \forall x \in X, y \geqslant x \text{ then } f_y = \overline{\pi}_{x \rightsquigarrow y}(f_x) \} \\ &= \{ f \in \prod_{x \in X} \overline{R}_{x^{\uparrow}} \mid f_{x_i} = \overline{\pi}_{x_0 \rightsquigarrow x_i}(f_{x_0}) \text{ for all } i \geqslant 1 \} \\ &\cong \{ r \in \overline{R}_{x_0^{\uparrow}} \mid r \bmod \mathfrak{m}_{x_0 \rightsquigarrow x_i} \in \overline{R}_{x_i^{\uparrow}} = k(\overline{T}_{j \neq i}) \text{ for all } i \geqslant 1 \} \\ &= \overline{R}_{x_0^{\uparrow}} \\ &= k[\bar{t}_{x_1}, \ldots, \bar{t}_{x_n}]_{(\langle \bar{t}_{x_1} \rangle \cup \cdots \cup \langle \bar{t}_{x_n} \rangle)^{\mathsf{c}}}. \end{aligned}$$

Ultimately though, as we will see, it will not be especially important which ring we choose to represent a finite 1-dimensional irreducible spectral space (or, that is, which subbasis we choose for the topology on a finite spectral space); note that for the purposes of Lewis's construction, we may always take the field K on which we apply our induction hypothesis to be the field of fractions of whichever 1-dimensional domain we are using as a "building block".

Note that whenever we apply either of the techniques of Lewis's construction, if B is the pre-existing ring having certain spectral structure created or assumed to exist from an earlier stage of our procedure, that the ring D that we obtain after having applied the technique in question can always be taken to be a subring of B. At the very first step of the construction the ring B is a finite product of fields corresponding to the minimal elements of X. Compare this to the observation 3.10 that the ring R_X given by Ershov's construction can always be taken to be a subring of the product $R_{x_1^\uparrow} \times \cdots \times R_{x_n^\uparrow} \subseteq k(T_{x_1^\uparrow}) \times \cdots \times k(T_{x_n^\uparrow})$ where $x_1, \ldots, x_n$ are the minimal elements of X. Indeed it is not difficult to convince oneself that, given a fixed poset X, the construction of L_X takes place entirely within the ring $k(\overline{T}_{x_1}) \times \cdots \times k(\overline{T}_{x_n})$ for $\overline{T}_{x_i}$ as given above.

Gluing of 1-dimensional domains

Encouraged by this apparent similarity between the respective constituent building blocks of each of our constructions, we proceed to investigate how far the similarity is maintained when we apply the operations by which the ring L_X is built up from a finite spectral space X.

We first investigate the process by which we glue the spectrum of a 1-dimensional domain onto a maximal ideal, as described from page 79 onwards. We take X to be a finite spectral space of dimension n occurring at the n-th stage of our iterative procedure 3.6, such that all the height-n points of X have a unique immediate predecessor in X. Let $X_{\mathsf{ht}\,n} = \{z_1, \ldots, z_m\}$ be the points of maximum height of X. For simplicity, let us suppose that there is in fact a common immediate predecessor y for the z_i, that is, that $X_{\mathsf{ht}\,n} = y^{\updownarrow}$. Let $X' = X \setminus X_{\mathsf{ht}\,n}$. We firstly choose a domain A having m maximal ideals, by either of the methods discussed in the preceding section, so that $\mathsf{Spec}\,A \cong y^{\uparrow}$. We then set $K = \mathcal{Q}(A)$ and take a K-algebra B with $\mathsf{Spec}\,B \cong X'$ as guaranteed by the inductive hypothesis of 3.6, and pick out $\mathfrak{m}_y$ the maximal ideal of B corresponding to the point y of X'. Finally, we take the ring L_X to be the fibre product of A and B as in the following diagram, where all the maps are the obvious ones,

$$\begin{array}{ccc} L_X = A \times_C B & \xrightarrow{\psi'} & A \\ \varphi' \big\downarrow & & \big\downarrow \varphi \\ B & \xrightarrow{\psi} & B/\mathfrak{m}_y \ = C; \end{array}$$

we then have that $\mathsf{Spec}\, L_X$ has the structure of X' with the point y replaced by the space $y^{\uparrow}$; that is, we have $\mathsf{Spec}\, L_X \cong X$.

Note that as the method is essentially topological in nature it doesn't matter which representation of a 1-dimensional irreducible space we use as our ring A, provided we may still obtain the relevant maps φ and ψ (—for example, we require that $\mathsf{Spec}\, \varphi$ maps the single point of $\mathsf{Spec}\, C$ to the minimal point of $\mathsf{Spec}\, A$).

We then compare this situation with the rings defined by the Ershov construction. Writing $Y = y^{\uparrow}$ to avoid notational difficulties, we are provided with rings R_X, $R_{X'}$ and R_Y whose respective prime spectra are homeomorphic to X, X' and Y. We will then compare the Ershov ring R_X with the fibre product of the rings R_Y and $R_{X'}$ over the appropriate maps: we want to take the quotient of $R_{X'}$ by the maximal ideal $\mathfrak{m}_y$ corresponding to the point y of X'; and we want an inclusion of R_Y into this quotient ring. To this end, we define R_X and R_Y over some chosen field k, and then define $R_{X'}$ to be the ring defined on X' over the base field $K = k(t_{z_1}, \dots, t_{z_n})$; note this is similar to how we chose B above to be a K-algebra for K the field of fractions of our ring A.

$$\begin{array}{ccc} & & R_Y \\ & & \big\downarrow \\ R_{X'} & \longrightarrow & R_{X'}/\mathfrak{m}_y \end{array}$$

We also choose to take the subbasis $\overline{\mathcal{B}}_X$, $\overline{\mathcal{B}}_{X'}$ etc. as defined in the previous section to be the subbasis used in the definition of all the components of our constructions, that is, we take the subbasis for the coarse lower topology on a finite poset that consists of the collection $\overline{\mathcal{B}}_X = \{(x^{\uparrow})^{\mathsf{c}} \mid x \in X\}$. This is to facilitate the comparison of the rings we obtain. We thus re-label the rings corresponding to the spaces X, X' and Y as $\overline{R}_X$, $\overline{R}_{X'}$ and $\overline{R}_Y$ respectively.

Every constituent component of the construction of $\overline{R}_{X'}$ will be marked with a prime mark, e.g. as $\overline{R}'_{x\uparrow}$, $\overline{V}'_{x\rightsquigarrow y}$, $\overline{\pi}'_{x\rightsquigarrow y}$ etc., to distinguish them from those defined in the construction of R_X, which will be unmarked, as $\overline{R}_{x\uparrow}$, $\overline{V}_{x\rightsquigarrow y}$, $\overline{\pi}_{x\rightsquigarrow y}$. It will be important to distinguish these objects as they will often be defined on the same points and specialisation relations, since X' is a sub-poset of X; however the "ambient" poset they are defined within will significantly affect the definitions, as we will see.

Our first observation is that we have:

Proposition 4.1. *For $\overline{R}_X$ the Ershov ring defined on X from the subbasis $\overline{\mathcal{B}}_X$ over some chosen field k, and for $\overline{R}_{X'}$ the Ershov ring defined on X' from the subbasis $\overline{\mathcal{B}}_{X'}$ over the field $k(\bar{t}_{z_1}, \ldots, \bar{t}_{z_n})$, we have that $\overline{R}_X\restriction_{X'} \subseteq \overline{R}_{X'}$.*

We remind ourselves of the definition of $\overline{R}_X\restriction_{X'}$ as immediately preceding theorem 3.11: it is simply the projection of $\overline{R}_X$ onto X'. Then, our claim is that we have $\overline{R}_{x\uparrow} \subseteq \overline{R}'_{x\uparrow}$ for every $x \in X'$, and all conditions of the form $\overline{\pi}'_{x\rightsquigarrow w}(f_x) = f_w$ for $x \rightsquigarrow w$ in X' are satisfied by $f \in \overline{R}_X$; note that as $\overline{\pi}'_{x\rightsquigarrow w}$ is not in general the same as the map $\overline{\pi}_{x\rightsquigarrow w}$, some justification is required. We unpick the definitions as follows. Given $x \in X'$, the definitions of $\overline{R}_{x\uparrow}$, $\overline{R}'_{x\uparrow}$ respectively are as follows:

$$\overline{R}_{x\uparrow} = \bigcap_{w \leftsquigarrow x \text{ in } X} \overline{V}_{x\rightsquigarrow w},$$

$$\overline{R}'_{x\uparrow} = \bigcap_{w \leftsquigarrow x \text{ in } X'} \overline{V}'_{x\rightsquigarrow w}.$$

