GRAPHS ASSOCIATED WITH SPORADIC GROUP GEOMETRIES, AND THE SEMISIMPLE ELEMENTS OF $E_8(2)$

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Contents

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List of Tables

List of Figures

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Graphs associated with sporadic group geometries, and the semisimple elements of $E_8(2)$
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This thesis studies collinearity graphs and commuting involution graphs associated with sporadic group geometries, and the conjugacy classes of semisimple elements in the exceptional Lie-type group $E_8(2)$.

First we construct plane-line collinearity graphs for the sporadic simple groups and their associated minimal and maximal 2-local parabolic geometries. For such a group G and geometry Γ, the plane-line collinearity graph takes all planes of the geometry as its vertices and joins two vertices with an edge if their planes are collinear in the geometry. We construct these graphs for the groups M_{23}, J_4, Fi_{22}, Fi_{23}, He, Co_3 and Co_2. Additionally we construct a variety of collinearity graphs associated with the minimal 2-local geometries of McL.

A second short study looks at the commuting involution graphs associated with the Baby Monster sporadic group. These are graphs which take a conjugacy class of involutions as its vertex set and joins two vertices with an edge if they commute. We detail information relating to two such graphs.

Finally, we study the conjugacy classes of semisimple elements in the exceptional group $E_8(2)$. This study is a joint work with Ali Aubad, John Ballantyne, Alexander McGaw, Peter Neuhaus, Peter Rowley and David Ward in which we determine the structure of centralisers for all such elements including information such as fixed-space dimensions and powering up maps. The ultimate aim is to determine all maximal subgroups of $E_8(2)$. This is a lengthy ongoing project and this study forms part of that effort.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

Throughout the years, group theorists have exhibited a wide variety of approaches to the study of finite groups. From combinatorial arguments concerning properties of families of abstract groups, to computational investigations in a particular individual group. It's a content rich field with many different available avenues of research. One such method concerns the study of combinatorial structures upon which a group acts. This approach not only reveals information relating to the group but the structures themselves are often interesting in their own right. Over the last forty years, a number of studies have made successful use of this approach where the structure involved was a graph or an incidence geometry. In this thesis we make two such studies. The first relates to the action of the sporadic simple groups on their associated parabolic geometries, and the construction of their collinearity graphs. The second study concerns the action of the Baby Monster sporadic simple group on its commuting involution graphs. We also present a third study which concerns the determination of a variety of information relating to the semisimple elements of the exceptional Lie-type group $E_8(2)$. This computational investigation forms part of a large collaborative project which aims to classify the maximal subgroups of $E_8(2)$, a problem which has remained open for more than thirty years.

This thesis has four main chapters which are preceded in Chapter 2 by an overview of background material, key definitions and a description of the notation and conventions adopted throughout this thesis. In Section 2.2 we give the famous classification of finite simple groups and introduce $\mathbb{ATLAS}$ notation which we will adopt as our standard. Section 2.3 introduces p-groups and defines particular kinds of p-groups which will be of key importance throughout this thesis. In Section 2.4 we explore the notion of a building, first introduced by Tits in the

1950s as a geometric framework for the study of groups of Lie type. His work inspired studies of geometries associated with sporadic groups, the subject of two our chapters. Finally in Section 2.5 we define the geometries we will be working with in this thesis, and discuss parabolic subgroups and how these relate to our sporadic group geometries. We adopt the notation introduced in the study of Buekenhout [Bue79] when describing our geometries.

In Chapter 3 we consider the work of Ronan and Smith on the maximal 2-local parabolic geometries associated with the sporadic simple groups [RS80] and the work of Ronan and Stroth on their minimal 2-local parabolic geometries [RS84]. The theory of buildings developed by Tits [Tit74] provides a geometric interpretation for the groups of Lie type. This work, together with Buekenhout's work on diagrams for group geometries [Bue79], inspired Ronan, Smith and Stroth to extend this geometric interpretation to the sporadic simple groups. Our approach here will be to study the action of the sporadic groups on their associated geometries. For such a group G and geometry Γ, it will be the case that $G \leq Aut(\Gamma)$. Thus our interest will concern the action of subgroups of G which act as stabilisers of varieties in Γ. We begin in Chapter 3 with an overview of previous studies, such as those by Rowley and Walker, which investigate the action of point stabilisers and construct the associated point-line collinearity graphs.

Definition 1.0.1 *Let $S = \{0, 1, \ldots, n-1\}$. An incidence geometry over S is a tuple $(\Gamma, \tau, *)$ where Γ is a set of varieties, τ is a mapping of Γ onto S, and $*$ is a symmetric and reflexive relation defined on Γ (the incidence relation), such that*

(TP) The restriction of τ to every maximal set of pairwise incident elements of Γ is a bijection onto S.

The property (TP) is known as the transversality property. When the incidence relation $*$ and the type map τ are clear from context, we will simply refer to the geometry as Γ.

Definition 1.0.2 *Let Γ be an incidence geometry. Then the point-line collinearity graph of Γ is the graph where the vertex set is the set of points in Γ, and edges are drawn between vertices where the associated points are collinear in Γ.*

Currently all point-line collinearity graphs associated with the sporadic simple groups and their 2-local parabolic geometries are known, with the exception of the graph relating to the Monster group $\mathbb{M}$. This is the subject of ongoing research. The main body of Chapter 3 is then devoted to a study of plane stabilisers and the construction of plane-line collinearity graphs for the sporadic simple groups M_{23}, J_4, Fi_{22}, Fi_{23}, He, Co_3 and Co_2.

Definition 1.0.3 *Let Γ be an incidence geometry. Then the plane-line collinearity graph of Γ is the graph where the vertex set consists of all planes in Γ, and edges are drawn between vertices where the associated planes intersect in a line in Γ.*

For each group we give a complete description of the graph, information relating to the orbits including sizes, stabilisers and representatives, a list of attached computer files and other assorted data. The computer files consist of group generators, orbit representatives, and code for procedures in the computer package MAGMA [BCP97]. Amongst these graph constructions we obtain results such as the following.

Theorem 1.0.4 *Let G be a sporadic simple group with a 2-local geometry Γ, and $\mathcal{G}(\Gamma)$ be the associated plane-line collinearity graph. Let h be a fixed plane of Γ with stabiliser G_h.*

(i) For $G \cong M_{23}$, $\mathcal{G}(\Gamma)$ has diameter 3 and consists of 9 G_h-orbits.

(ii) For $G \cong J_4$, $\mathcal{G}(\Gamma)$ has diameter 4 and consists of 20 G_h-orbits.

(iii) For $G \cong Fi_{22}$, $\mathcal{G}(\Gamma)$ has diameter 4 and consists of 14 G_h-orbits.

(iv) For $G \cong Fi_{23}$, $\mathcal{G}(\Gamma)$ has diameter 5 and consists of 303 G_h-orbits.

(v) For $G \cong He$, $\mathcal{G}(\Gamma)$ has diameter 7 and consists of 34 G_h-orbits.

(vi) For $G \cong Co_3$, $\mathcal{G}(\Gamma)$ has diameter 5 and consists of 53 G_h-orbits.

(vii) For $G \cong Co_2$, $\mathcal{G}(\Gamma)$ has diameter 2 and consists of 5 G_h-orbits.

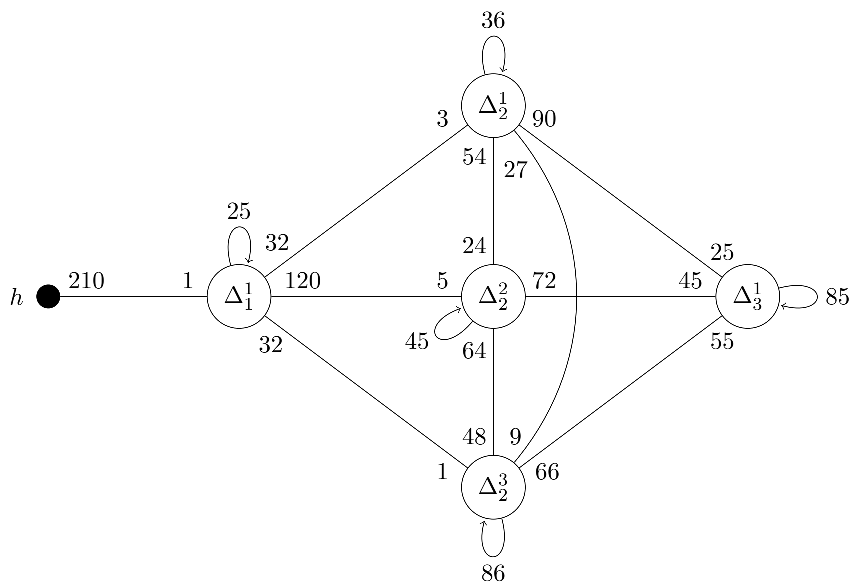


Figure 1.1: A Plane-Line Collinearity Graph for McL

Finally in Section 3.9 we summarise the information gained on collinearity graphs and list those groups and geometries for which the plane-line collinearity graphs remain yet to be constructed.

In Chapter 4 we construct a variety of collinearity graphs associated with the many geometries of the McLaughlin sporadic simple group McL. In [RS84], the authors describe four minimal 2-local geometries for McL which we examine in great detail. As with the graphs in Chapter 3, we give explicit details of the graphs, their orbits, representatives and stabilisers, together with a range of computer files.

In Chapter 5 we detail some additional information relating to the construction of commuting involution graphs associated with the Baby Monster group $\mathbb{B}$.

Definition 1.0.5 *Let G be a finite group and $X \subseteq G$. The commuting graph of G on X, denoted $\mathcal{C}(G, X)$, is the graph whose vertex set is X with vertices $x, y \in X$ joined by an edge if and only if $xy = yx$ and $x \neq y$.*

If X consists entirely of involutions, then $\mathcal{C}(G, X)$ is called a commuting involution graph.

In essence, commuting graphs first appeared in Brauer and Fowler's seminal paper 'On Groups of Even Order' [BF55], however the first example of a commuting involution graph is found in the work of Fischer on 3-transposition groups. Recent work on commuting involution graphs has been driven by Peter Rowley with the aim of determining all such graphs for the finite simple groups, amongst other interesting cases. In the case of sporadic simple groups, there are just three graphs which remain incomplete: two for the group $\mathbb{B}$ for the involution conjugacy classes $2C$ and $2D$ and one for the Monster group $\mathbb{M}$ and class $2B$.

We begin by considering the graph $\mathcal{C}(\mathbb{B}, 2C)$, whose construction was begun by Ben Wright in his PhD thesis [Wri11]. By studying the possible fusion maps for the centraliser of a $2C$ involution into $\mathbb{B}$, we determine the associated permutation character and thus the number of orbits in the commuting involution graph.

Definition 1.0.6 *Let G be a group, X be a set on which G acts, and let $x \in X$. Then the permutation rank of G is the number of distinct G_x-orbits in its action on X.*

We will prove the following result.

Theorem 1.0.7 *The permutation rank of $\mathbb{B}$ on $X = 2C$ is 163.*

Following this, we shift our attention in Section 5.3 to the graph $\mathcal{C}(\mathbb{B}, 2D)$, wherein we determine partial structure of the graph. Let $G = \mathbb{B}$, $X = 2D$ and $t \in X$. For G-conjugacy classes C, we define sets $X_C = \{x \in X | tx \in C\}$. These sets are either $C_G(t)$-orbits, or unions of $C_G(t)$-orbits. In this section we determine the size of all such sets and the location of a number of these within the graph structure of $\mathcal{C}(G, X)$.

In Chapter 6 we detail information relating to the semisimple elements of $E_8(2)$, one of the exceptional simple groups of Lie-type. Early studies of semisimple elements in Lie-type groups can be found in the papers of Steinberg [Ste63], Mizuno [Miz77], Carter [Car78], [Car81], and Deriziotis [Der81], [Der83]. Steinberg and Mizuno focus their attention on determining the conjugacy classes of semisimple elements, whilst the studies of Carter and Deriziotis focus on determining the structure of the centralisers of semisimple elements. These latter studies resulted in a general description of the centralisers for all groups of Lie type, with their twisted variants described in a later study by Deriziotis and Liebeck [DL85]. More recent work concerning semisimple elements may be found in the papers of Liebeck and Seitz [LS12], Fleischmann and Janiszczak [FJ93], and Fleischmann, Janiszczak and Knörr [FJK98].

Our aim in Chapter 6 is to give explicit descriptions of the conjugacy classes of semisimple elements of $E_8(2)$ and their centralisers. We give additional details including fixed-point space dimensions, powering-up maps, and find representatives for all 256 classes of semisimple elements.

Theorem 1.0.8 *The semisimple conjugacy classes of $E_8(2)$, together with the structure of their centralisers, dimensions of their fixed spaces and power maps, are given in Table 6.1.*

The study described in this chapter is a collaborative effort with Ali Aubad, John Ballantyne, Alexander McGaw, Peter Neuhaus, Peter Rowley and David Ward [ABM$^+$16], and forms part of an effort to classify the maximal subgroups of $E_8(2)$. Similar information was determined for semisimple elements of $E_7(2)$ by Ballantyne, Bates and Rowley [BBR15] where the maximal subgroups of $E_7(2)$ were fully classified. Vital to their efforts was the knowledge gained from their study of the semisimple elements. The collaborative project on $E_8(2)$ relies similarly on such data.

Chapter 2

Background Material

This chapter introduces the general background material and notation that is used throughout this thesis. For more information regarding group-theoretic definitions and results, we refer the reader to [Asc00], [Isa08] and [Suz82].

We begin by introducing general group notation in Section 2.1 which is used throughout this thesis. Section 2.2 introduces further notation from the Atlas of Finite Groups [CCN+09], henceforth referred to as the $\mathbb{ATLAS}$, along with information relating to the classification of finite simple groups. Section 2.3 includes background on p-groups, Section 2.4 concerns buildings associated with groups of Lie type, and finally Section 2.5 concludes this chapter with an introduction to geometries associated with the sporadic simple groups.

2.1 General Notation

Throughout this thesis, unless stated otherwise, G will denote a finite group. If H is a subgroup, proper subgroup, or normal subgroup of G, then we use the notation $H \leq G$, $H < G$, and $H \trianglelefteq G$ respectively. We use $|G|$ to denote the order of G, and $[G : H]$ to denote the index of a subgroup H in G. If p is a prime, we write $Syl_p(G)$ for the set of Sylow p-subgroups of G. The trivial group is denoted by 1, the cyclic group of order n by n, and the symmetric and alternating groups of degree n are denoted by $Sym(n)$ and $Alt(n)$ respectively. Additionally we adopt the convention that the dihedral group of order n is written $Dih(n)$.

For a given group G, we denote its centre and derived subgroup by $Z(G)$ and $G' = [G, G]$ respectively. Given elements $g, h \in G$, we use g^h to denote the conjugate of g by h, and g^G to denote the G-conjugacy class of g. The centraliser of g in G is denoted $C_G(g)$ and the normaliser of a subgroup $H \leq G$ is denoted $N_G(H)$. The commutator of g and h is written $[g,h] = g^{-1}h^{-1}gh$. If G acts on a set S and $x \in S$, we denote the stabiliser of x in G by G_x and the G-orbit of x by x^G. We also adopt the convention that all action is performed on the right, unless otherwise stated.

2.2 Finite Simple Groups and $\mathbb{ATLAS}$ Notation

2.2.1 Finite Simple Groups

A group G which has no non-trivial proper normal subgroups is called simple. A group that is not simple can be broken into two smaller groups, a normal subgroup and a quotient group, and the process can be repeated. If the group is also finite then this process will, by the Jordan-Hölder Theorem, eventually end in a unique decomposition of finite simple groups. Understanding these groups, therefore, is crucial to the study of finite group theory.

With this in mind, a huge effort was born in the twentieth century to classify all finite simple groups. This was a monumental task and took over one hundred mathematicians more than fifty years to complete, but their efforts were successful and resulted in the famous Classification Theorem for Finite Simple Groups.

Theorem 2.2.1 (The Classification Theorem for Finite Simple Groups)
Every finite simple group is isomorphic to one of the following:

(i) a cyclic group of prime order p;

(ii) an alternating group Alt(n), for $n \geq 5$*;*

(iii) a classical group:

$$\begin{array}{ll} \textit{linear}: & PSL_n(q),\ n \geq 2,\ \textit{except}\ PSL_2(2)\ \textit{and}\ PSL_2(3); \\ \textit{unitary}: & PSU_n(q),\ n \geq 3,\ \textit{except}\ PSU_3(2); \\ \textit{symplectic}: & PSP_{2n}(q),\ n \geq 2,\ \textit{except}\ PSp_4(2); \\ \textit{orthogonal}: & P\Omega_{2n+1}(q),\ n \geq 3,\ q\ \textit{odd}; \\ & P\Omega^+_{2n}(q),\ n \geq 4; \\ & P\Omega^-_{2n}(q),\ n \geq 4; \end{array}$$

where q is a power p^a *of a prime p;*

(iv) an exceptional group of Lie type:

$$G_2(q),\ q \geq 3;\ F_4(q);\ E_6(q);\ {}^2E_6(q);\ {}^3D_4(q);\ E_7(q);\ E_8(q)$$

where q is a prime power, or

$${}^2B_2(2^{2n+1}),\ n \geq 1;\ {}^2G_2(3^{2n+1}),\ n \geq 1;\ {}^2F_4(2^{2n+1}),\ n \geq 1$$

or the Tits group ${}^2F_4(2)'$;

(v) one of the 26 sporadic simple groups:

the five Mathieu groups :	$M_{11},\ M_{12},\ M_{22},\ M_{23},\ M_{24};$
the seven Leech lattice groups :	$Co_1,\ Co_2,\ Co_3,\ McL,\ HS,\ Suz,\ J_2;$
the three Fischer groups :	$Fi_{22},\ Fi_{23},\ Fi_{24}^{'};$
the five Monstrous groups :	$M,\ B,\ Th,\ HN,\ He;$
the six pariahs :	$J_1,\ J_3,\ J_4,\ O'N,\ Ly,\ Ru.$

There are a few repetitions found in the groups listed in Theorem 2.2.1. A complete list of these is given by the following isomorphisms:

$$\begin{aligned} PSL_2(4) \cong PSL_2(5) &\cong Alt(5), \\ PSL_2(7) &\cong PSL_3(2), \\ PSL_2(9) &\cong Alt(6), \\ PSL_4(2) &\cong Alt(8), \\ PSU_4(2) &\cong PSp_4(3). \end{aligned}$$

For a wonderfully written exposition on the history and proof of the classification theorem, we refer the reader to [Sol01]. A good overview of the finite simple groups and their properties can also be found in [Wil09].

2.2.2 ATLAS Notation

Groups are often characterised via extensions of their subgroups. This convention is used throughout the ATLAS and this thesis. Whilst this description does not always define a group up to isomorphism, it does give us meaningful information about the structure of the group.

Definition 2.2.2 *Given groups G and H, the direct product $G \times H$ is defined to set of ordered pairs (g, h), for $g \in G$ and $h \in H$, such that $(g, h)(g', h') = (gg', hh')$.*

Definition 2.2.3 *If G is a group with a normal subgroup N such that $G/N \cong H$, then we call G an extension of N by H, denoted $G = N.H$ or $G = NH$.*

In some literature this is also referred to as an upward extension of N by H, or alternatively as a downward extension of H by N.

Definition 2.2.4 *Let G, H and N be groups. We call G a split extension of N by H if G contains a normal subgroup N_0 complemented by a subgroup H_0 with $N \cong N_0$ and $H \cong H_0$. We denote such an extension $G = N : H$.*

A split extension is also known as an (internal) semidirect product. The structure of this product may be completely described by giving the homomorphism $\varphi : H \to Aut(N)$ which shows how H acts on N by conjugation. It may be defined as the set consisting of ordered pairs (n, h) with $n \in N$ and $h \in H$ such that $(n, h)(n', h') = (nn', h^{\varphi(n')}h')$. The notation $N^{\cdot}H$ is used to denote an extension which is not a split extension.

Definition 2.2.5 *If G is a group with subgroups H and K which commute element-wise and G is generated by H and K, then we call G the central product of H and K, denoted $G = H \circ K$.*

If $H \cap K = 1$ then G is simply the direct product of H and K.

2.3 p-Groups

Let p be a prime. A group P is called a p-group if every element of P has a power of p as its order. In the case of finite groups, an equivalent definition is $|P| = p^n$ for some $n \geq 0$. The exponent of P is the largest p^r such that p^r is the order of an element in P.

Definition 2.3.1 *An abelian p-group P of exponent p is called an elementary abelian p-group. The rank of P is $log_p|P|$.*

From this point forward, we use p^n to denote the elementary abelian p-group of rank n, that is the group $(p)^n$. In keeping with $\mathbb{ATLAS}$ notation, we write $[p^n]$ for an arbitrary p-group of order p^n.

It will often be the case that the p-groups we consider exhibit a richer structure. Let P be an arbitrary finite p-group and assume P has a chief series

$$1 = P_0 \lhd P_1 \lhd \cdots \lhd P_n = P$$

with elementary abelian sections P_i/P_{i-1} of rank k_i for $i = 1, \ldots, n$. Then we write $p^{k_1+\cdots+k_n}$ in place of P.

Definition 2.3.2 *A p-group P is called extra-special if its centre $Z(P)$ has order p and the quotient $P/Z(P)$ is a non-trivial elementary abelian p-group.*

For each prime p and positive integer n, there are exactly two extra-special groups of order p^{1+2n} up to isomorphism. These are of particular interest since they often appear in the centralisers of involutions in finite simple groups. For example, in the Monster group $\mathbb{M}$, the centraliser of a $2B$-involution has structure $2^{1+24}.Co_1$.

Before we give the full classification for extra-special p-groups, we require a few definitions. The following are all important examples of p-groups.

Definition 2.3.3 *The dihedral group $Dih(2n)$ is given by the generators and relations*

$$Dih(2n) = \langle\ x, y\ :\ x^n = y^2 = 1,\ x^y = x^{-1}\rangle.$$

The semi-dihedral or quasi-dihedral group $SD(2^n)$ is given by

$$SD(2^n) = \langle\ x, y\ :\ x^{2^{n-1}} = y^2 = 1,\ x^y = x^{2^{n-2}-1}\rangle.$$

The quaternion group Q_{4n} is given by

$$Q_{4n} = \langle\ x, y\ :\ x^{2n} = y^4 = 1,\ x^y = x^{-1},\ y^2 = x^n\rangle.$$

The modular p-group $Mod_n(p)$ is given by

$$Mod_n(p) = \langle\ x, y\ :\ x^{p^{n-1}} = y^p = 1,\ x^y = x^{1+p^{n-2}}\rangle.$$

And finally, we define

$$p_+^{1+2} = \langle\ x, y, z\ :\ x^p = y^p = z^p = 1, [x, z] = [y, z] = 1,\ [x, y] = z\rangle.$$

In the case $n = 3$, we also write $Mod_3(p) = p_-^{1+2}$. Let C_n denote the cyclic group of order n. Then the dihedral groups $Dih(2n)$ are of order $2n$, and are split extensions of C_n by C_2. The semi-dihedral groups $SD(2^n)$ are of order 2^n, and are split extensions of $C_{2^{n-1}}$ by C_2. The quaternion groups Q_{4n}, also called dicyclic groups, are of order $4n$, and are non-split extensions of C_{2n} by C_2. The groups $Mod_n(p)$ are of order p^n, and are split extensions of $C_{p^{n-1}}$ by C_p.

We now give a classification of all extra-special p-groups. In the interest of brevity, we shall denote the central product of n copies of a group H by H^n.

Theorem 2.3.4 *Every extra-special p-group P can be written as the central product of n copies of nonabelian subgroups of order p^3. Let $A = p_-^{1+2}$ and $B = p_+^{1+2}$.*

(i) If $p = 2$, then $|P| = 2^{2n+1}$ and either $P \cong Dih(8)^n$ or $P \cong Q_8 \circ Dih(8)^{n-1}$.

(ii) If $p > 2$, then $|P| = p^{2n+1}$ and either $P \cong A^n$ or $P \cong B^n$.

Also of interest to us is the notion of a p-core.

Definition 2.3.5 *Let G be a group and p be a prime. We define the p-core of G to be*

$$O_p(G) = \langle\, N \;:\; N \trianglelefteq G,\ N \text{ is a } p\text{-group}\rangle.$$

Clearly $O_p(G)$ is the largest normal p-subgroup of G. This will play a key role in our study of the parabolic geometries related to the sporadic simple groups. Of great importance for our studies here is the notion of a p-local subgroup.

Definition 2.3.6 *Let G be a group and p be a prime. A subgroup H of G is called p-local if $H = N_G(P)$ for some nonidentity p-subgroup P of G.*

The role of these subgroups is explained further in Section 2.5.2.

2.4 Buildings

The notion of a building was first introduced by Jacques Tits in a series of papers from the 1950s (see [Tit55a],[Tit55b],[Tit56]) to provide a geometric framework for understanding certain families of groups, in particular the finite groups of Lie type. His seminal book [Tit74] inspired many studies on geometries associated with groups, including those for the sporadic groups on which this thesis is based.

Before we give the definition of a building, we first recall some ideas relating to simplexes and complexes.

Definition 2.4.1 *A poset (partially ordered set) X is called a simplex of rank d if X is isomorphic to a poset formed from all subsets of a set of d elements, ordered with respect to containment.*

Whilst X is partially ordered, and thus two elements A and B in X may not be comparable, we still have a notion of extrema. For any two such elements A and B there is a greatest lower bound C, namely an element such that $C \subseteq A$, $C \subseteq B$, and C is the largest element which satisfies these conditions. In literature C is often denoted $A \cap B$. Using this idea we define the following structure.

Definition 2.4.2 *A poset Δ is called a complex if the following conditions hold:*

(i) For any two elements A and B, there is a greatest lower bound.

(ii) For any $A \in \Delta$, the subset of Δ consisting of all elements in A forms a simplex.

If any element of Δ is contained within some maximal element in Δ which is a simplex of rank d, then we say Δ is a complex of rank d.

As an example, consider a vector space V of dimension $d+1$ over a field k, and let Δ denote the set of all subspaces of V. We define the partial order $\leq$ on Δ to be reverse inclusion of subspaces, i.e. for all $A, B \in \Delta$, $A \leq B \Leftrightarrow B \subseteq A$. Clearly for any two elements A and B, the greatest lower bound $A \cap B$ is the subspace of V generated by A and B. Moreover, as Δ consists of all subspaces of V, the subset of Δ consisting of all elements in any $A \in \Delta$ is isomorphic to the poset on $(d - \dim(A))$ elements. Hence Δ is a complex of rank d.

Definition 2.4.3 *Let Δ be a complex. A subset Γ of Δ is called a subcomplex of Δ if for all $A \in \Gamma$ and $B \subseteq A$, we have that $B \in \Gamma$.*

We also require a notion of closeness in a complex. Consider our example of a complex of subspaces ordered by reverse inclusion. Suppose now that A and B are maximal elements of Δ, in other words, two subspaces of V of dimension 1. There is an element $C \in \Delta$ of rank $d-1$ which is contained in both A and B, namely a subspace of dimension 2 which contains the two subspaces of dimension 1. Thus we may conclude that A and B are close in some sense. We now introduce the notions of adjacency and connectivity within in a complex which make this idea more precise.

Definition 2.4.4 *Let Δ be a complex of rank d. We define the following:*

(i) Two maximal elements A and B in Δ are adjacent if $A \neq B$ and there exists $C \in \Delta$ of rank $d-1$ such that $C \subseteq A$ and $C \subseteq B$.

(ii) For any elements A and B in Δ we say A and B are connected if there exists a finite sequence $[C_1, \ldots, C_r]$ of maximal elements C_i of Δ such that $A \subseteq C_1$, $B \subseteq C_r$, and C_i and C_{i+1} are adjacent for $i = 1, \ldots, r-1$.

We say the complex Δ is connected if any two elements A and B are connected in this way.

We require one final definition before we are ready to give the definition of a building.

Definition 2.4.5 *Let Δ be a complex of rank d. If any element $A \in \Delta$ of rank $d-1$ is contained in precisely two maximal elements, then Δ is called a thin complex. If however there are at least three maximal elements containing any element of Δ of rank $d-1$, then Δ is called a thick complex.*

We're now able to give the main definition of this section.

Definition 2.4.6 *Let Δ be a complex of rank d. We call Δ a building of rank d if there is a collection $\mathcal{A}$ of subcomplexes of Δ such that the following conditions are satisfied:*

(i) Δ is a thick complex.

(ii) Every element of $\mathcal{A}$ is a connected, thin complex of rank d.

(iii) Given any two elements of Δ, there is an element of $\mathcal{A}$ which contains them both.

(iv) Let A and A' be two elements of Δ. If Γ and Γ' are elements of $\mathcal{A}$ which contain both A and A', then there exists an isomorphism $\phi : \Gamma \to \Gamma'$ such that for all $B \subseteq A$, $\phi(B) = B$ and for all $B' \subseteq A'$, $\phi(B') = B'$.

In literature, the elements of $\mathcal{A}$ are often called apartments and the maximal elements of Δ referred to as chambers. For more information on this subject we refer the reader to the books of Tits [Tit74] and Suzuki [Suz82].

We now show how the construction of a building relates to the study of groups. Let G be a finite group of Lie type and V be its natural G-module of dimension $d+1$ over a field k. We define a flag $\mathcal{F}$ of subspaces of rank r to be a set $\{V_1, \ldots, V_r\}$ of nonempty subspaces of V such that $0 \neq V_1 \subset V_2 \subset \cdots \subset V_r$. We will denote the set of all subspaces of V by $\mathcal{P}(V)$ and let $\Delta(\mathcal{P})$ denote the set of all flags of subspaces of V. Let $\Sigma = \{b_0, b_1, \ldots, b_d\}$ denote a basis for V. If $\mathcal{F} = \{V_0, V_1, \ldots, V_d\}$ is a flag such that for every V_i, there exists a subset Σ_i of Σ such that Σ_i is a basis for V_i, then we say that Σ supports $\mathcal{F}$, or alternatively $\mathcal{F}$ is supported by Σ. Let $\Sigma_{\Delta(\mathcal{P})}$ denote the set of all flags of subspaces of V which are supported by Σ. Then $\Delta(\mathcal{P})$ is a complex of rank d under the inclusion relation with the structure of a building, whose apartments are the complexes $\Sigma_{\Delta(\mathcal{P})}$ for each basis Σ of V. A full proof of this result may be found in [Suz82].

For each flag $\mathcal{F}$ in $\Delta(\mathcal{P})$, let $G_{\mathcal{F}}$ denote the stabiliser in G of $\mathcal{F}$. Let $\Delta = \{G_{\mathcal{F}} | \mathcal{F} \in \Delta(\mathcal{P})\}$, the collection of all such stabilisers, and partially order the set by reverse inclusion, denoted by $\subseteq$. Clearly if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two flags such that $\mathcal{F}_2$ is contained in $\mathcal{F}_1$, then it follows that $G_{\mathcal{F}_1} \subseteq G_{\mathcal{F}_2}$. Hence it follows that Δ and $\Delta(\mathcal{P})$ are isomorphic as complexes. Moreover, let $\mathcal{A}$ denote an apartment of $\Delta(\mathcal{P})$ and let $G_{\mathcal{A}}$ be the subcomplex of Δ corresponding to the stabilisers in G of elements of $\mathcal{A}$. Since $\Delta(\mathcal{P})$ is a building, and is isomorphic to Δ, it follows that Δ is a building with apartments given by the $G_{\mathcal{A}}$. We say that Δ is the building of G. Furthermore we note that G is the automorphism group of Δ.

We can work with Δ as one would work with a geometry. By identifying the elements of rank 1 in Δ with vertices and higher rank elements with their higher dimensional geometric analogues, we obtain the structure of an incidence geometry for the group G. These geometries provide us with a concrete physical entity which we can use to study and classify groups, including the sporadic simple groups. The next section explores this subject and gives further information relating to buildings for groups of Lie type.

2.5 Sporadic Group Geometries

A significant part of this thesis will be devoted to the study of various 2-local geometries associated with the sporadic simple groups. In this section we introduce the notation and basic definitions we will use throughout.

2.5.1 Definitions and Notation

Geometries, such as the Euclidean plane, are complicated objects involving concepts of length, continuity, angles and incidence. We obtain a simpler structure known as an incidence geometry by removing all such concepts and retaining only the information about which points lie on which lines. Even with this severe limitation, we are still able to prove fundamental results about its structure.

Definition 2.5.1 *Let $S = \{0, 1, \ldots, n-1\}$. An incidence geometry over S is a tuple $(\Gamma, \tau, *)$ where Γ is a set of varieties, τ is a mapping of Γ onto S, and $*$ is a symmetric and reflexive relation defined on Γ (the incidence relation), such that*

(TP) The restriction of τ to every maximal set of pairwise incident elements of Γ is a bijection onto S.

When the incidence relation $*$ and the type map τ are clear from context, we will simply refer to the geometry as Γ. Additionally, we define the rank of the geometry to be $|S|$. We refer to a (possibly empty) set of pairwise incident elements of Γ as a flag F, and define the type of F to be the image of F under τ. The transversality property (TP) indicates the importance of these sets. Clearly, such a flag F together with the restrictions of τ and $*$ to F form a geometry over $\tau(F)$.

Definition 2.5.2 *The residue geometry of a flag F in a geometry $\Gamma = (\Gamma, \tau, *)$ over S, is the geometry $\Gamma_F = (\Gamma_F, \tau_F, *_F)$ over the set $S_F = S \setminus \tau(F)$ such that*

(i) Γ_F is the set of elements of $\Gamma \setminus F$ incident with all elements of F;

(ii) τ_F and $*_F$ are the restrictions of τ and $*$ to Γ_F.

As an example, we consider the projective plane of order 2, also known as the Fano Plane. This is an incidence geometry of rank 2 consisting of seven points and seven lines. If F_1 is a flag consisting of a point p, then the residue geometry of F_1 simply consists of the three lines incident with p. Similarly, if F_2 is a flag consisting of just a line l, then the residue geometry of F_2 consists of the three points contained in l.

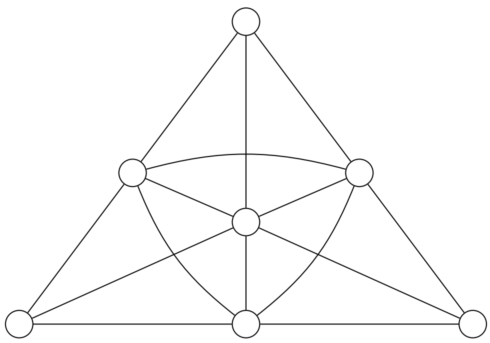


Figure 2.1: The Finite Projective Plane of Order 2

Given a geometry $(\Gamma, \tau, *)$, we may construct the incidence graph where the vertex set is Γ and two vertices $x, y \in \Gamma$ are joined by an edge if and only if $x * y$. Theoretically this is simple to construct but for larger geometries the process can be a computational nightmare. Often it is simpler and more useful to construct graph where, for example, we take the vertices to be just the points of the geometry with edges drawn between vertices where the associated points are collinear.

Definition 2.5.3 *Let Γ be an incidence geometry. Then the point-line collinearity graph of Γ is the graph where the vertex set is the set of points in Γ, and edges are drawn between vertices where the associated points are collinear in Γ.*

Alternatively, we may define our vertices to be any set of objects we choose in the geometry with incidence described by another type of object. In this thesis, we study the analogous idea of a plane-line collinearity graph with the obvious definition.

When it is not ambiguous, we will use $\mathcal{G}(\Gamma)$ to refer to a particular collinearity graph. In practice we are often dealing with larger groups and geometries for which it is more meaningful to consider our vertices to be orbits of vertices under the action of a vertex stabiliser. When describing the structure of these geometries and graphs, we will use the following notation and conventions.

Definition 2.5.4 *Given an incidence geometry* $(\Gamma, \tau, *)$ *over a set* $S = \{0, 1, \ldots, n-1\}$, *with* $i \in S$, $x \in \Gamma$ *and* $\Sigma \subseteq \Gamma$, *we define:*

$$\Gamma_i = \{y \in \Gamma | \tau(y) = i\} \text{ (the objects of type } i \text{ in } \Gamma\text{)}$$

$$\Gamma_x = \{y \in \Gamma | x * y\} \text{ (the residue geometry of } x\text{)}$$

$$\Gamma_\Sigma = \{y \in \Gamma | x * y, \forall x \in \Sigma\} \text{ (the residue geometry of } \Sigma\text{)}$$

In this thesis we will be considering geometries for which $G \leq Aut(\Gamma)$ is a sporadic simple group. We use G_x and G_Σ to denote respectively the subgroup of G fixing x and the subgroup fixing every object in Σ. Additionally for $x \in \Gamma$, we define

$$Q(x) = \{g \in G_x | g \text{ fixes every object in } \Gamma_x\}.$$

Clearly $Q(x)$ is a normal subgroup of G_x. Suppose $\mathcal{G}(\Gamma)$ is a collinearity graph for Γ with vertex set given by Γ_0, and let $x, y \in \Gamma_0$. We use $d(\cdot, \cdot)$ to denote the standard distance metric for a graph and define

$$\Delta_i(x) = \{y \in \Gamma_0 | d(x, y) = i\}.$$

We refer to $\Delta_i(x)$ as the i^{th} disk of $\mathcal{G}(\Gamma)$, and we use $\Delta_i^j(x)$ to denote the j^{th} orbit of $\Delta_i(x)$. For $x, y \in \Gamma_0$ we write $\{x, y\}^\perp = \Delta_1(x) \cap \Delta_1(y)$.

2.5.2 Parabolic Subgroups and 2-Local Geometries

As discussed in Section 2.4, for a finite simple group of Lie type there exists a natural geometry called a building. The theory of buildings provides a satisfying classification of the groups of Lie type in terms of these geometric structures; structures which may be defined using certain parabolic subgroups. This work, together with work by Buekenhout [Bue79] on geometries for the sporadic groups, inspired Ronan and Smith's study of 2-local parabolic geometries associated with the sporadic simple groups [RS80]. In their paper, the authors describe maximal 2-local geometries. A further study by Ronan and Stroth describes the minimal 2-local geometries [RS84].

Definition 2.5.5 *Let G be a finite simple group. If G is of Lie type, let p be the characteristic of the field over which G is defined, else if G is sporadic let p be any prime. Let B denote the normaliser in G of a Sylow p-subgroup. A subgroup $P \leq G$ is called a minimal parabolic subgroup of G if B is contained in a unique maximal subgroup of P.*

Moreover, every subgroup containing B is of the form $\langle P_1, \ldots, P_m \rangle$ where $P_1, \ldots, P_m$ are minimal parabolic subgroups. The conjugates of these subgroups are called the parabolic subgroups of G. In the Lie type case, we call B a Borel subgroup. In the case where G is of Lie type, the parabolic subgroups form the simplexes of a building of G (where the building itself consists of all such subgroups partially ordered by reverse inclusion). The situation is different in the case where G is a sporadic group, since it may not even be the case that G can be generated using minimal parabolic subgroups. To account for this we relax the definition slightly and instead of requiring our minimal parabolic subgroups to contain a Sylow p-subgroup, we ask that they are p-local.

Definition 2.5.6 *Let $P_1, \ldots, P_n$ be minimal parabolic subgroups of G. We call $\{P_1, \ldots, P_n\}$ a minimal parabolic system of rank n if $G = \langle P_1, \ldots, P_n \rangle$ and no proper subset generates G.*

If G is a group of Lie type in characteristic p with Lie rank at least 2, we find exactly one such system. Namely, the one which arises from the building of G. In the case where G is sporadic, the following are true [RS84]:

(i) Every sporadic group with the exception of J_1 and M_{11} admits a minimal parabolic system of rank at least 2 for $p = 2$;

(ii) G may have more than one minimal parabolic system for a given prime p.

For an example of the last remark, see the case $G = McL$ with $p = 2$. By considering the parabolic subgroups associated with a given minimal parabolic system of a group G, we find that the set of all such minimal parabolic systems can be partitioned into two sets; geometric and non-geometric.

Definition 2.5.7 *A minimal parabolic system is called geometric if for any two associated parabolic subgroups P and Q, we have that $P \cap Q$ is also a parabolic subgroup. A system which is not geometric is called non-geometric.*

Let $S = \{P_1, \ldots, P_n\}$ be a minimal parabolic system for a group G with respect to B, the normaliser of a Sylow p-subgroup. For $g \in G$, it is clear that $S^g = \{P_1^g, \ldots, P_n^g\}$ is a minimal parabolic system for G with respect to B^g. We define $P_\emptyset = B$ and

$$\Delta = \bigcup_{g \in G} \{P_I^g : I \subsetneq \{1, \ldots, n\}\}.$$

We find that Δ is a complex, as defined in Definition 2.4.2. We may now construct a similar setup to buildings in the following way. We define the vertices of Δ to be parabolic subgroup P_I^g for some $g \in G$ and $I \subseteq \{1, \ldots, n\}$ of cardinality $n-1$ and order with respect to reverse inclusion. The edges of Δ are parabolic subgroups P_I^g corresponding to $I \subseteq \{1, \ldots, n\}$ of cardinality $n-2$, and so forth. Lastly we define the conjugates of $P_\emptyset$ to be the chambers of the Δ.

The group G acts naturally on Δ by conjugation. We note that minimal parabolic systems for Lie type groups in their natural characteristic are always geometric, and such systems give rise to a building in the usual sense. All minimal parabolic geometries considered in this thesis will be associated with geometric minimal parabolic systems for $p = 2$.

We also consider maximal 2-local parabolic geometries, wherein we work with maximal parabolic subgroups. Given a minimal parabolic system $\{P_1, \ldots, P_n\}$, these are subgroups $P_{\{1,\ldots,n\}\setminus\{i\}}$ generated by $n-1$ minimal parabolic subgroups. For a group of Lie type G in characteristic p with an associated building Δ, the stabiliser P of a vertex in Δ, that is an element of rank 1, is a maximal parabolic subgroup which is p-constrained and contains a Sylow p-subgroup. In this thesis we will consider geometries for which the stabiliser of a vertex is a maximal 2-constrained 2-local subgroup, without the requirement that it contains a Sylow 2-subgroup.

