

ON THE SYMMETRIC SQUARE OF QUATERNIONIC PROJECTIVE SPACE

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Contents

Abstract	5
Declaration	6
Copyright Statement	7
Acknowledgements	8
1 Introduction	9
1.1 Notation	10
1.2 The Quaternionic Geometry Essentials	11
1.3 Symmetric Squares	14
1.4 The Main Results in Cohomology	15
2 Decompositions and Borel Space	18
2.1 The Decomposition of $SP^2(\mathbb{H}\mathbb{P}^n)$	18
2.2 Comparison of B_n and SP_n^2	21
2.3 Infinite Cases	24
3 The Classifying Space of $Pin(4)$	27
3.1 Pin and $Spin$ Groups	27
3.2 The Actions of $Spin(4)$, $Pin(4)$ and $Spin(5)$	29
3.3 Stiefel Manifolds and Grassmannians	30
3.4 $BSpin(4)$, $BPin(4)$ and $BSpin(5)$	31
3.5 On Vector Bundles over $BPin(4)$	33
4 The Cohomology Ring of $BPin(4)$	37

4.1	The Cohomology of Γ and Γ_n	37
4.1.1	The Integral Cohomology	40
4.1.2	The Mod 2 Cohomology	42
4.2	Configuration Space	43
4.2.1	The Configuration Space $C_2(\mathbb{H}\mathbb{P}^n)$	43
4.2.2	The Symmetric Orthogonalisation	44
4.3	The Cohomology of the Borel Space	45
5	The Classifying Space of $\mathbb{Z}/2 \times S^3$	52
5.1	The Quotient V_{n+1}/G_A	52
5.2	The Cohomology of $A_2 \subset SP^2(\mathbb{H}\mathbb{P}^2)$	54
5.3	The Map $B(\mathbb{Z}/2 \times S^3) \rightarrow BPin(4)$	59
6	The Thom Space $Th(\theta)$	63
6.1	$MPin(4)$	64
6.2	The Cohomology of $MPin(4)$	66
6.2.1	The Cohomology of $B/\widehat{\Delta}$	66
6.2.2	Thom Isomorphism	69
7	The Cohomology Ring I	77
7.1	$H^*(SP^2(\mathbb{H}\mathbb{P}^\infty); \mathbb{Z})$	78
7.1.1	$H^{4k}(SP^2(\mathbb{H}\mathbb{P}^\infty))$	79
7.1.2	$H^{4k+2}(SP^2(\mathbb{H}\mathbb{P}^\infty))$	84
7.1.3	$H^{4k+1}(SP^2(\mathbb{H}\mathbb{P}^\infty))$ and $H^{4k+3}(SP^2(\mathbb{H}\mathbb{P}^\infty))$	84
7.2	Product Structures	86
7.3	$H^*(SP^2(\mathbb{H}\mathbb{P}^\infty); \mathbb{Z}/2)$	90
8	The Cohomology Ring II	93
8.1	$H^*(SP^2(\mathbb{H}\mathbb{P}^n); \mathbb{Z})$	93
8.1.1	$H^*(SP^2(\mathbb{H}\mathbb{P}^n))$ for $* > 8n$	94
8.1.2	$H^{4k}(SP^2(\mathbb{H}\mathbb{P}^n))$	95
8.1.3	$H^{4k+2}(SP^2(\mathbb{H}\mathbb{P}^n))$	103
8.1.4	$H^{4k+1}(SP^2(\mathbb{H}\mathbb{P}^n))$ and $H^{4k+3}(SP^2(\mathbb{H}\mathbb{P}^n))$	103
8.2	Relation to Nakaoka's Work	105

8.3	Finale	110
	Bibliography	114
A	Lemmas	119
A.1	Gluing Lemma	119
A.2	Ladder Lemma	119
B	Background Materials	120
B.1	NDR	120
B.2	Semidirect Products	120
B.3	Serre Spectral Sequence	121
C	Examples	123
C.1	$H^5(SP^2(\mathbb{H}\mathbb{P}^n); \mathbb{Z})$ and $H^9(SP^2(\mathbb{H}\mathbb{P}^n); \mathbb{Z})$	123
C.2	$\theta_{m,1}$ and $\theta_{m,2}$	124

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The main purpose of this thesis is to calculate the integral cohomology ring of the symmetric square of quaternionic projective space, which has been an open problem since computations with symmetric squares were first proposed in the 1930's. The geometry of this particular case forms an essential part of the thesis, and unexpected results concerning two universal $Pin(4)$ bundles are also included. The cohomological computations involve a commutative ladder of long exact sequences, which arise by decomposing the symmetric square and the corresponding Borel space in compatible ways. The geometry and the cohomology of the configuration space of unordered pairs of distinct points in quaternionic projective space, and of the Thom space $MPin(4)$, also feature, and seem to be of independent interest.

Declaration

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At the time of writing this thesis, the world is facing great difficulties; I would like to quote the following, written by Jean Vanier.

Faced by social norms and laws,
is there not a personal conscience
within each of us
inviting us
to be agents of peace,
truth and justice?

Chapter 1

Introduction

The *symmetric products* of a space have attracted many researchers in mathematics over the last 80 years. The symmetric products of an algebraic curve or a space are subjects in algebraic geometry and topology, whilst the physicists may work on symmetric products orbifolds in relation to string theory. One of the classical contributions in this area of mathematical research is the work of Minoru Nakaoka [44–47] in the 1950’s, who was especially interested in the symmetric products and the cyclic products of n -spheres. In the 1960’s the *homology* groups of the symmetric products of a space were described by R. James Milgram [41], in which he completed the calculations begun by Morse, Smith and Richardson in the 1930’s. It is believed that the cohomology rings of the symmetric products, on the other hand, are very difficult to compute in general, and fully known only for a few special cases. Nevertheless, most recently, in 2015, Dmitry Gugnin submitted a paper [21] on the integral cohomology of the symmetric product of a space X modulo torsion $H^*(SP^n(X); \mathbb{Z})/\text{Tor}$, where X is connected and homotopy equivalent to a CW complex. One of the main results of this thesis is a computation of the whole cohomology ring, including torsion, of the symmetric square of quaternionic projective space; our method also works for the complex case, which we plan to discuss in a forthcoming paper [7]. The underlying geometry is beautiful and plays a vital role for our cohomological calculation.

The structure of the thesis is as follows. Chapter 1 contains notation, basics of quaternionic geometry, and a brief introduction to the symmetric square $SP^2(\mathbb{H}\mathbb{P}^n)$ of quaternionic projective space, followed by the formulation of the main theorems in cohomology. In Chapter 2, we decompose the symmetric square $SP^2(\mathbb{H}\mathbb{P}^n)$ and the

Borel space $B_n = S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n)$ into subspaces, and explain their relationships; we also describe the infinite dimensional cases when $n = \infty$, which are considered as colimits $X = \bigcup_n X_n$. Chapter 3 mainly focuses on the group $Pin(4)$; we define the group action on \mathbb{R}^4 , two models for the classifying space $BPin(4)$, and two universal vector bundles over $BPin(4)$. In Chapter 4, we compute the integral cohomology rings of $BPin(4)$, the Borel space B_n , and the configuration space of unordered pairs of distinct points in $\mathbb{H}\mathbb{P}^n$. The main aim of Chapter 5 is to compute the homomorphism induced by the map $\mathbb{Z}/2 \times S^3 \rightarrow Pin(4)$. Chapter 6 is concerned with the geometry and the cohomology of certain Thom spaces, including $MPin(4)$. In the last two Chapters, 7 and 8, we justify the main theorems in cohomology which are formulated in Chapter 1. In the final part of Chapter 8, we also compare our cohomological results with Nakaoka's work on the cyclic product of a finite connected simplicial complex. Some background materials, lemmas and examples, are included in Appendices.

Most of the geometry described in Chapters 1–6 comes from combining known results in Algebraic Topology. However, we believe that our study of the two universal $Pin(4)$ bundles, together with the Thom space $MPin(4)$ and its restrictions, are original; as are our applications of symmetric orthogonalisation. Chapters 4–8 contain original computations for the integral cohomology rings of the symmetric squares of quaternionic projective space (both finite and infinite), the configuration subspaces of unordered pairs of distinct points, and the associated Borel spaces for $\mathbb{H}\mathbb{P}^n$.

1.1 Notation

We begin by listing some notation that we shall often use below.

$\mathbb{H}\mathbb{P}^n$ = the quaternionic projective space of dimension n

S^n = the standard n -sphere

SP_n^2 = the symmetric square of $\mathbb{H}\mathbb{P}^n$

B_n = the Borel space $S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n)$

SP^2 = the infinite union $\bigcup_n SP_n^2$

$CP_p(X)$ = the p -fold cyclic product of a space X , for any prime p

$C_n(X)$ = the n -th unordered configuration space of distinct points in X

$V_{\mathbb{H}}(k, m)$ = the quaternionic Steifel manifold of orthonormal k -frames in \mathbb{H}^m

$V_{n+1} = V_{\mathbb{H}}(2, n + 1)$

$Gr_{\mathbb{H}}(k, m)$ = the quaternionic Grassmannian of k -dimensional subspaces of \mathbb{H}^m

G_A = the compact Lie group $\mathbb{Z}/2 \times S^3$

$(G_i)^m = \bigoplus_{i=1}^m G_i$, the direct sum of abelian groups G_i

$M_{\mathbb{H}}(n)$ = the space of $n \times n$ quaternionic matrices

$M_{\mathbb{R}}(n)$ = the space of $n \times n$ real matrices

\simeq homotopic (of maps) or homotopy equivalence (of spaces)

\cong homeomorphism (of spaces) or isomorphism (of groups or rings)

$H^*(X)$ = the reduced integral cohomology of X

$H^*(X_+)$ = the unreduced integral cohomology of X

$R[x_1, \dots, x_n]$ = the polynomial ring in variables x_1, \dots, x_n over $R = \mathbb{Z}$ or $\mathbb{Z}/2$

$R\langle x_1, \dots, x_n \rangle$ = the free R -module generated by x_1, \dots, x_n over $R = \mathbb{Z}$ or $\mathbb{Z}/2$

UCT = universal coefficient theorem

LES = long exact sequence

NDR = neighbourhood deformation retract

SSS = Serre spectral sequence

1.2 The Quaternionic Geometry Essentials

In this section we review some basics of quaternionic geometry, which has a long history. There are many references on the subject, and we have mainly used [3, 4, 20, 40, 50, 58].

The quaternions $\mathbb{H} \cong \mathbb{R}^4$ form a non-commutative real algebra; its elements are expressed as $q = a1 + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = ijk = -1$. For any quaternion $q \in \mathbb{H}$, the *norm* is the real number

$$|q| = |a1 + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2} \geq 0,$$

and satisfies the rule $|q_1q_2| = |q_1| |q_2|$; \mathbb{H} is a normed division algebra so that $q_1q_2 = 0$ implies $q_1 = 0$ or $q_2 = 0$. For $q = a1 + bi + cj + dk \in \mathbb{H}$, its (*quaternionic*) *conjugate* is $\bar{q} = a1 - bi - cj - dk$, and conjugation satisfies $\overline{qp} = \bar{p} \bar{q}$, $q\bar{q} = \bar{q}q = |q|^2 1$ where $p, q \in \mathbb{H}$. It is often the case that 1 is omitted from these expressions for convenience, for example by writing $a1 + bi + cj + dk$ as $a + bi + cj + dk$. Then $a = \text{Re}(q)$ is the real part of q , and quaternions with $\text{Re}(q) = 0$ are called *pure*; we write the set of all pure quaternions as $\text{Pure}(\mathbb{H})$. When dealing with quaternions, care must be taken due to their non-commutative multiplication. A right \mathbb{H} -*module* X means an abelian group with a right scalar multiplication

$$X \times \mathbb{H} \rightarrow X, \quad (x, q) \mapsto xq$$

which satisfies $(x + y)q = xq + yq$, $x(p + q) = xp + xq$, $x(qp) = (xq)p$ for any $x, y \in X$ and $p, q \in \mathbb{H}$. In this thesis the right \mathbb{H} -linear structure is always used, unless otherwise stated. We sometimes call such an X a right \mathbb{H} -vector space, and its elements can be written as column vectors, with scalars acting on the right. A right linear map $T: \mathbb{H}^n \rightarrow \mathbb{H}^m$ satisfies $T(uq) = (T(u))q$, for any $u \in \mathbb{H}^n$ and $q \in \mathbb{H}$, and T can be represented by an $m \times n$ quaternionic matrix acting on the left.

We now list some further quaternionic notions.

Definition 1.1. *The \mathbb{H} -inner product $\langle \cdot, \cdot \rangle$ is given by*

$$\langle \cdot, \cdot \rangle: \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{H}, \quad (u, v) \mapsto \langle u, v \rangle = u^*v,$$

where u and v are column vectors and u^* is the transpose-conjugate of u .

The \mathbb{H} -inner product has the following properties:

- $\langle u, u \rangle = \|u\|^2 \geq 0$, and $u = 0$ if $\langle u, u \rangle = 0$
- $\langle u, v \rangle = \overline{\langle v, u \rangle}$

- $\langle u + u', v \rangle = \langle u, v \rangle + \langle u', v \rangle$
- $\langle ua, v \rangle = \bar{a}\langle u, v \rangle, \quad \langle u, vb \rangle = \langle u, v \rangle b$ (\mathbb{H} -linear in the second argument)

for $a, b \in \mathbb{H}$, $u, u', v \in \mathbb{H}^n$. The standard *norm* on \mathbb{H}^n , or the *length* of $u \in \mathbb{H}^n$, is the non-negative real number given by

$$\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{u^*u} \geq 0.$$

The norm $\|\cdot\|: \mathbb{H}^n \rightarrow \mathbb{R}^{\geq 0}$ satisfies [20] the triangle inequality $\|u+v\| \leq \|u\| + \|v\|$.

Given some subspace $W \subset \mathbb{H}^n$, the *orthogonal complement* is

$$W^\perp = \{v \in \mathbb{H}^n \mid \langle u, v \rangle = 0, \forall u \in W\},$$

and u and v are called *orthogonal* if $\langle u, v \rangle = 0$.

Recall that the quaternionic projective space $\mathbb{H}\mathbb{P}^n$ is the space of lines in \mathbb{H}^{n+1} passing through the origin, with the following metric.

Definition 1.2 ([13]). *The chordal distance between two points of $\mathbb{H}\mathbb{P}^n$ is*

$$d([u], [v]) \stackrel{\text{def}}{=} \sqrt{1 - |\langle u, v \rangle|^2} \quad \text{with} \quad 0 \leq d([u], [v]) \leq 1$$

where u and v are unit vectors in \mathbb{H}^{n+1} .

This metric is well-defined because

$$\sqrt{1 - |\langle u, v \rangle|^2} = \sqrt{1 - |\langle ua, vb \rangle|^2}$$

for any unit quaternions a and b , and clearly satisfies

$$d([u], [v]) = d([v], [u]).$$

A matrix $A \in M_{\mathbb{H}}(n)$ is said to be invertible if $AB = BA = I$ for some $B \in M_{\mathbb{H}}(n)$. Let $GL_{\mathbb{H}}(n) \subset M_{\mathbb{H}}(n)$ be the subgroup of invertible $n \times n$ matrices with respect to matrix multiplication.

Definition 1.3. *For any $n \geq 1$, the (quaternionic) symplectic group is the compact Lie group*

$$Sp(n) = \{A \in GL_{\mathbb{H}}(n) \mid A^*A = AA^* = I\}$$

where I is the $n \times n$ identity matrix and A^* is the conjugate-transpose of A .

Example 1.4. For $n = 1$,

$$Sp(1) = \{q \in \mathbb{H} \mid |q| = 1\}$$

is the group of unit quaternions, which is naturally identified with the 3 dimensional sphere $S^3 \subset \mathbb{R}^4 \cong \mathbb{H}$.

A matrix $A \in M_{\mathbb{H}}(n)$ is called Hermitian if $A^* = A$. We list below some facts about quaternionic matrices.

Theorem 1.5 ([58]). For the quaternionic matrices A and B , the following hold.

1. $(\overline{A})^t = \overline{(A^t)}$
2. $(AB)^* = B^*A^*$
3. $\overline{AB} \neq \overline{A}\overline{B}$ in general
4. $(AB)^t \neq B^tA^t$ in general
5. $(AB)^{-1} = B^{-1}A^{-1}$ if A and B are invertible
6. $(A^*)^{-1} = (A^{-1})^*$ if A is invertible
7. $(\overline{A})^{-1} \neq \overline{(A^{-1})}$ in general
8. $(A^t)^{-1} \neq (A^{-1})^t$ in general

where A^t is the transpose of A and $(\overline{A})^t = A^*$.

1.3 Symmetric Squares

In this section we introduce the symmetric square $SP^2(M)$ of a topological space M . In our case, M is always a CW complex, and usually a manifold; so $SP^2(M)$ is also CW [2, Section 5.2] [47].

The n -th symmetric product $SP^n(M)$ of a space M is the quotient of the n -fold Cartesian product $M^n = M \times \cdots \times M$ by the action of the symmetric group of degree n , which permutes the coordinates. An element of $SP^n(M)$ is an unordered n -tuple $[x_1, x_2, \dots, x_n]$ of points $x_i \in M$. Even if M is a manifold, however, $SP^2(M)$ is usually not. Exceptions include the examples [28]

$$SP^n(S^1) \simeq S^1, \quad SP^n(\mathbb{C}) \cong \mathbb{C}^n, \quad SP^n(S^2) \cong \mathbb{C}P^n, \quad SP^n(\mathbb{R}P^2) \cong \mathbb{R}P^{2n}.$$

The n -th symmetric product $SP^n(M)$ of a manifold M is an orbifold [1, Example 1.13]; our main interest is when $n = 2$ and $M = \mathbb{H}\mathbb{P}^n$.

Definition 1.6. *The symmetric square $SP^2(\mathbb{H}\mathbb{P}^n)$ of quaternionic projective space $\mathbb{H}\mathbb{P}^n$ is the quotient*

$$SP^2(\mathbb{H}\mathbb{P}^n) = \mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n / \sim$$

of the Cartesian square, where $(x, y) \sim (y, x)$ for any $x, y \in \mathbb{H}\mathbb{P}^n$; that is

$$SP^2(\mathbb{H}\mathbb{P}^n) = \mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n / \mathbb{Z}/2.$$

The case $n = \infty$ may also be considered, and is important.

Following [2, Section 5.2], the sequence of inclusions $\mathbb{H}\mathbb{P}^n \subset \mathbb{H}\mathbb{P}^{n+1}$ for $n \geq 1$ gives inclusions $SP^2(\mathbb{H}\mathbb{P}^n) \subset SP^2(\mathbb{H}\mathbb{P}^{n+1})$, whose colimit is

$$SP^2(\mathbb{H}\mathbb{P}^\infty) \stackrel{\text{def}}{=} \bigcup_n SP^2(\mathbb{H}\mathbb{P}^n).$$

So $SP^2(\mathbb{H}\mathbb{P}^\infty)$ has the union or colimit topology.

1.4 The Main Results in Cohomology

In this section, we formulate our main theorem, which describes the integral cohomology ring of the symmetric square of quaternionic projective space.

For some infinite unions $X = \bigcup_n X_n$, the cohomology $H^*(X_n)$ is better described in terms of $H^*(X)$. For example, $H^*(\mathbb{H}\mathbb{P}_+^n)$ is usually expressed as the truncation $\mathbb{Z}[z]/(z^{n+1})$, $|z| = 4$ of $H^*(\mathbb{H}\mathbb{P}_+^\infty) \cong \mathbb{Z}[z]$. We shall apply the same approach to the symmetric square, by first computing the cohomology ring $H^*(SP^2(\mathbb{H}\mathbb{P}^\infty)_+)$, and then expressing $H^*(SP^2(\mathbb{H}\mathbb{P}^n)_+)$ as its truncation.

The same procedure gives $H^*(\mathbb{C}\mathbb{P}_+^n)$ as $\mathbb{Z}[c_1]/(c_1^{n+1})$.

In these descriptions, there are two possible choices of z , which differ by sign. In this thesis we follow [40, Theorem 5.6], so the natural map $q: \mathbb{C}\mathbb{P}^\infty \rightarrow \mathbb{H}\mathbb{P}^\infty$ induces $q^*(z) = c_1^2$ (not $-c_1^2$) in integral cohomology.

From now on, we often write:

- SP^2 for $SP^2(\mathbb{H}\mathbb{P}^\infty) = \bigcup_n SP^2(\mathbb{H}\mathbb{P}^n)$
- SP_n^2 for $SP^2(\mathbb{H}\mathbb{P}^n)$.

Our description of $H^*(SP_+^2) := H^*(SP_+^2; \mathbb{Z})$ is as follows.

Theorem 1.7. *The ring $H^*(SP_+^2)$ is isomorphic to*

$$\mathbb{Z} \left[\left(\frac{1}{2}\right)^{s-1} h^s, \left(\frac{1}{2}\right)^m g^\ell h^m, t_{i,j} : s, \ell, i \geq 1, m \geq 0, 1 \leq j < 2i \right] / \mathcal{I}$$

where $|g| = 4$, $|h| = 8$, $|t_{i,j}| = 4i + 2j + 1$, and

$$\mathcal{I} = \left(2t_{i,j}, t_{i,j}t_{k,l}, t_{i,j} \left(\frac{1}{2}\right)^{s-1} h^s, t_{i,j} \left(\frac{1}{2}\right)^m g^\ell h^m : k \geq 1, 1 \leq l < 2k \right).$$

The torsion-free product structure is indicated by the notation $\left(\frac{1}{2}\right)^{s-1} h^s, \left(\frac{1}{2}\right)^m g^\ell h^m$ as follows.

For any $s, s', \ell, \ell' \geq 1$ and $m, m' \geq 0$,

- $\left(\frac{1}{2}\right)^{s-1} h^s \cdot \left(\frac{1}{2}\right)^{s'-1} h^{s'} = 2 \cdot \left(\frac{1}{2}\right)^{s+s'-1} h^{s+s'}$
- $\left(\frac{1}{2}\right)^m g^\ell h^m \cdot \left(\frac{1}{2}\right)^{s-1} h^s = 2 \cdot \left(\frac{1}{2}\right)^{m+s} g^\ell h^{m+s}$
- $\left(\frac{1}{2}\right)^m g^\ell h^m \cdot \left(\frac{1}{2}\right)^{m'} g^{\ell'} h^{m'} = \left(\frac{1}{2}\right)^{m+m'} g^{\ell+\ell'} h^{m+m'}$.

For example, $g \cdot h = 2 \cdot \frac{1}{2} gh$ in $H^{12}(SP_+^2)$ and $h \cdot h = 2 \cdot \frac{1}{2} h^2$ in $H^{16}(SP_+^2)$.

To truncate the above description in Theorem 1.7, we consider the homomorphisms induced by the inclusions of subspaces $\mathbb{H}\mathbb{P}^n \subset \mathbb{H}\mathbb{P}^\infty$, to obtain the following.

Theorem 1.8. *For $n \geq 1$, the integral cohomology ring is*

$$H^*((SP_n^2)_+) \cong H^*(SP_+^2) / (\theta_{n,1}, \theta_{m,2}, t_{i,j} : m \geq n, i \geq n+1)$$

where polynomials $\theta_{m,1}$ and $\theta_{m,2}$ are given by

$$\begin{cases} \theta_{1,1} = g^2 - h \\ \theta_{1,2} = \frac{1}{2} gh \end{cases} \text{ for } m = 1, \text{ and } \begin{cases} \theta_{m,1} = g \theta_{m-1,1} - \theta_{m-1,2} \\ 2 \theta_{m,2} = h \theta_{m-1,1} \end{cases} \text{ for } m \geq 2.$$

If $* > 8n$, then the cohomology group $H^*((SP_n^2)_+)$ is zero for cellular reasons [47], and we will confirm this in Section 8.1.1.

As a typical example, we look at the case $n = 3$.

Example 1.9. *We have*

$$H^*((SP_3^2)_+) \cong H^*((SP_+^2)_+) / (\theta_{3,1}, \theta_{m,2}, t_{i,j} : m \geq 3, i \geq 4).$$

Table 1.1 displays this information in an alternative form.

Table 1.1: $H^* := H^*((SP_3^2)_+; \mathbb{Z})$ with generators

*	0	1	2	3	4
H^*	\mathbb{Z}	0	0	0	\mathbb{Z}
	1				g
*	5	6	7	8	9
H^*	0	0	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}$	0
			$t_{1,1}$	g^2, h	
*	10	11	12	13	14
H^*	0	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}/2$	0
		$t_{2,1}$	$g^3, \frac{1}{2}gh$	$t_{2,2}$	
*	15	16	17	18	19
H^*	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
	$t_{2,3}, t_{3,1}$	$\frac{1}{2}g^2h, \frac{1}{2}h^2$	$t_{3,2}$		$t_{3,3}$
*	20	21	22	23	24
H^*	\mathbb{Z}	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	\mathbb{Z}
	$\frac{1}{4}gh^2$	$t_{3,4}$		$t_{3,5}$	$\frac{1}{4}h^3$

For $* \geq 25$ the cohomology is 0. The relations are

$$g^4 = 4 \cdot \frac{1}{2}g^2h - \frac{1}{2}h^2, \quad \frac{1}{2}g^3h = 3 \cdot \frac{1}{4}gh^2,$$

$$\frac{1}{2}g^4h - 4 \cdot \frac{1}{4}g^2h^2 + \frac{1}{4}h^3 = 0,$$

$$t_{i,j} \alpha = 0 \text{ for any generator } \alpha \neq 1.$$

Chapter 2

Decompositions and Borel Space

The existence of a rational isomorphism $H^*(B_n; \mathbb{Q}) \cong H^*(SP^2(\mathbb{H}\mathbb{P}^n); \mathbb{Q})$, where B_n is the Borel space $B_n := S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n)$ is well known [1, page 38]. This will play an important part in our computation of the integral cohomology ring $H^*(SP^2(\mathbb{H}\mathbb{P}^n))$. In this Chapter we will show how the Borel space B_n and the symmetric square $SP^2(\mathbb{H}\mathbb{P}^n)$ are related geometrically. We decompose these spaces into subspaces, which yield a commutative ladder of cofiber sequences. Our cohomological computation of the symmetric square $SP^2(\mathbb{H}\mathbb{P}^n)$ is based on these commutative ladders.

2.1 The Decomposition of $SP^2(\mathbb{H}\mathbb{P}^n)$

In this section we describe subspaces of the symmetric square SP_n^2 through the chordal distance on $\mathbb{H}\mathbb{P}^n$,

$$d([u], [v]) = \sqrt{1 - |\langle u, v \rangle|^2}, \quad 0 \leq d([u], [v]) \leq 1$$

of Definition 1.2, where u and v are unit vectors in \mathbb{H}^{n+1} .

We shall often write an element of SP_n^2 as $[x, y]$ where $x, y \in \mathbb{H}\mathbb{P}^n$.

Definition 2.1. *There are subspaces of the symmetric square SP_n^2 given by*

$$\Delta_n = \{[u, v] \in SP_n^2 \mid d([u], [v]) = 0\}$$

$$N_n = \{[u, v] \in SP_n^2 \mid 0 \leq d([u], [v]) \leq 1/2\}$$

$$A_n = \{[u, v] \in SP_n^2 \mid d([u], [v]) = 1/2\}$$

$$L_n = \{[u, v] \in SP_n^2 \mid 1/2 \leq d([u], [v]) \leq 1\}$$

$$\Gamma_n = \{[u, v] \in SP_n^2 \mid d([u], [v]) = 1\}.$$

Having defined these subspaces, we may decompose SP_n^2 as

$$SP_n^2 = L_n \cup_{A_n} N_n \tag{2.2}$$

with

$$\Gamma_n \subset L_n, \quad \Delta_n \subset N_n, \quad A_n = L_n \cap N_n,$$

and the decomposition in turn yields a commutative diagram

$$\begin{array}{ccccc} A_n & \longrightarrow & L_n & \longrightarrow & L_n/A_n \\ \downarrow & & \downarrow & & \downarrow \cong \\ N_n & \longrightarrow & SP_n^2 & \longrightarrow & SP_n^2/N_n \end{array} \tag{2.3}$$

with a canonical homeomorphism $L_n/A_n \xrightarrow{\cong} SP_n^2/N_n$.

We should note that:

- there is a homeomorphism $\mathbb{H}\mathbb{P}^n \cong \Delta_n$ given by $z \mapsto [z, z] \in \Delta_n$ for any $z \in \mathbb{H}\mathbb{P}^n$; we call the subspace $\Delta_n \subset SP_n^2$ the *diagonal*
- the complement $SP_n^2 \setminus \Delta_n$ consists of unordered pairs of distinct points on $\mathbb{H}\mathbb{P}^n$.

Some additional properties and descriptions of subspaces of the symmetric square SP_n^2 come from [29, page 585], as follows. We refer the reader to Appendix B.1 for definition of NDR.

Proposition 2.4 ([29], Corollary 4.2). *Assume that M is a closed smooth manifold. A neighbourhood deformation retract NDR of the diagonal M in the symmetric square $SP^2(M)$ is homeomorphic to the fibrewise cone on the projectivised tangent bundle of M .*

The point about this result is that the $\mathbb{Z}/2$ action on $M \times M$ restricts to the fibrewise antipodal action on the normal disk bundle to the diagonal, which is isomorphic to the tangent disk bundle of M .

From [29, Proposition 4.1, Corollary 4.2], we describe below the case $M = \mathbb{H}\mathbb{P}^n$. Let τ be the tangent bundle of $\mathbb{H}\mathbb{P}^n$, and let $D(\tau)$ and $S(\tau)$ be the total spaces of the unit disk and sphere bundles. Then

- (i) the subspace $N_n \subset SP_n^2$ is homeomorphic to the total space of the fibrewise quotient bundle

$$C\mathbb{R}P^{4n-1} \longrightarrow D(\tau)/\mathbb{Z}/2 \longrightarrow \mathbb{H}\mathbb{P}^n, \quad (N_n \cong D(\tau)/\mathbb{Z}/2 \simeq \mathbb{H}\mathbb{P}^n \cong \Delta_n)$$

with fibre the cone $C\mathbb{R}P^{4n-1}$, the projection is the retraction of Proposition 2.4, and the section given by the cone points is its homotopy inverse,

- (ii) the subspace $A_n \subset N_n$ is homeomorphic to the boundary $S(\tau)/\mathbb{Z}/2$ of $D(\tau)/\mathbb{Z}/2$, which is the total space $\mathbb{R}\mathbb{P}(\tau)$ of the real projectivisation of τ with fibre $\mathbb{R}\mathbb{P}^{4n-1}$

$$\mathbb{R}\mathbb{P}^{4n-1} \longrightarrow \mathbb{R}\mathbb{P}(\tau) \longrightarrow \mathbb{H}\mathbb{P}^n.$$

Proposition 2.5 ([14], 5.4.4, page 114). *A pair (X, A) is an NDR if and only if the inclusion $A \hookrightarrow X$ is a closed cofibration.*

We then deduce that the inclusion $\Delta_n \hookrightarrow SP_n^2$ is a closed cofibration by Proposition 2.4 and Proposition 2.5. We shall confirm this property below by applying Illman's triangulation results [26, Theorem 3.6–3.8], which are useful in several ways.

Theorem 2.6 (Illman, 1978). *Let G be a finite group, M a smooth G -manifold, and M' a closed smooth G -submanifold. Then there exists a smooth equivariant triangulation of the pair (M, M') , and any triangulation is unique up to subdivision.*

We always use this theorem with $G = \mathbb{Z}/2$. It give us, for example

Lemma 2.7. *For any $n \geq 1$, the inclusions (1) $\Delta_n \hookrightarrow N_n$, and (2) $N_n \hookrightarrow SP_n^2$ are both cofibrations.*

Proof. (1) Given $N_n = N'_n/\mathbb{Z}/2$, where $N'_n \cong D(\tau)$. Apply Theorem 2.6 with $M = N'_n \subset \mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n$ and $M' = \Delta_n$. Then Δ_n may be realised as a $\mathbb{Z}/2$ equivariant simplicial subcomplex of N'_n . Factoring out $\mathbb{Z}/2$, which acts trivially on Δ_n , gives Δ_n as a simplicial (or CW) subcomplex of N_n , so (1) is true.

(2) Apply Theorem 2.6 with $M = \mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n$ and $M' = N'_n$. The rest of the proof is then the same as (1). \square

We deduce from Lemma 2.7 that the composition $\Delta_n \hookrightarrow SP_n^2$ is also a cofibration [2, page 100].

Lemma 2.8. *For $n \geq 1$, the map $s: SP_n^2/\Delta_n \rightarrow SP_n^2/N_n$ is a homotopy equivalence.*

Proof. The Gluing Lemma (Lemma A.1) [9, 10, 37] applies to the diagram

$$\begin{array}{ccc} \{\bullet\} & \xleftarrow{p} \Delta_n \xrightarrow{i} & SP_n^2 \\ \downarrow & & \downarrow \simeq \\ \{\bullet\} & \xleftarrow{p'} N_n \xrightarrow{i'} & SP_n^2 \end{array} \quad (2.9)$$

since i and i' are closed cofibrations, where p and p' are projections onto one-point spaces, so the corresponding map of pushouts (or colimits) $s: SP_n^2/\Delta_n \rightarrow SP_n^2/N_n$ is a homotopy equivalence. \square

Remark 2.10. *The homotopy equivalence $s: SP_n^2/\Delta_n \rightarrow SP_n^2/N_n$ combined with the homeomorphism of diagram (2.3) gives a canonical homotopy equivalence*

$$SP_n^2/\Delta_n \xrightarrow{\simeq} SP_n^2/N_n \xleftarrow{\cong} L_n/A_n. \quad (2.11)$$

Basically, this map involves collapsing $N_n/\Delta_n \subset SP_n^2/\Delta_n$, which is a contractible subcomplex.

2.2 Comparison of B_n and SP_n^2

In this section we compare the Borel space $B_n = S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n)$ and the symmetric square SP_n^2 . It is known that the space B_n and SP_n^2 are closely related, and the correspondence is well studied. For example, the following properties come from [8, page 371, Proposition 1.1].

Let $\Delta_n \cong \mathbb{H}\mathbb{P}^n$ be the diagonal $\Delta_n \subset SP_n^2$. Then the projection of pairs

$$(S^\infty \times (\mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n), S^\infty \times \Delta_n) \longrightarrow (\mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n, \Delta_n)$$

induces a map of orbit spaces

$$(S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n), \mathbb{R}\mathbb{P}^\infty \times \Delta_n) \longrightarrow (SP_n^2, \Delta_n), \quad (2.12)$$

and the induced homomorphism in Čech cohomology

$$\check{H}^*(SP_n^2, \Delta_n) \longrightarrow \check{H}^*(S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n), \mathbb{R}\mathbb{P}^\infty \times \Delta_n) \quad (2.13)$$

is an isomorphism for any coefficients. Since we are working with CW complexes, where Čech and singular cohomology agree, our approach in this section is to describe the geometry of B_n and SP_n^2 in enough detail to understand why the map of quotient spaces induced by (2.12) is a homotopy equivalence. Then (2.13) may be rewritten as an isomorphism

$$H^*(SP_n^2/\Delta_n) \longrightarrow H^*(S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n)/\Delta_n \times \mathbb{R}\mathbb{P}^\infty) \quad (2.14)$$

in singular cohomology. We will apply this in Chapter 7 and Chapter 8.

Our description of B_n and SP_n^2 includes the decompositions of these spaces as follows.

The Borel space B_n is related to the symmetric square SP_n^2 by a projection

$$\pi : B_n \rightarrow SP_n^2 \quad \text{given by} \quad [v, (z_1, z_2)] \mapsto [z_1, z_2] \quad (2.15)$$

where $v \in S^\infty$ and $z_1, z_2 \in \mathbb{H}\mathbb{P}^n$. This lifts the decomposition of $SP_n^2 = L_n \cup_{A_n} N_n$ to a decomposition $B_n = \widehat{L}_n \cup_{\widehat{A}_n} \widehat{N}_n$, where

$$\widehat{\Delta}_n = \pi^{-1}(\Delta_n), \quad \widehat{N}_n = \pi^{-1}(N_n), \quad \widehat{A}_n = \pi^{-1}(A_n), \quad \text{and} \quad \widehat{L}_n = \pi^{-1}(L_n).$$

So the following properties hold:

- $\widehat{A}_n = \widehat{L}_n \cap \widehat{N}_n$
- \widehat{N}_n is a closed neighbourhood of the diagonal $\widehat{\Delta}_n := \{[v, (z, z)]\} \subset B_n$
- \widehat{L}_n is the complement of the interior of \widehat{N}_n in B_n
- the decomposition yields the commutative diagram

$$\begin{array}{ccccc} \widehat{A}_n & \longrightarrow & \widehat{L}_n & \longrightarrow & \widehat{L}_n/\widehat{A}_n \\ \downarrow & & \downarrow & & \downarrow \cong \\ \widehat{N}_n & \longrightarrow & B_n & \longrightarrow & B_n/\widehat{N}_n \end{array} \quad (2.16)$$

with a canonical homeomorphism $\widehat{L}_n/\widehat{A}_n \xrightarrow{\cong} B_n/\widehat{N}_n$, which lifts the homeomorphism of (2.3).

We shall use the disk bundle $D^{4n} \rightarrow D(\tau) \rightarrow \mathbb{H}\mathbb{P}^n \cong \Delta_n$ to identify \widehat{N}_n with $S^\infty \times_{\mathbb{Z}/2} D(\tau) \subset B_n$. The projection of the disk bundle

$$D^{4n} \rightarrow S^\infty \times_{\mathbb{Z}/2} D(\tau) \rightarrow S^\infty \times_{\mathbb{Z}/2} \Delta_n \quad (2.17)$$

therefore gives a homotopy equivalence $\widehat{N}_n \simeq \widehat{\Delta}_n$, which is compatible with $N_n \simeq \Delta_n$ under projection π .

Remark 2.18. *The map*

$$\mathbb{R}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^n \xrightarrow{\cong} \widehat{\Delta}_n, \quad ([v], z) \mapsto [v, (z, z)]$$

is a homeomorphism, where $[v] \in \mathbb{R}\mathbb{P}^\infty$, $z \in \mathbb{H}\mathbb{P}^n$ and $[v, (z, z)] \in \widehat{\Delta}_n$.

Analogous to Lemma 2.7, there are some additional properties as follows.

Lemma 2.19. *For any $n \geq 1$, the inclusions (1) $\widehat{\Delta}_n \hookrightarrow \widehat{N}_n$ and (2) $\widehat{N}_n \hookrightarrow B_n$ are both cofibrations.*

Proof. (1) For each $n \geq 1$, there is a standard equivariant cell decomposition of S^n with respect to the antipodal action. These are compatible for increasing n , and their union is an equivariant CW structure on S^∞ . Now take the subcomplex $\Delta_n \subset N'_n$ from the proof of Lemma 2.7 and form the subcomplex $S^\infty \times_{\mathbb{Z}/2} \Delta_n \subset S^\infty \times_{\mathbb{Z}/2} N'_n$. These are homeomorphic to $\widehat{\Delta}_n \subset \widehat{N}_n$, so the inclusion map is a cofibration.

(2) Take the subcomplex $N'_n \subset \mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n$, and repeat the method of (1). \square

We deduce from Lemma 2.19 that the composition $\widehat{\Delta}_n \hookrightarrow B_n$ is also a cofibration.