Note that $\bigcap\limits_{w \leftsquigarrow x \text{ in } X} \overline{V}_{x \rightsquigarrow w} \subseteq \bigcap\limits_{w \leftsquigarrow x \text{ in } X'} \overline{V}_{x \rightsquigarrow w}$; so that we may satisfy our claim if it holds when we compare just those rings $\overline{V}_{x \rightsquigarrow w}$ and $\overline{V}'_{x \rightsquigarrow w}$ where x and w are in X'. We see the definitions:

$$\overline{V}_{x \rightsquigarrow w} = k(\overline{T}_{w^{\downarrow \mathsf{c}}}, \frac{\overline{T}_{x^{\downarrow \mathsf{c}}} \setminus \overline{T}_{w^{\mapsdown \mathsf{c}}}}{\bar{t}_x})[\bar{t}_x]_{\langle \bar{t}_x \rangle};$$

$$\overline{V}'_{x \rightsquigarrow w} = K(\overline{T}'_{w^{\downarrow \mathsf{c}}}, \frac{\overline{T}'_{x^{\downarrow \mathsf{c}}} \setminus \overline{T}'_{w^{\mapsdown \mathsf{c}}}}{\bar{t}_x})[\bar{t}_x]_{\langle \bar{t}_x \rangle},$$

where the hopefully suggestive notation is defined as $\overline{T}_{w^{\downarrow \mathsf{c}}} = \{\bar{t}_v \mid v \in (w^{\downarrow})^{\mathsf{c}}\}$; $\frac{\overline{T}_{x^{\downarrow \mathsf{c}}} \setminus \overline{T}_{w^{\mapsdown \mathsf{c}}}}{\bar{t}_x} = \{\frac{\bar{t}_v}{\bar{t}_x} \mid v \in (x^{\downarrow})^{\mathsf{c}} \setminus (w^{\mapsdown})^{\mathsf{c}},\ v \neq x\}$ where down-sets and complements are taken in X; the prime-marked versions have identical definitions except with down-sets and complements taken in X'. We then see that for all $w \in X$ such that $w \neq z_i$ we have that $\{z_1, \ldots, z_n\} \subseteq (w^{\downarrow})^{\mathsf{c}}$, so that $\bar{t}_{z_1}, \ldots, \bar{t}_{z_m}$ are units in $\overline{V}_{x \rightsquigarrow w}$ for all x and w in X'; thus justifying our particular choice of subbasis. But then we in fact see that $\overline{V}_{x \rightsquigarrow w}$ and $\overline{V}'_{x \rightsquigarrow w}$ define identical rings for all x and w in X', so $\bigcap\limits_{w \leftsquigarrow x \text{ in } X'} \overline{V}_{x \rightsquigarrow w} = \bigcap\limits_{w \leftsquigarrow x \text{ in } X'} \overline{V}'_{x \rightsquigarrow w} = \overline{R}'_{x^{\uparrow}}$ for all $x \in X'$. Furthermore note that $\overline{V}_{x \rightsquigarrow w} = \overline{V}'_{x \rightsquigarrow w}$ means that the maps $\overline{\pi}_{x \rightsquigarrow w}, \overline{\pi}'_{x \rightsquigarrow w}$ are likewise identical for all $x, w \in X'$, and so the condition on $\overline{R}_{X'}$ that $\overline{\pi}'_{x \rightsquigarrow w}(f_x) = f_w$ for all $f \in \overline{R}_{X'}$ is satisfied for all $f \in \overline{R}_X{\restriction}_{X'}$ by the analogous condition $\overline{\pi}_{x \rightsquigarrow w}(f_x) = f_w$ on $\overline{R}_X$. Hence we have $\overline{R}_X{\restriction}_{X'} \subseteq \overline{R}_{X'}$ as required. □

Now recall from 3.11 that, since $Y = y^{\uparrow}$ is up-closed, we have that $\overline{R}_X{\restriction}_Y = \overline{R}_Y$. Then, as we may identify $\overline{R}_Y$ with the subring $\{f_y \mid f \in \overline{R}_Y\}$, or equivalently, from the above equality, the subring $\{f_y \mid f \in \overline{R}_X\}$ of $\overline{R}_{y^{\uparrow}}$, then since $\overline{R}_{y^{\uparrow}} \subseteq \overline{R}'_{y^{\uparrow}}$, we may identify $\overline{R}_Y$ with a subring of $\overline{R'}_{y^{\uparrow}}$.

This suggests the following comparison with the construction of Lewis: observe that projection onto the y-component is the (surjective, since y is maximal in X') homomorphism $\mathsf{ev}_y : \overline{R}_{X'} \twoheadrightarrow \overline{R}'_{y^{\uparrow}}$. Suppose we wished to consider only those elements f of $\overline{R}_{X'}$ for which $f_y \in \overline{R}_Y$ (via the identification discussed in the previous paragraph). That is, we wish to consider a ring

$$\begin{aligned} R_0 &= \{f \in \overline{R}_{X'} \mid f_y \in \overline{R}_Y\} \\ &= \{f \in \overline{R}_{X'} \mid \mathsf{ev}_y(f) \in \overline{R}_Y\} \\ &= \{f \in \overline{R}_{X'} \mid f \bmod \ker(\mathsf{ev}_y) \in \overline{R}_Y\}, \end{aligned}$$

where we notice that $\ker(\mathsf{ev}_y)$ is the maximal ideal $\mathfrak{m}_y$ of $\overline{R}_{X'}$ corresponding to the point y in X'. That is, $R_0 = \{f' \in \overline{R}_{X'} \mid f'_y \in \overline{R}_Y\}$ is exactly the fibre product of $\overline{R}_{X'}$ and $\overline{R}_Y$ over $\mathfrak{m}_y$.

We may reverse the identification of $\overline{R}_Y$ with a subring of $\overline{R}_{y^\uparrow}$ by sending f_y to $\prod_{x \in Y} \overline{\pi}_{y \rightsquigarrow x}(f_y)$; thus we may define R_0 (isomorphically) on the whole of X by defining for each $f \in R_0$ that $f_{z_i} = \overline{\pi}_{y \rightsquigarrow z_i}(f_y)$. Since for every $f \in \overline{R}_X$ we have that $f_y \in \overline{R}_Y$, we have that $\overline{R}_X \subseteq R_0$. We find in fact that the inclusion is strict.

To show this, we examine in more detail the distinction between the rings $\overline{R}_{x^\uparrow}$ and $\overline{R}'_{x^\uparrow}$. In the case that $x \leqslant y$, we have that

$$\overline{R}_{x^\uparrow} = \bigcap_{w \leftrightsquigarrow x \text{ in } X} \overline{V}_{x \rightsquigarrow w} = \overline{R}'_{x^\uparrow} \cap \bigcap_i \overline{V}_{x \rightsquigarrow z_i}.$$

To analyse the rings $\overline{V}_{x \rightsquigarrow z_i}$ we first consider the set $W = (y^\downarrow)^{\mathsf{c}} \setminus y^{\updownarrow}$; the indeterminates corresponding to such points are always units in the definition of a ring $\overline{V}_{x \rightsquigarrow z_i}$; we label the corresponding set $\overline{T}_W$. Then we can see from the definition and from comparison with the similar situation in the preceding section that

$$\overline{V}_{x \rightsquigarrow z_i} = k(\overline{T}_W, \overline{T}_{j \neq i}, \frac{\overline{T}_{x^{\downarrow\mathsf{c}}} \setminus \overline{T}_{y^{\downarrow\mathsf{c}}}}{\bar{t}_{z_i}})[\bar{t}_{z_i}]_{\langle \bar{t}_{z_i} \rangle},$$

for $\overline{T}_{j \neq i} = \{\bar{t}_{z_j} \mid j \neq i\}$, where, as in the earlier section, we allow ourselves a different choice of uniformising element for the ring, choosing $\bar{t}_{z_i}$ as opposed to $\bar{t}_x$. Then, we can see that no element of the form $\dfrac{\bar{t}_w}{\bar{t}_{z_j}^2}$, say, for any $w \in (x^\downarrow)^{\mathsf{c}} \backslash (y^\downarrow)^{\mathsf{c}}$ (e.g. for $w = y$) can appear in $\overline{V}_{x \rightsquigarrow z_j}$, and so cannot appear in $\overline{R}_{x^\uparrow}$.

For $f \in R_0$ we have that $f_y \in \overline{R}_{y\uparrow}$, and that we have $f_y = \overline{\pi}_{x\rightsquigarrow y}(f_x)$, where

$$\overline{R}_{y\uparrow} = k[\bar{t}_{z_1}, \ldots, \bar{t}_{z_m}]_{(\langle\bar{t}_{z_1}\rangle\cup\cdots\cup\langle\bar{t}_{z_m}\rangle)^{\mathsf{c}}},$$

and

$$\overline{\pi}_{x\rightsquigarrow y} : \overline{V}_{x\rightsquigarrow y} \longrightarrow K_{x\rightsquigarrow y},$$

that is,

$$\overline{\pi}_{x\rightsquigarrow y} : k(\overline{T}_{y^{\downarrow\mathsf{c}}}, \frac{\overline{T}_{x^{\downarrow\mathsf{c}}} \setminus \overline{T}_{y^{\Upsilon\mathsf{c}}}}{\bar{t}_y})[\bar{t}_y]_{\langle\bar{t}_y\rangle} \to k(\overline{T}_{y^{\downarrow\mathsf{c}}}, \frac{\overline{T}_{x^{\downarrow\mathsf{c}}} \setminus \overline{T}_{y^{\Upsilon\mathsf{c}}}}{\bar{t}_y}),$$

so that $f_x = f_y + g$ where $g \in \mathfrak{m}_{x\rightsquigarrow y} = \bar{t}_y\overline{V}_{x\rightsquigarrow y}$. But note that in R_0, that is, in $\overline{R}'_{x\uparrow}$ such that $f \in \overline{R}_{X'}$ and $f_y \in \overline{R}_{y\uparrow}$, there is nothing to prevent $f_x = \dfrac{\bar{t}_y}{\bar{t}^2_{z_1}}$, say (and then $f_y = \overline{\pi}_{x\rightsquigarrow y}(\frac{\bar{t}_y}{\bar{t}^2_{z_1}}) = 0 \in \overline{R}_{y\uparrow}$), in which case $f_x \notin \overline{V}_{x\rightsquigarrow z_1}$ and so $f_x \notin \overline{R}_{x\uparrow}$; hence $\overline{R}_X \subsetneq R_0$.