To see how incidence geometries arise from such systems, consider the following. Let $S = \{P_1, \ldots, P_n\}$ be a minimal parabolic system for a group G with respect to B, the normaliser of a Sylow p-subgroup. Then we may form a geometry in the following way.

Let R be the set of all conjugates in G of the maximal parabolic subgroups $P_{\{1,\ldots,n\}\setminus\{i\}}$, $I = \{1, \ldots, n\}$, and define the map $\tau : R \to I$ by $\tau : P_{\{1,\ldots,n\}\setminus\{i\}} \mapsto i$. Additionally define the incidence relation $*$ by $P^g_{\{1,\ldots,n\}\setminus\{i\}} * P^h_{\{1,\ldots,n\}\setminus\{j\}}$ precisely when $i \neq j$ and there is some $k \in G$ such that $B^k \subseteq P^g_{\{1,\ldots,n\}\setminus\{i\}} \cap P^h_{\{1,\ldots,n\}\setminus\{j\}}$. As the transversality property holds, we conclude that the tuple $(R, \tau, *)$ is an incidence geometry. The rank of the geometry is equal to the rank of the minimal parabolic system S.

To each geometry is associated a diagram. For an extensive description of such diagrams, we refer the reader to [Bue79]. A hollow node ○ indicates that the 2-local subgroup P of G does not contain a Sylow 2-subgroup of G. A square node □ indicates that whilst there are no vertices in the geometry belonging to it, there are certain subgeometries in which objects can be associated to it. Adjacent to each node of a diagram we write $\frac{A}{B}$ to denote the vertex stabiliser given as a split extension $A : B$, where A acts trivially on the residue of the given vertex.

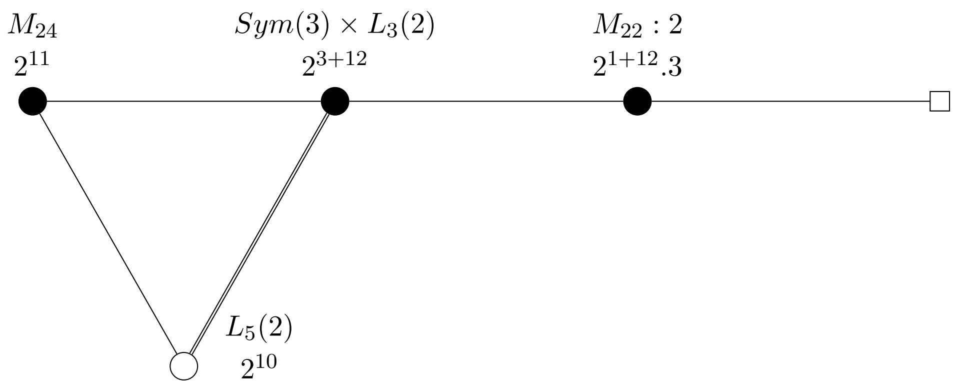


Figure 2.2: The Maximal 2-local Geometry for J_4

As an example, consider the above geometry. The $2^{11} : M_{24}$ vertex is shaded indicating that this subgroup contains a Sylow 2-subgroup of J_4, whilst the $2^{10} : L_5(2)$ is unshaded indicating that it does not contain a Sylow 2-subgroup of J_4. If we refer to the three shaded vertices as points, line and planes, reading left to right, we note that adjacent to the planes vertex is a square node. This indicates that this geometry has subgeometries which contain objects corresponding to a node here.

Any additional notation required will be explained in the relevant section.

Chapter 3

Plane-Line Collinearity Graphs

The classification of finite simple groups is arguably one of the most profound mathematical theorems ever proven, and perhaps one of the most intriguing. Every finite simple group belongs to one of three infinite families, else it is one of just twenty-six particular exceptions. These exceptions, the sporadic simple groups, have been studied by many mathematicians with great interest. The Mathieu groups, described by French mathematician Emile Mathieu in the mid-late nineteenth century, were the first sporadic groups to be discovered. They are remarkable groups: for example, with the exception of the symmetric and alternating groups, M_{12} and M_{24} are the only 5-transitive permutation groups. Almost a century passed before the discovery of the next sporadic group.

The five Mathieu groups are not just among the twenty-six sporadic simple groups; they are closely related to almost every sporadic group. They also arise as automorphism groups of particular discrete geometries. A natural question to ask then is: do the other sporadic groups also arise as automorphism groups of discrete geometries? And perhaps more meaningful: is there anything useful to be gained by studying the action of the sporadic groups on these associated geometries?

In this chapter we consider minimal and maximal 2-local parabolic geometries associated with the sporadic simple groups M_{23}, J_4, Fi_{22}, Fi_{23}, He, Co_3 and Co_2. For an introduction to the study of such geometries, we refer the reader to Section 2.5. Later in Chapter 4 we will construct a variety of collinearity graphs associated with the geometries of the group McL.

3.1 Literature Review

3.1.1 Point-Line Collinearity Graphs

Central to the study of the sporadic group geometries, are the structure of their point-line collinearity graphs. These are graphs $\mathcal{G}(\Gamma)$ for which the vertices are orbits of points under the action of a point-stabiliser in a group $G \leq Aut(\Gamma)$, with an edge drawn between two orbits if points in one orbit are collinear with points in the second. Many of these graphs have been determined and we summarise the results here. For those geometries studied later, a full description is given in the relevant section. Otherwise we refer the reader to papers where further information may be obtained.

The graph for Fi_{22} was calculated by Rowley and Walker in [RW96].

Theorem 3.1.1 *Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for the minimal 2-local geometry Γ for $G = Fi_{22}$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ has diameter 3.

(ii) $|\Delta_1(h)| = 154$ and $\Delta_1(h)$ is a G_h-orbit.

(iii) $|\Delta_2(h)| = 8,624$ and $\Delta_2(h)$ consists of two G_h-orbits.

(iv) $|\Delta_3(h)| = 133,376$ and $\Delta_3(h)$ consists of four G_h-orbits.

In another substantial study [RW11], [RW12a] and [RW12b], Rowley and Walker determined the point-line collinearity graph for the maximal 2-local geometry for Fi_{23}.

Theorem 3.1.2 *Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for the maximal 2-local geometry Γ for $G = Fi_{23}$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ has diameter 4.

(ii) $|\Delta_1(h)| = 506$ and $\Delta_1(h)$ is a G_h-orbit.

(iii) $|\Delta_2(h)| = 141,680$ and $\Delta_2(h)$ consists of two G_h-orbits.

(iv) $|\Delta_3(h)| = 29,233,920$ and $\Delta_3(h)$ consists of six G_h-orbits.

(v) $|\Delta_4(h)| = 166,371,328$ and $\Delta_4(h)$ consists of six G_h-orbits.

In a further study [RW10] determined the first three discs of the point-line collinearity graph for Fi_{24}'. This work was aided by the fact the graph for Fi_{23} embeds into the Fi_{24}' graph. The graph was later completed by Rowley and Wright [RW16].

Theorem 3.1.3 *Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for the maximal 2-local geometry Γ for $G = Fi_{24}'$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 5.*

(ii) $|\Delta_1(h)| = 1,518$ *and $\Delta_1(h)$ is a G_h-orbit.*

(iii) $|\Delta_2(h)| = 1,560,504$ *and $\Delta_2(h)$ consists of three G_h-orbits.*

(iv) $|\Delta_3(h)| = 1,400,874,432$ *and $\Delta_3(h)$ consists of ten G_h-orbits.*

(v) $|\Delta_4(h)| = 656,569,113,600$ *and $\Delta_4(h)$ consists of 46 G_h-orbits.*

(vi) $|\Delta_5(h)| = 1,845,442,396,160$ *and $\Delta_5(h)$ consists of 59 G_h-orbits.*

In their 2011 paper [RT11b], Rowley and Taylor studied the minimal 2-local parabolic geometries associated with the sporadic groups HN and Th. Their motivation for constructing the point-line collinearity graphs was that both graphs appear as full subgraphs of the point-line collinearity graph for the Monster group $\mathbb{M}$.

Theorem 3.1.4 *Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for the minimal parabolic geometry Γ for $G = HN$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 5.*

(ii) $|\Delta_1(h)| = 150$ *and $\Delta_1(h)$ is a G_h-orbit.*

(iii) $|\Delta_2(h)| = 17,760$ *and $\Delta_2(h)$ consists of three G_h-orbits.*

(iv) $|\Delta_3(h)| = 1,638,400$ *and $\Delta_3(h)$ consists of eight G_h-orbits.*

(v) $|\Delta_4(h)| = 68,721,664$ *and $\Delta_4(h)$ consists of fifty G_h-orbits.*

(vi) $|\Delta_5(h)| = 3,686,400$ *and $\Delta_5(h)$ consists of three G_h-orbits.*

Theorem 3.1.5 *Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for the minimal parabolic geometry Γ for $G = Th$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 5.*

(ii) $|\Delta_1(h)| = 270$ *and $\Delta_1(h)$ is a G_h-orbit.*

(iii) $|\Delta_2(h)| = 64,800$ *and $\Delta_2(h)$ consists of two G_h-orbits.*

(iv) $|\Delta_3(h)| = 15,060,480$ *and $\Delta_3(h)$ consists of six G_h-orbits.*

(v) $|\Delta_4(h)| = 858,497,006$ *and* $\Delta_4(h)$ *consists of twenty* G_h*-orbits.*

(vi) $|\Delta_5(h)| = 103,219,200$ *and* $\Delta_5(h)$ *consists of two* G_h*-orbits.*

The graphs for HN and Th were determined computationally using MAGMA. For both geometries, a point h has stabiliser $G_h \cong C_G(t)$ where t is an involution in the group G and is the unique non-trivial central element of $C_G(t)$. Using this fact, Rowley and Taylor identified the points of Γ with the conjugacy class t^G. Under this correspondence, two points h_1 and h_2 are incident in the geometry if their corresponding involutions t_1 and t_2 are such that $t_1 \in O_2(C_G(t_2))$. Similar correspondences will feature in our study of the plane-line collinearity graphs.

The graphs for the groups He and Suz were constructed by Rowley in an unpublished manuscript [Row15].

Theorem 3.1.6 *Let* $\mathcal{G}(\Gamma)$ *be the point-line collinearity graph for the minimal 2-local geometry* Γ *for* $G = He$ *and let* $h \in \mathcal{G}(\Gamma)$*. Then*

(i) $\mathcal{G}(\Gamma)$ *has diameter 4.*

(ii) $|\Delta_1(h)| = 90$ *and* $\Delta_1(h)$ *is a* G_h*-orbit.*

(iii) $|\Delta_2(h)| = 3,720$ *and* $\Delta_2(h)$ *consists of three* G_h*-orbits.*

(iv) $|\Delta_3(h)| = 24,960$ *and* $\Delta_3(h)$ *consists of six* G_h*-orbits.*

(v) $|\Delta_4(h)| = 384$ *and* $\Delta_4(h)$ *is a* G_h*-orbit.*

Theorem 3.1.7 *Let* $\mathcal{G}(\Gamma)$ *be the point-line collinearity graph for the minimal 2-local geometry* Γ *for* $G = Suz$ *and let* $h \in \mathcal{G}(\Gamma)$*. Then*

(i) $\mathcal{G}(\Gamma)$ *has diameter 4.*

(ii) $|\Delta_1(h)| = 60$ *and* $\Delta_1(h)$ *is a* G_h*-orbit.*

(iii) $|\Delta_2(h)| = 2,400$ *and* $\Delta_2(h)$ *consists of two* G_h*-orbits.*

(iv) $|\Delta_3(h)| = 53,760$ *and* $\Delta_3(h)$ *consists of four* G_h*-orbits.*

(v) $|\Delta_4(h)| = 349,184$ *and* $\Delta_4(h)$ *consists of five* G_h*-orbits.*

In a substantial study spanning three papers [RW07], [RW08a], [RW08b] Rowley and Walker exhausted many properties of the J_4 maximal 2-local geometry. They also described a computer-free construction of the point-line collinearity graph. To do so, they assumed some geometric data. For further information, see Section 3.3.

Theorem 3.1.8 *Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for the maximal 2-local geometry Γ for $G = J_4$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 3.*

(ii) *$|\Delta_1(h)| = 15,180$ and $\Delta_1(h)$ is a G_h-orbit.*

(iii) *$|\Delta_2(h)| = 36,071,728$ and $\Delta_2(h)$ consists of three G_h-orbits.*

(iv) *$|\Delta_3(h)| = 136,980,480$ and $\Delta_3(h)$ consists of two G_h-orbits.*

In their studies of the Leech lattice and involution geometries (see [Iva99],[IS02]), Ivanov and Shpectorov detail the point-line collinearity graph which is associated with both the maximal and minimal 2-local geometries for the group Co_2. This graph and its orbits are constructed and detailed thoroughly in papers by Rowley and Walker [RW04c], [RW94].

Theorem 3.1.9 *Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for the maximal 2-local geometry Γ for $G = Co_2$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 3.*

(ii) *$|\Delta_1(h)| = 462$ and $\Delta_1(h)$ is a G_h-orbit.*

(iii) *$|\Delta_2(h)| = 23,584$ and $\Delta_2(h)$ consists of two G_h-orbits.*

(iv) *$|\Delta_3(h)| = 22,528$ and $\Delta_3(h)$ is a G_h-orbit.*

Finally, in [RW04a] and [RW04b], Rowley and Walker determined the point-line collinearity graph for the minimal 2-local geometry of the Baby Monster group $\mathbb{B}$. Throughout the paper, they did not assume that that group was $\mathbb{B}$, only that Γ was a rank 4 geometry and G a subgroup of $Aut(\Gamma)$ with the following properties:

(i) Γ is a string geometry.

(ii) For $l \in \Gamma_1$, $|\Gamma_0(l)| = 3$ and two collinear points in Γ determine a unique line.

(iii) For $a \in \Gamma_0$ and $X \in \Gamma_3$, Γ_a is isomorphic to the Co_2-minimal parabolic geometry and Γ_X is isomorphic to a projective 3-space geometry (over $GF(2)$).

(iv) G acts flag transitively on Γ.

(v) For $a \in \Gamma_0$, $G_a \cong 2^{1+22}Co_2$, $Q(a) \cong 2^{1+22} = O_2(G_a)$ and $Z_1(a) = Z(G_a) = Z(Q(a)) = \mathbb{Z}_2$. Moreover $Q(a)/Z(Q(a))$ is isomorphic to the irreducible 22-dimensional $GF(2)$ Co_2-module which occurs as a composition factor in the Leech lattice reduced mod 2.

(vi) Let $l \in \Gamma_1$ and $X \in \Gamma_3$, then $G_l \cong 2^{2+10+20}(Sym(3) \times M_{22}.2)$ has a unique maximal normal subgroup of order 2^2 and $G_X \cong 2^{9+16+6+4}L_4(2)$ with $Q(X) = O_2(G_X) \cong 2^{9+16+6+4}$.

After much toil, they produced the point-line collinearity graph.

Theorem 3.1.10 *Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for the minimal 2-local geometry Γ of $\mathbb{B}$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 4.*

(ii) *$|\Delta_1(h)| = 93,150$ and $\Delta_1(h)$ is a G_h-orbit.*

(iii) *$|\Delta_2(h)| = 4,466,563,200$ and $\Delta_2(h)$ consists of three G_h-orbits.*

(iv) *$|\Delta_3(h)| = 7,702,219,980,800$ and $\Delta_3(h)$ consists of four G_h-orbits.*

(v) *$|\Delta_4(h)| = 4,000,762,036,224$ and $\Delta_4(h)$ is a G_h-orbit.*

3.1.2 Plane-Line Collinearity Graphs

Less well explored are the plane-line collinearity graphs associated with the sporadic simple groups. Naturally in order for such graphs to exist we require that the associated geometry has rank at least three. There are fourteen sporadic groups for which such a geometry exists, namely

$$M_{23},\ M_{24},\ Suz,\ He,\ McL,\ Co_3,\ Co_2,\ Co_1,\ Fi_{22},\ Fi_{23},\ Fi'_{24},\ J_4,\ \mathbb{B} \text{ and } \mathbb{M}.$$

For three of these groups, M_{24}, Suz and Co_1, the plane-line collinearity graph is known. The graph for M_{24} was constructed by Rowley in [Row04].

Theorem 3.1.11 *Let $\mathcal{G}(\Gamma)$ be the plane-line collinearity graph for the minimal 2-local geometry Γ of M_{24} and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 6.*

(ii) *$|\Delta_1(h)| = 14$ and $\Delta_1(h)$ is a G_h-orbit.*

(iii) $|\Delta_2(h)| = 168$ *and* $\Delta_2(h)$ *is a* G_h*-orbit.*

(iv) $|\Delta_3(h)| = 1,428$ *and* $\Delta_3(h)$ *consists of two* G_h*-orbits.*

(v) $|\Delta_4(h)| = 7,630$ *and* $\Delta_4(h)$ *consists of five* G_h*-orbits.*

(vi) $|\Delta_5(h)| = 2,016$ *and* $\Delta_5(h)$ *consists of three* G_h*-orbits.*

(vii) $|\Delta_6(h)| = 128$ *and* $\Delta_6(h)$ *is a* G_h*-orbit.*

The graph for the Suzuki group Suz was constructed by Yoshiara in [Yos88].

Theorem 3.1.12 *Let* $\mathcal{G}(\Gamma)$ *be the plane-line collinearity graph for the minimal 2-local geometry* Γ *of* Suz *and let* $h \in \mathcal{G}(\Gamma)$*. Then*

(i) $\mathcal{G}(\Gamma)$ *has diameter 4.*

(ii) $|\Delta_1(h)| = 54$ *and* $\Delta_1(h)$ *is a* G_h*-orbit.*

(iii) $|\Delta_2(h)| = 2,088$ *and* $\Delta_2(h)$ *consists of two* G_h*-orbits.*

(iv) $|\Delta_3(h)| = 26,496$ *and* $\Delta_3(h)$ *consists of two* G_h*-orbits.*

(v) $|\Delta_4(h)| = 106,496$ *and* $\Delta_4(h)$ *consists of three* G_h*-orbits.*

In [IS02], Shpectorov and Ivanov study involution geometries wherein they describe the plane-line collinearity graph for the maximal 2-local geometry of Co_1.

Theorem 3.1.13 *Let* $\mathcal{G}(\Gamma)$ *be the plane-line collinearity graph for the maximal 2-local geometry* Γ *of* Co_1 *and let* $h \in \mathcal{G}(\Gamma)$*. Then*

(i) $\mathcal{G}(\Gamma)$ *has diameter 4.*

(ii) $|\Delta_1(h)| = 270$ *and* $\Delta_1(h)$ *is a* G_h*-orbit.*

(iii) $|\Delta_2(h)| = 47,160$ *and* $\Delta_2(h)$ *consists of two* G_h*-orbits.*

(iv) $|\Delta_3(h)| = 2,971,200$ *and* $\Delta_3(h)$ *consists of three* G_h*-orbits.*

(v) $|\Delta_4(h)| = 43,602,944$ *and* $\Delta_4(h)$ *consists of four* G_h*-orbits.*

The rest of this chapter explores plane-line collinearity graphs for a variety of the sporadic simple groups. Each section is separated into three subsections: one defines the associated geometry, the next describes the shape and structure of the graph and its orbits, and finally one in which we describe the computer files attached to the study.

3.2 Plane-Line Graph for M_{23}

The first five of the sporadic simple groups to be discovered were the Mathieu groups, first described by Emile Mathieu in the 19th century (see [Mat61] and [Mat73]). Their significance is well established, and are examples of multiply-transitive permutation groups.

Definition 3.2.1 *A group is called k-transitive if there exists a set of elements on which the groups acts faithfully and k-transitively.*

Remark. Note that transitivity computed from a particular permutation representation may not give the maximal transitivity of the abstract group. For example, the Higman-Sims sporadic simple group HS has a 1-transitive representation of degree 100 and a 2-transitive representation of degree 176. Further, whilst k-transitivity of groups and graphs are related, they are not identical concepts.

The Mathieu groups M_{12} and M_{24} are the only 5-transitive groups besides $Sym(5)$ and $Alt(7)$. The groups M_{11} and M_{23} are 4-transitive, while M_{22} is 3-transitive.

When Mathieu first proposed the existence of these groups it was, at least initially, difficult to see that these groups were not just alternating groups in a different guise. Any such doubt was lifted by Witt in 1938 who constructed these groups explicitly as automorphism groups of Steiner systems (see [Wit38a] and [Wit38b]).

Definition 3.2.2 *A Steiner system with parameters t, k, n, written $S(t, k, n)$, is an n-element set S together with a family of k-element subsets (called blocks) with the property that each t-element subset of S is contained within exactly one block.*

The canonical example of such a system is the finite projective plane of order q. This is an $S(2, q + 1, q^2 + q + 1)$ system, since it is comprised of $q^2 + q + 1$ points, each line passes through $q + 1$ points, and pairs of distinct points lie on exactly one line.

The Mathieu groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24} arise as automorphism groups of the Steiner systems $S(4, 5, 11)$, $S(5, 6, 12)$, $S(3, 6, 22)$, $S(4, 7, 23)$ and $S(5, 8, 24)$ respectively. In each case, the group is the full automorphism group with the exception of M_{22} which is the unique index 2 subgroup of the full automorphism group of $S(3, 6, 22)$. It is these Steiner systems which give rise to geometries associated with the Mathieu groups, with only M_{23} and M_{24} exhibiting 2-local geometries of rank 3. Since the plane-line collinearity graph for M_{24} has already been determined in [Row04], we turn our attention to M_{23} and $S(4, 7, 23)$.

3.2.1 The Minimal 2-local Geometry

The minimal 2-local geometry for M_{23} was defined by Ronan and Stroth in [RS84].

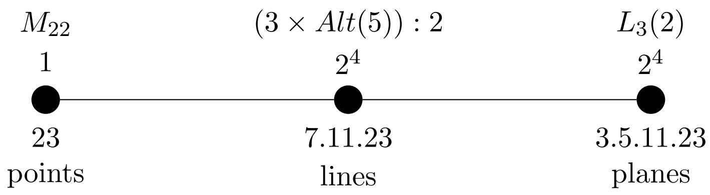


Figure 3.1: The Minimal 2-local Geometry for M_{23}

This geometry is defined on a Steiner system $S(4,7,23)$. Points of the geometry correspond to the 23 points of the Steiner system, the lines are trios (3-sets), and planes are one class of projective planes formed from the seven points of a heptad (7-set). The minimal parabolic system associated with this geometry is $\{2^4.L_2(2), 2^4.L_2(2), 2^4.Sym(5)\}$.

As a point of interest, the minimal geometry for $Alt(7)$ given in Figure 3.2 appears as a subgeometry with automorphism group $2^4 Alt(7)$. This subgeometry was first discovered by Neumaier [Neu84] and is the only known example of a geometry locally isomorphic to a finite building of type C_3 which is not covered by a building (see [Tit81]).

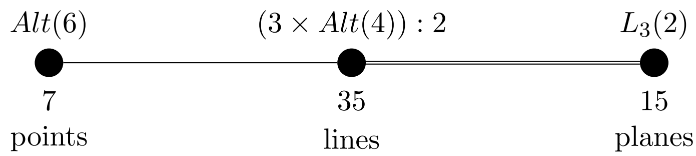


Figure 3.2: A Minimal 2-local Geometry for $Alt(7)$

We work in MAGMA with the 23-dimensional permutation representation of $G = M_{23}$ using the generators given on the Online Atlas. We construct the sets of points, lines and planes as G-sets using the following commands.

$> Points := GSet(G);$

$> Lines := GSet(G, \{\{1,2,3\}\});$

$> Planes := GSet(G, \{\{\{1,2,3\},\{1,5,11\},\{1,4,17\},\{3,5,4\},\{3,11,17\},$
$\{2,5,17\},\{2,4,11\}\}\});$

These commands, together with orbit representatives, are given in the computer file described at the end of this section.

3.2.2 The Graph

The point-line collinearity graph for M_{23} under the action of the point stabiliser M_{22} is a trivial construction, since any two points of the geometry lie on a shared line. The plane-line collinearity graph $\mathcal{G}(\Gamma)$ however requires a little more thought.

Theorem 3.2.3 *Let $\mathcal{G}(\Gamma)$ be the plane-line collinearity graph for the minimal parabolic geometry Γ for $G = M_{23}$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ has diameter 3.

(ii) $|\Delta_1(h)| = 98$ and $\Delta_1(h)$ consists of two G_h-orbits.

(iii) $|\Delta_2(h)| = 2,800$ and $\Delta_2(h)$ consists of five G_h-orbits.

(iv) $|\Delta_3(h)| = 896$ and $\Delta_3(h)$ is a G_h-orbit.

A graphical representation of $\mathcal{G}(\Gamma)$ is given in Figure 3.3 with adjacency detailed in Table 3.1.

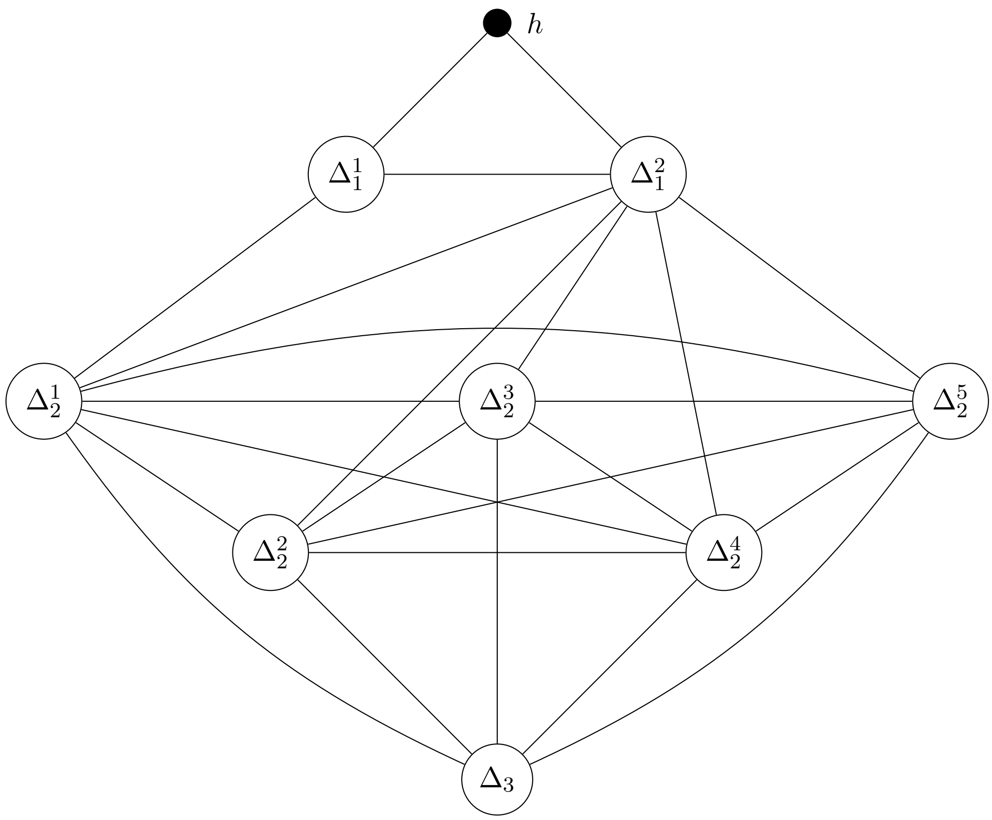


Figure 3.3: The Plane-Line Collinearity Graph of M_{23}

	h	$\Delta_1^1(h)$	$\Delta_1^2(h)$	$\Delta_2^1(h)$	$\Delta_2^2(h)$	$\Delta_2^3(h)$	$\Delta_2^4(h)$	$\Delta_2^5(h)$	$\Delta_3^1(h)$
h	–	14	84	0	0	0	0	0	0
$\Delta_1^1(h)$	1	13	12	72	0	0	0	0	0
$\Delta_1^2(h)$	1	2	15	8	20	20	16	16	0
$\Delta_2^1(h)$	0	3	2	21	4	4	8	32	24
$\Delta_2^2(h)$	0	0	5	4	15	10	20	20	24
$\Delta_2^3(h)$	0	0	5	4	10	15	12	44	8
$\Delta_2^4(h)$	0	0	3	6	15	9	18	21	26
$\Delta_2^5(h)$	0	0	1	8	5	11	7	48	18
$\Delta_3^1(h)$	0	0	0	9	9	3	13	27	37

Table 3.1: The Collapsed Adjacency Matrix for the Plane-Line Collinearity Graph of M_{23}

Lemma 3.2.4 *Each plane is incident with seven lines. Each line is incident with fifteen planes.*

Proof. Let h and l be an arbitrary plane and line respectively. Since h is a projective plane of order 2, by definition h is incident with seven lines. The line l is a trio of $S(4, 7, 23)$ wherein we have five heptads containing l. Each heptad gives rise to three projective planes containing l. □

Corollary 3.2.5 *For a fixed plane h, $|\Delta_1(h)| = 98$.*

This follows immediately from Lemma 3.2.4, since $|\Delta_1(h)| = 7 \times (15 - 1)$.

Theorem 3.2.6 *For any two planes h and x, $|h \cap x| \neq 0$.*

This result was proven by R. Noda in [Nod72]. Interestingly, the only Steiner systems in which any two blocks meet are projective planes $S(4, 7, 23)$ and $S(t, t + 1, 2t + 3)$. We refine this intersection further.

Lemma 3.2.7 *For any two planes h and x, $|h \cap x| = 1$, 3 or 7.*

Proof. Since a 4-set determines a unique heptad in $S(4, 7, 23)$, and by Theorem 3.2.6 two planes cannot have a null-intersection, clearly either $|h \cap x| = 7$ or $1 \leq |h \cap x| \leq 3$. The case $|h \cap x| = 2$ is easily discarded since any two points determine a unique line, and thus if h and x meet in two points, they must also meet in the unique line determined by these points and so $|h \cap x| \geq 3$. □

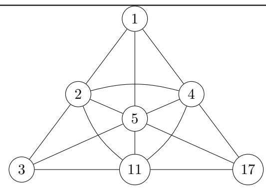
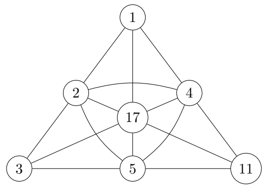
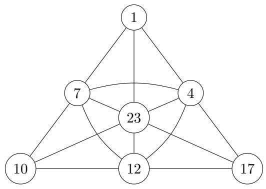
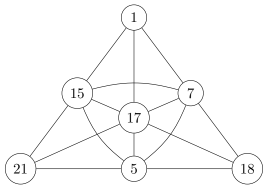
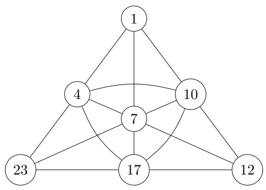
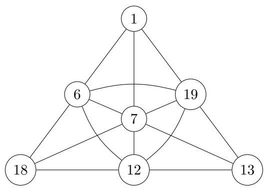
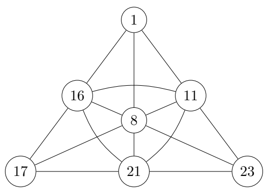
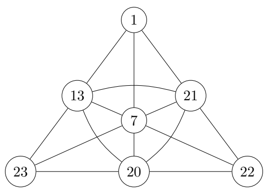
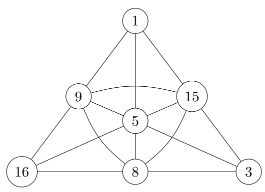
Orbit	Size	Representative x	$G_{h,x}$	$\lvert h \cap x \rvert$	$\lvert \{h \cap x\}^{\perp} \rvert$
h	1		$2^4 : L_3(2)$	7	98
$\Delta_1^1(h)$	14		$2^4 : Alt(4)$	7	25
$\Delta_1^2(h)$	84		$2^2.Dih(8)$	3	17
$\Delta_2^1(h)$	336		$Dih(8)$	3	5
$\Delta_2^2(h)$	336		$Dih(8)$	3	5
$\Delta_2^3(h)$	336		$Dih(8)$	1	5
$\Delta_2^4(h)$	448		$Sym(3)$	3	3
$\Delta_2^5(h)$	1344		2	1	1
$\Delta_3^1(h)$	896		3	3	0

Table 3.2: The G_h-orbits of the Plane-Line Collinearity Graph for M_{23}

Information relating to the G_h-orbits of the graph is given in Table 3.2.2. The orbits are easily determined in MAGMA by directly asking for the orbits under G_h of the set *planes* defined in Section 3.2.1. Additionally for a representative x in each orbit $\Delta_i^j(h)$, we give the stabiliser $G_{h,x}$, the size of the intersection $h \cap x$, and number of planes incident with both h and x, denoted $|\{h \cap x\}^\perp|$. Representatives $x \in \Delta_2^1(h)$ and $y \in \Delta_2^2(h)$ can be differentiated by calculating the stabiliser in $Q(h) = O_2(G_h) \cong 2^4$, with $Q(h)_x \cong 2^2$ and $Q(h)_y \cong 1$. In Table 3.2.2 we give the plane-line distributions of the lines incident with $x \in \Delta_i^j(h)$. These distributions are invariant under the action of $G_{h,x}$.

G_h-orbit	$G_{h,x}$-orbit sizes	Plane-Line Distribution
$\Delta_1^1(h)$	1	$1\ \Delta_0\ 2\ \Delta_1^1\ 12\ \Delta_1^2$
	6	$1\ \Delta_0\ 2\ \Delta_1^1\ 12\ \Delta_2^1$
$\Delta_1^2(h)$	1	$1\ \Delta_0\ 2\ \Delta_1^1\ 12\ \Delta_1^2$
	2	$3\ \Delta_1^2\ 6\ \Delta_2^2\ 6\ \Delta_2^3$
	4	$1\ \Delta_1^2\ 2\ \Delta_2^1\ 2\ \Delta_2^2\ 2\ \Delta_2^3\ 4\ \Delta_2^4\ 4\ \Delta_2^5$
$\Delta_2^1(h)$	1	$3\ \Delta_1^1\ 12\ \Delta_2^1$
	2	$1\ \Delta_1^2\ 2\ \Delta_2^1\ 2\ \Delta_2^2\ 2\ \Delta_2^3\ 4\ \Delta_2^4\ 4\ \Delta_2^5$
	4	$3\ \Delta_2^1\ 6\ \Delta_2^5\ 6\ \Delta_3^1$
$\Delta_2^2(h)$	1	$3\ \Delta_1^2\ 6\ \Delta_2^2\ 6\ \Delta_2^3$
	1	$3\ \Delta_2^2\ 12\ \Delta_2^5$
	1	$3\ \Delta_2^2\ 4\ \Delta_2^4\ 8\ \Delta_3^1$
	2	$1\ \Delta_1^2\ 2\ \Delta_2^1\ 2\ \Delta_2^2\ 2\ \Delta_2^3\ 4\ \Delta_2^4\ 4\ \Delta_2^5$
	2	$3\ \Delta_2^2\ 4\ \Delta_2^4\ 8\ \Delta_3^1$
$\Delta_2^3(h)$	1	$3\ \Delta_1^2\ 6\ \Delta_2^2\ 6\ \Delta_2^3$
	2	$1\ \Delta_1^2\ 2\ \Delta_2^1\ 2\ \Delta_2^2\ 2\ \Delta_2^3\ 4\ \Delta_2^4\ 4\ \Delta_2^5$
	4	$3\ \Delta_2^3\ 1\ \Delta_2^4\ 9\ \Delta_2^5\ 2\ \Delta_3^1$
$\Delta_2^4(h)$	1	$3\ \Delta_2^3\ 1\ \Delta_2^4\ 9\ \Delta_2^5\ 2\ \Delta_3^1$
	3	$1\ \Delta_1^2\ 2\ \Delta_2^1\ 2\ \Delta_2^2\ 2\ \Delta_2^3\ 4\ \Delta_2^4\ 4\ \Delta_2^5$
	3	$3\ \Delta_2^2\ 4\ \Delta_2^4\ 8\ \Delta_3^1$
$\Delta_2^5(h)$	1	$1\ \Delta_1^2\ 2\ \Delta_2^1\ 2\ \Delta_2^2\ 2\ \Delta_2^3\ 4\ \Delta_2^4\ 4\ \Delta_2^5$
	1	$3\ \Delta_2^2\ 12\ \Delta_2^5$
	1	$3\ \Delta_2^3\ 1\ \Delta_2^4\ 9\ \Delta_2^5\ 2\ \Delta_3^1$
	2	$3\ \Delta_2^1\ 6\ \Delta_2^5\ 6\ \Delta_3^1$
	2	$3\ \Delta_2^3\ 1\ \Delta_2^4\ 9\ \Delta_2^5\ 2\ \Delta_3^1$
$\Delta_3^1(h)$	1	$3\ \Delta_2^3\ 1\ \Delta_2^4\ 9\ \Delta_2^5\ 2\ \Delta_3^1$
	3	$3\ \Delta_2^1\ 6\ \Delta_2^5\ 6\ \Delta_3^1$
	3	$3\ \Delta_2^2\ 4\ \Delta_2^4\ 8\ \Delta_3^1$

Table 3.3: The Plane-Line Distribution in the Collinearity Graph for M_{23}

3.2.3 The Computer Files

The computer file **M23.txt** contains generators for M_{23} and representatives for all G_h-orbits.

3.3 Plane-Line Graph for J_4

The Janko groups comprise four of the sporadic simple groups. Discovered by Zvonimir Janko between 1965 and 1976, the groups are unlike the Mathieu groups, Conway groups, and Fischer groups in that they do not form any kind of series. The four groups do not feature as subgroups of each other; their connection is purely historical.

With the cyclic groups, alternating groups and groups of Lie type already well documented, the discovery of the J_4, the final sporadic group to be discovered, concluded the search for finite simple groups. Though this could only be truly said in hindsight once the Classification Theorem had been completed. Janko predicted the existence of J_4 in 1976 [Jan76][1] whilst studying simple groups exhibiting large extra-special 2-subgroups, a common attribute of most of the sporadic simple groups.

Theorem 3.3.1 *Let G be a non-abelian finite simple group which possesses an involution z such that $H = C_G(z)$ satisfies the following conditions:*

(i) The subgroup $E = O_2(H)$ is an extra-special 2-group of order 2^{13} and $C_H(E) \subseteq E$;

(ii) A Sylow 3-subgroup P of $O_{2,3}(H)$ has order 3 and $C_E(P) = Z(E) = \langle z \rangle$;

(iii) We have $H/O_{2,3}(H) \cong Aut(M_{22})$, $N_H(P) \neq C_H(P)$, and $P \subseteq (C_H(P))'$;

Then $G \cong J_4$, a finite simple group of order $2^{21}.3^3.5.7.11^3.23.29.31.37.43$.

The first construction and proof of the group's uniqueness was shown by Simon Norton and others in 1980 [Nor80] using computer calculations. They constructed the 112-dimensional modular representation of J_4 over $GF(2)$, the finite field of two elements, by exploiting the action of J_4 as a stabiliser of a certain 4995-dimensional subspace of the exterior square. Later, Ivanov [Iva92] and Aschbacher and Segev [AS91] provided computer-free proofs of uniqueness, followed by a computer-free construction in 1999 by Ivanov and Meierfrankenfeld [IM99].

Since J_4 is divisible by the primes 37 and 43, and the Monster sporadic group $\mathbb{M}$ is not, J_4 does not feature as a subquotient of $\mathbb{M}$ and is thus one of the sporadic simple groups known as the Pariahs (the others being J_1, J_3, Ly, $O'N$ and Ru).

[1] The title of this paper is incorrect. Subsequent work would show the full covering group of M_{22} is larger, possessing a centre of order 12, not 6.

The thirteen maximal subgroups of J_4 were uncovered by Kleidman and Wilson in [KW88]. These feature in the maximal 2-local geometry we now describe.

3.3.1 The Maximal 2-local Geometry

In [RS80], Ronan and Smith describe the maximal 2-local geometry Γ for $G = J_4$.

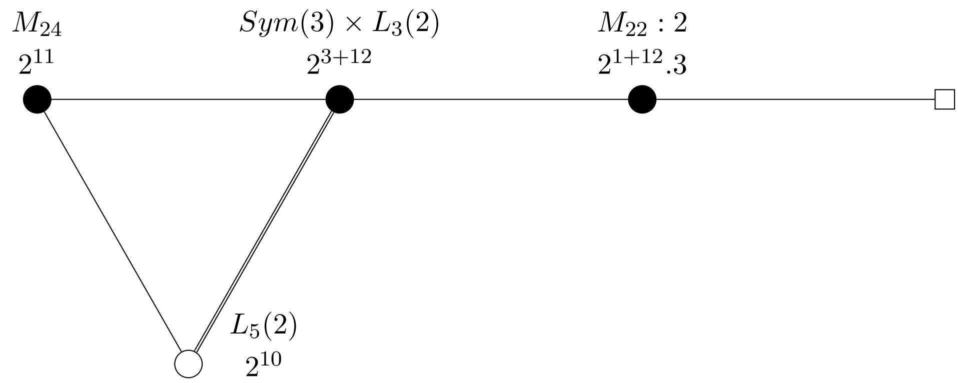


Figure 3.4: The Maximal 2-local Geometry for J_4

In the above geometry, we refer to the objects stabilised by the subgroup $2^{1+12}.3 : M_{22} : 2$ as planes, and the objects stabilised by $2^{3+12} : (Sym(3) \times L_3(2))$ as lines. The square node indicates that there are no objects in the geometry associated with that node, however, there are some subgeometries in which objects may be associated to it.