By making an analogy with Lemma 2.8, it follows that the map $b_n: B_n/\widehat{\Delta}_n \rightarrow B_n/\widehat{N}_n$ is a homotopy equivalence. As in Remark 2.10, b_n combines with the homeomorphism of diagram (2.16) to give a canonical homotopy equivalence

$$B_n/\widehat{\Delta}_n \xrightarrow{\cong} B_n/\widehat{N}_n \xleftarrow{\cong} \widehat{L}_n/\widehat{A}_n. \quad (2.20)$$

This is compatible with the equivalence (2.11) under the projection π , and comes from collapsing the contractible subcomplex $\widehat{N}_n/\widehat{\Delta}_n \subset B_n/\widehat{\Delta}_n$.

Notice that π is a double covering away from the diagonal, so $\mathbb{Z}/2$ acts freely on \widehat{A}_n and \widehat{L}_n . This leads to the following additional facts.

Remark 2.21. *The restricted projections $\pi: \widehat{A}_n \rightarrow A_n$ and $\pi: \widehat{L}_n \rightarrow L_n$, which are described by $[v, (z_1, z_2)] \mapsto [z_1, z_2]$ for $z_1 \neq z_2$, are homotopy equivalences, since they appear in fibrations $S^\infty \rightarrow \widehat{A}_n \rightarrow A_n$ and $S^\infty \rightarrow \widehat{L}_n \rightarrow L_n$, and S^∞ is contractible.*

Having described the decompositions of both SP_n^2 and B_n , we now display a very important commutative diagram

$$\begin{array}{ccccc}
 \Delta_n & \xrightarrow{i} & SP_n^2 & \longrightarrow & SP_n^2/\Delta_n \\
 \uparrow \pi & & \uparrow \pi & & \uparrow \pi_n \\
 \widehat{\Delta}_n & \xrightarrow{f} & B_n & \longrightarrow & B_n/\widehat{\Delta}_n
 \end{array} \tag{2.22}$$

where we now know that each row is a cofibre sequence.

Proposition 2.23. *For any $n \geq 1$, the map $\pi_n: B_n/\widehat{\Delta}_n \rightarrow SP_n^2/\Delta_n$ is a homotopy equivalence.*

Proof. By combining the equivalences (2.11) and (2.20), we need only prove that the quotient projection $\pi: \widehat{L}_n/\widehat{A}_n \rightarrow L_n/A_n$ is a homotopy equivalence. So consider the commutative diagram

$$\begin{array}{ccccc}
 \{\bullet\} & \xleftarrow{p} & \widehat{A}_n & \longrightarrow & \widehat{L}_n \\
 \downarrow & & \downarrow \pi & & \downarrow \pi \\
 \{\bullet\} & \xleftarrow{\widehat{p}} & A_n & \longrightarrow & L_n
 \end{array}$$

where p and \widehat{p} are projections onto one-point spaces. By Remark 2.21, both vertical maps π are homotopy equivalences. Also, the inclusions $A_n \hookrightarrow L_n$ and $\widehat{A}_n \hookrightarrow \widehat{L}_n$ are cofibrations, by the same arguments as Lemma 2.7 and 2.19¹. So we can apply the gluing Lemma to the diagram, and deduce that the induced map of pushouts (or colimits) is a homotopy equivalence, as we needed. \square

2.3 Infinite Cases

So far, our main focus has been on sequences of spaces X_n that are constructed from $\mathbb{H}\mathbb{P}^n$. Analogous to the case $SP^2 = \bigcup_n SP_n^2$ [2, Section 5.2], it is convenient to study versions that arise in the same way from $\mathbb{H}\mathbb{P}^\infty$, by forming $X = \bigcup_n X_n$ with the colimit topology. Examples include

$$\widehat{\Delta} = \bigcup_n \widehat{\Delta}_n \cong \mathbb{R}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty \quad \text{and} \quad B = \bigcup_n B_n. \tag{2.24}$$

Each of these is based on the definition of \mathbb{H}^∞ as $\bigcup_n \mathbb{H}^n$ where $\mathbb{H}^n \subset \mathbb{H}^{n+1}$ comes from including (q_1, q_2, \dots, q_n) as $(q_1, q_2, \dots, q_n, 0)$ for any $q_i \in \mathbb{H}$.

¹Or, alternatively, because A_n is the boundary of the smooth manifold L_n , and $\widehat{A}_n = \pi^{-1}(A_n)$ under the fibration $\pi|_{L_n}$ [14, page 117].

Proposition 2.25. *The inclusion $\Delta \hookrightarrow SP^2$ is a cofibration, where $\Delta \cong \mathbb{H}\mathbb{P}^\infty$.*

Proof. Illman's triangulation results may be applied to the $\mathbb{Z}/2$ -equivariant pairs

$$(\Delta_n, \mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n) \subset (\Delta_{n+1}, \mathbb{H}\mathbb{P}^{n+1} \times \mathbb{H}\mathbb{P}^{n+1}),$$

for any $n \geq 1$. After subdivision, if necessary, the commutative diagram

$$\begin{array}{ccc} \Delta_n & \xrightarrow{\subset} & \Delta_{n+1} \\ \downarrow \subset & & \downarrow \subset \\ SP_n^2 & \xrightarrow{\subset} & SP_{n+1}^2 \end{array} \quad (2.26)$$

of subcomplexes arises; so all four inclusions are cofibrations. Also, the square is a pullback because $\Delta_{n+1} \cap SP_n^2$ in SP_{n+1}^2 is homeomorphic to Δ_n . So by [33, Lemma 3.2], the map of colimits $\Delta \hookrightarrow SP^2$ is also a cofibration. \square

For future use, note from the considerations above that the diagram

$$\begin{array}{ccc} B_n/\widehat{\Delta}_n & \xrightarrow{\pi} & SP_n^2/\Delta_n \\ \downarrow \subset & & \downarrow \subset \\ B_{n+1}/\widehat{\Delta}_{n+1} & \xrightarrow{\pi} & SP_{n+1}^2/\Delta_{n+1} \end{array} \quad (2.27)$$

is also commutative for every n .

Remark 2.28. *The inclusion $f: \widehat{\Delta} \hookrightarrow B$ is also a cofibration, by starting with the pairs*

$$S^\infty \times (\Delta_n, \mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n) \subset S^\infty \times (\Delta_{n+1}, \mathbb{H}\mathbb{P}^{n+1} \times \mathbb{H}\mathbb{P}^{n+1}),$$

and following the method of Lemma 2.19.

Therefore, the maps i and f of the commutative diagram

$$\begin{array}{ccccc} \Delta & \xrightarrow{i} & SP^2 & \longrightarrow & SP^2/\Delta \\ \uparrow \pi & & \uparrow \pi & & \uparrow \pi_\infty \\ \widehat{\Delta} & \xrightarrow{f} & B & \longrightarrow & B/\widehat{\Delta} \end{array} \quad (2.29)$$

are closed cofibrations. This diagram is the colimit of the diagrams (2.22) as $n \rightarrow \infty$.

So using the gluing lemma one more time yields

Corollary 2.30. *The map $\pi_\infty: B/\widehat{\Delta} \rightarrow SP^2/\Delta$ in diagram (2.29) is a homotopy equivalence.*

Alternatively, this also follows from Proposition 2.25, Ladder Lemma A.2, and [33, Proposition 2.5].

We have now shown that π_∞^* is an isomorphism in the commutative ladder of long exact sequences induced by (2.29) with any coefficients.

As we shall see, the relationship between B and SP^2 is interesting in other ways. For example, there exist homotopy equivalences $B \simeq \Gamma = \bigcup_n \Gamma_n$ and $\widehat{\Delta} \simeq A = \bigcup_n A_n$; also, the quotient space SP^2/Δ can be viewed as a Thom space over Γ .

Chapter 3

The Classifying Space of $Pin(4)$

This chapter is mainly concerned with the classifying space $BPin(4)$. We will see that both the Borel space B and the space Γ are models for $BPin(4)$. Over the last 50 years, the Pin group and its subgroup $Spin$ have attracted many mathematicians and physicists; as a result there exist different approaches to the Pin and $Spin$ groups depending on their research interests. The standard definitions of these groups are usually given in terms of Clifford algebras; physicists may also work with gamma matrices, otherwise known as Dirac matrices. We shall exploit different interpretations and descriptions of the same group $Pin(4)$. We shall also describe two actions of $Pin(4)$ on \mathbb{R}^4 , and consider the associated vector bundles over $BPin(4)$.

3.1 Pin and $Spin$ Groups

In this section we review certain properties of the groups $Pin(n)$ and $Spin(n)$. These groups act naturally on \mathbb{R}^n , in a way that is usually defined in terms of multiplication in the Clifford algebra associated with \mathbb{R}^n . There are many references on these topics, which include [3, 19, 34, 49].

We shall recall some classical low dimensional results as follows.

Proposition 3.1. [50, Cor.13.60] *There are isomorphisms*

$$Spin(1) \cong O(1) \cong S^0 \cong \mathbb{Z}/2$$

$$Spin(2) \cong U(1) \cong S^1$$

$$Spin(3) \cong Sp(1) \cong S^3$$

$$Spin(4) \cong Sp(1) \times Sp(1) \cong S^3 \times S^3$$

$$Spin(5) \cong Sp(2)$$

of compact Lie groups.

From [5], we have some useful and important facts:

- the inclusion $Spin(4) \hookrightarrow Spin(5)$ can be described as the embedding of the diagonal matrices $Sp(1) \times Sp(1) \rightarrow Sp(2)$
- the normaliser of $Sp(1) \times Sp(1)$ in $Sp(2)$ consists of all matrices

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix},$$

where a and b are unit quaternions

- the normaliser of $Sp(1) \times Sp(1)$ in $Sp(2)$ is the wreath product of $Sp(1)$ and $\mathbb{Z}/2$; that is, the semidirect product of $Sp(1) \times Sp(1)$ with $\mathbb{Z}/2$ (Appendix B.2). In this context, $\mathbb{Z}/2$ is generated by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The Pin and $Spin$ groups are often described in terms of Clifford algebra. The real Clifford algebra Cl_n associated to \mathbb{R}^n with the orthonormal basis $\{e_1, \dots, e_n\}$ is spanned by

$$\{e_{i_1}e_{i_2}\dots e_{i_k} \mid 0 \leq k \leq n \text{ and } i_1 < \dots < i_k\}$$

so the dimension of Cl_n is 2^n . More precisely [19], given a vector space \mathbb{R}^{p+q} with a symmetric bilinear form $Q_b: \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \rightarrow \mathbb{R}$ and associated non-degenerate quadratic form $Q(v) = Q_b(v, v)$ where Q is given by

$$Q(x_1, \dots, x_{p+q}) = x_1^2 + \dots + x_p^2 - (x_{p+1}^2 + \dots + x_{p+q}^2),$$

then the Clifford algebra associated with \mathbb{R}^{p+q} and Q is a real algebra $Cl_{p,q}$ together with a linear map $i: \mathbb{R}^{p+q} \rightarrow Cl_{p,q}$ satisfying the condition¹ $i(v)^2 = Q(v) \cdot 1$ for any

¹Lawson and Michelson [32] use the condition $i(v)^2 = -Q(v)$ to define the Clifford algebras, and our $Cl_{p,q}$ is their $Cl_{q,p}$ as a consequence of using the opposite sign convention.

$v \in \mathbb{R}^{p+q}$. The standard notations for the Pin and $Spin$ groups associated to $Cl_{p,q}$ are $Pin(p, q)$ and $Spin(p, q)$, and they are given by [19]

$$Spin(p, q) = \{x \in Cl_{p,q}^0 \mid xvx^{-1} \in \mathbb{R}^{p+q} \text{ for any } v \in \mathbb{R}^{p+q}, N(x) = 1\}$$

$$Pin(p, q) = \{x \in Cl_{p,q} \mid xvt(x)N(x) \in \mathbb{R}^{p+q} \text{ for any } v \in \mathbb{R}^{p+q}, N(x) = \pm 1\},$$

in particular

$$Pin(0, n) = \{x \in Cl_{p,q} \mid xvt(x) \in \mathbb{R}^n \text{ for any } v \in \mathbb{R}^n, N(x) = 1\}.$$

Here, $Cl_{p,q}^0$ is the subalgebra generated by the monomials $\{e_{i_1}e_{i_2}e_{i_3}\dots e_{i_{2k}}\}$ of even grading, and $t: Cl_{p,q} \rightarrow Cl_{p,q}$ given by $t(e_{i_1}e_{i_2}\dots e_{i_k}) = e_{i_k}e_{i_{k-1}}\dots e_{i_1}$ is the *reversion* map. Also, $N(x) = x\bar{x}$ is the *Clifford norm* where \bar{x} is the *Clifford conjugation* of x , given by $\bar{e}_i = -e_i$, $\overline{e_{i_1}e_{i_2}\dots e_{i_k}} = (-1)^k e_{i_k}\dots e_{i_2}e_{i_1}$ where $1 \leq i_1 < i_2 < \dots < i_k \leq i_n$.

It is well known [11, 19, 50] that there exists an isomorphism $Spin(p, q) \cong Spin(q, p)$ whereas $Pin(p, q)$ and $Pin(q, p)$ are *not* isomorphic in general. For example [34], $Pin(0, 1) \cong \mathbb{Z}/4$ and $Pin(1, 0) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. However [6, 31], there *is* a less well-known isomorphism $Pin(0, 4) \cong Pin(4, 0)$; and also [6], the group $Pin(0, 4)$ is the semidirect product of $Spin(0, 4)$ with $\mathbb{Z}/2$. Often, $Pin(0, n)$ associated to the negative definite form on \mathbb{R}^n is called $Pin^-(n)$ whilst $Pin(n, 0)$ associated to the positive definite form on \mathbb{R}^n is called $Pin^+(n)$. We shall write $Pin(n)$ for $Pin(0, n)$ and $Spin(n)$ for $Spin(0, n) \cong Spin(n, 0)$.

By combining results from the above literature, we arrive at the following.

Remark 3.2. *The normaliser of $Sp(1) \times Sp(1)$ in $Sp(2)$ (or equivalently, of $Spin(4)$ in $Spin(5)$) is isomorphic to $Pin(4)$.*

3.2 The Actions of $Spin(4)$, $Pin(4)$ and $Spin(5)$

In this section we describe standard actions of $Spin(4) \subset Pin(4) \subset Spin(5)$ on \mathbb{R}^4 and \mathbb{R}^5 . The references include [5], which refers to the normaliser of $Sp(1) \times Sp(1)$ in $Sp(2)$, or equivalently, to the wreath product of $Sp(1)$ and $\mathbb{Z}/2$.

- Using the isomorphism $Spin(4) \cong Sp(1) \times Sp(1)$, the action of $Spin(4)$ on $\mathbb{R}^4 \cong \mathbb{H}$ can be given by

$$h \mapsto ahb^{-1} = ah\bar{b}$$

where $(a, b) \in Sp(1) \times Sp(1)$ and $h \in \mathbb{H}$

- using the isomorphism $Spin(5) \cong Sp(2)$, the action of $Spin(5)$ on \mathbb{R}^5 can be given by conjugation $AMA^{-1} = AMA^*$ on quaternionic Hermitian matrices of trace 0, $M = \begin{pmatrix} r & h \\ \bar{h} & -r \end{pmatrix}$, where $A \in Sp(2)$ and $(r, h) \in \mathbb{R}^5 \cong \mathbb{R} \oplus \mathbb{H}$
- using the isomorphism $Spin(4) \cong Sp(1) \times Sp(1)$ and the fact that $Pin(4)$ is the normaliser of $Sp(1) \times Sp(1)$ in $Sp(2)$, the action of $Pin(4)$ on $\mathbb{R}^4 \cong \mathbb{H}$ can be given by writing $Pin(4) = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid a, b \in Sp(1) \cong S^3 \}$. Calculating gives

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} r & h \\ \bar{h} & -r \end{pmatrix} \begin{pmatrix} \bar{a} & 0 \\ 0 & \bar{b} \end{pmatrix} = \begin{pmatrix} r & ah\bar{b} \\ b\bar{h}\bar{a} & -r \end{pmatrix},$$

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} r & h \\ \bar{h} & -r \end{pmatrix} \begin{pmatrix} 0 & \bar{b} \\ \bar{a} & 0 \end{pmatrix} = \begin{pmatrix} -r & a\bar{h}\bar{b} \\ bh\bar{a} & r \end{pmatrix}$$

where $r \in \mathbb{R}, h \in \mathbb{H} \cong \mathbb{R}^4$ and a, b are unit quaternions. So $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ acts on h by $h \mapsto a\bar{h}\bar{b}$, where $h \in \mathbb{H}$, and in particular, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acts on h by $h \mapsto \bar{h}$. As required, the action of $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is the action of $Spin(4)$, that is, $h \mapsto ah\bar{b}$.

3.3 Stiefel Manifolds and Grassmannians

In this section we summarise some basic properties of the quaternionic Steifel manifolds and Grassmannians; references include [22].

The quaternionic Steifel manifold $V_{\mathbb{H}}(k, m)$ is the space of orthonormal k -frames in \mathbb{H}^m , and the quaternionic Grassmannian $Gr_{\mathbb{H}}(k, m)$ is the space of k -dimensional subspaces of \mathbb{H}^m . These spaces have the structures of CW complexes. The infinite dimensional cases $m = \infty$ are considered as the colimits $V_{\mathbb{H}}(k, \infty) = \bigcup_m V_{\mathbb{H}}(k, m)$ and $Gr_{\mathbb{H}}(k, \infty) = \bigcup_m Gr_{\mathbb{H}}(k, m)$. Related to these spaces we have fibre bundles

$$Sp(k) \rightarrow V_{\mathbb{H}}(k, m) \rightarrow Gr_{\mathbb{H}}(k, m) \tag{3.3}$$

$$Sp(k) \rightarrow V_{\mathbb{H}}(k, \infty) \rightarrow Gr_{\mathbb{H}}(k, \infty) \tag{3.4}$$

where $Gr_{\mathbb{H}}(k, -)$ is identified with $V_{\mathbb{H}}(k, -)/Sp(k)$. We shall describe the action of $Sp(2)$ on $V_{\mathbb{H}}(2, -)$ in Section 3.4. The infinite dimensional Stiefel manifold $V_{\mathbb{H}}(k, \infty)$ is contractible [22], so the fibre bundle (3.4) is a model for the universal G -bundle $G \rightarrow EG \rightarrow BG$ where $G = Sp(2)$ acts freely on a contractible space EG .

Definition 3.5 ([40]). *The base space BG of the universal G -bundle $G \rightarrow EG \rightarrow BG$ is called a classifying space of G .*

For all our examples, BG can be chosen to be a CW complex, so it is unique up to homotopy equivalence [40, Cor 6.11]. In certain cases, it is necessary to consider more than one model at a time, and we need to take care with their relationship.

3.4 $BSpin(4)$, $BPin(4)$ and $BSpin(5)$

In this section we describe models for the classifying spaces of the compact Lie groups $Spin(4) \cong Sp(1) \times Sp(1)$, $Pin(4)$ and $Spin(5) \cong Sp(2)$.

For convenience we sometimes write:

- V_{n+1} for $V_{\mathbb{H}}(2, n+1)$
- V for $V_{\mathbb{H}}(2, \infty) = \bigcup_m V_{\mathbb{H}}(2, m)$.

The inclusions $Sp(1) \times Sp(1) \hookrightarrow Pin(4) \hookrightarrow Sp(2)$ induce the commutative diagrams

$$\begin{array}{ccccc}
 Sp(1) \times Sp(1) & \longrightarrow & Pin(4) & \longrightarrow & Sp(2) \\
 \downarrow & & \downarrow & & \downarrow \\
 V_{n+1} & \xlongequal{\quad\quad\quad} & V_{n+1} & \xlongequal{\quad\quad\quad} & V_{n+1} \\
 \downarrow & & \downarrow & & \downarrow \\
 V_{n+1}/Sp(1) \times Sp(1) & \longrightarrow & V_{n+1}/Pin(4) & \longrightarrow & V_{n+1}/Sp(2)
 \end{array}$$

and

$$\begin{array}{ccccc}
 Sp(1) \times Sp(1) & \longrightarrow & Pin(4) & \longrightarrow & Sp(2) \\
 \downarrow & & \downarrow & & \downarrow \\
 V & \xlongequal{\quad\quad\quad} & V & \xlongequal{\quad\quad\quad} & V \\
 \downarrow & & \downarrow & & \downarrow \\
 V/Sp(1) \times Sp(1) & \longrightarrow & V/Pin(4) & \longrightarrow & V/Sp(2)
 \end{array}$$

of bundle maps. The second diagram is the union of the first, over all n .

We write

$$\begin{aligned}
 Sp(2) &= \{A \in GL_{\mathbb{H}}(2) \mid AA^* = A^*A = I\} \\
 V &= \{(u \ v) \mid u, v \in \mathbb{H}^{\infty}, \langle u, v \rangle = 0, |u| = |v| = 1\},
 \end{aligned}$$

so $(u v)$ is in V if and only if $(u v)^*(u v) = I$. Then define the action $\phi: V \times Sp(2) \rightarrow V$ by the matrix product

$$\phi((u v), A) = (u v)A$$

for all $(u v) \in V$ and $A \in Sp(2)$. This action is well defined, because

$$((u v)A)^*(u v)A = A^*(u v)^*(u v)A = A^*A = I.$$

The same formulae clearly restrict to the case V_{n+1} .

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $(u v)A = (ua + vc \quad ub + vd) = (u v)$, then A is the identity matrix; that is, the action of $Sp(2)$ on V is free. Also, $Pin(4)$ and $Sp(1) \times Sp(1)$ act freely on the contractible space V since they are subgroups of $Sp(2)$. So we may state

Proposition 3.6. *The spaces $V/(Sp(1) \times Sp(1))$, $V/Pin(4)$, $V/Sp(2)$ are models for the classifying spaces $B(Sp(1) \times Sp(1))$, $BPin(4)$, $BSp(2)$, respectively.*

We should recall from (3.4) that $V/Sp(2)$ is identified with $Gr_{\mathbb{H}}(2, \infty)$.

If G is a semidirect product of N by H , then there is a short exact sequence (Appendix B.2), $1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$. Thus for the case $G = Pin(4)$, $N = Spin(4)$, $H = \mathbb{Z}/2$, we have the exact sequence

$$1 \rightarrow Spin(4) \rightarrow Pin(4) \rightarrow \mathbb{Z}/2 \rightarrow 1,$$

which gives rise to the fibration p of classifying spaces

$$BSpin(4) \rightarrow BPin(4) \xrightarrow{p} B\mathbb{Z}/2$$

by [43, Theorem 11.4].

Let $Pin(4)$ act on S^∞ by projecting onto $\mathbb{Z}/2$, and applying the antipodal map. Then we notice that $S^\infty \times_{Pin(4)} V$ is homotopy equivalent to $V/Pin(4)$, since there is a fibration $S^\infty \rightarrow S^\infty \times_{Pin(4)} V \rightarrow V/Pin(4)$. Also there is a $\mathbb{Z}/2$ -equivariant homotopy equivalence

$$V/(Sp(1) \times Sp(1)) \xrightarrow{\cong} (S^{4\infty+3} \times S^{4\infty+3})/(Sp(1) \times Sp(1)) \cong \mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty,$$

induced by including pairs of orthonormal vectors into all pairs of unit vectors. This yields an equivalence $S^\infty \times_{Pin(4)} V \xrightarrow{\cong} B = S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)$. Assembling these

we have a commutative diagram

$$\begin{array}{ccccc}
 B(Sp(1) \times Sp(1)) & \longrightarrow & V/Pin(4) & \longrightarrow & B\mathbb{Z}/2 \\
 \uparrow = & & \uparrow \simeq & & \uparrow \\
 V/(Sp(1) \times Sp(1)) & \longrightarrow & S^\infty \times_{Pin(4)} V & \longrightarrow & S^\infty/\mathbb{Z}/2 \\
 \downarrow \simeq & & \downarrow \simeq & & \parallel \\
 \mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty & \longrightarrow & B & \longrightarrow & S^\infty/\mathbb{Z}/2
 \end{array} \quad (3.7)$$

which shows that the Borel space B is an alternative model for $BPin(4)$. It also shows that we may assume the homotopy equivalence between them identifies $\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty$ with our model for $B(Sp(1) \times Sp(1))$, and identifies the natural projection $B \rightarrow \mathbb{R}\mathbb{P}^\infty$ with the map $BPin(4) \rightarrow B\mathbb{Z}/2$ obtained by projecting $Pin(4)$ onto $\mathbb{Z}/2$.

Remark 3.8. We may think of the space $S^\infty \times_{Pin(4)} V$ in diagram (3.7) as $\widehat{\Gamma}$, because it is homeomorphic to $S^\infty \times_{\mathbb{Z}/2} V/(Sp(1) \times Sp(1))$. Then our two models for $BPin(4)$ are related by

$$B \leftarrow \widehat{\Gamma} \xrightarrow{\pi} \Gamma := V/Pin(4),$$

where the map $\widehat{\Gamma} \rightarrow \Gamma$ is the restriction map $\pi|_{\widehat{\Gamma}}$ of π (2.15), and $\widehat{\Gamma} \hookrightarrow B$ lies above the inclusion of the subspace $\Gamma \subset SP^2$.

3.5 On Vector Bundles over $BPin(4)$

This section concerns vector bundles over $BPin(4)$ that are closely related to certain $Spin(m)$ bundles. Some useful properties of the universal $Spin(m)$ bundles when $m = 3, 4, 5$ were described in [5] whilst the characteristics of Pin^\pm structures on vector bundles were studied in [30].

First we shall describe the vector bundle θ that is associated to the action of $Pin(4)$ on \mathbb{R}^4 as defined in Section 3.2. Recall that V stands for $V_{\mathbb{H}}(2, \infty)$ as in Section 3.3, and that

$$\Gamma = V/Pin(4) = V/\sim \quad (3.9)$$

is a model for $BPin(4)$, where $(u, v) \sim (ua, vb) \sim (v, u)$ for $a, b \in S^3$.

Definition 3.10. The real 4-plane vector bundle

$$\mathbb{R}^4 \rightarrow \mathbb{R}(\theta) \rightarrow BPin(4) \quad (3.11)$$

has total space $\mathbb{R}(\theta)$ given by

$$\mathbb{R}(\theta) = V \times_{Pin(4)} \mathbb{R}^4 = V \times \mathbb{R}^4 / \sim,$$

where $((u, v), h) \sim ((v, u), \bar{h}) \sim ((ua, vb), \bar{a}hb)$ for $h \in \mathbb{H} \cong \mathbb{R}^4$, and the projection map is given by

$$V \times_{Pin(4)} \mathbb{R}^4 \rightarrow V/Pin(4), \quad [(u, v), h] \mapsto [u, v].$$

We shall confirm below (and again in Section 6.2.2) that the vector bundle θ is the universal $Pin^-(4)$ bundle. Since $Pin^+(4)$ and $Pin^-(4)$ are isomorphic Lie groups as mentioned in Section 3.1, Γ is also a model for $BPin^+(4)$. The following question then arises naturally: “What is the $Pin^+(4)$ action on \mathbb{R}^4 , that gives rise to the universal $Pin^+(4)$ bundle over Γ ?”

Proposition 3.12. *The $Pin^+(4)$ action on $\mathbb{R}^4 \cong \mathbb{H}$ can be given by*

- $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ acts on $h \in \mathbb{H}$ by $h \mapsto ah\bar{b}$
- $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ acts on $h \in \mathbb{H}$ by $h \mapsto -a\bar{h}\bar{b}$,

and in particular, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acts on h by $h \mapsto -\bar{h}$.

Proof. Let $\rho_+ : Pin(4) \rightarrow O(4)$ be the stated representation, and $\mathbf{r} \in O(4)$ be reflection across the hyperplane e_1^\perp . Then the preimage $\rho_+^{-1}(\{I, \mathbf{r}\})$ consists of the matrices

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}.$$

This subgroup is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, so it follows from [30, page 180] that ρ_+ is the standard representation of $Pin^+(4)$, (and not $Pin^-(4)$). \square

Remark 3.13. *Let $\rho_- : Pin(4) \rightarrow O(4)$ be the representation of $Pin(4)$ in Section 3.2. Then applying the same procedure gives*

$$\rho_-^{-1}(\{I, \mathbf{r}\}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

which is isomorphic to $\mathbb{Z}/4$. So by [30, page 180], ρ_- gives the $Pin^-(4)$ action, and θ of Definition 3.10 must be the universal $Pin^-(4)$ bundle.

We may also check the action of $Pin^+(4)$ of Proposition 3.12 against Kirby and Taylor's characterisation of Pin^\pm structures on a given vector bundle. To do this, we write $Pin^-(\xi)$, $Pin^+(\xi)$ and $Spin(\xi)$ respectively for the sets of all possible Pin^- , Pin^+ and $Spin$ structures on an n -plane vector bundle ξ . By definition, each of these structures consists of a homotopy class of lifts of its classifying map from $BO(n)$ to $BPin^\pm(n)$ or $BSpin(n)$.

Lemma 3.14 ([30], Lemma 1.7). *Let ξ be a vector bundle $\mathbb{R}^n \rightarrow E \rightarrow X$. Then there exist natural bijections*

$$Pin^-(\xi) \leftrightarrow Spin(\xi \oplus \det \xi) \quad (3.15)$$

$$Pin^+(\xi) \leftrightarrow Spin(\xi \oplus 3 \det \xi) \quad (3.16)$$

where $\det \xi$ is the determinant line bundle associated to ξ .

Before considering the case Pin^+ (3.16), it is helpful to interpret (3.15) first, in terms of our Definition 3.10. Our computations in Section 3.2 show that the pull back of the universal $Spin(5)$ bundle along $BPin(4) \rightarrow BSpin(5)$ is a Whitney sum. It is isomorphic to $\theta \oplus \lambda$, where λ is the line bundle coming from the action of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on \mathbb{R} by $r \mapsto -r$. This action defines $\det \theta$. In other words, our $Pin^-(4)$ structure on θ corresponds to the pull back $Spin(5)$ structure on $\theta \oplus \det \theta$ under (3.15).

We shall take an analogous approach to the case Pin^+ . Let q be a pure quaternion (so $\bar{q} = -q$), and let h be any quaternion $h \in \mathbb{H} \cong \mathbb{R}^4$. Then, we find below that the $Pin^+(4)$ action as in Proposition 3.12 agrees with the action of $Pin(4)$ on $\mathbb{R}^4 \cong \mathbb{H}$ deduced from conjugation on quaternionic Skew-Hermitian matrices

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \bar{q} & h \\ -\bar{h} & q \end{pmatrix} \begin{pmatrix} \bar{a} & 0 \\ 0 & \bar{b} \end{pmatrix} = \begin{pmatrix} \bar{q} & ah\bar{b} \\ -b\bar{h}\bar{a} & q \end{pmatrix}$$

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} \bar{q} & h \\ -\bar{h} & q \end{pmatrix} \begin{pmatrix} 0 & \bar{b} \\ \bar{a} & 0 \end{pmatrix} = \begin{pmatrix} q & -a\bar{h}\bar{b} \\ bh\bar{a} & \bar{q} \end{pmatrix}$$

for $\mathbb{R}^7 \cong \mathbb{R}^3 \oplus \mathbb{R}^4 \cong Pure(\mathbb{H}) \oplus \mathbb{H}$. This illustrates (3.16), by showing how the pull back of the universal $Spin(7)$ bundle along $BPin(4) \rightarrow BSpin(7)$ splits as the universal $Pin^+(4)$ bundle plus 3 copies of its determinant line bundle.

Remark 3.17. *The action of $Pin^-(4)$ of Section 3.2 has occasionally occurred in the literature, for example in Dupont [16, chapter 6, page 48]. On the other hand, we have not been able to find any reference to the action of $Pin^+(4)$ of Proposition 3.12.*

Chapter 4

The Cohomology Ring of $BPin(4)$

In this chapter we describe the cohomology ring of $BPin(4)$ via two models B and Γ . Fred W. Roush [52] computed the integral cohomology of the Borel space B , however, his result has not been published. We shall extend his work and describe the integral cohomology of B_n . To compute the cohomology of Γ with \mathbb{Z} and $\mathbb{Z}/2$ coefficients, we consider the fibre bundle $\mathbb{R}P^4 \rightarrow \Gamma \rightarrow Gr_{\mathbb{H}}(2, \infty)$. We also describe the integral cohomology of Γ_n , and therefore the integral cohomology ring of the configuration space of unordered pairs of distinct points in $\mathbb{H}P^n$. Our calculation uses a quaternionic version of a symmetric orthogonalisation procedure that originally appeared in the literature [35] of chemical physics.

4.1 The Cohomology of Γ and Γ_n

In this section we shall compute the cohomology rings of Γ and Γ_n .

From [5, 49] we have that $Spin(5)/Spin(4) \cong S^4$ and $Sp(2)/Pin(4) \cong \mathbb{R}P^4$. Then analogous to the complex case in [57] there is a commutative diagram

$$\begin{array}{ccccc}
 Sp(1) \times Sp(1) & \longrightarrow & Sp(2) & \longrightarrow & Sp(2)/Sp(1) \times Sp(1) \cong S^4 \\
 \downarrow & & \parallel & & \downarrow \\
 Pin(4) & \longrightarrow & Sp(2) & \longrightarrow & Sp(2)/Pin(4) \cong \mathbb{R}P^4
 \end{array} \tag{4.1}$$

of fiber bundles, which in turn induces two commutative diagrams

$$\begin{array}{ccccc}
 S^4 & \longrightarrow & V_{n+1}/Sp(1) \times Sp(1) & \longrightarrow & Gr_{\mathbb{H}}(2, n+1) \\
 \downarrow c & & \downarrow c'_n & & \parallel \\
 \mathbb{R}P^4 & \longrightarrow & \Gamma_n & \longrightarrow & Gr_{\mathbb{H}}(2, n+1)
 \end{array} \tag{4.2}$$

$$\begin{array}{ccccc}
 S^4 & \longrightarrow & V/Sp(1) \times Sp(1) & \longrightarrow & Gr_{\mathbb{H}}(2, \infty) \\
 \downarrow c & & \downarrow c' & & \parallel \\
 \mathbb{R}P^4 & \longrightarrow & \Gamma & \longrightarrow & Gr_{\mathbb{H}}(2, \infty)
 \end{array} \tag{4.3}$$

of fibre bundles, where $\Gamma = \bigcup_n \Gamma_n$ and $Gr_{\mathbb{H}}(2, \infty) = \bigcup_n Gr_{\mathbb{H}}(2, n+1)$ are the colimits, and c, c'_n, c' are double coverings. So for any diagram

$$\begin{array}{ccccc}
 \mathbb{R}P^4 & \longrightarrow & \Gamma_n & \xrightarrow{\ell_n} & Gr_{\mathbb{H}}(2, n+1) \\
 \parallel & & \downarrow & & \downarrow \\
 \mathbb{R}P^4 & \longrightarrow & \Gamma & \xrightarrow{\ell} & Gr_{\mathbb{H}}(2, \infty)
 \end{array} \tag{4.4}$$

mapping into the colimit, the upper bundle is the restriction, and therefore the pull back, of the lower bundle.

We should recall from Section 3.4 that the space Γ_n is identified with $V_{n+1}/Pin(4) = V_{n+1}/\sim$, where $(u, v) \sim (v, u) \sim (ua, vb)$ for $a, b \in S^3$. For the infinite case we take colimits and identify Γ with $V/Pin(4) = \bigcup_n V_{n+1}/Pin(4)$. Similarly, the space $V_{n+1}/Sp(1) \times Sp(1)$ is V_{n+1}/\sim , where $(u, v) \sim (ua, vb)$ for $a, b \in S^3$. For the infinite case we take the colimit $V/Sp(1) \times Sp(1) = \bigcup_n V_{n+1}/Sp(1) \times Sp(1)$.

We shall use the fibre bundles from (4.4) to compute the cohomology rings of Γ and Γ_n . As input we need the following standard facts.

From [22, page 212–214],

$$H^*(\mathbb{R}P_+^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[a], \quad H^*(\mathbb{R}P_+^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[a]/(a^{n+1}) \tag{4.5}$$

where $|a| = 1$, and

$$H^*(\mathbb{R}P_+^\infty; \mathbb{Z}) \cong \mathbb{Z}[w]/(2w), \quad |w| = 2 \tag{4.6}$$

$$H^*(\mathbb{R}P_+^{2k}; \mathbb{Z}) \cong \mathbb{Z}[w]/(2w, w^{k+1}), \quad |w| = 2 \tag{4.7}$$

$$H^*(\mathbb{R}P_+^{2k+1}; \mathbb{Z}) \cong \mathbb{Z}[w, v]/(2w, w^{k+1}, v^2, wv), \quad |w| = 2, \quad |v| = 2k+1. \tag{4.8}$$

Reduction mod 2 is the injective ring homomorphism

$$\rho: H^*(\mathbb{R}P^\infty; \mathbb{Z}) \rightarrow H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \quad \text{given by} \quad w \mapsto a^2, \tag{4.9}$$

for $* > 0$. The unique non-zero element of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ is called the 1st *Stiefel-Whitney class* of the tautological line bundle λ over $\mathbb{R}P^\infty$, written $a = w_1(\lambda)$.

For any $n \geq 1$, a real n -plane bundle ζ over the base space X , has Stiefel-Whitney classes $w_i(\zeta) \in H^i(X_+; \mathbb{Z}/2)$, $i \geq 0$, which satisfy the following [36, 40, 42].

$$(1) \quad w_0(\zeta) = 1, \text{ and } w_i(\zeta) = 0 \text{ when } i > n \quad (4.10)$$

$$(3) \quad w_i(\zeta \oplus \mathbb{R}^1) = w_i(\zeta) \text{ where } \mathbb{R}^1 \text{ is the trivial line bundle over } X \quad (4.11)$$

$$(4) \quad w_i(\zeta \oplus \eta) = \sum_{j=0}^i w_j(\zeta) \cdot w_{i-j}(\eta), \quad (4.12)$$

for any other real vector bundle η over X . The *total Stiefel-Whitney class* $w(\zeta)$ is defined to be $\sum_{i \geq 0} w_i(\zeta) \in H^*(X_+; \mathbb{Z}/2)$. So (4.12) implies the formula

$$w(\zeta \oplus \eta) = w(\zeta) \cdot w(\eta). \quad (4.13)$$

Similarly, a quaternionic vector bundle κ over X has *Symplectic Pontryagin classes* $p_i(\kappa) \in H^{4i}(X_+)$, $i \geq 0$, with analogous properties [40]. It is important to note that $\rho(p_i(\kappa)) = w_{4i}(\kappa)$ in $H^{4i}(X_+; \mathbb{Z}/2)$.

Other essential input includes the cohomology rings for $Gr_{\mathbb{H}}(2, n+1)$, and the model $Gr_{\mathbb{H}}(2, \infty)$ for the classifying space $BSp(2)$.