It might be contended that by choosing the ring $\overline{R}_{X'}$ to be constructed over a different field than the rings $\overline{R}_X$ and $\overline{R}_Y$, that we are examining a somewhat artificial situation. However I believe the similarity drawn out between the conditions by which the ring R_X is defined; the role of maximal ideals which correspond to evaluating functions at maximal points of a space; and the fibre product method of Lewis's construction, is sufficiently evident to be of interest.

Joining of maximal points

We likewise consider the joining of maximal points. Suppose as in the exposition of Lewis's iterative construction that from the poset X we have obtained a poset X' having the following properties: firstly, that we may obtain X from X' by joining certain maximal points of X' to a single point; and secondly, each of these maximal points of X' under consideration has a unique immediate predecessor in X'. In Lewis's construction, we then apply the joining of maximal points procedure to a ring B (—in fact, a K-algebra for any choice of

K) having spectrum homeomorphic to X', in order to obtain a ring L_X with spectrum homeomorphic to X. We compare the rings B, L_X with the rings $R_{X'}$, R_X provided by the Ershov construction.

For simplicity suppose we are joining a single set of maximal points in X' to a single point in X. Then, if $z_{1,1}, \dots, z_{1,m}$ are the maximal points of X' corresponding to the maximal point z_1 of X, each with immediate predecessor $y_1, \dots, y_m$ in X' respectively, and if $\mathfrak{n}_{1,1}, \dots, \mathfrak{n}_{1,m}$ are the maximal ideals of B corresponding to the maximal points in question, then we map the field K diagonally into, and the ring B surjectively onto, the product $C = B/\mathfrak{n}_{1,1} \times \cdots \times B/\mathfrak{n}_{1,m}$.

To carry out the joining procedure we then take the fibre product over these maps, giving the ring $L_X = K \times_C B \cong \{b \in B \mid b \bmod \mathfrak{n}_{1,1} = \cdots = b \bmod \mathfrak{n}_{1,m} = a \in K\}$, such that $\mathsf{Spec}\, L_X \cong X$.

$$\begin{array}{ccc}
L_X = K \times_C B & \xrightarrow{\ \psi'\ } & K \\
\Big\downarrow{\scriptstyle \varphi'} & & \Big\downarrow{\scriptstyle \varphi} \\
B & \xrightarrow{\ \psi\ } & B/\mathfrak{n}_{1,1} \times \cdots \times B/\mathfrak{n}_{1,m} = C
\end{array}$$

We now compare the rings $R_{X'}$, R_X defined from X', X by the Ershov construction. The definitions differ in two ways: firstly $R_{X'}$ is defined as a product over points in X', that is,

$$R_{X'} \subseteq \prod_{x \in X'} R'_{x^{\uparrow}} = \Big(\prod_{x \in z_1^{\downarrow}} R'_{x^{\uparrow}}\Big) \times R'_{z_{1,1}^{\uparrow}} \times \cdots \times R'_{z_{1,m}^{\uparrow}},$$

whereas

$$R_X \subseteq \Big(\prod_{x \in z_1^{\downarrow}} R_{x^{\uparrow}}\Big) \times R_{z_1^{\uparrow}};$$

and where the definition of $R_{X'}$ has compatibility criteria

$$\pi'_{y_1 \rightsquigarrow z_{1,1}}(f_{y_1}) = f_{z_{1,1}}$$
$$\pi'_{y_2 \rightsquigarrow z_{1,2}}(f_{y_2}) = f_{z_{1,2}}$$
$$\vdots$$
$$\pi'_{y_m \rightsquigarrow z_{1,m}}(f_{y_m}) = f_{z_{1,m}},$$

these correspond to the criteria

$$\pi_{y_1 \rightsquigarrow z_1}(f_{y_1}) = \pi_{y_2 \rightsquigarrow z_1}(f_{y_2}) = \cdots = \pi_{y_m \rightsquigarrow z_1}(f_{y_m}) = f_{z_1}$$

in R_X.

The situation then looks somewhat reminiscent of the description of our fibre product just above. We may think one way to proceed is, defining by φ_i the isomorphism of fields $\varphi_i : R'_{z_{1,i}^{\uparrow}} = k(t_{z_{1,i}}) \to k(t_{z_1}) = R_{z_1^{\uparrow}}$ given by $t_{z_{1,i}} \mapsto t_{z_1}$, then taking

$$R_0 = \{f \in R_{X'} \mid \varphi_1(f_{z_{1,1}}) = \cdots = \varphi_m(f_{z_{1,m}})\},$$

we can identify R_0 with a product

$$R_0 \subseteq \prod_{x \in z_1^{\updownarrow}} R'_{x^{\uparrow}} \times R_{z_1^{\uparrow}}$$

by defining $f_{z_1} = \varphi_i(f_{z_{1,i}}) \in R_{z_1^{\uparrow}}$

The difficulty is that elements in $R'_{x^{\uparrow}}$ may refer to any of the indeterminates $t_{z_{1,1}}, \ldots, t_{z_{1,m}}$ where those in $R_{x^{\uparrow}}$ only refer to the indeterminate t_{z_1}, and the stipulation that $\varphi_1(f_{z_{1,1}}) = \cdots = \varphi_m(f_{z_{1,m}})$ is not a strong enough condition on f to allow us to define a map on the indeterminates $t_{z_{1,i}}$ that would allow us to compare an element $f \in R_0$ with an element $f' \in R_X$. However, allowing ourselves to be slightly more "impressionistic" for a moment (—and allowing ourselves to notice, with reference to the analogous argument of the previous section, that R_0 is indeed the fibre product $\{f \in R_{X'} \mid f \ \mathsf{mod}\ \mathsf{ker}(\mathsf{ev}_{z_{1,1}}) = \cdots =$

$f \bmod \ker(\mathsf{ev}_{z_{1,m}})\}$, at least under the identification of each field $R_{z_{1,i}^{\uparrow}} = k(t_{z_{1,i}})$ with $k(t_{z_1})$), we see that there are in some sense a great deal more elements in R_0 than in R_X, in exactly the sense that R_0 has indeterminates $t_{z_{1,1}}, \ldots, t_{z_{1,m}}$ where R_X has only t_{z_1}.

We now leave impressionism strictly behind us for the remainder of this thesis.

Comparison of constructions on a general spectral space

We now proceed to explore how the 'Ershov ring' R_X associated to a finite spectral space X may be defined in the case that X is no longer finite. Recall we began our construction of R_X in an identical way to how we began the construction of the Hochster ring H_X (—initially, the ring A_X) associated to a general spectral space X: we identified a suitable subbasis $\mathcal{B}_X$ for the topology of X and formed corresponding sets of indeterminates T_X and characteristic functions χ taking values in the polynomial ring $k[T_X]$ over some chosen field k.

Our finite construction began to proceed along distinct lines to that of Hochster when, instead of extending to elements defined by having some property with respect to an 'index' of valuations, we specified valuation *rings* $V_{x \rightsquigarrow y}$ corresponding to each specialisation $x \rightsquigarrow y$ in our space X and took the condition on our final ring R_X to be "compatibility" of values taken by functions at points x and y via the canonical quotient maps corresponding to these valuation rings. We recall that the definition of $V_{x \rightsquigarrow y}$ in our original exposition (on page 98) made reference to the order relation on X via the sets $T_{y^{\uparrow}}$, $T_{x^{\updownarrow}}$ and so on. However, as we will now see, we can without difficulty extend the definition of $V_{x \rightsquigarrow y}$ beyond the finite setting, in fact using concepts we have already defined:

Let X be any spectral space, and take $\mathcal{B}_X \subseteq \mathring{\mathcal{K}}(X)$ an open subbasis for X, $T_X = \{t_U\}_{U \in \mathcal{B}_X}$ a collection of indeterminates indexed by $\mathcal{B}_X$, and characteristic functions $\chi_U : X \to k[T_X]$ defined by

$$\chi_U(x) = \begin{cases} t_U & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

Recall that given a point $x \in X$ we defined the set of indeterminates $T_x \subseteq T_X$ to be the set $T_x = \{t_U \in T_X \mid x \in U\}$. We note that in the case that X is finite and $\mathcal{B}_X$ is the basis $\mathcal{B}_X = \{y^{\downarrow} \mid y \in X\}$ as chosen at the start of our exposition of Ershov's construction then T_x is exactly the set $T_{x^{\uparrow}}$; furthermore as noted at this stage in both the Hochster and the Ershov procedure, given a specialisation $x \rightsquigarrow y$ (—in any spectral space X) we have that $T_y \subseteq T_x$. Thus we make a general definition of the valuation ring $V_{x \rightsquigarrow y}$ corresponding to the specialisation $x \rightsquigarrow y$ in a general spectral space X (—as our definitions are consistent with those previously given in the finite case we make no change to the notation of the objects defined):