Let h be a fixed plane in Γ. The subgroup $G_h \cong 2^{1+12}.3 : M_{22} : 2$ is a maximal subgroup of G and is the centraliser of an involution in the conjugacy class $2A$. Thus, we may identify our planes with involutions in the G-class $2A$. The twenty orbits of this class under G_h were determined by Rowley and Taylor in [RT11a]. In their study, Rowley and Taylor enumerate many properties of the orbits including the size of each orbit, the centraliser in $Q(h) = O_{2,3}(G_h) \cong 2^{1+12}.3$ for representatives x of each orbit, and the numbers q_{2A} and q_{2B} of $2A$ and $2B$ elements in $C_{Q(h)}(x)$. These properties, along with the order of hx and the dimension of its fixed space on the natural G-module are enough to distinguish between all twenty G_h-orbits. Thus it only remains for us to determine adjacency to construct the plane-line collinearity graph.

We work within the 112-dimensional representation of G over $GF(2)$, using the generators given on the Online Atlas. We calculate the group $Q(h)$ by taking random elements of G_h having order 21 or 33 and taking elements q which are respectively their 7th and 11th powers. Since $M_{22} : 2$ contains elements of orders 7 and 11 but no elements of orders 21 nor 33, these

q must lie in $Q(h)$ and are sufficient to generate it. Our fixed plane h corresponds to the unique central involution in $Q(h)$.

3.3.2 The Graph

The main result of this section is the construction of the plane-line collinearity graph.

Theorem 3.3.2 *Let $\mathcal{G}(\Gamma)$ be the plane-line collinearity graph for the maximal 2-local geometry Γ for J_4 and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ has diameter 4.

(ii) $|\Delta_1(h)| = 1,386$ and $\Delta_1(h)$ is a G_h-orbit.

(iii) $|\Delta_2(h)| = 1,079,760$ and $\Delta_2(h)$ consists of four G_h-orbits.

(iv) $|\Delta_3(h)| = 490,659,840$ and $\Delta_3(h)$ consists of nine G_h-orbits.

(v) $|\Delta_4(h)| = 2,580,480,000$ and $\Delta_4(h)$ consists of five G_h-orbits.

Lemma 3.3.3 *Each plane is incident with 231 lines. Each line is incident with seven planes.*

Proof. Let h and l be an arbitrary plane and line respectively. As described in [RW07] the residue geometry of a plane is a rank 2 geometry of duads (lines) and hexads (points) defined on the Steiner system $S(3, 6, 22)$, wherein we find 231 duads. The residue geometry of a point is a rank 2 geometry of trios (lines) and sextets (planes) defined on the Steiner system $S(5, 8, 24)$, wherein each trio is contained in seven sextets [Cur76]. □

Corollary 3.3.4 *For a fixed plane h, $|\Delta_1(h)| = 1,386$.*

Proof. This follows immediately from Lemma 3.3.3 since $231 \times (7 - 1) = 1,386$. □

A brief consideration of the list of orbit sizes given in [RT11a] quickly reveals that $\Delta_1(h)$ is a single G_h-orbit and $x \in \Delta_1(h)$ if and only if xh lies in the orbit denoted by $2A2$. With this information, we can now determine adjacency within the plane-line collinearity graph.

We do this directly by calculating the entire orbit $\Delta_1(h)$ and moving this around the graph by conjugation to each of the twenty orbit representatives. In each case, we use the G_h-orbit invariants mentioned earlier to determine where the 1,386 neighbours lie and record the results.

A pictorial representation of the complete plane-line collinearity graph is given in Figure 3.5 and the collapsed adjacency matrix is given in Table 3.4.

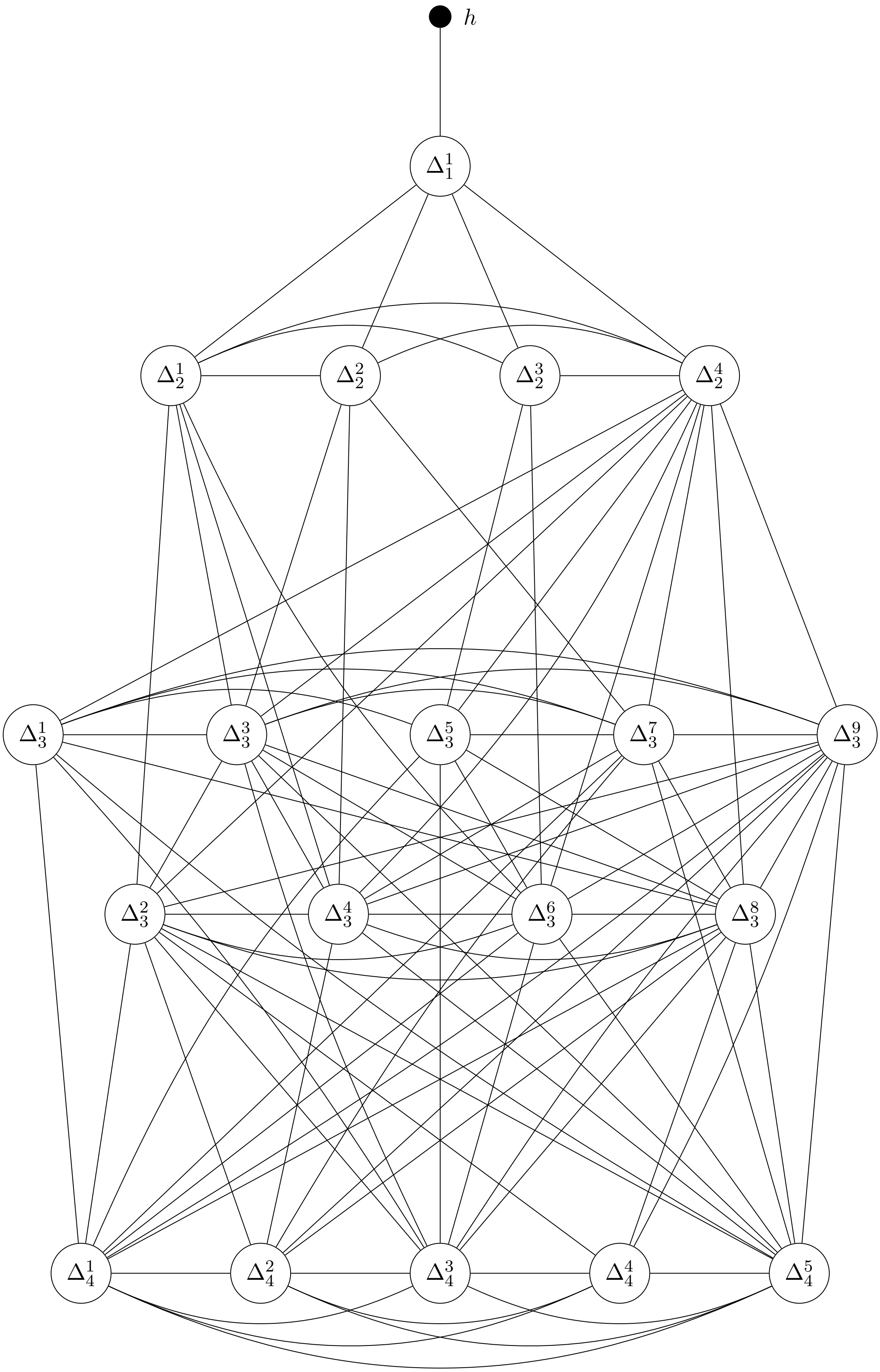


Figure 3.5: The Plane-Line Collinearity Graph of J_4

	h	Δ_1^1	Δ_2^1	Δ_2^2	Δ_2^3	Δ_2^4	Δ_3^1	Δ_3^2	Δ_3^3	Δ_3^4	Δ_3^5	Δ_3^6	Δ_3^7	Δ_3^8	Δ_3^9	Δ_4^1	Δ_4^2	Δ_4^3	Δ_4^4	Δ_4^5
h	-	1386	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	1	65	240	320	120	640	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	3	75	16	12	96	0	384	192	224	0	384	0	0	0	0	0	0	0	0
Δ_2^2	0	7	28	35	0	196	0	0	672	112	0	0	336	0	0	0	0	0	0	0
Δ_2^3	0	9	72	0	57	288	0	0	0	0	576	384	0	0	0	0	0	0	0	0
Δ_2^4	0	1	12	14	6	57	64	96	192	48	80	96	144	384	192	0	0	0	0	0
Δ_3^1	0	0	0	0	0	7	21	0	84	0	14	0	28	84	28	112	0	336	0	672
Δ_3^2	0	0	1	0	0	2	0	71	28	20	0	16	0	128	32	32	192	288	256	320
Δ_3^3	0	0	1	2	0	8	32	56	75	12	0	16	32	256	64	0	0	192	0	640
Δ_3^4	0	0	7	2	0	12	0	240	72	45	0	32	48	192	64	0	288	0	0	384
Δ_3^5	0	0	0	0	3	20	32	0	0	0	83	48	48	576	0	192	0	384	0	0
Δ_3^6	0	0	6	0	1	12	0	96	48	16	24	63	0	192	208	48	0	288	0	384
Δ_3^7	0	0	0	3	0	18	32	0	96	24	24	0	69	384	32	128	192	0	0	384
Δ_3^8	0	0	0	0	0	1	2	16	16	2	6	4	8	155	24	40	144	280	288	400
Δ_3^9	0	0	0	0	0	3	4	24	24	4	0	26	4	144	81	16	48	288	192	528
Δ_4^1	0	0	0	0	0	0	8	12	0	0	6	3	8	120	8	33	96	228	384	480
Δ_4^2	0	0	0	0	0	0	0	24	0	3	0	0	4	144	8	32	163	192	384	432
Δ_4^3	0	0	0	0	0	0	4	18	6	0	2	3	0	140	24	38	96	255	352	448
Δ_4^4	0	0	0	0	0	0	0	11	0	0	0	0	0	99	11	44	132	242	363	484
Δ_4^5	0	0	0	0	0	0	4	10	10	1	0	2	2	100	22	40	108	224	352	511

Table 3.4: The Collapsed Adjacency Matrix for the Plane-Line Collinearity Graph of J_4

Orbit	X_C	Size	g	G_{h,h^g}	$\|Q(h)_{h^g}\|$	$dim(C_V(h^g))$	q_{2A}	q_{2B}
h	$1A$	1	–	$2^{1+12}.3.M_{22}:2$	$2^{13}\cdot 3$	62	1,387	2,772
Δ_1^1	$2A2$	$2\cdot 3^2\cdot 7\cdot 11$	$(xy)^{18}x(xy)^{19}$	$2^{1+11}.2^5:Sym(5)$	2^{12}	62	747	1,364
Δ_2^1	$2A1$	$2^5\cdot 3^2\cdot 5\cdot 7\cdot 11$	$(xy)^{19}x(xy)^{14}x(xy)^4$	$2^{1+7}.2^4.Sym(4)\times 2$	2^8	62	107	84
Δ_2^2	$2B1$	$2^7\cdot 3^2\cdot 5\cdot 11$	$(xy)^{17}x(xy)^{32}y$	$2^{1+6}.2^3:L_3(2)\times 2$	2^7	56	71	56
Δ_2^3	$2B2$	$2^4\cdot 3\cdot 5\cdot 7\cdot 11$	$(xy)^{10}x(xy)^2x(xy)^{34}x(xy)^{25}$	$2^{1+8}.3.2^4.Sym(4)\times 2$	$2^9\cdot 3$	56	139	180
Δ_2^4	$4A$	$2^8\cdot 3^2\cdot 5\cdot 7\cdot 11$	$xy^2xyxy^2xy^2xyxy^2x$	$2^6.2^3:Sym(4)\times 2$	2^6	32	33	30
Δ_3^1	$3A$	$2^{14}\cdot 3^2\cdot 5\cdot 11$	x	$2^3:L_3(2)\times 2$	1	40	0	0
Δ_3^2	$4B1$	$2^{12}\cdot 3^3\cdot 5\cdot 7\cdot 11$	$yxy^3xy^2xy^2$	$2^2.2^5:2^2$	2^2	32	1	2
Δ_3^3	$4B2$	$2^{11}\cdot 3^3\cdot 5\cdot 7\cdot 11$	y^2xyxyx	$2^5.2^5:Dih(8)$	2^2	32	3	0
Δ_3^4	$4B3$	$2^{10}\cdot 3^2\cdot 5\cdot 7\cdot 11$	$yxyxy^2xy^3xy^2x$	$2^3.2^4:Sym(4)\times 2$	2^3	32	3	4
Δ_3^5	$4B4$	$2^{10}\cdot 3^2\cdot 5\cdot 7\cdot 11$	$xy^2xyxy^2xy^3xy^3$	$2^4.2^5:Dih(12)$	2^4	32	9	6
Δ_3^6	$4C1$	$2^{11}\cdot 3^2\cdot 5\cdot 7\cdot 11$	$xyxy^2xy^2xyxy^2$	$2^3.2^4:Sym(4)$	2^3	28	3	4
Δ_3^7	$4C2$	$2^{11}\cdot 3^2\cdot 5\cdot 7\cdot 11$	$y^2xy^3xy^2xy^2xy^3x$	$2^3.2^4:Sym(4)$	2^3	28	7	0
Δ_3^8	$6B$	$2^{15}\cdot 3^3\cdot 5\cdot 7\cdot 11$	xy^2xy^2x	$2^3:2^3$	1	22	0	0
Δ_3^9	$6C$	$2^{14}\cdot 3^2\cdot 5\cdot 7\cdot 11$	xy^2xyx	$2^3:Sym(4)\times 2$	1	20	0	0
Δ_4^1	$5A$	$2^{15}\cdot 3^2\cdot 5\cdot 7\cdot 11$	xy^3xyxy	$2^3:Sym(4)$	1	24	0	0
Δ_4^2	$8C$	$2^{15}\cdot 3^3\cdot 5\cdot 7\cdot 11$	xyx	$2^3:Dih(8)$	1	16	0	0
Δ_4^3	$10A$	$2^{16}\cdot 3^3\cdot 5\cdot 7\cdot 11$	$yxyx$	2^{2+3}	1	12	0	0
Δ_4^4	$11B$	$2^{20}\cdot 3^3\cdot 5\cdot 7$	y^2	$Dih(22)$	1	12	0	0
Δ_4^5	$12B$	$2^{17}\cdot 3^2\cdot 5\cdot 7\cdot 11$	y	$2\times Dih(8)$	1	12	0	0

Table 3.5: The G_h-orbits of the Plane-Line Collinearity Graph for J_4

Information relating to the G_h-orbits is given in Table 3.5. In the first two columns we give the G_h-orbit label, and the class X_C such that for $z \in \Delta_i^j(h)$, we have zh lies in the G-conjugacy class C. Next we give the size of the orbit, and the element used to conjugate the fixed involution h to an orbit representative. We give the stabiliser in G_h of each representative, and the size of the stabiliser in $Q(h) = O_{2,3}(G) \cong 2^{1+12}.3$. The final three columns give q_{2A} and q_{2B}, the two G_h-orbit invariants detailed earlier, together with $dim(C_V(h^g))$, the dimension of the fixed space of h^g on the natural G-module V.

3.3.3 The Computer Files

There are two computer files attached to this section.

J4Gens.txt

This file contains generators for J_4 as a group of 112×112 matrices over $GF(2)$. The generators are taken from the Online Atlas.

J4Reps.txt

This file contains representatives for each of the G_h-orbits of the plane-line collinearity graph, given as involutions conjugate to the fixed involution h. The entry dij is the representative for Δ_i^j. Thus to call the representative for Δ_2^3, the user would need simply to type $d23$. The file **J4Gens.txt** should be loaded before loading the representatives.

3.4 Plane-Line Graph for Fi_{22}

The sporadic simple groups Fi_{22}, Fi_{23} and Fi'_{24} were discovered in the 1970s [Fis71] by Bernd Fischer while investigating *3-transposition groups.*

Definition 3.4.1 *A group G is called a 3-transposition group if it is generated by a conjugacy class of involutions, called Fischer transpositions, and the product of any two transpositions has order at most 3.*

The symmetric groups $Sym(n)$ are an example of such groups, where the conjugacy class of transpositions generates the group and the product of any distinct pair gives an element of order 2 or 3. In addition to this, Fischer required that $G' = G''$ and that any 2-subgroups and 3-subgroups normal in G were also central.

Fischer's sporadic groups are named due to their connection with the previously discovered Mathieu groups. The group Fi_{22} is generated by a conjugacy class of size 3510. A maximal set of commuting 3-transpositions, known as a *base*, has size 22. There are 1024 3-transpositions, called *anabasic*, which do not commute with any within a given base. The remaining 2364, called *hexadic*, commute with exactly 6 in the base. These sets of 6 transpositions form the blocks of an $S(3, 6, 22)$ Steiner system whose automorphisms generate the Mathieu group M_{22}. For Fi_{23}, the 3-transpositions form bases of size 23, 7 of which commute with a non-basic transposition. This gives rise to an $S(4, 7, 23)$ Steiner system and hence M_{23}. Similarly, the bases of Fi_{24} have size 24, 8 of which commute with a non-basic transposition, giving an $S(5, 8, 24)$ Steiner system and its automorphism group M_{24}. In this last case, the group Fi_{24} is not simple. The third of Fischer's sporadic groups is the derived group Fi'_{24} of index 2.

The group Fi_{22} has order $64, 561, 751, 654, 400 = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ and features twelve conjugacy classes of maximal subgroups described by Wilson in [Wil84].

- $2.U_6(2)$
- $2^6 : Sp_6(2)$
- $2^{5+8} : (Sym(3) \times A_6)$
- $\Omega_7(3)$
- $2^{2+8} : (U_4(2) : 2)$
- $3^{1+6} : 2^{3+4} : 3^2 : 2$
- $\Omega_8^+(2) : Sym(3)$
- $U_4(3) : 2 \times Sym(3)$
- $Sym(10)$
- $2^{10} : M_{22}$
- ${}^2F_4(2)'$
- M_{12}

We will see that two of these maximal subgroups feature as stabilisers in the minimal 2-local geometry for Fi_{22}.

3.4.1 The Minimal 2-Local Geometry

Fischer's sporadic simple group Fi_{22} is generated by its conjugacy class of involutions of size 3510, the so-called Fischer transpositions. The minimal 2-local geometry for $G = Fi_{22}$ described in [RS84] is defined in terms of these transpositions.

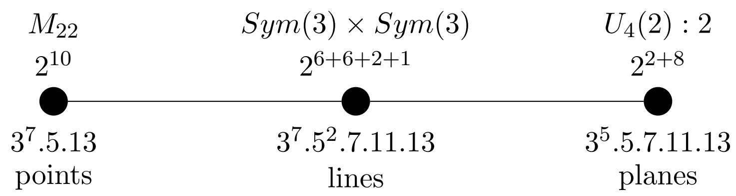


Figure 3.6: The Minimal 2-local Geometry for Fi_{22}

The points of this geometry are maximal sets of pairwise commuting Fischer transpositions of size 22. The planes of the geometry arise as duads (pairs of transpositions) and the lines as triduads (triples of distinct duads). The point-line collinearity graph associated with this geometry is described in Theorem 3.1.1. Incidence in the geometry is given by set containment. Two planes meet in a line if their duads lie in a shared triduad, and two lines are coplanar if they intersect in a duad.

The minimal parabolic system associated with this geometry is

$$\{2^{10}.L_2(2), 2^{6+6+2+1}.Sym(5), 2^{2+8}.L_2(2)\}.$$

The maximal subgroups of Fi_{22} with shape $2^{5+8} : (Sym(3) \times Alt(6))$ and $2^6 : Sp_6(2)$ stabilise subgeometries which are buildings of shape $C_2 \times A_1$ and C_3 respectively [Wil84].

When exploring this geometry we use the permutation representation of G on a set of 3510 points given on the Online Atlas. The conjugacy class consisting of the 3510 Fischer transpositions is easily calculated and, using a random element search, two commuting transpositions are quickly found. Call this pair h. We may ask directly for the stabiliser subgroup $G_h \cong 2^{2+8} : (U_4(2) : 2)$ and the conjugates of h in $G = Fi_{22}$. In this way we obtain a set Γ_2 of all 1,216,215 planes. Orbits of G_h on Γ_2 can also be determined directly and we find there are fourteen such orbits.

3.4.2 The Graph

The main result of this section is the construction of the plane-line collinearity graph for Fi_{22}.

Theorem 3.4.2 *Let $\mathcal{G}(\Gamma)$ be the plane-line collinearity graph for the minimal 2-local geometry Γ for Fi_{22} and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 4.*

(ii) *$|\Delta_1(h)| = 270$ and $\Delta_1(h)$ is a G_h-orbit.*

(iii) *$|\Delta_2(h)| = 40,392$ and $\Delta_2(h)$ consists of four G_h-orbits.*

(iv) *$|\Delta_3(h)| = 686,080$ and $\Delta_3(h)$ consists of five G_h-orbits.*

(v) *$|\Delta_4(h)| = 489,472$ and $\Delta_4(h)$ consists of three G_h-orbits.*

Lemma 3.4.3 *Each plane is incident with 135 lines. Each line is incident with three planes.*

Proof. The residue geometry of a plane is explored in detail by Rowley and Walker [RW96]. In their study, the authors show that a plane is incident with 135 lines. Clearly each line is incident with three planes since, by definition, a triduad contains three duads. □

Corollary 3.4.4 *For a fixed plane h, $|\Delta_1(h)| = 270$.*

Proof. This follows immediately from Lemma 3.4.3 since $135 \times (3 - 1) = 270$. □

After the orbit consisting of the fixed plane h, the next smallest has size 270. Thus, it must necessarily be the case that this is the orbit $\Delta_1(h)$ and hence $\Delta_1(h)$ is a single G_h-orbit.

Adjacency in the graph is determined in the same way described in the construction of the J_4 plane-line collinearity graph. In other words, we take the orbit $\Delta_1(h)$ and move it around the graph via conjugation.

The next page gives a graphical depiction of the plane-line collinearity graph and its associated collapsed adjacency matrix. Information relating to the G_h-orbits is given after in Table 3.7.

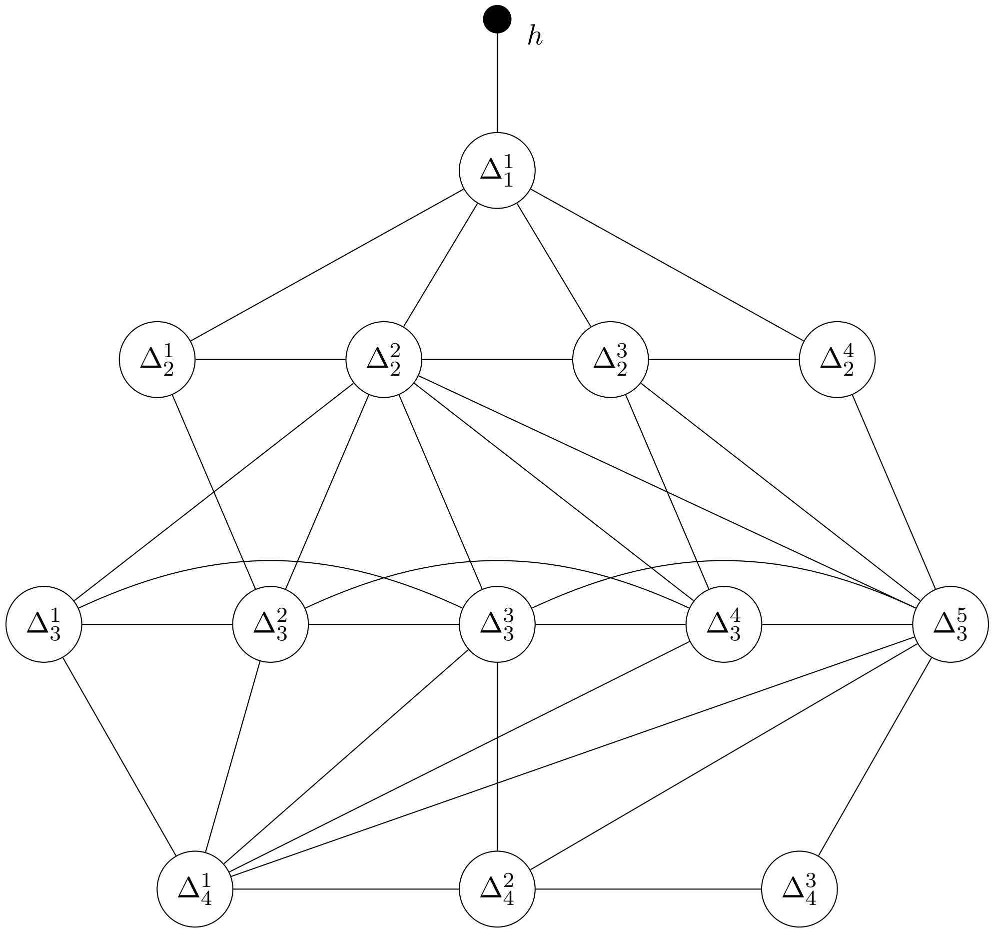


Figure 3.7: The Plane-Line Collinearity Graph of Fi_{22}

	h	Δ_1^1	Δ_2^1	Δ_2^2	Δ_2^3	Δ_2^4	Δ_3^1	Δ_3^2	Δ_3^3	Δ_3^4	Δ_3^5	Δ_4^1	Δ_4^2	Δ_4^3
h	–	270	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	1	25	64	128	48	4	0	0	0	0	0	0	0	0
Δ_2^1	0	15	15	180	0	0	0	60	0	0	0	0	0	0
Δ_2^2	0	1	6	61	6	0	32	28	96	24	16	0	0	0
Δ_2^3	0	3	0	48	21	6	0	0	0	96	96	0	0	0
Δ_2^4	0	3	0	0	72	3	0	0	0	0	192	0	0	0
Δ_3^1	0	0	0	27	0	0	27	54	54	0	0	108	0	0
Δ_3^2	0	0	1	14	0	0	32	39	32	24	0	128	0	0
Δ_3^3	0	0	0	9	0	0	6	6	99	36	18	84	12	0
Δ_3^4	0	0	0	6	3	0	0	12	96	33	24	96	0	0
Δ_3^5	0	0	0	8	6	1	0	0	96	48	51	32	26	2
Δ_4^1	0	0	0	0	0	0	10	20	70	30	5	125	10	0
Δ_4^2	0	0	0	0	0	0	0	0	96	0	39	96	36	3
Δ_4^3	0	0	0	0	0	0	0	0	0	0	135	0	135	0

Table 3.6: The Collapsed Adjacency Matrix for the Plane-Line Collinearity Graph of Fi_{22}

In Table 3.7 below we give information relating to the G_h-orbits. In the column $G_{h,x}$ we give the stabiliser in G_h of a representative $x \in \Delta_i^j(h)$ for each orbit. The following column gives the size of this stabiliser and finally we give the stabiliser of x in $Q(h) = O_2(G_h) \cong 2^{2+8}$. This information is sufficient to distinguish all fourteen orbits.

Orbit	Size	$G_{h,x}$	$\|G_{h,x}\|$	$\|Q(h)_x\|$
h	1	$2^{2+8}.U_4(2):2$	$2^{17} \cdot 3^4 \cdot 5$	2^{10}
$\Delta_1^1(h)$	270	$2^{2+7}.2^4 Sym(4)$	$2^{16} \cdot 3$	2^9
$\Delta_2^1(h)$	1,152	$2^{1+4}.Sym(6) \times 2$	$2^{10} \cdot 3^2 \cdot 5$	2^5
$\Delta_2^2(h)$	34,560	$2^4.2^2 Sym(4)$	$2^9 \cdot 3$	2^4
$\Delta_2^3(h)$	4,320	$2^{2+4}.2^4 Dih(12)$	$2^{12} \cdot 3$	2^6
$\Delta_2^4(h)$	360	$2^{2+6}.2(Alt(4) \times Alt(4))2$	$2^{14} \cdot 3^2$	2^8
$\Delta_3^1(h)$	40,960	$3^3(Sym(6) \times 2)$	$2^4 \cdot 3^4$	1
$\Delta_3^2(h)$	69,120	$2^3.2^2 Sym(4)$	$2^8 \cdot 3$	2^3
$\Delta_3^3(h)$	368,640	$3^2(Dih(8) \times 2)$	$2^4 \cdot 3^2$	1
$\Delta_3^4(h)$	138,240	$2^2.2^4(3 \times Dih(8))$	$2^7 \cdot 3$	2^2
$\Delta_3^5(h)$	69,120	$2^2.2^3 Sym(4)$	$2^8 \cdot 3$	2^2
$\Delta_4^1(h)$	442,368	$Sym(5)$	$2^3 \cdot 3 \cdot 5$	1
$\Delta_4^2(h)$	46,080	$2(Alt(4) \times Alt(4))2.2$	$2^7 \cdot 3^2$	2
$\Delta_4^3(h)$	1,024	$2.U_4(2)$	$2^7 \cdot 3^4 \cdot 5$	2

Table 3.7: The G_h-orbits of the Plane-Line Collinearity Graph for Fi_{22}

3.4.3 The Computer Files

There are two computer files attached to this section.

Fi22Gens.txt

This file contains generators for Fi_{22} as a permutation group on 3510 points. The generators are taken from the Online Atlas. This file should be loaded first.

Fi22Reps.txt

This file contains representatives for each of the G_h-orbits of the plane-line collinearity graph, given as duads of two Fischer transpositions. The duad dij is the representative for Δ_i^j. Thus to call the representative for Δ_2^3, the user would need simply to type $d23$.

3.5 Plane-Line Graph for Fi_{23}

The group $G = Fi_{23}$ is the second of three sporadic simple groups introduced by Fischer. In this section we detail the plane-line collinearity graph associated with this group.

3.5.1 The Minimal 2-Local Geometry

The minimal 2-local geometry Γ associated with Fi_{23} was introduced in [RS84].

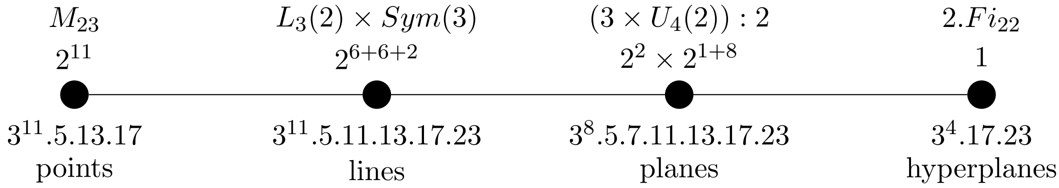


Figure 3.8: The Minimal 2-local Geometry for Fi_{23}

As with the minimal 2-local geometry for Fi_{22}, the objects of this geometry are defined in terms of Fischer transpositions. In this group, this is a conjugacy class of size 31,671. The hyperplanes of the geometry correspond to the Fischer transpositions. The points are identified with bases, that is maximal pairwise commuting sets of transpositions, of size 23. The planes are classes of projective planes formed from the seven points of a heptad, whilst the lines are tri-transpositions, known as triads. The point-line collinearity graph associated with this geometry is described in Theorem 3.1.2.

The minimal parabolic system associated with this geometry is described in [RS84]. We work with the permutation representation of the group over the set of 31,671 points for all calculations.

The hyperplanes, under the action of $2.Fi_{22}$, split into three orbits of sizes 1, 3510 and 28,160. The hyperplane-plane collinearity graph is a simple construction.

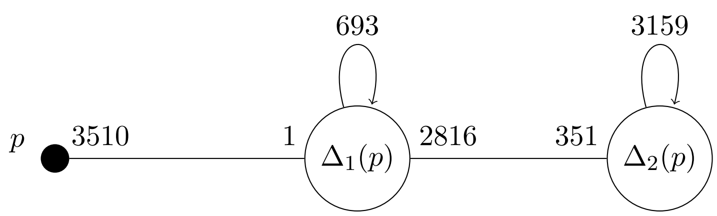


Figure 3.9: The Hyperplane-Plane Collinearity Graph for Fi_{23}

We will see that the plane-line collinearity graph is far more intricate.

3.5.2 The Graph

The plane-line collinearity graph is significantly larger than previous examples boasting a total of 303 orbits.

Theorem 3.5.1 *Let $\mathcal{G}(\Gamma)$ be the plane-line collinearity graph for the minimal 2-local geometry Γ of $G = Fi_{23}$, and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 5.*

(ii) *$|\Delta_1(h)| = 810$ and $\Delta_1(h)$ is a G_h-orbit.*

(iii) *$|\Delta_2(h)| = 462,096$ and $\Delta_2(h)$ consists of 11 G_h-orbits.*

(iv) *$|\Delta_3(h)| = 163,964,928$ and $\Delta_3(h)$ consists of 53 G_h-orbits.*

(v) *$|\Delta_4(h)| = 12,038,144,000$ and $\Delta_4(h)$ consists of 234 G_h-orbits.*

(vi) *$|\Delta_5(h)| = 637,009,920$ and $\Delta_5(h)$ consists of 3 G_h-orbits.*

Lemma 3.5.2 *Each plane is incident with 135 lines. Each line is incident with seven planes.*

Proof. The residue geometry of a hyperplane is isomorphic to the Fi_{22} minimal 2-local parabolic geometry described in Section 3.4.1. There we find that each plane contains 135 lines. In the Fi_{23} geometry Γ, the lines are described by heptads and the planes by triads. Each heptad contains seven triads, and thus each line is incident with seven planes. □

Corollary 3.5.3 *For a fixed plane h, $|\Delta_1(h)| = 810$.*

Proof. This follows immediately from Lemma 3.5.2 since $135 \times (7 - 1) = 810$. □

With Fi_{23} we're in the fortunate position that tri-transpositions can be placed in one-to-one correspondence with the conjugacy class $h^G = 2C$, with $G_h \cong 2^2 \times 2^{1+8} : (3 \times U_4(2)) : 2$. Fortunately, these orbits and their sizes have already been determined by Paul Taylor [Tay11], thus it remains only for us to classify the adjacency within the graph. Considering the orbit sizes, we have a single option for $\Delta_1(h)$. As with Fi_{22}, we move this orbit around the graph via conjugation.

The subsequent pages detail the G_h-orbits of the graph with the first two columns detailing the correspondence between our graph-theoretic notation used here and the group-theoretic notation used in Taylor's paper. The collapsed adjacency matrix is given in the computer files.

Table 3.8: The G_h-orbits of the Plane-Line Collinearity Graph for Fi_{23}

Orbit	X_C	Size	$G_{h,x}$	$\|G_{h,x}\|$	$\|Q(h)_x\|$
h	$1A$	1	$2^2 \times 2^{1+8}.(3 \times U_4(2)) : 2$	$2^{18} \cdot 3^5 \cdot 5$	2^{11}
$\Delta_1^1(h)$	$2C1$	810	$2^{4+6}.(2^4.Alt(4)) : 2$	$2^{17} \cdot 3$	2^{10}
$\Delta_2^1(h)$	$2A$	180	$2^{4+5}.(3 \times 2.(Alt(4) \times Alt(4)).2) : 2$	$2^{16} \cdot 3^3$	2^9
$\Delta_2^2(h)$	$2B1$	540	$2^{5+4}.(2.(Alt(4) \times Alt(4)).2) : 2$	$2^{16} \cdot 3^2$	2^9
$\Delta_2^3(h)$	$2B2$	3,456	$2^6.(2 \times Sym(6))$	$2^{11} \cdot 3^2 \cdot 5$	2^6
$\Delta_2^4(h)$	$2B3$	4,320	$2^7.(3 \times 2^4.Sym(3)) : 2$	$2^{13} \cdot 3^2$	2^7
$\Delta_2^5(h)$	$2C2$	12,960	$2^{2+6}.(2^2 \times Sym(4))$	$2^{13} \cdot 3$	2^8
$\Delta_2^6(h)$	$2C3$	12,960	$2^7.(2^4.Sym(3)) : 2$	$2^{13} \cdot 3$	2^7
$\Delta_2^7(h)$	$2C4$	12,960	$2^7.(2^4.Sym(3)) : 2$	$2^{13} \cdot 3$	2^7
$\Delta_2^8(h)$	$2C5$	103,680	$2^5.(2^2 \times Sym(4))$	$2^{10} \cdot 3$	2^5
$\Delta_2^9(h)$	$4B1$	103,680	$2^5.(2^2 \times Sym(4))$	$2^{10} \cdot 3$	2^5
$\Delta_2^{10}(h)$	$4B2$	103,680	$2^5.(2^2 \times Sym(4))$	$2^{10} \cdot 3$	2^5
$\Delta_2^{11}(h)$	$4B3$	103,680	$2^5.(2^2 \times Sym(4))$	$2^{10} \cdot 3$	2^5
$\Delta_3^1(h)$	$3C1$	73,728	$(3 \times Sym(6)) : 2$	$2^5 \cdot 3^3 \cdot 5$	1
$\Delta_3^2(h)$	$3C2$	122,880	$2.(3^3 \times Sym(4)) : 2$	$2^5 \cdot 3^4$	2
$\Delta_3^3(h)$	$3C3$	368,640	$2^2.(3 \times 3^2 : 2^2) : 2$	$2^5 \cdot 3^3$	2^2
$\Delta_3^4(h)$	$4A1$	103,680	$2^5.(2^2 \times Sym(4))$	$2^{10} \cdot 3$	2^5
$\Delta_3^5(h)$	$4A2$	103,680	$2^3.(2^4.Alt(4)) : 2$	$2^{10} \cdot 3$	2^3
$\Delta_3^6(h)$	$4A3$	414,720	$2^3.(2^2 \times Sym(4))$	$2^8 \cdot 3$	2^3
$\Delta_3^7(h)$	$4A4$	414,720	$2^3.(2^2 \times Sym(4))$	$2^8 \cdot 3$	2^3
$\Delta_3^8(h)$	$4B4$	414,720	$2^3.(Dih(8) \times Alt(4))$	$2^8 \cdot 3$	2^3
$\Delta_3^9(h)$	$4B5$	1,244,160	2^{2+6}	2^8	2^3
$\Delta_3^{10}(h)$	$4B6$	1,658,880	$2.(2^2 \times Sym(4))$	$2^6 \cdot 3$	2
$\Delta_3^{11}(h)$	$4B7$	1,658,880	$2.(2^2 \times Sym(4))$	$2^6 \cdot 3$	2
$\Delta_3^{12}(h)$	$4C1$	103,680	$2^3.(2^4.Alt(4)) : 2$	$2^{10} \cdot 3$	2^3
$\Delta_3^{13}(h)$	$4C2$	103,680	$2^5.(2^2 \times Sym(4))$	$2^{10} \cdot 3$	2^5
$\Delta_3^{14}(h)$	$4C3$	207,360	$2^3.(2.(2^2 \times Alt(4)).2)$	$2^9 \cdot 3$	2^3
$\Delta_3^{15}(h)$	$4C4$	207,360	$2^4.(2^2 \times Sym(4))$	$2^9 \cdot 3$	2^4
$\Delta_3^{16}(h)$	$4C5$	414,720	$2^3.(2^2 \times Sym(4))$	$2^8 \cdot 3$	2^3
$\Delta_3^{17}(h)$	$4C6$	414,720	$2^3.(2^2 \times Sym(4))$	$2^8 \cdot 3$	2^3
$\Delta_3^{18}(h)$	$4C7$	1,244,160	2^{3+5}	2^8	2^3
$\Delta_3^{19}(h)$	$4C8$	1,244,160	2^{3+5}	2^8	2^3
$\Delta_3^{20}(h)$	$4C9$	1,244,160	2^{3+5}	2^8	2^3
$\Delta_3^{21}(h)$	$4C10$	1,244,160	2^{3+5}	2^8	2^3
$\Delta_3^{22}(h)$	$4C11$	1,244,160	2^{3+5}	2^8	2^3
$\Delta_3^{23}(h)$	$4C12$	1,244,160	2^{3+5}	2^8	2^4
$\Delta_3^{24}(h)$	$4D1$	1,244,160	2^{2+6}	2^8	2^3
$\Delta_3^{25}(h)$	$4D2$	1,244,160	2^{2+6}	2^8	2^4
$\Delta_3^{26}(h)$	$4D3$	9,953,280	2^5	2^5	2
$\Delta_3^{27}(h)$	$6E1$	1,105,920	$(3 \times 2 \times Sym(4)) : 2$	$2^5 \cdot 3^2$	1