Assume $R = \mathbb{Z}$ or $\mathbb{Z}/2$. From [40], we have

$$H^*(Gr_{\mathbb{H}}(2, \infty)_+; R) \cong R[p_1, p_2]$$

where $|p_i| = 4i$ for $i = 1, 2$, and $p_i = p_i(\xi)$ for the tautological quaternionic 2-plane bundle ξ over $Gr_{\mathbb{H}}(2, \infty)$. Also

$$H^*(Gr_{\mathbb{H}}(2, n+1)_+; R) \cong R[p_1, p_2, \bar{p}_1, \dots, \bar{p}_{n-1}] / \mathcal{I}$$

where $\bar{p}_j \in H^{4j}(Gr_{\mathbb{H}}(2, n+1); R)$ is $p_j(\xi^\perp)$ and the ideal \mathcal{I} is given by the relation

$$(1 + p_1 + p_2)(1 + \bar{p}_1 + \bar{p}_2 + \dots + \bar{p}_{n-1}) = 1.$$

This equation arises from $\xi \oplus \xi^\perp \cong \mathbb{H}^{n+1}$, where \mathbb{H}^{n+1} is the trivial $n+1$ -plane quaternionic bundle over the Grassmannian. Equivalently, from [48]

$$H^*(Gr_{\mathbb{H}}(2, n+1)_+; R) \cong R[p_1, p_2] / (q_n, q_{n+1}) \quad (4.14)$$

where q_j is the j -th complete symmetric polynomial in z_1, z_2 for $p_1 = z_1 + z_2$, $p_2 = z_1 z_2$.

4.1.1 The Integral Cohomology

To compute the cohomology of Γ and Γ_n it is helpful to recall from the literature some results on the Serre spectral sequence (Appendix B.3) for a fibration $F \xrightarrow{i} E \xrightarrow{p} M$ when the fundamental group of the base M is trivial. We assume below that R is a commutative ring with unity, and that the spectral sequence is denoted by

$$E_2^{*,*} = H^*(M_+; H^*(F_+; R)) \Rightarrow H^*(E_+; R).$$

Lemma 4.15 ([40], Lemma 2.15). *Suppose the local coefficient ring is trivial and the base space M and the fibre F of the fibration are path connected, and either $H^*(M_+; R)$ or $H^*(F_+; R)$ is a finitely generated free R -module in each dimension. Then the product $E_2^{p,0} \otimes E_2^{0,q} \rightarrow E_2^{p,q}$ induces an isomorphism*

$$E_2^{*,*} \cong H^*(M_+; R) \otimes_R H^*(F_+; R)$$

for any $p, q \geq 0$.

Theorem 4.16 ([40], Theorem 4.2). *Let the base space M and the fibre F of the fibration $F \xrightarrow{i} E \xrightarrow{p} M$ be path connected. Suppose that the local coefficient ring is trivial and all differentials are zero, $d_k = 0$, $k \geq 2$, that is $E_2 = E_\infty$. Then*

$$\begin{aligned} p^* : H^*(M_+; R) &\rightarrow H^*(E_+; R) \quad \text{is a monomorphism, and} \\ i^* : H^*(E_+; R) &\rightarrow H^*(F_+; R) \quad \text{is an epimorphism.} \end{aligned}$$

Lemma 4.15 and Theorem 4.16 are applicable to the bundle $\mathbb{R}\mathbb{P}^4 \xrightarrow{i} \Gamma \xrightarrow{\ell} Gr_{\mathbb{H}}(2, \infty)$ with $R = \mathbb{Z}$, since the base $Gr_{\mathbb{H}}(2, \infty)$ is simply connected, its integral cohomology is the polynomial ring $\mathbb{Z}[p_1, p_2]$, and $H^*(\mathbb{R}\mathbb{P}^4) = 0$ in odd dimensions by (4.7).

The homotopy exact sequence of the bundle shows that $i_* : \pi_1(\mathbb{R}\mathbb{P}^4) \cong \mathbb{Z}/2 \rightarrow \pi_1(\Gamma)$ is an isomorphism.

Proposition 4.17. *The integral cohomology ring of Γ is*

$$H^*(\Gamma_+) \cong \mathbb{Z}[\tilde{x}, \tilde{y}, w]/(2w, \tilde{x}w + w^3), \quad |\tilde{x}| = 4, |\tilde{y}| = 8, |w| = 2.$$

Proof. We shall consider the SSS for the bundle $\mathbb{R}\mathbb{P}^4 \xrightarrow{i} \Gamma \xrightarrow{\ell} Gr_{\mathbb{H}}(2, \infty)$, where $E_2^{p,q} \cong H^p(Gr_{\mathbb{H}}(2, \infty)_+; H^q(\mathbb{R}\mathbb{P}_+^4))$. Then

$$E_2^{*,*} \cong H^*(Gr_{\mathbb{H}}(2, \infty)_+) \otimes_{\mathbb{Z}} H^*(\mathbb{R}\mathbb{P}_+^4)$$

by Lemma 4.15, and $E_2 = E_\infty$ for dimensional reasons. Then i^* is an epimorphism and ℓ^* is a monomorphism by Theorem 4.16.

The group $Pin(4)$ is the semidirect product of $Spin(4)$ by $\mathbb{Z}/2$, so there are maps $\mathbb{Z}/2 \hookrightarrow Pin(4) \rightarrow \mathbb{Z}/2$, which yields maps $B\mathbb{Z}/2 \rightarrow BPin(4) \rightarrow B\mathbb{Z}/2$ of classifying spaces, whose composition is the identity. Let f be the map $\Gamma = BPin(4) \rightarrow B\mathbb{Z}/2 = \mathbb{RP}^\infty$. Then we define $w := f^*(w)$ in $H^2(\Gamma_+)$, where $2w = 0$, $|w| = 2$, and note that all powers w^k are non-zero, because they pull back non-zero to $H^{2k}(\mathbb{RP}_+^\infty)$. Also, $f_*: \pi_1(\Gamma) \rightarrow \pi_1(\mathbb{RP}^\infty)$ must be non-zero, and therefore an isomorphism. It follows that $(f \circ i)_*: \pi_1(\mathbb{RP}^4) \rightarrow \pi_1(\mathbb{RP}^\infty)$ is an isomorphism as well, and therefore $i^*(w^k) = w^k$ in $H^{2k}(\mathbb{RP}_+^4)$ for $k = 1, 2$.

We can now say that w and w^2 represent the classes $1 \otimes w$ and $1 \otimes w^2$ in $E_\infty^{0,2}$ and $E_\infty^{0,4}$ respectively, and have order 2. Similarly, $\ell^*(p_1)$ and $\ell^*(p_2)$ represent the classes $p_1 \otimes 1$ and $p_2 \otimes 1$ in $E_\infty^{4,0}$ and $E_\infty^{8,0}$, respectively. Therefore $\ell^*(p_1)^i \ell^*(p_2)^j w^k$ represents $p_1^i p_2^j \otimes w^k$ in $E_\infty^{*,*}$, and $E_\infty \cong H^*(\Gamma_+)$ additively. Let us consider the product structure, remembering that $w^3 \neq 0$, which in turn suggests that $w^3 = \ell^*(p_1) w$ because $H^6(\Gamma_+) \cong E_\infty^{4,2} \cong \mathbb{Z}/2$ on generator $p_1 \otimes w$. We now complete the proof by setting $\tilde{x} := \ell^*(p_1)$ and $\tilde{y} := \ell^*(p_2)$. □

Remark 4.18. *We should note that the generator \tilde{x} in Proposition 4.17 can be replaced by $x := \tilde{x} + w^2$; in this case, the ideal $(2w, \tilde{x}w + w^3)$ is given by $(2w, xw)$.*

Also, it follows from [5] that the bundle $\mathbb{RP}^4 \xrightarrow{i} \Gamma \xrightarrow{\ell} Gr_{\mathbb{H}}(2, \infty)$ is the projectivisation of the universal $Spin(5)$ vector bundle χ_5 over $BSp(2)$. So $\Gamma \xrightarrow{f} \mathbb{RP}^\infty$ classifies the tautological line bundle λ over $\mathbb{RP}(\chi_5) = \Gamma$.

We shall now express the cohomology of Γ_n in terms of $H^*(\Gamma_+)$.

Proposition 4.19. *For any $n \geq 1$,*

$$H^*((\Gamma_n)_+) \cong H^*(\Gamma_+)/\langle h_n, h_{n+1} \rangle$$

where $|h_i| = 4i$.

Proof. Apply $H^*(-)$ to the commutative diagram (4.4) of fibre bundles

$$\begin{array}{ccccc} \mathbb{RP}^4 & \xrightarrow{i_n} & \Gamma_n & \xrightarrow{\ell_n} & Gr_{\mathbb{H}}(2, n+1) \\ \parallel & & \downarrow \gamma & & \downarrow r \\ \mathbb{RP}^4 & \xrightarrow{i} & \Gamma & \xrightarrow{\ell} & Gr_{\mathbb{H}}(2, \infty). \end{array}$$

The homomorphism r^* is surjective where the kernel is given by the ideal (q_n, q_{n+1}) by (4.14). The base space $Gr_{\mathbb{H}}(2, n+1)$ is simply connected and its cohomology is a finitely generated free \mathbb{Z} -module in each dimension. Thus both i^* and i_n^* are epimorphisms whilst ℓ^* and ℓ_n^* are monomorphisms. The homomorphism ℓ^* is given by $\ell^*(p_1) =: \tilde{x}$, $\ell^*(p_2) =: \tilde{y}$, and so $\gamma^*(\ell^*(q_n)) = 0 = \gamma^*(\ell^*(q_{n+1}))$. The SSS for Γ_n is analogous to that for Γ , therefore

$$H^*((\Gamma_n)_+) \cong H^*(\Gamma_+)/(\ell^*(q_n), \ell^*(q_{n+1}))$$

where $h_n := \ell^*(q_n)$, $h_{n+1} := \ell^*(q_{n+1})$ as required. \square

Example 4.20. For $n = 2$,

$$H^*((\Gamma_2)_+) \cong H^*(\Gamma_+)/(\tilde{x}^2 - \tilde{y}, \tilde{x}^3 - 2\tilde{x}\tilde{y}).$$

4.1.2 The Mod 2 Cohomology

The mod 2 cohomology $H^*(\Gamma_+; \mathbb{Z}/2)$ plays an important role in Section 6.2.2 and 7.2. Compared to the integral cohomology ring, the $\mathbb{Z}/2$ -cohomology is relatively easy to describe by using the Leray-Hirsch theorem and a well-known result on product structure for the cohomology of projective bundles [57, Theorem 2.3]. The additive structure of $H^*(\Gamma_+; \mathbb{Z}/2)$ can also be read off from Proposition 4.17 by applying the Universal Coefficient Theorem.

Theorem 4.21 (Leray-Hirsch [22]). *Let $F \xrightarrow{i} E \xrightarrow{p} X$ be a fibre bundle such that, for some commutative ring R*

(i) $H^m(F_+; R)$ is a finitely generated free R -module for each m

(ii) there exist elements $c_j \in H^*(E_+; R)$ such that the $i^*(c_j)$ form a basis for $H^*(F_+)$.

Then the map

$$\varphi: H^*(X_+; R) \otimes_R H^*(F_+; R) \rightarrow H^*(E_+; R), \quad \sum_{k,j} x_k \otimes i^*(c_j) \mapsto \sum_{k,j} p^*(x_k) c_j,$$

is an isomorphism of R -modules. In other words, $H^*(E_+; R)$ is a free $H^*(X_+; R)$ -module, with basis the c_j .

Lemma 4.22. *The ring $H^*(\Gamma_+; \mathbb{Z}/2)$ is given by*

$$H^*(\Gamma_+; \mathbb{Z}/2) \cong H^*(Gr_{\mathbb{H}}(2, \infty)_+; \mathbb{Z}/2)[a]/(a^5 - p_1a),$$

where $|a| = 1$, $H^*(Gr_{\mathbb{H}}(2, \infty)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[p_1, p_2]$, $|p_i| = 4i$.

Proof. We shall use the Leray-Hirsch theorem for projective bundles of vector bundles and consider $\mathbb{R}\mathbb{P}^4 \xrightarrow{i} \Gamma \xrightarrow{\ell} Gr_{\mathbb{H}}(2, \infty)$. We define $a := f^*(a)$ in $H^1(\Gamma_+; \mathbb{Z}/2)$ by using the map $f: \Gamma \rightarrow \mathbb{R}\mathbb{P}^\infty$ from the proof of Proposition 4.17. It follows that $i^*(a) = a$, so $H^*(\mathbb{R}\mathbb{P}^4_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[a]/(a^5)$ has basis $i^*(a)$, $i^*(a^2)$, $i^*(a^3)$, $i^*(a^4)$ as required. Note from Remark 4.18 that a is $w_1(\lambda)$, by definition.

In order to find the multiplicative structure, we can apply the reduction mod 2 to Proposition 4.17, or use the fact χ_5 has only one non-zero Stiefel-Whitney class $w_4(\chi_5) = p_1$ in $H^4(Gr_{\mathbb{H}}(2, \infty)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2$. So applying [57, Theorem 2.3] gives $a^5 = \sum_{j=0}^4 w_{5-j}(\chi_5) a^j = p_1 a$, as required. \square

Remark 4.23. *Analogous to $H^*(\Gamma_+; \mathbb{Z}/2)$, we also have*

$$H^*((\Gamma_n)_+; \mathbb{Z}/2) \cong H^*(Gr_{\mathbb{H}}(2, n+1)_+; \mathbb{Z}/2)[a]/(a^5 - p_1a), \quad (4.24)$$

where $|a| = 1$, $p_1 \in H^4(Gr_{\mathbb{H}}(2, n+1)_+; \mathbb{Z}/2)$.

Proof. Again, this follows from the Leray-Hirsch theorem and [57, Theorem 2.3]. \square

4.2 Configuration Space

In this section we shall show that the second configuration space $C_2(\mathbb{H}\mathbb{P}^n)$ and the space Γ_n are homotopy equivalent. To establish the homotopy equivalence we shall use a quaternionic version of the symmetric orthogonalisation process.

4.2.1 The Configuration Space $C_2(\mathbb{H}\mathbb{P}^n)$

The n -th unordered configuration space¹ $C_n(X)$ of a topological space X is the quotient $C_n(X) = \tilde{C}_n(X)/\Sigma_n$ by the natural action of the n -th symmetric group Σ_n , where $\tilde{C}_n(X) = \{(x_1, x_2, \dots, x_n) \mid x_i \neq x_j \text{ for } i \neq j\}$ is the space of ordered n -tuples of distinct points of X .

¹The space $C_n(X)$ is sometimes called the n -th braid space [12, 27] of X .

During the late 60's - 70's, Samuel Feder [17] worked on the mod 2 cohomology of $C_2(\mathbb{R}\mathbb{P}^n)$ and $C_2(\mathbb{C}\mathbb{P}^n)$; the integral and the mod 2 cohomology rings of $C_2(\mathbb{C}\mathbb{P}^n)$ are also described by Tsutomu Yasui in [57]. More recently Jesús González and Peter Landweber computed $H^*(C_2(\mathbb{R}\mathbb{P}^n)_+; \mathbb{Z})$ as groups [18], and also as rings [15] together with Carlos Domínguez. By analogy with the complex case and the real case in [17, 57], we have a fibration ξ ,

$$C_2(\mathbb{H}\mathbb{P}^1) \rightarrow C_2(\mathbb{H}\mathbb{P}^n) \rightarrow Gr_{\mathbb{H}}(2, n+1),$$

and there is a homotopy equivalence $C_2(\mathbb{H}\mathbb{P}^1) \simeq \mathbb{R}\mathbb{P}^4$, given by the fact that $C_2(\mathbb{H}\mathbb{P}^1)$ is an open 4-disk bundle over $\mathbb{R}\mathbb{P}^4$ [27, Lemma 2.1]. We may also notice that there is a subspace of $C_2(\mathbb{H}\mathbb{P}^n)$, which consists of all pairs of lines (m, l) through the origin in \mathbb{H}^{n+1} with the inner product $\langle m, l \rangle = 0$; this is the space Γ_n of Definition 2.1.

4.2.2 The Symmetric Orthogonalisation

The idea of the (Löwdin) symmetric orthogonalisation originates in a paper [35] by Per-Olov Löwdin. The process was used in the literature of quantum chemistry [38], and still seems less known to mathematicians than the non-symmetrical Gram-Schmidt orthogonalisation. Given any $n \times k$ complex matrix X of linearly independent unit column vectors x_1, \dots, x_k , let M be the positive definite Hermitian matrix X^*X . Then M has a unique positive definite Hermitian square root, and $V = XM^{-1/2}$ is Löwdin's $n \times k$ matrix of orthogonalised columns. The process does not appear in the literature over the quaternions, but is just as valid, by applying [58, Corollary 6.2]. We make it explicit for the case $k = 2$ below, by introducing the 2×2 matrix $M^{-1/2}$ over \mathbb{H} .

Definition 4.25. For any linearly independent unit vectors $x, y \in \mathbb{H}^{n+1}$, let M be the Hermitian matrix $M = \begin{pmatrix} x^*x & x^*y \\ y^*x & y^*y \end{pmatrix} = \begin{pmatrix} 1 & x^*y \\ y^*x & 1 \end{pmatrix}$, and define the associated Hermitian matrix N to be

$$N = N_{x,y} = \begin{pmatrix} \frac{C+D}{2CD} & \frac{-h}{(C+D)CD} \\ \frac{-\bar{h}}{(C+D)CD} & \frac{C+D}{2CD} \end{pmatrix}$$

where $h = x^*y \in \mathbb{H}$, $\bar{h} = y^*x \in \mathbb{H}$, $|h| = \sqrt{h\bar{h}} = \sqrt{\bar{h}h} \in \mathbb{R}$, $|h| < 1$,
 $C = (1 + |h|)^{1/2}$, $D = (1 - |h|)^{1/2}$.

If x, y are orthogonal, then M is the identity matrix, and so is N .

Simple computations show that $N^2 = M^{-1}$, and that N transforms x, y to an orthonormal pair u, v by $(u \ v) = (x \ y)N$. The most important property is that the process is symmetrical; if we start with $(y \ x)$ then we get $(v \ u)$. Note also that N is well defined, because $|h| < 1$, and depends continuously on x and y .

Proposition 4.26. *For any $n \geq 1$, there is a homotopy equivalence $\Gamma_n \simeq C_2(\mathbb{H}\mathbb{P}^n)$, and therefore an isomorphism $H^*((\Gamma_n)_+) \cong H^*(C_2(\mathbb{H}\mathbb{P}^n)_+)$.*

Proof. Let unit column vectors $x, y \in \mathbb{H}^{n+1}$, with $x \neq y$, represent elements $[x], [y] \in \mathbb{H}\mathbb{P}^n$ respectively. Then an element of $C_2(\mathbb{H}\mathbb{P}^n)$ can be expressed as an $(n+1) \times 2$ matrix $(x \ y)$, up to the action of $\text{Pin}(4)$. So consider the maps

$$\mathfrak{s}: \Gamma_n \rightarrow C_2(\mathbb{H}\mathbb{P}^n), \quad \mathfrak{o}: C_2(\mathbb{H}\mathbb{P}^n) \rightarrow \Gamma_n$$

given by $\mathfrak{s}(u \ v) = (u \ v)$ and $\mathfrak{o}(x \ y) = (x \ y)N_{x,y}$, where we factor out by $\text{Pin}(4)$ on both sides. If $(x' \ y') := (x \ y)N$, then for any $a, b \in S^3$, we have $|x^*y| = |(xa)^*yb| = |y^*x|$, $(xa \ yb)N_{xa,yb} = (x'a \ y'b)$ and $(y \ x)N_{y,x} = (y' \ x')$. Clearly $\mathfrak{o} \circ \mathfrak{s} = 1_{\Gamma_n}$, the identity map on Γ_n . We need to prove that $\mathfrak{s} \circ \mathfrak{o}$ is homotopic to $1_{C_2(\mathbb{H}\mathbb{P}^n)}$.

For $0 \leq t \leq 1$, and any unit vectors $x, y \in \mathbb{H}^{n+1}$, define the 2×2 matrix

$$N_t = N_{x,y,t} = \begin{pmatrix} \left(\frac{C+D}{2CD} - 1\right)t + 1 & \frac{-h}{(C+D)CD}t \\ \frac{-\bar{h}}{(C+D)CD}t & \left(\frac{C+D}{2CD} - 1\right)t + 1 \end{pmatrix}$$

where

$$h = x^*y \in \mathbb{H}, \quad \bar{h} = y^*x \in \mathbb{H}, \quad |h| \in \mathbb{R},$$

$$C = (1 + |h|)^{1/2}, \quad D = (1 - |h|)^{1/2}$$

which varies continuously with t . Then M_0 is the 2×2 identity matrix, and M_1 is the matrix N of Definition 4.25. So the map $(x \ y) \mapsto (x \ y)M_t$ induces a homotopy $1_{C_2(\mathbb{H}\mathbb{P}^n)} \simeq \mathfrak{s} \circ \mathfrak{o}$. \square

We will use this process later in Proposition 6.4.

4.3 The Cohomology of the Borel Space

In this section we shall describe the integral cohomology ring $H^*((B_n)_+)$, by extending the calculation of $H^*(B_+)$ in [52]. There is, of course, an isomorphism $H^*(B_+) \cong$

$H^*(\Gamma_+)$ because $B \simeq \Gamma$ are both models for $B\text{Pin}(4)$, by Remark 3.8. To compute the cohomology, we first recall some useful results regarding group cohomology.

Let $H^i(G, A)$ be the i -th cohomology of a finite group G with coefficients in a G -module A , where we can think of a G -module A as an abelian group together with an action of G . From the topological point of view, this is the same as $H^i(BG; A)$, where $\pi_1(BG) = G$ acts on A . We shall consider only the case when G is the cyclic group C_k of order k with generator τ , and use [24, Proposition 7.1] to define the cohomology groups $H^*(G, A)$ as follows.

Definition 4.27. *Let $\alpha, \beta: A \rightarrow A$ be the C_k -homomorphisms given by*

$$\alpha(a) = (\tau - 1)a, \quad \beta(a) = (\tau^{k-1} + \tau^{k-2} + \cdots + \tau + 1)a, \quad a \in A.$$

Then $H^0(C_k, A) = \ker \alpha$, and for $n \geq 1$

$$H^{2n-1}(C_k, A) = \ker \beta / \text{im } \alpha, \quad H^{2n}(C_k, A) = \ker \alpha / \text{im } \beta. \quad (4.28)$$

The integral cohomology ring of the Borel space $B = S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)$ can be computed by using the Serre spectral sequence (Appendix B.3), and by applying (4.28) with $k = 2$, $A = \mathcal{H}^*((\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)_+)$, where $\mathbb{Z}/2$ acts on $H^*((\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)_+) \cong \mathbb{Z}[z_1, z_2]$ by swapping z_1 and z_2 . Before starting the computation, we recall that $H^*((\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)_+) \cong \mathbb{Z}[z_1, z_2]$ where $|z_i| = 4$, and z_1, z_2 are the pull-backs of $z \in H^4(\mathbb{H}\mathbb{P}^\infty)$ under the two projections.

Theorem 4.29 ([52]). *The Serre spectral sequence of the fibre bundle*

$$\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty \rightarrow S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty) \xrightarrow{p} S^\infty / \mathbb{Z}/2 = \mathbb{R}\mathbb{P}^\infty \quad (4.30)$$

in integral cohomology can be described as follows.

$$\begin{aligned} E_2^{0,*} &= \text{invariant classes of } H^*((\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)_+) \text{ under interchange of the factors} \\ &= \text{the polynomial ring } \mathbb{Z}[z_1 + z_2, z_1 z_2] \\ E_2^{2i, 8q} &= \mathbb{Z}/2 \quad \text{for } i > 0 \text{ with the generator } w^i(z_1 z_2)^q \end{aligned}$$

and all other $E_2^{p,q}$ are zero. The relation $w(z_1 + z_2) = 0$ holds in $E_2^{2,4}$. Moreover $E_2 = E_\infty$ and $E_\infty^{,*} \cong H^*(B_+)$ as graded rings.*

Therefore, Theorem 4.29 can be summarised as

$$H^*(B_+) \cong \mathbb{Z}[x, y, w]/(2w, wx), \quad |x| = 4, |y| = 8, |w| = 2 \quad (4.31)$$

where $x, y \in H^*(B_+)$ represent $z_1 + z_2, z_1z_2 \in E_\infty^{0,*}$ respectively, and $w \in H^2(B_+)$ represents $w \in E_\infty^{2,0}$. We should note that the generator x in (4.31) could be replaced by $x + w^2$ by amending the ideal of relations. Indeed, setting $\tilde{x} := x + w^2$ and $\tilde{y} := y$ yields $H^*(B_+) \cong \mathbb{Z}[x, y, w]/(2w, xw) \cong \mathbb{Z}[\tilde{x}, \tilde{y}, w]/(2w, \tilde{x}w + w^3) \cong H^*(\Gamma_+)$, which confirms Roush's result (4.31) and agrees with Proposition 4.17, our computation of $H^*(BPin(4)_+)$ using the other model Γ for $BPin(4)$.

The proof of Theorem 4.29 above proceeds by applying (4.28) with $A = \mathcal{H}^*((\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)_+)$. In other words, we have the E_2 page of the spectral sequence

$$E_2^{p,q} = H^p((\mathbb{R}\mathbb{P}^\infty)_+, \mathcal{H}^q((\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)_+)).$$

For example, $E_2^{0,q} = H^0(\mathbb{R}\mathbb{P}^\infty_+, \mathcal{H}^q((\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)_+)) = \ker \alpha$, which means that $E_2^{0,*}$ is the invariant classes of $H^*(\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)_+$; that is, the polynomial ring $\mathbb{Z}[z_1 + z_2, z_1z_2]$. Then, we also have that $E_2^{2,4} = 0$ because

$$H^2(\mathbb{R}\mathbb{P}^\infty, \mathcal{H}^4((\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty))) = \ker \alpha / \text{im } \beta,$$

and $\text{im } \beta = \ker \alpha = \mathbb{Z}\langle z_1 + z_2 \rangle$ for $(p, q) = (2, 4)$. So $w(z_1 + z_2) = 0$ holds in the product structure on $E_2^{*,*}$.

All differentials are zero because all non-zero classes are in even dimensions, hence $E_2 = E_\infty$. By diagram (3.7) and Remark 3.8, we may take $w := p^*(w)$ in $H^2(B)$ to be a representative of order 2 for w in $E_\infty^{2,0} \cong H^2(\mathbb{R}\mathbb{P}^\infty)$ where p is the projection $B \xrightarrow{p} \mathbb{R}\mathbb{P}^\infty$. Powers of w are therefore representatives of order 2 for all of $E_\infty^{*,0} \cong H^*(\mathbb{R}\mathbb{P}^\infty)$.

Applying diagram (3.7) twice more, the elements \tilde{x} and \tilde{y} of Proposition 4.17 may be pulled back to $H^4(B)$ and $H^8(B)$ respectively, and restrict to $z_1 + z_2$ and z_1z_2 in $H^*(\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)$ because they are given by $\ell^*(p_1)$ and $\ell^*(p_2)$ under $\ell: \Gamma \rightarrow BSp(2)$. So they represent $z_1 + z_2$ and z_1z_2 in $E_\infty^{0,*}$. Monomials in w, \tilde{x} and \tilde{y} therefore give order 2 representatives for all generators in $E_\infty^{p,q}$ with $p > 0$. This means that all the extension problems are trivial, thus $E_\infty \cong H^*(B)$ additively.

To show $E_\infty \cong H^*(B_+)$ multiplicatively, it is enough to identify a representative x for $z_1 + z_2 \in E_\infty^{0,4}$ such that $w(z_1 + z_2) = 0$, where $H^4(B) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ and $H^6(B) \cong \mathbb{Z}/2$ on w^3 . This we shall now do.

The following facts about the subgroups of $Sp(2)$ are useful, and we shall also use them in Section 5.3.

For $a \in S^3$, the composition of the subgroup inclusions

$$S^3 \hookrightarrow \mathbb{Z}/2 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2) \quad (4.32)$$

is given by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in Sp(2)$ and therefore agrees with

$$S^3 \hookrightarrow S^3 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2)$$

where the first subgroup is the diagonal.

Let τ be the generator of $\mathbb{Z}/2$ and consider the composition of inclusions

$$\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2), \quad (4.33)$$

which is given by $\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in Sp(2)$.

Remark 4.34. *The map (4.33), which is given by $\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in Sp(2)$ can also be rewritten as $\tau \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in terms of the basis of eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which agrees with the composition $\mathbb{Z}/2 = O(1) \rightarrow U(1) \rightarrow Sp(1) \rightarrow Sp(2)$. So both homomorphisms induce the same map on the cohomology of classifying spaces.*

The inclusion map $\mathbb{R}P^\infty \xrightarrow{i} B$ induces the homomorphism i^* , given by $i^*(w^k) = w^k$, $k \geq 1$. Also, the maps $\mathbb{R}P^\infty \xrightarrow{i} B \xrightarrow{\ell} BSp(2)$ induce the homomorphisms i^*, ℓ^* such that $(i^* \circ \ell^*)(p_1) = w^2$, where $p_1 \in H^4(BSp(2))$; this follows from Remark 4.34. Then, $i^*(w \ell^*(p_1)) = w^3$ and $i^*(w(\ell^*(p_1) + w^2)) = w^3 + w^3 = 0$. Because $H^6(B) \cong \mathbb{Z}/2$ on w^3 , we may now identify $\ell^*(p_1) + w^2$ as the required representative for $z_1 + z_2$.

The integral cohomology ring of $B_n = S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}P^n \times \mathbb{H}P^n)$ can be computed analogously, by combining Theorem 4.29 and (4.28) for $k = 2$, $A = H^*((\mathbb{H}P^n \times \mathbb{H}P^n)_+)$, together with the cohomology $H^*((\mathbb{H}P^n \times \mathbb{H}P^n)_+) \cong \mathbb{Z}[z_1, z_2]/(z_1^{n+1}, z_2^{n+1})$. This means that we have the spectral sequence for the bundle

$$\mathbb{H}P^n \times \mathbb{H}P^n \rightarrow S^\infty \times_{\mathbb{Z}/2} (\mathbb{H}P^n \times \mathbb{H}P^n) \rightarrow \mathbb{R}P^\infty$$

where

$$\begin{aligned} E_2^{p,q} &= H^p(\mathbb{R}P^\infty_+, \mathcal{H}^q((\mathbb{H}P^n \times \mathbb{H}P^n)_+)) \\ E_2^{0,*} &= \text{invariant classes of } H^*((\mathbb{H}P^n \times \mathbb{H}P^n)_+) \text{ under interchange of the factors} \\ &= \mathbb{Z}[z_1 + z_2, z_1 z_2]/\mathcal{I}_n \quad \text{where the ideal } \mathcal{I}_n \text{ is described below} \\ E_2^{2i,8q} &= \mathbb{Z}/2 \quad \text{for } i > 0, q \leq n \text{ with the generator } w^i(z_1 z_2)^q \end{aligned}$$

and other $E_2^{p,q}$ are zero. The relation $E_2^{2,4} = w(z_1 + z_2) = 0$ holds for the same reason as the the case B . Also, again all differentials are zero for dimensional reason, and $E_2 = E_\infty \cong H^*((B_n)_+)$ as rings.

Theorem 4.35. *For any $n \geq 1$, there exist elements $f_{n,1}$ and $f_{n,2}$ in the cohomology ring $H^*(B_+) \cong \mathbb{Z}[x, y, w]/(2w, wx)$ such that*

$$H^*((B_n)_+) \cong H^*(B_+)/(\mathfrak{f}_{n,1}, \mathfrak{f}_{n,2}, y^{n+1}), \quad (4.36)$$

where $|\mathfrak{f}_{n,1}| = 4(n+1)$, $|\mathfrak{f}_{n,2}| = 4(n+2)$; also $\mathfrak{f}_{n,1}, \mathfrak{f}_{n,2}$ are given by

$$\begin{cases} \mathfrak{f}_{1,1} = x^2 - 2y \\ \mathfrak{f}_{1,2} = xy \end{cases} \text{ for } n = 1, \text{ and } \begin{cases} \mathfrak{f}_{n,1} = x \mathfrak{f}_{n-1,1} - \mathfrak{f}_{n-1,2} \\ \mathfrak{f}_{n,2} = y \mathfrak{f}_{n-1,1} \end{cases} \text{ for } n \geq 2. \quad (4.37)$$

Proof. It is left to show that the ideal \mathcal{I}_n is given by $(\mathfrak{f}_{n,1}, \mathfrak{f}_{n,2}, y^{n+1})$ in $H^*(B)$. Any invariant polynomial in $\mathbb{Z}[z_1, z_2]/(z_1^{n+1}, z_2^{n+1})$ may be lifted to a unique polynomial in $\mathbb{Z}[z_1, z_2]$ that has no monomial containing z_1^{n+1} or z_2^{n+1} , or any higher powers of z_1 or z_2 . The same is true after interchanging z_1 and z_2 , so the lifts of the invariants are actually invariant in $\mathbb{Z}[z_1, z_2]$, and must lie in $\mathbb{Z}[z_1 + z_2, z_1 z_2]$. So the ideal \mathcal{I}_n above is the kernel of the composition

$$\mathbb{Z}[z_1 + z_2, z_1 z_2] \xrightarrow{\subset} \mathbb{Z}[z_1, z_2] \rightarrow \mathbb{Z}[z_1, z_2]/(z_1^{n+1}, z_2^{n+1}),$$

and contains $r_{n,1} := z_1^{n+1} + z_2^{n+1}$ and $r_{n,2} := z_1 z_2 (z_1^n + z_2^n)$. So

$$\mathcal{I}_n \supseteq (r_{n,1}, r_{n,2}, (z_1 z_2)^{n+1}) := \mathcal{R}_n,$$

and we need to prove the opposite inclusion.

For this we note that

$$r_{n+1,1} = (z_1 + z_2) r_{n,1} - r_{n,2} \quad \text{and} \quad r_{n+1,2} = z_1 z_2 r_{n,1} \quad (4.38)$$

hold in $\mathbb{Z}[z_1 + z_2, z_1 z_2]$, so $r_{n+1,1}$ and $r_{n+1,2}$ lie in \mathcal{R}_n . Repeating the process shows that $r_{n+k,1}, r_{n+k,2}$ also lie in \mathcal{R}_n for any $k \geq 0$. Now let $g(z_1, z_2) \in \mathcal{I}_n$, which means that $g(z_1, z_2) = g(z_2, z_1)$ and every monomial in g is divisible by z_1^{n+1} or z_2^{n+1} . The monomials must therefore

- (i) either occur in pairs such as $z_1^i z_2^{n+1+k} + z_1^{n+1+k} z_2^i$

(ii) or be of the form $(z_1 z_2)^{n+1+k}$,

for some $i, k \geq 0$. These can be written as $(z_1^i + z_2^i)r_{n+k,1} - r_{n+k+i,1}$ and $(z_1 z_2)^k (z_1 z_2)^{n+1}$ respectively. Because they are both contained in \mathcal{R}_n , so is $g(z_1, z_2)$. It follows that $\mathcal{I}_n \subset \mathcal{R}_n$, and they equal as required.

Finally, we remember from the justification for (4.31) that $z_1 + z_2$ and $z_1 z_2$, when considered as elements in $E_\infty^{0,*}$, have unique representatives x and y in $H^*(B)$ that are pull-backs from $H^*(BSp(2))$. So $r_{n,1}$ and $r_{n,2}$ in $\mathbb{Z}[z_1 + z_2, z_1 z_2]$ have corresponding representatives $f_{n,1}(x, y)$ and $f_{n,2}(x, y)$ in $H^*(B)$, and $(z_1 z_2)^{n+1}$ has representative y^{n+1} . Then by (4.38) the expressions $f_{n,1} = x f_{n-1,1} - f_{n-1,2}$ and $f_{n,2} = y f_{n-1,1}$ hold in $H^*(B)$ for $n \geq 2$, where $f_{1,1} = x^2 - 2y$ and $f_{1,2} = xy$.

□

Example 4.39. For $n = 3$,

$$H^*((B_3)_+) \cong H^*(B_+)/ (x^4 - 4x^2y + 2y^2, x^3y - 3xy^2, y^4).$$

We shall observe the following Remarks.

Remark 4.40. For any $n \geq 1$, there is an expression for $f_{n,i}$, $i = 1, 2$

$$f_{n,i} = \lambda_k y^k + \sum_{\ell+2m=n+i} \lambda_{\ell,m} x^\ell y^m$$

where $\lambda_k = \pm 2$, $\lambda_{\ell,m} \in \mathbb{Z}$, $2k = n + i$. This expression comes from (4.37).

Remark 4.41. From Theorem 4.35, for any $n \geq 1$ we have that

$$H^r((B_n)_+) \cong H^r(B_+) \quad \text{if } r \leq 4n + 3$$

$$H^r(B_n) \cong (\mathbb{Z}/2)^{n+1} \quad \text{if } r = 8n + 2\ell, \ell = 1, 2, 3, \dots$$

$$H^r(B_n) = 0 \quad \text{if } r \text{ is odd.}$$

Also, for $x^{n+1} \in H^{4(n+1)}(B_n)$ there is an expression

$$x^{n+1} = \sum_{a,b} \lambda_{a,b} x^a y^b, \quad \lambda_{a,b} \in \mathbb{Z} \tag{4.42}$$

where $b \geq 1$, $a + 2b = n + 1$.

For future use we include the following Lemmas.

Lemma 4.43. *For any $k \geq 2$, there is an expression*

$$2y^k = \sum_m \mu_m f_{m,2} \in H^{8k}(B) \quad (4.44)$$

where $m \geq k-1$, $\mu_m \in \mathbb{Z}\langle x^i y^j \rangle$, $j \geq 0$, $i \geq 1$.

Proof. We use the relation $y f_{n,2} = xy f_{n,1} - y f_{n+1,1} = x f_{n+1,2} - f_{n+2,2}$, which comes from (4.37).

For $k = 2$, it is clear that $2y^2 = x f_{1,2} - y f_{1,1} = x f_{1,2} - f_{2,2}$.

For $k \geq 3$, we have that

$$\begin{aligned} 2y^k &= y^{k-2}(x f_{1,2} - f_{2,2}) \\ &= xy^{k-3}y f_{1,2} - y^{k-3}y f_{2,2} \\ &= xy^{k-3}(x f_{2,2} - f_{3,2}) - y^{k-3}(x f_{3,2} - f_{4,2}) \\ &\vdots \\ &= \sum_{m=k-1}^{2k-2} \mu_m f_{m,2}, \text{ where } \mu_m \in \mathbb{Z}\langle x^i y^j \rangle, j \geq 0, i \geq 1. \end{aligned}$$

□

Lemma 4.45. *For $n \geq 1$, $f_{n,1} = f_{n,2} = 0$ implies that $f_{m,2} = 0$ for all $m \geq n$.*

Proof. Suppose $f_{n,1} = f_{n,2} = 0$, then

$$\begin{aligned} f_{n+1,2} &= y f_{n,1} = 0 \\ f_{n+2,2} &= y f_{n+1,1} = xy f_{n,1} - y f_{n,2} = 0 \\ f_{n+3,2} &= y f_{n+2,1} = xy f_{n+1,1} - y f_{n+1,2} = 0 \\ f_{n+4,2} &= y f_{n+3,1} = xy f_{n+2,1} - y f_{n+2,2} = 0 \\ &\vdots \\ f_{n+k,2} &= y f_{n+k-1,1} = xy f_{n+k-2,1} - y f_{n+k-2,2} = 0 \end{aligned}$$

for any $k \geq 1$, by using the relation $f_{n+1,1} = x f_{n,1} - f_{n,2}$.