Firstly, in the case that $y = x$ we define the ring $V_{x \rightsquigarrow x}$ to be the improper valuation ring

$$V_{x \rightsquigarrow x} = k(T_x).$$

In the case that $y \neq x$ we have that $T_y \subsetneq T_x$ (since distinct points are topologically distinguishable, and so separated by some subbasis set); we then take some U_0 such that $x \in U_0$, $y \notin U_0$ (i.e. such that $t_{U_0} \in T_x \backslash T_y$), then, writing t_0 for the indeterminate t_{U_0} we define the set $\frac{T_x^{\wedge} \backslash T_y}{t_0} = \{\frac{t_U}{t_0} \mid t_U \in T_x \setminus T_y,\, U \neq U_0\}$. We then define the discrete valuation ring $V_{x \rightsquigarrow y}$ to be

$$V_{x \rightsquigarrow y} = k(T_y, \frac{T_x^{\wedge} \setminus T_y}{t_0})[t_0]_{\langle t_0 \rangle},$$

in a manner practically identical to the finite case. Note that the choice of U_0 in fact in no way affects the definition of the ring $V_{x \rightsquigarrow y}$: for any choice of U'

such that $x \in U'$, $y \notin U'$, the indeterminate $t_{U'} \in T_x \setminus T_y$ has the same value as t_0 (since $\frac{t_{U'}}{t_0}$ is a unit in $V_{x \rightsquigarrow y}$ so has value 0), and so $t_{U'}$ in fact has value 1, and is hence a uniformiser for the valuation ring; that is, we could replace U_0 by U' in all our definitions and we would obtain an identical valuation ring $V_{x \rightsquigarrow y}$.

We again have a unique maximal ideal $\mathfrak{m}_{x \rightsquigarrow y} = t_0 V_{x \rightsquigarrow y}$ and canonical quotient map $\pi_{x \rightsquigarrow y} : V_{x \rightsquigarrow y} \to V_{x \rightsquigarrow y}/\mathfrak{m}_{x \rightsquigarrow y} = K_{x \rightsquigarrow y}$.

The comparison of valuation rings

Recall that in chapter 2 we defined a discrete valuation $\mathsf{v}_{x \rightsquigarrow y} : k(T_x) \to \mathbb{Z} \cup \{\infty\}$ corresponding to each specialisation $x \rightsquigarrow y$ in a spectral space X. We have just seen how in analogy with Ershov's construction described in the preceding chapter, we may define a valuation ring $V_{x \rightsquigarrow y}$ corresponding to each specialisation $x \rightsquigarrow y$ in X, for a general spectral space X. We see in fact that $V_{x \rightsquigarrow y}$ as so defined is the valuation ring of the valuation $\mathsf{v}_{x \rightsquigarrow y}$.

Recall the valuation $\mathsf{v}_{x \rightsquigarrow y}$ was defined first on $k[T_x]$ by

- $\mathsf{v}_{x \rightsquigarrow y}(t_U) = \begin{cases} 1 & y \notin U, \text{ that is, } t_U \notin T_y; \\ 0 & y \in U, \text{ or, } t_U \in T_y; \end{cases}$
- $\mathsf{v}_{x \rightsquigarrow y}(t_1^{\alpha_1} \dots t_n^{\alpha_n}) = \sum \alpha_i$ for those i such that $\mathsf{v}_{x \rightsquigarrow y}(t_i) = 1$; and
- $\mathsf{v}_{x \rightsquigarrow y}(\sum \lambda_i m_i) = \mathsf{min}(\{\mathsf{v}_{x \rightsquigarrow y}(m_i)\})$ for m_i distinct monomials in T_x and $\lambda_i \in k^\times$;

It then has a uniquely defined extension to a discrete valuation $\mathsf{v}_{x \rightsquigarrow y} : k(T_x) \to \mathbb{Z} \cup \{\infty\}$.

Then we see

Proposition 4.2. *Given $\mathsf{v}_{x \rightsquigarrow y} : k(T_x) \to \mathbb{Z} \cup \{\infty\}$, $V_{x \rightsquigarrow y} \subseteq k(T_x)$ as defined above, we have that $V_{x \rightsquigarrow y}$ is the valuation ring of the discrete valuation $\mathsf{v}_{x \rightsquigarrow y}$, that is, $V_{x \rightsquigarrow y} = \{r \in k(T_x) \mid \mathsf{v}_{x \rightsquigarrow y}(r) \geqslant 0\}$.*

Let $V = \{r \in k(T_x) \mid \mathsf{v}_{x \rightsquigarrow y}(r) \geqslant 0\}$ be the valuation ring of $\mathsf{v}_{x \rightsquigarrow y}$. For $t_U \in T_y$ we have that $\mathsf{v}_{x \rightsquigarrow y}(t_U) = 0$, so that t_U is a unit in the valuation ring; then to simplify our argument let $k' = k(T_y)$, then $V_{x \rightsquigarrow y} = k'(\frac{T_x^\wedge \setminus T_y}{t_0})[t_0]_{\langle t_0 \rangle}$ and we need to show that $k'(\frac{T_x^\wedge \setminus T_y}{t_0})[t_0]_{\langle t_0 \rangle} = V$. This modification allows us to treat $t_U \in T_y$ as coefficients in our base field.

Since $\mathsf{v}_{x \rightsquigarrow y}$ is a valuation on $k(T_x) = k(T_y)(T_x \setminus T_y) = k'(T_x \setminus T_y)$, every $r \in V$ is of the form $\frac{f}{g}$ for $f, g \in k'[T_x \setminus T_y]$, $g \neq 0$.

Take some $\frac{f}{g} \in V$. We pick a term of least degree of g, which we will write as $g_0 = \gamma_0 t_1^{\alpha_1} \dots t_n^{\alpha_n}$ for $\gamma_0 \in (k')^\times$ and $t_i \in T_x \setminus T_y$ (where of course we may have $n = 0$ so that $g_0 = \gamma_0$).

Let

$$g' = \frac{g}{g_0}.$$

We claim $g' \in k'(\frac{T_x^\wedge \setminus T_y}{t_0})[t_0] \setminus \langle t_0 \rangle$. For convenience we write $k'(\frac{T_x^\wedge \setminus T_y}{t_0})[t_0] = R'$. Writing g termwise, we have

$$g' = \frac{\gamma_d m_d}{g_0} + \dots + \frac{\gamma_1 m_1}{g_0} + \frac{g_0}{g_0}$$

for monomials m_i and coefficients $\gamma_i \in (k')^\times$, where each m_i has degree greater or equal to the degree of g_0 (and where in describing g_0 as a term of g we implicitly mean that each monomial m_i is distinct from the monomial of g_0).

Then we write

$$m_i = m_i' m_i''$$

where m_i'' has degree equal to the degree of g_0, so that

$$g' = \frac{\gamma_d m_d''}{g_0} m_{d'}' + \dots + \frac{\gamma_1 m_1''}{g_0} m_1' + 1.$$

Then

$$\frac{\gamma_i m_i''}{g_0} = \frac{\gamma_i {t'_1}^{\beta_1} \dots {t'_{n'}}^{\beta_{n'}}}{\gamma_0 t_1^{\alpha_1} \dots t_n^{\alpha_n}},$$

where $\sum_{j'} \beta_{j'} = \sum_j \alpha_j = N$. Note for all $t_U \neq t_0$ we have $\dfrac{t_U}{t_0} \in R'$ and $\dfrac{t_0}{t_U} \in R'$; furthermore $\dfrac{t_0}{t_0} = 1 \in R'$.

Then

$$\begin{aligned} \frac{\gamma_i m_i''}{g_0} &= \Big(\frac{t_0^N}{t_0^N}\Big) \frac{\gamma_i {t_1'}^{\beta_1} \dots {t_{n'}'}^{\beta_{n'}}}{\gamma_0 t_1^{\alpha_1} \dots t_n^{\alpha_n}} \\ &= \frac{\gamma_i}{\gamma_0} \big(\frac{t_1'}{t_0}\big)^{\beta_1} \cdot \dots \cdot \big(\frac{t_{n'}'}{t_0}\big)^{\beta_{n'}} \cdot \big(\frac{t_0}{t_1}\big)^{\alpha_1} \cdot \dots \cdot \big(\frac{t_0}{t_n}\big)^{\alpha_n} \in R' \end{aligned}$$

for each i, and $m_i' \in k[T_x \setminus T_y] \subseteq R'$, so that $g' \in R'$. Furthermore as each monomial m_i is distinct from the monomial of g_0, we do not obtain any new constant terms; so g' has constant term equal to 1, and so lies outside the ideal generated by t_0.

Writing f termwise, we have

$$\frac{f}{g} = \frac{f}{g_0 g'} = \frac{\lambda_{d'} M_{d'}}{g_0 g'} + \dots + \frac{\lambda_1 M_1}{g_0 g'} + \frac{\lambda_0 M_0}{g_0 g'}.$$

for monomials M_j and coefficients $\lambda_j \in k^\times$.