Orbit	X_C	Size	$G_{h,x}$	$\|G_{h,x}\|$	$\|Q(h)_x\|$
$\Delta_3^{28}(h)$	$6E2$	1,105,920	$2^2.(3 \times Dih(12)) : 2$	$2^5 \cdot 3^2$	2^2
$\Delta_3^{29}(h)$	$6G1$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	2^2
$\Delta_3^{30}(h)$	$6G2$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	2
$\Delta_3^{31}(h)$	$6K1$	1,105,920	$2.(3^2 : Dih(8)) : 2$	$2^5 \cdot 3^2$	2
$\Delta_3^{32}(h)$	$6K3$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	1
$\Delta_3^{33}(h)$	$6K4$	3,317,760	$2^2(2 \times Dih(12))$	$2^5 \cdot 3$	2^2
$\Delta_3^{34}(h)$	$6K5$	3,317,760	$2^2(2 \times Dih(12))$	$2^5 \cdot 3$	2
$\Delta_3^{35}(h)$	$6K6$	3,317,760	$2^2(2 \times Dih(12))$	$2^5 \cdot 3$	1
$\Delta_3^{36}(h)$	$6L1$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	1
$\Delta_3^{37}(h)$	$6L2$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	2
$\Delta_3^{38}(h)$	$6M1$	3,317,760	$2^2(2 \times Dih(12))$	$2^5 \cdot 3$	2^2
$\Delta_3^{39}(h)$	$6M2$	3,317,760	$2^2(2 \times Dih(12))$	$2^5 \cdot 3$	2
$\Delta_3^{40}(h)$	$6M5$	9,953,280	2^{2+3}	2^5	1
$\Delta_3^{41}(h)$	$6M6$	9,953,280	2^{2+3}	2^5	2^2
$\Delta_3^{42}(h)$	$8A1$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	1
$\Delta_3^{43}(h)$	$8A2$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	2
$\Delta_3^{44}(h)$	$8A3$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	1
$\Delta_3^{45}(h)$	$8A4$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	2^3
$\Delta_3^{46}(h)$	$8A5$	9,953,280	2^{2+3}	2^5	1
$\Delta_3^{47}(h)$	$8A6$	9,953,280	2^{2+3}	2^5	2^2
$\Delta_3^{48}(h)$	$8B1$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	1
$\Delta_3^{49}(h)$	$8B2$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	2^2
$\Delta_3^{50}(h)$	$8B3$	9,953,280	2^{2+3}	2^5	1
$\Delta_3^{51}(h)$	$8B4$	9,953,280	2^{2+3}	2^5	2
$\Delta_3^{52}(h)$	$8B5$	9,953,280	2^{2+3}	2^5	2^2
$\Delta_3^{53}(h)$	$8B6$	9,953,280	2^{2+3}	2^5	1
$\Delta_4^{1}(h)$	$3A1$	1,536	$2^2.U_4(2) : 2$	$2^9 \cdot 3^4 \cdot 5$	2^2
$\Delta_4^{2}(h)$	$3A2$	23,040	$2^2.(3 \times 2.(Alt(4) \times Alt(4)).2) : 2$	$2^9 \cdot 3^3$	2^2
$\Delta_4^{3}(h)$	$3B$	81,920	$(3 \times 3^3 : Sym(4)) : 2$	$2^4 \cdot 3^5$	1
$\Delta_4^{4}(h)$	$3D1$	983,040	$(3 \times 3^{1+2} : 2) : 2$	$2^2 \cdot 3^4$	1
$\Delta_4^{5}(h)$	$3D2$	2,949,120	$(3^2 \times Sym(3)) : 2$	$2^2 \cdot 3^3$	1
$\Delta_4^{6}(h)$	$5A1$	1,327,104	$2 \times Sym(5)$	$2^4 \cdot 3 \cdot 5$	2
$\Delta_4^{7}(h)$	$5A2$	6,635,520	$2 \times Sym(4)$	$2^4 \cdot 3$	1
$\Delta_4^{8}(h)$	$5A3$	6,635,520	$2^2.Dih(12)$	$2^4 \cdot 3$	2
$\Delta_4^{9}(h)$	$6A1$	69,120	$2^2.(2.(Alt(4) \times Alt(4)).2) : 2$	$2^9 \cdot 3^2$	2^2
$\Delta_4^{10}(h)$	$6A2$	69,120	$2^2.(2.(Alt(4) \times Alt(4)).2) : 2$	$2^9 \cdot 3^2$	2^2
$\Delta_4^{11}(h)$	$6A3$	276,480	$2^2.(3 \times 2^2 \times Alt(4)) : 2$	$2^7 \cdot 3^2$	2^2
$\Delta_4^{12}(h)$	$6B$	1,474,560	$(3 \times 3^2 : 2^2) : 2$	$2^3 \cdot 3^3$	1
$\Delta_4^{13}(h)$	$6C1$	110,592	$2^2 \times Sym(6)$	$2^6 \cdot 3^2 \cdot 5$	2
$\Delta_4^{14}(h)$	$6C2$	138,240	$2^2.(2.(Alt(4) \times Alt(4)).2)$	$2^8 \cdot 3^2$	2^2
$\Delta_4^{15}(h)$	$6C3$	414,720	$2^{2+3}.Sym(3)$	$2^8 \cdot 3$	2^2
$\Delta_4^{16}(h)$	$6C4$	414,720	$2^{2+3}.Sym(3)$	$2^8 \cdot 3$	2^2
$\Delta_4^{17}(h)$	$6C5$	829,440	$2^2.(2^2 \times Sym(4))$	$2^7 \cdot 3$	2^2

Orbit	X_C	Size	$G_{h,x}$	$\|G_{h,x}\|$	$\|Q(h)_x\|$
$\Delta_4^{18}(h)$	$6C6$	1,658,880	$2.(2^2 \times Sym(4))$	$2^6 \cdot 3$	2
$\Delta_4^{19}(h)$	$6D1$	2,211,840	$(3 \times 3 : Dih(8)) : 2$	$2^4 \cdot 3^2$	1
$\Delta_4^{20}(h)$	$6D2$	2,211,840	$(3^2 : Dih(8)) : 2$	$2^4 \cdot 3^2$	1
$\Delta_4^{21}(h)$	$6E3$	2,211,840	$Dih(12) \times Dih(12)$	$2^4 \cdot 3^2$	2
$\Delta_4^{22}(h)$	$6E4$	2,211,840	$Dih(12) \times Dih(12)$	$2^4 \cdot 3^2$	2
$\Delta_4^{23}(h)$	$6F$	6,635,520	$2^3.Sym(3)$	$2^4 \cdot 3$	1
$\Delta_4^{24}(h)$	$6H$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	1
$\Delta_4^{25}(h)$	$6I1$	414,720	$2^{2+3}.Sym(3)$	$2^8 \cdot 3$	2^2
$\Delta_4^{26}(h)$	$6I2$	829,440	$2^6.Sym(3)$	$2^7 \cdot 3$	2^2
$\Delta_4^{27}(h)$	$6I3$	829,440	$2^6.Sym(3)$	$2^7 \cdot 3$	2^2
$\Delta_4^{28}(h)$	$6I4$	1,244,160	2^{3+5}	2^8	2^2
$\Delta_4^{29}(h)$	$6I5$	1,658,880	$2^5.Sym(3)$	$2^6 \cdot 3$	2
$\Delta_4^{30}(h)$	$6I6$	1,658,880	$2^5.Sym(3)$	$2^6 \cdot 3$	2
$\Delta_4^{31}(h)$	$6I7$	9,953,280	2^5	2^5	2
$\Delta_4^{32}(h)$	$6J1$	8,847,360	$3^2 : 2^2$	$2^2 \cdot 3^2$	1
$\Delta_4^{33}(h)$	$6J2$	8,847,360	$3^2 : 2^2$	$2^2 \cdot 3^2$	1
$\Delta_4^{34}(h)$	$6K2$	2,211,840	$Dih(12) \times Dih(12)$	$2^4 \cdot 3^2$	2
$\Delta_4^{35}(h)$	$6K7$	6,635,520	$2^3.Sym(3)$	$2^4 \cdot 3$	2
$\Delta_4^{36}(h)$	$6L3$	6,635,520	$2^3.Sym(3)$	$2^4 \cdot 3$	2
$\Delta_4^{37}(h)$	$6L4$	19,906,560	2^4	2^4	2
$\Delta_4^{38}(h)$	$6M3$	6,635,520	$2^3.Sym(3)$	$2^4 \cdot 3$	2
$\Delta_4^{39}(h)$	$6M4$	6,635,520	$2^3.Sym(3)$	$2^4 \cdot 3$	2
$\Delta_4^{40}(h)$	$6N1$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{41}(h)$	$6N2$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{42}(h)$	$6O1$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{43}(h)$	$6O2$	79,626,240	2^2	2^2	1
$\Delta_4^{44}(h)$	$7A1$	2,654,208	$Sym(5)$	$2^3 \cdot 3 \cdot 5$	1
$\Delta_4^{45}(h)$	$7A2$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{46}(h)$	$7A3$	39,813,120	$Dih(8)$	2^3	1
$\Delta_4^{47}(h)$	$8A7$	19,906,560	2^4	2^4	1
$\Delta_4^{48}(h)$	$8A8$	19,906,560	2^4	2^4	2
$\Delta_4^{49}(h)$	$8A9$	19,906,560	2^4	2^4	2
$\Delta_4^{50}(h)$	$8A10$	19,906,560	2^4	2^4	1
$\Delta_4^{51}(h)$	$8B7$	19,906,560	2^4	2^4	1
$\Delta_4^{52}(h)$	$8B8$	19,906,560	2^4	2^4	2
$\Delta_4^{53}(h)$	$8B9$	19,906,560	2^4	2^4	1
$\Delta_4^{54}(h)$	$8B10$	19,906,560	2^4	2^4	2
$\Delta_4^{55}(h)$	$8C1$	39,813,120	2^3	2^3	1
$\Delta_4^{56}(h)$	$8C2$	39,813,120	2^3	2^3	1
$\Delta_4^{57}(h)$	$9A$	17,694,720	$Dih(18)$	$2 \cdot 3^2$	1
$\Delta_4^{58}(h)$	$9B1$	4,423,680	$3^2 : Dih(8)$	$2^3 \cdot 3^2$	1
$\Delta_4^{59}(h)$	$9B2$	13,271,040	$Sym(4)$	$2^3 \cdot 3$	1
$\Delta_4^{60}(h)$	$9B3$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1

Orbit	X_C	Size	$G_{h,x}$	$\lvert G_{h,x}\rvert$	$\lvert Q(h)_x\rvert$
$\Delta_4^{61}(h)$	$9C1$	8,847,360	$Sym(3) \times Sym(3)$	$2^2 \cdot 3^2$	1
$\Delta_4^{62}(h)$	$9C2$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{63}(h)$	$9D1$	17,694,720	$Dih(18)$	$2 \cdot 3^2$	1
$\Delta_4^{64}(h)$	$9D2$	53,084,160	$Sym(3)$	$2 \cdot 3$	1
$\Delta_4^{65}(h)$	$9E1$	53,084,160	$Sym(3)$	$2 \cdot 3$	1
$\Delta_4^{66}(h)$	$9E2$	159,252,480	2	2	1
$\Delta_4^{67}(h)$	$10A1$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	2
$\Delta_4^{68}(h)$	$10A2$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	2
$\Delta_4^{69}(h)$	$10A3$	19,906,560	$2 \times Dih(8)$	2^4	2
$\Delta_4^{70}(h)$	$10A4$	19,906,560	$2 \times Dih(8)$	2^4	1
$\Delta_4^{71}(h)$	$10B1$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	2
$\Delta_4^{72}(h)$	$10B2$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	2
$\Delta_4^{73}(h)$	$10B3$	19,906,560	$2 \times Dih(8)$	2^4	2
$\Delta_4^{74}(h)$	$10B4$	19,906,560	$2 \times Dih(8)$	2^4	1
$\Delta_4^{75}(h)$	$10B5$	39,813,120	2^3	2^3	2
$\Delta_4^{76}(h)$	$10B6$	39,813,120	2^3	2^3	1
$\Delta_4^{77}(h)$	$10C1$	19,906,560	$2 \times Dih(8)$	2^4	2
$\Delta_4^{78}(h)$	$10C2$	19,906,560	$2 \times Dih(8)$	2^4	2
$\Delta_4^{79}(h)$	$10C3$	39,813,120	2^3	2^3	1
$\Delta_4^{80}(h)$	$10C4$	39,813,120	2^3	2^3	2
$\Delta_4^{81}(h)$	$11A$	159,252,480	2	2	1
$\Delta_4^{82}(h)$	$12A1$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	1
$\Delta_4^{83}(h)$	$12A2$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	2
$\Delta_4^{84}(h)$	$12A3$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	1
$\Delta_4^{85}(h)$	$12A4$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	2
$\Delta_4^{86}(h)$	$12A5$	9,953,280	2^{2+3}	2^5	1
$\Delta_4^{87}(h)$	$12A6$	9,953,280	2^{2+3}	2^5	2
$\Delta_4^{88}(h)$	$12B1$	6,635,520	$2 \times Sym(4)$	$2^4 \cdot 3$	1
$\Delta_4^{89}(h)$	$12B2$	6,635,520	$2 \times Sym(4)$	$2^4 \cdot 3$	1
$\Delta_4^{90}(h)$	$12B3$	19,906,560	$2 \times Dih(8)$	2^4	1
$\Delta_4^{91}(h)$	$12B4$	19,906,560	$2 \times Dih(8)$	2^4	1
$\Delta_4^{92}(h)$	$12C1$	3,317,760	$2^{2+3}.3$	$2^5 \cdot 3$	1
$\Delta_4^{93}(h)$	$12C2$	9,953,280	2^{2+3}	2^5	1
$\Delta_4^{94}(h)$	$12C3$	39,813,120	2^3	2^3	1
$\Delta_4^{95}(h)$	$12D1$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{96}(h)$	$12D2$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{97}(h)$	$12E1$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	1
$\Delta_4^{98}(h)$	$12E2$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	1
$\Delta_4^{99}(h)$	$12E3$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	1
$\Delta_4^{100}(h)$	$12E4$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	1
$\Delta_4^{101}(h)$	$12F1$	39,813,120	2^3	2^3	1
$\Delta_4^{102}(h)$	$12F2$	39,813,120	2^3	2^3	1
$\Delta_4^{103}(h)$	$12G1$	9,953,280	2^{2+3}	2^5	1

Orbit	X_C	Size	$G_{h,x}$	$\|G_{h,x}\|$	$\|Q(h)_x\|$
$\Delta_4^{104}(h)$	$12G2$	9,953,280	2^{2+3}	2^5	1
$\Delta_4^{105}(h)$	$12G3$	9,953,280	2^{2+3}	2^5	1
$\Delta_4^{106}(h)$	$12G4$	9,953,280	2^{2+3}	2^5	1
$\Delta_4^{107}(h)$	$12G5$	39,813,120	2^3	2^3	1
$\Delta_4^{108}(h)$	$12H1$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	1
$\Delta_4^{109}(h)$	$12H2$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	1
$\Delta_4^{110}(h)$	$12H3$	39,813,120	2^3	2^3	1
$\Delta_4^{111}(h)$	$12H4$	39,813,120	2^3	2^3	1
$\Delta_4^{112}(h)$	$12I1$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	2
$\Delta_4^{113}(h)$	$12I2$	3,317,760	$2^4.Sym(3)$	$2^5 \cdot 3$	1
$\Delta_4^{114}(h)$	$12I3$	9,953,280	2^{2+3}	2^5	2
$\Delta_4^{115}(h)$	$12I4$	9,953,280	2^{2+3}	2^5	2
$\Delta_4^{116}(h)$	$12I5$	9,953,280	2^{2+3}	2^5	1
$\Delta_4^{117}(h)$	$12I6$	9,953,280	2^{2+3}	2^5	1
$\Delta_4^{118}(h)$	$12I7$	19,906,560	2^4	2^4	2
$\Delta_4^{119}(h)$	$12I8$	19,906,560	2^4	2^4	1
$\Delta_4^{120}(h)$	$12I9$	19,906,560	2^4	2^4	2
$\Delta_4^{121}(h)$	$12I10$	19,906,560	2^4	2^4	1
$\Delta_4^{122}(h)$	$12J1$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	2
$\Delta_4^{123}(h)$	$12J2$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	1
$\Delta_4^{124}(h)$	$12J3$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	1
$\Delta_4^{125}(h)$	$12J4$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	2
$\Delta_4^{126}(h)$	$12J5$	39,813,120	2^3	2^3	1
$\Delta_4^{127}(h)$	$12J6$	39,813,120	2^3	2^3	1
$\Delta_4^{128}(h)$	$12K1$	39,813,120	2^3	2^3	1
$\Delta_4^{129}(h)$	$12K2$	39,813,120	2^3	2^3	1
$\Delta_4^{130}(h)$	$12K3$	39,813,120	2^3	2^3	1
$\Delta_4^{131}(h)$	$12K4$	39,813,120	2^3	2^3	1
$\Delta_4^{132}(h)$	$12L1$	39,813,120	2^3	2^3	1
$\Delta_4^{133}(h)$	$12L2$	39,813,120	2^3	2^3	1
$\Delta_4^{134}(h)$	$12L3$	39,813,120	2^3	2^3	1
$\Delta_4^{135}(h)$	$12L4$	39,813,120	2^3	2^3	1
$\Delta_4^{136}(h)$	$12M1$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{137}(h)$	$12M2$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{138}(h)$	$12M3$	79,626,240	2^2	2^2	1
$\Delta_4^{139}(h)$	$12M4$	79,626,240	2^2	2^2	1
$\Delta_4^{140}(h)$	$12N1$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	1
$\Delta_4^{141}(h)$	$12N2$	13,271,040	$2 \times Dih(12)$	$2^3 \cdot 3$	2
$\Delta_4^{142}(h)$	$12N3$	39,813,120	2^3	2^3	2
$\Delta_4^{143}(h)$	$12N4$	39,813,120	2^3	2^3	1
$\Delta_4^{144}(h)$	$12N5$	39,813,120	2^3	2^3	1
$\Delta_4^{145}(h)$	$12N6$	39,813,120	2^3	2^3	1
$\Delta_4^{146}(h)$	$12O1$	79,626,240	2^2	2^2	1

Orbit	X_C	Size	$G_{h,x}$	$\|G_{h,x}\|$	$\|Q(h)_x\|$
$\Delta_4^{147}(h)$	$12O2$	79,626,240	2^2	2^2	1
$\Delta_4^{148}(h)$	$12O3$	79,626,240	2^2	2^2	1
$\Delta_4^{149}(h)$	$12O4$	79,626,240	2^2	2^2	1
$\Delta_4^{150}(h)$	$13A1$	53,084,160	$Sym(3)$	$2 \cdot 3$	1
$\Delta_4^{151}(h)$	$13B1$	53,084,160	$Sym(3)$	$2 \cdot 3$	1
$\Delta_4^{152}(h)$	$14A1$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{153}(h)$	$14A2$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{154}(h)$	$14A3$	79,626,240	2^2	2^2	1
$\Delta_4^{155}(h)$	$14A4$	79,626,240	2^2	2^2	1
$\Delta_4^{156}(h)$	$14B1$	39,813,120	$Dih(8)$	2^3	1
$\Delta_4^{157}(h)$	$14B2$	39,813,120	$Dih(8)$	2^3	1
$\Delta_4^{158}(h)$	$14B3$	79,626,240	2^2	2^2	1
$\Delta_4^{159}(h)$	$14B4$	79,626,240	2^2	2^2	1
$\Delta_4^{160}(h)$	$15A1$	13,271,040	$Sym(4)$	$2^3 \cdot 3$	1
$\Delta_4^{161}(h)$	$15A2$	39,813,120	$Dih(8)$	2^3	1
$\Delta_4^{162}(h)$	$15A3$	79,626,240	2^2	2^2	1
$\Delta_4^{163}(h)$	$15B1$	53,084,160	$Sym(3)$	$2 \cdot 3$	1
$\Delta_4^{164}(h)$	$15B2$	159,252,480	2	2	1
$\Delta_4^{165}(h)$	$17A$	318,504,960	1	1	1
$\Delta_4^{166}(h)$	$18A1$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{167}(h)$	$18A2$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{168}(h)$	$18B1$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{169}(h)$	$18B2$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{170}(h)$	$18B3$	79,626,240	2^2	2^2	1
$\Delta_4^{171}(h)$	$18B4$	79,626,240	2^2	2^2	1
$\Delta_4^{172}(h)$	$18C1$	79,626,240	2^2	2^2	1
$\Delta_4^{173}(h)$	$18C2$	79,626,240	2^2	2^2	1
$\Delta_4^{174}(h)$	$18D$	159,252,480	2	2	1
$\Delta_4^{175}(h)$	$18E1$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{176}(h)$	$18E2$	39,813,120	$Dih(8)$	2^3	1
$\Delta_4^{177}(h)$	$18E3$	39,813,120	$Dih(8)$	2^3	1
$\Delta_4^{178}(h)$	$18E4$	79,626,240	2^2	2^2	1
$\Delta_4^{179}(h)$	$18F1$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{180}(h)$	$18F2$	79,626,240	2^2	2^2	1
$\Delta_4^{181}(h)$	$18G1$	159,252,480	2	2	1
$\Delta_4^{182}(h)$	$18G2$	159,252,480	2	2	1
$\Delta_4^{183}(h)$	$18H1$	159,252,480	2	2	1
$\Delta_4^{184}(h)$	$18H2$	159,252,480	2	2	1
$\Delta_4^{185}(h)$	$20A1$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{186}(h)$	$20A2$	26,542,080	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{187}(h)$	$20A3$	79,626,240	2^2	2^2	1
$\Delta_4^{188}(h)$	$20A4$	79,626,240	2^2	2^2	1
$\Delta_4^{189}(h)$	$20B1$	79,626,240	2^2	2^2	1

Orbit	X_C	Size	$G_{h,x}$	$\|G_{h,x}\|$	$\|Q(h)_x\|$
$\Delta_4^{190}(h)$	$20B2$	79,626,240	2^2	2^2	1
$\Delta_4^{191}(h)$	$20B3$	79,626,240	2^2	2^2	1
$\Delta_4^{192}(h)$	$20B4$	79,626,240	2^2	2^2	1
$\Delta_4^{193}(h)$	$21A1$	159,252,480	2	2	1
$\Delta_4^{194}(h)$	$21A2$	159,252,480	2	2	1
$\Delta_4^{195}(h)$	$22A$	159,252,480	2	2	1
$\Delta_4^{196}(h)$	$24A1$	79,626,240	2^2	2^2	1
$\Delta_4^{197}(h)$	$24A2$	79,626,240	2^2	2^2	1
$\Delta_4^{198}(h)$	$24A3$	79,626,240	2^2	2^2	1
$\Delta_4^{199}(h)$	$24A4$	79,626,240	2^2	2^2	1
$\Delta_4^{200}(h)$	$24B1$	79,626,240	2^2	2^2	1
$\Delta_4^{201}(h)$	$24B2$	79,626,240	2^2	2^2	1
$\Delta_4^{202}(h)$	$24B3$	79,626,240	2^2	2^2	1
$\Delta_4^{203}(h)$	$24B4$	79,626,240	2^2	2^2	1
$\Delta_4^{204}(h)$	$24C1$	79,626,240	2^2	2^2	1
$\Delta_4^{205}(h)$	$24C2$	79,626,240	2^2	2^2	1
$\Delta_4^{206}(h)$	$24C3$	79,626,240	2^2	2^2	1
$\Delta_4^{207}(h)$	$24C4$	79,626,240	2^2	2^2	1
$\Delta_4^{208}(h)$	$26A1$	159,252,480	2	2	1
$\Delta_4^{209}(h)$	$26A2$	159,252,480	2	2	1
$\Delta_4^{210}(h)$	$26B1$	159,252,480	2	2	1
$\Delta_4^{211}(h)$	$26B2$	159,252,480	2	2	1
$\Delta_4^{212}(h)$	$27A$	318,504,960	1	1	1
$\Delta_4^{213}(h)$	$28A1$	159,252,480	2	2	1
$\Delta_4^{214}(h)$	$28A2$	159,252,480	2	2	1
$\Delta_4^{215}(h)$	$30A1$	39,813,120	$Dih(8)$	2^3	1
$\Delta_4^{216}(h)$	$30A2$	39,813,120	$Dih(8)$	2^3	1
$\Delta_4^{217}(h)$	$30A3$	79,626,240	2^2	2^2	1
$\Delta_4^{218}(h)$	$30A4$	79,626,240	2^2	2^2	1
$\Delta_4^{219}(h)$	$30B1$	79,626,240	2^2	2^2	1
$\Delta_4^{220}(h)$	$30B2$	79,626,240	2^2	2^2	1
$\Delta_4^{221}(h)$	$30B3$	79,626,240	2^2	2^2	1
$\Delta_4^{222}(h)$	$30B4$	79,626,240	2^2	2^2	1
$\Delta_4^{223}(h)$	$30C1$	159,252,480	2	2	1
$\Delta_4^{224}(h)$	$30C2$	159,252,480	2	2	1
$\Delta_4^{225}(h)$	$36A1$	159,252,480	2	2	1
$\Delta_4^{226}(h)$	$36A2$	159,252,480	2	2	1
$\Delta_4^{227}(h)$	$36B1$	159,252,480	2	2	1
$\Delta_4^{228}(h)$	$36B2$	159,252,480	2	2	1
$\Delta_4^{229}(h)$	$39A$	318,504,960	1	1	1
$\Delta_4^{230}(h)$	$39B$	318,504,960	1	1	1
$\Delta_4^{231}(h)$	$42A1$	159,252,480	2	2	1
$\Delta_4^{232}(h)$	$42A2$	159,252,480	2	2	1

Orbit	X_C	Size	$G_{h,x}$	$\|G_{h,x}\|$	$\|Q(h)_x\|$
$\Delta_4^{233}(h)$	$60A1$	159,252,480	2	2	1
$\Delta_4^{234}(h)$	$60A2$	159,252,480	2	2	1
$\Delta_5^{1}(h)$	$13A2$	159,252,480	2	2	1
$\Delta_5^{2}(h)$	$13B2$	159,252,480	2	2	1
$\Delta_5^{3}(h)$	$35A$	318,504,960	1	1	1

3.5.3 The Computer Files

There are three computer files attached to this section.

Fi23Gens.txt

This file contains generators for Fi_{23} as a permutation group on 31,671 points. The generators are taken from the Online Atlas. This file should be loaded first.

Fi23Reps.txt

This file contains representatives for each of the G_h-orbits of the plane-line collinearity graph, given as involutions in $2C_G$. The entry dij is the representative for Δ_i^j. Thus to call the representative for Δ_2^3, the user would need simply to type $d23$.

Fi23Adjacency.pdf

This file contains the collapsed adjacency matrix for the plane-line collinearity graph of Fi_{23} over 90 pages.

3.6 Plane-Line Graph for He

The sporadic simple group He was discovered by Dieter Held [Hel69] in 1969 during an investigation of simple groups related to the Mathieu group M_{24}. In particular, those simple groups featuring involutions whose centraliser is isomorphic to that of an involution in M_{24}. The group He was first constructed by John McKay and Graham Higman computationally as a permutation group with point stabiliser $Sp_4(4) : 2$. Unfortunately, this work was never published.

This group is of particular interest to the study of the Monster group $\mathbb{M}$ since He centralises an element of order 7 in $\mathbb{M}$. This prime turns out to play a special role in the group. Whilst the smallest faithful complex representation of the group has dimension 51, the smallest representation of He over any field is its 50-dimensional representation over the field of 7 elements. The smallest faithful permutation representation of the group, the representation we shall be using in this study, is a rank 5 action on a set of 2058 points.

The automorphism group of He has shape $He : 2$ and features as a maximal subgroup of Fischer's sporadic simple group Fi'_{24}. In fact, Fi'_{24} has two conjugacy classes of maximal subgroups of this shape. The two classes are fused by an outer automorphism of Fi'_{24}.

The group has order $4,030,387,200 = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$, and contains eleven conjugacy classes of maximal subgroups:

- $Sp_4(4) : 2$
- $2^2.L_3(4).Sym(3)$
- $2^6 : 3.Sym(6)$
- $2^6 : 3.Sym(6)$
- $2^{1+6}.L_3(2)$
- $7^2 : 2.L_2(7)$
- $3.Sym(7)$
- $7^{1+2} : (3 \times Sym(3))$
- $Sym(4) \times L_3(2)$
- $7 : 3 \times L_3(2)$
- $5^2 : 4.Alt(4)$

The maximal subgroup $2^{1+6}.L_3(2)$ appears in our minimal 2-local geometry as the stabiliser of a plane.

3.6.1 The Minimal 2-local Geometry

The minimal 2-local geometry for He was described by Ronan and Stroth in [RS84].

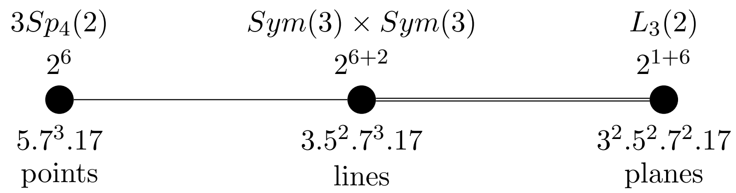


Figure 3.10: The Minimal 2-local Geometry for He

Is well known that the centraliser of an involution in the G-conjugacy class $2B$ has shape $2^{1+6}.L_3(2)$, where the unique central involution is the nontrivial element of $Z(O_2(C_G(x))) = Z(2^{1+6}) = 2$. These are known as central involutions, meaning that the involutions are central in some Sylow 2-subgroup of G. Note that given any two central involutions in He, their product is also a central involution. Furthermore, elementary abelian subgroups consisting solely of central involutions are known as pure central elementary abelian subgroups. Lines here may be put into correspondence with a G-conjugacy class of pure central elementary abelian subgroups of order 2^2.

The point-line collinearity graph associated with this geometry is given in Theorem 3.1.6.

Since the planes can be put into one-to-one correspondence with the central involutions of G, our G_h orbits will therefore be orbits of involutions under the action of an involution centraliser. Fortunately, these orbits and their sizes have already been determined in a paper by Bates, Rowley and Taylor [BRT15, Table 28], in which they also provide representatives.

We work with the permutation representation of G on a set of size 2058, and the representatives provided by Bates, Rowley and Taylor. We can obtain the set of lines by executing the following command:

$$> SubgroupClasses(G : IsElementaryAbelian := true, OrderEqual := 4);$$

We see that there are two choices of pure central elementary abelian subgroups of order 2^2, each of the correct size. This is because we in fact have two isomorphic geometries for G, with certain orbits fusing together under the action of the automorphism group of G. Our choice of which set to use will be explained further in the next subsection.

3.6.2 The Graph

The main result of this section is the structure of the plane-line collinearity graph for He.

Theorem 3.6.1 *Let $\mathcal{G}(\Gamma)$ be the plane-line collinearity graph for the minimal 2-local geometry Γ for $G = He$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ *has diameter 7.*

(ii) $|\Delta_1(h)| = 14$ *and* $\Delta_1(h)$ *is a* G_h*-orbit.*

(iii) $|\Delta_2(h)| = 168$ *and* $\Delta_2(h)$ *is a* G_h*-orbit.*

(iv) $|\Delta_3(h)| = 1,428$ *and* $\Delta_3(h)$ *consists of two* G_h*-orbits.*

(v) $|\Delta_4(h)| = 9,422$ *and* $\Delta_4(h)$ *consists of five* G_h*-orbits.*

(vi) $|\Delta_5(h)| = 52,640$ *and* $\Delta_5(h)$ *consists of nine* G_h*-orbits.*

(vii) $|\Delta_6(h)| = 113,960$ *and* $\Delta_6(h)$ *consists of twelve* G_h*-orbits.*

(viii) $|\Delta_7(h)| = 9,792$ *and* $\Delta_7(h)$ *consists of three* G_h*-orbits.*

Lemma 3.6.2 *Each plane is incident with 7 lines. Each line is incident with 3 planes.*

Proof. Each line is associated with a pure central elementary abelian subgroup of size 2^2 which contain three central involutions. Each central involution lies in seven such conjugate subgroups. This can be proven computationally or read straight from the diagram of the geometry. Alternatively we direct the reader to the paper of Mason and Smith [MS82]. □

Corollary 3.6.3 *For a fixed plane h,* $|\Delta_1(h)| = 14$.

Proof. This follows immediately from Lemma 3.6.2 since $7 \times (3 - 1) = 14$. □

When we are considering the size of the G_h-orbits given in [BRT15] we find that the two smallest orbits, with the exception of the fixed orbit containing h, are of size 14. Thus we have two options for $\Delta_1(h)$. These two options correspond to the two isomorphic geometries for G. The choice of either orbit results in the same plane-line collinearity graph, though we do not prove this here. In the automorphism group of G, these two orbits fuse into a single orbit.

For the purpose of this study, we choose the first orbit of size 14 listed in [BRT15] to be our orbit $\Delta_1(h)$. The G_h-orbits and their properties, and the collapsed adjacency matrix for the plane-line collinearity graph are given on the following pages.

Orbit	X_C	Size	g	G_{h,h^g}	$\|G_{h,h^g}\|$
h	$1A$	1	–	$2^{1+6}.L_3(2)$	21,504
$\Delta_1^1(h)$	$2B3$	14	$b^2(ab)^2b^4ab^5ab^3ab^3$	$2^{1+5}.Sym(4)$	1536
$\Delta_2^1(h)$	$2B2$	168	$ab^5ab^3ab^3ab^5$	$2^4.Dih(8)$	128
$\Delta_3^1(h)$	$2A$	84	$(ab)^2b^5(ab)^2$	$2^{1+4}.Dih(8)$	256
$\Delta_3^2(h)$	$4C1$	1,344	ab^5ab^5	$2.Dih(8)$	16
$\Delta_4^1(h)$	$2B4$	14	$b^2ab^3(ab)^2b^4ab^3ab^3$	$2^{1+5}.Sym(4)$	1,536
$\Delta_4^2(h)$	$3B$	2,688	b^2	$Dih(8)$	8
$\Delta_4^3(h)$	$4C3$	672	$b^2(ab)^2b^2(ab)^3$	$2^2.Dih(8)$	32
$\Delta_4^4(h)$	$4C6$	672	$(ab)^4b(ab)^2$	$2^2.Dih(8)$	32
$\Delta_4^5(h)$	$6B1$	5,376	$b^2(ab)^2b^2ab$	2^2	4
$\Delta_5^1(h)$	$4B1$	1,344	$b^2ab^3ab^3ab^3$	2^4	16
$\Delta_5^2(h)$	$4B2$	1,344	$b^2ab^5(ab)^2b^4$	2^4	16
$\Delta_5^3(h)$	$4B3$	112	$ab^3(ab)^4b^4(ab)^2$	$2^3.Sym(4)$	192
$\Delta_5^4(h)$	$4B4$	112	$(ab)^3b^4(ab)^4b^2$	$2^3.Sym(4)$	192
$\Delta_5^5(h)$	$4C4$	672	$b^2(ab)^4b^2(ab)^2$	$2^2.Dih(8)$	32
$\Delta_5^6(h)$	$4C5$	672	$b^2ab^3(ab)^4b^2(ab)^2$	$2^2.Dih(8)$	32
$\Delta_5^7(h)$	$6A$	5,376	$ab^3ab^3(ab)^4$	2^2	4
$\Delta_5^8(h)$	$17B$	21,504	ab	1	1
$\Delta_5^9(h)$	$21A$	21,504	ab^3ab^3	1	1
$\Delta_6^1(h)$	$2B1$	168	$(ab)^2$	$2^4.Dih(8)$	128
$\Delta_6^2(h)$	$3A$	896	b^4ab^3	$Sym(4)$	24
$\Delta_6^3(h)$	$5A$	5,376	$(ab)^3$	2^2	4
$\Delta_6^4(h)$	$8A1$	5,376	$(ab)^2b^2(ab)^2b^2$	4	4
$\Delta_6^5(h)$	$8A2$	5,376	$ab^3(ab)^2b^2ab$	4	4
$\Delta_6^6(h)$	$10A1$	5,376	$b^2(ab)^2$	2^2	4
$\Delta_6^7(h)$	$10A2$	5,376	$b^2ab^3ab^3$	2^2	4
$\Delta_6^8(h)$	$12B1$	10,752	$b^2ab^3(ab)^2$	2	2
$\Delta_6^9(h)$	$12B2$	10,752	$b^4(ab)^2$	2	2
$\Delta_6^{10}(h)$	$15A$	21,504	$b^2(ab)^3$	1	1
$\Delta_6^{11}(h)$	$17A$	21,504	$ab^3(ab)^2$	1	1
$\Delta_6^{12}(h)$	$21B$	21,504	b^2ab	1	1
$\Delta_7^1(h)$	$4C2$	1,344	$b^4ab^3ab^3(ab)^3b^2$	$2.Dih(8)$	16
$\Delta_7^2(h)$	$6B2$	5,376	ab^3ab	2^2	4
$\Delta_7^3(h)$	$7C$	3,072	$b^2ab^3ab^3(ab)^2$	7	7

Table 3.9: The G_h-orbits of the Plane-Line Collinearity Graph for He

Table 3.10: The Collapsed Adjacency Matrix for the Plane-Line Collinearity Graph of He

	h	Δ_1^1	Δ_2^1	Δ_3^1	Δ_3^2	Δ_4^1	Δ_4^2	Δ_4^3	Δ_4^4	Δ_4^5	Δ_5^1	Δ_5^2	Δ_5^3	Δ_5^4	Δ_5^5	Δ_5^6	Δ_5^7
h	–	14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	1	1	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	1	3	2	8	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	4	1	0	1	0	8	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	1	0	3	0	4	0	2	4	0	0	0	0	0	0	0
Δ_4^1	0	0	0	6	0	0	0	0	0	0	0	0	0	8	0	0	0
Δ_4^2	0	0	0	0	2	0	1	0	0	2	0	0	0	0	1	0	0
Δ_4^3	0	0	0	1	0	0	0	1	0	8	0	2	0	0	0	2	0
Δ_4^4	0	0	0	0	4	0	0	0	1	0	0	0	1	0	0	0	8
Δ_4^5	0	0	0	0	1	0	1	1	0	2	1	0	0	0	0	0	0
Δ_5^1	0	0	0	0	0	0	0	0	0	4	1	0	0	0	1	0	0
Δ_5^2	0	0	0	0	0	0	0	1	0	0	0	2	0	1	0	1	0
Δ_5^3	0	0	0	0	0	0	0	0	6	0	0	0	0	0	0	0	0
Δ_5^4	0	0	0	0	0	1	0	0	0	0	0	12	0	1	0	0	0
Δ_5^5	0	0	0	0	0	0	4	0	0	0	2	0	0	0	0	0	0
Δ_5^6	0	0	0	0	0	0	0	2	0	0	0	2	0	0	0	1	0
Δ_5^7	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
Δ_5^8	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
Δ_5^9	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1
Δ_6^1	0	0	0	0	0	0	0	0	0	0	0	8	0	0	0	4	0
Δ_6^2	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
Δ_6^3	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1
Δ_6^4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
Δ_6^5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_6^6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
Δ_6^7	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	2
Δ_6^8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_6^9	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
Δ_6^{10}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_6^{11}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_6^{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_7^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_7^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_7^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	Δ_5^8	Δ_5^9	Δ_6^1	Δ_6^2	Δ_6^3	Δ_6^4	Δ_6^5	Δ_6^6	Δ_6^7	Δ_6^8	Δ_6^9	Δ_6^{10}	Δ_6^{11}	Δ_6^{12}	Δ_7^1	Δ_7^2	Δ_7^3
h	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^2	0	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^5	4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^1	0	0	0	0	4	0	0	0	4	0	0	0	0	0	0	0	0
Δ_5^2	0	0	1	0	0	0	0	0	0	0	8	0	0	0	0	0	0
Δ_5^3	0	0	0	8	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^5	0	0	0	0	0	0	0	8	0	0	0	0	0	0	0	0	0
Δ_5^6	0	0	1	0	0	8	0	0	0	0	0	0	0	0	0	0	0
Δ_5^7	4	4	0	0	1	0	0	1	2	0	0	0	0	0	0	0	0
Δ_5^8	2	2	0	0	1	1	0	0	0	2	0	2	1	1	0	0	0
Δ_5^9	2	3	0	0	0	0	1	0	0	1	1	0	1	2	0	0	0
Δ_6^1	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_6^2	0	0	0	1	6	0	0	6	0	0	0	0	0	0	0	0	0
Δ_6^3	4	0	0	1	2	0	0	0	1	0	0	4	0	0	0	0	0
Δ_6^4	4	0	0	0	0	1	0	0	0	0	2	0	0	4	0	2	0
Δ_6^5	0	4	0	0	0	0	0	0	0	2	0	4	0	4	0	0	0
Δ_6^6	0	0	0	1	0	0	0	2	1	0	4	0	4	0	0	0	0
Δ_6^7	0	0	0	0	1	0	0	1	1	0	0	4	0	4	0	0	0
Δ_6^8	4	2	0	0	0	0	1	0	0	3	0	2	2	0	0	0	0
Δ_6^9	0	2	0	0	0	1	0	2	0	0	4	0	0	2	1	1	0
Δ_6^{10}	2	0	0	0	1	0	1	0	1	1	0	4	3	1	0	0	0
Δ_6^{11}	1	1	0	0	0	0	0	1	0	1	0	3	3	2	0	1	1
Δ_6^{12}	1	2	0	0	0	1	1	0	1	0	1	1	2	2	0	1	1
Δ_7^1	0	0	0	0	0	0	0	0	0	0	8	0	0	0	2	4	0
Δ_7^2	0	0	0	0	0	2	0	0	0	0	2	0	4	4	1	1	0
Δ_7^3	0	0	0	0	0	0	0	0	0	0	0	0	7	7	0	0	0

Table 3.9 lists information relating to the G_h-orbits of the plane-line collinearity graph. The first column lists the orbits whilst the second column lists their corresponding name given in [BRT15]. The third column gives the size of the orbit whilst the fourth column gives an element which conjugates the fixed plane h to a representative of the given orbit. The elements a and b are the standard generators of G given in the computer files where the fixed plane $h = (ab^2)^6$. The final two columns give the structure and size of the stabiliser of each orbit representative in G_h.

3.6.3 The Computer Files

There are two computer files attached to the work on He.

HeGens.txt

This file contains generators from the Online Atlas for the Held sporadic simple group. This should be loaded before the subsequent file.

HeReps.txt

This file contains representatives for the G_h-orbits of the plane-line collinearity graph for He. The representatives are given as dij referring the representative for the orbit Δ_i^j. Thus the representative for the orbit Δ_4^5 is named $d45$. The user should load the file **HeGens.txt** before loading the representatives.

3.7 Plane-Line Graph for Co_3

The three Conway sporadic simple groups are related strongly to the Leech lattice. This lattice was first constructed during the search for a solution to the sphere packing problem: given a sphere, how many spheres of equal size can touch it? Clearly in one-dimensional Euclidean space, the answer is two. In two dimensions, we may have six. In three, the number increases to twelve. In eight-dimensional space, the answer is 240. The root system of E_8, an exceptional group of Lie type, describes this unique rigid solution. The only other dimension for which the answer is known is 24. Here a given sphere may be touched by 196,560 other spheres. The Leech lattice describes the unique rigid solution.

The Leech lattice Λ exhibits a lot of symmetry and its group of automorphisms is the Conway group Co_0 of order $8,315,553,613,086,720,000$. However, Co_0 is not simple having a central subgroup of order 2. The quotient of Co_0 by its centre gives Co_1, the first of Conway's sporadic simple groups. The other two groups are described as stabilisers of lattice vectors.

Recall the definition of the Steiner system $S(5,8,24)$ on the 24-element set S given in Section 3.2. Let C denote the binary Golay code, i.e. the set consisting of the empty set, S, all octads, complements of octads and dodecads of S. Additionally, for x and y in $\mathbb{Z}^{24}$, let $(x \cdot y)$ denote the standard inner product.