□

Chapter 5

The Classifying Space of $\mathbb{Z}/2 \times S^3$

In Section 2.1 we viewed the subspace $A_n \subset SP^2(\mathbb{H}\mathbb{P}^n)$

$$A_n \stackrel{\text{def}}{=} \{[u], [v] \mid ([u], [v]) \sim ([v], [u]), d = 1/2, \|u\| = \|v\| = 1\}$$

as the total space of the (real) projectivisation of the tangent bundle of $\mathbb{H}\mathbb{P}^n$. In this chapter we shall identify A_n with the quotient of the quaternionic Stiefel manifold V_{n+1} by the action of the subgroup $\mathbb{Z}/2 \times S^3$ of $Pin(4)$. The main aim of the Chapter is to compute the homomorphism induced by the map $B(\mathbb{Z}/2 \times S^3) \rightarrow BPin(4)$ of classifying spaces in integral cohomology.

We will usually write G_A for the group $\mathbb{Z}/2 \times S^3$.

5.1 The Quotient V_{n+1}/G_A

Theorem 5.1. *The quotient space V_{n+1}/G_A is homeomorphic to the total space A_n of the (real) projectivisation $\mathbb{R}P(\tau_{\mathbb{H}\mathbb{P}^n})$ of the tangent bundle of $\mathbb{H}\mathbb{P}^n$*

$$\mathbb{R}P^{4n-1} \longrightarrow A_n \longrightarrow \mathbb{H}\mathbb{P}^n$$

where V_{n+1} is the quaternionic Stiefel manifold, of ordered pairs of orthonormal vectors in \mathbb{H}^{n+1} .

Proof. We define the action of $\mathbb{Z}/2$ on V_{n+1} by swapping the pair of vectors, $(u, v) \mapsto (v, u)$ and the action of S^3 on V_{n+1} by $(v, u) \mapsto (va, ua)$. Then the quotient of V_{n+1} by the action of $G_A := \mathbb{Z}/2 \times S^3$ is given by

$$V_{n+1}/G_A = V_{n+1}/\sim$$

where $(u, v) \sim (v, u) \sim (va, ua)$ for $a \in S^3$. By [42], the tangent space $T_{[h]}$ at any point $[h] \in \mathbb{H}\mathbb{P}^n$ may be thought of as the space of quaternionic linear transformations $\alpha: [h] \rightarrow [h]^\perp$. This works by identifying a transformation α with the line through $(h, \alpha(h))$; this line is generated over \mathbb{H} by $h + \alpha(h)$ in \mathbb{H}^{n+1} . In other words, α is identified with its own graph, and $T_{[h]}$ is a $4n$ -dimensional real vector space (but is not quaternionic [49, page 68]). We write the total space of the tangent bundle as $\mathbb{R}(\tau_{\mathbb{H}\mathbb{P}^n}) = \bigcup_{[h]} T_{[h]}$, and its unit sphere bundle as $S(\tau_{\mathbb{H}\mathbb{P}^n}) = \bigcup_{[h]} S_{[h]}$, where $S_{[h]}$ denotes the transformations that have $\|\alpha(h)\| = 1$.

We first define a homeomorphism V_{n+1}/S^3 to $S(\tau_{\mathbb{H}\mathbb{P}^n})$ by $(u, v) \mapsto \alpha_{u,v}$, where

$$\alpha_{u,v}: [(u+v)/\sqrt{2}] \rightarrow [(u+v)/\sqrt{2}]^\perp$$

is given by

$$\alpha_{u,v}((u+v)q/\sqrt{2}) = \alpha_{u,v}((u+v)/\sqrt{2})q = (u-v)q/\sqrt{2}$$

for any $q \in \mathbb{H}$. The map is well defined because $(ua, va) \mapsto \alpha_{ua,va}$, and we have $\alpha_{ua,va}((u+v)a/\sqrt{2}) = (u-v)a/\sqrt{2}$ for any $a \in S^3$. So $\alpha_{ua,va}$ is the same function as $\alpha_{u,v}$. We have used the fact that $(u+v)^*(u-v) = 0$, and that $\alpha_{u,v}$ corresponds to a unit tangent vector because it maps unit vectors to unit vectors.

The map is homeomorphism because we can describe its inverse.

Suppose $[h] \in \mathbb{H}\mathbb{P}^n$, then any unit tangent vector in $S_{[h]}$ corresponds to a linear transformation $\alpha: [h] \rightarrow [h]^\perp$, which is given by $\alpha(h) = k$ for some $k \in [h]^\perp \subset \mathbb{H}^{n+1}$ with $\|k\| = 1$. Then define the inverse function from $S(\tau_{\mathbb{H}\mathbb{P}^n})$ to V_{n+1}/S^3 by

$$\alpha \mapsto ((h+k)/\sqrt{2}, (h-k)/\sqrt{2}).$$

It is well defined because $\alpha(hq) = kq$ for any $q \in S^3$.

Both of these maps are continuous.

Finally, we factor out by the action of $\mathbb{Z}/2$, which acts by $(u, v) \sim (v, u)$ and $\alpha \sim -\alpha$. The previous homeomorphism and its inverse are equivariant, so they pass to orbit spaces. \square

Corollary 5.2. *We have a homeomorphism $A \cong V/G_A$ by defining $A := \bigcup_n A_n$ and $V/G_A := \bigcup_n V_{n+1}/G_A$.*

Proof. We can check that the above homeomorphisms are compatible when we include \mathbb{H}^{n+1} in \mathbb{H}^{n+2} . So we get a homeomorphism of colimits. \square

5.2 The Cohomology of $A_2 \subset SP^2(\mathbb{H}\mathbb{P}^2)$

Corollary 5.2 implies that the colimit $A = \bigcup_n A_n$ is a model for the classifying space $B(\mathbb{Z}/2 \times S^3) = BG_A$. So the quotient $V/G_A \rightarrow V/Pin(4)$ is a model for the map $f: BG_A \rightarrow BPin(4)$ induced by the inclusion of the subgroup $G_A \rightarrow Pin(4)$, and we have a commutative diagram

$$\begin{array}{ccc} A_n = V_{n+1}/\mathbb{Z}/2 \times S^3 & \longrightarrow & V_{n+1}/Pin(4) = \Gamma_n \\ \downarrow & & \downarrow \\ B(\mathbb{Z}/2 \times S^3) & \xrightarrow{f} & BPin(4) \end{array} \quad (5.3)$$

The map f induces the homomorphism f^* , which is important for our cohomological computation of $SP^2(\mathbb{H}\mathbb{P}^n)$. In this section we will describe the integral cohomology ring of A_n where $n = 2$, by considering the Serre spectral sequence for the fibre bundle

$$\mathbb{R}\mathbb{P}^7 \xrightarrow{i} A_2 \rightarrow \mathbb{H}\mathbb{P}^2.$$

The homotopy exact sequence of this bundle shows that i_* is an isomorphism on $\pi_1(-)$ so that $\pi_1(A_2) = \mathbb{Z}/2$.

Since A_2 is a projectivisation by Theorem 5.1, it has a canonical line bundle over it, which restricts to the tautological line bundle λ on each fibre. So, in mod 2 cohomology, its 1st Stiefel-Whitney class $a \in H^1(A_2; \mathbb{Z}/2)$ pulls back to $i^*(a) = a \in H^1(\mathbb{R}\mathbb{P}^7; \mathbb{Z}/2)$. This abuse of notation is not necessarily confusing, because a is the only non-zero element of $H^1(A_2; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

It helps to note that a may also be defined in $H^1(A_2; \mathbb{Z}/2)$ by considering the map $A_2 \xrightarrow{g} \mathbb{R}\mathbb{P}^\infty$, given by the composition of the map $A_2 \hookrightarrow A = BG_A$ of diagram (5.3) and the map $r_1: BG_A \rightarrow B(\mathbb{Z}/2) \simeq \mathbb{R}\mathbb{P}^\infty$ of classifying spaces induced by the projection $G_A \rightarrow \mathbb{Z}/2$. So we have $a := r_1^*(a)$ in $H^1(A; \mathbb{Z}/2) = H^1(BG_A; \mathbb{Z}/2)$, which restricts to the element $g^*(a)$ in $H^1(A_2; \mathbb{Z}/2)$. It must agree with a as first defined, because g_* is non-zero on $\pi_1(-)$, so g represents the line bundle λ .

Remark 5.4. *We can describe an integral element $w \in H^2(A) = H^2(BG_A)$ in exactly the same way, by pulling back $w \in H^2(\mathbb{R}\mathbb{P}^\infty)$ along r_1^* . This element restricts to $w := g^*(w)$ in $H^2(A_2)$. Then reduction mod 2 satisfies $\rho(w) = a^2$ in both $H^2(A_2; \mathbb{Z}/2)$ and $H^2(A; \mathbb{Z}/2)$, because the same equation is true in $H^2(\mathbb{R}\mathbb{P}^\infty; \mathbb{Z}/2)$.*

We can now describe the mod 2 cohomology ring $H^*(A_2; \mathbb{Z}/2)$, which helps to compute the integral cohomology. We apply the Leray-Hirsch Theorem 4.21, which tells us that the mod 2 cohomology looks additively like that of a product bundle if certain conditions are satisfied. In this case the conditions hold because the elements a, a^2, \dots, a^7 in $H^*(A_2; \mathbb{Z}/2)$ are enough to map onto $H^*(\mathbb{R}P^7; \mathbb{Z}/2)$. So the Theorem gives that the mod 2 cohomology of A_2 is a free $H^*((\mathbb{H}P^2)_+; \mathbb{Z}/2)$ -module

$$H^*((A_2)_+; \mathbb{Z}/2) \cong H^*((\mathbb{H}P^2)_+; \mathbb{Z}/2)\langle 1, a, a^2, \dots, a^7 \rangle. \quad (5.5)$$

Let $\pi: A_2 \rightarrow \mathbb{H}P^2$ be the bundle projection. Then the isomorphism (5.5) includes the fact that $\pi^*: H^*(\mathbb{H}P^2; \mathbb{Z}/2) \rightarrow H^*(A_2; \mathbb{Z}/2)$ is a monomorphism, its image is a direct summand, and it induces the module structure. So we set $z := \pi^*(z)$ in $H^4(A_2; \mathbb{Z}/2)$, where $H^*((\mathbb{H}P^2)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[z]/(z^3)$, and z is the mod 2 reduction of the first symplectic Pontryagin class of the tautological line bundle over $\mathbb{H}P^2$.

Since A_2 is the projectivisation of $\tau_{\mathbb{H}P^2}$, multiplication in $H^*(A_2; \mathbb{Z}/2)$ involves its Stiefel-Whitney classes [57, Theorem 2.3]. According to Szczarba [55, Corollary 2.3], the total Stiefel-Whitney class is

$$w(\tau_{\mathbb{H}P^2}) = (1 + z)^3 = 1 + z + z^2 \quad \text{in } H^*(\mathbb{H}P^2; \mathbb{Z}/2)$$

so we have that

$$a^8 = \sum_{j=1}^8 w_j(\tau_{\mathbb{H}P^2}) a^{8-j} = za^4 + z^2$$

in $H^8(A_2; \mathbb{Z}/2)$. Note that $a^{11} = za^7 + z^2a^3 \neq 0$, and $a^{12} = 0$. To summarise:

Proposition 5.6. *There is an isomorphism of graded rings*

$$H^*((A_2)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[z, a]/(z^3, a^8 + za^4 + z^2)$$

where $|a| = 1$ and $|z| = 4$.

The integral cohomology of A_2 is more delicate compared to the $\mathbb{Z}/2$ -cohomology. However, considering the Serre spectral sequence (SSS) for the associated tangent sphere bundle will lead to Proposition 5.15.

We shall write τ for $\tau_{\mathbb{H}P^2}$.

Let $S(\tau) := A'_2$ be the total space of the tangent sphere bundle and let $D(\tau)$ be the total space of the corresponding disk bundle $D^8 \rightarrow D(\tau) \xrightarrow{p} \mathbb{H}P^2$. Then the

commutative diagram of (homotopy) cofibre sequences

$$\begin{array}{ccccc} S(\tau) & \longrightarrow & D(\tau) & \longrightarrow & D(\tau)/S(\tau) \\ \parallel & & p \downarrow \simeq & & \downarrow \cong \\ S(\tau) & \xrightarrow{p} & \mathbb{H}\mathbb{P}^2 & \longrightarrow & \text{Th}(\tau) \end{array}$$

induces the long exact sequence in cohomology

$$\cdots \leftarrow H^*(S(\tau)_+) \leftarrow H^*(\mathbb{H}\mathbb{P}_+^2) \xleftarrow{\iota^*} H^*(\text{Th}(\tau)) \leftarrow \cdots \quad (5.7)$$

This is a version of the *Gysin sequence* [22] for $S(\tau)$.

Since τ is orientable, there is a Thom class $t \in H^8(\text{Th}(\tau); \mathbb{Z})$, which, by definition (see Section 6.1 for the definition of a Thom space), pulls back along the zero section to the Euler class $e(\tau) \in H^8(\mathbb{H}\mathbb{P}_+^2)$; that is

$$\iota^*(t) = e(\tau) = \chi z^2 \quad (5.8)$$

where $\chi = 3$ is the Euler characteristic of $\mathbb{H}\mathbb{P}^2$.

For the tangent sphere bundle $S^7 \rightarrow A_2' \rightarrow \mathbb{H}\mathbb{P}^2$, the E_2 page of the SSS is

$$\begin{aligned} E_2^{p,q} &\cong H^p(\mathbb{H}\mathbb{P}_+^2; H^q(S_+^7; \mathbb{Z})) \\ &= \begin{cases} \mathbb{Z} & \text{for } p = 0, 4, 8 \text{ and } q = 0, 7 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

because the local coefficients are trivial, and it converges to $H^*(A_2')$. We call this spectral sequence SS_1 .

Similarly, for $\mathbb{R}\mathbb{P}^7 \xrightarrow{i} A_2 \xrightarrow{\pi} \mathbb{H}\mathbb{P}^2$, the E_2 page is

$$\begin{aligned} E_2^{p,q} &\cong H^p(\mathbb{H}\mathbb{P}_+^2; H^q(\mathbb{R}\mathbb{P}_+^7; \mathbb{Z})) \\ &= \begin{cases} \mathbb{Z} & \text{for } p = 0, 4, 8 \text{ and } q = 0, 7 \\ \mathbb{Z}/2 & \text{for } p = 0, 4, 8 \text{ and } q = 2, 4, 6 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and it converges to $H^*((A_2)_+)$. We call this spectral sequence SS_2 .

In order to apply the Leray-Hirsch Theorem above, we used that i^* is an epimorphism in mod 2 cohomology. This forces all differentials to be 0 in the spectral sequence $E_2^{p,q} \cong H^p(\mathbb{H}\mathbb{P}_+^2; H^q(\mathbb{R}\mathbb{P}_+^7; \mathbb{Z}/2))$, which converges to $H^*((A_2)_+; \mathbb{Z}/2)$ (and is the reason why (5.5) is true). We call this spectral sequence SS_3 .

In all three spectral sequences, the only possible non-zero differentials are d_4 and d_8 . The fact that all differentials $d_4: \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$ are trivial in SS_3 therefore implies that the corresponding differentials d_4 in SS_2 are zero, by naturality of reduction mod 2. Also for dimensional reasons all the other d_4 s in SS_2 are trivial. So it remains to look at the only other possible non-trivial differential $d_8: E_8^{0,7} \rightarrow E_8^{8,0}$ in SS_2 , which may be written as $d_8: H^7(\mathbb{RP}_+^7; \mathbb{Z}) \rightarrow H^8(\mathbb{HP}_+^2; \mathbb{Z})$, and therefore as $d_8: \mathbb{Z} \rightarrow \mathbb{Z}$.

Let v, g and z^2 be generators

$$v \in H^7(\mathbb{RP}_+^7; \mathbb{Z}) \cong \mathbb{Z}, \quad g \in H^7(S_+^7; \mathbb{Z}) \cong \mathbb{Z}, \quad z^2 \in H^8(\mathbb{HP}_+^2; \mathbb{Z}) \cong \mathbb{Z}.$$

Factoring out the action of $\mathbb{Z}/2$ on the fibres of $S(\tau)$ gives a bundle map from the sphere bundle of τ to its projectivisation. On each fibre it is projection $\sigma: S^7 \rightarrow \mathbb{RP}^7$. The bundle map induces a map of spectral sequences from SS_2 to SS_1 , which works on the E_2 pages by applying $\sigma^*: H^7(\mathbb{RP}_+^7) \rightarrow H^7(S_+^7)$. This is actually given by $v \mapsto 2g$, which can be justified by considering the long exact sequence in cohomology induced by the cofibre sequence

$$S^7 \xrightarrow{\sigma} \mathbb{RP}^7 \rightarrow \mathbb{RP}^8,$$

or by the geometric fact that σ has degree 2. Note σ^* is the identity in dimension 0.

One of the properties of the Gysin sequence [39, Example 5.C, page 143] tells us that d'_8 is given by (5.8), in the form $d'_8(1 \otimes g) = 3(z^2 \otimes 1)$. So we can compute

$$d_8(1 \otimes v) = \sigma^*(d_8(1 \otimes v)) \tag{5.9}$$

$$= d'_8(\sigma^*(1 \otimes v)) \tag{5.10}$$

$$= d'_8(1 \otimes 2g) \tag{5.11}$$

$$= 2d'_8(1 \otimes g) \tag{5.12}$$

$$= 2 \cdot 3(z^2 \otimes 1) \tag{5.13}$$

$$= 6(z^2 \otimes 1). \tag{5.14}$$

Also, $E_9 = E_\infty$ for SS_2 .

Assembling all the above together we have the following.

Proposition 5.15. *The integral cohomology ring of A_2 is described by Table 5.1.*

Table 5.1: $H^* := H^*(A_2)$

*	0	1	2	3	4	5	6	7
H^*	\mathbb{Z}	0	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	0
	1		w		\tilde{z}, w^2		$w\tilde{z}, w^3$	
*	8	9	10	11	12	13	14	15
H^*	$\mathbb{Z}/2 \oplus \mathbb{Z}/6$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	\mathbb{Z}	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	\mathbb{Z}
	$w^2\tilde{z}, \tilde{z}^2$		$w^3\tilde{z}, w\tilde{z}^2$	m	$w^2\tilde{z}^2$		$w^3\tilde{z}^2$	$m\tilde{z}$

$$\text{where } w^4 = w^2\tilde{z} + 3\tilde{z}^2 \neq 0, \quad \text{and } w^6 = 0. \quad (5.16)$$

Proof. In the spectral sequence SS_2 , (5.14) shows that the E_∞ term differs from the E_2 term only because $E_2^{0,7}$ has become 0, and $E_2^{8,0}$ has become $\mathbb{Z}/6$ on generator z^2 . So we have to check the extension problems and the multiplicative structure.

From Remark 5.4, we have the element w in $H^2(A_2)$, which represents the generator of $E_\infty^{0,2} \cong \mathbb{Z}/2$. Also, we have $\tilde{z} := \pi^*(z) \in H^4(A_2)$, which represents the generator of $E_\infty^{4,0} \cong \mathbb{Z}$. Therefore every element $w^i\tilde{z}^j$ of $H^{2i+4j}(A_2)$ represents the corresponding generator of $E_\infty^{4j,2i}$ for $0 \leq i \leq 3$ and $0 \leq j \leq 2$, and has order 2. So every extension problem in which they arise is trivial. The rest of the non-zero groups in E_∞ are $E_\infty^{4,7} \cong E_\infty^{8,7} \cong \mathbb{Z}$, in dimensions 11 and 15, where there are no extension problems.

To find the relation (5.16), we first reduce mod 2 and look back at Proposition 5.6. Since $\rho(w^4) = a^8 = za^4 + z^2$, we must have $w^4 = w^2\tilde{z} + k\tilde{z}^2$ for some odd integer k . Hence $w^6 = w^4\tilde{z} + kw^2\tilde{z}^2 = 0$. Also, w^4 has order 2 and \tilde{z}^2 has order 6, so $k \equiv 3 \pmod{6}$. \square

The appearance of an element of order 6 is interesting, and suggests that further work should be done on $H^*(A_n)$ for larger n . In particular, since $\chi(\mathbb{H}\mathbb{P}^n) = n+1$, the corresponding SSS with coefficient in $\mathbb{Z}[1/2]$ shows that

$$H^*(A_n; \mathbb{Z}[1/2]) \cong H^*(\mathbb{H}\mathbb{P}^n; \mathbb{Z}[1/2]) / ((n+1)p_1^n).$$

So for any odd prime p , each of $H^{4n}(A_n; \mathbb{Z}[1/2])$ and $H^{4n}(A_n)$ contains a summand isomorphic to \mathbb{Z}/p^m , where p^m is the highest power of p dividing $(n+1)$.

5.3 The Map $B(\mathbb{Z}/2 \times S^3) \rightarrow BPin(4)$

For this section, we should remember our notation $p_i \in H^{4i}(BSp(2))$ for the symplectic Pontryagin class, where $i = 1$ and 2 , and $H^*(BSp(2)_+) \cong \mathbb{Z}[p_1, p_2]$. Restriction to $BS^3 = BSp(1) = \mathbb{H}\mathbb{P}^\infty$ acts by mapping p_1 to z and p_2 to 0 , and restriction to $BS^3 \times BS^3 = \mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty$ by mapping p_1 to $z_1 + z_2$ and p_2 to $z_1 z_2$, where $z_1 = z \otimes 1$ and $z_2 = 1 \otimes z$. Then $H^*(B(S^3 \times S^3)_+) \cong H^*((\mathbb{H}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)_+) \cong \mathbb{Z}[z_1, z_2]$.

For BG_A we need to define generators more carefully. Of course we know that $H^*(BG_A) \cong \mathbb{Z}[w, z]/(2w)$, and we have already defined w in Remark 5.4. So we must describe $z \in H^4(BG_A)$ in terms of the model $A = V/G_A$. We shall use the map $\pi_\infty: V/G_A \rightarrow \mathbb{H}\mathbb{P}^\infty$, defined as in Theorem 5.1 by

$$\pi_\infty([u, v]) = [(u + v)/\sqrt{2}]$$

where (u, v) lies in V .

Lemma 5.17. *There is a homotopy commutative triangle*

$$\begin{array}{ccc} A = V/G_A & & \\ \downarrow e & \searrow \pi_\infty & \\ \mathbb{R}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty & \xrightarrow{r_2} & \mathbb{H}\mathbb{P}^\infty \end{array}$$

where e is a homotopy equivalence and r_2 is projection onto the second factor.

Proof. We define e by first considering the maps

$$V \longleftarrow S^\infty \times V \longrightarrow S^\infty \times S^{4\infty+3}.$$

Each space is contractible, and is acted on freely by G_A . The element τ of order 2 acts antipodally on S^∞ , and diagonally on $S^\infty \times V$, and S^3 acts only on the second factors, in the usual way (S^3 acts trivially on S^∞). The left map is projection onto the second factor, and the right map is defined by $(s, (u, v)) \mapsto (s, (u + v)/\sqrt{2})$. So both maps are homotopy equivalences and equivariant. The induced maps of orbit spaces are also homotopy equivalences. They combine to give the triangle. \square

The Lemma states that π_∞ is a model for r_2 , and allows us to use $e^{-1}i_1: \mathbb{R}\mathbb{P}^\infty \rightarrow A$ and $e^{-1}i_2: \mathbb{H}\mathbb{P}^\infty \rightarrow A$ as models for the maps induced by the subgroup inclusions $\mathbb{Z}/2 \rightarrow G_A$ and $S^3 \rightarrow G_A$.

Corollary 5.18. *The element $\tilde{z} \in H^4(A_2)$ arises by pulling back z from $H^4(BG_A)$.*

Proof. This follows from the commutative diagram

$$\begin{array}{ccc} A_2 & \xrightarrow{\pi} & \mathbb{H}\mathbb{P}^2 \\ \downarrow & & \downarrow \\ A & \xrightarrow{\pi_\infty} & \mathbb{H}\mathbb{P}^\infty \end{array}$$

by applying $H^4(-)$. □

Note that the map g of Remark 5.4 can also be constructed from the homotopy equivalence of Lemma 5.17 by applying the projection r_1 instead of r_2 .

Recall (4.32), (4.33), and Remark 4.34. For $a \in S^3$, the composition of the subgroup inclusions $S^3 \hookrightarrow \mathbb{Z}/2 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2)$ is given by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in Sp(2)$ and therefore agrees with $S^3 \hookrightarrow S^3 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2)$ where the first subgroup is the diagonal. Let τ be the generator of $\mathbb{Z}/2$ and consider the composition of inclusions $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2 \times S^3 \rightarrow Pin(4) \rightarrow Sp(2)$, which is given by $\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in Sp(2)$. This may be rewritten as $\tau \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in terms of the basis of eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which agrees with the composition $\mathbb{Z}/2 = O(1) \rightarrow U(1) \rightarrow Sp(1) \rightarrow Sp(2)$.

Proposition 5.19. *Let \tilde{x}, \tilde{y} be the generators for $H^*(BPin(4)_+)$ given in Proposition 4.17. Then, the homomorphism $f^*: H^*(BPin(4)) \rightarrow H^*(BG_A)$ induced by the inclusion $G_A \rightarrow Pin(4)$ satisfies*

$$f^*(\tilde{x}) = 2z + w^2 \tag{5.20}$$

$$f^*(\tilde{y}) = z^2 + w^2z, \tag{5.21}$$

where z and w are defined as in Remark 5.4 and Lemma 5.17.

Proof. Given $\ell: BPin(4) \rightarrow BSp(2)$, then by definition $\tilde{x} = \ell^*(p_1)$ and $\tilde{y} = \ell^*(p_2)$.

We first show (5.20). Since there are no cross terms in $H^4(BG_A)$,

$$H^4(BSp(2)) \xrightarrow{\ell^*} H^4(BPin(4)) \xrightarrow{f^*} H^4(BG_A)$$

$$\text{acts by } p_1 \longmapsto \tilde{x} \longmapsto Az + Bw^2,$$

where $A \in \mathbb{Z}$ and $B = 0$ or 1 .

To find A , we consider

$$BSp(2) \xleftarrow{\ell} BPin(4) \xleftarrow{\Delta} B(S^3 \times S^3) \xleftarrow{\Delta} BS^3$$

and apply $H^4(-)$. The induced homomorphisms are given by $p_1 \mapsto \tilde{x} \mapsto z_1 + z_2 \mapsto 2z$, so $A = 2$ by comparison with (4.32).

To find B , we consider

$$BSp(2) \leftarrow BSp(1) \leftarrow BU(1) \leftarrow B\mathbb{Z}/2,$$

where $BSp(1) = \mathbb{H}\mathbb{P}^\infty$, $BU(1) = \mathbb{C}\mathbb{P}^\infty$ and $B\mathbb{Z}/2 = \mathbb{R}\mathbb{P}^\infty$, and apply $H^4(-)$. The induced homomorphisms are given by $p_1 \mapsto z \mapsto c^2 \mapsto w^2$ [40], where $c \in H^2(\mathbb{C}\mathbb{P}^\infty) \cong \mathbb{Z}$ is the first Chern class of the tautological complex line bundle over $\mathbb{C}\mathbb{P}^\infty$. So $B = 1$ in $\mathbb{Z}/2$, by comparison with (4.33).

We now show (5.21). Suppose $f^*(\tilde{y}) = Cz^2 + Dw^2z + Ew^4$, where $C \in \mathbb{Z}$ and $D, E = 0$ or 1 , and let us consider

$$BSp(2) \xleftarrow{\ell} BPin(4) \leftarrow B(S^3 \times S^3) \xleftarrow{\Delta} BS^3.$$

Then the induced homomorphisms in $H^8(-)$ are $p_2 \mapsto \tilde{y} \mapsto z_1z_2 \mapsto z^2$, so $C = 1$ by comparison with (4.32).

To find D and E , we consider the commutative diagram

$$\begin{array}{ccccc} V_{n+1}/G_A & \xrightarrow{f_n} & V_{n+1}/Pin(4) & \longrightarrow & V_{n+1}/Sp(2) \\ \downarrow & & \downarrow & & \downarrow \\ BG_A & \xrightarrow{f} & BPin(4) & \xrightarrow{\ell} & BSp(2) \end{array}$$

where $V_{n+1}/G_A \cong A_n$, and $V_{n+1}/Pin(4)$ is identified with Γ_n . For $n = 2$, we get

$$\begin{array}{ccccc} A_2 & \xrightarrow{f_2} & \Gamma_2 & \longrightarrow & Gr_{\mathbb{H}}(2, 3) \\ \downarrow & & \downarrow & & \downarrow \\ BG_A & \xrightarrow{f} & BPin(4) & \xrightarrow{\ell} & BSp(2) \end{array}$$

to which we may apply $H^8(-)$ and the results of Section 5.2.

We recall that $H^8((A_2)_+) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/6$ from Proposition 5.15 with corresponding generators are $w^2\tilde{z}$ and \tilde{z}^2 , which pull back from w^2z and z^2 in $H^8(BG_A)$ by Remark 5.4 and Corollary 5.18. Also, $w^4 = w^2\tilde{z} + 3\tilde{z}^2$ from (5.1). We saw that $\tilde{x}^2 = \tilde{y} \in H^8(\Gamma_2)$ in Example 4.20, and $f^*(\tilde{x}) = 2z + w^2$ implies that $f^*(\tilde{x}^2) = f^*(\tilde{x})^2 = (2z + w^2)^2 = 4z^2 + w^4$. Hence by the commutativity of the diagram, we deduce that

$$\tilde{z}^2 + Dw^2\tilde{z} + Ew^4 = 4\tilde{z}^2 + w^4$$

in $H^8(A_2)$. So $(1 + 3E)\tilde{z}^2 + (D + E)w^2\tilde{z} = 7\tilde{z}^2 + w^2\tilde{z}$, and

$$1 + 3E \equiv 1 \pmod{6} \quad \text{and} \quad D + E \equiv 1 \pmod{2}.$$

Therefore, we have that $D = 1$ and $E = 0$, as we wanted. □

Corollary 5.22. *Given $H^*(B_+) \cong \mathbb{Z}[x, y, w]/(2w, wx)$, then*

$$f^*(x^m) = (2z)^m, \quad f^*(y^m) = (z^2 + w^2z)^m = \sum_{k=0}^m \binom{m}{k} (z^2)^{m-k} (w^2z)^k.$$

in $H^{4m}(B)$ and $H^{8m}(B)$ for any $m \geq 1$.

Chapter 6

The Thom Space $\text{Th}(\theta)$

In this Chapter, we will show that the quotient L/A is a Thom space $\text{Th}(\theta)$, where θ is the universal vector bundle over $B\text{Pin}(4)$, which is sometimes denoted by $M\text{Pin}(4)$. The homotopy equivalences $L/A \simeq SP^2/\Delta \simeq B/\widehat{\Delta}$ of Chapter 2 induce isomorphisms in cohomology, which are important in our calculation of $H^*(SP_n^2)$. In earlier chapters we have shown the interconnections between the Borel space and the symmetric square.

Recall from Section 2.2 the commutative diagram

$$\begin{array}{ccccc}
 \Delta_n & \xrightarrow{i} & SP_n^2 & \longrightarrow & SP_n^2/\Delta_n \\
 \uparrow \pi & & \uparrow \pi & & \simeq \uparrow \\
 \widehat{\Delta}_n & \xrightarrow{f} & B_n & \longrightarrow & B_n/\widehat{\Delta}_n
 \end{array} \tag{6.1}$$

where i and f are cofibrations and $\pi: B_n \rightarrow SP_n^2$ is the canonical projection. Also, from the decomposition $SP_n^2 = L_n \cup_{A_n} N_n$ in Section 2.1, and the natural homeomorphism $SP_n^2/N_n \cong L_n/A_n$, we have a commutative diagram

$$\begin{array}{ccccc}
 N_n & \xrightarrow{\iota} & SP_n^2 & \longrightarrow & SP_n^2/N_n \\
 \uparrow & & \uparrow & & \cong \uparrow \\
 A_n & \xrightarrow{j} & L_n & \longrightarrow & L_n/A_n
 \end{array}$$

where ι and j are also cofibrations.

Applying $H^*(-)$ to either diagram yields a ladder of long exact sequences (LES) in cohomology. For the infinite dimensional cases, we consider the spaces $X = \bigcup_n X_n$ as colimits.

6.1 $MPin(4)$

A real n -plane vector bundle ξ over X has the structure group $O(n)$ when a continuous choice of inner product is given each fibre. After making this choice, the Thom space $\text{Th}(\xi) := D(\xi)/S(\xi)$ is constructed by taking the unit disk bundle $D(\xi)$ and collapsing its boundary, the unit sphere bundle $S(\xi)$, to a single point ∞ . Equivalently, $\text{Th}(\xi)$ is homeomorphic to the mapping cone $C(p) = X \cup_p CS(\xi)$ of the projection, $S(\xi) \xrightarrow{p} X$, where ∞ corresponds to the cone point.

Now remember that for the model $\Gamma = V/Pin(4)$ for the classifying space $BPin(4)$, the corresponding action of $Pin(4)$ on $\mathbb{R}^4 \cong \mathbb{H}$ is given in Section 3.2.

Remark 6.2. *The universal 4-disk and 3-sphere bundles associated to θ are given by*

$$D^4 \rightarrow D(\theta) \rightarrow \Gamma \quad \text{for } |h| \leq 1 \quad \text{and} \quad S^3 \rightarrow S(\theta) \rightarrow \Gamma \quad \text{for } |h| = 1,$$

where $D(\theta) = V \times_{Pin(4)} D^4$ and $S(\theta) = V \times_{Pin(4)} S^3$.

Lemma 6.3. *There exists a homeomorphism g between the total space of the sphere bundle $S(\theta)$ and the quotient space V/G_A , where $G_A = \mathbb{Z}/2 \times S^3$.*

Proof. For $a, b, h \in S^3$ and $(u, v) \in V$, we have that

$$S(\theta) = \{((u, v), h) \mid ((u, v), h) \sim ((ua, vb), \bar{a}hb) \sim ((v, u), \bar{h})\}$$

and $V/G_A = \{(u, v) \mid (u, v) \sim (ub, vb) \sim (v, u)\}.$

So we define $g: S(\theta) = V \times_{Pin(4)} S^3 \rightarrow V/G_A$ on equivalence classes by sending $[(u, v), h]$ to $[uh, v]$. For $g^{-1}: V/G_A \rightarrow V \times_{Pin(4)} S^3$, we send $[u, v]$ to $[(u, v), 1]$.

These are well-defined, continuous, and inverse to each other. □

In the subsequent parts, so long as the meaning is clear we use the notation $S(\theta)$ and $D(\theta)$ for $S^3 \rightarrow S(\theta) \rightarrow \Gamma_n$ and $D^4 \rightarrow D(\theta) \rightarrow \Gamma_n$ respectively, as well as infinite dimensional cases.

Proposition 6.4. *For any $n \geq 1$, there exist homeomorphisms*

$$S(\theta) \cong A_n \quad \text{and} \quad D(\theta) \cong L_n,$$

that are consistent with the inclusions $S(\theta) \hookrightarrow D(\theta)$ and $A_n \hookrightarrow L_n$.

Proof. For any $n \geq 1$, consider the open disk bundle $\mathring{D}^4 \rightarrow \mathring{D}(\theta) \rightarrow \Gamma_n$, where $\mathring{D} = \{h \in \mathbb{H} \mid |h| < 1\}$ and $\mathring{D}(\theta) = V_{n+1} \times_{P\text{in}(4)} \mathring{D}^4$. We first show a certain map

$$\epsilon: \mathring{D}(\theta) \rightarrow SP_n^2 \setminus \Delta_n$$

is a homeomorphism. If $h \in \mathbb{H}$ has $|h| < 1$, then ϵ is given by $[(u, v), h] \mapsto (u \ v)N_h^{-1}$ where $N_h^{-1} = \begin{pmatrix} (C+D)/2 & h/(C+D) \\ \bar{h}/(C+D) & (C+D)/2 \end{pmatrix}$ and $C = (1 + |h|)^{1/2}$, $D = (1 - |h|)^{1/2}$ (the matrix N is the symmetric orthogonalisation operator for $(x \ y)$ as in Definition 4.25). The map ϵ is well-defined because $[(u, v), h] \sim [(v, u), \bar{h}] \sim [(ua, vb), \bar{a}hb]$ and $N_h^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N_{\bar{h}}^{-1}$, $N_h^{-1} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} N_{\bar{a}hb}^{-1}$, so $(u \ v)N_h^{-1} \sim (v \ u)N_{\bar{h}}^{-1} \sim (ua \ vb)N_{\bar{a}hb}^{-1}$.

For any pair of unit column vectors $x, y \in \mathbb{H}^{n+1}$, $d([x], [y]) \neq 0$ (chordal distance d as in Definition 2.1) representing an element of $SP_n^2 \setminus \Delta_n$, the inverse of ϵ ,

$$\epsilon^{-1}: SP_n^2 \setminus \Delta_n \rightarrow \mathring{D}(\theta)$$

is given by $[[x], [y]] \mapsto [(\tilde{x}, \tilde{y}), h]$ where $(\tilde{x} \ \tilde{y}) = (x \ y)N$ and $h = x^*y \in \mathbb{H}$, so ϵ^{-1} is also well-defined because $|\bar{h}| = |h| = |x^*y| = |(xa)^*(yb)|$.

Recalling Section 4.2.2, ϵ maps $D(\theta)_{\sqrt{3}/2}$ homeomorphically onto A_n and $D(\theta)_{\leq \sqrt{3}/2}$ homeomorphically onto L_n where

$$\begin{aligned} D(\theta)_{\sqrt{3}/2} &:= \{[(u, v), h] \mid |h| = \sqrt{3}/2\} \\ D(\theta)_{\leq \sqrt{3}/2} &:= \{[(u, v), h] \mid |h| \leq \sqrt{3}/2\}. \end{aligned}$$

Clearly $D(\theta)_{\sqrt{3}/2} \cong S(\theta)$ and $D(\theta)_{\leq \sqrt{3}/2} \cong D(\theta)$. □

Corollary 6.5. *For any $n \geq 1$, the space $L_n/A_n \cong SP_n^2/N_n$ is homeomorphic to the Thom space $\text{Th}(\theta)$ over Γ_n .*

Proof. This follows from the homeomorphism of pairs given by Proposition 6.4. □

Proposition 6.6. *The space $L/A \cong SP^2/N$ is homeomorphic to the Thom space $M\text{Pin}(4)$ over Γ .*

Proof. As n increases, the homeomorphism α and their restrictions to $D(\theta)_{\leq \sqrt{3}/2}$ and $D(\theta)_{\sqrt{3}/2}$ are compatible, so we get homeomorphisms $S(\theta) \cong A$ and $D(\theta) \cong L$. □

6.2 The Cohomology of $MPin(4)$

The main aim of this section is to describe the cohomology ring

$$H^*(MPin(4)) \cong H^*(B/\widehat{\Delta}) \cong H^*(SP^2/\Delta)$$

with \mathbb{Z} and $\mathbb{Z}/2$ coefficients. For example, we shall show that

$$H^*(B/\widehat{\Delta}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} & \text{if } * = 4k > 4 \\ \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 & \text{if } * = 4k + 1, 4k + 3, k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

We use the ladder of long exact sequences

$$\begin{array}{ccccccccccc} \longleftarrow & H^{*+1}(SP^2/\Delta) & \xleftarrow{\delta} & H^*(\Delta) & \xleftarrow{i^*} & H^*(SP^2) & \xleftarrow{s^*} & H^*(SP^2/\Delta) & \longleftarrow & & \\ & \cong \downarrow & & \downarrow & & \downarrow \pi^* & & \downarrow \cong & & & \\ \longleftarrow & H^{*+1}(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^*(\widehat{\Delta}) & \xleftarrow{f^*} & H^*(B) & \xleftarrow{\beta^*} & H^*(B/\widehat{\Delta}) & \longleftarrow & & \end{array} \quad (6.7)$$

that comes from applying $H^*(-) := H^*(-; \mathbb{Z})$ to (2.29). As input we have that $H^*(B_+; \mathbb{Z}) \cong \mathbb{Z}[x, y, w]/(xw, 2w)$, $|x| = 4$, $|y| = 8$, $|w| = 2$ from (4.31), and the cohomology of the diagonals $H^*(\Delta_+; \mathbb{Z}) \cong H^*(\mathbb{H}\mathbb{P}_+^\infty; \mathbb{Z}) \cong \mathbb{Z}[z]$, $|z| = 4$ and $H^*(\widehat{\Delta}_+; \mathbb{Z}) \cong H^*((\mathbb{R}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)_+; \mathbb{Z}) \cong \mathbb{Z}[w, z]/(2w)$, $|z| = 4$, $|w| = 2$, which come from Section 2.3. Also, from Section 5.3

$$f^*(x) = 2z, \quad f^*(y) = z^2 + w^2z \quad (6.8)$$

in $H^*(\widehat{\Delta})$, where $x = \tilde{x} + w^2$ and $y = \tilde{y}$. It follows from the right hand side of diagram (3.7), and the description of $w \in H^2(B)$ in (4.31), that we may also use the same symbol w in both $H^2(B)$ and $H^2(\widehat{\Delta})$, as a way of simplifying the notation.