Note as $\mathsf{v}_{x \rightsquigarrow y}(f) \geqslant \mathsf{v}_{x \rightsquigarrow y}(g)$ then every monomial M_j has degree greater than or equal to the degree of g_0. Then write

$$M_j = M_j' M_j''$$

where M_j'' has degree equal to the degree of g_0, so that

$$\frac{f}{g} = \frac{\lambda_{d'} M_{d'}''}{g_0} \frac{M_{d'}'}{g'} + \dots + \frac{\lambda_1 M_1''}{g_0} \frac{M_1'}{g'} + \frac{\lambda_0 M_0''}{g_0} \frac{M_0'}{g'}.$$

Then to verify our first claim, we use exactly the procedure we have just demonstrated to observe that, as g_0 and each M_j'' have the same degree, then $\frac{\lambda_j M_j''}{g_0}$ lies in R'. Furthermore, $M_j' \in k[T_x \setminus T_y] \subseteq R'$. Then let $f_j' = \frac{\lambda_j M_j'' M_j'}{g_0} \in R'$. Then

$$\frac{f}{g} = \frac{f_{d'}' + \dots + f_1' + f_0'}{g'},$$

with $f_{d'}' + \dots + f_1' + f_0' \in R'$ and $g' \in R' \setminus \langle t_0 \rangle$; so $\frac{f}{g} \in R'_{\langle t_0 \rangle} = V_{x \rightsquigarrow y}$.

Conversely note that the elements with non-zero value in $k'(\frac{T_x^\wedge \setminus T_y}{t_0})[t_0]$ are exactly those in the ideal generated by t_0; therefore we may divide by any element outside this ideal without changing the value assigned, so that every element in $k'(\frac{T_x^\wedge \setminus T_y}{t_0})[t_0]_{\langle t_0 \rangle} = V_{x \rightsquigarrow y}$ has value $\geqslant 0$ and so $V = V_{x \rightsquigarrow y}$ as claimed.

□

The next stage of Ershov's construction defines a ring $R_{x^\uparrow}$ for each $x \in X$, being the intersection of the valuation rings

$$R_{x^\uparrow} = \bigcap_{y \leftsquigarrow x} V_{x \rightsquigarrow y}.$$

We maintain this definition (and notation) in the general case, noting that in general the ring $R_{x^\uparrow}$ is now no longer a finite intersection of discrete valuation rings.

We note that for all elements h of Hochster's ring H_X, by proposition 2.15 we have that $\mathsf{v}_{x \rightsquigarrow y}(h_x) \geqslant 0$ for all specialisations $x \rightsquigarrow y$ in X; that is, for every $h \in H_X$ we have that h_x lies within the valuation ring of $\mathsf{v}_{x \rightsquigarrow y}$ for all specialisations $x \rightsquigarrow y$, which is the ring $V_{x \rightsquigarrow y}$ by the preceding proposition. But then h_x is in the intersection of the valuation rings $V_{x \rightsquigarrow y}$ for $y \leftsquigarrow x$, so that $h_x \in R_{x^\uparrow}$ at every $x \in X$ for all $h \in H_X$.

Then $H_X \subseteq \prod_{x \in X} R_{x^\uparrow}$. The Ershov ring R_X is also originally defined as a subring of $\prod_{x \in X} R_{x^\uparrow}$, being the ring

$$R_X = \{f \in \prod_{x \in X} R_{x^\uparrow} \mid \forall x \in X, y \geqslant x \text{ then } f_y = \pi_{x \rightsquigarrow y}(f_x)\}.$$

We see that in our updated general setting we still have

$$V_{y \rightsquigarrow z} = k(T_{z^\uparrow}, \frac{T_y^\wedge \setminus T_z}{t_{0_y}})[t_{0_y}]_{\langle t_{0_y} \rangle} \subseteq k(T_y) \subseteq k(T_y, \frac{T_x^\wedge \setminus T_y}{t_{0_x}}) = K_{x \rightsquigarrow y}$$

(where $t_{0_y} \in T_y \setminus T_z$, $t_{0_x} \in T_x \setminus T_y$ are the indeterminates corresponding to our chosen basis sets U_{0_y} such that $y \in U_{0_y}$, $z \notin U_{0_y}$ and U_{0_x} such that $x \in U_{0_x}$,

$y \notin U_{0_x}$ respectively), for any $x \rightsquigarrow y$ and $y \rightsquigarrow z$ in X, so that for all $x \in X$ and for all y' with $x \rightsquigarrow y'$ we have $R_{y'\uparrow} \subseteq K_{x \rightsquigarrow y'}$ and it again makes sense to compare $\pi_{x \rightsquigarrow y'}(f_x) \in K_{x \rightsquigarrow y'}$ with $f_{y'} \in R_{y'\uparrow}$; therefore we may essentially maintain our definition for R_X as above in the case that X is any spectral space, giving the general Ershov ring R_X corresponding to a spectral space X, defined as

$$R_X = \{f \in \prod_{x \in X} R_{x\uparrow} \mid \forall x \in X, y \leftsquigarrow x \text{ then } f_y = \pi_{x \rightsquigarrow y}(f_x)\}.$$

The comparison of admissibility conditions

We now consider how the condition of admissibility to R_X as given above applies to elements $h \in H_X \subseteq \prod_{x \in X} R_{x\uparrow}$. In fact we have for all $h \in H_X$ that $\pi_{x \rightsquigarrow y}(h_x) = h_y$ for all $x \in X$ and $x \rightsquigarrow y$; that is,

Theorem 4.3. *Given any spectral space X and the rings H_X as defined on page 68 and R_X the general Ershov ring as defined above, we have $H_X \subseteq R_X$.*

We have that $H_X \subseteq \prod_{x \in X} R_{x\uparrow}$; we must show that for all $h \in H_X$ we have $\pi_{x \rightsquigarrow y}(h_x) = h_y$ for all $x \in X$ and $x \rightsquigarrow y$. We prove this by an induction on the construction of H_X from the rings A_i as defined on page 68.

Given $h \in H_X$ we have that for some $n \in \mathbb{N}$, that h is contained in the ring A_n obtained at the n-th stage of our inductive definition.

We apply an induction to the construction of the ring A_{i+1} from the ring A_i as follows: if A_i is such that for all $f \in A_i$ we have that $\pi_{x \rightsquigarrow y}(f_x) = f_y$, we prove that for all $h' \in A_{i+1}$ we have that $\pi_{x \rightsquigarrow y}(h'_x) = h'_y$.

Our base case is the ring $A_0 = A_X$ defined on page 54; we show that for all $f \in A_X \subseteq \prod_{x \in X} R_{x\uparrow}$ we have that $\pi_{x \rightsquigarrow y}(f_x) = f_y$.

Given a general $f = \sum_i \lambda_i m_i$ in A_X for $\lambda_i \in k^\times$ and m_i monomials in $k[\mathcal{X}_X]$ then

$$f_x = \left(\lambda_1 m_1\right)_x + \cdots + \left(\lambda_d m_d\right)_x,$$

where

$$\left(\lambda_i m_i\right)_x = \left(\lambda_i \chi_{U_1}^{\alpha_1} \chi_{U_2}^{\alpha_2} \dots \chi_{U_N}^{\alpha_N}\right)_x = \begin{cases} \lambda_i t_{U_1}^{\alpha_1} t_{U_2}^{\alpha_2} \dots t_{U_N}^{\alpha_N} & x \in U_j \text{ for all } j \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_y = \left(\lambda_1 m_1\right)_y + \dots + \left(\lambda_d m_d\right)_y,$$

for

$$\left(\lambda_i m_i\right)_y = \left(\lambda_i \chi_{U_1}^{\alpha_1} \chi_{U_2}^{\alpha_2} \dots \chi_{U_N}^{\alpha_N}\right)_y = \begin{cases} \lambda_i t_{U_1}^{\alpha_1} t_{U_2}^{\alpha_2} \dots t_{U_N}^{\alpha_N} & y \in U_j \text{ for all } j \\ 0 & \text{otherwise.} \end{cases}$$

Note that as $x \rightsquigarrow y$ then we have $y \in U_j$ for all U_j in some particular term of f only if we have $x \in U_j$ for all those U_j, so that for any term $\lambda_i m_i$ for which $\left(\lambda_i m_i\right)_y \neq 0$ we necessarily have $\left(\lambda_i m_i\right)_x \neq 0$.

We examine $\pi_{x \rightsquigarrow y}(f_x) = \pi_{x \rightsquigarrow y}(\left(\lambda_1 m_1\right)_x) + \dots + \pi_{x \rightsquigarrow y}(\left(\lambda_d m_d\right)_x)$ termwise, considering those terms for which $\left(\lambda_i m_i\right)_x \neq 0$. Observe that for an indeterminate $t_U \in T_x$, then $y \in U \iff t_U \in T_y \iff t_U \in k(T_y)$, so that $\pi_{x \rightsquigarrow y}(t_U) = t_U$; conversely $y \notin U \iff \mathsf{v}_{x \rightsquigarrow y}(t_U) > 0 \iff t_U \in \mathfrak{m}_{x \rightsquigarrow y} \iff \pi_{x \rightsquigarrow y}(t_U) = 0$. Then we have

$$\begin{aligned} \pi_{x \rightsquigarrow y}(\left(\lambda_i m_i\right)_x) = \pi_{x \rightsquigarrow y}(\lambda_i t_{U_1}^{\alpha_1} t_{U_2}^{\alpha_2} \dots t_{U_N}^{\alpha_N}) &= \lambda_i \pi_{x \rightsquigarrow y}(t_{U_1}^{\alpha_1}) \cdot \ldots \cdot \pi_{x \rightsquigarrow y}(t_{U_N}^{\alpha_N}) \\ &= \begin{cases} \lambda_i t_{U_1}^{\alpha_1} \cdot \ldots \cdot t_{U_N}^{\alpha_N} & \text{if } y \in U_j \text{ for all } j \\ 0 & \text{otherwise} \end{cases} \\ &= \left(\lambda_i m_i\right)_y, \end{aligned}$$

thus $\pi_{x \rightsquigarrow y}(f_x) = f_y$ as required.