Definition 3.7.1 *The vector $x = (x_1, \ldots, x_{24}) \in \mathbb{Z}^{24}$ is in Λ if and only if*

(i) each coordinate x_i is congruent modulo 2, to some n;

(ii) the set of i such that x_i takes any given value modulo 4 form a C-set;

(iii) and

$$\sum_{i=1}^{24} x_i \equiv 4n \ (mod\ 8).$$

It may be further observed that for vectors x and y in Λ, the inner product $(x \cdot y)$ is a multiple of 8 and $(x \cdot x)$ is a multiple of 16. We call a vector $x \in \Lambda$ a vector of type n, if $(x \cdot x) = 16n$. The automorphism group of a vector of type 2 is the sporadic simple group Co_2, whilst the third of Conway's sporadic groups Co_3 arises as the automorphism group of a vector of type 3.

3.7.1 The Maximal 2-Local Geometry

The maximal 2-local geometry for $G = Co_3$ is given in [RS84].

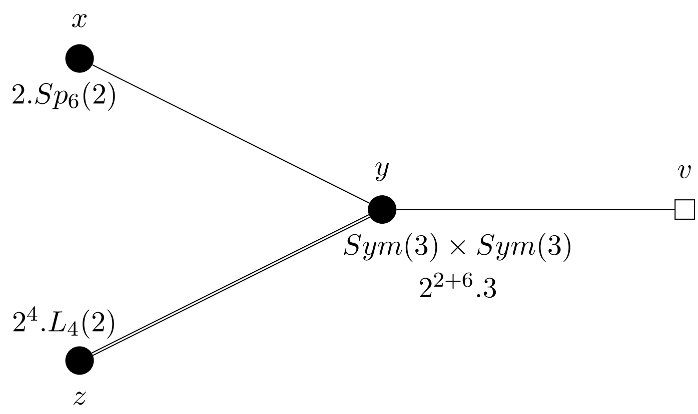


Figure 3.11: The Maximal 2-local Geometry for Co_3

The points, lines and planes are represented by vertices x, y and z respectively. There are 170,775 points, 17,931,375 lines, and 1,536,975 planes. The points may be put into one-to-one correspondence with the involutions in the G-conjugacy class $2A$, with centraliser $2.Sp_6(2)$. These are central involutions, which were defined and described earlier in Section 3.6.1. The lines and planes of our geometry correspond to pure central elementary abelian subgroups of orders 2^2 and 2^4 respectively. For further information, we refer the reader to the paper of Maginnis and Onofrei [MO07b].

Remark. Additionally, there is also a conjugacy class of pure central elementary abelian subgroups of order 2^3. These do not correspond to objects in the geometry and are each contained in a unique subgroup 2^4.

We work within the permutation representation of G on a set of 276 points. The order of G is $495,766,656,000 = 2^{10}.3^7.5^3.7.11.23$ which is small enough to lend itself well to direct computations. We are therefore able to put our hands directly on the required classes of elementary abelian subgroups with the command:

$$> SubgroupClasses(G : IsElementaryAbelian := true, OrderDividing := 16);$$

With the lines and planes in our grasp, we are in a position to investigate the plane-line collinearity graph.

3.7.2 The Graph

We construct the plane-line collinearity graph in this section.

Theorem 3.7.2 *Let $\mathcal{G}(\Gamma)$ be the plane-line collinearity graph for the maximal 2-local geometry Γ for $G = Co_3$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ *has diameter 5.*

(ii) $|\Delta_1(h)| = 70$ *and* $\Delta_1(h)$ *is a* G_h*-orbit.*

(iii) $|\Delta_2(h)| = 3,080$ *and* $\Delta_2(h)$ *consists of three* G_h*-orbits.*

(iv) $|\Delta_3(h)| = 91,904$ *and* $\Delta_3(h)$ *consists of thirteen* G_h*-orbits.*

(v) $|\Delta_4(h)| = 1,138,944$ *and* $\Delta_4(h)$ *consists of thirty* G_h*-orbits.*

(vi) $|\Delta_5(h)| = 302,976$ *and* $\Delta_5(h)$ *consists of five* G_h*-orbits.*

Lemma 3.7.3 *Each plane is incident with 35 lines. Each line is incident with 3 planes.*

Proof. For a fixed plane h, the residue geometry of h is a projective space $PG(3,2)$, wherein we find 15 points and 35 lines incident with h. For a fixed line l, the residue geometry of l is a digon, a complete bipartite graph defined on the three points of l and the three planes incident with l. □

Further information relating to the residue geometries and their properties may be obtained from [MO07b].

Corollary 3.7.4 *For a fixed plane h,* $|\Delta_1(h)| = 70$.

Proof. This follows immediately from Lemma 3.7.3 since $35 \times (3-1) = 70$. □

Calculating the orbits of the planes under the action of $G_h \cong 2^4.L_4(2)$ reveals that there are 53 orbits in total. With the exception of the orbit consisting of the fixed plane h, the smallest orbit is of size 70 and consists solely of planes which intersect h in a pure central elementary abelian subgroup of order 2^2. Thus it follows that these planes form the disc $\Delta_1(h)$ and that $\Delta_1(h)$ consists of a single G_h-orbit. Once again, we explore the adjacency of the graph by conjugating these neighbours to a representative of each orbit and determining where they lie.

Information relating to the G_h-orbits of the plane-line collinearity graph and the full collapsed adjacency matrix are given upon the subsequent pages. There, $Q(h) = O_2(G_h) \cong 2^4$.

Table 3.11: The G_h-orbits of the Plane-Line Collinearity Graph for Co_3

Orbit	Size	g	G_{h,h^g}	$\lvert G_{h,h^g}\rvert$	$\lvert Q(h)_{h^g}\rvert$
h	1	$-$	$2^4.L_4(2)$	$2^{10}\cdot 3^2\cdot 5\cdot 7$	2^4
$\Delta_1^1(h)$	70	$a(ab)^3a^2(ab)^7a$	$2^4.(2^4:3:Sym(3))$	$2^9\cdot 3^2$	2^4
$\Delta_2^1(h)$	840	$(ab)^7$	$2^3.(Sym(4)\times 2)$	$2^7\cdot 3$	2^3
$\Delta_2^2(h)$	1,120	$a(ab)^{13}b^2(ab)^{11}$	$2^2.(Alt(4):Sym(3))$	$2^5\cdot 3^2$	2^2
$\Delta_2^3(h)$	1,120	$a(ab)^6b^2(ab)^3$	$2^2.(Alt(4):Sym(3))$	$2^5\cdot 3^2$	2^2
$\Delta_3^1(h)$	10,080	$a(ab)^{13}a(ab)^8$	2^{2+3}	2^5	2
$\Delta_3^2(h)$	10,080	ab^2ab	2^{2+3}	2^5	2
$\Delta_3^3(h)$	960	$a(ab)^8a(ab)^5a^2$	$2\times L_2(7)$	$2^4\cdot 3\cdot 7$	2
$\Delta_3^4(h)$	6,720	$(ab)^8a^2b$	$2\times Sym(4)$	$2^4\cdot 3$	2^2
$\Delta_3^5(h)$	6,720	$b^3a(ab)^6$	$2\times Sym(4)$	$2^4\cdot 3$	2
$\Delta_3^6(h)$	13,440	a^2b	$Sym(4)$	$2^3\cdot 3$	1
$\Delta_3^7(h)$	4,480	$a^2b^2a^2$	$Alt(4):Sym(3)$	$2^3\cdot 3^2$	1
$\Delta_3^8(h)$	4,480	b	$Alt(4):Sym(3)$	$2^3\cdot 3^2$	1
$\Delta_3^9(h)$	13,440	$a(ab)^6a$	$Sym(4)$	$2^3\cdot 3$	1
$\Delta_3^{10}(h)$	2,688	$b(ab)^9b^2(ab)^{10}$	$Sym(5)$	$2^3\cdot 3\cdot 5$	1
$\Delta_3^{11}(h)$	13,440	b^3a	$Sym(4)$	$2^3\cdot 3$	1
$\Delta_3^{12}(h)$	4,480	ab^2a	$Alt(4):Sym(3)$	$2^3\cdot 3^2$	1
$\Delta_3^{13}(h)$	896	$b^2(ab)^6a(ab)^7$	$Alt(5):Sym(3)$	$2^3\cdot 3^2\cdot 5$	1
$\Delta_4^1(h)$	26,880	aba^2	$Dih(12)$	$2^2\cdot 3$	2
$\Delta_4^2(h)$	40,320	$(ab)^2a^2$	$Dih(8)$	2^3	1
$\Delta_4^3(h)$	80,640	$a(ab)^{10}$	2^2	2^2	1
$\Delta_4^4(h)$	4,480	a^2b^2a	$(Sym(3)\times Sym(3)):2$	$2^3\cdot 3^2$	1
$\Delta_4^5(h)$	80,640	a	2^2	2^2	1
$\Delta_4^6(h)$	13,440	$a(ab)^9b(ab)^4$	$Sym(4)$	$2^3\cdot 3$	1
$\Delta_4^7(h)$	40,320	$a(ab)^{12}$	$Dih(8)$	2^3	1
$\Delta_4^8(h)$	26,880	ab^3a^2	$Dih(12)$	$2^2\cdot 3$	1
$\Delta_4^9(h)$	40,320	aba	$Dih(8)$	2^3	1

Orbit	Size	g	G_{h,h^g}	$\lvert G_{h,h^g}\rvert$	$\lvert Q(h)_{h^g}\rvert$
$\Delta_4^{10}(h)$	8,960	$a^2ba(ab)^{12}$	$3 : Sym(3) \cdot 2$	$2^2 \cdot 3^2$	1
$\Delta_4^{11}(h)$	26,880	$(ab)^{12}a^2$	$Dih(12)$	$2^2 \cdot 3$	2
$\Delta_4^{12}(h)$	80,640	$(ab)^4a^2$	2^2	2^2	1
$\Delta_4^{13}(h)$	26,880	$a(ab)^2$	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{14}(h)$	40,320	$a^2(ab)^2a$	$Dih(8)$	2^3	1
$\Delta_4^{15}(h)$	26,880	ab^2a^2	$Dih(12)$	$2^2 \cdot 3$	1
$\Delta_4^{16}(h)$	13,440	$a^2(ab)^3a$	$Sym(4)$	$2^3 \cdot 3$	1
$\Delta_4^{17}(h)$	2,688	$a^2(ab)^7a$	$Sym(5)$	$2^3 \cdot 3 \cdot 5$	1
$\Delta_4^{18}(h)$	896	$a(ab)^6b^2(ab)^4$	$Alt(6)$	$2^3 \cdot 3^2 \cdot 5$	1
$\Delta_4^{19}(h)$	13,440	$a(ab)^7a$	$Sym(4)$	$2^3 \cdot 3$	1
$\Delta_4^{20}(h)$	13,440	$(ab)^3a$	$Sym(4)$	$2^3 \cdot 3$	1
$\Delta_4^{21}(h)$	1,920	$a(ab)^9b(ab)^{12}$	$L_2(7)$	$2^3 \cdot 3 \cdot 7$	1
$\Delta_4^{22}(h)$	26,880	$(ab)^9a^2$	$Alt(4)$	$2^2 \cdot 3$	1
$\Delta_4^{23}(h)$	53,760	a^2ba	$Sym(3)$	$2 \cdot 3$	1
$\Delta_4^{24}(h)$	53,760	a^2b^3a	$Sym(3)$	$2 \cdot 3$	1
$\Delta_4^{25}(h)$	53,760	ab^3a	$Sym(3)$	$2 \cdot 3$	1
$\Delta_4^{26}(h)$	17,920	$(ab)^9a$	$3^2 : 2$	$2 \cdot 3^2$	1
$\Delta_4^{27}(h)$	161,280	$(ab)^7a$	2	2	1
$\Delta_4^{28}(h)$	53,760	a^2ba^2	$Sym(3)$	$2 \cdot 3$	1
$\Delta_4^{29}(h)$	53,760	ba^2	$Sym(3)$	$2 \cdot 3$	1
$\Delta_4^{30}(h)$	53,760	ab	$Sym(3)$	$2 \cdot 3$	1
$\Delta_5^{1}(h)$	161,280	$(ab)^2$	2	2	1
$\Delta_5^{2}(h)$	53,760	$a(ab)^2a(ab)^9$	$Sym(3)$	$2 \cdot 3$	1
$\Delta_5^{3}(h)$	53,760	$(ab)^2a$	$Sym(3)$	$2 \cdot 3$	1
$\Delta_5^{4}(h)$	32,256	$(ab)^6a$	$Dih(10)$	$2 \cdot 5$	1
$\Delta_5^{5}(h)$	1,920	$b^2(ab)^{12}b^2(ab)^2$	$2^3 : 7 : 3$	$2^3 \cdot 3 \cdot 7$	1

Table 3.12: The Collapsed Adjacency Matrix for the Plane-Line Collinearity Graph of Co_3

	z	Δ_1^1	Δ_2^1	Δ_2^2	Δ_2^3	Δ_3^1	Δ_3^2	Δ_3^3	Δ_3^4	Δ_3^5	Δ_3^6	Δ_3^7	Δ_3^8	Δ_3^9	Δ_3^{10}	Δ_3^{11}	Δ_3^{12}	Δ_3^{13}
z	–	70	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	1	1	36	16	16	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	3	3	0	0	12	12	8	24	8	0	0	0	0	0	0	0	0
Δ_2^2	0	1	0	0	1	18	0	0	0	18	12	4	4	12	0	0	0	0
Δ_2^3	0	1	0	1	0	0	18	0	0	18	0	0	0	0	12	12	4	4
Δ_3^1	0	0	1	2	0	0	1	0	0	2	0	0	0	0	0	0	0	0
Δ_3^2	0	0	1	0	2	1	0	0	0	2	0	0	0	0	0	0	0	0
Δ_3^3	0	0	7	0	0	0	0	7	0	0	0	0	0	0	0	0	0	0
Δ_3^4	0	0	3	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0
Δ_3^5	0	0	1	3	3	3	3	0	0	1	0	0	0	0	0	0	0	0
Δ_3^6	0	0	0	1	0	0	0	0	0	0	4	0	0	1	0	0	0	0
Δ_3^7	0	0	0	1	0	0	0	0	0	0	0	4	1	0	0	0	0	0
Δ_3^8	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
Δ_3^9	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
Δ_3^{10}	0	0	0	0	5	0	0	0	0	0	0	0	0	0	0	5	0	0
Δ_3^{11}	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
Δ_3^{12}	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
Δ_3^{13}	0	0	0	0	5	0	0	0	0	0	0	0	0	0	0	0	5	0
Δ_4^1	0	0	0	0	0	3	0	1	3	0	0	2	0	0	0	0	0	0
Δ_4^2	0	0	0	0	0	2	1	0	0	0	4	0	0	4	0	0	0	0
Δ_4^3	0	0	0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0
Δ_4^4	0	0	0	0	0	9	0	0	6	0	0	4	0	0	0	0	0	0
Δ_4^5	0	0	0	0	0	1	0	0	0	0	0	0	1	3	0	0	0	0
Δ_4^6	0	0	0	0	0	6	0	0	0	1	0	0	0	4	0	0	0	0
Δ_4^7	0	0	0	0	0	1	0	0	2	0	0	0	0	0	0	0	0	0
Δ_4^8	0	0	0	0	0	3	0	0	1	3	0	3	0	3	1	0	3	0
Δ_4^9	0	0	0	0	0	2	0	0	0	1	0	0	0	0	0	4	0	0

	Δ_4^1	Δ_4^2	Δ_4^3	Δ_4^4	Δ_4^5	Δ_4^6	Δ_4^7	Δ_4^8	Δ_4^9	Δ_4^{10}	Δ_4^{11}	Δ_4^{12}	Δ_4^{13}	Δ_4^{14}	Δ_4^{15}	Δ_4^{16}	Δ_4^{17}	Δ_4^{18}
z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	8	8	8	4	8	8	4	8	8	0	0	0	0	0	0	0	0	0
Δ_3^2	0	4	0	0	0	0	0	0	0	8	8	8	8	8	8	8	4	0
Δ_3^3	28	0	0	0	0	0	0	0	0	0	28	0	0	0	0	0	0	0
Δ_3^4	12	0	0	4	0	0	12	4	0	0	12	0	4	0	16	0	0	0
Δ_3^5	0	0	0	0	0	2	0	12	6	0	0	0	12	6	0	2	0	2
Δ_3^6	0	12	6	0	0	0	0	0	0	0	4	0	6	12	0	8	4	0
Δ_3^7	12	0	18	4	0	0	0	18	0	0	0	0	0	0	0	0	0	0
Δ_3^8	0	0	0	0	18	0	0	0	0	4	0	0	18	0	0	12	0	0
Δ_3^9	0	12	0	0	18	4	0	6	0	0	0	0	0	0	0	0	0	0
Δ_3^{10}	0	0	0	0	0	0	0	10	0	10	0	0	0	0	20	0	0	0
Δ_3^{11}	0	0	0	0	0	0	0	0	12	0	0	6	6	12	12	0	0	0
Δ_3^{12}	0	0	0	0	0	0	0	18	0	0	0	18	0	0	0	0	0	4
Δ_3^{13}	0	0	0	0	0	0	0	0	0	30	0	0	30	0	0	0	0	0
Δ_4^1	3	0	6	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0
Δ_4^2	0	6	4	0	0	0	0	0	0	0	0	8	0	4	0	0	1	0
Δ_4^3	2	2	4	0	7	0	6	1	0	0	2	4	1	0	0	0	0	0
Δ_4^4	0	0	0	0	0	0	9	0	0	0	0	0	0	0	6	0	0	0
Δ_4^5	0	0	7	0	0	0	0	1	2	0	0	6	1	2	0	0	0	0
Δ_4^6	0	0	0	0	0	0	0	0	6	0	0	12	0	0	0	0	0	1
Δ_4^7	0	0	12	1	0	0	4	0	8	0	0	0	0	0	2	0	0	0
Δ_4^8	0	0	3	0	3	0	0	3	0	1	0	3	4	0	0	0	0	0
Δ_4^9	0	0	0	0	4	2	8	0	8	0	0	4	0	0	4	0	0	0

	Δ_4^{19}	Δ_4^{20}	Δ_4^{21}	Δ_4^{22}	Δ_4^{23}	Δ_4^{24}	Δ_4^{25}	Δ_4^{26}	Δ_4^{27}	Δ_4^{28}	Δ_4^{29}	Δ_4^{30}	Δ_5^1	Δ_5^2	Δ_5^3	Δ_5^4	Δ_5^5
z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^5	6	6	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^6	0	0	0	8	4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^7	0	0	0	0	0	12	0	0	0	0	0	0	0	0	0	0	0
Δ_3^8	0	0	0	0	0	0	12	4	0	0	0	0	0	0	0	0	0
Δ_3^9	12	0	0	0	0	0	0	0	12	4	0	0	0	0	0	0	0
Δ_3^{10}	0	20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^{11}	0	4	0	0	0	0	0	0	12	0	4	0	0	0	0	0	0
Δ_3^{12}	12	0	0	0	0	0	0	4	0	0	0	12	0	0	0	0	0
Δ_3^{13}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^1	0	0	0	0	0	2	6	0	0	2	6	2	30	0	0	0	0
Δ_4^2	4	0	0	4	4	0	0	0	8	0	0	8	8	0	0	0	0
Δ_4^3	0	0	0	0	4	2	4	0	8	2	2	0	10	4	2	0	0
Δ_4^4	0	0	0	0	0	0	0	0	0	0	0	0	36	0	0	0	0
Δ_4^5	0	0	0	4	4	0	0	0	2	0	6	10	10	8	0	2	0
Δ_4^6	0	0	0	0	0	0	0	0	0	4	12	12	12	0	0	0	0
Δ_4^7	0	0	0	4	0	4	4	0	4	0	0	4	16	4	0	0	0
Δ_4^8	0	0	0	0	0	6	0	0	6	6	6	0	6	6	0	0	0
Δ_4^9	0	1	0	0	4	4	4	0	8	4	0	0	4	4	0	0	0

	z	Δ_1^1	Δ_2^1	Δ_2^2	Δ_2^3	Δ_3^1	Δ_3^2	Δ_3^3	Δ_3^4	Δ_3^5	Δ_3^6	Δ_3^7	Δ_3^8	Δ_3^9	Δ_3^{10}	Δ_3^{11}	Δ_3^{12}	Δ_3^{13}
Δ_4^{10}	0	0	0	0	0	0	9	0	0	0	0	0	2	0	3	0	0	3
Δ_4^{11}	0	0	0	0	0	0	3	1	3	0	2	0	0	0	0	0	0	0
Δ_4^{12}	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0
Δ_4^{13}	0	0	0	0	0	0	3	0	1	3	3	0	3	0	0	3	0	1
Δ_4^{14}	0	0	0	0	0	0	2	0	0	1	4	0	0	0	0	4	0	0
Δ_4^{15}	0	0	0	0	0	0	3	0	4	0	0	0	0	0	2	6	0	0
Δ_4^{16}	0	0	0	0	0	0	6	0	0	1	8	0	4	0	0	0	0	0
Δ_4^{17}	0	0	0	0	0	0	15	0	0	0	20	0	0	0	0	0	0	0
Δ_4^{18}	0	0	0	0	0	0	0	0	0	15	0	0	0	0	0	0	20	0
Δ_4^{19}	0	0	0	0	0	0	0	0	0	3	0	0	0	12	0	0	4	0
Δ_4^{20}	0	0	0	0	0	0	0	0	0	3	0	0	0	0	4	4	0	0
Δ_4^{21}	0	0	0	0	0	0	0	0	0	7	0	0	0	0	0	0	0	0
Δ_4^{22}	0	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0
Δ_4^{23}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
Δ_4^{24}	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
Δ_4^{25}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
Δ_4^{26}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0
Δ_4^{27}	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0
Δ_4^{28}	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
Δ_4^{29}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
Δ_4^{30}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
Δ_5^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	Δ_4^1	Δ_4^2	Δ_4^3	Δ_4^4	Δ_4^5	Δ_4^6	Δ_4^7	Δ_4^8	Δ_4^9	Δ_4^{10}	Δ_4^{11}	Δ_4^{12}	Δ_4^{13}	Δ_4^{14}	Δ_4^{15}	Δ_4^{16}	Δ_4^{17}	Δ_4^{18}
Δ_4^{10}	0	0	0	0	0	0	0	3	0	0	0	9	3	0	0	0	0	0
Δ_4^{11}	4	0	6	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0
Δ_4^{12}	0	4	4	0	6	2	0	1	2	1	0	4	1	2	0	0	0	0
Δ_4^{13}	0	0	3	0	3	0	0	4	0	1	0	3	3	0	0	0	0	0
Δ_4^{14}	0	4	0	0	4	0	0	0	0	0	0	4	0	0	0	2	0	0
Δ_4^{15}	0	0	0	1	0	0	3	0	6	0	0	0	0	0	3	0	0	0
Δ_4^{16}	0	0	0	0	0	0	0	0	0	0	0	0	0	6	0	0	0	0
Δ_4^{17}	0	15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{18}	0	0	0	0	0	15	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{19}	0	12	0	0	0	0	0	0	0	0	0	12	0	3	0	0	0	0
Δ_4^{20}	0	0	0	0	0	0	0	0	3	0	0	0	0	0	4	0	0	0
Δ_4^{21}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7	0	0
Δ_4^{22}	0	6	0	0	12	0	6	0	0	0	0	0	0	0	0	4	0	0
Δ_4^{23}	0	3	6	0	6	0	0	0	3	0	1	3	3	3	0	0	0	0
Δ_4^{24}	1	0	3	0	0	0	3	3	3	0	0	0	0	0	0	0	0	0
Δ_4^{25}	3	0	6	0	0	0	3	0	3	0	0	3	3	0	0	1	0	0
Δ_4^{26}	0	0	0	0	0	0	0	0	0	1	3	9	9	18	3	0	0	1
Δ_4^{27}	0	2	4	0	1	0	1	1	2	0	1	5	0	3	0	0	0	0
Δ_4^{28}	1	0	3	0	0	1	0	3	3	0	0	6	0	3	3	0	0	0
Δ_4^{29}	3	0	3	0	9	3	0	3	0	0	0	3	0	0	1	0	0	0
Δ_4^{30}	1	6	0	0	15	3	3	0	0	0	0	12	3	0	0	1	0	0
Δ_5^1	5	2	5	1	5	1	4	1	1	0	2	4	0	0	2	1	0	0
Δ_5^2	0	0	6	0	12	0	3	3	3	0	1	0	0	0	0	0	0	0
Δ_5^3	0	0	3	0	0	0	0	0	0	3	9	0	3	9	6	1	1	0
Δ_5^4	0	0	0	0	5	0	0	0	0	5	5	0	5	5	5	5	0	0
Δ_5^5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	Δ_4^{19}	Δ_4^{20}	Δ_4^{21}	Δ_4^{22}	Δ_4^{23}	Δ_4^{24}	Δ_4^{25}	Δ_4^{26}	Δ_4^{27}	Δ_4^{28}	Δ_4^{29}	Δ_4^{30}	Δ_5^{1}	Δ_5^{2}	Δ_5^{3}	Δ_5^{4}	Δ_5^{5}
Δ_4^{10}	0	0	0	0	0	0	0	2	0	0	0	0	0	0	18	18	0
Δ_4^{11}	0	0	0	0	2	0	0	2	6	0	0	0	12	2	18	6	0
Δ_4^{12}	2	0	0	0	2	0	2	2	10	4	2	8	8	0	0	0	0
Δ_4^{13}	0	0	0	0	6	0	6	6	0	0	0	6	0	0	6	6	0
Δ_4^{14}	1	0	0	0	4	0	0	8	12	4	0	0	0	0	12	4	0
Δ_4^{15}	0	2	0	0	0	0	0	2	0	6	2	0	12	0	12	6	0
Δ_4^{16}	0	0	1	8	0	0	4	0	0	0	0	4	12	0	4	12	0
Δ_4^{17}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	20	0	0
Δ_4^{18}	0	0	0	0	0	0	0	20	0	0	0	0	0	0	0	0	0
Δ_4^{19}	0	0	0	0	4	0	0	0	0	0	12	4	0	4	0	0	0
Δ_4^{20}	0	0	0	0	0	12	0	0	0	0	4	0	0	12	12	12	0
Δ_4^{21}	0	0	0	0	0	28	28	0	0	0	0	0	0	0	0	0	0
Δ_4^{22}	0	0	0	1	0	0	12	0	12	0	0	0	12	0	0	0	1
Δ_4^{23}	1	0	0	0	3	0	3	3	12	3	0	6	6	1	3	0	0
Δ_4^{24}	0	3	1	0	0	15	13	0	9	3	0	0	9	0	0	3	0
Δ_4^{25}	0	0	1	6	3	13	6	0	0	0	3	0	9	3	0	3	0
Δ_4^{26}	0	0	0	0	9	0	0	0	0	0	3	9	0	3	0	0	0
Δ_4^{27}	0	0	0	2	4	3	0	0	10	4	7	3	6	6	2	1	0
Δ_4^{28}	0	0	0	0	3	3	0	0	12	0	6	1	6	9	3	3	0
Δ_4^{29}	3	1	0	0	0	0	3	1	21	6	0	0	9	0	0	0	0
Δ_4^{30}	1	0	0	0	6	0	0	3	9	1	0	2	0	0	1	0	2
Δ_5^{1}	0	0	0	2	2	3	3	0	6	2	3	0	9	1	4	1	0
Δ_5^{2}	1	3	0	0	1	0	3	1	18	9	0	0	3	0	3	0	0
Δ_5^{3}	0	3	0	0	3	0	0	0	6	3	0	1	12	3	1	3	0
Δ_5^{4}	0	5	0	0	0	5	5	0	5	5	0	0	5	0	5	0	0
Δ_5^{5}	0	0	0	14	0	0	0	0	0	0	0	56	0	0	0	0	0

In Table 3.11, we give the size of each orbit $\Delta_i^j(h)$ and an element g which conjugates the fixed subgroup h to a representative of the associated orbit. The elements a and b are the standard generators of G of orders 3 and 4 respectively, with h being given in the computer files. The final three columns give the stabiliser in G_h of each representative and its order, and the stabiliser in $Q(h) = O_2(G_h) \cong 2^4$.

3.7.3 The Computer Files

There are two computer files attached to the work on Co_3.

Co3Gens.txt

This file contains generators from the Online Atlas for the Co_3 sporadic simple group. This should be loaded before the subsequent file.

Co3Reps.txt

This file contains representatives for the G_h-orbits of the plane-line collinearity graph for Co_3. The representatives are given as dij referring the representative for the orbit Δ_i^j. Thus the representative for the orbit Δ_4^5 is named $d45$. The user should load the file **Co3Gens.txt** before loading the representatives.

3.8 Plane-Line Graph for Co_2

The group $G = Co_2$ is the second of the three sporadic simple groups introduced by Conway. In this section we detail a plane-line collinearity graph associated with this group.

3.8.1 The Maximal 2-Local Geometry

The maximal 2-local geometry Γ associated with Co_2 was introduced in [RS80].

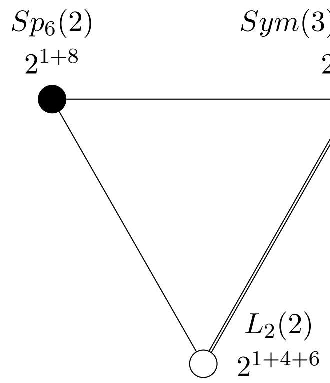


Figure 3.12: The Maximal 2-local Geometry for Co_2

We refer to the $2^{10}.M_{22} : 2$ vertices as points, the $2^{4+10} : (Sym(3) \times Sym(5))$ vertices as lines, and the $2^{1+8}.Sp_6(2)$ vertices as planes. The point-line collinearity graph is described in Theorem 3.1.9. In this geometry we're in the fortunate situation that the planes can be put into one-to-one correspondence with the involutions of the conjugacy class $2A$ in G.

In [BRT15], the authors determine the five orbits of $2A$-involutions under the stabiliser $G_h \cong 2^{1+8}.Sp_6(2)$ and provide representatives for each orbit. Thus it remains only for us to classify adjacency.

Lemma 3.8.1 *Each plane is incident with 315 lines. Each line is incident with 5 planes.*

Proof. A proof of this lemma follows from a brief examination of the maximal parabolic subgroups $G_h \cong 2^{1+8}.Sp_6(2)$ and $G_l \cong 2^{4+10} : (Sym(3) \times Sym(5))$, respectively the stabilisers of a plane and a line. Let G_h and G_l be such that they share the same Sylow 2-subgroup of G and let $M = G_h \cap G_l$. We find that $[G_h : M] = 315$ and $[G_l : M] = 5$. Concrete copies of these groups are given in the computer files in Section 3.8.3. □

Lemma 3.8.2 *For a fixed plane h, $|\Delta_1(h)| = 1,260$.*

Proof. This follows immediately from Lemma 3.8.1 since $315 \times (5 - 1) = 1,260$. □

3.8.2 The Graph

The main result of this section is the structure of the plane-line collinearity graph for Co_2.

Theorem 3.8.3 *Let $\mathcal{G}(\Gamma)$ be the plane-line collinearity graph for the maximal 2-local geometry Γ for $G = Co_2$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ has diameter 2.

(ii) $|\Delta_1(h)| = 1,260$ and $\Delta_1(h)$ is a G_h-orbit.

(iii) $|\Delta_2(h)| = 55,664$ and $\Delta_2(h)$ consists of three G_h-orbits.

The plane-line collinearity graph is given in Figure 3.13 along with the adjacency, and properties relating to the G_h-orbits are listed in Table 3.13. Representatives for each orbit are given in terms of a conjugating element, where a and b are the standard generators for Co_2 of orders 2 and 5 respectively and a is itself the fixed $2A$-involution h. Finally we have the stabiliser of each representative in G_h and the stabiliser in $Q(h) = O_2(G_h) \cong 2^{1+8}$.

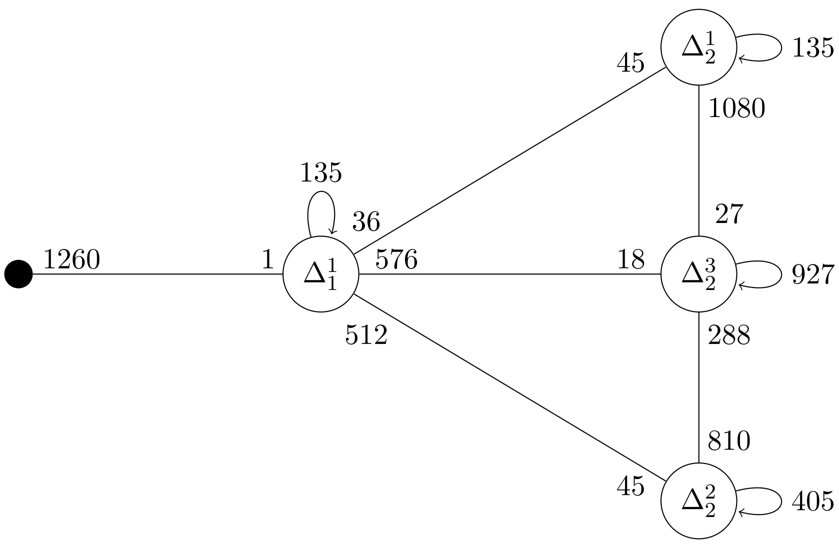


Figure 3.13: The Plane-Line Collinearity Graph of Co_2

Orbit	Size	g	G_{h,h^g}	$\lvert Q(h)_{h^g}\rvert$
h	1	–	$2^{1+8}.Sp_6(2)$	2^9
$\Delta_1^1(h)$	1,260	bab^2ab	$2^{1+6}.(2^{1+4} \times 2^2) : (Sym(3) \times Sym(3))$	2^7
$\Delta_2^1(h)$	1,008	b^3abab^3	$2^5.2^5 : Sym(6)$	2^5
$\Delta_2^2(h)$	14,336	bab	$U_4(2) : 2$	1
$\Delta_2^3(h)$	40,320	b	$2^2.(2^{1+4} \times 2^2) : (Sym(3) \times Sym(3))$	2^2

Table 3.13: The G_h-orbits of the Plane-Line Collinearity Graph for Co_2

3.8.3 The Computer Files

There are two computer files attached to the work on Co_2.

Co2Gens.txt

This file contains generators from the Online Atlas for the Co_2 sporadic simple group as permutation group of size 2300. This should be loaded before the subsequent file.

Co2Subgroups.txt

This file contains generators for the maximal parabolic subgroups used in Lemma 3.8.1, here named $Gh \cong 2^{1+8}.Sp_6(2)$ and $Gl \cong 2^{4+10} : (Sym(3) \times Sym(5))$. They are constructed such that they share the same Sylow 2-subgroup in G.

3.9 Remaining Cases

With the collinearity graphs associated with McL dealt with in the next chapter, there remain five groups for which plane-line collinearity graphs associated with 2-local geometries are currently unknown, namely Co_1, Fi'_{24}, $\mathbb{B}$ and $\mathbb{M}$, and the minimal 2-local case for Co_2.

The minimal 2-local geometries for Co_2 and Co_1 are given in [RS84].

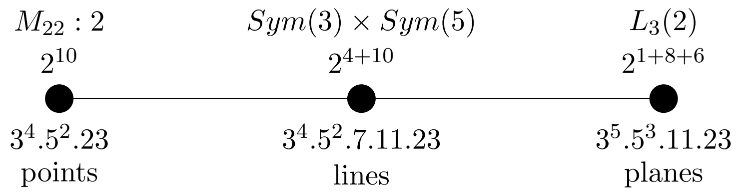


Figure 3.14: The Minimal 2-local Geometry for Co_2

Both of these geometries are defined in terms of sections of the Leech lattice, and may be extracted from the maximal 2-local geometries associated with these groups described in [RS80].

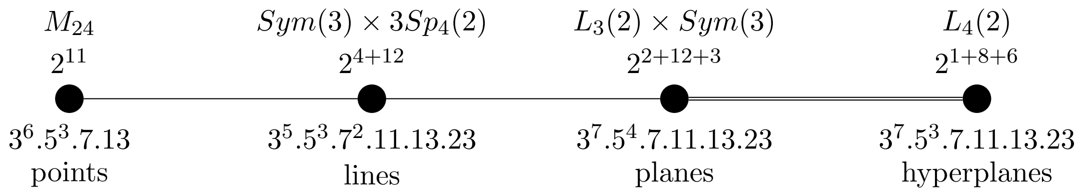
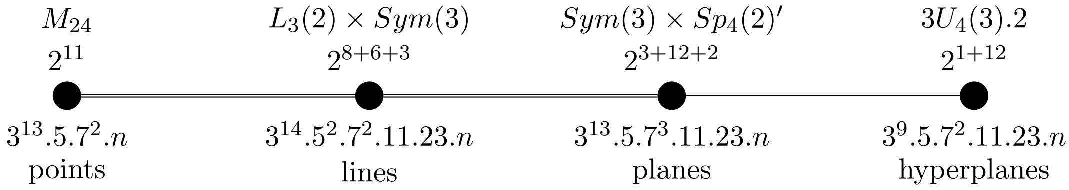


Figure 3.15: The Minimal 2-local Geometry for Co_1

The minimal geometry associated with Fi'_{24} is given in Figure 3.16.

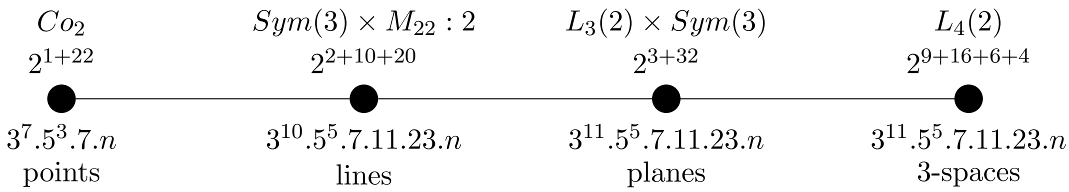


$n = 13.17.29$

Figure 3.16: The Minimal 2-local Geometry for Fi'_{24}

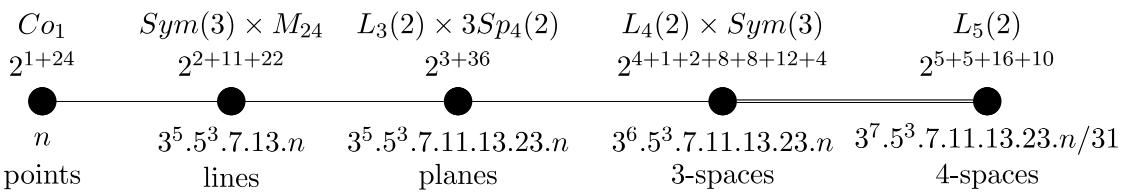
Here we are in a more fortunate position as the hyperplanes may be put into one-to-one correspondence with the involutions in the conjugacy class $2B_{Fi'_{24}}$, where the centraliser of such an involution is a subgroup of shape $2^{1+12}.3.U_4(3).2$. The orbits of $2B$ involutions are described in [Tay11], where the author also provides representatives for these orbits. Thus all that remains is to determine adjacency to achieve the graph.

Finally we have the geometries associated with the monstrous sporadic groups, $\mathbb{B}$ and $\mathbb{M}$.



$n = 13.17.19.31.47$

Figure 3.17: The Minimal 2-local Geometry for $\mathbb{B}$



$n = 3^{11}.5^5.7^4.11.13^2.17.19.29.31.41.47.59.71$

Figure 3.18: The Minimal 2-local Geometry for $\mathbb{M}$

The geometries associated with the groups $\mathbb{B}$ and $\mathbb{M}$ are incredibly large and not much is known about them, which unfortunately renders them intractable for now.

Chapter 4

Graphs Associated with McL

This chapter differs from the other studies of the previous chapter in that we explore a variety of graphs associated with multiple different geometries. Here our group of interest is the McLaughlin sporadic simple group McL.

First discovered by Jack McLaughlin in 1969 [McL69], McL appeared as an index 2 subgroup of the automorphism group of a certain graph having 275 vertices. It features as a stabiliser of a triangle in the Leech lattice and thus appears within the sporadic groups Co_1, Co_2 and Co_3. Additionally its automorphism group $McL : 2$ is found in the Lyons sporadic group Ly which contains $3.McL : 2$ as a maximal subgroup.

The group has order $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 = 898,128,000$ and contains twelve conjugacy classes of maximal subgroups.

- $U_4(3)$
- M_{22}
- M_{22}
- $U_3(5)$
- $3^{1+4} : 2.Sym(5)$
- $3^4 : M_{10}$
- $L_3(4) : 2$
- $2.Alt(8)$
- $2^4 : Alt(7)$
- $2^4 : Alt(7)$
- M_{11}
- $5^{1+2} : 3 : 8$

For the subgroups M_{22} and $2^4 : Alt(7)$ we have two classes of maximal subgroups each, fused under the outer automorphism. These subgroups will be of particular interest in our study of the minimal 2-local geometries.

4.1 The First Minimal 2-Local Geometry for McL

In this section we describe all collinearity graphs associated with a minimal 2-local geometry for $G = McL$. This geometry was first described in [RS84].

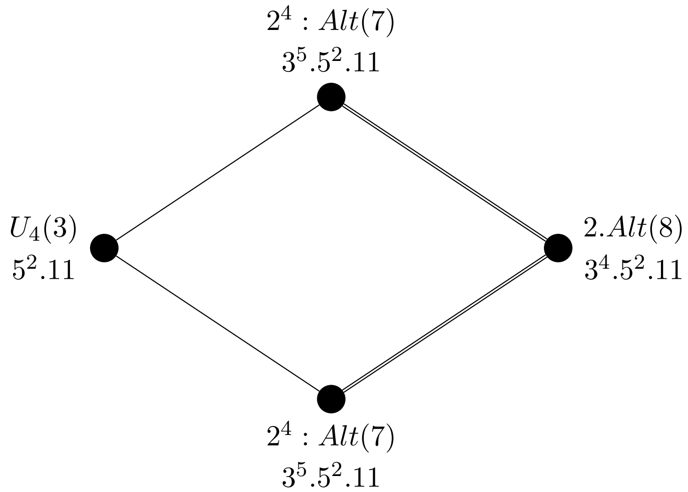


Figure 4.1: A Minimal 2-local Geometry for McL

For ease, we will refer to the $U_4(3)$ vertices as points, the $2^4 : Alt(7)$ vertices as lines and the $2.Alt(8)$ vertices as planes. We may consider the points to the points of the 275-dimensional permutation representation of G. The lines then correspond to two classes of heptads.