6.2.1 The Cohomology of $B/\widehat{\Delta}$

We shall start with the case $* = 4k$.

Lemma 6.9. *If $k = 1$, then $H^{4k}(B/\widehat{\Delta}) := H^{4k}(B/\widehat{\Delta}; \mathbb{Z}) = 0$.*

Proof. We have the exact sequence

$$\cdots \longleftarrow H^4(\widehat{\Delta}) \xleftarrow{f^*} H^4(B) \xleftarrow{\beta^*} H^4(B/\widehat{\Delta}) \xleftarrow{\delta} H^3(\widehat{\Delta}) = 0$$

where $H^4(B) \cong \mathbb{Z}/2\langle w^2 \rangle \oplus \mathbb{Z}\langle x \rangle$, $H^4(\widehat{\Delta}) \cong \mathbb{Z}/2\langle w^2 \rangle \oplus \mathbb{Z}\langle z \rangle$. We also have the images $f^*(w^2) = w^2$, $f^*(x) = 2z$, which yield that $\ker f^* = 0$, and therefore $H^4(B/\widehat{\Delta}) \cong \ker f^* = 0$. \square

Lemma 6.10. *If $k \geq 2$, then the cohomology group $H^{4k}(B/\widehat{\Delta}) := H^{4k}(B/\widehat{\Delta}; \mathbb{Z})$ is torsion free, that is, $H^{4k}(B/\widehat{\Delta}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$.*

Proof. From the exact sequence

$$0 \longleftarrow H^{4k+1}(B/\widehat{\Delta}) \xleftarrow{\delta} H^{4k}(\widehat{\Delta}) \xleftarrow{f^*} H^{4k}(B) \xleftarrow{\beta^*} H^{4k}(B/\widehat{\Delta}) \longleftarrow 0$$

we have $H^{4k}(B/\widehat{\Delta}) \cong \ker f^*$, so it is sufficient to show that $\ker f^*$ is torsion free. Suppose, for contradiction, that $\ker f^* \subset H^{4k}(B)$ has a nonzero torsion element σ ; that is, $2\sigma = 0$ and $f^*(\sigma) = 0$. Then

$$\begin{aligned} f^*(\sigma) &= f^* \left(\sum_{a,b} \lambda_{a,b} w^a y^b \right), \lambda_{a,b} \in \{0, 1\}, a \geq 1, b \geq 0, a + 4b = 2k \\ &= f^*(w^{\alpha_1} y^{\beta_1} + w^{\alpha_2} y^{\beta_2} + \cdots + w^{\alpha_j} y^{\beta_j}) \\ &\quad [1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_j, \beta_1 > \beta_2 > \cdots > \beta_j \geq 0] \\ &= w^{\alpha_1} z^{2\beta_1} + \text{other terms} \neq 0. \end{aligned}$$

This is a contradiction. Thus, for $k \geq 2$, $\ker f^* \cong H^{4k}(B/\widehat{\Delta})$ is torsion free. \square

Lemma 6.11. *For any $k \geq 0$, $H^{4k+2}(B/\widehat{\Delta}; \mathbb{Z}) = 0$.*

Proof. From the exact sequence

$$\longleftarrow H^{4k+2}(\widehat{\Delta}) \xleftarrow{f^*} H^{4k+2}(B) \xleftarrow{\beta^*} H^{4k+2}(B/\widehat{\Delta}) \xleftarrow{\delta} H^{4k+1}(\widehat{\Delta}) = 0$$

we have $H^{4k+2}(B/\widehat{\Delta}) \cong \ker f^*$, so it is sufficient to show that $\ker f^* = 0$. From (4.31), $H^{4k+2}(B; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ with basis elements $w^i y^j$, where $i + 4j = 2k + 1$. By (5.22), the images $f^*(w^i y^j)$ are given by $w^i (z^2 + w^2 z)^j$. Suppose, for contradiction, that $\ker f^* \neq 0$ and let σ be a nonzero element of $\ker f^* \subset H^{4k+2}(B)$. Then

$$\begin{aligned} f^*(\sigma) &= f^* \left(\sum_{a,b} \lambda_{a,b} w^a y^b \right), \lambda_{a,b} \in \{0, 1\}, a \geq 1, b \geq 0, a + 4b = 2k + 1 \\ &= f^*(w^{\alpha_1} y^{\beta_1} + w^{\alpha_2} y^{\beta_2} + \cdots + w^{\alpha_j} y^{\beta_j}) \\ &\quad [1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_j, \beta_1 > \beta_2 > \cdots > \beta_j \geq 0] \\ &= w^{\alpha_1} z^{2\beta_1} + \text{other terms} \neq 0. \end{aligned}$$

This is a contradiction. Thus, $H^{4k+2}(B/\widehat{\Delta}) \cong \ker f^* = 0$. \square

Lemma 6.12. *If $k \geq 1$, then*

$$H^{4k+1}(B/\widehat{\Delta}; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2.$$

Proof. Case $k = 1$. We have the exact sequence

$$0 \longleftarrow H^5(B/\widehat{\Delta}) \xleftarrow{\delta} H^4(\widehat{\Delta}) \xleftarrow{f^*} H^4(B) \xleftarrow{\beta^*} H^4(B/\widehat{\Delta}) = 0$$

where $H^4(B) \cong \mathbb{Z}/2\langle w^2 \rangle \oplus \mathbb{Z}\langle x \rangle$ and $H^4(\widehat{\Delta}) \cong \mathbb{Z}/2\langle w^2 \rangle \oplus \mathbb{Z}\langle z \rangle$ where $|w| = 2$, $|x| = |z| = 4$. Also, $(\delta \circ f^*)(x) = \delta(f^*(x)) = \delta(2z) = 0$, and $(\delta \circ f^*)(w^2) = \delta(w^2) = 0$. Clearly δ is surjective and $\delta(z) \neq 0$. Thus $H^5(B/\widehat{\Delta}) \cong \mathbb{Z}/2 = \langle \delta(z) \rangle$.

Case $k \geq 2$. We have the exact sequence

$$0 \longleftarrow H^{4k+1}(B/\widehat{\Delta}) \xleftarrow{\delta} H^{4k}(\widehat{\Delta}) \xleftarrow{f^*} H^{4k}(B) \xleftarrow{\beta^*} H^{4k}(B/\widehat{\Delta}) \longleftarrow 0$$

where $H^{4k}(\widehat{\Delta}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 \oplus \mathbb{Z}$ with minimal generating set $w^a z^b$, z^k for $a+2b = 2k$. Clearly, δ is surjective from the exact sequence. We shall show that $\delta(z^k) \neq 0$ and $2\delta(z^k) = 0$.

If $k \geq 2$ is even, then a minimal generating set for $H^{4k}(B)$ is $y^{k/2}$, $w^i y^j$, $x^\alpha y^\beta$, x^k where $i + 4j = 2k$, $i \geq 4$, $j \geq 0$, $\alpha + 2\beta = k$. The images of these elements under the homomorphism f^* are given by $f^*(y^{k/2}) = (z^2 + w^2 z)^{k/2} = z^k + w^k z^{k/2} +$ other terms, $f^*(w^i y^j) = w^i (z^2 + w^2 z)^j$, $f^*(x^\alpha y^\beta) = (2z)^\alpha z^{2\beta}$, $f^*(x^k) = (2z)^k$, where $k/2 > j$. Then, the preimage $(f^*)^{-1}(z^k)$ is empty, due to the fact $k/2 > j$; thus $\delta(z^k) \neq 0$. Furthermore, $(\delta \circ f^*)(y^{k/2}) = \delta((z^2 + w^2 z)^{k/2}) = \delta\left(z^k + \sum_{a,b} \lambda_{a,b} w^a z^b\right) = 0$, $\lambda_{a,b} \in \{0, 1\}$, implies that $\delta(z^k) = \delta\left(\sum_{a,b} \lambda_{a,b} w^a z^b\right)$, from which we deduce that $2\delta(z^k) = 0$.

An analogous argument is applied to the case $k = \text{odd}$, and the only major difference is that there is no element $y^{k/2}$ for $H^{4k}(B)$. If $k \geq 3$ is odd, then a minimal generating set for $H^{4k}(B)$ is $w^i y^j$, $x^\alpha y^\beta$, x^k where $i + 4j = 2k$, $i \geq 2$, $j \geq 0$, $\alpha + 2\beta = k$, $\alpha \neq 0$. Then from the images of these elements under f^* , it is clear that the preimage $(f^*)^{-1}(z^k)$ is empty; so again $\delta(z^k) \neq 0$. Furthermore, the image of $xy^m \in H^{4k}(B)$, $m = \frac{k-1}{2}$ under the composition $\delta \circ f^*$ is given by $(\delta \circ f^*)(xy^m) = \delta(2z(z^2 + w^2 z)^m) = \delta(2z^{2m+1}) = 2\delta(z^{2m+1}) = 0$. \square

We shall now proceed to the $(4k + 3)$ -th cohomology.

Lemma 6.13. *If $k = 0$, then $H^{4k+3}(B/\widehat{\Delta}; \mathbb{Z}) = 0$.*

Proof. It follows from the exact sequence

$$0 \longleftarrow H^3(B/\widehat{\Delta}) \xleftarrow{\delta} H^2(\widehat{\Delta}) \xleftarrow{f^*} H^2(B) \xleftarrow{\beta^*} H^2(B/\widehat{\Delta}) = 0$$

where $H^2(\widehat{\Delta}) \cong \mathbb{Z}/2\langle w \rangle$, $H^2(B) \cong \mathbb{Z}/2\langle w \rangle$ and $f^*(w) = w$. Clearly δ is surjective and the composition $(\delta \circ f^*)(w) = \delta(w) = 0$. \square

Lemma 6.14. *If $k \geq 1$, then*

$$H^{4k+3}(B/\widehat{\Delta}; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2.$$

Proof. We have the exact sequence

$$H^{4k+3}(B) = 0 \xleftarrow{\beta^*} H^{4k+3}(B/\widehat{\Delta}) \xleftarrow{\delta} H^{4k+2}(\widehat{\Delta}) \xleftarrow{f^*} H^{4k+2}(B) \longleftarrow 0$$

where $H^{4k+2}(\widehat{\Delta}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ with basis $w^i z^j$, $i + 2j = 2k + 1$, $i \geq 1$, $j \geq 0$ and $H^{4k+2}(B) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ with basis $w^\alpha y^\beta$, $\alpha + 4\beta = 2k + 1$, $\alpha \geq 1$, $\beta \geq 0$. Clearly δ is surjective, and its image is the quotient $\mathbb{Z}/2$ vector space, as required. Note that $(\delta \circ f^*)(w^\alpha y^\beta) = \delta(w^\alpha (z^2 + zw^2)^\beta) = 0$ in $H^{4k+3}(B/\widehat{\Delta}; \mathbb{Z})$. \square

Example 6.15. *Case $k = 1$. We have*

$$0 \longleftarrow H^7(B/\widehat{\Delta}) \xleftarrow{\delta} H^6(\widehat{\Delta}) \xleftarrow{f^*} H^6(B) \longleftarrow 0$$

where $H^6(B) \cong \mathbb{Z}/2\langle w^3 \rangle$, $H^6(\widehat{\Delta}) \cong \mathbb{Z}/2\langle w^3 \rangle \oplus \mathbb{Z}/2\langle wz \rangle$ and $f^*(w^3) = w^3$. Then $(\delta \circ f^*)(w^3) = \delta(w^3) = 0$. Clearly δ is surjective, and $H^7(B/\widehat{\Delta}; \mathbb{Z}) \cong \mathbb{Z}/2$ is generated by $\delta(wz)$.

6.2.2 Thom Isomorphism

There is a useful reference book [53] for this section; we shall list some basics related to the Thom isomorphism below.

Given the Thom space $\text{Th}(\xi)$ of an n -plane bundle $\mathbb{R}^n \rightarrow \mathbb{R}(\xi) \rightarrow X$,

- (1) a unique Thom isomorphism $H^*(X_+; \mathbb{Z}/2) \cong H^{*+n}(\text{Th}(\xi); \mathbb{Z}/2)$ always exists over $\mathbb{Z}/2$ coefficients, and is given by relative cup product with the $\mathbb{Z}/2$ Thom class in $H^n(\text{Th}(\xi); \mathbb{Z}/2)$

- (2) the existence of a Thom isomorphism $H^*(X_+; \mathbb{Z}) \cong H^{*+n}(\text{Th}(\xi); \mathbb{Z})$ is equivalent to the vanishing of the first Stiefel-Whitney class $w_1(\xi)$ of ξ in $H^1(X; \mathbb{Z}/2)$, and to the existence of a Thom class t (or $-t$) in $H^n(\text{Th}(\xi))$
- (3) the reduced cohomology groups $H^*(\text{Th}(\xi); R)$ are trivial for $* < n$ and any coefficient ring R
- (4) if ξ is orientable, the Euler class $e = e(\xi) \in H^n(X; \mathbb{Z})$ may be defined [23, page 88] to be the restriction of the Thom class $t \in H^n(\text{Th}(\xi); \mathbb{Z})$ to the zero section; it can also be defined as the class et corresponding to t^2 under the Thom isomorphism.

By a Theorem of René Thom [42, page 91], for $i \leq n = |t|$ we have that

$$\text{Sq}^i t = w_i(\xi) t \quad (6.16)$$

in $H^{n+i}(\text{Th}(\xi); \mathbb{Z}/2)$, where $w_i(\xi)$ is the i -th Stiefel-Whitney class. This implies that the mod 2 Euler class $e = w_n(\xi)$ since the Steenrod square Sq^k with $k = |t|$ acts by $\text{Sq}^k t = t^2 = w_k(\xi) t$.

We shall apply (1) – (4) to our case, $\text{Th}(\theta) = MPin(4)$ and $X = BPin(4)$, where θ is the universal $Pin(4)$ vector bundle, $\mathbb{R}^4 \rightarrow \mathbb{R}(\theta) \rightarrow BPin(4)$ as in Definition 3.10, and $BPin(4) = \Gamma$.

- (1) There exists the Thom isomorphism τ

$$\tau: H^*(BPin(4)_+; \mathbb{Z}/2) \xrightarrow{\cong} H^{*+4}(MPin(4); \mathbb{Z}/2) \quad (6.17)$$

defined by $\tau(x) = xt$ where t is the unique Thom class

$$t \in H^4(MPin(4); \mathbb{Z}/2). \quad (6.18)$$

- (2) There is no such isomorphism (6.17) over \mathbb{Z} , because $w_1(\theta)$ is non-zero (6.21) (in other words, the bundle θ is not orientable).
- (3) The bundle θ is a 4-plane bundle, so

$$H^*(MPin(4); \mathbb{Z}/2) \cong H^*(B/\widehat{\Delta}; \mathbb{Z}/2) = 0 \quad \text{if } * < 4. \quad (6.19)$$

(4) The Euler class $e = e(\theta) \in H^4(BPin(4); \mathbb{Z}/2)$ is

$$e \stackrel{\text{def}}{=} j^*(t), \quad j^*: H^*(MPin(4); \mathbb{Z}/2) \rightarrow H^*(BPin(4); \mathbb{Z}/2)$$

where j^* is induced by the inclusion of the zero section $BPin(4) \xrightarrow{j} MPin(4)$.

So $e = w_n(\theta)$.

By Lemma 4.22 we have

$$H^*(BPin(4)_+; \mathbb{Z}/2) \cong H^*(BSp(2)_+; \mathbb{Z}/2)[a]/(a^5 - p_1a),$$

where $H^*(BSp(2)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[p_1, p_2]$, $|p_i| = 4i$, $|a| = 1$.

Lemma 6.20. *Let τ be the Thom isomorphism (6.17)*

$$\tau: H^*(BPin(4)_+; \mathbb{Z}/2) \rightarrow H^{*+4}(MPin(4); \mathbb{Z}/2).$$

Then the relation

$$t^2 = (p_1 + a^4)t$$

holds in $H^8(MPin(4); \mathbb{Z}/2)$.

Proof. Consider the map $BPin(4) \xrightarrow{\ell} BSp(2) \cong BSpin(5)$ and the universal $Spin(5)$ vector bundle χ_5 over $BSpin(5)$. It follows from Section 3.2 that $\ell^*(\chi_5) = \theta \oplus \lambda$ where θ is a 4-plane bundle and λ is a non-trivial line bundle. This agrees with results in [30] and [5], although $Pin(4)$ was not introduced in [5]; instead the wreath product of S^3 and $\mathbb{Z}/2$, and the normaliser of $Spin(4)$ in $Spin(5)$ were used.

Let $w(-)$ be the total Stiefel-Whitney class and $p_1 := \ell^*(p_1)$ in $H^4(BPin(4); \mathbb{Z}/2)$ be the mod 2 symplectic Pontryagin class. Because $w(\chi_5) = 1 + p_1$ in $H^*(BSp(2))$, we have that $w(\ell^*(\chi_5)) = w(\theta \oplus \lambda) = w(\theta)w(\lambda) \Rightarrow 1 + p_1 = w(\theta)(1 + a)$, so $w(\theta)$ is given by

$$\begin{aligned} w(\theta) &= 1 + w_1(\theta) + w_2(\theta) + w_3(\theta) + w_4(\theta) \\ &= 1 + a + a^2 + a^3 + (a^4 + p_1) + a^5 + p_1a + \cdots \end{aligned}$$

Equating terms with the same dimensions yields

$$w_1(\theta) = a \tag{6.21}$$

$$w_2(\theta) = a^2 \tag{6.22}$$

$$w_3(\theta) = a^3 \tag{6.23}$$

$$w_4(\theta) = a^4 + p_1 \tag{6.24}$$

$$0 = a^5 + p_1a \tag{6.25}$$

and hence that $t^2 = et = w_4(\theta)t = (a^4 + p_1)t$. \square

From [30], the obstructions to putting Pin^\pm and $Spin$ structures on an n -plane bundle $\xi, \mathbb{R}^n \rightarrow E \rightarrow X$ are

- $w_2(\xi)$ for a Pin^+ structure
- $w_1^2(\xi) + w_2(\xi)$ for a Pin^- structure
- $w_1(\xi)$ and $w_2(\xi)$ for a $Spin$ structure,

where $w_i(\xi) \in H^i(X; \mathbb{Z}/2)$ is the i -th Stiefel-Whitney class. These are consistent with our computations, because from (6.21) and (6.22), we have that $(w_1(\theta))^2 = a^2 = w_2(\theta)$.

This confirms that θ is indeed the universal $Pin(4)$ bundle.

We also recall a form of the Universal Coefficient Theorem (UCT) that is very useful for our computations.

Theorem 6.26 ([54], page 66). *For any abelian group G and any $m \geq 1$, there exists an isomorphism*

$$H^m(X, G) \cong H^m(X, \mathbb{Z}) \otimes G \oplus \text{Tor}(H^{m+1}(X, \mathbb{Z}), G). \quad (6.27)$$

For any finitely generated abelian groups G and G' , the torsion product $\text{Tor}(G, G')$ may be built up from the properties [54]

$$\text{Tor}(G, G') \cong \text{Tor}(G', G), \quad \text{Tor}(G, \mathbb{Z}) \cong \text{Tor}(\mathbb{Z}, G) = 0,$$

$$\text{Tor}(\mathbb{Z}/p, \mathbb{Z}/q) \cong \mathbb{Z}/\text{gcd}(p, q) \text{ for any } p, q \geq 2,$$

$$\text{if } G = \bigoplus_i G_i, \quad G' = \bigoplus_i G'_i, \quad \text{then } \text{Tor}(G, G') \cong \bigoplus_{i,j} \text{Tor}(G_i, G'_j).$$

Lemma 6.28. *If $* = 4k + 1, 4k + 3, k \geq 1$, then the reduction mod 2 homomorphism*

$$\rho: H^*(B/\widehat{\Delta}; \mathbb{Z}) \rightarrow H^*(B/\widehat{\Delta}; \mathbb{Z}/2)$$

is an isomorphism.

Proof. Apply formula (6.27) for $G = \mathbb{Z}/2, X = B/\widehat{\Delta}, m = 4k + 3, 4k + 1$.

Case $* = 4k + 3$. From Section 6.2, $H^{4(k+1)}(B/\widehat{\Delta}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ is torsion free, and $H^{4k+3}(B/\widehat{\Delta}; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ by Lemma 6.14. By applying (6.27)

we have an isomorphism $H^{4k+3}(B/\widehat{\Delta}; \mathbb{Z}) \otimes \mathbb{Z}/2 \cong H^{4k+3}(B/\widehat{\Delta}; \mathbb{Z}/2)$, and therefore $\rho: H^{4k+3}(B/\widehat{\Delta}; \mathbb{Z}) \xrightarrow{\cong} H^{4k+3}(B/\widehat{\Delta}; \mathbb{Z}/2)$.

Case $* = 4k + 1$. The cohomology $H^{4k+2}(B/\widehat{\Delta}; \mathbb{Z}) = 0$ by Lemma 6.11, and $H^{4k+1}(B/\widehat{\Delta}; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ by Lemma 6.12. By applying (6.27) we have an isomorphism $H^{4k+1}(B/\widehat{\Delta}; \mathbb{Z}) \otimes \mathbb{Z}/2 \cong H^{4k+1}(B/\widehat{\Delta}; \mathbb{Z}/2)$, and therefore deduce $\rho: H^*(B/\widehat{\Delta}; \mathbb{Z}) \xrightarrow{\cong} H^*(B/\widehat{\Delta}; \mathbb{Z}/2)$. \square

We shall further describe $H^*(B/\widehat{\Delta}; \mathbb{Z}/2)$ for $* = 4k + 1, 4k + 3$.

Proposition 6.29. *If $* = 4k + 1, 4k + 3$, then*

$$H^*(B/\widehat{\Delta}; \mathbb{Z}/2) \cong \underbrace{\mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2}_{\ell+1 \text{ summands}} =: (\mathbb{Z}/2)^{\ell+1}$$

where $\ell = \frac{k-1}{2}$ when k is odd, and $\ell = \frac{k}{2} - 1$ when k is even.

Proof. We use the Thom isomorphism

$$H^*(\Gamma_+; \mathbb{Z}/2) \cong H^{*+4}(B/\widehat{\Delta}; \mathbb{Z}/2). \quad (6.30)$$

First observe that $H^{4m}(\Gamma; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 = (\mathbb{Z}/2)^{m+1}$, because a basis is given by the $4m$ dimensional classes

$$p_1^m, p_1^{m-1}a^4, p_1^{m-2}p_2, p_1^{m-3}p_2a^4, p_1^{m-4}p_2^2, \cdots, p_1p_2^{\ell-1}a^4, p_2^\ell, \quad \text{if } m = 2\ell$$

$$p_1^m, p_1^{m-1}a^4, p_1^{m-2}p_2, p_1^{m-3}p_2a^4, \cdots, p_1p_2^\ell, p_2^\ell a^4, \quad \text{if } m = 2\ell + 1,$$

in $H^*(Gr_{\mathbb{H}}(2, \infty); \mathbb{Z}/2)\langle 1, a, a^2, a^3, a^4 \rangle$, where $|a| = 1$, $|p_i| = 4i$, $a^5 = p_1a$.

Then, multiplying each basis element of $H^{4m}(\Gamma; \mathbb{Z}/2)$ by a with the relation $a^5 = p_1a$, yields

$$H^{4m+1}(\Gamma; \mathbb{Z}/2) = \begin{cases} (\mathbb{Z}/2)^{\frac{m}{2}+1} & \text{if } m = 2\ell \\ (\mathbb{Z}/2)^{\frac{1}{2}(m+1)} & \text{if } m = 2\ell + 1 \end{cases}$$

with basis elements of the form $p_1^\lambda p_2^\mu a$. Multiplying by a^2 then gives basis elements $p_1^\lambda p_2^\mu a^3$ for $H^{4m+3}(\Gamma; \mathbb{Z}/2)$.

We may now write for both $* = 4m + 1$ and $* = 4m + 3$

$$H^*(\Gamma; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\ell+1}$$

where $\ell = \frac{m}{2}$ if m is even, and $\ell = \frac{1}{2}(m - 1)$ if m is odd.

For $* = 4k + 1, 4k + 3$ with $k = m + 1$, we deduce, by the Thom isomorphism (6.30), that there is an isomorphism

$$H^*(B/\widehat{\Delta}; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\ell+1},$$

where $\ell = \frac{k-1}{2}$ if k is odd, and $\ell = \frac{1}{2}(k-2)$ if k is even. \square

We shall say more about $H^*(B/\widehat{\Delta}; \mathbb{Z}/2)$ later in Section 7.2.

Corollary 6.31. *By Lemma 6.28, if $* = 4k + 1, 4k + 3$, then there is an isomorphism*

$$H^*(B/\widehat{\Delta}; \mathbb{Z}) \cong (\mathbb{Z}/2)^{\ell+1}$$

of abelian groups, where $\ell = \frac{k-1}{2}$ if k is odd, and $\ell = \frac{k}{2} - 1$ if k is even.

Example 6.32. *Corollary 6.31 may be written in an alternative way as follows.*

$$\begin{aligned} H^*(B/\widehat{\Delta}; \mathbb{Z}) &\cong \mathbb{Z}/2 && \text{for } * = 5, 7, 9, 11 \\ H^*(B/\widehat{\Delta}; \mathbb{Z}) &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 && \text{for } * = 13, 15, 17, 19 \\ H^*(B/\widehat{\Delta}; \mathbb{Z}) &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 && \text{for } * = 21, 23, 25, 27 \\ &\vdots && \vdots \\ H^*(B/\widehat{\Delta}; \mathbb{Z}) &\cong (\mathbb{Z}/2)^a, && a = 1, 2, 3, \dots \\ &&& \text{for } * = 8a - 3, \quad 8a - 1, \quad 8a + 1, \quad 8a + 3. \end{aligned}$$

Now recall from (6.7) the LES

$$\dots \leftarrow H^{*+1}(B/\widehat{\Delta}) \xleftarrow{\delta} H^*(\widehat{\Delta}) \xleftarrow{f^*} H^*(B) \leftarrow H^*(B/\widehat{\Delta}) \leftarrow \dots$$

Lemma 6.33. *For any $m \geq 1$, there exists an expression*

$$\delta(w^{2m}z^m) = \sum_{0 \leq j < 2i} \lambda_{j,i} \delta(w^j z^i), \quad \lambda_{j,i} \in \{0, 1\}, \quad 2i + j = 4m \quad (6.34)$$

in $H^{8m+1}(B/\widehat{\Delta}; \mathbb{Z})$.

Proof. This follows directly from the fact that $(\delta \circ f^*)(y^m) = \delta((z^2 + w^2z)^m) = 0$ in the 4-term exact sequence

$$0 \longleftarrow H^{8m+1}(B/\widehat{\Delta}) \xleftarrow{\delta} H^{8m}(\widehat{\Delta}) \xleftarrow{f^*} H^{8m}(B) \longleftarrow H^{8m}(B/\widehat{\Delta}) \longleftarrow 0.$$

For example, $\delta(w^6z^3) = \delta(z^6) + \delta(w^2z^5) + \delta(w^4z^4)$ when $m = 3$. \square

Corollary 6.35. *Any element $\delta(w^b z^m) \in H^*(B/\widehat{\Delta})$, $b > 2m$, $* = 2b + 4m + 1$ can be expressed as*

$$\delta(w^b z^m) = \sum_{0 \leq j < 2i} \lambda_{j,i} \delta(w^j z^i), \quad \lambda_{j,i} \in \{0, 1\}.$$

Proof. Let $(2m, m)$ denote $\delta(w^{2m} z^m)$ in $H^{8m+1}(B/\widehat{\Delta})$. For any $m \geq 1$, we have

$$(2m, m) = (b_1, a_1) + (b_2, a_2) + \cdots + (b_i, a_i) + \cdots + (b_r, a_r)$$

where $b_i < 2a_i$ by Lemma 6.33. So

$$(2m + \beta, m) = (b_1 + \beta, a_1) + (b_2 + \beta, a_2) + \cdots + (b_r + \beta, a_r);$$

and if $b_i + \beta \geq 2a_i$, then the process may be iterated until it terminates. \square

Example 6.36. *Given the expressions $(12, 6) = (0, 12) + (4, 10) + (8, 8)$, $(10, 5) = (0, 10) + (2, 9) + (8, 6)$, $(8, 4) = (0, 8)$, $(6, 3) = (0, 6) + (2, 5) + (4, 4)$, $(4, 2) = (0, 4)$, $(2, 1) = (0, 2)$. Then, $(b, m) = (22, 3)$ can be written as*

$$\begin{aligned} (22, 3) &= (6 + 16, 3) \\ &= (16, 6) + (18, 5) + (20, 4) \\ &= (12 + 4, 6) + (10 + 8, 5) + (8 + 12, 4) \\ &= (4, 12) + (8, 10) + (12, 8) + (8, 10) + (10, 9) + (16, 6) + (12, 8) \\ &= (4, 12) + (10, 9) + (16, 6) \\ &= (4, 12) + (10, 9) + (12 + 4, 6) \\ &= (4, 12) + (10, 9) + (4, 12) + (8, 10) + (12, 8) \\ &= (10, 9) + (8, 10) + (12, 8), \end{aligned}$$

and the process has terminated.

Proposition 6.37. *If $* = 4k + 3$, $4k + 1$, $k \geq 1$, then a basis for the $\mathbb{Z}/2$ vector space $H^*(B/\widehat{\Delta}; \mathbb{Z})$ is given by $\delta(w^j z^i)$ where $i \geq 1$, $0 \leq j < 2i$.*

Proof. If $* = 4k + 3$, $4k + 1$, $k \geq 1$ then we have the LES

$$0 = H^*(B) \longleftarrow H^*(B/\widehat{\Delta}) \xleftarrow{\delta} H^{*-1}(\widehat{\Delta}) \longleftarrow H^{*-1}(B) \longleftarrow$$

with the obvious surjection δ . By Corollary 6.35 for any $\delta(w^b z^a) \in H^*(B/\widehat{\Delta})$ with $b \geq 2a$, there exists an expression $\delta(w^b z^a) = \sum_{0 \leq j < 2i} \lambda_{j,i} \delta(w^j z^i)$, $\lambda_{j,i} \in \{0, 1\}$. Also, from Corollary 6.31 and Example 6.32, for the positive integers $m' = 1, 2, 3, 4, \dots$

$$H^{*+1}(B/\widehat{\Delta}; \mathbb{Z}) \cong (\mathbb{Z}/2)^{m'}$$

where $*$ = $8m' - 4$, $8m' - 2$, $8m'$, $8m' + 2$. It is easy to check that the number of images $\delta(w^j z^i)$ with $1 \leq i$, $0 \leq j < 2i$ is equal to m' , by considering generators $w^j z^i$ for $H^*(\widehat{\Delta})$ with $1 \leq i$, $0 \leq j < 2i$ for each dimension $*$ = $8m' - 4$, $8m' - 2$, $8m'$, $8m' + 2$.

They are:

$$\text{for } * = 8m' - 4 : w^{2m'-2} z^{m'}, w^{2m'-4} z^{m'+1}, \dots, w^2 z^{2m'-2}, z^{2m'-1}$$

$$\text{for } * = 8m' - 2 : w^{2m'-1} z^{m'}, w^{2m'-3} z^{m'+1}, \dots, w^3 z^{2m'-2}, w z^{2m'-1}$$

$$\text{for } * = 8m' : w^{2m'-2} z^{m'+1}, \dots, w^2 z^{2m'-1}, z^{2m'}$$

$$\text{for } * = 8m' + 2 : w^{2m'-1} z^{m'+1}, \dots, w^3 z^{2m'-1}, w z^{2m'}$$

There are indeed m' of these in each case, and the other generators all have $j \geq 2i$. So δ is a surjection from the $\mathbb{Z}/2$ vector space they generate (after tensoring with $\mathbb{Z}/2$ in cases $8m' - 4$ and $8m'$) onto the $\mathbb{Z}/2$ vector space $H^*(B/\widehat{\Delta}; \mathbb{Z})$. Since both vector space have dimension m' , the restriction of δ is an isomorphism, as we wanted. \square

Chapter 7

The Cohomology Ring I

The main aim of this chapter is to justify Theorem 1.7. As we mentioned earlier in Chapter 2, for a quotient orbifold such as the symmetric product $SP^n(M)$, there exists a rational isomorphism [1, page 38] between the orbit space $\mathfrak{X} = M^n/\Sigma_n$ and the associated Borel space \mathfrak{B} in cohomology

$$H^*(\mathfrak{B}; \mathbb{Q}) \cong H^*(\mathfrak{X}; \mathbb{Q}). \quad (7.1)$$

This well known fact is quite useful for our computation, from which we can immediately deduce that the numbers of \mathbb{Z} -summands in the integral cohomology of the symmetric square and the associated Borel construction are the same.

Another relevant result comes from Nakaoka's work [46], through which we have an isomorphism $H^*(SP^2(X); \mathbb{Z}) \cong H^*(X; \mathbb{Z})$ for $* \leq r + 1$ where X is a connected CW complex space such that its homology is $H_*(X; \mathbb{Z}) = 0$ for $* < r$. Applying this to our case yields

$$H^*(SP^2; \mathbb{Z}) = 0 \quad \text{if } * = 1, 2, 3, 5, \quad \text{and} \quad H^4(SP^2; \mathbb{Z}) \cong \mathbb{Z}. \quad (7.2)$$

We shall reprove these results in passing; the integral cohomology ring $H^*(SP^2)$ when $* > 5$ is our primary interest.

7.1 $H^*(SP^2(\mathbb{H}\mathbb{P}^\infty); \mathbb{Z})$

Our main focus in this section is to describe $H^*(SP^2) := H^*(SP^2(\mathbb{H}\mathbb{P}^\infty); \mathbb{Z})$. For example, we shall show that

$$H^*(SP^2) \cong \begin{cases} \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} & \text{if } * = 4k \\ \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 & \text{if } * = 4k + 1 \geq 13, \\ & \text{or } * = 4k + 3 \geq 7 \\ 0 & \text{otherwise.} \end{cases}$$

We use the ladder of long exact sequences

$$\begin{array}{ccccccc} \longleftarrow H^{*+1}(SP^2/\Delta) & \xleftarrow{\delta} & H^*(\Delta) & \xleftarrow{i^*} & H^*(SP^2) & \xleftarrow{s^*} & H^*(SP^2/\Delta) \longleftarrow \\ & \cong \downarrow & \downarrow \pi_\Delta^* & & \downarrow \pi^* & & \cong \downarrow \\ \longleftarrow H^{*+1}(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^*(\widehat{\Delta}) & \xleftarrow{f^*} & H^*(B) & \xleftarrow{\beta^*} & H^*(B/\widehat{\Delta}) \longleftarrow \end{array} \quad (7.3)$$

that comes from applying $H^*(-) := H^*(-; \mathbb{Z})$ to (2.29), where π_Δ^* is induced by $\pi|_\Delta$.

Remark 7.4. *It is occasionally helpful to write the upper and lower coboundary homomorphisms in this ladder as δ_u and δ_l respectively.*

As input we have

$$H^*(B_+; \mathbb{Z}) \cong \mathbb{Z}[x, y, w]/(xw, 2w) \text{ with } |x| = 4, |y| = 8, |w| = 2$$

from (4.31), and the cohomology of the diagonals

$$H^*(\Delta_+; \mathbb{Z}) \cong H^*(\mathbb{H}\mathbb{P}_+^\infty; \mathbb{Z}) \cong \mathbb{Z}[z] \text{ with } |z| = 4$$

and

$$\begin{aligned} H^*(\widehat{\Delta}_+; \mathbb{Z}) &\cong H^*((\mathbb{R}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty)_+; \mathbb{Z}) \\ &\cong \mathbb{Z}[w, z]/(2w) \text{ with } |z| = 4, |w| = 2, \end{aligned}$$

which come from Section 2.3. The homomorphism f^* from (6.8) given by $f^*(x) = 2z$, $f^*(y) = z^2 + w^2z$, $f^*(w) = w$, and the restricted homomorphism $\pi_\Delta^*(z^k) = z^k$, which comes from the projection map $\mathbb{R}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^\infty \rightarrow \mathbb{H}\mathbb{P}^\infty$, are important and used throughout this chapter and the next. We should also recall Proposition 6.37 which states that if $* = 4k + 3$, $4k + 1$, $k \geq 1$ then $H^*(B/\widehat{\Delta}; \mathbb{Z}) = \mathbb{Z}/2 \langle \delta(w^j z^i) \rangle$ where $i \geq 1$, $0 \leq j < 2i$ and $|\delta(w^j z^i)| = 4i + 2j + 1$.

7.1.1 $H^{4k}(SP^2(\mathbb{H}\mathbb{P}^\infty))$

We shall begin with the $4k$ -th cohomology $H^{4k}(SP^2) := H^{4k}(SP^2(\mathbb{H}\mathbb{P}^\infty); \mathbb{Z})$.