We now make the assumption for the purposes of an induction that A_i is such that for all $f \in A_i$ we have that $\pi_{x \rightsquigarrow y}(f_x) = f_y$.

Every element of A_{i+1} arises from some finite extension of A_i by elements $\frac{f}{g}$ for $f, g \in A_i$ satisfying the condition of 2.13. Therefore per the discussion preceding theorem 2.18, it suffices to show that if we have $\pi_{x\rightsquigarrow y}(f_x) = f_y$ for every $f \in A_i$ then we have that $\pi_{x\rightsquigarrow y}(h'_x) = h'_y$ for all $h' \in A_{i+1}[\frac{f}{g}]$ for some $f, g \in A_i$ satisfying the condition of 2.13.

We consider a general element h' in $A_i[\frac{f}{g}]$,

$$h' = f_n\big(\frac{f}{g}\big)^n + \cdots + f_1\big(\frac{f}{g}\big) + f_0$$

for $f_0, \ldots, f_n \in A_i$. We have

$$\begin{aligned}\pi_{x\rightsquigarrow y}(h'_x) &= \pi_{x\rightsquigarrow y}\Big((f_n)_x\big(\frac{f}{g}\big)_x^n + \cdots + (f_1)_x\big(\frac{f}{g}\big)_x + (f_0)_x\Big)\\ &= (f_n)_y\pi_{x\rightsquigarrow y}\Big(\big(\frac{f}{g}\big)_x\Big)^n + \cdots + (f_1)_y\pi_{x\rightsquigarrow y}\Big(\big(\frac{f}{g}\big)_x\Big) + (f_0)_y,\end{aligned}$$

by our inductive assumption and the properties of ring homomorphisms.

Then $\pi_{x\rightsquigarrow y}(h'_x) = h'_y \iff \pi_{x\rightsquigarrow y}\big(\big(\frac{f}{g}\big)_x\big) = \big(\frac{f}{g}\big)_y$; hence we reduce to considering $h = \frac{f}{g} \in A_i[\frac{f}{g}]$.

If $h_x = 0$ then $f_x = 0$ and so $f_y = \pi_{x\rightsquigarrow y}(f_x) = 0$ so that $h_y = 0$, in which case we have $\pi_{x\rightsquigarrow y}(h_x) = \pi_{x\rightsquigarrow y}(0) = 0 = h_y$ as required.

Otherwise assume $h_x = \frac{f_x}{g_x} \neq 0$.

In the case $\pi_{x\rightsquigarrow y}(h_x) = 0$ we have equality:

$$\begin{aligned}\pi_{x\rightsquigarrow y}(h_x) = 0 \iff h_x \in \mathfrak{m}_{x\rightsquigarrow y} &\iff \mathsf{v}_{x\rightsquigarrow y}(h_x) > 0\\ &\iff \mathsf{v}_{x\rightsquigarrow y}(f_x) > \mathsf{v}_{x\rightsquigarrow y}(g_x)\\ &\iff \mathsf{v}_{x\rightsquigarrow y}(f_x) > 0\\ &\iff f_y = 0\\ &\iff h_y = 0.\end{aligned}$$

Alternatively,

$$\begin{aligned} \pi_{x\rightsquigarrow y}(h_x) \neq 0 &\iff h_x \notin \mathfrak{m}_{x\rightsquigarrow y} \iff \mathsf{v}_{x\rightsquigarrow y}(h_x) = 0 \\ &\iff \mathsf{v}_{x\rightsquigarrow y}(f_x) = \mathsf{v}_{x\rightsquigarrow y}(g_x). \end{aligned}$$

Then, since $h_x \neq 0$ so that $f_x, g_x \neq 0$, then by the condition of 2.13 we must have $\mathsf{v}_{x\rightsquigarrow y}(f_x) = \mathsf{v}_{x\rightsquigarrow y}(g_x) = 0$, so that $\frac{1}{g_x} \in V_{x\rightsquigarrow y}$. Then $\pi_{x\rightsquigarrow y}(\frac{1}{g_x}) = \frac{1}{\pi_{x\rightsquigarrow y}(g_x)} = \frac{1}{g_y}$, so that $\pi_{x\rightsquigarrow y}(h_x) = \pi_{x\rightsquigarrow y}(\frac{f_x}{g_x}) = \pi_{x\rightsquigarrow y}(f_x) \cdot \pi_{x\rightsquigarrow y}(\frac{1}{g_x}) = \frac{f_y}{g_y} = h_y$ as required.

Then for every $h' \in A_i[\frac{f}{g}]$ we have that $\pi_{x\rightsquigarrow y}(h'_x) = h'_y$; hence by induction every finite extension of A_i by admissible pairs f, g has this property. But then as every element in A_{i+1} arises from some such finite extension of A_i by admissible pairs, we have $\pi_{x\rightsquigarrow y}(h'_x) = h'_y$ for all $h' \in A_{i+1}$, which completes the induction.

Thus we have that for all $n \in \mathbb{N}$, the ring A_n has the property that for all $h \in A_n$ we have that $\pi_{x\rightsquigarrow y}(h_x) = h_y$; thus this property holds for all $h \in H_X$ and the theorem is proved. □

Two obvious questions strike us at this point. We have generalised the ring R_X from the finite setting to be defined on any spectral space X. When X is finite, we have seen that $\mathsf{Spec}\, R_X \cong X$. However firstly this result was proved by an induction on $|X|$, and furthermore one part of the proof relied on the fact that the ring $R_{x_0^\uparrow}$ was a finite intersection of discrete valuations rings for x_0 a minimum element of X, in order to argue about the structure of the non-zero primes of R_X. Since both these arguments are no longer relevant in the general setting, we may not expect that we still have that $\mathsf{Spec}\, X \cong X$ for X any spectral space; however whether this does in fact hold remains a question to be decided one way or the other.

We have further seen that for any spectral space X, the Hochster ring H_X is contained in the general Ershov ring R_X. An obvious question to consider is whether we in fact have $H_X = R_X$; note that if this is the case then indeed we must have $\mathsf{Spec}\, R_X \cong X$ since we know $\mathsf{Spec}\, H_X \cong X$ for all X.

We revisit a basic example from earlier.

Given $X = \mathsf{Spec}\,\mathbb{Z}$ then we saw (page 59) that the ring A_X initially assigned to X by the Hochster construction was the ring

$$A_X = \{ \prod_{\mathfrak{q} \in \mathsf{Spec}\,\mathbb{Z}} \mathsf{ev}_{\mathfrak{q}}(f) \mid f \in k[\mathcal{X}_X]\}$$

where $k[\mathcal{X}_X]\}$ consists of polynomial expressions over k of functions

$$\chi_p(\mathfrak{q}) = \begin{cases} t_p & \mathfrak{q} \neq \langle p \rangle \\ 0 & \mathfrak{q} = \langle p \rangle \end{cases}$$

for positive primes p; we observed that A_X can just be viewed as the ring of all polynomials in $k[T_X] = k[t_2, t_3, t_5, t_7, \ldots] = R_{\langle 0 \rangle}$ together with its projections $R_{\langle 0 \rangle} \mapsto R_{\mathfrak{p}}$ given by $f \mapsto f \bmod t_p$ for p the prime number such that $\mathfrak{p} = \langle p \rangle$ (—a notational convention we will assume from now on).

We consider when $f, g \in A_X$ are such that $\mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(f) \geqslant \mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(g)$ for all $\mathfrak{p}$ and $\mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(f) = \mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(g) \implies \mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(g) = \infty$ or 0. If g has non-zero constant term then $\mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(g) = 0$ for all $\mathfrak{p} \in X$ so the inequality holds. Alternatively if g has constant term equal to zero, we have $g_{\mathfrak{p}} = 0 \iff g \in \langle t_p \rangle$; then if $\mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(f) \geqslant \mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(g)$ for all $\mathfrak{p}$ then for any p such that $g \in \langle t_p \rangle$ we must have $f \in \langle t_p \rangle$; furthermore, as $\mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(g)$ is the greatest power of t_p dividing g and as $\mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(g) > 0$ then we must have $\mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(f) > \mathsf{v}_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}}(g)$ and so f is divisible by some higher power of t_p. Then we can take $m_g = \prod t_p^{\alpha}$ for α the greatest power of t_p dividing g, over all p such that $g \in \langle t_p \rangle$. Then we have elements $f', g' \in A_X$ where $g' = \dfrac{g}{m_g}$ has non-zero constant term, such that $\frac{f}{g} = \frac{f'}{g'}$ (since, as the power of t_p dividing f was always strictly greater than that dividing g, we have for $f' = \dfrac{f}{m_g}$ that $f'_{\mathfrak{p}} = 0 \iff f_{\mathfrak{p}} = 0$). Thus the admissible elements $\frac{f}{g}$ of

A_X are exactly those for which g has non-zero constant term, and so hence the ring A_1 obtained at the first stage of our extension is the ring $k[t_2, t_3, t_5, t_7, \ldots]$ localised at the set of all g with non-zero constant term, that is, the set of those g not lying in any principal ideal $\langle t_p \rangle$.