When we work with the planes, we use a different description of the geometry given in [MO07a]. In their study, Maginnis and Onofrei note that $2.Alt(8)$ is the centraliser of an involution in G and thus we may regard the planes as involutions. Further, with this correspondence, the lines may be regarded as two classes of elementary abelian subgroups of order 2^4. Incidence is given by containment.

4.1.1 The Graphs

We first consider the point-line collinearity graph. Let p denote a fixed point. The 275 points split into three orbits under the action of $G_p \cong U_4(3)$ of sizes 1, 112 and 162. Two points are incident if there exists a heptad which contains them both. We choose one of the classes of heptads to acts as our lines. The choice is not important since the point-line collinearity graphs will be isomorphic to one another.

After generating the group using the generators given on the Online Atlas, we may generate the class of points and a class of lines using the following code:

$> points := GSet(G);$

$> lines := GSet(G, \{\{23, 48, 88, 142, 157, 211, 235\}\});$

A short investigation reveals that given a point p, there are 162 other points which are found to share a heptad with p. Thus our collinearity graph is easily found to have the structure given in the following theorem.

Theorem 4.1.1 *Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for the minimal parabolic geometry Γ for $G = McL$ and let $p \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 2.*

(ii) *$|\Delta_1(p)| = 162$ and $\Delta_1(p)$ is a G_p-orbit.*

(iii) *$|\Delta_2(p)| = 112$ and $\Delta_2(p)$ is a G_p-orbit.*

A graphical representation of the graph is given in Figure 4.2.

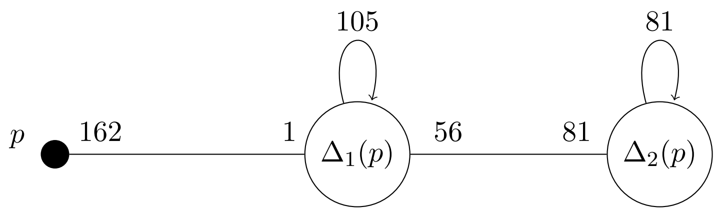


Figure 4.2: A Point-Line Collinearity Graph for McL

Next we consider the line-point collinearity graph. We now let our vertices be given by orbits of heptads l under the action of the line stabiliser $G_l \cong 2^4 : Alt(7)$. There are thirteen such orbits. Here, two heptads are incident if they intersect in a single point. Given a heptad l, there are 3122 such heptads which form three orbits under G_l. We use the same setup once more to construct the line-point collinearity graph.

Theorem 4.1.2 *Let $\mathcal{G}(\Gamma)$ be the line-point collinearity graph for the minimal parabolic geometry Γ for $G = McL$ and let $l \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 2.*

(ii) *$|\Delta_1(l)| = 3,122$ and $\Delta_1(l)$ consists of three G_l-orbits.*

(iii) *$|\Delta_2(l)| = 19,152$ and $\Delta_2(l)$ consists of nine G_l-orbits.*

The next page gives the G_l-orbits and the collapsed adjacency matrix for the graph.

	l	Δ_1^1	Δ_1^2	Δ_1^3	Δ_2^1	Δ_2^2	Δ_2^3	Δ_2^4	Δ_2^5	Δ_2^6	Δ_2^7	Δ_2^8	Δ_2^9
l	–	210	672	2240	0	0	0	0	0	0	0	0	0
Δ_1^1	1	41	96	416	0	6	114	192	384	528	432	528	384
Δ_1^2	1	30	185	500	36	0	150	90	0	360	450	330	990
Δ_1^3	1	39	150	445	18	9	132	171	180	414	396	456	711
Δ_2^1	0	0	216	360	6	35	0	240	105	240	570	360	990
Δ_2^2	0	9	0	144	28	64	9	624	492	264	912	360	216
Δ_2^3	0	57	240	704	0	3	166	108	72	312	384	296	780
Δ_2^4	0	24	36	228	16	52	27	430	378	452	618	420	441
Δ_2^5	0	48	0	240	7	41	18	378	344	552	498	576	420
Δ_2^6	0	33	72	276	8	11	39	226	276	519	444	564	654
Δ_2^7	0	27	90	264	19	38	48	309	249	444	497	426	711
Δ_2^8	0	33	66	304	12	15	37	210	288	564	426	525	642
Δ_2^9	0	16	132	316	22	6	65	147	140	436	474	428	940

Table 4.1: The Collapsed Adjacency Matrix for a Line-Point Collinearity Graph of McL

In Table 4.2 we give the size of each orbit, a representative heptad from the given orbit, and the size of its stabiliser in G_h together with the order of this subgroup. Since these orbits will reappear in graphs given later in this chapter, we also attribute a label to each for ease of reference.

Label	Orbit	Size	Representative, m	$G_{l,m}$	$\|G_{l,m}\|$
l	l	1	$\{23, 48, 88, 142, 157, 211, 235\}$	$2^4 : Alt(7)$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$
$O1$	Δ_1^1	210	$\{5, 56, 116, 134, 157, 228, 245\}$	$2^3.Sym(4)$	$2^6 \cdot 3$
$O2$	Δ_1^2	672	$\{23, 30, 52, 65, 84, 91, 223\}$	$Alt(5)$	$2^2 \cdot 3 \cdot 5$
$O3$	Δ_1^3	2240	$\{5, 31, 48, 52, 55, 147, 179\}$	$3 : Sym(3)$	$2 \cdot 3^2$
$O4$	Δ_2^1	112	$\{70, 89, 113, 141, 152, 239, 257\}$	$Alt(6)$	$2^3 \cdot 3^2 \cdot 5$
$O5$	Δ_2^2	140	$\{55, 107, 113, 118, 184, 244, 255\}$	$2^2.(Alt(4) \times 3) : 2$	$2^5 \cdot 3^2$
$O6$	Δ_2^3	420	$\{53, 62, 128, 130, 142, 158, 235\}$	$2^2.Sym(4)$	$2^5 \cdot 3$
$O7$	Δ_2^4	1680	$\{80, 112, 118, 170, 201, 226, 248\}$	$Sym(4)$	$2^3 \cdot 3$
$O8$	Δ_2^5	1680	$\{3, 55, 71, 111, 134, 201, 206\}$	$Sym(4)$	$2^3 \cdot 3$
$O9$	Δ_2^6	3360	$\{77, 91, 132, 134, 179, 230, 240\}$	$2.Sym(3)$	$2^2 \cdot 3$
$O10$	Δ_2^7	3360	$\{26, 31, 90, 152, 183, 240, 244\}$	$Dih(12)$	$2^2 \cdot 3$
$O11$	Δ_2^8	3360	$\{78, 147, 170, 195, 206, 254, 266\}$	$Dih(12)$	$2^2 \cdot 3$
$O12$	Δ_2^9	5040	$\{4, 112, 114, 146, 158, 189, 203\}$	$Dih(8)$	2^3

Table 4.2: The G_l-orbits of a Line-Point Collinearity Graph for McL

We now study the plane-line and line-plane collinearity graphs associated with this geometry. As in [MO07a], we consider the planes to be involutions of G and the lines to be a class of elementary abelian subgroups of order 2^4. Two lines are said to be adjacent if their intersection is a subgroup of size 2, whilst two points are adjacent if they lie in the same elementary abelian subgroup, i.e. they commute.

First we consider the plane-line collinearity graph. Let h be a fixed involution of G, and so $G_h \cong 2.Alt(8)$. The orbits of h^G under G_h were determined by Bates, Rowley and Taylor in [BRT15] wherein they found the conjugacy class splits into six orbits. The commuting graph on these orbits has the following structure.

Theorem 4.1.3 *Let $\mathcal{G}(\Gamma)$ be the plane-line collinearity graph for the minimal parabolic geometry Γ for $G = McL$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ *has diameter 3.*

(ii) $|\Delta_1(h)| = 210$ *and* $\Delta_1(h)$ *is a* G_h*-orbit.*

(iii) $|\Delta_2(h)| = 14,000$ *and* $\Delta_2(h)$ *consists of three* G_h*-orbits.*

(iv) $|\Delta_3(h)| = 8,064$ *and* $\Delta_3(h)$ *is a* G_h*-orbit.*

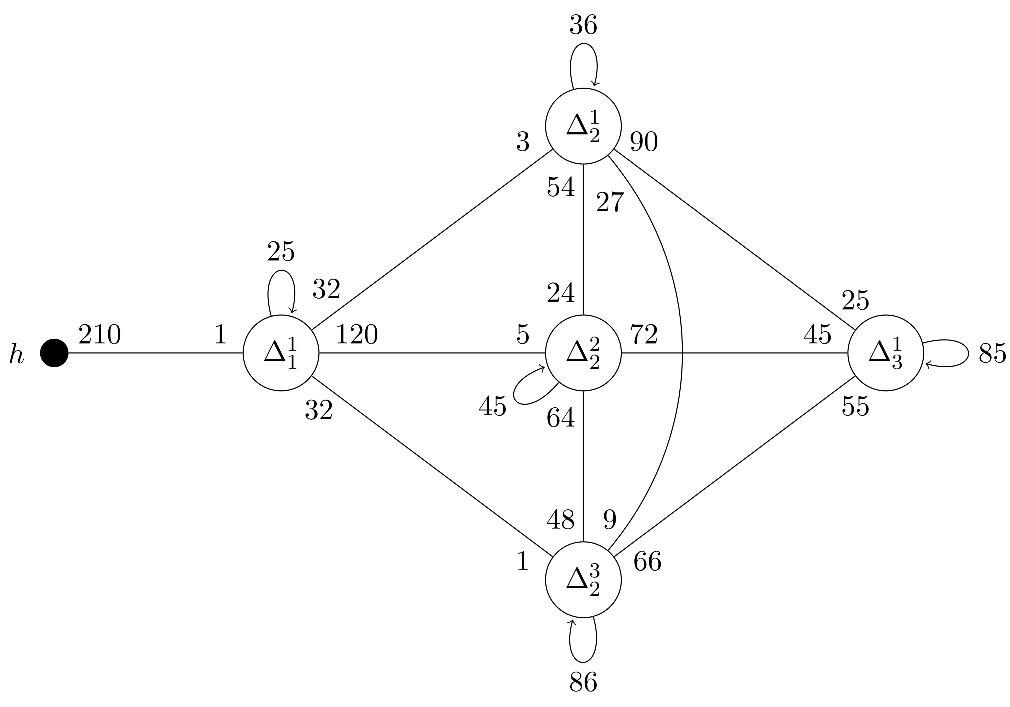


Figure 4.3: A Plane-Line Collinearity Graph for McL

The orbits Δ_1^1, Δ_2^1, Δ_2^2, Δ_2^3 and Δ_3^1 are of sizes 210, 2240, 5040, 8064 and 6720 respectively.

Finally, we determine the line-plane collinearity graph in the same fashion as before. The G_l orbits were determined earlier in the construction of the line-point collinearity graph so it remains for us to classify the adjacency within the graph. Given an elementary abelian subgroup l of order 2^4, there are unsurprisingly 210 subgroups which meet l in a subgroup of order 2.

Theorem 4.1.4 *Let $\mathcal{G}(\Gamma)$ be the line-plane collinearity graph for the minimal parabolic geometry Γ for $G = McL$ and let $l \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ has diameter 3.

(ii) $|\Delta_1(l)| = 210$ and $\Delta_1(l)$ is a G_l-orbit.

(iii) $|\Delta_2(l)| = 11,312$ and $\Delta_2(l)$ consists of six G_l-orbits.

(iv) $|\Delta_3(l)| = 10,752$ and $\Delta_3(l)$ consists of five G_l-orbits.

The collapsed adjacency matrix associated with this graph is given in Table 4.3 along with labels corresponding to the orbits described in Table 4.2.

Label	Orbit	l	Δ_1^1	Δ_2^1	Δ_2^2	Δ_2^3	Δ_2^4	Δ_2^5	Δ_2^6	Δ_3^1	Δ_3^2	Δ_3^3	Δ_3^4	Δ_3^5
l	l	–	210	0	0	0	0	0	0	0	0	0	0	0
$O1$	Δ_1^1	1	25	6	18	16	48	48	48	0	0	0	0	0
$O5$	Δ_2^1	0	9	0	9	0	0	96	0	12	0	12	0	72
$O6$	Δ_2^2	0	9	3	6	0	0	48	48	0	12	24	48	12
$O2$	Δ_2^3	0	5	0	0	5	60	30	30	0	30	0	20	30
$O9$	Δ_2^4	0	3	0	0	12	39	24	36	0	18	12	12	54
$O10$	Δ_2^5	0	3	4	6	6	24	37	18	3	21	13	24	51
$O11$	Δ_2^6	0	3	0	6	6	36	18	45	0	6	24	24	42
$O4$	Δ_3^1	0	0	15	0	0	0	90	0	0	0	15	0	90
$O7$	Δ_3^2	0	0	0	3	12	36	42	12	0	24	6	36	39
$O8$	Δ_3^3	0	0	1	6	0	24	26	48	1	6	14	24	60
$O3$	Δ_3^4	0	0	0	9	6	18	36	36	0	27	18	33	27
$O12$	Δ_3^5	0	0	2	1	4	36	34	28	2	13	20	12	58

Table 4.3: The Collapsed Adjacency Matrix for a Line-Plane Collinearity Graph of McL

4.2 The Second Minimal 2-Local Geometry for McL

This second geometry is also described in [RS84].

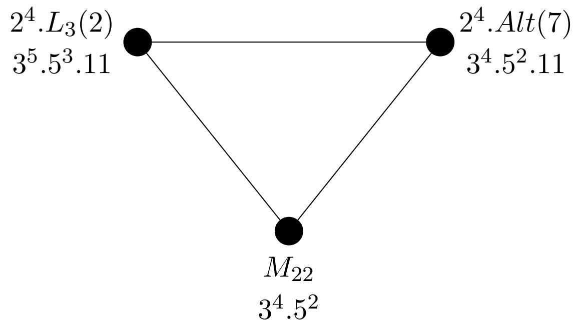


Figure 4.4: A Minimal 2-local Geometry for McL

For ease, we will refer to the M_{22} vertices as points, the $2^4.L_3(2)$ vertices as lines, and the $2^4.Alt(7)$ vertices as planes. We work with the 2025-dimensional permutation representation of the group and so the points of the geometry correspond to the points of the representation, whilst the lines and planes correspond to duads and heptads.

4.2.1 The Graphs

First we consider the point-line graph. Our 2025 points will be given by the set upon which the permutation representation of degree 2025 acts. The lines ($2^2.L_3(2)$ vertices) correspond to a conjugacy class of duads of these points. Two points are said to be adjacent if they share a duad. Under the action of $G_p \cong M_{22}$, the points split into four orbits of sizes 1, 330, 462 and 1232. We generate the points and lines using the following code:

$> points := GSet(G);$

$> lines := GSet(G, \{\{1, 3\}\});$

A brief survey of these sets confirms that given a point p, there are 330 points which share a duad with p. The point-line collinearity graph is easily determined.

Theorem 4.2.1 *Let $\mathcal{G}(\Gamma)$ be the point-line collinearity graph for the minimal parabolic geometry Γ for $G = McL$ and let $p \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ has diameter 2.

(ii) $|\Delta_1(p)| = 330$ and $\Delta_1(p)$ is a G_p-orbit.

(iii) $|\Delta_2(p)| = 1,694$ and $\Delta_2(p)$ consists of two G_p-orbits.

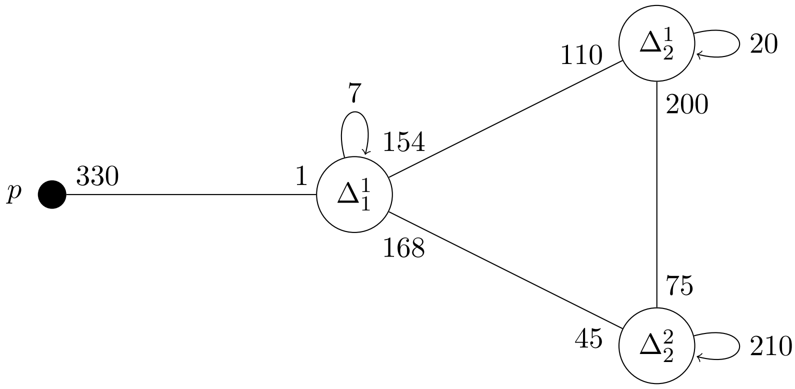


Figure 4.5: A Point-Line Collinearity Graph for McL

Whilst we are considering these orbits, we also construct the point-plane graph. Recall planes here are a class of heptads. We of course have the same four orbits of points but two points will now be adjacent if there exists a heptad which contains them both. We construct the set of planes by executing the following code:

$> planes := GSet(G, \{\{1, 444, 717, 929, 1333, 1545, 1893\}\});$

We find that given a point p, there are 462 points which share a heptad with p. The point-plane graph is easily determined.

Theorem 4.2.2 *Let $\mathcal{G}(\Gamma)$ be the point-plane collinearity graph for the minimal parabolic geometry Γ for $G = McL$ and let $p \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 2.*

(ii) *$|\Delta_1(p)| = 462$ and $\Delta_1(p)$ is a G_p-orbit.*

(iii) *$|\Delta_2(p)| = 1,562$ and $\Delta_2(p)$ consists of two G_p-orbits.*

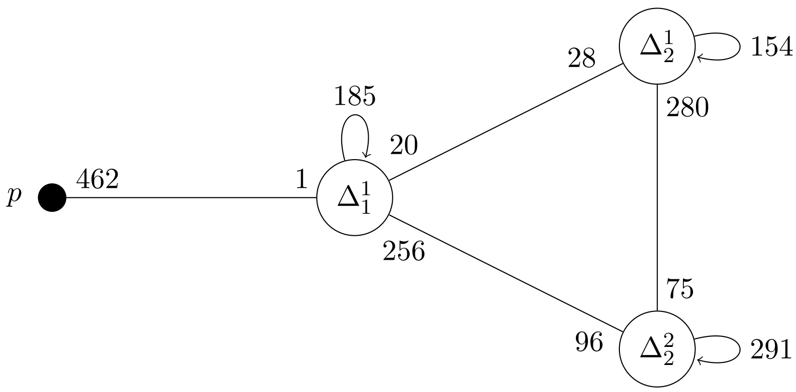


Figure 4.6: A Point-Plane Collinearity Graph for McL

We next shift our focus and consider the planes to be our vertices and we consider the plane-point and plane-line collinearity graphs. First let us consider the plane-point graph. Given that our planes may be considered as a conjugacy class of heptads and our points are the 2025 points of the permutation representation, we see that this graph is precisely the line-point collinearity graph described in Theorem 4.1.2.

The plane-line collinearity graph considers orbits of heptads as its vertices and two heptads are considered to be adjacent if they meet in a duad. Given a heptad h, there are 420 heptads adjacent to it. We previously determined the orbits in Section 4.1, thus it remains only for us to classify adjacency.

Theorem 4.2.3 *Let $\mathcal{G}(\Gamma)$ be the plane-line collinearity graph for the minimal parabolic geometry Γ for $G = McL$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ *has diameter 3.*

(ii) $|\Delta_1(h)| = 420$ *and* $\Delta_1(h)$ *is a* G_h*-orbit.*

(iii) $|\Delta_2(h)| = 18,354$ *and* $\Delta_2(h)$ *consists of eight* G_h*-orbits.*

(iv) $|\Delta_3(h)| = 3,500$ *and* $\Delta_3(h)$ *consists of three* G_h*-orbit.*

The collapsed adjacency matrix for the graph is given in Table 4.4, together with labels corresponding to the orbits in Table 4.2.

Label	Orbit	h	Δ_1^1	Δ_2^1	Δ_2^2	Δ_2^3	Δ_2^4	Δ_2^5	Δ_2^6	Δ_2^7	Δ_2^8	Δ_3^1	Δ_3^2	Δ_3^3
l	h	–	420	0	0	0	0	0	0	0	0	0	0	0
$O6$	Δ_1^1	1	65	12	6	48	112	24	24	32	96	0	0	0
$O4$	Δ_2^1	0	45	15	0	0	0	60	90	0	180	0	30	0
$O1$	Δ_2^2	0	12	0	18	0	96	96	0	96	24	6	48	24
$O2$	Δ_2^3	0	30	0	0	60	90	0	45	15	180	0	0	0
$O3$	Δ_2^4	0	21	0	9	27	96	63	45	42	90	0	9	18
$O9$	Δ_2^5	0	3	2	6	0	42	97	48	78	78	2	22	42
$O10$	Δ_2^6	0	3	3	0	9	30	48	96	57	90	3	45	36
$O11$	Δ_2^7	0	4	0	6	3	28	78	57	94	84	0	30	36
$O12$	Δ_2^8	0	8	4	1	24	40	52	60	56	161	0	12	2
$O5$	Δ_3^1	0	0	0	9	0	0	48	72	0	0	39	180	72
$O7$	Δ_3^2	0	0	2	6	0	12	44	90	60	36	15	95	60
$O8$	Δ_3^3	0	0	0	3	0	24	84	72	72	6	6	60	93

Table 4.4: The Collapsed Adjacency Matrix for a Plane-Line Collinearity Graph of McL

Next up we have the line-point collinearity graph. The vertices here are represented by a class of duads, under the stabiliser $G_l \cong 2^4 : L_3(2)$. Two lines are adjacent if their duads intersect in a point.

Theorem 4.2.4 *Let $\mathcal{G}(\Gamma)$ be the line-point collinearity graph for the minimal parabolic geometry Γ for $G = McL$ and let $l \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ *has diameter 3.*

(ii) $|\Delta_1(l)| = 658$ *and* $\Delta_1(l)$ *consists of four* G_l*-orbits.*

(iii) $|\Delta_2(l)| = 187,740$ *and* $\Delta_2(l)$ *consists of 109* G_l*-orbits.*

(iv) $|\Delta_3(l)| = 145,726$ *and* $\Delta_3(l)$ *consists of 79* G_l*-orbit.*

The collapsed adjacency matrix for this graph is given in the computer files, along with representatives and code used to determine adjacency.

Finally we have the line-plane collinearity graph. We once again use the same orbits, however there are now just 28 neighbours. This is easily read from Figure 4.4.

Theorem 4.2.5 *Let $\mathcal{G}(\Gamma)$ be the line-plane collinearity graph for the minimal parabolic geometry Γ for $G = McL$ and let $l \in \mathcal{G}(\Gamma)$. Then*

(i) $\mathcal{G}(\Gamma)$ *has diameter 5.*

(ii) $|\Delta_1(l)| = 28$ *and* $\Delta_1(l)$ *consists of two* G_l*-orbits.*

(iii) $|\Delta_2(l)| = 560$ *and* $\Delta_2(l)$ *consists of six* G_l*-orbits.*

(iv) $|\Delta_3(l)| = 9,184$ *and* $\Delta_3(l)$ *consists of 21* G_l*-orbits.*

(v) $|\Delta_4(l)| = 104,832$ *and* $\Delta_4(l)$ *consists of 70* G_l*-orbits.*

(vi) $|\Delta_5(l)| = 219,520$ *and* $\Delta_5(l)$ *consists of 93* G_l*-orbits.*

Since the collapsed adjacency matrix would be mostly empty, we have not provided a complete table. Instead, the computer files provide representatives for these orbits and a speedy procedure which determines the neighbours of a given vertex.

4.3 The Third and Fourth Minimal 2-Local Geometries

With two geometries despatched, there remain two minimal 2-local geometries associated with McL. This section deals with those two geometries. Once more, these are given in [RS84].

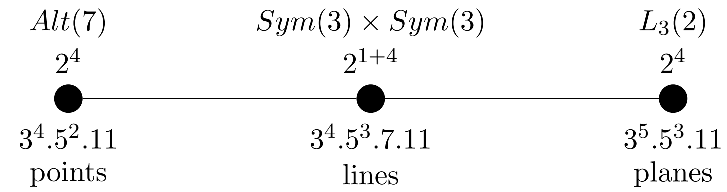


Figure 4.7: A Minimal 2-local Geometry for McL

The point-line collinearity graph associated with this geometry was constructed by Rowley in [Row15], and is isomorphic to the line-plane collinearity graph for the first of our minimal 2-local geometries described in Section 4.1. For full details of this graph, see Theorem 4.1.4.

The plane-line collinearity graph is once again described in terms of the 193 orbits of duads under $G_h \cong 2^4.L_3(2)$. From the diagram we quickly deduce that a plane is incident with three lines and a line meets with seven planes. Thus, $|\Delta_1(h)| = 14$.

Theorem 4.3.1 *Let $\mathcal{G}(\Gamma)$ be the plane-line collinearity graph for the minimal parabolic geometry Γ for $G = McL$ and let $h \in \mathcal{G}(\Gamma)$. Then*

(i) *$\mathcal{G}(\Gamma)$ has diameter 7.*

(ii) *$|\Delta_1(h)| = 14$ and $\Delta_1(h)$ is a G_h-orbit.*

(iii) *$|\Delta_2(h)| = 168$ and $\Delta_2(h)$ consists of two G_h-orbits.*

(iv) *$|\Delta_3(h)| = 1,680$ and $\Delta_3(h)$ consists of seven G_h-orbits.*

(v) *$|\Delta_4(h)| = 15,190$ and $\Delta_4(h)$ consists of 20 G_h-orbits.*

(vi) *$|\Delta_5(h)| = 121,184$ and $\Delta_5(h)$ consists of 67 G_h-orbits.*

(vii) *$|\Delta_6(h)| = 194,992$ and $\Delta_6(h)$ consists of 93 G_h-orbits.*

(viii) *$|\Delta_7(h)| = 896$ and $\Delta_7(h)$ consists of two G_h-orbits.*

Since the collapsed adjacency matrix would be mostly empty, we do not provide a complete table. Instead, orbit representatives and procedures for determining adjacency are provided in the computer files.

Finally, we describe the graphs associated with the last of the four geometries.

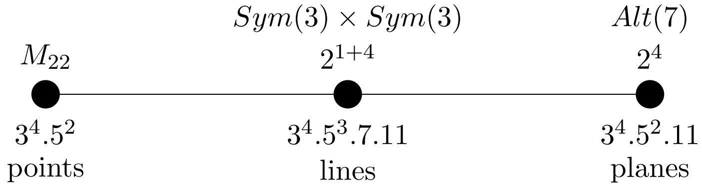


Figure 4.8: A Minimal 2-local Geometry for McL

The point-line collinearity graph associated with this geometry is identical to the point-line collinearity graph for the second minimal geometry described by Theorem 4.2.1 in Section 4.2.

Whilst, the plane-line collinearity graph for this geometry is precisely the line-plane collinearity graph described by Theorem 4.1.4 in Section 4.1.

4.4 Computer Files

Here we describe the computer files attached to this chapter.

McLGens275.txt

This file contains generators for McL as a permutation group on 275 points. These generators are taken from the Online Atlas.

McLGens2025.txt

This file contains generators for McL as a permutation group on 2025 points. These generators are taken from the Online Atlas.

McLCode1.txt

This file contains representatives for the line-point collinearity graph described by Theorem 4.2.4 in Section 4.2. Additionally this file contains the procedure used to determine adjacency within the graph.

McLCode2.txt

This file contains representatives for the line-plane collinearity graph described by Theorem 4.2.5 in Section 4.2. Additionally this file contains the procedure used to determine adjacency within the graph.

McLCode3.txt

This file contains representatives for the plane-line collinearity graph described by Theorem 4.3.1 in Section 4.3. Additionally this file contains the procedure used to determine adjacency within the graph.

McLAdjacency.txt

The collapsed adjacency matrix for the line-point collinearity graph described by Theorem 4.2.4 in Section 4.2 is given across 40 pages in this file.

Chapter 5

Commuting Graphs for $\mathbb{B}$

In this chapter we detail information relating to commuting involution graphs associated with the Baby Monster sporadic simple group $\mathbb{B}$.

5.1 Introduction

Over recent years there has been an effort to construct all commuting involution graphs associated with finite simple groups.

Definition 5.1.1 *Let G be a finite group and $X \subseteq G$. The commuting graph of G on X, denoted $\mathcal{C}(G, X)$, is the graph whose vertex set is X with vertices $x, y \in X$ joined by an edge if and only if $xy = yx$ and $x \neq y$.*

If X consists entirely of involutions, then $\mathcal{C}(G, X)$ is called a *commuting involution graph.* We will be interested in the case when X is a conjugacy class of involutions.

In essence, commuting graphs first appeared in Brauer and Fowler's seminal 1955 paper 'On Groups of Even Order' [BF55]. Here, the group was of even order and the vertex set consisted of all non-identity elements of the group. Among their many discoveries are the two following results concerning involutions.

Lemma 5.1.2 *Let G be a finite group of even order with more than one class of involutions. If x and y are two non-conjugate involutions then there exists an involution which commutes with both x and y.*

Theorem 5.1.3 *Let G be a finite group of even order with more than one class of involutions. For any two involutions x and y, we have $d(x, y) \leq 3$.*

The first explicit example of a commuting involution graph is found in the work of Fischer on 3-transposition groups. These are groups generated by a set of involutions S such that S is the union of conjugacy classes and for all $g, h \in S$, the product gh has at most order 3. Fischer's work on 3-transposition groups and the study of their commuting involution graphs led to a classification of all almost simple 3-transposition groups and the discovery of Fi_{22}, Fi_{23} and Fi'_{24}, three of the sporadic simple groups.

More recent work on commuting involution graphs, primarily those associated with finite simple groups, has been driven by Peter Rowley. The ultimate aim is to determine all such graphs for the finite simple groups, amongst other interesting cases, for each of the conjugacy classes of involutions. Much of the work has been completed by Rowley and three of his former students, D. Bundy, C. Bates and S. Perkins. Together they have written four papers [BBPR03b], [BBPR04], [BBPR03a] and [BBPR07] which cover the cases where G is a symmetric group, a special linear group, a finite Coxeter group, or one of (a subset of) the sporadic simple groups.

Theorem 5.1.4 *Let $G \cong Sym(n)$ and X be a conjugacy class of involutions. Then $\mathcal{C}(G, X)$ is either disconnected or it is connected and the diameter at most 4, with equality in exactly three cases.*

Theorem 5.1.5 *Let $G \cong SL_3(p)$ and X be a conjugacy class of involutions. Then $\mathcal{C}(G, X)$ is connected with diameter 3 and the following hold:*

(i) If p is even then

$$|\Delta_1(t)| = 2p^2 - p - 2,$$
$$|\Delta_2(t)| = 2p^2(p-1),$$
$$|\Delta_3(t)| = p^3(p-1).$$

(ii) If p is odd then

$$|\Delta_1(t)| = p(p+1),$$
$$|\Delta_2(t)| = (p^2-1)(p^2+2),$$
$$|\Delta_3(t)| = (p+1)(p-1)^2.$$

Similar results are known for the projective special linear groups. A full list of results and proofs are given in [BBPR04].

Theorem 5.1.6 *Let G be a finite Coxeter group and X be a conjugacy class of involutions. Then*

(i) If $G \cong B_n$ or D_n, then $\mathcal{C}(G, X)$ is disconnected, or connected with diameter at most 5.

(ii) If $G \cong E_6$, then $\mathcal{C}(G, X)$ is connected of diameter at most 5.

(iii) If $G \cong E_7$ or E_8, then $\mathcal{C}(G, X)$ is connected of diameter at most 4.

(iv) If $G \cong F_4, H_3$ or H_4, then $\mathcal{C}(G, X)$ is disconnected, or connected with diameter 2.

(v) If $G \cong I_n$, then $\mathcal{C}(G, X)$ is disconnected.

This is a highly abridged form of the complete result which can be found in [BBPR03a].

Theorem 5.1.7 *Let H be a sporadic simple group and $H \leq G \leq Aut(H)$. Let X be a conjugacy class of involutions of G. Then for (H, X) not equal to $(J_4, 2B)$, $(Fi'_{24}, 2B)$, $(Fi'_{24}, 2D)$, $(\mathbb{B}, 2C)$, $(\mathbb{B}, 2D)$ or $(\mathbb{M}, 2B)$, the graph $\mathcal{C}(G, X)$ has diameter at most 4.*

More recent work by Taylor has shown that for the first three excluded cases above, the diameter of the graph is 3. Additionally, Rowley also determined that the diameter of the graph for the case $(H, X) = (\mathbb{M}, 2B)$ is 3.

Whilst Rowley and his collaborators have calculated many of the graphs associated to the sporadic simple groups, the cases left undetermined were J_4 with the class $2B$, Fi'_{24} with the classes $2B$ and $2D$, the Baby Monster group $\mathbb{B}$ with the classes $2C$ and $2D$, and finally the Monster $\mathbb{M}$ with the class $2B$. However, recent work by Rowley and Taylor calculated the remaining graphs for J_4 and Fi'_{24}, and another of Rowley's students B. Wright partially determined the graph for $\mathbb{B}$ with the class $2C$, see [Wri11].

And so there are only three commuting involution graphs which remain to be completed for the sporadic simple groups, namely $\mathbb{B}$ with the classes $2C$ and $2D$, and $\mathbb{M}$ with the class $2B$.

5.2 Permutation Rank for $(\mathbb{B}, 2C)$

In this section we uncover some additional information relating to the commuting involution graph for $(\mathbb{B}, 2C)$, the construction of which was begun in [Wri11].

In investigating the structure of the commuting involution graph for $(\mathbb{B}, 2C)$, it would be beneficial to know the number of orbits we expect to find within the graph. This section determines the precise number of orbits by obtaining the permutation rank for the group.

Let $G = \mathbb{B}$ and let M be the maximal subgroup $(2^2 \times F_4(2)) : 2$, the centraliser of an involution in $2C$. The character tables for both of these groups are known and are fully described in the GAP character table library. Let 1_M be the trivial character of M and induce up to G. We denote the induced character by 1_M^G. This character 1_M^G is the permutation character of G associated with the action of G on the cosets of M. By taking the inner product of 1_M^G with itself we obtain the permutation rank of G.

$$\text{Permutation Rank of } G = \langle 1_M^G, 1_M^G \rangle$$

The challenges here lies not in calculating the inner product, which is a simple calculation, but in finding the correct permutation character. To do this we employ a method described by Breuer in [Bre99]. First we load the character tables for G and M and calculate the possible fusion maps for M in G.

$> b := CharacterTable(\text{“}B\text{”});$

$> maxes := Maxes(b);$

$> m := CharacterTable(maxes[6]);$

$> fus := PossibleClassFusions(m, b);$

The command *Maxes* constructs a set containing the GAP labels for all stored maximal subgroups of G. Here our required subgroup M is the sixth maximal subgroup listed in the set. The set *fus* of class fusions contains 64 different possible fusion maps. We narrow this down by determining which maps give rise to a potential permutation character.

$> cand := Set(List(fus, x \to Induced(m, b, [TrivialCharacter(m)], x)[1]));$

Fortunately, the set *cand* consisting of all possibilities for the permutation character contains just one entry. Thus this entry must be the permutation character we seek.

1_M^G = (156849238149120000, 1609085288448, 60622848000, 539630848, 468418560, 22239360, 174960, 48435200, 22007808, 661504, 1289472, 483840, 460800, 285952, 19968, 13456, 16640, 15400, 0, 405504, 113760, 24192, 2160, 5472, 5104, 720, 5376, 2160, 240, 532, 120, 24192, 6144, 6144, 2688, 3840, 1152, 2400, 896, 384, 480, 224, 272, 96, 48, 0, 54, 448, 200, 148, 0, 80, 0, 0, 2592, 432, 800, 576, 240, 360, 64, 240, 96, 0, 60, 112, 40, 0, 48, 0, 96, 44, 0, 10, 3, 120, 22, 22, 8, 8, 10, 0, 96, 96, 16, 16, 32, 16, 8, 8, 1, 6, 12, 6, 6, 0, 4, 0, 120, 0, 0, 8, 12, 12, 0, 6, 0, 0, 3, 0, 0, 0, 0, 24, 24, 24, 24, 0, 0, 0, 12, 0, 0, 0, 8, 0, 6, 0, 3, 1, 0, 4, 4, 2, 2, 0, 10, 4, 4, 2, 2, 0, 0, 0, 0, 0, 4, 4, 4, 4, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 4, 4, 0, 2, 0, 3, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)

Taking the inner product of 1_M^G with itself gives us the required permutation rank.

$$\text{Permutation Rank of } G = \langle 1_M^G, 1_M^G \rangle = 163.$$

Thus the G-conjugacy class $2C$ splits into 163 orbits under the action of the subgroup M. Further investigation reveals that this permutation character is only consistent with two of our 64 possible fusion maps.

5.3 Graph for $(\mathbb{B}, 2D)$

For the remainder of this chapter $G = \mathbb{B}$, X denotes the G-conjugacy class $2D$, and t is a fixed element in X. We partition X using the following notation. For a conjugacy class C in G, we define $X_C = \{x \in X | tx \in C\}$. These sets will either be $C_G(t)$-orbits or unions of $C_G(t)$-orbits. In general, if an element $x \in X_C$ lies within a particular disc $\Delta_i(t)$, then usually the entirety of X_C can be found within the same disc with limited exceptions.

We determine the size of the sets X_C using the GAP command

$$> ClassMultiplicationCoefficient(tbl, 5, j, 5)$$

where *tbl* is the character table for G stored in the GAP library and j runs through 1 to 184 corresponding to the conjugacy classes of G. The two number 5's denote $2D$, the fifth conjugacy class of G. In the case of $(\mathbb{B}, 2C)$ this returned 77 non-zero class structure constants meaning that only 77 of the X_C are non-empty. In the case of $(\mathbb{B}, 2D)$, there are 150 non-empty X_C. The non-empty X_C and their sizes are recorded in Table 5.1.

Table 5.1: The Class Structure Constants for $\mathcal{C}(\mathbb{B}, 2D)$.