For $k = 1$, the cohomology group $H^4(SP^2) \cong \mathbb{Z}$ from (7.2) is expected by Nakaoka's work [46]. We shall confirm his result by using the commutative ladder

$$\begin{array}{ccccccc}
 \longleftarrow & H^5(SP^2/\Delta) & \longleftarrow & H^4(\Delta) & \xleftarrow{i^*} & H^4(SP^2) & \xleftarrow{s^*} & H^4(SP^2/\Delta) & \longleftarrow & 0 \\
 & \cong \downarrow & & \downarrow & & \downarrow \pi^* & & \cong \downarrow & & \\
 0 & \longleftarrow & H^5(B/\widehat{\Delta}) & \longleftarrow & H^4(\widehat{\Delta}) & \longleftarrow & H^4(B) & \longleftarrow & H^4(B/\widehat{\Delta}) & \longleftarrow & 0
 \end{array} \tag{7.5}$$

induced by (2.29). Here, $H^4(\Delta) \cong \mathbb{Z}$, and by Lemma 6.9 we have that $H^4(B/\widehat{\Delta}) = 0$. The rational isomorphism $H^*(SP^2; \mathbb{Q}) \cong H^*(B; \mathbb{Q})$ from (7.1) and $H^4(B) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ from (4.31), suggest that the cohomology group $H^4(SP^2) \cong \mathbb{Z} \oplus \text{torsion}$. However, the LES

$$\dots \leftarrow \mathbb{Z} \cong H^4(\Delta) \xleftarrow{i^*} H^4(SP^2) \xleftarrow{s^*} H^4(SP^2/\Delta) = 0$$

(of the upper row) in the ladder (7.5), implies that $\ker i^* \cong \text{im } s^* = 0$. Then, the 4-th cohomology group $H^4(SP^2)$ must be torsion free, which confirms the result (7.2).

In general, we have the following.

Lemma 7.6. *For any $k \geq 1$, the integral cohomology $H^{4k}(SP^2)$ is torsion free, that is, $H^{4k}(SP^2) \cong \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$.*

Proof. We have the ladder

$$\begin{array}{ccccccc}
 \longleftarrow & H^{4k+1}(SP^2/\Delta) & \longleftarrow & H^{4k}(\Delta) & \xleftarrow{i^*} & H^{4k}(SP^2) & \xleftarrow{s^*} & H^{4k}(SP^2/\Delta) & \longleftarrow & 0 \\
 & \cong \downarrow & & \downarrow & & \downarrow \pi^* & & \cong \downarrow & & \\
 \longleftarrow & H^{4k+1}(B/\widehat{\Delta}) & \longleftarrow & H^{4k}(\widehat{\Delta}) & \longleftarrow & H^{4k}(B) & \longleftarrow & H^{4k}(B/\widehat{\Delta}) & \longleftarrow & 0
 \end{array} \tag{7.7}$$

where $H^{4k}(\Delta) \cong \mathbb{Z}$, and the cohomology $H^{4k}(B/\widehat{\Delta})$ is torsion free by Lemma 6.10. Then, the canonical isomorphism $H^*(SP^2/\Delta) \cong H^*(B/\widehat{\Delta})$ yields

$$\dots \leftarrow \mathbb{Z} \xleftarrow{i^*} H^{4k}(SP^2) \xleftarrow{s^*} \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \leftarrow 0$$

which shows that $H^{4k}(SP^2)$ must be torsion free. \square

Let us describe the cohomology ladder in more detail, this time paying attention to generators and the maps between them. We shall use various inputs mentioned at the beginning of this chapter.

Case $k = 1$. Let us describe the ladder of exact sequences

$$\begin{array}{ccccccc}
 \dots & \longleftarrow & H^5(SP^2/\Delta) & \xleftarrow{\delta} & H^4(\Delta) & \xleftarrow{i^*} & H^4(SP^2) \cong \mathbb{Z} \longleftarrow 0 \\
 & & \downarrow \cong & & \downarrow \pi_\Delta^* & & \downarrow \pi^* \\
 0 & \longleftarrow & H^5(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^4(\widehat{\Delta}) & \xleftarrow{f^*} & H^4(B) \longleftarrow 0
 \end{array} \tag{7.8}$$

Here, $H^4(\Delta) \cong \mathbb{Z}\langle z \rangle$, and by Proposition 6.37 we have $H^5(B/\widehat{\Delta}) \cong \mathbb{Z}/2\langle \delta(z) \rangle$. The homomorphism π_Δ^* sends $z \mapsto z$, and the isomorphism $H^5(SP^2/\Delta) \cong H^5(B/\widehat{\Delta})$ yields that $\text{im } i^* \cong \ker \delta = \mathbb{Z}\langle 2z \rangle$. Furthermore, f^* is an injection, π^* is an injection because it is a rational isomorphism, and $f^*(x) = 2z$. These facts lead to the following.

Definition 7.9. *Given the homomorphism $\pi^*: H^4(SP^2) \rightarrow H^4(B)$, we define the generator*

$$g \in H^4(SP^2) \cong \mathbb{Z}$$

as the unique preimage of $x \in H^4(B)$ under π^* . So $i^*(g) = 2z$ in $H^4(\Delta)$.

Case $k = 2$. Let us describe the cohomology ladder

$$\begin{array}{cccccccc}
 \longleftarrow & H^9(SP^2/\Delta) & \xleftarrow{\delta} & H^8(\Delta) & \xleftarrow{i^*} & H^8(SP^2) & \xleftarrow{s^*} & H^8(SP^2/\Delta) \longleftarrow 0 \\
 & \cong \downarrow & & \downarrow \pi_\Delta^* & & \downarrow \pi^* & & \cong \downarrow \\
 0 & \longleftarrow & H^9(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^8(\widehat{\Delta}) & \xleftarrow{f^*} & H^8(B) & \xleftarrow{\beta^*} & H^8(B/\widehat{\Delta}) \longleftarrow 0
 \end{array}$$

This time, we have $H^8(B) \cong \mathbb{Z}\langle x^2 \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}/2\langle w^4 \rangle$, and $H^8(\widehat{\Delta}) \cong \mathbb{Z}\langle z^2 \rangle \oplus \mathbb{Z}/2\langle w^2z \rangle \oplus \mathbb{Z}/2\langle w^4 \rangle$. From the ladder, $H^8(B/\widehat{\Delta}) \cong \ker f^*$, and $\ker f^* \cong \mathbb{Z}$, generated by $x^2 - 4y$, which follows from (6.8). Then, the canonical isomorphism $H^8(SP^2/\Delta) \cong H^8(B/\widehat{\Delta})$ implies that $\ker i^* \cong H^8(SP^2/\Delta) \cong \mathbb{Z}$. So we may choose a generator σ for $\ker i^*$ such that $\pi^*(\sigma) = x^2 - 4y \in H^8(B)$, and a generator $\bar{\sigma}$ for $H^8(SP^2/\Delta)$ such that $s^*(\bar{\sigma}) = \sigma$. Input from Proposition 6.37 gives an isomorphism $H^9(B/\widehat{\Delta}) \cong \mathbb{Z}/2\langle \delta(z^2) \rangle$. By an analogous argument to the 4-th cohomology case, we then have that $\text{im } i^* \cong \ker \delta = \mathbb{Z}\langle 2z^2 \rangle$, where $H^8(\Delta) \cong \mathbb{Z}\langle z^2 \rangle$. So there is a short exact sequence

$$0 \leftarrow \mathbb{Z}\langle 2z^2 \rangle \xleftarrow{i^*} H^8(SP^2) \xleftarrow{s^*} \mathbb{Z}\langle \bar{\sigma} \rangle \leftarrow 0, \tag{7.10}$$

which must be split. This proves that $H^8(SP^2) \cong \mathbb{Z}\langle \alpha \rangle \oplus \mathbb{Z}\langle \sigma \rangle$, where $i^*(\alpha) = 2z^2$, so $f^*\pi^*(\alpha) = 2z^2$. But also $f^*(2y) = 2f^*(y) = 2(z^2 + w^2z) = 2z^2$, and therefore $f^*(\pi^*(\alpha) - 2y) = 0$, which means that $\pi^*(\alpha) - 2y = m(x^2 - 4y) = m\pi^*(\sigma)$ for some integer m . Thus $\pi^*(\alpha - m\sigma) = 2y$, and π^* is injective because it is a rational isomorphism. This finally leads to the following.

Definition 7.11. Given the homomorphism $\pi^*: H^8(SP^2) \rightarrow H^8(B)$, we define the generator

$$h \in H^8(SP^2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

as the unique preimage of $2y \in H^8(B)$ under π^* . So $i^*(h) = 2z^2$ in $H^8(\Delta)$.

Definitions 7.9 and Definition 7.11 can be generalised as we shall see below, which leads to basis elements

$$(1/2)^{s-1}h^s, (1/2)^u g^t h^u \quad \text{where } s, t \geq 1, u \geq 0$$

for $H^{4k}(SP^2)$.

Case $k = 2m \geq 2$ for $H^{4k}(SP^2)$.

From (4.31), we have $H^{4(2m)}(B) \cong (\mathbb{Z}/2)^m \oplus (\mathbb{Z})^{m+1} = \mathbb{Z}/2\langle w^a y^b \rangle \oplus \mathbb{Z}\langle x^c y^d \rangle$ where $4 \leq a \leq 4m$, $0 \leq b \leq m-1$, $0 \leq c \leq 2m$, $0 \leq d \leq m$; explicitly, the generators are

$$\underbrace{w^{4m}, w^{4m-4}y, w^{4m-8}y^2, \dots, w^4 y^{m-1}}_m, \underbrace{y^m, x^2 y^{m-1}, \dots, x^{2m-2}y, x^{2m}}_{m+1}.$$

Then, we observe that $\ker f^*$ contains the subgroup $(\mathbb{Z})^m = \mathbb{Z}\langle k_0, k_1, \dots, k_{m-1} \rangle$ where

$$\begin{aligned} k_0 &= x^{2m} - 2^{2m}y^m, & k_1 &= x^{2m-2}y - 2^{2m-2}y^m, & k_2 &= x^{2m-4}y^2 - 2^{2m-4}y^m \\ & \dots & k_j &= x^{2(m-j)}y^j - 2^{2(m-j)}y^m, & \dots & k_{m-1} &= x^2 y^{m-1} - 2^2 y^m \end{aligned}$$

for $0 \leq j \leq m-1$. So $k_j = (x^2 - 4y)p_j(x, y)$ for some polynomial $p_j(x, y)$.

Actually, this subgroup is the whole kernel. For let $g(x, y) + h(w, y)$ be

$$\sum_{0 \leq i \leq m} g_i x^{2(m-i)} y^i + \sum_{0 \leq j \leq m-1} h_j w^{4(m-j)} y^j, \quad g_i, h_j \in \mathbb{Z}$$

in $\ker f^* \subseteq H^{8m}(B)$. Then

$$g(2z, z^2 + w^2 z) + h(w, z^2 + w^2 z) = 0 \tag{7.12}$$

in $\mathbb{Z}[z, w]/(2w)$. Equating coefficients of $w^{4m}, w^{4m-2}z, w^{4m-4}z^2, \dots, w^{2m+2}z^{m-1}$ shows that $h_0, h_1, \dots, h_{m-1} \equiv 0 \pmod{2}$, so $h(w, y) = 0$. Equating coefficients of z^{2m} shows that $\mathfrak{g} := \sum_{i=0}^m 2^{2(m-i)} g_i = 0$, so $g(x, y)|_{x^2=4y} = \mathfrak{g} y^m = 0$. Therefore $x^2 - 4y$ divides $g(x, y)$, and $g(x, y) + h(w, y)$ lies in the subgroup we have given.

By analogy with the case $m = 1$, the canonical isomorphism $H^{8m}(SP^2/\Delta) \cong H^{8m}(B/\widehat{\Delta})$ implies that $\ker i^* \cong H^{8m}(SP^2/\Delta) \cong \mathbb{Z}^m$, and we may choose basis elements $\sigma_0, \sigma_1, \dots, \sigma_{m-1}$ for $\ker i^*$ with $\pi^*(\sigma_i) = k_i$ for $i = 0, 1, \dots, m-1$. Similarly there are basis elements $k'_0, k'_1, \dots, k'_{m-1}$ for $H^{8m}(SP^2/\Delta)$ such that $s^*(k'_i) = \sigma_i$.

Proposition 6.37 gives an isomorphism $H^{8m+1}(B/\widehat{\Delta}) \cong \mathbb{Z}/2\langle \delta(2z^{2m}) \rangle$. So $\text{im } i^* \cong \ker \delta = \mathbb{Z}\langle 2z^{2m} \rangle$, where $H^{8m}(\Delta) \cong \mathbb{Z}\langle z^{2m} \rangle$. We then get a short exact sequence

$$0 \leftarrow \mathbb{Z}\langle 2z^{2m} \rangle \xleftarrow{i^*} H^{8m}(SP^2) \xleftarrow{s^*} \mathbb{Z}\langle k'_0, k'_1, \dots, k'_{m-1} \rangle \leftarrow 0 \quad (7.13)$$

which must be split. This proves that $H^{8m}(SP^2) \cong \mathbb{Z}\langle \alpha_m \rangle \oplus \mathbb{Z}\langle \sigma_0, \sigma_1, \dots, \sigma_{m-1} \rangle$ where $i^*(\alpha_m) = 2z^{2m}$, so $f^* \pi^*(\alpha_m) = 2z^{2m}$. But also $f^*(2y^m) = 2z^{2m}$, and therefore $\pi^*(\alpha_m) - 2y^m := \kappa$ lies in $\ker f^*$, and $\kappa = \pi^*(\sigma)$ for some σ in $\text{im } s^*$. Finally, $\pi^*(\alpha_m - \sigma) = 2y^m$, and π^* is injective because it is a rational isomorphism. This leads to the following.

Definition 7.14. For any $m \geq 1$, we define the generator

$$(1/2)^{m-1}h^m \in H^{8m}(SP^2)$$

as the unique preimage of $2y^m$ under π^* . So $i^*((1/2)^{m-1}h^m) = 2z^{2m}$.

Remark 7.15. The notation we have used for this generator is important for the multiplicative structure later; it refers to the fact that $2^{m-1} \cdot (1/2)^{m-1}h^m = h^m$. This is true because both map to $2^m y^m$ under the monomorphism π^* .

Having defined the generator $(1/2)^{m-1}h^m$ as in definition 7.14, it has now become more natural and convenient for us to choose the alternative basis $2y^m, x^{2(m-j)}y^j$, where $0 \leq j \leq m-1$, for $\text{im } \pi^*$.

Definition 7.16. For $0 \leq j \leq m-1$, we define the generator

$$(1/2)^j g^{2(m-j)} h^j \in H^{8m}(SP^2)$$

as the unique preimage of $x^{2(m-j)}y^j$ under π^* . So $i^*((1/2)^j g^{2(m-j)} h^j) = 2^{2(m-j)} z^{2m}$.

The notation indicates that $2^j \cdot (1/2)^j g^{2(m-j)} h^j = g^{2(m-j)} h^j$ in $H^{8m}(SP^2)$.

We shall repeat a similar process for $H^{4k}(SP^2)$, $k = \text{odd}$.

Case $k = 2m + 1 \geq 3$.

We have $H^{4(2m+1)}(B) \cong (\mathbb{Z}/2)^{m+1} \oplus \mathbb{Z}^{m+1} = \mathbb{Z}/2\langle w^a y^b \rangle \oplus \mathbb{Z}\langle x^c y^d \rangle$, where $2 \leq a \leq 4m+2$, $0 \leq b \leq m$, $1 \leq c \leq 2m+1$, $0 \leq d \leq m$, explicitly; the generators are

$$\underbrace{w^{4m+2}, w^{4m-2}y, w^{4m-6}y^2, \dots, w^2y^m}_{m+1}, \underbrace{xy^m, x^3y^{m-1}, \dots, x^{2m-1}y, x^{2m+1}}_{m+1}.$$

Then, we observe that $\ker f^*$ contains the subgroup $\mathbb{Z}^m = \mathbb{Z}\langle k_0, k_1, \dots, k_{m-1} \rangle$ where

$$\begin{aligned} k_0 &= x^{2m+1} - 2^{2m}xy^m, & k_1 &= x^{2m-1}y - 2^{2m-2}xy^m, & k_2 &= x^{2m-3}y^2 - 2^{2m-4}xy^m \\ & \dots & k_j &= x^{2(m-j)+1}y^j - 2^{2(m-j)}xy^m, & \dots & & k_{m-1} &= x^3y^{m-1} - 2^2xy^m \end{aligned}$$

for $0 \leq j \leq m-1$, and $k_j = (x^2 - 4y)p_j(x, y)$ for some polynomial $p_j(x, y)$.

As in the case $k = 2m$, we may prove that this subgroup is actually the whole of $\ker f^*$, and that there is a short exact sequence

$$0 \leftarrow \mathbb{Z}\langle 2z^{2m+1} \rangle \xleftarrow{i^*} H^{8m+4}(SP^2) \xleftarrow{s^*} \mathbb{Z}\langle k'_0, k'_1, \dots, k'_{m-1} \rangle \leftarrow 0 \quad (7.17)$$

which must be split. This proves that $H^{8m+4}(SP^2) \cong \mathbb{Z}\langle \alpha_m \rangle \oplus \mathbb{Z}^m$ where $i^*(\alpha_m) = 2z^{2m+1}$, so $f^*\pi^*(\alpha_m) = 2z^{2m+1}$. By an analogous argument to the case $k = 2m$, we may change α_m so that its image under π^* is xy^m . Also, π^* is injective because it is a rational isomorphism. This leads to the following.

Definition 7.18. For any $m \geq 1$, we define the generator

$$(1/2)^m gh^m \in H^{8m+4}(SP^2)$$

as the unique preimage of $xy^m \in H^{8m+4}(B)$ under π^* . So $i^*((1/2)^m gh^m) = 2z^{2m+1}$.

The notation indicates that $2^m \cdot (1/2)^m gh^m = gh^m$ in $H^{8m+4}(SP^2)$.

As before, it is now become more natural and convenient for us to choose the alternative basis $x^{2(m-j)+1}y^j$ instead of $x^{2(m-j)+1}y^j - 2^{2(m-j)}xy^m$ for $\text{im } \pi^*$.

Definition 7.19. We define the generator

$$(1/2)^j g^{2(m-j)+1} h^j \in H^{8m+4}(SP^2), \quad 0 \leq j \leq m-1$$

as the unique preimage of $x^{2(m-j)+1}y^j$ under π^* .

So $i^*((1/2)^j g^{2(m-j)+1} h^j) = 2^{2(m-j)+1} z^{2m+1}$.

Finally, summarising the above results we arrive at the following.

Proposition 7.20. For any $k \geq 1$, a basis for $H^{4k}(SP^2)$ is given by

$$(1/2)^{s-1}h^s, \quad (1/2)^m g^\ell h^m, \quad |g| = 4, |h| = 8, \quad m \geq 0, \ell, s \geq 1$$

where

$$(1/2)^{s-1}h^s := (\pi^*)^{-1}(2y^s), \quad (1/2)^m g^\ell h^m := (\pi^*)^{-1}(x^\ell y^m)$$

for $\pi^*: H^*(SP^2) \rightarrow H^*(B)$.

7.1.2 $H^{4k+2}(SP^2(\mathbb{H}\mathbb{P}^\infty))$

We now proceed to $H^{4k+2}(SP^2) := H^{4k+2}(SP^2(\mathbb{H}\mathbb{P}^\infty); \mathbb{Z})$.

Proposition 7.21. For any $k \geq 0$, $H^{4k+2}(SP^2) = 0$.

Proof. We have the commutative ladder

$$\begin{array}{ccccccc} \longleftarrow 0 & = & H^{4k+2}(\Delta) & \xleftarrow{i^*} & H^{4k+2}(SP^2) & \xleftarrow{\cong s^*} & H^{4k+2}(SP^2/\Delta) & \longleftarrow 0 \\ & & \downarrow & & \downarrow \pi^* & & \downarrow \cong & \\ \longleftarrow & & H^{4k+2}(\widehat{\Delta}) & \xleftarrow{f^*} & H^{4k+2}(B) & \xleftarrow{\beta^*} & H^{4k+2}(B/\widehat{\Delta}) & \longleftarrow 0 \end{array}$$

where $H^{4k+2}(B) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ with basis $w^i y^j$ where $i + 4j = 2k + 1$, $j \geq 0$, $i \geq 1$ from (4.31). The image $f^*(w^i y^j) = w^i (z^2 + w^2 z)^j$ confirms that $\ker f^* = 0$. Then, by the isomorphisms $\ker f^* \cong H^{4k+2}(B/\widehat{\Delta}) \cong H^*(SP^2/\Delta) \cong H^{4k+2}(SP^2)$, we must have $H^{4k+2}(SP^2) = 0$. \square

7.1.3 $H^{4k+1}(SP^2(\mathbb{H}\mathbb{P}^\infty))$ and $H^{4k+3}(SP^2(\mathbb{H}\mathbb{P}^\infty))$

We shall now describe $H^*(SP^2) := H^*(SP^2(\mathbb{H}\mathbb{P}^\infty); \mathbb{Z})$ for $* = 4k + 1, 4k + 3$.

For $* = 1, 3, 5$, the cohomology group $H^*(SP^2) = 0$ is given by (7.2), which comes from Nakaoka [46]. We shall confirm his results for $* = 3, 5$ by using our cohomology ladder. This works as below.

Lemma 7.22. If $* = 1, 3, 5, 9$, then $H^*(SP^2(\mathbb{H}\mathbb{P}^\infty)) = 0$.

Proof. Case $* = 3$. The ladder

$$\begin{array}{ccccccc} 0 & \longleftarrow & H^3(SP^2) & \xleftarrow{s^*} & H^3(SP^2/\Delta) & \xleftarrow{\delta} & H^2(\Delta) = 0 \\ & & \downarrow \pi^* & & \downarrow \cong & & \downarrow \\ 0 & = & H^3(B) & \xleftarrow{\beta^*} & H^3(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^2(\widehat{\Delta}) & \longleftarrow \end{array}$$

shows that $H^3(SP^2/\Delta) \xrightarrow{s^*} H^3(SP^2)$ is an isomorphism, and also, $H^3(B/\widehat{\Delta}) = 0$ by Lemma 6.13. Then, using the canonical isomorphism $H^3(B/\widehat{\Delta}) \cong H^3(SP^2/\Delta)$, we confirm that $H^3(SP^2) = 0$.

Case $* = 5$. We have the ladder

$$\begin{array}{ccccccc} 0 & \longleftarrow & H^5(SP^2) & \xleftarrow{s^*} & H^5(SP^2/\Delta) & \xleftarrow{\delta} & H^4(\Delta) \longleftarrow H^4(SP^2) \\ & & \downarrow \pi^* & & \cong \downarrow & & \downarrow & \downarrow \\ \longleftarrow & 0 = H^5(B) & \xleftarrow{\beta^*} & H^5(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^4(\widehat{\Delta}) & \longleftarrow & H^4(B) \end{array}$$

where $H^4(\Delta) \cong \mathbb{Z}\langle z \rangle$, and $H^5(B/\widehat{\Delta}) \cong \mathbb{Z}/2\langle \delta(z) \rangle$ by Proposition 6.37. Then, since $s^*(\delta(z)) = 0$ and s^* is a surjection, we confirm that $H^5(SP^2) = 0$.

Case $* = 9$. We have the ladder

$$\begin{array}{ccccccc} 0 & \longleftarrow & H^9(SP^2) & \xleftarrow{s^*} & H^9(SP^2/\Delta) & \xleftarrow{\delta} & H^8(\Delta) \longleftarrow \\ & & \downarrow \pi^* & & \cong \downarrow & & \downarrow \\ 0 = H^9(B) & \xleftarrow{\beta^*} & H^9(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^8(\widehat{\Delta}) & \longleftarrow & \end{array}$$

where $H^8(\Delta) \cong \mathbb{Z}\langle z^2 \rangle$, and $H^9(B/\widehat{\Delta}) \cong \mathbb{Z}/2\langle \delta(z^2) \rangle$ by Proposition 6.37.

So, $H^9(SP^2) = 0$ follows from the fact that $s^*(\delta(z^2)) = 0$, and that s^* is a surjection. \square

In general, $H^{4k+1}(SP^2)$ has the following property.

Lemma 7.23. *For any $k \geq 1$, the LES*

$$H^{4k+1}(\Delta) \xleftarrow{i^*} H^{4k+1}(SP^2) \xleftarrow{s^*} H^{4k+1}(SP^2/\Delta) \xleftarrow{\delta} H^{4k}(\Delta) \longleftarrow \dots$$

is given by

$$0 \xleftarrow{i^*} (\mathbb{Z}/2)^m \xleftarrow{s^*} (\mathbb{Z}/2)^m \oplus \mathbb{Z}/2 \xleftarrow{\delta} \mathbb{Z} \longleftarrow \dots$$

for some $m \geq 0$, and $\ker s^* \cong \mathbb{Z}/2\langle \delta(z^k) \rangle$.

Proof. We use the notation of Remark 7.4. It is clear that $H^{4k}(\Delta) \cong \mathbb{Z}\langle z^k \rangle$ and $H^{4k+1}(\Delta) = 0$. By Proposition 6.37, we have that $H^{4k+1}(B/\widehat{\Delta}) \cong (\mathbb{Z}/2)^{m+1} = \mathbb{Z}/2\langle \delta_l(w^j z^i) \rangle$, where $i \geq 1$, $0 \leq j < 2i$. So $H^{4k+1}(SP^2/\Delta) \cong (\mathbb{Z}/2)^{m+1}$. If $j \geq 1$, then $\delta_l(w^j z^i)$ cannot lie in the image of δ_u , so the only non-zero element in its image is $\delta_u(z^k)$, which must generate $\ker s^*$. \square

Corollary 7.24. *For any $k \geq 3$, we have $H^{4k+1}(SP^2) \cong \mathbb{Z}/2\langle \delta(w^j z^i) \rangle$, where $i \geq 2$, $2 \leq j < 2i$.*

On the other hand, $H^{4k+3}(SP^2)$ has the following property.

Lemma 7.25. *For any $k \geq 0$, there exists an isomorphism*

$$H^{4k+3}(SP^2) \cong H^{4k+3}(SP^2/\Delta). \quad (7.26)$$

Proof. It follows from the exact sequence

$$0 = H^{4k+3}(\Delta) \xleftarrow{i^*} H^{4k+3}(SP^2) \xleftarrow{s^*} H^{4k+3}(SP^2/\Delta) \xleftarrow{\delta} H^{4k+2}(\Delta) = 0$$

that the homomorphism s^* is an isomorphism for any $k \geq 0$. □

Corollary 7.27. *For any $k \geq 0$, there exists an isomorphism*

$$H^{4k+3}(SP^2) \cong H^{4k+3}(B/\widehat{\Delta}).$$

Assembling the properties of the $(4k + 1)$ st and $(4k + 3)$ -rd cohomology groups described above, we may now introduce the following.

Definition 7.28. *For $* = 4k + 3 \geq 7$, $4k + 1 \geq 13$, we define a basis for $H^*(SP^2)$ by*

$$t_{i,j} := u^*(\delta(w^j z^i)), \quad 1 \leq j < 2i, \quad 1 \leq i, \quad |t_{i,j}| = 4i + 2j + 1$$

where $\delta(w^j z^i) \in H^*(B/\widehat{\Delta})$, and $u^* := s^* \circ (\pi_r^*)^{-1}$ in the commutative ladder

$$\begin{array}{ccccccc} \longleftarrow & H^*(SP^2) & \xleftarrow{s^*} & H^*(SP^2/\Delta) & \longleftarrow & H^{*-1}(\Delta) & \longleftarrow \\ & \downarrow \pi^* & & \cong \downarrow \pi_r^* & & \downarrow & \\ \longleftarrow & H^*(B) & \longleftarrow & H^*(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^{*-1}(\widehat{\Delta}) & \longleftarrow \end{array}$$

7.2 Product Structures

In this section we will describe the multiplicative structures for $H^*(SP^2)$.

If $* = 4k$, then the product structures

- $(1/2)^{s-1} h^s \cdot (1/2)^{s'-1} h^{s'} = 2 \cdot (1/2)^{s+s'-1} h^{s+s'}$
- $(1/2)^m g^\ell h^m \cdot (1/2)^{s-1} h^s = 2 \cdot (1/2)^{m+s} g^\ell h^{m+s}$

$$\bullet (1/2)^m g^\ell h^m \cdot (1/2)^{m'} g^{\ell'} h^{m'} = (1/2)^{m+m'} g^{\ell+\ell'} h^{m+m'}.$$

for $(1/2)^m g^\ell h^m$, $(1/2)^{s-1} h^s$, $s, \ell \geq 1$, $m \geq 0$, $|g| = 4$, $|h| = 8$, come naturally from Proposition 7.20.

The product structure between the torsion elements $t_{i,j}$ can be deduced relatively easy, too.

Proposition 7.29. *The relation*

$$t_{i,j} t_{k,\ell} = 0$$

holds for any of the torsion elements $t_{i,j}$ of Definition 7.28.

Proof. The dimension of any product $t_{i,j} t_{k,\ell}$ is given by

$$\begin{aligned} |t_{i,j} t_{k,\ell}| &= (4i + 2j + 1) + (4k + 2\ell + 1) \\ &= 2(2i + 2k + j + \ell + 1). \end{aligned}$$

By Lemma 7.6 and Proposition 7.21, when $*$ = even, the cohomology group is either $H^*(SP^2) \cong (\mathbb{Z})^r$ or 0. Thus, any product $t_{i,j} t_{k,\ell}$ of torsion elements must be zero, since it is also a torsion element. \square

The remaining products $t_{i,j} (1/2)^m g^\ell h^m$, $t_{i,j} (1/2)^{s-1} h^s$ are more delicate. Our strategy is to use the mod 2 reduction homomorphism,

$$\rho: H^*(X; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Z}/2).$$

From Section 7.1, if $*$ $\neq 4k$ then the cohomology group $H^*(SP^2)$ is either 0 or $(\mathbb{Z}/2)^r$, so ρ is injective for $*$ $\neq 4k$. The dimensions of $t_{i,j} (1/2)^m g^\ell h^m$ and $t_{i,j} (1/2)^{s-1} h^s$ are not equal to $4k$, so

$$\begin{aligned} \rho(t_{i,j} \cdot (1/2)^m g^\ell h^m) = 0 &\text{ implies } t_{i,j} \cdot (1/2)^m g^\ell h^m = 0, \\ \rho(t_{i,j} \cdot (1/2)^{s-1} h^s) = 0 &\text{ implies } t_{i,j} \cdot (1/2)^{s-1} h^s = 0. \end{aligned}$$

We shall use the cohomology ladders

$$\begin{array}{ccccccc} H^*(\Delta) & \xleftarrow{i^*} & H^*(SP^2) & \xleftarrow{s^*} & H^*(SP^2/\Delta) & \xleftarrow{\delta} & H^{*-1}(\Delta) & (7.30) \\ \downarrow \rho & & \downarrow \rho & & \downarrow \rho & & \downarrow \rho & \\ H^*(\Delta; \mathbb{Z}/2) & \xleftarrow{i_2^*} & H^*(SP^2; \mathbb{Z}/2) & \xleftarrow{s_2^*} & H^*(SP^2/\Delta; \mathbb{Z}/2) & \xleftarrow{\delta} & H^{*-1}(\Delta; \mathbb{Z}/2) \end{array}$$

and

$$\begin{array}{ccccccc}
 H^*(\widehat{\Delta}) & \xleftarrow{f^*} & H^*(B) & \xleftarrow{\quad} & H^*(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^{*-1}(\widehat{\Delta}) \\
 \downarrow \rho & & \downarrow \rho & & \downarrow \rho & & \downarrow \rho \\
 H^*(\widehat{\Delta}; \mathbb{Z}/2) & \xleftarrow{f_2^*} & H^*(B; \mathbb{Z}/2) & \xleftarrow{\quad} & H^*(B/\widehat{\Delta}; \mathbb{Z}/2) & \xleftarrow{\delta} & H^{*-1}(\widehat{\Delta}; \mathbb{Z}/2)
 \end{array} \quad (7.31)$$

which arise because ρ is a natural transformation, and also the ladder

$$\begin{array}{ccccccc}
 H^*(\Delta; \mathbb{Z}/2) & \xleftarrow{i_2^*} & H^*(SP^2; \mathbb{Z}/2) & \xleftarrow{s_2^*} & H^*(SP^2/\Delta; \mathbb{Z}/2) & \xleftarrow{\delta} & H^{*-1}(\Delta; \mathbb{Z}/2) \\
 \downarrow \pi_\Delta^* & & \downarrow \pi^* & & \cong \downarrow & & \downarrow \\
 H^*(\widehat{\Delta}; \mathbb{Z}/2) & \xleftarrow{f_2^*} & H^*(B; \mathbb{Z}/2) & \xleftarrow{\quad} & H^*(B/\widehat{\Delta}; \mathbb{Z}/2) & \xleftarrow{\delta} & H^{*-1}(\widehat{\Delta}; \mathbb{Z}/2)
 \end{array} \quad (7.32)$$

that comes from applying $H^*(-; \mathbb{Z}/2)$ to (2.29).

Using (6.27), we have

$$H^{4k}(B; \mathbb{Z}/2) \cong H^{4k}(B; \mathbb{Z}) \otimes \mathbb{Z}/2 \oplus \text{Tor}(H^{4k+1}(B, \mathbb{Z}), \mathbb{Z}/2) = H^{4k}(B; \mathbb{Z}) \otimes \mathbb{Z}/2$$

since $H^{4k+1}(B; \mathbb{Z}) = 0$. Also $H^*(\widehat{\Delta}_+; \mathbb{Z}/2) \cong \mathbb{Z}/2\langle a, z \rangle$, where $|a| = 1$, $|z| = 4$, and $\rho(w) = a^2$. So f_2^* is given by

$$f_2^*(x) = 0, \quad f_2^*(y) = z^2 + a^4 z. \quad (7.33)$$

Lemma 7.34. *If $* = 4k$, then $\text{im } i_2^* = 0$.*

Proof. Since $H^{4k}(\Delta; \mathbb{Z}/2) \cong \mathbb{Z}/2\langle z^k \rangle$, it is enough to prove that z^k does not lie in $\text{im } i_2^*$.

$$\begin{array}{ccc}
 H^{4k}(\Delta; \mathbb{Z}/2) & \xleftarrow{i_2^*} & H^{4k}(SP^2; \mathbb{Z}/2) \\
 \downarrow & & \downarrow \\
 H^*(\widehat{\Delta}; \mathbb{Z}/2) & \xleftarrow{f_2^*} & H^{4k}(B; \mathbb{Z}/2)
 \end{array} \quad (7.35)$$

This follows at once from (7.33). □

Lemma 7.36. *For $k \geq 1$, there is a split short exact sequence*

$$0 \leftarrow H^{4k+1}(SP^2; \mathbb{Z}/2) \xleftarrow{s_2^*} H^{4k+1}(SP^2/\Delta; \mathbb{Z}/2) \xleftarrow{\delta} H^{4k}(\Delta; \mathbb{Z}/2) \leftarrow 0 \quad (7.37)$$

given by

$$0 \xleftarrow{i_2^*} (\mathbb{Z}/2)^m \xleftarrow{s_2^*} (\mathbb{Z}/2)^m \oplus \mathbb{Z}/2 \xleftarrow{\delta} \mathbb{Z}/2 \leftarrow 0$$

for some $m \geq 0$ where $\ker s_2^* \cong \mathbb{Z}/2\langle \delta(z^k) \rangle$.

Proof. It follows by applying mod 2 reduction to the LES described in Lemma 7.23, together with Lemma 7.34. \square

Lemma 7.38. *For $*$ = $4k$, $4k + 2$, $4k + 3$, $k \geq 1$, there exists an isomorphism*

$$H^*(SP^2; \mathbb{Z}/2) \cong H^*(SP^2/\Delta; \mathbb{Z}/2).$$

Proof. Case $*$ = $4k + 2$, $4k + 3$. We have the exact sequence

$$H^*(\Delta; \mathbb{Z}/2) \xleftarrow{i_2^*} H^*(SP^2; \mathbb{Z}/2) \xleftarrow{s_2^*} H^*(SP^2/\Delta; \mathbb{Z}/2) \xleftarrow{\delta} H^{*-1}(\Delta; \mathbb{Z}/2)$$

where $H^{4k+3}(\Delta; \mathbb{Z}/2) = H^{4k+2}(\Delta; \mathbb{Z}/2) = H^{4k+1}(\Delta; \mathbb{Z}/2) = 0$, so clearly, s_2^* is an isomorphism.

Case $*$ = $4k$. From the LES

$$\longleftarrow H^{4k}(\Delta; \mathbb{Z}/2) \xleftarrow{i_2^*} H^{4k}(SP^2; \mathbb{Z}/2) \xleftarrow{s_2^*} H^{4k}(SP^2/\Delta; \mathbb{Z}/2) \longleftarrow 0$$

s_2^* is injective. On the other hand, by Lemma 7.34, $\text{im } i_2^* = 0$, so s_2^* is surjective, too. \square

The $\mathbb{Z}/2$ cohomology of symmetric squares was studied by Nakaoka in 1950's, and Lemma 7.34 and 7.38 agree with Nakaoka [47, Lemma 11.3].

Proposition 7.39. *The relations $t_{i,j}(1/2)^m g^\ell h^m = 0$ and $t_{i,j}(1/2)^{s-1} h^s = 0$ hold in $H^*(SP^2)$.*

Proof. By Lemma 4.22, $H^*(\Gamma_+; \mathbb{Z}/2) \cong H^*(Gr_{\mathbb{H}}(2, \infty); \mathbb{Z}/2)[a]/(a^5 - p_1 a)$. Let τ be the Thom isomorphism $\tau: H^*(\Gamma_+; \mathbb{Z}/2) \longrightarrow \tilde{H}^{*+4}(\text{Th}(\theta); \mathbb{Z}/2)$, and t the Thom class $t \in H^4(\text{Th}(\theta); \mathbb{Z}/2) \cong \mathbb{Z}/2$. Then for $*$ ≥ 4 , any generator of $H^*(\text{Th}(\theta); \mathbb{Z}/2)$ can be expressed as $t p_1^\ell p_2^m a^s$ where $\ell, m, s \geq 0$. In particular, we have an expression $t p_1^\ell p_2^m a^s$ where $\ell, m \geq 0$ and $s \geq 1$ for generators with odd dimensions. Suppose that

$$H^{4k}(\text{Th}(\theta); \mathbb{Z}/2) = \mathbb{Z}/2\langle \nu_i \rangle, \quad H^*(\text{Th}(\theta); \mathbb{Z}/2) = \mathbb{Z}/2\langle \zeta_\kappa \rangle, \quad * = 4k + 1, 4k + 3, \quad k \geq 1,$$

then, for any basis elements ν_ι and ζ_κ

$$\begin{aligned}
 \nu_\iota \zeta_\kappa &= (t p_1^\ell p_2^m a^s)(t p_1^i p_2^j a^u), \quad u \geq 1 \\
 &= t^2 p_1^{\ell+i} p_2^{m+j} a^{s+u} \\
 &= (p_1 + a^4) t p_1^{\ell+i} p_2^{m+j} a^{s+u} \\
 &= (p_1^{1+\ell+i} p_2^{m+j} a^{s+u} + p_1^{\ell+i} p_2^{m+j} a^{4+s+u}) t \\
 &= (p_1^{1+\ell+i} p_2^{m+j} a^{s+u} + p_1^{\ell+i} p_2^{m+j} a^5 a^{s+u-1}) t \\
 &= (p_1^{1+\ell+i} p_2^{m+j} a^{s+u} + p_1^{1+\ell+i} p_2^{m+j} a^{s+u}) t \\
 &= 0,
 \end{aligned}$$

where $t^2 = (p_1 + a^4) t$, by Lemma 6.20.