We now examine the general Ershov ring R_X associated to $X = \mathsf{Spec}\,\mathbb{Z}$. The set $T_\mathfrak{q}$ for each $\mathfrak{q} \in X$ is $T_\mathfrak{q} = \{t_p \mid \mathfrak{p} \in D(p)\} = \{t_p \mid \mathfrak{q} \neq \langle p \rangle\} = T_X \setminus \{t_q\}$ where $\mathfrak{q} = \langle q \rangle$; we have $T_{\langle 0 \rangle} = T_X$. For $\mathfrak{q} \neq 0$ let us write $T_\mathfrak{q} = T_{\neq q}$. Then $V_{\mathfrak{p} \rightsquigarrow \mathfrak{p}} = k(T_{\neq p})$ and for $\mathfrak{p} \neq \langle 0 \rangle$ we have $R_{\mathfrak{p}\uparrow} = V_{\mathfrak{p} \rightsquigarrow \mathfrak{p}} = k(T_{\neq p})$; we see

$$V_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}} = k(T_{\neq p})[t_p]_{\langle t_p \rangle},$$

and then

$$R_{\langle 0 \rangle\uparrow} = \bigcap_{\mathfrak{p} \in X} V_{\langle 0 \rangle \rightsquigarrow \mathfrak{p}} = k[T_X]_{(\bigcup \langle t_p \rangle)^{\mathsf{c}}}.$$

Then we have

$$\begin{aligned}
R_X &= \{f \in \prod_{\mathfrak{p} \in X} R_{\mathfrak{p}\uparrow} \mid \forall \mathfrak{p} \in X, \mathfrak{q} \leftsquigarrow \mathfrak{p} \text{ then } f_\mathfrak{q} = \pi_{\mathfrak{p} \rightsquigarrow \mathfrak{q}}(f_\mathfrak{p})\} \\
&= \{f \in \prod_{\mathfrak{p} \in X} R_{\mathfrak{p}\uparrow} \mid f_\mathfrak{q} = \pi_{\langle 0 \rangle \rightsquigarrow \mathfrak{q}}(f_{\langle 0 \rangle}) \text{ for all } \mathfrak{p} \in X\} \\
&\cong \{r \in R_{\langle 0 \rangle\uparrow} \mid (r \bmod t_p k(T_{\neq p})[t_p]_{\langle t_p \rangle}) \in k(T_{\neq p}) \text{ for all } p\} \\
&= R_{\langle 0 \rangle\uparrow} \\
&= k[T_X]_{(\bigcup \langle t_p \rangle)^{\mathsf{c}}}.
\end{aligned}$$

But this is exactly the ring A_1 obtained at the first stage of our extension procedure, and so (since $H_X \subseteq R_X$) we have $A_1 = H_X = R_X$, and furthermore we must have that $\mathsf{Spec}\, R_X \cong X$.

We now proceed to give an example showing that the general conclusion to both of our questions is negative.

The example of $X = \big(\mathsf{Spec}\,\mathbb{Z}\big)_{\mathsf{inv}}$

We examine the ring R_X where $X = \big(\mathsf{Spec}\,\mathbb{Z}\big)_{\mathsf{inv}}$. We first pick a subbasis for X. We may take the complements of a subbasis for the space $\mathsf{Spec}\,\mathbb{Z}$. Let $D_{\mathbb{Z}}(n)$ denote the basic open set of $\mathsf{Spec}\,\mathbb{Z}$ given by $D_{\mathbb{Z}}(n) = \{\mathfrak{p} \mid n \notin \mathfrak{p}\}$. Write $D_{\mathsf{inv}}(n)$ for its complement, $\mathsf{Spec}\,\mathbb{Z} \setminus D_{\mathbb{Z}}(n) = \mathcal{V}_{\mathbb{Z}}(n) = \{\mathfrak{p} \mid n \in \mathfrak{p}\}$. The sets $D_{\mathsf{inv}}(p)$ for $p \in \mathbb{N}$ prime form an open subbasis for $X = \big(\mathsf{Spec}\,\mathbb{Z}\big)_{\mathsf{inv}}$; observe $D_{\mathsf{inv}}(p) = \mathcal{V}_{\mathbb{Z}}(p) = \{\mathfrak{q} \in X \mid p \in \mathfrak{q}\} = \{\langle p\rangle\}$. We take a set of indeterminates $T_X = \{t_{D_{\mathsf{inv}}(p)} \mid p \in \mathbb{N} \text{ prime}\}$ where we will write t_p for $t_{D_{\mathsf{inv}}(p)}$; the set of indeterminates $T_{\mathfrak{q}}$ corresponding to a point $\mathfrak{q} \in X$ is the set $T_{\mathfrak{q}} = \{t_p \mid \mathfrak{q} \in D_{\mathsf{inv}}(p)\} = \{t_p \mid p \in \mathfrak{q}\} = \{t_q\}$ where q is the positive prime number such that $\mathfrak{q} = \langle q\rangle$; except where $\mathfrak{q} = \langle 0\rangle$, in which case $T_{\langle 0\rangle} = \{t_p \mid p \in \langle 0\rangle\} = \emptyset$. In X our non-trivial specialisations are $\mathfrak{p} \rightsquigarrow \langle 0\rangle$ for $\mathfrak{p} \neq \langle 0\rangle$; we examine the corresponding valuation ring $V_{\mathfrak{p}\rightsquigarrow\langle 0\rangle}$. This involves picking some element in $T_{\mathfrak{p}} \setminus T_{\langle 0\rangle}$ and dividing any remaining indeterminates in $T_{\mathfrak{p}} \setminus T_{\langle 0\rangle}$ by this element; but $T_{\mathfrak{p}} \setminus T_{\langle 0\rangle} = T_{\mathfrak{p}} \setminus \emptyset = \{t_p\}$ for p the positive prime such that $\mathfrak{p} = \langle p\rangle$, so there is nothing to do. Then

$$V_{\mathfrak{p}\rightsquigarrow\langle 0\rangle} = k[t_p]_{\langle t_p\rangle},$$

and we see

$$R_{\mathfrak{p}\uparrow} = \bigcap_{\mathfrak{q}\leftsquigarrow\mathfrak{p}} V_{\mathfrak{p}\rightsquigarrow\mathfrak{q}} = V_{\mathfrak{p}\rightsquigarrow\langle 0\rangle} = k[t_p]_{\langle t_p\rangle};$$

and

$$R_{\langle 0\rangle\uparrow} = k(T_{\langle 0\rangle}) = k(\emptyset) = k.$$

Then $R_X = \{f \in \prod_{\mathfrak{p}\in X} R_{\mathfrak{p}\uparrow} \mid f_{\mathfrak{p}} \bmod \langle t_p\rangle = f_{\langle 0\rangle}\}$.

We show that there exist $f, g \in R_X$ with $z(g) \subseteq z(f)$ but where $f \notin \mathsf{rad}\, g$ (note this condition is necessary in all cases to have $\mathsf{Spec}\, R_X \cong X$; the fact that it was also sufficient in the case of those rings built up in chapter 2 does not concern us at present). Define

$$f = \begin{cases} f_{\mathfrak{p}} = t_p \in R_{\mathfrak{p}\uparrow} \text{ for } \mathfrak{p} \neq \langle 0 \rangle \\ f_{\langle 0 \rangle} = 0 \in R_{\langle 0 \rangle}, \end{cases}$$

and

$$g = \begin{cases} g_{\mathfrak{p}} = (t_p)^p \in R_{\mathfrak{p}\uparrow} \text{ for } \mathfrak{p} \neq \langle 0 \rangle \\ g_{\langle 0 \rangle} = 0 \in R_{\langle 0 \rangle}. \end{cases}$$

Clearly $f, g \in R_X$, and $z(g) = z(f) = \{\langle 0 \rangle\}$. Now $f \in \mathsf{rad}\, g \iff \exists r \in R_X$, $f^n = rg$ for some power $n \in \mathbb{N}$. But $f^n = rg \iff f^n_{\mathfrak{p}} = r_{\mathfrak{p}} g_{\mathfrak{p}}$ at all $\mathfrak{p} \in X$, in which case then we would have $t^n_p = r_{\mathfrak{p}} t^p_p$ at all $\mathfrak{p}$, so that $r_{\mathfrak{p}} = t^{n-p}_p$. But to have $r \in R_X$ we must have $r_{\mathfrak{p}} \in V_{\mathfrak{p} \rightsquigarrow \langle 0 \rangle}$ for all $\mathfrak{p}$, that is we must have that $\mathsf{v}_{\mathfrak{p} \rightsquigarrow \langle 0 \rangle}(r_{\mathfrak{p}}) = \mathsf{v}_{\mathfrak{p} \rightsquigarrow \langle 0 \rangle}(t^{n-p}_p) = (n-p)\mathsf{v}_{\mathfrak{p} \rightsquigarrow \langle 0 \rangle}(t_p) = n - p \geqslant 0$ for all p. But p is unbounded, and so there is no n so that $f^n = rg$ for any $r \in R_X$. □

Thus $\mathsf{Spec}\, R_X \ncong X$, and *a fortiori* $H_X \subsetneq R_X$. We could in fact see directly this latter fact, since we know that all $h \in H_X$ have finite image as functions $X \to k(T_X)$ (that is, the set $\{h_x\}_{x \in X}$ is a finite subset of $k(T_X)$), and for both our functions f, g considered in our example we have that the sets $\{f_x\}_{x \in X}$ and $\{g_x\}_{x \in X}$ are infinite, so that $f, g \in R_X \setminus H_X$.