Class C	Structure Constant $\lvert X_C \rvert$	Factorisation
1A	1	1
2A	100800	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7$
2B	4540455	$3^5 \cdot 5 \cdot 37 \cdot 101$
2C	313528320	$2^{12} \cdot 3^7 \cdot 5 \cdot 7$
2D	728611200	$2^7 \cdot 3^5 \cdot 5^2 \cdot 937$
3A	5426380800	$2^{20} \cdot 3^2 \cdot 5^2 \cdot 23$
3B	375809638400	$2^{31} \cdot 5^2 \cdot 7$
4A	3902653440	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 37 \cdot 109$
4B	10317300480	$2^8 \cdot 3^4 \cdot 5 \cdot 191 \cdot 521$
4C	172788940800	$2^{17} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 31$
4D	236191334400	$2^{14} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 113$
4E	360706867200	$2^{17} \cdot 3^6 \cdot 5^2 \cdot 151$
4F	707412787200	$2^{15} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 1523$
4G	1404622356480	$2^{14} \cdot 3^3 \cdot 5 \cdot 7 \cdot 257 \cdot 353$
4H	6866966937600	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$
4J	5707608883200	$2^{27} \cdot 3^5 \cdot 5^2 \cdot 7$
5A	3950027735040	$2^{28} \cdot 3^3 \cdot 5 \cdot 109$
5B	29222957481984	$2^{34} \cdot 3^5 \cdot 7$
6A	198180864000	$2^{23} \cdot 3^3 \cdot 5^3 \cdot 7$
6B	1268357529600	$2^{28} \cdot 3^3 \cdot 5^2 \cdot 7$
6C	7408000696320	$2^{22} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 89$
6D	13529146982400	$2^{33} \cdot 3^2 \cdot 5^2 \cdot 7$
6E	19025362944000	$2^{28} \cdot 3^4 \cdot 5^3 \cdot 7$
6F	20547391979520	$2^{28} \cdot 3^7 \cdot 5 \cdot 7$
6G	20293720473600	$2^{32} \cdot 3^3 \cdot 5^2 \cdot 7$
6H	32402571264000	$2^{22} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 109$
6I	30440580710400	$2^{31} \cdot 3^4 \cdot 5^2 \cdot 7$
6J	121762322841600	$2^{33} \cdot 3^4 \cdot 5^2 \cdot 7$
6K	81174881894400	$2^{34} \cdot 3^3 \cdot 5^2 \cdot 7$
7A	129879811031040	$2^{37} \cdot 3^3 \cdot 5 \cdot 7$
8A	3805072588800	$2^{28} \cdot 3^4 \cdot 5^2 \cdot 7$
8B	22988980224000	$2^{25} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 29$
8C	22988980224000	$2^{25} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 29$
8D	34245653299200	$2^{28} \cdot 3^6 \cdot 5^2 \cdot 7$
8E	45993814917120	$2^{24} \cdot 3^4 \cdot 5 \cdot 7 \cdot 967$
8F	91321742131200	$2^{31} \cdot 3^5 \cdot 5^2 \cdot 7$

Class C	Structure Constant $\lvert X_C \rvert$	Factorisation
8G	30440580710400	$2^{31} \cdot 3^4 \cdot 5^2 \cdot 7$
8H	91321742131200	$2^{31} \cdot 3^5 \cdot 5^2 \cdot 7$
8I	91321742131200	$2^{31} \cdot 3^5 \cdot 5^2 \cdot 7$
8J	239719573094400	$2^{28} \cdot 3^6 \cdot 5^2 \cdot 7^2$
8K	365286968524800	$2^{33} \cdot 3^5 \cdot 5^2 \cdot 7$
8M	730573937049600	$2^{34} \cdot 3^5 \cdot 5^2 \cdot 7$
9A	649399055155200	$2^{37} \cdot 3^3 \cdot 5^2 \cdot 7$
9B	974098582732800	$2^{36} \cdot 3^4 \cdot 5^2 \cdot 7$
10A	91321742131200	$2^{31} \cdot 3^5 \cdot 5^2 \cdot 7$
10B	357676823347200	$2^{29} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 47$
10C	308464551198720	$2^{34} \cdot 3^3 \cdot 5 \cdot 7 \cdot 19$
10D	292229574819840	$2^{35} \cdot 3^5 \cdot 5 \cdot 7$
10E	456608710656000	$2^{31} \cdot 3^5 \cdot 5^3 \cdot 7$
10F	730573937049600	$2^{34} \cdot 3^5 \cdot 5^2 \cdot 7$
11A	1558557732372480	$2^{39} \cdot 3^4 \cdot 5 \cdot 7$
12A	45660871065600	$2^{30} \cdot 3^5 \cdot 5^2 \cdot 7$
12B	81174881894400	$2^{34} \cdot 3^3 \cdot 5^2 \cdot 7$
12C	124806380912640	$2^{30} \cdot 3^4 \cdot 5 \cdot 7 \cdot 41$
12D	152202903552000	$2^{31} \cdot 3^4 \cdot 5^3 \cdot 7$
12E	190253629440000	$2^{29} \cdot 3^4 \cdot 5^4 \cdot 7$
12F	243524645683200	$2^{34} \cdot 3^4 \cdot 5^2 \cdot 7$
12G	273965226393600	$2^{31} \cdot 3^6 \cdot 5^2 \cdot 7$
12H	464218855833600	$2^{29} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 61$
12I	426168129945600	$2^{32} \cdot 3^4 \cdot 5^2 \cdot 7^2$
12J	730573937049600	$2^{34} \cdot 3^5 \cdot 5^2 \cdot 7$
12K	487049291366400	$2^{35} \cdot 3^4 \cdot 5^2 \cdot 7$
12L	905607276134400	$2^{29} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 17$
12M	487049291366400	$2^{35} \cdot 3^4 \cdot 5^2 \cdot 7$
12N	974098582732800	$2^{36} \cdot 3^4 \cdot 5^2 \cdot 7$
12O	730573937049600	$2^{34} \cdot 3^5 \cdot 5^2 \cdot 7$
12P	730573937049600	$2^{34} \cdot 3^5 \cdot 5^2 \cdot 7$
12Q	730573937049600	$2^{34} \cdot 3^5 \cdot 5^2 \cdot 7$
12R	730573937049600	$2^{34} \cdot 3^5 \cdot 5^2 \cdot 7$
12S	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
13A	3409345039564800	$2^{35} \cdot 3^4 \cdot 5^2 \cdot 7^2$
14A	81174881894400	$2^{34} \cdot 3^3 \cdot 5^2 \cdot 7$
14B	974098582732800	$2^{36} \cdot 3^4 \cdot 5^2 \cdot 7$
14C	974098582732800	$2^{36} \cdot 3^4 \cdot 5^2 \cdot 7$

Class C	Structure Constant $\|X_C\|$	Factorisation
14D	1461147874099200	$2^{35} \cdot 3^5 \cdot 5^2 \cdot 7$
14E	730573937049600	$2^{34} \cdot 3^5 \cdot 5^2 \cdot 7$
15A	1753377448919040	$2^{36} \cdot 3^6 \cdot 5 \cdot 7$
15B	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
16A	243524645683200	$2^{34} \cdot 3^4 \cdot 5^2 \cdot 7$
16B	243524645683200	$2^{34} \cdot 3^4 \cdot 5^2 \cdot 7$
16C	1461147874099200	$2^{35} \cdot 3^5 \cdot 5^2 \cdot 7$
16D	1461147874099200	$2^{35} \cdot 3^5 \cdot 5^2 \cdot 7$
16E	1461147874099200	$2^{35} \cdot 3^5 \cdot 5^2 \cdot 7$
16H	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
17A	8766887244595200	$2^{36} \cdot 3^6 \cdot 5^2 \cdot 7$
18A	974098582732800	$2^{36} \cdot 3^4 \cdot 5^2 \cdot 7$
18B	3896394330931200	$2^{38} \cdot 3^4 \cdot 5^2 \cdot 7$
18C	3896394330931200	$2^{38} \cdot 3^4 \cdot 5^2 \cdot 7$
18D	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
18E	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
18F	3896394330931200	$2^{38} \cdot 3^4 \cdot 5^2 \cdot 7$
19A	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
20A	426168129945600	$2^{32} \cdot 3^4 \cdot 5^2 \cdot 7^2$
20B	584459149639680	$2^{36} \cdot 3^5 \cdot 5 \cdot 7$
20C	730573937049600	$2^{34} \cdot 3^5 \cdot 5^2 \cdot 7$
20D	2009078326886400	$2^{32} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
20E	2435246456832000	$2^{35} \cdot 3^4 \cdot 5^3 \cdot 7$
20F	2191721811148800	$2^{34} \cdot 3^6 \cdot 5^2 \cdot 7$
20G	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
20I	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
20J	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
21A	6818690079129600	$2^{36} \cdot 3^4 \cdot 5^2 \cdot 7^2$
22B	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
24A	1948197165465600	$2^{37} \cdot 3^4 \cdot 5^2 \cdot 7$
24B	1948197165465600	$2^{37} \cdot 3^4 \cdot 5^2 \cdot 7$
24C	1948197165465600	$2^{37} \cdot 3^4 \cdot 5^2 \cdot 7$
24D	1948197165465600	$2^{37} \cdot 3^4 \cdot 5^2 \cdot 7$
24E	1461147874099200	$2^{35} \cdot 3^5 \cdot 5^2 \cdot 7$
24F	1948197165465600	$2^{37} \cdot 3^4 \cdot 5^2 \cdot 7$
24G	1461147874099200	$2^{35} \cdot 3^5 \cdot 5^2 \cdot 7$
24I	1948197165465600	$2^{37} \cdot 3^4 \cdot 5^2 \cdot 7$
24J	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$

Class C	Structure Constant $\lvert X_C\rvert$	Factorisation
24K	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
24L	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
25A	11689182992793600	$2^{38} \cdot 3^5 \cdot 5^2 \cdot 7$
26A	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
26B	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
27A	11689182992793600	$2^{38} \cdot 3^5 \cdot 5^2 \cdot 7$
28A	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
28B	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
28C	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
28D	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
28E	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
30A	974098582732800	$2^{36} \cdot 3^4 \cdot 5^2 \cdot 7$
30B	3896394330931200	$2^{38} \cdot 3^4 \cdot 5^2 \cdot 7$
30C	3896394330931200	$2^{38} \cdot 3^4 \cdot 5^2 \cdot 7$
30D	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
30E	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
30F	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
33A	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
34A	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
34B	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
34C	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
35A	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
36A	1948197165465600	$2^{37} \cdot 3^4 \cdot 5^2 \cdot 7$
36B	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
36C	11689182992793600	$2^{38} \cdot 3^5 \cdot 5^2 \cdot 7$
39A	11689182992793600	$2^{38} \cdot 3^5 \cdot 5^2 \cdot 7$
40A	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
40B	2922295748198400	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7$
40C	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
42A	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
42B	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
42C	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
55A	11689182992793600	$2^{38} \cdot 3^5 \cdot 5^2 \cdot 7$
56A	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
56B	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
60A	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
60B	5844591496396800	$2^{37} \cdot 3^5 \cdot 5^2 \cdot 7$
60C	11689182992793600	$2^{38} \cdot 3^5 \cdot 5^2 \cdot 7$

The following result is proven in [BBPR07].

Lemma 5.3.1 *Let $x \in X$ and $z = tx$. Suppose z has order m. Then the following hold:*

(i) $x \in \Delta_1(t)$ *if and only if* $m = 2$;

(ii) If $m \geq 4$ *is even and* $z^{m/2} \in X$, *then* $x \in \Delta_2(t)$;

(iii) If $C_{C_G(z)}(x) \cap X = \emptyset$ *then* $d(t, x) \geq 3$. *In particular, if* $C_{C_G(z)}(x)$ *has odd order, then* $d(t, x) \geq 3$;

(iv) Suppose m *is odd and assume there does not exist any elements* $g \in G$ *of order* $2m$ *such that* $g^2 = z$ *and* $g^m \in X$, *then* $d(t, x) \geq 3$.

Using Lemma 5.3.1 part (i) immediately gives us that $\Delta_1(t) = X_{2A} \cup X_{2B} \cup X_{2C} \cup X_{2D}$, and thus $|\Delta_1(t)| = 1,046,780,775$. Using the second part of the lemma we can quickly determine some of the constituents of the second disk. For example, consider the class $6E$. The $\mathbb{ATLAS}$ tells us that the cube of an element in $6E$ is contained in $2C$. Hence, by the lemma, we conclude that $X_{6E} \subseteq \Delta_2(t)$.

The fourth part of the lemma allows us to quickly show some sets cannot lie in the first two disks. For example, consider the class $11A$. There are two classes of elements of order 22 in $\mathbb{B}$, namely $22A$ and $22B$. They both power to $11A$, and their eleventh powers lie in $2A$ and $2B$ respectively. Since neither of these power to $2C$, we conclude that for $x \in X_{11A}$, $d(t, x) \geq 3$.

Applying Lemma 5.3.1 in this fashion immediately yields Table 5.2.

$\Delta_1(t)$	$\Delta_2(t)$	$d(t, x) \geq 3$
$2A$, $2B$, $2C$, $2D$	$4E$, $4H$, $4J$, $6E$, $6H$, $6I$, $6J$, $8G$, $8J$, $8K$, $8M$, $10E$, $10F$, $12F$, $12J$, $12N$, $12P$, $12Q$, $12R$, $12S$, $14E$, $16H$, $18D$, $18E$, $20G$, $20J$, $24I$, $24J$, $24L$, $28E$, $30E$, $30F$, $36C$, $60C$	$11A$, $13A$, $17A$, $19A$, $21A$, $33A$, $35A$, $39A$, $55A$

Table 5.2: The Location of some X_C in the Commuting Involution Graph $\mathcal{C}(\mathbb{B}, 2D)$

Chapter 6

The Semisimple Elements of $E_8(2)$

In this chapter we study the conjugacy classes of semisimple elements in the finite simple group $E_8(2)$, their associated centralisers, fixed space dimensions and power maps. This study emerged from work on a collaborative effort to classify the maximal subgroups of $E_8(2)$, a problem which has been open for several decades.

The work described in this chapter is a collaborative effort with Ali Aubad, John Ballantyne, Alexander McGaw, Peter Neuhaus, Peter Rowley and David Ward [ABM+16].

6.1 Introduction

Since before the classification was complete, many authors have expended considerable time examining the properties of finite simple groups. Such properties include finding representations of the groups, studying subgroup structure and conjugacy classes, and determining complex character tables. Particular attention has been given to the sporadic groups and other small simple groups, with a huge amount of such information amassed in the $\mathbb{ATLAS}$.

Finite groups of Lie type have also received intense scrutiny. They were among the first groups to be considered in mathematics with $PSL_2(q)$ constructed by Galois in the 1830s (for a full list of the finite simple groups of Lie type, we refer the reader to Theorem 2.2.1). Approaches to the study of these groups have varied from investigations using the theory of linear algebraic groups ([LS90],[LS03]), to more computational studies ([KW90],[PS06]). One such study by Kleidman and Wilson [KW90] determines the maximal subgroups of $E_6(2)$ and $Aut(E_6(2))$.

Theorem 6.1.1 *Let H be a maximal subgroup of $E_6(2)$. Then H has one of the following structures:*

(i) $2^{16} : \Omega_{10}^{+}(2)$

(ii) $2^{5+20} : (L_2(2) \times L_5(2))$

(iii) $2_{+}^{1+20} : L_6(2)$

(iv) $[2^{29}] : (L_2(2) \times L_3(2) \times L_3(2))$

(v) $F_4(2)$

(vi) $L_2(2) \times L_6(2)$

(vii) $3.(U_3(2) \times L_3(4)).Sym(3)$

(viii) $L_3(8) : 3$

(ix) $(L_3(2) \times L_3(2) \times L_3(2)) : Sym(3)$

(x) $L_3(2) \times G_2(2)$

(xi) $7^3 : 3^{1+2} : 2Alt(4)$

(xii) $(7 \times {}^3D_4(2)) : 3$

(xiii) $G_2(2)$

The maximal subgroups of $Aut(E_6(2))$ and related $E_6(2)$ fusion data may be found in [KW90]. Of particular relevance for our efforts in this chapter is the work by Ballantyne, Bates and Rowley [BBR15] on the maximal subgroups of $E_7(2)$. In their paper the authors also determine information relating to the conjugacy classes of semisimple elements. Here we reproduce their main result: the determination of the maximal subgroups of $E_7(2)$.

Theorem 6.1.2 *Let H be a maximal subgroup of $E_7(2)$. Then H has one of the following structures:*

(i) $2^{1+32} : \Omega_{12}^{+}(2)$

(ii) $[2^{53}] : (Sym(3) \times L_3(2) \times L_4(2))$

(iii) $[2^{42}] : (Sym(3) \times \Omega_{10}^{+}(2))$

(iv) $[2^{42}] : L_7(2)$

(v) $E_6(2) : 2$

(vi) $U_8(2) : 2$

(vii) $3^{\cdot}(U_3(2) \times U_6(2)) : Sym(3)$

(viii) $(L_2(8) \times {}^3D_4(2)) : 3$

(ix) $L_2(128) : 7$

(x) $3^7 : (2 \times Sp_6(2))$

(xi) $[2^{47}] : (Sym(3) \times L_6(2))$

(xii) $[2^{50}] : (L_3(2) \times L_5(2))$

(xiii) $2^{27} : E_6(2)$

(xiv) $Sym(3) \times \Omega_{12}^{+}(2)$

(xv) $3^{\cdot 2}E_6(2) : Sym(3)$

(xvi) $L_8(2) : 2$

(xvii) $(L_3(2) \times L_6(2)) : 2$

(xviii) $((Sym(3))^3 \times \Omega_8^{+}(2)) : Sym(3)$

(xix) $U_3(3) : 2 \times Sp_6(2)$

Note that maximal subgroups of all shapes above exist in $E_7(2)$. In addition to this information, the authors also describe the centralisers, powers ups, fixed space dimensions and Brauer character values for semisimple elements of $E_7(2)$.

A key resource for our efforts in this chapter will be the work of Frank Lübeck. In [Lüb], Lübeck gives a parametrisation of the conjugacy classes of $E_8(2)$. This is obtained by considering the group as a group of fixed points under a Frobenius morphism inside a connected reductive algebraic group G. Each element $g \in G$ exhibits a Jordan decomposition $g = su$ into semisimple and unipotent constituents.

In [Lüb] Lübeck gives, for each of the 1156 conjugacy classes of $E_8(2)$, the order of the centraliser $C_G(g)$, information relating to the centraliser $C_G(s)$ of the semisimple part s under the restricted Frobenius morphism, and the number of rational points in the radical of the centraliser of s. Throughout this chapter, we will make frequent reference to the information listed in the expansive table of [Lüb]. In the interest of brevity, we will refer to the information relating to the conjugacy class listed in row n of the table as Lübeck number n.

We now give the main result of this chapter. Throughout V denotes the natural 248-dimensional G-module.

Theorem 6.1.3 *The semisimple conjugacy classes of $E_8(2)$, together with the structure of their centralisers, dimensions of their fixed spaces on V and power maps, are given in Table 6.1.*

Conjugacy Class	Lübeck Number	$C_G(x)$	$\|C_G(x)\|$	$\dim(C_V(x))$	Powers
$1A$	1	$E_8(2)$	$\|E_8(2)\|$	248	-
$3A$	294	$3 \times E_7(2)$	$2^{63} \cdot 3^{12} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43 \cdot 73 \cdot 127$	134	-
$3B$	376	$3 \times \Omega_{14}^-(2)$	$2^{42} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 43$	92	-
$3C$	147	$3.(^2E_6(2) \times U_3(2)).3$	$2^{39} \cdot 3^{13} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	86	-
$3D$	258	$3 \times U_9(2)$	$2^{36} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 43$	80	-
$5A$	480	$5 \times \Omega_{12}^-(2)$	$2^{30} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31$	68	-
$5B$	247	$SU_5(4)$	$2^{20} \cdot 3^2 \cdot 5^5 \cdot 13 \cdot 17 \cdot 41$	48	-
$7A$	441	$7 \times E_6(2)$	$2^{36} \cdot 3^6 \cdot 5^2 \cdot 7^4 \cdot 13 \cdot 17 \cdot 31 \cdot 73$	80	-
$7B$	516	$7 \times L_3(2) \times {}^3D_4(2)$	$2^{15} \cdot 3^5 \cdot 7^4 \cdot 13$	38	-
$9A$	560	$9 \times \Omega_{10}^-(2)$	$2^{20} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	48	3C
$9B$	656	$9 \times Sym(3) \times {}^3D_4(2)$	$2^{13} \cdot 3^7 \cdot 7^2 \cdot 13$	34	3C
$9C$	580	$9 \times Sym(3) \times U_5(2)$	$2^{11} \cdot 3^8 \cdot 5 \cdot 11$	30	3C
$9D$	366	$9 \times Sym(3) \times U_3(8)$	$2^{10} \cdot 3^7 \cdot 7 \cdot 19$	28	3C
$11A$	679	$11 \times U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11^2$	28	-
$13A$	712	$13 \times {}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13^2$	32	-
$13B$	709	$13 \times U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13^2$	20	-
$15A$	540	$15 \times \Omega_{10}^+(2)$	$2^{20} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 17 \cdot 31$	48	3B,5A
$15B$	636	$5 \times 3^2 : 2 \times \Omega_8^-(2)$	$2^{13} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 17$	34	3A,5A
$15C$	686	$15 \times U_5(2)$	$2^{10} \cdot 3^6 \cdot 5^2 \cdot 11$	28	3D,5A
$15D$	621	$5 \times GU_3(2) \times L_4(2)$	$2^9 \cdot 3^6 \cdot 5^2 \cdot 7$	26	3C,5A
$15E$	600	$15 \times L_2(4) \times U_4(2)$	$2^8 \cdot 3^6 \cdot 5^3$	24	3B,5A
$15F$	706	$15 \times U_3(4)$	$2^6 \cdot 3^2 \cdot 5^3 \cdot 13$	20	3B,5B
$15G$	695	$15 \times L_2(16)$	$2^4 \cdot 3^2 \cdot 5^2 \cdot 17$	16	3D,5B
$17AB$	738	$17 \times \Omega_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17^2$	32	-
$17CD$	693	$17 \times L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17^2$	16	-
$19A$	823	$19 \times 3^{\cdot}PGU_3(2)$	$2^3 \cdot 3^4 \cdot 19$	14	-

Table 6.1: The Conjugacy Classes of Semisimple Elements in $E_8(2)$

Conjugacy Class	Lübeck Number	$C_G(x)$	$\lvert C_G(x)\rvert$	$\dim(C_V(x))$	Powers
$21A$	610	$21 \times L_6(2)$	$2^{15} \cdot 3^5 \cdot 5 \cdot 7^3 \cdot 31$	38	3A,7A
$21B$	720	$21 \times {}^3D_4(2)$	$2^{12} \cdot 3^5 \cdot 7^3 \cdot 13$	32	3A,7B
$21C$	728	$21 \times \Omega_8^-(2)$	$2^{12} \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 17$	32	3B,7A
$21D$	469	$7 \times 3.(3^2 : Q_8 \times L_3(4)) : 3$	$2^9 \cdot 3^6 \cdot 5 \cdot 7^2$	26	3C,7A
$21E$	594	$21 \times L_3(2) \times L_2(8)$	$2^6 \cdot 3^4 \cdot 7^3$	20	3A,7B
$21F$	697	$7 \times L_3(2) \times 3_+^{1+2} : 2Alt(4)$	$2^6 \cdot 3^5 \cdot 7^2$	20	3C,7B
$21G$	760	$21 \times 3 \times L_2(8)$	$2^3 \cdot 3^4 \cdot 7^2$	14	3B,7B
$21H$	826	$21 \times 3_+^{1+2} : 2Alt(4)$	$2^3 \cdot 3^5 \cdot 7$	14	3D,7B
$31ABC$	672	$31 \times L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31^2$	28	-
$31D$	857	31^2	31^2	8	-
$33AB$	768	$33 \times U_4(2)$	$2^6 \cdot 3^5 \cdot 5 \cdot 11$	20	3D,11A
$33CD$	748	$11 \times Sym(3) \times 3_+^{1+2} : 2Alt(4)$	$2^4 \cdot 3^5 \cdot 11$	16	3C,11A
$33E$	811	$33 \times 3_+^{1+2} : 2Alt(4)$	$2^3 \cdot 3^5 \cdot 11$	14	3A,11A
$33F$	790	$33 \times 3 \times Sym(3)^2$	$2^2 \cdot 3^4 \cdot 11$	12	3B,11A
$35A$	778	$35 \times U_4(2)$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7$	20	5A,7A
$39A$	762	$13 \times Sym(3) \times L_2(8)$	$2^4 \cdot 3^4 \cdot 7 \cdot 13$	14	3A,13A
$39B$	820	$13 \times 3_+^{1+2} : 2Alt(4)$	$2^3 \cdot 3^4 \cdot 13$	14	3C,13A
$39C$	872	195	$3 \cdot 5 \cdot 13$	8	3B,13B
$41AB$	864	205	$5 \cdot 41$	8	-
$43ABC$	837	$129 \times Sym(3)$	$2 \cdot 3^2 \cdot 43$	10	-
$45A$	773	$45 \times L_4(2)$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7$	20	3C,5A,9A,15D
$45B$	798	$45 \times 3 \times Alt(5)$	$2^2 \cdot 3^4 \cdot 5^2$	12	3C,5A,9A,15D
$45C$	853	$45 \times 3 \times Sym(3)$	$2 \cdot 3^4 \cdot 5$	10	3C,5A,9C,15D
$51AB$	783	$51 \times L_4(2)$	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 17$	20	3B,17AB
$51CD$	764	$51 \times Sym(3) \times Alt(5)$	$2^3 \cdot 3^3 \cdot 5 \cdot 17$	14	3A,17AB
$51EF$	832	$17 \times GU_3(2)$	$2^3 \cdot 3^4 \cdot 17$	14	3C,17AB

Conjugacy Class	Lübeck Number	$C_G(x)$	$\lvert C_G(x)\rvert$	$\dim(C_V(x))$	Powers
$51GH$	870	255	$3 \cdot 5 \cdot 17$	8	3D,17CD
$55A$	877	165	$3 \cdot 5 \cdot 11$	8	5A,11A
$57AB$	823	$19 \times 3^{\cdot} PGU_3(2)$	$2^3 \cdot 3^4 \cdot 19$	14	3C,19A
$57C$	861	$3 \times 19 \times 9$	$3^3 \cdot 19$	8	3A,19A
$57DE$	863	57×3	$3^2 \cdot 19$	8	3D,19A
$63ABC$	754	$63 \times Sym(3) \times L_3(2)$	$2^4 \cdot 3^4 \cdot 7^2$	16	3C,7B,9B,21F
$63D$	802	$63 \times Alt(5)$	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	12	3C,7A,9A,21D
$63E$	843	$63 \times 7 \times Sym(3)$	$2 \cdot 3^3 \cdot 7^2$	10	3C,7A,9B,21D
$63FGH$	849	$63 \times 3 \times Sym(3)$	$2 \cdot 3^4 \cdot 7$	10	3C,7B,9D,21F
$65ABCD$	800	$65 \times Alt(5)$	$2^2 \cdot 3 \cdot 5^2 \cdot 13$	12	5B,13B
$65EF$	858	13×5^2	$5^2 \cdot 13$	8	5A,13B
$73ABCD$	814	$73 \times L_3(2)$	$2^3 \cdot 3 \cdot 7 \cdot 73$	14	-
$85AB$	804	$85 \times Sym(3)^2$	$2^2 \cdot 3^2 \cdot 5 \cdot 17$	12	5A,17AB
$85CDEF$	870	255	$3 \cdot 5 \cdot 17$	8	5B,17CD
$91ABC$	817	$91 \times L_3(2)$	$2^3 \cdot 3 \cdot 7^2 \cdot 13$	14	7B,13A
$91D$	865	91×7	$7^2 \cdot 13$	8	7A,13A
$93ABC$	808	$93 \times L_3(2)$	$2^3 \cdot 3^2 \cdot 7 \cdot 31$	14	3A,31ABC
$93DEF$	788	$93 \times Alt(5)$	$2^2 \cdot 3^2 \cdot 5 \cdot 31$	12	3B,31ABC
$99AB$	841	$99 \times Sym(3)$	$2 \cdot 3^3 \cdot 11$	10	3C,9C,11A,33CD
$99CD$	867	99×3	$3^3 \cdot 11$	8	3C,9A,11A,33CD
$105AB$	829	$35 \times GU_3(2)$	$2^3 \cdot 3^4 \cdot 5 \cdot 7$	14	3C,5A,7A,15D,21D,35A
$105C$	794	$105 \times Sym(3)^2$	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	12	3B,5A,7A,15A,21C,35A
$105D$	851	$105 \times 3 \times Sym(3)$	$2 \cdot 3^3 \cdot 5 \cdot 7$	10	3A,5A,7A,15B,21A,35A
$117ABC$	845	$117 \times Sym(3)$	$2 \cdot 3^3 \cdot 13$	10	3C,9B,13A,39B
$119AB$	878	357	$3 \cdot 7 \cdot 17$	8	7A,17AB
$127ABCDEFGHI$	835	$127 \times Sym(3)$	$2 \cdot 3 \cdot 127$	10	-

Conjugacy Class	Lübeck Number	$C_G(x)$	$\|C_G(x)\|$	$\dim(C_V(x))$	Powers
$129ABCDEF$	837	$129 \times Sym(3)$	$2 \cdot 3^2 \cdot 43$	10	3D,43ABC
$129GHI$	859	129×3	$3^2 \cdot 43$	8	3A,43ABC
$129JKLMNO$	859	129×3	$3^2 \cdot 43$	8	3B,43ABC
$151ABCDE$	868	151	151	8	-
$153AB$	879	153	$3^2 \cdot 17$	8	3C,9A,17AB,51EF
$155ABC$	876	465	$3 \cdot 5 \cdot 31$	8	5A,31ABC
$165AB$	877	165	$3 \cdot 5 \cdot 11$	8	3D,5A,11A,15C,33AB,55A
$171ABCDEF$	847	$171 \times Sym(3)$	$2 \cdot 3^3 \cdot 19$	10	3C,9D,19A,57AB
$195ABCD$	872	195	$3 \cdot 5 \cdot 13$	8	3B,5B,13B,15F,39C,65ABCD
$205ABCDEFGH$	864	205	$5 \cdot 41$	8	5B,41AB
$217ABCDEF$	839	$217 \times Sym(3)$	$2 \cdot 3 \cdot 7 \cdot 31$	10	7A,31ABC
$219ABCD$	875	219	$3 \cdot 73$	8	3A,73ABCD
$241ABCDEFGHIJ$	866	241	241	8	-
$255ABCD$	855	$255 \times Sym(3)$	$2 \cdot 3^2 \cdot 5 \cdot 17$	10	3A,5A,15B,17AB,51CD,85AB
$255EF$	860	255×3	$3^2 \cdot 5 \cdot 17$	8	3B,5A,15A,17AB,51AB,85AB
$255GHIJKLMN$	870	255	$3 \cdot 5 \cdot 17$	8	3D,5B,15G,17CD,51GH,85CDEF
$273ABC$	873	273	$3 \cdot 7 \cdot 13$	8	3A,7B,13A,21B,39A,91ABC
$315AB$	871	315	$3^2 \cdot 5 \cdot 7$	8	3C,5A,7A,9A,15D,21D,35A,45A,63D,105AB
$331ABCDEFGHIJK$	869	331	331	8	-
$357ABCD$	878	357	$3 \cdot 7 \cdot 17$	8	3B,7A,17AB,21C,51AB,119AB
$381ABCDEFGHI$	874	381	$3 \cdot 127$	8	3A,127ABCDEFGHI
$465ABCDEF$	876	465	$3 \cdot 5 \cdot 31$	8	3B,5A,15A,31ABC,93DEF,155ABC
$511ABCDEFGH$	862	511	$7 \cdot 73$	8	7A,73ABCD
$651ABCDEF$	880	651	$3 \cdot 7 \cdot 31$	8	3A,7A,21A,31ABC,93ABC,217ABCDEF

6.2 Preliminary Results

For the rest of the chapter we set $G = E_8(2)$, and recall that

$$|G| = 2^{120} \cdot 3^{13} \cdot 5^5 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 31^2 \cdot 41 \cdot 43 \cdot 73 \cdot 127 \cdot 151 \cdot 241 \cdot 331.$$

When working with G computationally, we use the 248-dimensional representation over $GF(2)$. However, to make this representation more efficient to work with, we realise it as a subgroup of $GL(248, 2)$. To do this, we first have MAGMA produce G as an object in the "GrpLie" category and construct its adjoint representation.

```
> H := GroupOfLieType("E8", GF(2));
> f := AdjointRepresentation(H);
> Q := Codomain(f);
```

Here, Q will be the group $GL(248, 2)$. Next we construct generators for H by taking elements which correspond to the fundamental roots.

```
> Hgens := [ ];
> for i := 1 to 8 do;
> Append(~ Hgens, elt < H| < i, 1 >>);
> Append(~ Hgens, elt < H| < 120 + i, 1 >>);
> end for;
```

Our generators are currently considered elements of H. We now map them into the matrix group Q using the adjoint representation f, and finally generate $E_8(2)$.

```
> Ggens := [ ];
> for h in Hgens do;
> Append(~ Ggens, f(h));
> end for;
> G := sub < Q|Ggens >;
```

We use this copy of G, sitting inside $GL(248, 2)$, for all calculations. However, we caution that even with this representation of the group, the sheer enormity of $|G|$ makes direct calculations inadvisable and indeed will almost certainly cause MAGMA to crash. Additionally, in our investigations we construct subgroups of G as subgroups of $GL(248, 2)$.

Throughout this chapter we will require information relating to various subgroups of G. This information is summarised in the next two results.

Theorem 6.2.1 *The following groups are isomorphic to a subgroup of G:*

(i) $\Omega_{16}^{+}(2)$;

(ii) $Sym(3) \times E_7(2)$;

(iii) $L_3(2) \times E_6(2)$;

(iv) $SU_5(4)$;

(v) $PGU_5(4)$;

(vi) ${}^3D_4(2) \times {}^3D_4(2)$;

(vii) $U_5(2) \times U_5(2)$;

(viii) $L_5(2) \times L_5(2)$;

(ix) $U_3(4) \times U_3(4)$;

(x) $U_3(16)$.

Proof. Full details may be gained from the 𝔸𝕋𝕃𝔸𝕊 [CCN+09] or from Liebeck and Seitz [LS12]. □

Lemma 6.2.2 *Let P be a Sylow 3-subgroup of G. Then P has exponent 9.*

Proof. See [ABM+16], where P is constructed explicitly. □

The first step in our attempt to explore the conjugacy classes of semisimple elements is to determine the number of G-conjugacy classes for elements of each given order. By Steinberg [Ste68], we know that G has $2^8 = 256$ conjugacy classes of semisimple elements.

Lemma 6.2.3 *Lower bounds for the number of semisimple conjugacy classes of G are listed in Table 6.2.*

The lower bounds on the number of semisimple conjugacy classes were obtained using a procedure we shall describe shortly. Before that, we require two MAGMA functions. The first computes the Brauer character of a given element in $GL(n, k)$, where k is a finite field of characteristic 2.

$> function\ BrauerCharacter(g);$
$> p := CharacteristicPolynomial(g);$
$> R, S < w > := RootsInSplittingField(p);$
$> k := \#R;\ o := Order(w);$

We now need to map elements of the finite field k to suitable roots of unity in the complex field. Below we deal with the case that S has prime order separately to avoid the situation

Element Order, o	Lower Bound, ℓ	Powering Parameter, p
1	1	1
3	4	1
5	2	1
7	2	1
9	4	1
11	1	1
13	2	1
15	7	1
17	4	1
21	8	1
31	4	1
33	6	1
35	1	1
39	3	1
43	3	1
45	3	1
51	8	1
55	1	1
63	8	1
65	6	1
73	4	1
85	6	1
91	4	1
93	6	1
99	4	3
105	4	1
117	3	1
119	2	7
127	9	15
129	15	1
151	5	1
153	2	9
155	3	5
165	2	5
195	4	1
217	6	1
219	4	3
255	14	1
273	3	1
315	2	1
381	9	1
511	8	1

Table 6.2: Lower Bounds on the Number of Conjugacy Classes in $E_8(2)$

where $w = 1$ and so does not generate the multiplicative group of S.

```
> if IsPrime(#S) then
> Q<x> := CyclotomicField(#S − 1);
> f := pmap< S−>Q | [2^i−>x^i : i in [1..#S − 1]] >;
> else
> Q<x> := CyclotomicField(o);
> f := pmap< S−>Q | [w^i−>x^i : i in [1..o]] >;
> end if;
```

Finally we sum the relative powers of unity and return the Brauer Character.

```
> c := 0;
> for j := 1 to k do
> v := R[j, 2] ∗ f(R[j, 1]);
> c := c + v;
> end for;
> return [c, #S];
> end function;
```

The second function we require generates an element of order ℓ in a given finite group G, if such an element exists.

```
> function Element(G, ℓ);
> x := Id(G);
> repeat
> r := Random(G);  o := Order(r);
> if o mod ℓ eq 0 then
> k := IntegerRing()!(o/ℓ);
> x := r^k;
> end if;
> until Order(x) eq ℓ;
> return x;
> end function;
```

We're now in a position to give the function used to verify Lemma 6.2.3. This function gives sets of tuples detailing the Brauer character, field extension of the character, and the dimension of the fixed space for elements of a given order ℓ in G. It runs until a known lower bound b is attained.

$> function\ LowerBound(G, \ell, b);$
$> SET := \{\ \};$
$> repeat$
$> g := Element(G, \ell);$
$> x := BrauerCharacter(g);$
$> y := [x[1], x[2], Dimension(Eigenspace(g, 1))];$
$> if\ y\ in\ SET\ eq\ false\ then\ SET := SET\ join\ \{Y\};$
$> end\ if;$
$> until\ \#SET\ eq\ b;$
$> return\ SET;$
$> end\ function;$

Occasionally we need to power up elements and then take Brauer Characters to distinguish between classes. Our final function in this section deals with this situation using the additional parameter p, representing the power of the element g which needs to be taken. The required value of p for a given g is given in Table 6.2. To construct this function we simply replace the the following two lines in the above function:

$> function\ LowerBound(G, \ell, b);$
$> x := BrauerCharacter(g);$

with the following two lines respectively in their respective positions of the code:

$> function\ LowerBoundPower(G, \ell, b, p);$
$> x := BrauerCharacter(g\hat{\ }p);$

We note the conjugacy classes listed in Lemma 6.2.3 account for 197 of the 256 G-conjugacy classes of semisimple elements. In the proceeding sections of this chapter, these lower bounds will allow for easy determination of the number of conjugacy classes for most orders.

6.3 Prime Order Elements

In this section we study the elements of prime order. Later in Section 6.4 we employ our findings to determine information relating to the semisimple elements of composite order. In the interest of brevity we write classes, instead of G-conjugacy classes.

Lemma 6.3.1 *There are four classes of elements of order 3 in G, with centralisers as given in Table 6.1.*

Proof. The centralisers of elements of order 3 are given in Table 3 of [Sei82]. By their orders, they are easily paired with their respective entries in Lübeck's list [Lüb]. □

G-class	Lübeck Number	Centraliser
3A	294	$3 \times E_7(2)$
3B	376	$3 \times \Omega_{14}^{-}(2)$
3C	147	$3.({}^2E_6(2) \times U_3(2)).3$
3D	258	$3 \times U_9(2)$

Table 6.3: The Conjugacy Classes of Elements of Order 3 in $E_8(2)$

Lemma 6.3.2 *There are two classes of elements of order 5 in G, with centralisers as given in Table 6.1.*

Proof. From Lemma 6.2.3 we know there at least two classes of elements of order 5. By Theorem 6.2.1 (iv) there exists $H_1 \leq G$ such that $H_1 \cong SU_5(4)$. Since $|H_1|$ is divisible by 5^5, it follows that for $P \in Syl_5(H_1)$, we have that $P \in Syl_5(G)$. Calculating centralisers in H_1 of nontrivial elements of P reveals that they are all divisible by 5^4. There are only two classes satisfying this on Lübeck's list, namely numbers 247 and 480. Hence there are exactly two classes of elements of order 5.

It is well known that the center of $SU_5(4)$ is a cyclic group of order 5. Let $x \in Z(H_1)$. Then, $H_1 \leq C_G(x)$. By Theorem 6.2.1 (i), there exists $H_2 \leq G$ such that $H_2 \cong \Omega_{16}^{+}(2)$, and furthermore we can find $y \in H_2$ of order 5 such that $C_{H_2}(y) \cong 5 \times \Omega_{12}^{-}(2)$. Since the orders of the subgroups H_1 and $C_{H_2}(y)$ match those on Lübeck's list, centralise elements of order 5 in G, and clearly cannot be subgroups of one another, we conclude the statement. □

G-class	Lübeck Number	Centraliser
5A	480	$5 \times \Omega_{12}^{-}(2)$
5B	247	$SU_5(4)$

Table 6.4: The Conjugacy Classes of Elements of Order 5 in $E_8(2)$

Lemma 6.3.3 *There are two classes of elements of order 7 in G, with centralisers as given in Table 6.1.*

Proof. From Lemma 6.2.3 we know there are at least two classes of elements of order 7. By Theorem 6.2.1 (*vi*) there exists $H_1 \leq G$ such that $H_1 \cong {}^3D_4(2) \times {}^3D_4(2)$. Since $|H_1|$ is divisible by 7^4, it follows that for $P \in Syl_7(H_1)$, we have that $P \in Syl_7(G)$ and that P must be elementary abelian of order 7^4. The only entries on Lübeck's list divisible by 7^4 are numbers 441 and 576. Hence there are exactly two classes of elements of order 7.

By Theorem 6.2.1 (*iii*) there exists $H_2 \in G$ such that $H_2 \cong L_3(2) \times E_6(2)$. Inside H_1, we can find $x \in H_1$ of order 7 with $C_{H_1}(x) \cong 7 \times L_3(2) \times {}^3D_4(2)$. Inside H_2 we can find $y \in H_2$ of order 7 with $C_{H_2}(y) \cong 7 \times E_6(2)$. Since the orders of $C_{H_1}(x)$ and $C_{H_2}(y)$ match the orders on Lübeck's list, they centralise elements of order 7, and they cannot be subgroups of one another (to see that $C_{H_1}(x) \nleq C_{H_2}(y)$ we refer the reader to [KW90]) we conclude the statement. □

G-class	Lübeck Number	Centraliser
7A	441	$7 \times E_6(2)$
7B	516	$7 \times L_3(2) \times {}^3D_4(2)$

Table 6.5: The Conjugacy Classes of Elements of Order 7 in $E_8(2)$

Lemma 6.3.4 *There is a unique class of elements of order 11 in G, with centraliser structure* $11 \times U_5(2)$.

Proof. By Theorem 6.2.1 (*vii*) there exists $H \leq G$ such that $H \cong U_5(2) \times U_5(2)$. Since $|H|$ is divisible by 11^2, it follows that for $P \in Syl_{11}(H)$, we have that $P \in Syl_{11}(G)$ and that P must be elementary abelian of order 11^2. Since number 679 on Lübeck's list is the only centraliser divisible by 11^2, we conclude that there is a unique class of elements of order 11 in G. Moreover, there is an element $x \in H$ with centraliser $C_H(x) \cong 11 \times U_5(2)$ whose order matches the entry on Lübeck's list. Hence, $C_G(x) \cong 11 \times U_5(2)$. □

Lemma 6.3.5 *There are two classes of elements of order 13 in G, with centralisers as given in Table 6.1.*

G-class	Lübeck Number	Centraliser
11A	679	$11 \times U_5(2)$

Table 6.6: The Conjugacy Class of Elements of Order 11 in $E_8(2)$

Proof. From Lemma 6.2.3 we know there are at least two classes of elements of order 13. By Theorem 6.2.1 (vi) there exists $H_1 \leq G$ such that $H_1 \cong {}^3D_4(2) \times {}^3D_4(2)$. Since $|H_1|$ is divisible by 13^2, it follows that for $P \in Syl_{13}(H_1)$, we have that $P \in Syl_{13}(G)$ and that P must be elementary abelian of order 13^2. There are only two entries on Lübeck's list divisible by 13^2, namely numbers 709 and 712, and so we conclude there are exactly two classes of elements of order 13 in G.

By Theorem 6.2.1 (ix) there exists $H_2 \leq G$ such that $H_2 \cong U_3(4) \times U_3(4)$. Inside H_1 we can find $x \in H_1$ of order 13 with $C_{H_1}(x) \cong 13 \times {}^3D_4(2)$. Inside H_2 we can find $y \in H_2$ of order 13 with $C_{H_2}(y) \cong 13 \times U_3(4)$. Since the orders of the subgroups $C_{H_1}(x)$ and $C_{H_2}(y)$ match those on Lübeck's list, centralise elements of order 13 in G, and cannot be subgroups of one another, we conclude the statement. □

G-class	Lübeck Number	Centraliser
13A	712	$13 \times {}^3D_4(2)$
13B	709	$13 \times U_3(4)$

Table 6.7: The Conjugacy Classes of Elements of Order 13 in $E_8(2)$

Lemma 6.3.6 *There are four classes of elements of order 17 in G, with centralisers as given in Table 6.1.*

Proof. From Lemma 6.2.3 we know there are at least four classes of elements of order 17. By Theorem 6.2.1 (x) there exists $H_1 \leq G$ such that $H_1 \cong U_3(16)$. Since $|H_1|$ is divisible by 17^2, it follows that for $P \in Syl_{17}(H_1)$, we have that $P \in Syl_{17}(G)$ and that P is elementary abelian of order 17^2. A total of four centralisers on Lübeck's list are divisible by 17^2, namely numbers 693 and 738, each with multiplicity two. Hence there are exactly four classes of elements of order 17 in G.