Suppose that $\alpha \in \{(1/2)^m g^\ell h^m, (1/2)^{s-1} h^s\}$, and consider the mod 2 reductions $\rho(\alpha), \rho(t_{i,j}) \in H^*(SP^2; \mathbb{Z}/2)$. Then, there are expressions

$$\rho(\alpha) = \sum_{\iota} s_2^*(\lambda_\iota \nu_\iota) \quad \text{and} \quad \rho(t_{i,j}) = \sum_{\kappa} s_2^*(\mu_\kappa \zeta_\kappa)$$

where $\lambda_\iota, \mu_\kappa \in \{0, 1\}$, by Lemmas 7.38 and 7.36. The product $\nu_r \zeta_r = 0$ for any ν_ι and ζ_κ , hence

$$\sum_{\iota} \lambda_\iota \nu_\iota \sum_{\kappa} \mu_\kappa \zeta_\kappa = 0,$$

which then implies that $\rho(\alpha)\rho(t_{i,j}) = \rho(\alpha t_{i,j}) = 0$. \square

Remark 7.40. *Any product*

$$(t p_1^\lambda p_2^\mu a^\nu)(t p_1^\alpha p_2^\beta a^\gamma), \quad \gamma \geq 1$$

in $H^*(\text{Th}(\theta); \mathbb{Z}/2)$ vanishes; the only non-zero products are of the form

$$(t p_1^\lambda p_2^\mu)(t p_1^\alpha p_2^\beta).$$

7.3 $H^*(SP^2(\mathbb{H}\mathbb{P}^\infty); \mathbb{Z}/2)$

We finish this chapter by bringing together certain properties of $H^*(SP^2; \mathbb{Z}/2)$ and $H^*(\text{Th}(\theta); \mathbb{Z}/2)$ from above, and using them to start improving our understanding of the role of the Thom isomorphism.

By Lemma 7.36, there is a split short exact sequence

$$0 \leftarrow H^{4k+1}(SP^2; \mathbb{Z}/2) \xleftarrow{s_2^*} H^{4k+1}(SP^2/\Delta; \mathbb{Z}/2) \xleftarrow{\delta} H^{4k}(\Delta; \mathbb{Z}/2) \leftarrow 0$$

where $\ker s_2^* \cong \mathbb{Z}/2\langle \delta(z^k) \rangle$, and by Lemma 7.38, if $* \neq 4k + 1$ then

$$s_2^*: H^*(SP^2/\Delta; \mathbb{Z}/2) \rightarrow H^*(SP^2; \mathbb{Z}/2)$$

is an isomorphism. Then we may write

$$H^*(SP^2; \mathbb{Z}/2) \cong H^*(SP^2/\Delta; \mathbb{Z}/2)/(\delta(z^k))$$

where the only non-vanishing products in $H^*(\text{Th}(\theta); \mathbb{Z}/2) \cong H^*(SP^2/\Delta; \mathbb{Z}/2)$ are of the form $(t p_1^\lambda p_2^\mu)(t p_1^\alpha p_2^\beta)$, by Corollary 7.40.

For $H^*(\text{Th}(\theta); \mathbb{Z}/2)$ as described in the proof of Proposition 7.39, we write

$$H^*(\Gamma_+; \mathbb{Z}/2) t := H^*(\text{Th}(\theta); \mathbb{Z}/2)$$

where t is the unique Thom class in $H^4(\text{Th}(\theta); \mathbb{Z}/2)$ and

$$H^*(\Gamma_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[p_1, p_2]\langle 1, a, a^2, a^3, a^4 \rangle$$

with the relations $a^5 = p_1 a$, $t^2 = (a^4 + p_1) t$.

Recall from (4.14) that

$$H^*(Gr_{\mathbb{H}}(2, n+1)_+; \mathbb{Z}/2) \cong H^*(Gr_{\mathbb{H}}(2, \infty)_+; \mathbb{Z}/2)/(\mathfrak{q}_n, \mathfrak{q}_{n+1}) \quad (7.41)$$

where $H^*(Gr_{\mathbb{H}}(2, \infty)_+; \mathbb{Z}/2) \cong \mathbb{Z}/2[p_1, p_2]$, $|p_i| = 4i$, and from [48] that \mathfrak{q}_n is the n -th complete symmetric polynomial in z_1, z_2 , by setting $p_1 = z_1 + z_2$, $p_2 = z_1 z_2$. Also,

$$H^*((\Gamma_n)_+; \mathbb{Z}/2) \cong H^*(Gr_{\mathbb{H}}(2, n+1)_+; \mathbb{Z}/2)\langle 1, a, a^2, a^3, a^4 \rangle$$

from (4.24), and in particular, the homomorphisms

$$H^*(Gr_{\mathbb{H}}(2, n+1)_+; \mathbb{Z}/2) \rightarrow H^*((\Gamma_n)_+; \mathbb{Z}/2), \quad H^*(Gr_{\mathbb{H}}(2, \infty)_+; \mathbb{Z}/2) \rightarrow H^*(\Gamma_+; \mathbb{Z}/2)$$

induced by the projections are injective. Combining with the Thom isomorphism τ , these may be summarised as a diagram

$$\begin{array}{ccccc} H^*(Gr_{\mathbb{H}}(2, n+1)_+; \mathbb{Z}/2) & \longrightarrow & H^*((\Gamma_n)_+; \mathbb{Z}/2) & \xrightarrow[\cong]{\tau} & H^{*+4}(\text{Th}(\theta_n); \mathbb{Z}/2) & (7.42) \\ \eta_1^* \uparrow & & \eta_2^* \uparrow & & \eta_3^* \uparrow & \\ H^*(Gr_{\mathbb{H}}(2, \infty)_+; \mathbb{Z}/2) & \longrightarrow & H^*(\Gamma_+; \mathbb{Z}/2) & \xrightarrow[\cong]{\tau} & H^{*+4}(\text{Th}(\theta); \mathbb{Z}/2) & \end{array}$$

where η_i^* , $i = 1, 2, 3$ are surjective, and induced by the inclusions.

In particular, observe that in $H^{(4n+1)+4}(\text{Th}(\theta); \mathbb{Z}/2)$, we have $\ker \eta_3^* \cong \mathbb{Z}/2\langle \mathfrak{q}_n a t \rangle$.

Remark 7.43. *We have the identifications*

$$(1) \delta(z) = at, \quad (2) \delta(z^{n+1}) = q_n at,$$

where q_n , $|q_n| = 4n$, is the n -th complete symmetric polynomial as in (7.41).

(1) follows from the fact $H^5(\text{Th}(\theta); \mathbb{Z}/2) \cong \mathbb{Z}/2$.

(2) follows from (7.42) together with the commutative ladder

$$\begin{array}{ccccc} \longleftarrow & H^{*+1}(SP_n^2/\Delta_n; \mathbb{Z}/2) & \xleftarrow{\delta} & H^*(\Delta_n; \mathbb{Z}/2) & \xleftarrow{i_2^*} & H^*(SP_n^2; \mathbb{Z}/2) & (7.44) \\ & \uparrow \eta^* & & \uparrow \zeta^* & & \uparrow & \\ \longleftarrow & H^{*+1}(SP^2/\Delta; \mathbb{Z}/2) & \xleftarrow{\delta} & H^*(\Delta; \mathbb{Z}/2) & \xleftarrow{i_2^*} & H^*(SP^2; \mathbb{Z}/2) \end{array}$$

where $H^*(SP^2/\Delta; \mathbb{Z}/2) \cong H^*(\text{Th}(\theta); \mathbb{Z}/2)$, $H^*(SP_n^2/\Delta_n; \mathbb{Z}/2) \cong H^*(\text{Th}(\theta_n); \mathbb{Z}/2)$, and the vertical maps are induced by inclusions. Note that for all r , there is an isomorphism $H^r(X; \mathbb{Z}/2) \cong \varprojlim H^r(X_n; \mathbb{Z}/2)$, which comes from [22, Proposition 3F.5] and Section 2.3, where $X_n = SP_n^2$, Δ_n , SP_n^2/Δ_n and $X = \bigcup_n X_n$. The kernel of ζ^* is generated by z^{n+1} because $H^*(\Delta_n; \mathbb{Z}/2) \cong \mathbb{Z}/2[z]/(z^{n+1})$. Also, from Lemma 7.34, if $* = 4k$ then $0 = \text{im } i_2^* = \ker \delta$. Then, observe that for $H^{4(n+1)+1}(SP^2/\Delta; \mathbb{Z}/2)$ we have $\ker \eta^* \cong \mathbb{Z}/2\langle \delta(z^{n+1}) \rangle$. Therefore, it must be that $\delta(z^{n+1}) = q_n at$.

Chapter 8

The Cohomology Ring II

In this chapter, we mainly focus on the computation of the integral cohomology ring $H^*(SP^2(\mathbb{H}\mathbb{P}^n))$. We shall also check our results with M. Nakaoka's work [44, 47], in which he studied the mod p cohomology of p -fold cyclic products¹ $CP_p(X)$.

8.1 $H^*(SP^2(\mathbb{H}\mathbb{P}^n); \mathbb{Z})$

We shall prove Theorem 1.8 in this Section. Nakaoka's work included a computation of $H^*(CP_p(S^n))$ as an abelian group for any n -sphere S^n . For $p = 2$, we may compare our results on $H^*(SP^2(\mathbb{H}\mathbb{P}^1))$ with his computations of $H^*(CP_2(S^4))$, as follows.

Theorem 8.1 ([47]).

$$H^*(SP^2(\mathbb{H}\mathbb{P}^1)_+) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, 4, 8 \\ \mathbb{Z}/2 & \text{if } * = 7 \\ 0 & \text{otherwise.} \end{cases}$$

Nakaoka also proved [46] that if X is a connected CW complex such that its integral homology is $H_*(X_+) = 0$ for $0 < * < r$, then

$$H^*(SP^m(X)_+) \cong H^*(X_+) \quad \text{for } * \leq r + 1$$

holds for any $m \geq 1$. Taking $X = \mathbb{H}\mathbb{P}^n$ and $r = 4$ gives an isomorphism

$$H^*(SP^m(\mathbb{H}\mathbb{P}^n)) \cong H^*(\mathbb{H}\mathbb{P}^n) \quad \text{for } * \leq 5, \tag{8.2}$$

¹Nakaoka used the notation $Z_p(X)$

with any $m \geq 1$.

We build on his work, and on our results for $H^*(SP^2)$ from Chapter 7, and describe the cohomology ring $H^*(SP_n^2) := H^*(SP^2(\mathbb{H}\mathbb{P}^n); \mathbb{Z})$.

We shall use the commutative ladder of LES

$$\begin{array}{ccccccc}
 \longleftarrow & H^*(\Delta_n) & \xleftarrow{i^*} & H^*(SP_n^2) & \xleftarrow{s^*} & H^*(SP_n^2/\Delta_n) & \xleftarrow{\delta} & H^{*-1}(\Delta_n) & \longleftarrow \\
 & \downarrow & & \downarrow \pi^* & & \downarrow \cong & & \downarrow & \\
 \longleftarrow & H^*(\widehat{\Delta}_n) & \xleftarrow{f^*} & H^*(B_n) & \xleftarrow{\beta^*} & H^*(B_n/\widehat{\Delta}_n) & \xleftarrow{\delta} & H^{*-1}(\widehat{\Delta}_n) & \longleftarrow
 \end{array} \quad (8.3)$$

that comes from applying $H^*(-)$ to (2.22).

As input we have that for any $n \geq 1$

$$H^*((B_n)_+) \cong H^*(B_+)/(\mathfrak{f}_{n,1}, \mathfrak{f}_{n,2}, y^{n+1})$$

where $H^*(B_+) \cong \mathbb{Z}[x, y, w]/(2w, wx)$ from (4.36). We also have the integral cohomology of the diagonal spaces

$$H^*((\Delta_n)_+) \cong H^*(\mathbb{H}\mathbb{P}_+^n) \cong \mathbb{Z}[z]/(z^{n+1}), \quad |z| = 4$$

from Section 2.1, and

$$H^*(\widehat{\Delta}_n)_+ \cong H^*((\mathbb{R}\mathbb{P}^\infty \times \mathbb{H}\mathbb{P}^n)_+) \cong \mathbb{Z}[w, z]/(2w, z^{n+1}), \quad |z| = 4, |w| = 2, \quad (8.4)$$

which comes from Remark 2.18.

8.1.1 $H^*(SP^2(\mathbb{H}\mathbb{P}^n))$ for $* > 8n$

As a simple check on our computations we first confirm that the cohomology group $H^*(SP_n^2)$ is zero for $* > 8n$, as required by the cell structure [47].

Assume $* > 8n$, and recall from Remark 4.41 that the cohomology group of the Borel space is

$$H^*(B_n) \cong (\mathbb{Z}/2)^{n+1}, \quad n \geq 1$$

for $* = 8n + 2\ell$, $\ell = 1, 2, 3, \dots$, and $H^*(B_n) = 0$ when $* = \text{odd}$. There are four cases to consider, $* = 4k$, $4k + 1$, $4k + 2$ and $4k + 3$.

If $* = 4k > 8n$, then we have the ladder

$$\begin{array}{ccccccc}
 0 & \longleftarrow & i^* & H^{4k}(SP_n^2) & \xleftarrow{s^*} & H^{4k}(SP_n^2/\Delta_n) & \xleftarrow{\delta} & H^{4k-1}(\Delta_n) & = 0 \\
 & & & \downarrow \pi^* & & \downarrow \cong & & & \\
 H^{4k}(\widehat{\Delta}_n) & \xleftarrow{f^*} & H^{4k}(B_n) & \xleftarrow{\beta^*} & H^{4k}(B_n/\widehat{\Delta}_n) & \xleftarrow{\delta} & H^{4k-1}(\widehat{\Delta}_n) & = 0
 \end{array}$$

where $H^{4k}(B_n) \cong (\mathbb{Z}/2)^{n+1} = \mathbb{Z}/2\langle w^a y^b \rangle$ for $2k - 4n \leq a \leq 2k$, $0 \leq b \leq n$, $a + 4b = 2k$ and $H^{4k}(\widehat{\Delta}_n) \cong (\mathbb{Z}/2)^{n+1} = \mathbb{Z}/2\langle w^c z^d \rangle$ for $2k - 2n \leq c \leq 2k$, $0 \leq d \leq n$, $2c + 4d = 4k$.

We have $H^{4k}(B_n/\widehat{\Delta}_n) \cong \ker f^* = 0$, because $f^*(\sum_a \lambda_a w^a y^b) = 0$ implies that

$$\sum_{a \leq 2k} \lambda_a w^a (z^2 + w^2 z)^b = 0,$$

so equating coefficients of w^{2k} , $w^{2k-2}z$, \dots in succession gives $\lambda_{2k} = \lambda_{2k-4} = \dots = 0$.

But $H^{4k}(SP_n^2/\Delta_n)$ is isomorphic to the kernel of f^* , so

$$H^{4k}(SP_n^2/\Delta_n) \cong H^{4k}(SP_n^2) = 0, \quad k > 2n.$$

The remaining three cases are similar. If $*$ = $4k + 2$, then $\ker f^* = 0$ again, and

$$\ker f^* \cong H^{4k+2}(B_n/\widehat{\Delta}_n) \cong H^{4k+2}(SP_n^2/\Delta_n) \cong H^{4k+2}(SP_n^2).$$

If $*$ = $4k + 1$, $4k + 3$, then we have the exact sequence

$$H^*(B_n) = 0 \longleftarrow H^*(B_n/\widehat{\Delta}_n) \xleftarrow{\delta} H^{*-1}(\widehat{\Delta}_n) \xleftarrow{f^*} H^{*-1}(B_n) \longleftarrow 0$$

where $H^{*-1}(B_n) \cong H^{*-1}(\widehat{\Delta}_n) \cong (\mathbb{Z}/2)^{n+1}$, so f^* is an isomorphism.

Therefore, $0 = H^*(B_n/\widehat{\Delta}_n) \cong H^*(SP_n^2/\Delta_n) \cong H^*(SP_n^2)$.

8.1.2 $H^{4k}(SP^2(\mathbb{H}\mathbb{P}^n))$

In the previous section we confirmed that $H^*(SP_n^2) = 0$ if $*$ > $8n$. In this section we shall describe the integral cohomology ring $H^{4*}(SP_n^2)$.

We always assume $n \geq 1$.

Lemma 8.5. *For any $k \leq 2n$, the cohomology group $H^{4k}(B_n/\widehat{\Delta}_n)$ is torsion free; that is, $H^{4k}(B_n/\widehat{\Delta}_n) \cong \mathbb{Z}^m$ for some m .*

Proof. From the LES

$$\longleftarrow H^{4k}(\widehat{\Delta}_n) \xleftarrow{f^*} H^{4k}(B_n) \xleftarrow{\beta^*} H^{4k}(B_n/\widehat{\Delta}_n) \xleftarrow{\delta} H^{4k-1}(\widehat{\Delta}_n) = 0$$

we see that $H^{4k}(B_n/\widehat{\Delta}_n)$ is isomorphic to $\ker f^*$. So we need to show that $\ker f^*$ is torsion free.

Case $k \leq n$.

If $k \leq n$, then the cohomology ring $H^*(B_n)$ is isomorphic to $H^*(B)$ by Remark 4.41, and clearly $H^{4k}(\widehat{\Delta}_n) \cong H^{4k}(\widehat{\Delta})$. So, computing $\ker f^*$ is the same as finding the kernel of f_∞^* in the diagram

$$\begin{array}{ccccccc} \longleftarrow & H^{4k}(\widehat{\Delta}_n) & \xleftarrow{f^*} & H^{4k}(B_n) & \longleftarrow & H^{4k}(B_n/\widehat{\Delta}_n) & \xleftarrow{\delta} & H^{4k-1}(\widehat{\Delta}_n) = 0 \\ & \cong \uparrow & & \cong \uparrow & & \uparrow & & \\ \longleftarrow & H^{4k}(\widehat{\Delta}) & \xleftarrow{f_\infty^*} & H^{4k}(B) & \longleftarrow & H^{4k}(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^{4k-1}(\widehat{\Delta}) = 0 \end{array}$$

which we found to be torsion free in Section 7.1.1.

Case $k \geq n + 1$.

From (4.36), we have $H^*((B_n)_+) \cong H^*(B_+)/(\mathfrak{f}_{n,1}, \mathfrak{f}_{n,2}, y^{n+1})$, and from (4.42), any $x^k \in H^{4k}(B_n)$, $k \geq n + 1$ can be expressed as

$$x^k = \sum_{\ell, m} \lambda_{\ell, m} x^\ell y^m \quad \text{where} \quad \lambda_{\ell, m} \in \mathbb{Z}, \ell \geq 0, 0 < m \leq n;$$

in other words, any element in $H^{4k}(B_n)$ can be written as

$$\sum_{a, b} \lambda_{a, b} w^a y^b + \sum_{\ell, m} \lambda_{\ell, m} x^\ell y^m$$

where $\lambda_{a, b} \in \{0, 1\}$, $\lambda_{\ell, m} \in \mathbb{Z}$, $b, \ell \geq 0$, $a, m > 0$. So every torsion element has the form $u := \sum_j \lambda_j w^{a_j} y^{b_j}$, where $\lambda_j \in \{0, 1\}$, $2k = a_1 > a_2 > \dots$, $0 = b_1 < b_2 < \dots$, and $a_j + 4b_j = 2k$.

We know that $f^*(w^a y^b) = w^a (z^2 + w^2 z)^b$ in $H^{4k}(\widehat{\Delta}_n)$, so

$$f^*(u) = \sum_j \lambda_j w^{a_j} (z^2 + w^2 z)^{b_j} = 0$$

implies that $\lambda_1 = \lambda_2 = \dots = 0$, by equating coefficients of $w^{2k}, w^{2k-2}z, \dots$ in succession. Therefore $u = 0$, and there are no non-zero torsion elements in $\ker f^*$. \square

For future reference, note that

$$f^*(2y^{k/2}) = 2(z^2 + w^2 z)^{k/2} = 2z^k = 0, \quad (\text{when } k \text{ is even}) \quad (8.6)$$

$$f^*(x^\ell y^m) = (2z)^\ell (z^2 + w^2 z)^m = 2^\ell z^{\ell+2m} = 0, \quad (\ell > 0). \quad (8.7)$$

by (8.4), Remark 4.18 and Proposition 5.19. So, $2y^{k/2}, x^\ell y^m \in \ker f^*$.

Since $B_n/\widehat{\Delta}_n$ is connected, we also have $H^0((B_n/\widehat{\Delta}_n)_+) \cong \mathbb{Z}$.

Proposition 8.8. *Given $f^*: H^{4k}(B_n) \rightarrow H^{4k}(\widehat{\Delta}_n)$ as above; if $n+1 \leq k \leq 2n$, then any element of $\ker f^*$ can be written as*

$$\lambda_k 2y^{k/2} + \sum_{a,b} \mu_{a,b} x^a y^b, \quad \lambda_k, \mu_{a,b} \in \mathbb{Z}, \quad a+2b=k.$$

Proof. Suppose $\kappa = Ay^{k/2} + \sum_{a,b} B_{a,b} x^a y^b + \sum_{i,j} C_{i,j} w^i y^j$ is a non-zero element of $\ker f^*$, then

$$\begin{aligned} f^*(\kappa) &= f^* \left(Ay^{k/2} + \sum_{a,b} B_{a,b} x^a y^b + \sum_{i,j} C_{i,j} w^i y^j \right) \\ &= A(z^2 + w^2 z)^{k/2} + \sum_{i,j} C_{i,j} w^i (z^2 + w^2 z)^j \\ &= 0 \end{aligned}$$

where $a+2b=k$, $i+4j=2k$, $i>0$, $j \geq 0$ for some $A, B_{a,b}, C_{i,j} \in \mathbb{Z}$. If $A=0$ then $f^*(\kappa) = \sum_{i,j} C_{i,j} w^i (z^2 + w^2 z)^j = 0$ implies that $C_{2k,0} = C_{2k-4,1} = \dots = 0$ by equating coefficients of $w^{2k}, w^{2k-2}z, \dots$ in succession. If $A \neq 0$ then $C_{i,j} = 0$ by equating coefficients of $w^{k+2}z^{\frac{k}{2}-1}, w^{k+4}z^{\frac{k}{2}-2}, \dots$ in succession. This, in turn, yields that A must be even. \square

Lemma 8.9. *For any $n \geq 1$, the cohomology group $H^{4k}(SP_n^2)$ is torsion free.*

Proof. Case $k \leq n$.

From the LES

$$\longleftarrow H^{4k}(\Delta_n) \xleftarrow{i^*} H^{4k}(SP_n^2) \xleftarrow{s^*} H^{4k}(SP_n^2/\Delta_n) \xleftarrow{\delta} H^{4k-1}(\Delta_n) = 0 \quad (8.10)$$

s^* is injection and $H^{4k}(SP_n^2/\Delta_n) \cong \ker i^*$, and also, by Lemma 8.5, the cohomology group $H^{4k}(B_n/\widehat{\Delta}_n) \cong H^{4k}(SP_n^2/\Delta_n)$ is torsion free. Then (8.10) is given by

$$\longleftarrow \mathbb{Z} \xleftarrow{i^*} H^{4k}(SP_n^2) \longleftarrow \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \longleftarrow 0$$

which shows that $H^{4k}(SP_n^2)$ must be torsion free.

Case $k \geq n+1$.

From the exact sequence

$$0 \xleftarrow{i^*} H^{4k}(SP_n^2) \xleftarrow{s^*} H^{4k}(SP_n^2/\Delta_n) \xleftarrow{\delta} 0$$

we have that $s^*: H^{4k}(SP_n^2/\Delta_n) \xrightarrow{\cong} H^{4k}(SP_n^2)$ is an isomorphism, and Lemma 8.5 shows that $H^{4k}(B_n/\widehat{\Delta}_n) \cong H^{4k}(SP_n^2/\Delta_n)$ is torsion free. \square

Now let us consider the commutative ladder

$$\begin{array}{ccccccc}
 \longleftarrow & H^{4k}(\widehat{\Delta}) & \xleftarrow{f^*} & H^{4k}(B) & \xleftarrow{\beta^*} & H^{4k}(B/\widehat{\Delta}) & \longleftarrow H^{4k-1}(\widehat{\Delta}) = 0 \\
 & \downarrow d^* & & \downarrow b^* & & \downarrow q^* & \downarrow = \\
 \longleftarrow & H^{4k}(\widehat{\Delta}_n) & \xleftarrow{f^*} & H^{4k}(B_n) & \xleftarrow{\beta^*} & H^{4k}(B_n/\widehat{\Delta}_n) & \longleftarrow H^{4k-1}(\widehat{\Delta}_n) = 0
 \end{array} \quad (8.11)$$

If $k \leq n$, then b^* is an isomorphism by Remark 4.41, and clearly $H^{4k}(\widehat{\Delta}_n) \cong H^{4k}(\widehat{\Delta})$. So q^* is an isomorphism, by applying the Five Lemma [22, page 129] to the ladder (8.11). This, in turn, implies that $\phi_1^*: H^{4k}(SP^2/\Delta) \rightarrow H^{4k}(SP_n^2/\Delta_n)$ is an isomorphism in the commutative ladder

$$\begin{array}{ccccccc}
 \longleftarrow & H^{4k}(\Delta) & \xleftarrow{i^*} & H^{4k}(SP^2) & \xleftarrow{s^*} & H^{4k}(SP^2/\Delta) & \longleftarrow H^{4k-1}(\Delta) = 0 \\
 & \downarrow & & \downarrow \phi^* & & \downarrow \phi_1^* & \downarrow = \\
 \longleftarrow & H^{4k}(\Delta_n) & \xleftarrow{i^*} & H^{4k}(SP_n^2) & \xleftarrow{s^*} & H^{4k}(SP_n^2/\Delta_n) & \longleftarrow H^{4k-1}(\Delta_n) = 0
 \end{array} \quad (8.12)$$

by appealing to the diagram (2.27). Now (8.12) and the Five Lemma show that

Remark 8.13. For $k \leq n$, $\phi^*: H^{4k}(SP^2) \rightarrow H^{4k}(SP_n^2)$ is an isomorphism.

Lemma 8.14. For $n+1 \leq k \leq 2n$, additive generators for $H^{4k}(SP_n^2)$ may be chosen to be of the form

$$\left(\frac{1}{2}\right)^m g^\ell h^m \quad \text{and} \quad \left(\frac{1}{2}\right)^{\frac{k}{2}-1} h^{k/2} \quad (\text{if } k \text{ is even}) \quad (8.15)$$

where $|h| = 8$, $|g| = 4$, $\ell \geq 1$, $m \geq 0$, $\ell + 2m = k$.

Proof. If $n+1 \leq k \leq 2n$ then we have the commutative ladder

$$\begin{array}{ccccccc}
 0 = & H^{4k}(\Delta_n) & \longleftarrow & H^{4k}(SP_n^2) & \xleftarrow{s^*} & H^{4k}(SP_n^2/\Delta_n) & \longleftarrow 0 \\
 & \downarrow & & \downarrow \pi_n^* & & \cong \downarrow & \\
 & H^{4k}(\widehat{\Delta}_n) & \xleftarrow{f^*} & H^{4k}(B_n) & \xleftarrow{\beta^*} & H^{4k}(B_n/\widehat{\Delta}_n) & \longleftarrow 0
 \end{array}$$

where $H^{4k}(B_n/\widehat{\Delta}_n) \cong \ker f^*$ and $H^{4k}(SP_n^2) \cong H^{4k}(SP_n^2/\Delta_n)$. For $n+1 \leq k \leq 2n$, any element of $\ker f^*$ can be written as

$$\lambda_m 2y^m + \sum_{a,b} \mu_{a,b} x^a y^b, \quad \lambda_m, \mu_{a,b} \in \mathbb{Z}, \quad 2m = a + 2b = k \geq 2$$

by Proposition 8.8. Using Lemma 8.9 and (7.1), we define $(\frac{1}{2})^{m-1} h^m$, $(\frac{1}{2})^b g^a h^b$ as preimages $(\pi_n^*)^{-1}(2y^m)$ and $(\pi_n^*)^{-1}(x^a y^b)$ corresponding to the additive generators for $\ker f^*$, respectively. \square

Furthermore, combining Proposition 7.20 with Lemma 8.14 yields the following statement.

Corollary 8.16. *For $4(n+1) \leq 4^*$, the homomorphism*

$$\phi^*: H^{4^*}(SP^2) \rightarrow H^{4^*}(SP_n^2) \quad (8.17)$$

induced by the inclusion is a surjection.

Alternatively, it is possible to prove Corollary 8.16 by showing that the restriction homomorphism $H^*(SP_{n+1}^2) \rightarrow H^*(SP_n^2)$ is surjective for any $n \geq 1$, and referring to [22, page 313–314 and Theorem 3F.8] for the properties of inverse limit.

To describe $\ker \phi^*$, we use the commutative diagram

$$\begin{array}{ccc} H^{4^*}((B_n)_+) & \xleftarrow{\pi_n^*} & H^{4^*}((SP_n^2)_+) \\ b^* \uparrow & & \uparrow \phi^* \\ H^{4^*}(B_+) & \xleftarrow{\pi^*} & H^{4^*}(SP_+^2) \end{array} \quad (8.18)$$

where $b^*: H^{4^*}(B_+) \rightarrow H^{4^*}((B_n)_+)$ is a surjection by Remark 4.41. The homomorphisms π^* and π_n^* are injective, because there exists the rational isomorphism in cohomology between the Borel space and the symmetric square (7.1) and $H^{4^*}(SP_n^2)$ and $H^{4^*}(SP^2)$ are torsion free by Lemma 8.9 and 7.6.

Definition 8.19. *We define $\theta_{n,1}$ in $H^{4n+4}(SP^2)$ and $\theta_{n,2}$ in $H^{4n+8}(SP^2)$ by the unique preimages*

$$(\pi^*)^{-1}(f_{n,1}) =: \theta_{n,1} \quad \text{and} \quad (\pi^*)^{-1}(f_{n,2}) =: \theta_{n,2}.$$

Example 8.20. *Case $n = 2$.*

From Proposition 7.20, $H^{4^}((B_2)_+) \cong H^{4^*}(B_+)/(\mathfrak{f}_{2,1}, \mathfrak{f}_{2,2}, y^3)$, where $\mathfrak{f}_{2,1} = x^3 - 3xy$ and $\mathfrak{f}_{2,2} = x^2y - 2y^2$. Also, $\mathfrak{f}_{2,1} = \pi^*(g^3 - 3 \cdot \frac{1}{2}gh)$ and $\mathfrak{f}_{2,2} = \pi^*(\frac{1}{2}g^2h - \frac{1}{2}h^2)$ by Proposition 7.20, so $\theta_{2,1} := g^3 - 3 \cdot \frac{1}{2}gh$ and $\theta_{2,2} := \frac{1}{2}g^2h - \frac{1}{2}h^2$.*

Combining Theorem 4.35 and Definition 8.19 yields the following.

Remark 8.21. *The polynomials $\theta_{n,1}$ and $\theta_{n,2}$ are given by*

$$\left\{ \begin{array}{l} \theta_{1,1} = g^2 - h \\ \theta_{1,2} = \frac{1}{2}gh \end{array} \right. \quad \text{for } n = 1, \quad \text{and} \quad \left\{ \begin{array}{l} \theta_{n,1} = g \theta_{n-1,1} - \theta_{n-1,2} \\ 2 \theta_{n,2} = h \theta_{n-1,1} \end{array} \right. \quad \text{for } n \geq 2.$$

For further low dimensional examples, see Appendix C.2.

Remark 8.22. For some fixed $n \geq 1$, $\theta_{m,2;m \geq n} = 0$ implies that $(\frac{1}{2})^{k-1}h^k = 0$ for all $k \geq n+1$. Because $\pi^*((\frac{1}{2})^{k-1}h^k) = 2y^k$ by Proposition 7.20, and for any $k \geq 2$, there is an expression $2y^k = \sum_m \mu_m f_{m,2}$, where $\mu_m \in \mathbb{Z}\langle x^i y^j \rangle$, $j \geq 0$, $i \geq 1$, and $k-1 \leq m \leq 2(k-1)$ by Lemma 4.43. Therefore, if $\theta_{m,2;m \geq n} = 0$ then $(\frac{1}{2})^{k-1}h^k = \sum_m \mu'_m \theta_{m,2} = 0$ where $\mu'_m \in \mathbb{Z}\langle (\frac{1}{2})^j g^i h^j \rangle$, $j \geq 0$, $i \geq 1$.

Lemma 8.23. Suppose κ is any element of $\ker \phi^* \subset H^{4*}(SP_+^2)$. Then the image $\pi^*(\kappa)$ can be expressed as

$$\pi^*(\kappa) = \lambda_1 f_{n,1} + \lambda_2 f_{n,2} + \lambda_3 y^{n+1} \quad (8.24)$$

where $\lambda_3 \in \mathbb{Z}\langle 2y^s, x^m y^\ell \rangle$, and $\lambda_1, \lambda_2 \in \mathbb{Z}\langle y^s, x^m y^\ell \rangle$, $s, m \geq 1$, $\ell \geq 0$.

Proof. Recall $H^*((B_n)_+) \cong H^*(B_+)/(\mathfrak{f}_{n,1}, \mathfrak{f}_{n,2}, y^{n+1})$ from Theorem 4.35. So any element $\kappa' \in \ker b^*$ where $b: B_n \hookrightarrow B$ can be written as $\kappa' = \lambda'_1 f_{n,1} + \lambda'_2 f_{n,2} + \lambda'_3 y^{n+1}$ where $\lambda'_1, \lambda'_2, \lambda'_3 \in H^*(B_+)$. Also, $\ker \phi^* \subset \ker b^*$ from the diagram (8.18), so an expression of the form (8.24) is valid.

If $\lambda_1, \lambda_2 \in \mathbb{Z}/2\langle w^i y^j \rangle$, $i \geq 1$, $j \geq 0$, then $\lambda_1 f_{n,1} = \lambda_2 f_{n,2} = 0$ because $wx = 0$ and $2w = 0$. We may therefore assume that no monomials of λ_1 and λ_2 involve w . So the same is true for λ_3 , because the image of π^* does not involve w either. In fact, λ_3 lies in $\mathbb{Z}\langle 2y^s, x^m y^\ell \rangle$, because the coefficient of y^k in $f_{n,1}, f_{n,2}$ is always 2 or 0. \square

Example 8.25. $n = 2$.

$$\begin{aligned} \pi^*(\frac{1}{2}g^3h) &= x^3y = 3x f_{2,2} - 2y f_{2,1} \\ \pi^*(\frac{1}{4}g^2h^2) &= x^2y^2 = y f_{2,2} + 2y^3 \end{aligned}$$

Lemma 8.26. There exist expressions

$$y^i f_{n,1} = \sum_{m_i} \mu_{m_i} f_{m_i,2}$$

where $\mu_m = \pm x y^k$, $0 \leq k \leq i-2$, $m = n+2, n+4, \dots, n'-1$, $n' = n+2i-1$, and

$$y^j f_{n,2} = \sum_{m_j} \mu_{m_j} f_{m_j,2}$$

where $\mu_m = \pm x y^k$, $0 \leq k \leq j-1$, $m = n+1, n+3, \dots, n''-1$, and $n'' = n+2j$.

Proof. We use the relations $f_{n-1,2} = x f_{n-1,1} - f_{n,1}$, $f_{n,2} = y f_{n-1,1}$ from Theorem 4.35. For any $n \geq 1$ and $i \geq 1$ we then have that

$$\begin{aligned}
y^i f_{n,1} &= y^{i-1} f_{n+1,2} \\
&= xy^{i-1} f_{n+1,1} - y^{i-2} f_{n+3,2} \\
&= xy^{i-1} f_{n+1,1} - (xy^{i-2} f_{n+3,1} - y^{i-3} f_{n+5,2}) \\
&\quad \vdots \\
&= xy^{i-1} f_{n+1,1} - xy^{i-2} f_{n+3,1} + \cdots + (-1)^{i-2} xy f_{n'-2,1} + (-1)^{i-1} f_{n',2} \\
&= xy^{i-2} f_{n+2,2} - xy^{i-3} f_{n+4,2} + \cdots + (-1)^{i-2} x f_{n'-1,2} + (-1)^{i-1} f_{n',2} \\
&= \sum_m \mu_m f_{m,2} + (-1)^{i-1} f_{n',2}
\end{aligned}$$

where $\mu_m = \pm xy^k$, $0 \leq k \leq i-2$, $m = n+2, n+4, \dots, n'-1$, and $n' = n+2i-1$.

Similarly, for any $n \geq 1$ and $j \geq 1$ we have that

$$\begin{aligned}
y^j f_{n,2} &= y^j (x f_{n,1} - f_{n+1,1}) \\
&= xy^j f_{n,1} - y^{j-1} f_{n+2,2} \\
&= xy^j f_{n,1} - y^{j-1} (x f_{n+2,1} - f_{n+3,1}) \\
&= xy^j f_{n,1} - y^{j-1} x f_{n+2,1} + y^{j-2} f_{n+4,2} \\
&\quad \vdots \\
&= xy^j f_{n,1} - xy^{j-1} f_{n+2,1} + \cdots + (-1)^{j-1} xy f_{n''-2,1} + (-1)^j f_{n'',2} \\
&= xy^{j-1} f_{n+1,2} - xy^{j-2} f_{n+3,2} + \cdots + (-1)^{j-1} x f_{n''-1,2} + (-1)^j f_{n'',2} \\
&= \sum_m \mu_m f_{m,2} + (-1)^j f_{n'',2}
\end{aligned}$$

where $\mu_m = \pm xy^k$, $0 \leq k \leq j-1$, $m = n+1, n+3, \dots, n''-1$, and $n'' = n+2j$. \square

Using the relation $f_{n,2} = x f_{n,1} - f_{n+1,1}$ also yields the following.

Remark 8.27. For $k \geq 1$, $m \geq 1$, there is an expression

$$x^m y^k = x^{m-1} y^{k-1} f_{1,2} = \sum_{m_k} \mu_{m_k} f_{m_k,2},$$

where $m_k \geq k$, $\mu_{m_k} \in \mathbb{Z}[x]$, $k \leq i \leq m+2k-2$.

Example 8.28. For $m \geq 1$, we have that

$$\begin{aligned} x^m y &= x^{m-1} f_{1,2} \\ x^m y^2 &= x^{m-1} y f_{1,2} \\ &= x^{m-1} (xy f_{1,1} - y f_{2,1}) \\ &= x^{m-1} (x f_{2,2} - f_{3,2}). \end{aligned}$$

We can now describe $\ker \phi^*$ in dimension $4*$.

Proposition 8.29. For any $n \geq 1$, there is an isomorphism

$$H^{4*}((SP_n^2)_+) \cong H^{4*}(SP_+^2)/I$$

where $I = (\theta_{n,1}, \theta_{m,2:m \geq n})$.

Proof. First we note that $(f_{n,1}, f_{m,2:m \geq n}) \subset \ker b^*$, by Lemma 4.45; it then follows that $I \subseteq \ker \phi^*$, by the commutative diagram (8.18).