This suggests a possible modification to our definition of the general Ershov ring R_X: let

$$\begin{aligned} \widehat{R}_X &= \{f \in R_X \mid f \text{ has finite image as a function } X \to k(T_X)\} \\ &= \{f \in R_X \mid \text{ the set } \{f_x\}_{x \in X} \text{ is a finite subset of } k(T_X)\}. \end{aligned}$$

Note that in the case that X is finite this modification does not affect the ring defined, that is for finite X we have $\widehat{R}_X = R_X$.

Clearly, as remarked, since every element of the ring H_X has finite image as a function $X \to k(T_X)$, we still have $H_X \subseteq \widehat{R}_X$. We claim that, in the case of the example just given, at least—that is, in the case for $X = \big(\mathsf{Spec}\,\mathbb{Z}\big)_{\mathsf{inv}}$—the ring $\widehat{R}_X$ is in fact the Hochster ring H_X constructed from X.

Proposition 4.4. *For $X = \big(\mathsf{Spec}\,\mathbb{Z}\big)_{\mathsf{inv}}$ and for H_X as defined in chapter 2, $\widehat{R}_X$ as defined above, then $H_X = \widehat{R}_X$.*

Maintaining the subbasis and corresponding indeterminates etc. as in the preceding example, at the first stage of Hochster's construction we define the ring $A_X = \{\prod_{\mathfrak{p}\in X} \mathsf{ev}_{\mathfrak{p}}(f) \mid f \in k[\mathcal{X}_X]\} \subseteq \prod_{\mathfrak{p}\in X} k[T_{\mathfrak{p}}]$, being the evaluation at each point $\mathfrak{p} \in X$ of polynomial expressions formed over k from the characteristic functions χ_p taking value t_p at $\mathfrak{p} = \langle p\rangle$ and zero elsewhere. It is easy to see that

$$A_X = \{f \in (k \times k[t_2] \times k[t_3] \times k[t_5] \times \dots) \mid f \text{ contains finitely many distinct indeterminates and the constant term at each co-ordinate is equal}\}.$$

We consider for which $f, g \in A_X$ do we have $\mathsf{v}_{\mathfrak{p}\rightsquigarrow\langle 0\rangle}(f_{\mathfrak{p}}) \geqslant \mathsf{v}_{\mathfrak{p}\rightsquigarrow\langle 0\rangle}(g_{\mathfrak{p}})$ for all $\mathfrak{p} \in X$ and $\mathsf{v}_{\mathfrak{p}\rightsquigarrow\langle 0\rangle}(f_{\mathfrak{p}}) = \mathsf{v}_{\mathfrak{p}\rightsquigarrow\langle 0\rangle}(g_{\mathfrak{p}}) \implies \mathsf{v}_{\mathfrak{p}\rightsquigarrow\langle 0\rangle}(g_{\mathfrak{p}}) = \infty$ or 0.

In the case that g has non-zero constant term then $\mathsf{v}_{\mathfrak{p}\rightsquigarrow\langle 0\rangle}(g_{\mathfrak{p}}) = 0$ at all $\mathfrak{p} \in X$, so that the inequality holds; note that g is only non-constant in finitely many places. Alternatively if g has constant term equal to zero, then for our inequality to hold we must have that $f_{\mathfrak{p}} = 0$ whenever $g_{\mathfrak{p}} = 0$ (in particular f has constant term equal to zero since $g_{\langle 0\rangle} = 0$), furthermore wherever $g_{\mathfrak{p}} \neq 0$ (that is, where $g_{\mathfrak{p}} = \alpha_d t_p^d + \cdots + \alpha_2 t_p^2 + \alpha_1 t_p$), where necessarily we have that $\mathsf{v}_{\mathfrak{p}\rightsquigarrow\langle 0\rangle}(g_{\mathfrak{p}}) = \mathsf{min}(\{i \mid \alpha_i \neq 0\}) > 0$, we must have either $f_{\mathfrak{p}} = 0$, so that $\frac{f_{\mathfrak{p}}}{g_{\mathfrak{p}}} = 0$, or, for $f_{\mathfrak{p}} = \beta_D t_p^D + \cdots + \beta_2 t_p^2 + \beta_1 t_p$, as we must have $\mathsf{v}_{\mathfrak{p}\rightsquigarrow\langle 0\rangle}(f_{\mathfrak{p}}) = \mathsf{min}(\{j \mid \beta_j \neq 0\}) > \mathsf{v}_{\mathfrak{p}\rightsquigarrow\langle 0\rangle}(g_{\mathfrak{p}})$, we may cancel out a factor of $\alpha_i t_p^i$ from both $f_{\mathfrak{p}}$ and $g_{\mathfrak{p}}$ (for i the least degree of a term of $g_{\mathfrak{p}}$) so that $\frac{f_{\mathfrak{p}}}{g_{\mathfrak{p}}} = \frac{f'_{\mathfrak{p}}}{g'_{\mathfrak{p}}}$ where $g'_{\mathfrak{p}} = \dfrac{1}{\alpha_i t_p^i} g_{\mathfrak{p}}$ has constant term 1 (but where the constant term of $f'_{\mathfrak{p}}$ is still zero, so that $\frac{f'_{\mathfrak{p}}}{g'_{\mathfrak{p}}} \bmod \langle t_p\rangle = 0$). Then clearly we can define functions f', g' in A_X so that

$\dfrac{f'}{g'} = \dfrac{f}{g}$ and g' has constant term 1 (—we simply need to define $g'_{\mathfrak{q}}$ in the case that $f'_{\mathfrak{q}} = 0$; the natural choice is to define $g'_{\mathfrak{q}} = 1$).

That is the admissible functions $\dfrac{f}{g}$ for which $\dfrac{f}{g}$ is not already in A_X are those for any $f \in A_X$ (—that is, a member of the product of polynomials rings which has equal constant term at each co-ordinate and only finitely many non-constant values) and for g with non-zero constant term (and again only finitely many non-constant values). Taking $B = \{\dfrac{f}{g} \mid$ for those admissible f and g just described$\}$, we set $A_1 = A_X[B] \subseteq \prod_{\mathfrak{p}\in X} k(T_{\mathfrak{p}})$.

The ring $\widehat{R}_X$ consists of all $h \in (k \times k[t_2]_{\langle t_2 \rangle} \times k[t_3]_{\langle t_3 \rangle} \times k[t_5]_{\langle t_5 \rangle} \times \ldots)$ such that for $h_{\mathfrak{p}} = \frac{f}{g} = \frac{f_0+\alpha}{g_0+\beta}$ and $h_{\mathfrak{q}} = \frac{f'}{g'} = \frac{f'_0+\gamma}{g'_0+\delta}$ where f_0, g_0, f'_0, g'_0 have constant term equal to zero, that $\frac{\alpha}{\beta} = \frac{\gamma}{\delta}$, and that f_0, g_0 etc. are non-zero for only finitely many $\mathfrak{p} \in X$ (that is, there are only finitely many non-zero f_0's, g_0's); another way of stating this latter property is that $h_{\mathfrak{p}}$ is different to $\frac{\alpha}{\beta}$ only at finitely many $\mathfrak{p}$. Clearly every such h is obtained at the first stage of our extension of the ring A_X to A_1, and so (since $H_X \subseteq \widehat{R}_X$) we have $A_1 = H_X = \widehat{R}_X$ as claimed. □

The two examples we have seen lead us to four interrelated conjectures.

(1) $\mathsf{Spec}\, \widehat{R}_X \cong X$;

(2) $H_X = \widehat{R}_X$;

(3) If X has finitely many minimal elements then $R_X = \widehat{R}_X$;

(4) If X has finitely many minimal elements then $R_X = H_X$.

Cautionary note must however be taken that the examples thus far considered are fairly elementary, in two different respects. Firstly the specialisation structure of the respective spaces $\mathsf{Spec}\, \mathbb{Z}$ and $(\mathsf{Spec}\, \mathbb{Z})_{\mathsf{inv}}$ is almost as simple as it can be, outside of a finite setting. Secondly, the respective topologies on each space are at the "extremes" of the range of possible spectral topologies having such a specialisation structure: the topology on $\mathsf{Spec}\,\mathbb{Z}$ is the coarse lower topology on its specialisation poset, and the topology on $(\mathsf{Spec}\,\mathbb{Z})_{\mathsf{inv}}$ is the fine lower topology. This may obscure nuances which arise in either the definition of H_X or of R_X and $\widehat{R}_X$ on a spectral space with more sophisticated topological structure. It is with some considerable regret that due to the short time between the discovery of these lines of inquiry and the necessary completion date of this thesis, no concrete further progress has been able to be made on these questions.

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