By Theorem 6.2.1 (i) there exists $H_2 \leq G$ such that $H_2 \cong \Omega_{16}^+(2)$. Inside H_1 we can find $x \in H_1$ of order 17 with $C_{H_1}(x) \cong 17 \times L_2(16)$. Inside H_2 we can find $y \in H_2$ of order 17 with $C_{H_2}(y) \cong 17 \times \Omega_8^-(2)$. Since the orders of the subgroups $C_{H_1}(x)$ and $C_{H_2}(y)$ match those on Lübeck's list, centralise elements of order 17 in G, and cannot be subgroups of one another, the lemma follows. □

G-class	Lübeck Number	Centraliser
17AB	738	$17 \times \Omega_8^-(2)$
17CD	693	$17 \times L_2(16)$

Table 6.8: The Conjugacy Classes of Elements of Order 17 in $E_8(2)$

Lemma 6.3.7 *There is a unique class of elements of order 19 in G, with centraliser structure $3{\cdot}PGU_3(2) \times 19$.*

Proof. Since $|E_7(2)|$ is divisible by 19 and $E_7(2)$ has a single conjugacy class of elements of order 19 by [BBR15], we conclude that there is a single class in G. Thus there exists $x \in 19A$ such that $x \in C_G(3C) \cong 3.(^2E_6(2) \times U_3(2)).3$. The centraliser of x in $3.^2E_6(2)$ has shape 3×19 and so we see that $C_G(x)$ must contain a subgroup with structure $19 \times 3.U_3(2).3$. A comparison with Lübeck's list confirms that this is indeed the full centraliser $C_G(x)$. Work of Seitz [Sei82] shows this may be refined to $19 \times 3.PGU_3(2)$. There are five extensions of this shape. However, there are eleven remaining centralisers divisible by 19 on Lübeck's list and this eliminates two of these extensions.

Of the three remaining possibilities for $3.PGU_3(2)$, one contains precisely two classes of elements of order 9, each self-centralising. This would then imply G contains exactly two classes of elements of order 171 with centraliser order $3^2.19$. However, since $E_7(2)$ contains elements of order 171 with centraliser order $3^2.19$, and $C_G(3A) \cong Sym(3) \times E_7(2)$, we deduce that there exists elements of order 171 in G with centraliser order at least $2.3^3.19$. Hence this extension is not the one we seek.

Another possibility for $3.PGU_3(2)$ contains precisely three classes of elements of order 3, two with centraliser order $2^3.3^4$ and one with 3^3. This would then imply G contains exactly three classes of elements of order 57, two with centraliser order $2^3.3^4.19$ and one with $3^3.19$. Recall that for $g \in 3D$, $C_G(g) \cong \langle g \rangle \times U_9(2)$. Since $U_9(2)$ contains elements h of order 57 such that $h^{19} = g$, and the centraliser of such an element in $U_9(2)$ has order 3.19, we deduce that there are elements h of order 57 in G with $|C_G(h)| = 3^2.19$. Hence this extension is ruled out.

Therefore, by elimination, the correct extension of $3.PGU_3(2)$ is identified. □

G-class	Lübeck Number	Centraliser
19A	823	$19 \times 3^{\cdot}PGU_3(2)$

Table 6.9: The Conjugacy Class of Elements of Order 19 in $E_8(2)$

Lemma 6.3.8 *There are four classes of elements of order 31 in G, with centralisers as given in Table 6.1.*

Proof. From Lemma 6.2.3 we know there are at least four classes of elements of order 31. By Theorem 6.2.1 (*viii*) there exists $H \leq G$ such that $H \cong L_5(2) \times L_5(2)$. Since $|L_5(2)|$ is divisible by 31, it follows that for $P \in Syl_{31}(H)$, we have that $P \in Syl_{31}(G)$ and that P is elementary abelian of order 31^2. Since there are only four centralisers on Lübeck's list divisible by 31^2, namely numbers 672 and 857, we conclude there are exactly four classes of elements of order 31 in G. Inside H we can find elements x and y of order 31 with $C_H(x) \cong 31^2$ and $C_H(y) \cong 31 \times L_5(2)$. Since their orders match those on Lübeck's list, we conclude the centralisers are as given in Table 6.1. □

G-class	Lübeck Number	Centraliser
31ABC	672	$31 \times L_5(2)$
31D	857	31^2

Table 6.10: The Conjugacy Classes of Elements of Order 31 in $E_8(2)$

Lemma 6.3.9 *There are two classes of elements of order 41 in G, both centralised by a cyclic subgroup of order 205.*

Proof. Having accounted for number 247 of Lübeck's list as being a centraliser of a $5B$ element, there remain 10 centralisers divisible by 41, all of which have order 205. These must correspond to centralisers of elements of order 41 or 205. By Theorem 6.2.1 there exists $H \leq G$ such that $H \cong PGU_5(4)$ and we observe that H has eight conjugacy classes of elements of order 41. Since all G-conjugacy classes of such elements will be of equal size, we see that there are 8, 4, 2 or 1 G-conjugacy classes of elements of order 41. Furthermore, each G-conjugacy class of elements of order 41 is the union of fifth powers of G-conjugacy classes of elements of order 205. Consequently, it must be the case that the ten G-conjugacy classes with centralisers of order 205 must consist of two classes of elements of order 41 and eight of elements of order 205. □

G-class	Lübeck Number	Centraliser
41AB	864	205

Table 6.11: The Conjugacy Classes of Elements of Order 41 in $E_8(2)$

Lemma 6.3.10 *There are three classes of elements of order 43 in G, all with centraliser structure* $3 \times Sym(3) \times 43$.

Proof. From Lemma 6.2.3 we know there are at least three classes of elements of order 43. Since a Sylow 43-subgroup of G is contained in $E_7(2) \leq G$, [BBR15] shows there are exactly three such classes. By Theorem 6.2.1 (ii) there exists $H \leq G$ such that $H \cong Sym(3) \times E_7(2)$. Since the centraliser of an element of order 43 in $E_7(2)$ has structure 3×43, we have that for $x \in H$ of order 43, $C_H(x) \cong 3 \times Sym(3) \times 43$. A consideration of Lübeck's list yields the result. □

G-class	Lübeck Number	Centraliser
43ABC	837	$3 \times Sym(3) \times 43$

Table 6.12: The Conjugacy Classes of Elements of Order 43 in $E_8(2)$

Lemma 6.3.11 *There are four classes of elements of order 73 in G, all with centraliser structure* $73 \times L_3(2)$.

Proof. From Lemma 6.2.3 we know there are at least four classes of elements of order 73. Since a Sylow 73-subgroup of G is contained in $E_7(2) \leq G$, [BBR15] shows there are exactly four such classes. By Theorem 6.2.1 (iii) there exists $H \leq G$ such that $H \cong L_3(2) \times E_6(2)$. Since elements of order 73 are self-centralising in $E_6(2)$, the now familiar argument of centraliser orders using Lübeck's list yields the statement. □

G-class	Lübeck Number	Centraliser
73ABCD	814	$73 \times L_3(2)$

Table 6.13: The Conjugacy Classes of Elements of Order 73 in $E_8(2)$

Lemma 6.3.12 *There are nine classes of elements of order 127 in G, all with centraliser structure* $Sym(3) \times 127$.

Proof. From Lemma 6.2.3 we know there are at least nine classes of elements of order 127. Since a Sylow 127-subgroup of G is contained in $E_7(2) \leq G$, [BBR15] shows there are exactly nine such classes. By Theorem 6.2.1 (ii) there exists $H \leq G$ such that $H \cong Sym(3) \times E_7(2)$.

Since an element of order 127 is self-centralising in $E_7(2)$ [BBR15], it follows that for $x \in H$ of order 127, $C_H(x) \cong Sym(3) \times 127$. A consideration of Lübeck's list yields the result. □

G-class	Lübeck Number	Centraliser
127ABCDEFGHI	835	$Sym(3) \times 127$

Table 6.14: The Conjugacy Classes of Elements of Order 127 in $E_8(2)$

Lemma 6.3.13 *In G, there are five classes of elements of order 151, ten classes of elements of order 241, and eleven classes of elements of order 331, all of which are self-centralising.*

Proof. For $p \in \{151, 241, 331\}$ the only centralisers on Lübeck's list whose orders are divisible by p are the cyclic subgroups of order p; these are numbers 868, 866 and 869 respectively. □

G-class	Lübeck Number	Centraliser
151ABCDE	868	151
241ABCDEFGHIJ	866	241
331ABCDEFGHIJK	869	331

Table 6.15: The Conjugacy Classes of Elements of Orders 151, 241 and 331 in $E_8(2)$

6.4 Composite Order Elements

In this section we determine the centralisers of semisimple elements in G of composite order o, and realise the bounds on the number of G-conjugacy classes given in Table 6.2. Indeed, assume that $o = pm$ for some prime number p and $(p, m) = 1$. It was shown in Section 6.3 that all centralisers of elements of order p, with the exception of a $5B$ element, have the form $p \times H$ for some subgroup H. By considering all such elements of order p in G and elements m in the respective groups H, we may obtain upper bounds on the number of G-conjugacy classes of elements of composite order o. We begin by considering the elements of order o with $p = 5$.

6.4.1 Elements of order $5m$

In Section 6.3 we determined that there are precisely two classes of elements of order 5 in G, $5A$ and $5B$, whose elements respectively have centralisers $5 \times \Omega_{12}^-(2)$ and $SU_5(4)$. We determine the centralisers of various elements whose orders are multiples of 5 by working directly within these centralisers. Elements of orders 205 omitted here as they were dealt with in Section 6.3.

Elements of order 15

Since every element of order 15 in G will be contained in a subgroup with shape $5 \times \Omega_{12}^-(2)$ or $SU_5(4)$, we may assume, up to conjugation by a suitable element of G, that all elements of order 15 can be written in the form 5 multiplied by an element of order 3 in either $\Omega_{12}^-(2)$ or $SU_5(4)$. There are five conjugacy classes of elements of order 3 in $\Omega_{12}^-(2)$ and two such classes in $SU_5(4)$. Thus we deduce that G must contain at most seven conjugacy classes of elements of order 15. As this is the lower bound given in Table 6.2, we see that G has precisely seven classes of elements of order 15. Furthermore, since the centraliser of an element of order 15 must be a subgroup within its respective centraliser of an element of order 5, we may fully determine the centralisers of elements of order 15 by finding the centralisers of the elements of order 3 in $\Omega_{12}^-(2)$ and $SU_5(4)$.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
$\Omega_{12}^-(2)$	$3A$	$3 \times \Omega_{10}^+(2)$	$15 \times \Omega_{10}^+(2)$	$15A$
	$3B$	$3 \times Sym(3) \times \Omega_8^-(2)$	$15 \times Sym(3) \times \Omega_8^-(2)$	$15B$
	$3C$	$3 \times U_5(2)$	$15 \times U_5(2)$	$15C$
	$3D$	$GU_3(2) \times L_4(2)$	$5 \times GU_3(2) \times L_4(2)$	$15D$
	$3E$	$3 \times L_2(4) \times U_4(2)$	$15 \times L_2(4) \times U_4(2)$	$15E$
$SU_5(4)$	$3A$	$15 \times U_3(4)$	$15 \times U_3(4)$	$15F$
	$3B$	$15 \times L_2(16)$	$15 \times L_2(16)$	$15F$

Table 6.16: The Conjugacy Classes of Elements of Order 15 in $E_8(2)$

Elements of order 35

As the order of $SU_5(4)$ is not divisible by 7, it follows that all elements of order 35 power to an element within the G-class $5A$. Since there is only one conjugacy class of elements of order 7 in $\Omega_{12}^-(2)$, we conclude that there is precisely one conjugacy class of elements of order 35 in G. Its centraliser structure is easily determined.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
$\Omega_{12}^-(2)$	$7A$	$7 \times U_4(2)$	$35 \times U_4(2)$	$35A$

Table 6.17: The Conjugacy Classes of Elements of Order 35 in $E_8(2)$

Elements of order 45

Since $SU_5(4)$ does not contain any elements of order 9, it follows that all elements of order 45 power to an element within the G-class $5A$. There are three conjugacy classes of elements of order 9 in $\Omega^-_{12}(2)$ and thus we deduce that G must contain at most three conjugacy classes of elements of order 9. As this is the lower bound given in Table 6.2, we see that G has precisely three classes of elements of order 9.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
	$9A$	$9 \times L_4(2)$	$45 \times L_4(2)$	$45A$
$\Omega^-_{12}(2)$	$9B$	$9 \times 3 \times Alt(5)$	$45 \times 3 \times Alt(5)$	$45B$
	$9C$	$9 \times 3 \times Sym(3)$	$45 \times 3 \times Sym(3)$	$45C$

Table 6.18: The Conjugacy Classes of Elements of Order 45 in $E_8(2)$

Elements of order 55

As the order of $SU_5(4)$ is not divisible by 11, it follows that all elements of order 55 power to an element within the G-class $5A$. Since there is only one conjugacy class of elements of order 11 in $\Omega^-_{12}(2)$, we conclude that there is precisely one conjugacy class of elements of order 55 in G. Its centraliser structure is easily determined.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
$\Omega^-_{12}(2)$	$11A$	33	165	$55A$

Table 6.19: The Conjugacy Classes of Elements of Order 55 in $E_8(2)$

Elements of order 65

There are two conjugacy classes of elements of order 13 in $\Omega^-_{12}(2)$ and four such classes in $SU_5(4)$. Thus we deduce that G must contain at most six conjugacy classes of elements of order 65. As this is the lower bound given in Table 6.2, we see that G has precisely six classes of elements of order 65. Their respective centralisers are given in Table 6.20.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
$\Omega_{12}^-(2)$	$13AB$	65	5×65	$65EF$
$SU_5(4)$	$13ABCD$	$65 \times Alt(5)$	$65 \times Alt(5)$	$65ABCD$

Table 6.20: The Conjugacy Classes of Elements of Order 65 in $E_8(2)$

Elements of order 85

There are two conjugacy classes of elements of order 17 in $\Omega_{12}^-(2)$ and four such classes in $SU_5(4)$. Thus we deduce that G must contain at most six conjugacy classes of elements of order 85. As this is the lower bound given in Table 6.2, we see that G has precisely six classes of elements of order 85. Again, a simple computational investigation yields the structure of their centralisers.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
$\Omega_{12}^-(2)$	$17AB$	$17 \times Sym(3)^2$	$85 \times Sym(3)^2$	$85AB$
$SU_5(4)$	$17ABCD$	255	255	$85CDEF$

Table 6.21: The Conjugacy Classes of Elements of Order 85 in $E_8(2)$

Elements of order 105

Since the order of $SU_5(4)$ is not divisible by 21, it follows that all elements of order 105 power to an element within the G-class $5A$. There are four conjugacy classes of elements of order 21 in $\Omega_{12}^-(2)$ and thus we deduce that G must contain at most four conjugacy classes of elements of order 105. As this is the lower bound given in Table 6.2, we see that G has precisely four classes of elements of order 105.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
$\Omega_{12}^-(2)$	$21AB$	$7 \times GU_3(2)$	$35 \times GU_3(2)$	$105AB$
	$21C$	$21 \times Sym(3)^2$	$105 \times Sym(3)^2$	$105C$
	$21D$	$21 \times 3 \times Sym(3)$	$105 \times 3 \times Sym(3)$	$105D$

Table 6.22: The Conjugacy Classes of Elements of Order 105 in $E_8(2)$

Elements of order 155

Since the order of $SU_5(4)$ is not divisible by 31, it follows that all elements of order 155 power to an element within the G-class $5A$. There are three conjugacy classes of elements of order 31 in $\Omega_{12}^-(2)$ and thus we deduce that G must contain at most three conjugacy classes of elements of order 155. As this is the lower bound given in Table 6.2, we see that G has precisely three classes of elements of order 155.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
$\Omega_{12}^-(2)$	$31ABC$	93	465	$155ABC$

Table 6.23: The Conjugacy Classes of Elements of Order 155 in $E_8(2)$

Elements of order 165

As the order of $SU_5(4)$ is not divisible by 33, it follows that all elements of order 165 power to an element within the G-class $5A$. There are two conjugacy classes of elements of order 33 in $\Omega_{12}^-(2)$ and thus we deduce that G must contain at most two conjugacy classes of elements of order 165. As this is the lower bound given in Table 6.2, we see that G has precisely two classes of elements of order 165.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
$\Omega_{12}^-(2)$	$33AB$	33	165	$165AB$

Table 6.24: The Conjugacy Classes of Elements of Order 165 in $E_8(2)$

Elements of order 195

As $\Omega_{12}^-(2)$ does not contain any elements of order 39, it follows that all elements of order 195 power to an element within the G-class $5B$. There are four conjugacy classes of elements of order 39 in $SU_5(4)$ and thus we deduce that G must contain at most four conjugacy classes of elements of order 195. As this is the lower bound given in Table 6.2, we see that G has precisely four classes of elements of order 195, all of which are self-centralising.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
$SU_5(4)$	$39ABCD$	195	195	$195ABCD$

Table 6.25: The Conjugacy Classes of Elements of Order 195 in $E_8(2)$

Elements of order 255

There are six conjugacy classes of elements of order 51 in $\Omega_{12}^-(2)$ and eight such classes in $SU_5(4)$. Thus we deduce that G must contain at most fourteen conjugacy classes of elements of order 255. As this is the lower bound given in Table 6.2, we see that G has precisely fourteen classes of elements of order 255.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
$\Omega_{12}^-(2)$	$51ABCD$	$51 \times Sym(3)$	$255 \times Sym(3)$	$255ABCD$
	$51EF$	51×3	255×3	$255EF$
$SU_5(4)$	$51ABCDEFGH$	255	255	$255GHIJKLMN$

Table 6.26: The Conjugacy Classes of Elements of Order 255 in $E_8(2)$

Elements of order 315

As the order of $SU_5(4)$ is not divisible by 63, it follows that all elements of order 315 power to an element within the G-class $5A$. There are two conjugacy classes of elements of order 63 in $\Omega_{12}^-(2)$ and thus we deduce that G must contain at most two conjugacy classes of elements of order 315. As this is the lower bound given in Table 6.2, we see that G has precisely two classes of elements of order 315, all of which are self-centralising.

Subgroup H	H-class of h	$C_H(h)$	$C_G(5 \times h)$	G-class
$\Omega_{12}^-(2)$	$63AB$	63	315	$315AB$

Table 6.27: The Conjugacy Classes of Elements of Order 315 in $E_8(2)$

6.4.2 Elements of order $7m$

In Section 6.3 we determined that there are precisely two classes of elements of order 7 in G, $7A$ and $7B$, whose elements respectively have centralisers $7 \times E_6(2)$ and $7 \times L_3(2) \times {}^3D_4(2)$. We determine the centralisers of various elements whose orders are multiples of 7 by working directly within these centralisers.

Elements of order 21

There are three conjugacy classes of elements of order 3 in $E_6(2)$ and five such classes in $L_3(2) \times {}^3D_4(2)$. Thus we deduce that G must contain at most eight conjugacy classes of elements of order 21. As this is the lower bound given in Table 6.2, we see that G has precisely eight classes of elements of order 21.

Elements of order 91

There is one conjugacy class of elements of order 13 in $E_6(2)$ and three such classes in $L_3(2) \times {}^3D_4(2)$. Thus we deduce that G must contain at most four conjugacy classes of elements of order 91. As this is the lower bound given in Table 6.2, we see that G has precisely four classes of elements of order 91.

Subgroup H	H-class of h	$C_H(h)$	$C_G(7 \times h)$	G-class
$E_6(2)$	$3A$	$3 \times L_6(2)$	$21 \times L_6(2)$	$21A$
	$3B$	$3 \times \Omega_8^-(2)$	$21 \times \Omega_8^-(2)$	$21C$
	$3C$	$3.(3^2 : Q_8 \times L_3(4)) : 3$	$7 \times 3.(3^2 : Q_8 \times L_3(4)) : 3$	$21D$
$L_3(2) \times {}^3D_4(2)$	$3A$	$3 \times {}^3D_4(2)$	$21 \times {}^3D_4(2)$	$21B$
	$3B$	$3 \times L_3(2) \times L_2(8)$	$21 \times L_3(2) \times L_2(8)$	$21E$
	$3C$	$L_3(2) \times 3_+^{1+2} : 2Alt(4)$	$7 \times L_3(2) \times 3_+^{1+2} : 2Alt(4)$	$21F$
	$3D$	$3^2 \times L_2(8)$	$21 \times 3 \times L_2(8)$	$21G$
	$3E$	$3 \times 3_+^{1+2} : 2Alt(4)$	$21 \times 3_+^{1+2} : 2Alt(4)$	$21H$

Table 6.28: The Conjugacy Classes of Elements of Order 21 in $E_8(2)$

Subgroup H	H-class of h	$C_H(h)$	$C_G(7 \times h)$	G-class
$E_6(2)$	$13A$	91	7×91	$91D$
$L_3(2) \times {}^3D_4(2)$	$13ABC$	$13 \times L_3(2)$	$91 \times L_3(2)$	$91ABC$

Table 6.29: The Conjugacy Classes of Elements of Order 91 in $E_8(2)$

Elements of order 119

As the order of $L_3(2) \times {}^3D_4(2)$ is not divisible by 17, it follows that all elements of order 119 power to an element within the G-class $7A$. There are two conjugacy classes of elements of order 17 in $E_6(2)$ and thus we deduce that G must contain at most two conjugacy classes of elements of order 119. As this is the lower bound given in Table 6.2, we see that G has precisely two classes of elements of order 119.

Subgroup H	H-class of h	$C_H(h)$	$C_G(7 \times h)$	G-class
$E_6(2)$	$17AB$	51	357	$119AB$

Table 6.30: The Conjugacy Classes of Elements of Order 119 in $E_8(2)$

Elements of order 217

As the order of $L_3(2) \times {}^3D_4(2)$ is not divisible by 31, it follows that all elements of order 217 power to an element within the G-class $7A$. There are six conjugacy classes of elements of order 31 in $E_6(2)$ and thus we deduce that G must contain at most six conjugacy classes of elements of order 217. As this is the lower bound given in Table 6.2, we see that G has precisely six classes of elements of order 217.

Subgroup H	H-class of h	$C_H(h)$	$C_G(7 \times h)$	G-class
$E_6(2)$	$31ABCDEF$	$31 \times Sym(3)$	$217 \times Sym(3)$	$217ABCDEF$

Table 6.31: The Conjugacy Classes of Elements of Order 217 in $E_8(2)$

Elements of order 511

As the order of $L_3(2) \times {}^3D_4(2)$ is not divisible by 73, it follows that all elements of order 511 power to an element within the G-class $7A$. There are eight conjugacy classes of elements of order 73 in $E_6(2)$ and thus we deduce that G must contain at most eight conjugacy classes of elements of order 511. As this is the lower bound given in Table 6.2, we see that G has precisely eight classes of elements of order 511.

Subgroup H	H-class of h	$C_H(h)$	$C_G(7 \times h)$	G-class
$E_6(2)$	$73ABCDEFGH$	73	511	$511ABCDEFGH$

Table 6.32: The Conjugacy Classes of Elements of Order 511 in $E_8(2)$

6.4.3 Elements of order $11m$

In Section 6.3 we determined that there is a unique conjugacy classes of elements of order 11 in G, whose elements have centraliser $11 \times U_5(2)$. Here we determine the centralisers of elements of order 33 and 99.

Elements of order 33

There are six conjugacy classes of elements of order 3 in $U_5(2)$ and thus we deduce that G must contain at most six conjugacy classes of elements of order 33. As this is the lower bound given in Table 6.2, we see that G has precisely six classes of elements of order 33.

Subgroup H	H-class of h	$C_H(h)$	$C_G(11 \times h)$	G-class
$U_5(2)$	$3AB$	$3 \times U_4(2)$	$33 \times U_4(2)$	$33AB$
	$3CD$	$Sym(3) \times 3_+^{1+2} : 2Alt(4)$	$11 \times Sym(3) \times 3_+^{1+2} : 2Alt(4)$	$33CD$
	$3E$	$3 \times 3_+^{1+2} : 2Alt(4)$	$33 \times 3_+^{1+2} : 2Alt(4)$	$33E$
	$3F$	$3^2 \times Sym(3)^2$	$33 \times 3 \times Sym(3)^2$	$33F$

Table 6.33: The Conjugacy Classes of Elements of Order 33 in $E_8(2)$

Elements of order 99

There are four conjugacy classes of elements of order 9 in $U_5(2)$ and thus we deduce that G must contain at most four conjugacy classes of elements of order 99. As this is the lower bound given in Table 6.2, we see that G has precisely four classes of elements of order 99.

Subgroup H	H-class of h	$C_H(h)$	$C_G(11 \times h)$	G-class
$U_5(2)$	$9AB$	$9 \times Sym(3)$	$99 \times Sym(3)$	$99AB$
	$9CD$	9×3	99×3	$99CD$

Table 6.34: The Conjugacy Classes of Elements of Order 99 in $E_8(2)$

6.4.4 Elements of order $13m$

In Section 6.3 we determined that there are precisely two classes of elements of order 13 in G, namely $13A$ and $13B$, whose elements respectively have centralisers $13 \times {}^3D_4(2)$ and $13 \times U_3(4)$. We determine the centralisers of various elements whose orders are multiples of 13 by working directly within these centralisers.

Elements of order 39

There are two conjugacy class of elements of order 3 in ${}^3D_4(2)$ and one such class in $U_3(4)$. Thus we deduce that G must contain at most three conjugacy classes of elements of order 39. As this is the lower bound given in Table 6.2, we see that G has precisely three classes of elements of order 39.

Subgroup H	H-class of h	$C_H(h)$	$C_G(13 \times h)$	G-class
${}^3D_4(2)$	$3A$	$3 \times L_2(8)$	$39 \times L_2(8)$	$39A$
	$3B$	$3^{1+2}_+ : 2Alt(4)$	$13 \times 3^{1+2}_+ : 2Alt(4)$	$39B$
$U_3(4)$	$3A$	15	195	$39C$

Table 6.35: The Conjugacy Classes of Elements of Order 39 in $E_8(2)$

Elements of order 117

Since the order of $U_3(4)$ is not divisible by 9, it follows that all elements of order 117 power to an element within the G-class $13A$. There are three conjugacy classes of elements of order 9 in ${}^3D_4(2)$ and thus we deduce that G must contain at most three conjugacy classes of elements of order 117. As this is the lower bound given in Table 6.2, we see that G has precisely three classes of elements of order 117.

Subgroup H	H-class of h	$C_H(h)$	$C_G(13 \times h)$	G-class
${}^3D_4(2)$	$9ABC$	$9 \times Sym(3)$	$117 \times Sym(3)$	$117ABC$

Table 6.36: The Conjugacy Classes of Elements of Order 117 in $E_8(2)$

Elements of order 273

Since the order of $U_3(4)$ is not divisible by 21, it follows that all elements of order 273 power to an element within the G-class $13A$. There are three conjugacy classes of elements of order 21 in ${}^3D_4(2)$ and thus we deduce that G must contain at most three conjugacy classes of elements of order 273. As this is the lower bound given in Table 6.2, we see that G has precisely three classes of elements of order 273.

Subgroup H	H-class of h	$C_H(h)$	$C_G(13 \times h)$	G-class
${}^3D_4(2)$	$21ABC$	21	273	$273ABC$

Table 6.37: The Conjugacy Classes of Elements of Order 273 in $E_8(2)$

6.4.5 Elements of order $17m$

In Section 6.3 we determined that there are precisely four classes of elements of order 17 in G, namely $17AB$ and $17CD$, whose elements respectively have centralisers $17 \times \Omega_8^-(2)$ and $17 \times L_2(16)$. We determine the centralisers of various elements whose orders are multiples of 17 by working directly within these centralisers.

Elements of order 51

There are three conjugacy class of elements of order 3 in $\Omega_8^-(2)$ and one such class in $L_2(16)$. Thus, accounting for the fact each group lends itself to two centralisers of elements of order 17, we deduce that G must contain at most eight conjugacy classes of elements of order 51. As this is the lower bound given in Table 6.2, we see that G has precisely eight classes of elements of order 51.

Subgroup H	H-class of h	$C_H(h)$	$C_G(17 \times h)$	G-class
$\Omega_8^-(2)$	$3A$	$3 \times L_4(4)$	$51 \times L_4(2)$	$51AB$
	$3B$	$3 \times Sym(3) \times Alt(5)$	$51 \times Sym(3) \times Alt(5)$	$51CD$
	$3C$	$GU_3(2)$	$17 \times GU_3(2)$	$51EF$
$L_2(16)$	$3A$	15	255	$51GH$

Table 6.38: The Conjugacy Classes of Elements of Order 51 in $E_8(2)$

Elements of order 153

Since the order of $L_2(16)$ is not divisible by 9, it follows that all elements of order 153 power to an element within the G-classes $17A$ and $17B$. There is a unique conjugacy class of elements of order 9 in $\Omega_8^-(2)$ and thus we deduce that G must contain at most two conjugacy classes of elements of order 153. As this is the lower bound given in Table 6.2, we see that G has precisely two classes of elements of order 153, with one class powering to $17A$ and one to $17B$.

Subgroup H	H-class of h	$C_H(h)$	$C_G(17 \times h)$	G-class
$\Omega_8^-(2)$	$9A$	9	153	$153AB$

Table 6.39: The Conjugacy Classes of Elements of Order 153 in $E_8(2)$

6.4.6 Remaining Cases

In this section we deal with the remaining elements of composite order.

Elements of order 93

In Section 6.3 we determined that there are precisely four conjugacy classes of elements of order 31 in G, namely $31ABC$ and $31D$, with respective centralisers $31 \times L_5(2)$ and 31^2. Clearly elements of order 93 must power to elements within the classes $31ABC$. There are two conjugacy classes of elements of order 3 in $L_5(2)$ and thus it follows, accounting for all three centralisers, that there are at most six classes of elements of order 93 in G. Since this is the lower bound obtained in Table 6.2, we conclude that there are precisely six classes of elements of order 93 in G.

Subgroup H	H-class of h	$C_H(h)$	$C_G(31 \times h)$	G-class
$L_5(2)$	$3A$	$3 \times L_3(2)$	$93 \times L_3(2)$	$93ABC$
	$3B$	$3 \times Alt(5)$	$93 \times Alt(5)$	$93DEF$

Table 6.40: The Conjugacy Classes of Elements of Order 93 in $E_8(2)$

Elements of order 129

In Section 6.3 we determined that there are precisely three conjugacy classes of elements of order 43 in G with centraliser $129 \times Sym(3) \cong 43 \times 3 \times Sym(3)$. Clearly all elements of order 129 power to these three classes. Since $3 \times Sym(3)$ contains five conjugacy classes of elements of order 129, we find that there are at most fifteen classes of elements of order

129 in G. Since this is the lower bound obtained in Table 6.2, we conclude that there are precisely fifteen classes of elements of order 129 in G.

Subgroup H	H-class of h	$C_H(h)$	$C_G(43 \times h)$	G-class
$3 \times Sym(3)$	$3AB$	$3 \times Sym(3)$	$129 \times Sym(3)$	$129ABCDEF$
	$3CDE$	3^2	129×3	$129GHIJKLMNO$

Table 6.41: The Conjugacy Classes of Elements of Order 129 in $E_8(2)$

Elements of order 219

In Section 6.3 we determined that there are precisely four conjugacy classes of elements of order 73 in G with centraliser $73 \times L_3(2)$. Clearly all elements of order 219 power to these four classes. Since $L_3(2)$ contains a unique class of elements of order 3, we find that there are at most four classes of elements of order 219 in G. Since this is the lower bound obtained in Table 6.2, we conclude that there are precisely four classes of elements of order 219 in G.

Subgroup H	H-class of h	$C_H(h)$	$C_G(73 \times h)$	G-class
$L_3(2)$	$3A$	3	219	$219ABCD$

Table 6.42: The Conjugacy Classes of Elements of Order 219 in $E_8(2)$

Elements of order 381

In Section 6.3 we determined that there are precisely nine conjugacy classes of elements of order 127 in G with centraliser $127 \times Sym(3)$. Clearly all elements of order 381 power to these nine classes. Since $Sym(3)$ contains a unique class of elements of order 3, we find that there are at most nine classes of elements of order 381 in G. Since this is the lower bound obtained in Table 6.2, we conclude that there are precisely nine classes of elements of order 381 in G.

Subgroup H	H-class of h	$C_H(h)$	$C_G(127 \times h)$	G-class
$Sym(3)$	$3A$	3	381	$381ABCDEFGHI$

Table 6.43: The Conjugacy Classes of Elements of Order 381 in $E_8(2)$

Elements of order 9 and 63

In Section 6.3 we described the four centralisers of elements of order 3 in G. We now consider the G-classes corresponding to Lübeck numbers 366, 560, 580 and 656. Since the classes of prime order elements are already determined, these must correspond to elements of composite order. Each of their centralisers are divisible by 3^7 and thus, by a consideration of the order of centralisers for prime order elements, we conclude that these must be centralisers of elements of order 3^i for some $i > 1$. Since the exponent of a Sylow 3-subgroup S is 9 by Lemma 6.2.2, we deduce these are centralisers of elements of order 9. Furthermore, since the centraliser in S of an element of order 9 has order at least 3^4, their centralisers can be identified by looking within known maximal subgroups of G.

G-class	Lübeck Number	Centraliser	Maximal Subgroup
$9A$	560	$9 \times \Omega_{10}^-(2)$	$\Omega_{16}^+(2)$
$9B$	656	$9 \times Sym(3) \times {}^3D_4(2)$	$({}^3D_4(2))^2.6$
$9C$	580	$9 \times Sym(3) \times U_5(2)$	$(U_5(2))^2.4$
$9D$	366	$9 \times Sym(3) \times U_3(8)$	$Sym(3) \times E_7(2)$

Table 6.44: The Conjugacy Classes of Elements of Order 9 in $E_8(2)$

To determine the number of classes of elements of order 63, we follow the same approach used earlier in this section by considering an element of order 63 as a product of an element of order 9 and one of order 7 in one of $\Omega_{10}^-(2)$, $Sym(3) \times {}^3D_4(2)$, or $Sym(3) \times U_3(8)$. We do not include the subgroup $Sym(3) \times U_5(2)$ since it contains no elements of order 7. We deduce that there are eight classes of elements of order 63 in G with centralisers given in Table 6.45.

Subgroup H	H-class of h	$C_H(h)$	$C_G(9 \times h)$	G-class
$\Omega_{10}^-(2)$	$7A$	$7 \times Alt(5)$	$63 \times Alt(5)$	$63D$
$Sym(3) \times {}^3D_4(2)$	$7ABC$	$7 \times Sym(3) \times L_3(2)$	$63 \times Sym(3) \times L_3(2)$	$63ABC$
	$7D$	$7^2 \times Sym(3)$	$63 \times 7 \times Sym(3)$	$63E$
$Sym(3) \times U_3(8)$	$7ABC$	$7 \times 3 \times Sym(3)$	$63 \times 3 \times Sym(3)$	$63FGH$

Table 6.45: The Conjugacy Classes of Elements of Order 63 in $E_8(2)$

Since there are no further classes on Lübeck's list divisible by 3^4, we conclude that there are no further classes of elements of order 9 nor 63 and thus G contains precisely four classes of elements of order 9 and eight classes of elements of order 63.

Elements of order 57 and 171

In Section 6.3 we determined that there is a unique class of elements of order 19 in G with centraliser $19 \times H \cong 19 \times 3^{\cdot}PGU_3(2)$. In H there are five conjugacy classes of elements of order 3 and six classes of elements of order 9. Since there are eleven centralisers on Lübeck's list divisible by 19 which we have not yet accounted for, and they cannot correspond to elements of order 357, 465 and 651, we conclude these are centralisers of elements of orders 57 and 171. Their structures are easily determined.

Subgroup H	H-class of h	$C_H(h)$	$C_G(19 \times h)$	G-class
$3^{\cdot}PGU_3(2)$	$3AB$	$3^{\cdot}PGU_3(2)$	$19^{\cdot}PGU_3(2)$	$57AB$
	$3C$	3×9	57×9	$57C$
	$3DE$	3^2	57×3	$57DE$
	$9ABCDEF$	$9 \times Sym(3)$	$171 \times Sym(3)$	$171ABCDEF$

Table 6.46: The Conjugacy Classes of Elements of Orders 57 and 171 in $E_8(2)$

Elements of order 357, 465 and 651

There remain now sixteen classes on Lübeck's list unaccounted for. These include six classes with centraliser order 651, four classes with centraliser order 357, and six classes with centraliser order 651. The proper factors of 357, 465 and 651 have already been accounted for and thus we conclude that these classes correspond to self-centralising elements of orders 465, 357 and 651. This concludes the identification of all G-conjugacy classes of semisimple elements.

6.5 Fixed-Point Spaces and Powering Up Maps

In this section we determine the fixed-point spaces on the natural G-module V and powering up maps for the semisimple elements of G. We describe the methods used to obtain the dimensions of fixed-points spaces in Section 6.5.1, and explore powering up maps in Section 6.5.2. This information combined with the centralisers detailed in the previous two sections is sufficient to determine representatives for each of the conjugacy classes of semisimple elements.

6.5.1 Fixed-Point Spaces

Dimensions of fixed-point spaces are particularly useful information to have in our bank of knowledge due to their nature as a conjugacy class invariant. For an element x in G, we may determine the dimension of its fixed-point space on V in MAGMA using the command **Dimension(Eigenspace(x,1))**. This is a computationally simple command taking a matter of seconds. Let x and y be elements in G of orders 155 and 205 respectively. By considering the orders of centralisers for elements of order 5, we deduce that x must power to an element of $5A$ whilst y powers into $5B$. By taking appropriate powers we obtain representatives for $5A$ and $5B$ and find their respective fixed-point space dimensions by using our MAGMA command.

Other cases are more problematic. Consider the six conjugacy classes for elements of order 33. Executing the **LowerBoundPower** command described in Section 6.2 yields representatives for the six classes. We find four different fixed-point space dimensions with two repeated twice. These repeated dimensions must belong to classes $33AB$ and $33CD$, whilst the unique fixed-point space dimensions 12 and 14 correspond to classes $33E$ and $33F$ in some order. Let x be the representative with fixed-point space dimension 12 and y the representative with dimension 14. By considering the eleventh powers of x and y we find that x powers into $3B$ whilst y powers into $3A$. The centraliser of a $3A$ element is of the form $3 \times E_7(2)$, and the centraliser of an element of order 11 in $E_7(2)$ has the form $11 \times 3_+^{1+2} : 2Alt(4)$, by [BBR15]. Considering the centraliser structure determined in Section 6.4, we deduce that y is contained in $33E$ and thus elements of $33E$ has fixed-point space dimension 14, whilst x must therefore lie in $33F$ and such elements have fixed-point space dimension 12.

6.5.2 Powering Up Maps

With the dimensions of fixed-point spaces determined, information relating to the powering-up maps for each conjugacy class follows smoothly. The representatives gained in the work of the previous subsection were used to fill in much of the information and the conjugacy classes of the power-ups were often fully determined by fixed-point space dimension and Lagrange's theorem on the order of the centralisers. Where this was not sufficient to distinguish between classes, the **BrauerCharacter** procedure given in Section 6.2 often delivered the required result. However, there were some cases where a more detailed approach was required.

Consider the primes $p = 241, 331$. In each case, a Sylow p-subgroup P in G is cyclic of order p. Moreover, the automorphism group of a cyclic group of order 241 (resp. 331) is cyclic of order 240 (resp. 330) and intersects G in a cyclic subgroup of order 24 (resp. 30). It follows that the fusion of non-trivial elements of P is determined uniquely by these cyclic subgroups and is given by

$$241A^7 = 241B, \quad 241B^7 = 241C, \quad 241C^7 = 241D,$$
$$241D^7 = 241E, \quad 241E^7 = 241F, \quad 241F^7 = 241G,$$
$$241G^7 = 241H, \quad 241H^7 = 241I, \quad 241I^7 = 241J,$$

and

$$331A^3 = 331B, \quad 331B^3 = 331C, \quad 331C^3 = 331D, \quad 331D^3 = 331E,$$
$$331E^3 = 331F, \quad 331F^3 = 331G, \quad 331G^3 = 331H, \quad 331H^3 = 331I,$$
$$331I^3 = 331J, \quad 331J^3 = 331K.$$

We may calculate the power-up maps for elements of orders 57 and 171 directly inside the centraliser of an element of order 19. We see that $57A^{20} = 57B$, $57D^{20} = 57E$, $171A^{20} = 171B$, $171B^{20} = 171C$, $171C^{20} = 171D$, $171D^{20} = 171E$ and $171E^{20} = 171F$. A similar approach may be used for elements of orders 205, 357, 465 and 651. We illustrate this for elements of order 205.

We know that an element of order 205 powers up into the centraliser of a $5B$ element, which has shape $SU_5(4)$. There are eight classes of elements of order 205 in G and eight classes of elements of order 41 in $SU_5(4)$. Thus fusion of elements of order 205 in G is fully determined by elements of order 41 in $SU_5(4)$. Consider $w = xy \in SU_5(4)$ where x is the central element of order 5 and y has order 41. Taking powers w^{6k} for $k \in \mathbb{N}$ and using fusion of classes within $SU_5(4)$, we deduce that

$$205A^{11} = 205B, \quad 205B^{11} = 205C, \quad 205C^{11} = 205D, \quad 205D^{11} = 205E,$$
$$205E^{11} = 205F, \quad 205F^{11} = 205G, \quad 205G^{11} = 205H.$$

We repeat these same arguments for the remainder of the semisimple elements of the group G to obtain a full list of power maps. The completed information is given in Table 6.1.

For a full list of representatives for the conjugacy classes of $E_8(2)$, together with files containing all procedures used in this study, we refer the reader to [ABM$^+$16].

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