Any element $\kappa \in \ker \phi^* \subset H^{4*}(SP_+^2)$ can be described by

$$\begin{aligned} \pi^*(\kappa) &= \lambda_1 f_{n,1} + \lambda_2 f_{n,2} + \lambda_3 y^{n+1} \text{ by Lemma 8.23} \\ &= \lambda' f_{n,1} + \sum_m \lambda'_m f_{m,2:m \geq n} \text{ by Lemmas 8.26, 4.43, and Remark 8.27} \end{aligned}$$

where $\lambda', \lambda'_m \in \mathbb{Z}\langle 2y^s, x^m y^\ell \rangle$. So,

$$\begin{aligned} \kappa &= (\pi^*)^{-1} \left(\lambda' f_{n,1} + \sum_m \lambda'_m f_{m,2:m \geq n} \right) \\ &= \gamma \theta_{n,1} + \sum_m \gamma_m \theta_{m,2:m \geq n} \end{aligned}$$

where $\gamma, \gamma_m \in H^{4*}(SP_+^2)$. Thus $\kappa \in I$ and $\ker \phi^* \subseteq I$. □

Example 8.30. Case $n = 2$.

$$H^{4*}((SP_2^2)_+) \cong H^{4*}(SP_+^2)/(\theta_{2,1}, \theta_{m,2:m \geq 2}),$$

and the elements $g^4 - 3\frac{1}{2}g^2h$, $\frac{1}{2}g^3h$, and $\frac{1}{4}g^2h^2$ in $\ker \phi^*$, for example, can be expressed as

$$g^4 - 3\frac{1}{2}g^2h = g\theta_{2,1}, \quad \frac{1}{2}g^3h = 3g\theta_{2,2} - h\theta_{2,1}, \quad \frac{1}{4}g^2h^2 = g^2\theta_{2,2} - g\theta_{3,2}.$$

8.1.3 $H^{4k+2}(SP^2(\mathbb{H}\mathbb{P}^n))$

Lemma 8.31. *For any $n \geq 1$, $H^{4k+2}(SP_n^2) = 0$.*

Proof. The proof is similar to the infinite case (Proposition 7.21). We have the commutative ladder

$$\begin{array}{ccccccc} 0 & = & H^{4k+2}(\Delta_n) & \longleftarrow & H^{4k+2}(SP_n^2) & \xleftarrow{\cong s^*} & H^{4k+2}(SP_n^2/\Delta_n) & \longleftarrow & 0 \\ & & \downarrow & & \downarrow \pi^* & & \cong \downarrow & & \\ \longleftarrow & & H^{4k+2}(\widehat{\Delta}_n) & \xleftarrow{f^*} & H^{4k+2}(B_n) & \xleftarrow{\beta^*} & H^{4k+2}(B_n/\widehat{\Delta}_n) & \longleftarrow & 0 \end{array}$$

where $H^{4k+2}(B_n) = \mathbb{Z}/2\langle w^i y^j \rangle$ for $0 \leq j \leq n$, $i \geq 1$, $i + 4j = 2k + 1$.

If $f^*\left(\sum_{i,j} \lambda_{i,j} w^i y^j\right) = \sum_{i,j} \lambda_{i,j} w^i (z^2 + w^2 z)^j = 0$, then $\lambda_{i,j} = 0$ by equating the coefficients of w^{2k+1} , $w^{2k-3}z, \dots$, which implies that $\ker f^* = 0$. Therefore

$$0 = \ker f^* \cong H^{4k+2}(B_n/\widehat{\Delta}_n) \cong H^{4k+2}(SP_n^2/\Delta_n) \cong H^{4k+2}(SP_n^2),$$

as required. □

8.1.4 $H^{4k+1}(SP^2(\mathbb{H}\mathbb{P}^n))$ and $H^{4k+3}(SP^2(\mathbb{H}\mathbb{P}^n))$

We shall describe $H^*(SP_n^2)$ for $* = 4k + 1, 4k + 3$.

Lemma 8.32. *For $* = 4k + 1, 4k + 3$,*

$$q^*: H^*(B/\widehat{\Delta}) \rightarrow H^*(B_n/\widehat{\Delta}_n)$$

is an isomorphism if $k \leq n$, and a surjection if $k \geq n + 1$.

Proof. If $* \leq 4n + 3$, then by Remark 4.41, the cohomology ring $H^*(B)$ is isomorphic to $H^*(B_n; \mathbb{Z})$, and we know that $H^*(\widehat{\Delta}) \cong H^*(\widehat{\Delta}_n)$.

Then, for $* = 4k + 1, 4k + 3$, $k \leq n$, the commutative ladder

$$\begin{array}{ccccccc} 0 & = & H^*(B_n) & \longleftarrow & H^*(B_n/\widehat{\Delta}_n) & \xleftarrow{\delta} & H^{*-1}(\widehat{\Delta}_n) & \longleftarrow & H^{*-1}(B_n) & \longleftarrow & 0 \\ & & \uparrow & & \uparrow q^* & & \cong \uparrow d^* & & \cong \uparrow b^* & & \\ 0 & = & H^*(B) & \longleftarrow & H^*(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^{*-1}(\widehat{\Delta}) & \longleftarrow & H^{*-1}(B) & \longleftarrow & 0 \end{array}$$

shows that q^* is an isomorphism.

If $* = 4k + 1, 4k + 3, k \geq n + 1$, then we have the ladder

$$\begin{array}{ccccc} H^*(B_n) = 0 & \longleftarrow & H^*(B_n/\widehat{\Delta}_n) & \xleftarrow{\delta} & H^{*-1}(\widehat{\Delta}_n) \cong \mathbb{Z}[w, z]/(2w, z^{n+1}) \\ & & \uparrow q^* & & \uparrow d^* \\ H^*(B) = 0 & \longleftarrow & H^*(B/\widehat{\Delta}) & \xleftarrow{\delta} & H^{*-1}(\widehat{\Delta}) \cong \mathbb{Z}[w, z]/(2w) \end{array}$$

where d^* and both homomorphisms δ are surjective. Therefore, q^* is surjective. \square

Remark 8.33. *Both of the above two proofs are actually special cases of the Five Lemma, as applied in (8.11). So we may continue as in (8.12), and replace $B/\widehat{\Delta}$ with SP^2/Δ and $B_n/\widehat{\Delta}_n$ with SP_n^2/Δ_n , then use a version of the Five Lemma again.*

Proposition 8.34. *For $* = 4k + 1, 4k + 3$, the homomorphism*

$$\phi^*: H^*(SP^2) \rightarrow H^*(SP_n^2)$$

is an isomorphism if $k \leq n$, and surjective if $k \geq n + 1$.

Proof. We consider four cases.

Case 1. If $* = 4k + 3$ with $k \leq n$, then we have

$$\begin{array}{ccccccc} H^{4k+3}(\Delta_n) = 0 & \longleftarrow & H^{4k+3}(SP_n^2) & \xleftarrow{\cong} & H^{4k+3}(SP_n^2/\Delta_n) & \longleftarrow & H^{4k+2}(\Delta_n) = 0 \\ & & \uparrow \phi^* & & \cong \uparrow & & \\ H^{4k+3}(\Delta) = 0 & \longleftarrow & H^{4k+3}(SP^2) & \xleftarrow{\cong} & H^{4k+3}(SP^2/\Delta) & \longleftarrow & H^{4k+2}(\Delta) = 0 \end{array}$$

by Lemma 8.32 and Remark 8.33. Thus ϕ^* is an isomorphism.

Case 2. If $* = 4k + 3, k \geq n + 1$, then we have

$$\begin{array}{ccccccc} 0 & \longleftarrow & H^{4k+3}(SP_n^2) & \xleftarrow{\cong} & H^{4k+3}(SP_n^2/\Delta_n) & \longleftarrow & 0 \\ & & \uparrow \phi^* & & \uparrow \phi_1^* & & \\ 0 & \longleftarrow & H^{4k+3}(SP^2) & \xleftarrow{\cong} & H^{4k+3}(SP^2/\Delta) & \longleftarrow & 0 \end{array}$$

where ϕ_1^* is a surjection by Remark 8.33. Thus ϕ^* is a surjection.

Case 3. If $* = 4k + 1, k \leq n$, then we have

$$\begin{array}{ccccccc} H^{4k+1}(\Delta_n) = 0 & \longleftarrow & H^{4k+1}(SP_n^2) & \longleftarrow & H^{4k+1}(SP_n^2/\Delta_n) & \longleftarrow & H^{4k}(\Delta_n) \\ & & \uparrow \phi^* & & \cong \uparrow & & \cong \uparrow \\ H^{4k+1}(\Delta) = 0 & \longleftarrow & H^{4k+1}(SP^2) & \longleftarrow & H^{4k+1}(SP^2/\Delta) & \longleftarrow & H^{4k}(\Delta) \end{array}$$

by Lemma 8.32 and Remark 8.33. Thus ϕ^* is an isomorphism.

Case 4. If $* = 4k + 1$, $k \geq n + 1$, then we have

$$\begin{array}{ccccccc}
 0 & \longleftarrow & H^{4k+1}(SP_n^2) & \xleftarrow{\cong} & H^{4k+1}(SP_n^2/\Delta_n) & \longleftarrow & H^{4k}(\Delta_n) = 0 \\
 & & \uparrow \phi^* & & \uparrow \phi_1^* & & \uparrow \phi_2^* \\
 0 & \longleftarrow & H^{4k+1}(SP^2) & \xleftarrow{s^*} & H^{4k+1}(SP^2/\Delta) & \xleftarrow{\delta} & H^{4k}(\Delta) \longleftarrow
 \end{array}$$

where ϕ_1^* is surjective by Lemma 8.32 and Remark 8.33. Thus ϕ^* is a surjection. \square

Example 8.35. *The groups $H^5(SP_n^2; \mathbb{Z}) \cong H^9(SP_n^2; \mathbb{Z}) = 0$, $\forall n \geq 1$, are two trivial cases (see Appendix C.1).*

Proposition 8.34 and Definition 7.28 for $t_{i,j}$, and the fact $z^{n+1} = 0$ for $H^*(\mathbb{H}\mathbb{P}^n)$, yields the following.

Corollary 8.36. *For $* = 4k + 1, 4k + 3$, $\ker \phi^* = \mathbb{Z}/2\langle t_{i,j} : i \geq n + 1 \rangle$.*

We can now complete our description of $\ker \phi^*$ by combining Proposition 8.29 and Corollary 8.36 with the product structure for $H^*(SP^2)$ obtained in Section 7.2. the outcome is that we may write

$$H^*((SP_n^2)_+) \cong H^*(SP_+^2)/(\theta_{n,1}, \theta_{m,2:m \geq n}, t_{i,j:i \geq n+1})$$

for any $n \geq 1$, and therefore complete the proof of Theorem 1.8.

8.2 Relation to Nakaoka's Work

In this section, we shall compare our computations with Nakaoka's work [44, 47] on the cohomology of the p -fold cyclic products $CP_p(X)$. The results of ([47], Chapter II) were stated for a prime number p and a connected finite simplicial complex X . We shall apply them to the particular case $p = 2$ and $X = \mathbb{H}\mathbb{P}^n$, because there is a homeomorphism $CP_2(X) \cong SP^2(X)$. It is convenient to work with $H^*(SP_n^2/\Delta_n; \mathbb{Z}/2)$ rather than $H^*(SP_n^2, \Delta_n; \mathbb{Z}/2)$ for $* \geq 1$, but they are isomorphic [36, page 143].

Nakaoka introduced two types of elements, $\Phi_0^*(x_1 \times x_2)$ and $E_s(x)$ in [44, 47], which were defined via additive homomorphisms

$$\Phi_0^*: H^*(X \times X; \mathbb{Z}/2) \rightarrow H^*(SP^2(X)/\Delta_n; \mathbb{Z}/2) \quad (8.37)$$

$$E_s: H^q(X; \mathbb{Z}/2) \rightarrow H^{q+s}(SP^2(X)/\Delta_n; \mathbb{Z}/2), \quad s \geq 1 \quad (8.38)$$

for $x, x_1, x_2 \in H^*(X; \mathbb{Z}/2)$, where $\Phi_0^*(x \times y) = \Phi_0^*(y \times x)$.

The multiplicative properties of these elements ([47], Theorem 11.8) are:

$$\Phi_0^*(x_1 \times x_2)\Phi_0^*(y_1 \times y_2) = \Phi_0^*(x_1y_1 \times x_2y_2 + x_1y_2 \times x_2y_1) \quad (8.39)$$

$$E_s(x)E_r(y) = 0 \quad \text{for } s, r \geq 1 \quad (8.40)$$

$$E_s(x)\Phi_0^*(x_1 \times x_2) = 0 \quad \text{for } s \geq 1. \quad (8.41)$$

Theorem 8.42. ([47], Theorem 9.9) *The homomorphism*

$$E_s: H^q(X; \mathbb{Z}/2) \rightarrow H^{q+s}(SP^2(X)/\Delta_n; \mathbb{Z}/2)$$

is injective for $1 \leq s \leq q$.

By applying Nakaoka's theorems and lemmas to our case, we have the following reformulations of his results.

Theorem 8.43. ([47], Theorem 10.8)

For $z^k \in H^{4k}(\mathbb{H}\mathbb{P}^n; \mathbb{Z}/2)$, $1 \leq k \leq n$,

$$\Phi_0^*(z^k \times z^k) = E_{4k}(z^k)$$

holds.

Theorem 8.44. ([47], Theorem 11.2)

For any $m > 0$, a basis of $H^m(SP_n^2/\Delta_n; \mathbb{Z}/2)$ is given by

$$E_r(z^d), 1 \leq r \leq 4d, \quad \Phi_0^*(z^p \times z^q), p \neq q$$

for $z^d \in H^{4d}(\mathbb{H}\mathbb{P}^n; \mathbb{Z}/2)$, $r + 4d = m$ and $z^p, z^q \in H^(\mathbb{H}\mathbb{P}^n; \mathbb{Z}/2)$, $4(p + q) = m$.*

Lemma 8.45. ([47], Lemma 11.3)

The sequence

$$0 \rightarrow H^{m-1}(\Delta_n; \mathbb{Z}/2) \rightarrow H^m(SP_n^2/\Delta_n; \mathbb{Z}/2) \xrightarrow{s^*} H^m(SP_n^2; \mathbb{Z}/2) \rightarrow 0$$

is exact for $m > 1$.

Theorem 8.46. ([47], Theorem 11.4)

A basis of $H^m(SP_n^2; \mathbb{Z}/2)$ is given by

$$s^*E_r(z^d), 2 \leq r \leq 4d, \quad \Phi_0^*(z^p \times z^q), p \neq q$$

for $z^d \in H^{4d}(\mathbb{H}\mathbb{P}^n; \mathbb{Z}/2)$, $r + 4d = m$ and $z^p, z^q \in H^(\mathbb{H}\mathbb{P}^n; \mathbb{Z}/2)$, $4(p + q) = m$.*

Note that $H^*(\mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n; \mathbb{Z}/2) \rightarrow H^*(SP_n^2; \mathbb{Z}/2)$ is the composition $i^* \circ \Phi_0^*$

$$H^*(\mathbb{H}\mathbb{P}^n \times \mathbb{H}\mathbb{P}^n; \mathbb{Z}/2) \xrightarrow{\Phi_0^*} H^*(SP_n^2/\Delta_n; \mathbb{Z}/2) \xrightarrow{i^*} H^*(SP_n^2; \mathbb{Z}/2)$$

where i^* is the inclusion homomorphism ([47], Lemma 2.14).

We may now complete our cohomological calculations by checking that they agree with Nakaoka's work. This means we must first use Theorem 6.26 (UCT) to translate our integral calculations into $H^*(SP_n^2; \mathbb{Z}/2)$. From (6.27), for any abelian group G and any $m \geq 1$, there is an isomorphism

$$H^m(Y; G) \cong H^m(Y; \mathbb{Z}) \otimes G \oplus \text{Tor}(H^{m+1}(Y; \mathbb{Z}), G).$$

We take $G = \mathbb{Z}/2$ and $Y = SP_n^2$.

$$\text{Case } * = 1, 2, 3, 5, 9 : H^*(SP_n^2; \mathbb{Z}/2) \cong \text{Tor}(H^{*+1}(SP_n^2; \mathbb{Z}), \mathbb{Z}/2) = 0. \quad (8.47)$$

$$\text{Case } * = 4k + 3 \geq 7, 4k + 1 \geq 13 : H^*(SP_n^2; \mathbb{Z}/2) \cong H^*(SP_n^2; \mathbb{Z}) \otimes \mathbb{Z}/2. \quad (8.48)$$

$$\text{Case } * = 4k + 2, k \geq 1 : H^{4k+2}(SP_n^2; \mathbb{Z}/2) \cong \text{Tor}(H^{4k+3}(SP_n^2; \mathbb{Z}), \mathbb{Z}/2). \quad (8.49)$$

$$\text{Case } * = 4k : H^{4k}(SP_n^2; \mathbb{Z}/2) \cong H^{4k}(SP_n^2; \mathbb{Z}) \otimes \mathbb{Z}/2 \oplus H^{4k+1}(SP_n^2; \mathbb{Z}/2). \quad (8.50)$$

We shall explain (8.48) (8.49) (8.50), and introduce our notations for the mod 2 generators.

Let ρ be the mod 2 reduction homomorphism, $H^*(SP_n^2; \mathbb{Z}) \xrightarrow{\rho} H^*(SP_n^2; \mathbb{Z}/2)$.

From Sections 8.1.2 and 8.1.3, we have that $H^{4k+4}(SP_n^2; \mathbb{Z})$ is torsion free and $H^{4k+2}(SP_n^2; \mathbb{Z}) = 0$, which yield the isomorphism (8.48). So, if $* = 4k + 1, 4k + 3$, then ρ is an isomorphism on $H^*(SP_n^2; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 = \mathbb{Z}/2\langle t_{i,j} \rangle$ by Section 8.1.4. We write $b_{i,2j}$ for $\rho(t_{i,j})$, $1 \leq i, 1 \leq j < 2i$.

The facts that $H^{4k}(SP_n^2; \mathbb{Z})$ is torsion free and $H^{4k+1}(SP_n^2; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ as above, yield the isomorphism (8.50). We may write the generators corresponding to the first summand of (8.50) as $\rho((\frac{1}{2})^{s-1}h^s)$, $\rho((\frac{1}{2})^m g^\ell h^m)$ where $s, \ell \geq 1, m \geq 0$. Also, we have $H^{4k+1}(SP_n^2; \mathbb{Z})$ as a $\mathbb{Z}/2$ vector space on the elements $t_{i,j}$, so there exists an isomorphism $H^{4k+1}(SP_n^2; \mathbb{Z}/2) \cong \text{Tor}(H^{4k+1}(SP_n^2; \mathbb{Z}), \mathbb{Z}/2)$. We may now define a monomorphism

$$\sigma : H^{4k+1}(SP_n^2; \mathbb{Z}/2) \rightarrow H^{4k}(SP_n^2; \mathbb{Z}/2) \quad (8.51)$$

of $\mathbb{Z}/2$ vector spaces, by composing with inclusion of the second summand (8.50). Then, we write the generators corresponding to the second summand of (8.50) as $b_{i,2j-1}$ for $\sigma(b_{i,2j})$ where $i, j \geq 1, 2i + j = 2k$.

The vanishing cohomology $H^{4k+2}(SP_n^2; \mathbb{Z}) = 0$ from Lemma 8.31 yields (8.49). We have $H^{4k+3}(SP_n^2; \mathbb{Z}/2)$ as a $\mathbb{Z}/2$ vector space on the $b_{i,2j}$, so there exists an isomorphism

$$\sigma' : H^{4k+3}(SP_n^2; \mathbb{Z}/2) \cong \text{Tor}(H^{4k+3}(SP_n^2; \mathbb{Z}), \mathbb{Z}/2) \xrightarrow{\cong} H^{4k+2}(SP_n^2; \mathbb{Z}/2) \quad (8.52)$$

and we write $b_{i,2j-1}$ for the corresponding generators $\sigma'(b_{i,2j})$ where $i, j \geq 1, 2i + j = 2k + 1$.

Our generators and Nakaoka's both involve the groups $H^*(SP_n^2/\Delta_n)$, so it is very important to consider them at this point. From Section 8.1.4, if $* = 4k + 1$, or $4k + 3$, then $s^* : H^*(SP_n^2/\Delta_n) \rightarrow H^*(SP_n^2)$ splits, or it is an isomorphism. Also, we have used the canonical isomorphism $\pi^*|_{SP_n^2/\Delta_n} : H^*(SP_n^2/\Delta_n) \rightarrow H^*(B_n/\widehat{\Delta}_n)$ to describe elements in $H^*(SP^2)$. In the case of mod 2, the same construction gives new elements $(\pi^*)^{-1}\delta(a^j z^i)$ in $H^{4i+j+1}(SP_n^2/\Delta_n; \mathbb{Z}/2)$, where $\rho(w) = a^2 \in H^2(\mathbb{R}\mathbb{P}^\infty; \mathbb{Z}/2)$.

Amongst these elements we are especially interested in the ones that arise from $\delta(a^{2j} z^i)$, which are the mod 2 reductions $\rho(\delta(w^j z^i))$. They determine the integral basis elements $t_{i,j} := u^*(\delta(w^j z^i))$ as in Proposition 8.34 and Definition 7.28.

The products $\rho(t_{i,j})\rho(t_{k,\ell}) = 0$, $\rho(t_{i,j})\rho((\frac{1}{2})^m g^\ell h^m) = 0$, $\rho(t_{i,j})\rho((\frac{1}{2})^{s-1} h^s) = 0$, and Nakaoka's results (8.40) (8.41), together with the fact $z^i = 0, i > n$, yield a bijection

$$s^* E_{2j+1}(z^i) \leftrightarrow b_{i,2j} := \rho(t_{i,j}), \quad 1 \leq j < 2i, 1 \leq i \leq n, \quad (8.53)$$

between Nakaoka's basis elements and our own in $H^{4i+2j+1}(SP_n^2; \mathbb{Z}/2)$, which is also consistent with Corollary 7.40.

Case $j = 0, 1 \leq i \leq n$. Nakaoka's result $s^* E_1(z^i) = 0$ is equivalent to (the mod 2 reduction of) our $s^*(\delta(z^i)) = 0$ under the bijection $E_1(z^i) \leftrightarrow \delta(z^i)$.

For $4 \leq * \leq 7$, we have $H^*(SP_n^2; \mathbb{Z}/2) \cong \mathbb{Z}/2, n \geq 1$, so it is clear that

$$\rho(g) = \Phi_0^*(z \times 1), \quad b_{1,1} = s^* E_2(z), \quad b_{1,2} = s^* E_3(z)$$

hold; then $\rho(g) = \Phi_0^*(z \times 1)$ and (8.39) yield

$$\rho(g^i) = (\Phi_0^*(z \times 1))^i = \Phi_0^*(z^i \times 1) + \text{other terms} \quad (8.54)$$

where $\Phi_0^*(z^i \times 1) \neq 0$ if $i \leq n$. Then we have a bijection

$$\rho(g^i) \leftrightarrow \Phi_0^*(z^i \times 1)$$

by choosing the term $\Phi_0^*(z^i \times 1)$ as a representative.

We will show $\rho(h) = s^*E_4(z)$. For $n \geq 2$, $H^8(SP_n^2; \mathbb{Z}/2) \cong \mathbb{Z}/2\langle \rho(h) \rangle \oplus \mathbb{Z}/2\langle \rho(g^2) \rangle$, and $\rho(g^2) = (\Phi_0^*(z \times 1))^2 = \Phi_0^*(z^2 \times 1) + s^*E_4(z)$ by (8.54) (8.39). This suggests that either $\rho(h) = s^*E_4(z)$ or $\rho(h) = \Phi_0^*(z^2 \times 1)$. If $\rho(h) = \Phi_0^*(z^2 \times 1)$ then $\rho(g)\rho(h) = \Phi_0^*(z^2 \times 1)\Phi_0^*(z \times 1) \neq 0$, but $\rho(g)\rho(h) = \rho(gh) = \rho(2 \cdot \frac{1}{2}gh) = 0$, so we have a contradiction. Therefore $\rho(h) = s^*E_4(z)$.

Combining $\sigma, \sigma': H^*(SP_n^2; \mathbb{Z}/2) \rightarrow H^{*-1}(SP_n^2; \mathbb{Z}/2)$, $*$ = $4k + 1, 4k + 3$ from (8.51)(8.52) and $s^*E_{2j+1}(z^i) \leftrightarrow b_{i,2j}$ from (8.53) implies a bijection

$$s^*E_{2j}(z^i) \leftrightarrow b_{i,2j-1},$$

which agrees with Nakaoka's (8.40), (8.41) and Corollary 7.40.

For $n \geq 3$, we have $H^{12}(SP_n^2; \mathbb{Z}/2) \cong \mathbb{Z}/2\langle \rho(g^3) \rangle \oplus \mathbb{Z}/2\langle \rho(\frac{1}{2}gh) \rangle \oplus \mathbb{Z}/2\langle b_{2,3} \rangle$ where $\rho(g^3) \leftrightarrow \Phi_0^*(z^3 \times 1)$ and $b_{2,3} \leftrightarrow s^*E_4(z^2)$. So we obtain a bijection by defining

$$\rho(\frac{1}{2}gh) \leftrightarrow \Phi_0^*(z^2 \times z).$$

Then, the products $\rho((\frac{1}{2})^{s-1}h^s) \rho((\frac{1}{2})^{s'-1}h^{s'}) = 0$, $\rho((\frac{1}{2})^m g^\ell h^m) \rho((\frac{1}{2})^{s-1}h^s) = 0$ and (8.40)(8.41) suggest that $\rho((\frac{1}{2})^{s-1}h^s) = \sum_{\kappa, \ell} \lambda_{\kappa, \ell} s^*E_\kappa(z^\ell)$, $\lambda_{\kappa, \ell} \in \{0, 1\}$; so we have a bijection

$$\rho((\frac{1}{2})^{i-1}h^i) \leftrightarrow s^*E_{4i}(z^i)$$

by choosing $s^*E_{4i}(z^i)$ as a representative.

On the other hand, the product $\rho((\frac{1}{2})^m g^\ell h^m) \rho((\frac{1}{2})^{m'} g^{\ell'} h^{m'}) \neq 0$ and Nakaoka's results (8.39) (8.40) (8.41) suggest that $\rho((\frac{1}{2})^m g^\ell h^m) = \Phi_0^*(z^{\ell+m} \times z^m) + \text{other terms}$; we now have a bijection

$$\rho((\frac{1}{2})^m g^\ell h^m) \leftrightarrow \Phi_0^*(z^{\ell+m} \times z^m)$$

by choosing $\Phi_0^*(z^{\ell+m} \times z^m)$ as a representative.

It is possible to compute $H^*(SP_n^2)$ from $H^*(SP_N^2)$ with an arbitrarily large N . In fact, truncation has been our own approach to compute $H^*(SP_n^2)$ throughout this

thesis. In the $\mathbb{Z}/2$ cohomology case, it is clear that putting $z^{n+1} = 0$ into Nakaoka's basis elements for $H^*(SP_N^2; \mathbb{Z}/2)$ yields $H^*(SP_n^2; \mathbb{Z}/2)$. We include Table 8.2 and 8.3, to compare our basis elements with Nakaoka's for $n = 3$ and $n = N \geq 6$ in low dimensions.

In this section we have described a basis for $H^*(SP_n^2; \mathbb{Z}/2)$ and $H^*(SP_n^2/\Delta_n; \mathbb{Z}/2)$ using Theorem 6.26 (UCT) in order to check our results. On the other hand, in Chapter 6, we showed the isomorphism $H^*(SP_n^2/\Delta_n; \mathbb{Z}/2) \cong H^*(\text{Th}(\theta_n); \mathbb{Z}/2)$, which comes from $\text{Th}(\theta_n) \cong SP_n^2/N_n \simeq SP_n^2/\Delta_n$. It is interesting to compare Nakaoka's basis elements and our basis elements for

$$H^*(\text{Th}(\theta_n); \mathbb{Z}/2) \cong H^*(\Gamma_n; \mathbb{Z}/2) t$$

where t is the Thom class in $H^4(\text{Th}(\theta_n); \mathbb{Z}/2)$.

We compare them for $n = 2$ as in Table 8.1.

8.3 Finale

It is a general belief that the integral cohomology ring is harder to compute than $\mathbb{Z}/2$ -cohomology ring. In Section 8.2, $\mathbb{Z}/2$ -cohomology was used to check that our integral calculations were consistent with Nakaoka's work. Nakaoka compared [47] his elements with the action of the Steenrod square operation Sq^1 ; we have started to work on the connections between Sq^1 and our generators, and think it should be possible to find formulae for their action. As we write this thesis, we are also considering the relationship between our cohomological computation and the cohomology of the infinite symmetric product $SP^\infty(X) = \bigcup_n SP^n(X)$. The most recent research related to the cohomology of the symmetric products $SP^n(X)$ includes a 2015 paper [21] by Dmitry Gugin of Moscow State University in 2015. For $n \geq 2$, the structure of the integral cohomology ring $H^*(SP^n(X))$ remains an open problem except for a few basic cases.

Many of our methods have complex versions that apply to the case $SP^2(\mathbb{C}\mathbb{P}^n)$, and will be discussed in forthcoming work [7]. Currently, we are also working on the integral cohomology rings of the symmetric squares of the octonionic projective line and plane; we aim to complete our computations by spring 2016.

Table 8.1: $H^* := H^*(SP_2^2/\Delta_2; \mathbb{Z}/2)$

$*$	H^*	our basis	Nakaoka's basis
4	$\mathbb{Z}/2$	t	$\Phi_0^*(z \times 1)$
5	$\mathbb{Z}/2$	at	$E_1(z)$
6	$\mathbb{Z}/2$	a^2t	$E_2(z)$
7	$\mathbb{Z}/2$	a^3t	$E_3(z)$
8	$\mathbb{Z}/2$	a^4t	$E_4(z)$
	\oplus		
	$\mathbb{Z}/2$	p_1t	$\Phi_0^*(z^2 \times 1)$
9	$\mathbb{Z}/2$	p_1at	$E_1(z^2)$
10	$\mathbb{Z}/2$	p_1a^2t	$E_2(z^2)$
11	$\mathbb{Z}/2$	p_1a^3t	$E_3(z^2)$
12	$\mathbb{Z}/2$	p_1^2t	$\Phi_0^*(z^2 \times z)$
	\oplus		
	$\mathbb{Z}/2$	p_1a^4t	$E_4(z^2)$
13	$\mathbb{Z}/2$	$p_1^2at (= p_1a^5t)$	$E_5(z^2)$
14	$\mathbb{Z}/2$	$p_1^2a^2t (= p_1a^6t)$	$E_6(z^2)$
15	$\mathbb{Z}/2$	$p_1^2a^3t (= p_1a^7t)$	$E_7(z^2)$
16	$\mathbb{Z}/2$	$p_1^2a^4t (= p_1a^8t)$	$E_8(z^2)$

$$p_1^2 - p_2 = 0, \quad p_1^3 = 0$$

$$a^5 = p_1a, \quad t^2 = (a^4 + p_1)t.$$

Table 8.2: $H^* := H^*(SP_3^2; \mathbb{Z}/2)$

*	H^*	our generators $b_{i,2j} := \rho(t_{i,j})$	Nakaoka's basis $\Phi_0^*(x \times y) = \Phi_0^*(y \times x)$ $s^*E_{4i}(z^i) = \Phi_0^*(z^i \times z^i)$
4	$\mathbb{Z}/2$	$\rho(g)$	$\Phi_0^*(z \times 1)$
5	0	-	-
6	$\mathbb{Z}/2$	$b_{1,1}$	$s^*E_2(z)$
7	$\mathbb{Z}/2$	$b_{1,2}$	$s^*E_3(z)$
8	$\mathbb{Z}/2$	$\rho(h)$	$s^*E_4(z)$
	\oplus		
	$\mathbb{Z}/2$	$\rho(g^2)$	$\Phi_0^*(z^2 \times 1)$
9	0	-	-
10	$\mathbb{Z}/2$	$b_{2,1}$	$s^*E_2(z^2)$
11	$\mathbb{Z}/2$	$b_{2,2}$	$s^*E_3(z^2)$
12	$\mathbb{Z}/2$	$\rho(g^3)$	$\Phi_0^*(z^3 \times 1)$
	\oplus		
	$\mathbb{Z}/2$	$\rho(\frac{1}{2}gh)$	$\Phi_0^*(z^2 \times z)$
	\oplus		
	$\mathbb{Z}/2$	$b_{2,3}$	$s^*E_4(z^2)$
13	$\mathbb{Z}/2$	$b_{2,4}$	$s^*E_5(z^2)$
14	$\mathbb{Z}/2$	$b_{3,1}$	$s^*E_2(z^3)$
	\oplus		
	$\mathbb{Z}/2$	$b_{2,5}$	$s^*E_6(z^2)$
15	$\mathbb{Z}/2$	$b_{3,2}$	$s^*E_3(z^3)$
	\oplus		
	$\mathbb{Z}/2$	$b_{2,6}$	$s^*E_7(z^2)$
16	$\mathbb{Z}/2$	$\rho(\frac{1}{2}g^2h)$	$\Phi_0^*(z^3 \times z)$
	\oplus		
	$\mathbb{Z}/2$	$\rho(\frac{1}{2}h^2)$	$s^*E_8(z^2)$
	\oplus		
	$\mathbb{Z}/2$	$b_{3,3}$	$s^*E_4(z^3)$
17	$\mathbb{Z}/2$	$b_{3,4}$	$s^*E_5(z^3)$
18	$\mathbb{Z}/2$	$b_{3,5}$	$s^*E_6(z^3)$
19	$\mathbb{Z}/2$	$b_{3,6}$	$s^*E_7(z^3)$
20	$\mathbb{Z}/2$	$\rho(\frac{1}{4}gh^2)$	$\Phi_0^*(z^3 \times z^2)$
	\oplus		
	$\mathbb{Z}/2$	$b_{3,7}$	$s^*E_8(z^3)$
21	$\mathbb{Z}/2$	$b_{3,8}$	$s^*E_9(z^3)$
22	$\mathbb{Z}/2$	$b_{3,9}$	$s^*E_{10}(z^3)$
23	$\mathbb{Z}/2$	$b_{3,10}$	$s^*E_{11}(z^3)$
24	$\mathbb{Z}/2$	$\rho(\frac{1}{4}h^3)$	$s^*E_{12}(z^3)$

$$\rho(g^4) = \rho(\frac{1}{2}h^2), \rho(g^5) = 0, \rho(\frac{1}{2}g^3h) = \rho(\frac{1}{4}gh^2)$$

$$\rho(\frac{1}{2}g^4h) = \rho(\frac{1}{4}g^2h^2) = \rho(\frac{1}{4}h^3).$$

Analogous to the integral cohomology (Section8.1),

$$H^*(SP_n^2; \mathbb{Z}/2) \cong H^*(SP^2; \mathbb{Z}/2), \quad \text{if } * < 4(n+1).$$

Table 8.3: $H^* := H^*(SP_N^2; \mathbb{Z}/2)$ with $N \geq 6$ for $* \leq 24$

*	H^*	our generators $b_{i,2j} := \rho(t_{i,j})$	Nakaoka's basis $\Phi_0^*(x \times y) = \Phi_0^*(y \times x)$ $s^*E_{4i}(z^i) = \Phi_0^*(z^i \times z^i)$
$* \leq 15$	$H^*(SP_N^2) \cong H^*(SP_3^2)$		
16	$\mathbb{Z}/2$	$\rho(g^4)$	$\Phi_0^*(z^4 \times 1)$
	\oplus		
	$\mathbb{Z}/2$	$\rho(\frac{1}{2}g^2h)$	$\Phi_0^*(z^3 \times z)$
	\oplus		
	$\mathbb{Z}/2$	$\rho(\frac{1}{2}h^2)$	$s^*E_8(z^2)$
	\oplus		
	$\mathbb{Z}/2$	$b_{3,3}$	$s^*E_4(z^3)$
17	$\mathbb{Z}/2$	$b_{3,4}$	$s^*E_5(z^3)$
18	$(\mathbb{Z}/2)^2$	$b_{3,5}, b_{4,1}$	$s^*E_6(z^3), s^*E_2(z^4)$
19	$(\mathbb{Z}/2)^2$	$b_{3,6}, b_{4,2}$	$s^*E_7(z^3), s^*E_3(z^4)$
20	$\mathbb{Z}/2$	$\rho(g^5)$	$\Phi_0^*(z^5 \times 1)$
	\oplus		
	$\mathbb{Z}/2$	$\rho(\frac{1}{2}g^3h)$	$\Phi_0^*(z^4 \times z)$
	\oplus		
	$\mathbb{Z}/2$	$\rho(\frac{1}{4}gh^2)$	$\Phi_0^*(z^3 \times z^2)$
	\oplus		
	$\mathbb{Z}/2$	$b_{4,3}$	$s^*E_4(z^4)$
	\oplus		
	$\mathbb{Z}/2$	$b_{3,7}$	$s^*E_8(z^3)$
21	$(\mathbb{Z}/2)^2$	$b_{3,8}, b_{4,4}$	$s^*E_9(z^3), s^*E_5(z^4)$
22	$(\mathbb{Z}/2)^3$	$b_{3,9}, b_{4,5}, b_{5,1}$	$s^*E_{10}(z^3), s^*E_6(z^4), s^*E_2(z^5)$
23	$(\mathbb{Z}/2)^3$	$b_{3,10}, b_{4,6}, b_{5,2}$	$s^*E_{11}(z^3), s^*E_7(z^4), s^*E_3(z^5)$
24	$\mathbb{Z}/2$	$\rho(g^6)$	$\Phi_0^*(z^6 \times 1)$
	\oplus		
	$\mathbb{Z}/2$	$\rho(\frac{1}{2}g^4h)$	$\Phi_0^*(z^5 \times z)$
	\oplus		
	$\mathbb{Z}/2$	$\rho(\frac{1}{4}g^2h^2)$	$\Phi_0^*(z^4 \times z^2)$
	\oplus		
	$\mathbb{Z}/2$	$\rho(\frac{1}{4}h^3)$	$s^*E_{12}(z^3)$
	\oplus		
	$\mathbb{Z}/2$	$b_{4,7}$	$s^*E_8(z^4)$
	\oplus		
	$\mathbb{Z}/2$	$b_{5,3}$	$s^*E_4(z^5)$

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Appendix A

Lemmas

A.1 Gluing Lemma

The following is known as the gluing lemma.

Lemma A.1 ([9, 10, 37]). *Assume given a commutative diagram of topological spaces*

$$\begin{array}{ccccc} X & \xleftarrow{h} & A & \xrightarrow{g} & Y \\ \beta \downarrow & & \downarrow \alpha & & \downarrow \gamma \\ X' & \xleftarrow{h'} & A' & \xrightarrow{g'} & Y' \end{array}$$

where h and h' are cofibrations. If α, β, γ are homotopy equivalences, then so is their pushout

$$X \cup_A Y \rightarrow X' \cup_{A'} Y'.$$

A.2 Ladder Lemma

The following is called the ladder lemma.

Lemma A.2 ([37], Lemma 2.1.10). *Assume given a commutative diagram*

$$\begin{array}{ccccccc} X_0 & \xrightarrow{f_0} & X_1 & \longrightarrow & \dots & \longrightarrow & X_i & \xrightarrow{f_i} & X_{i+1} & \longrightarrow & \dots \\ \downarrow a_0 & & \downarrow a_1 & & & & \downarrow a_i & & \downarrow a_{i+1} & & \\ X'_0 & \xrightarrow{f'_0} & X'_1 & \longrightarrow & \dots & \longrightarrow & X'_i & \xrightarrow{f'_i} & X'_{i+1} & \longrightarrow & \dots \end{array}$$

in which f_i and f'_i are cofibrations. If the maps a_i are homotopy equivalences, then so is their colimit

$$\operatorname{colim} X_i \longrightarrow \operatorname{colim} X'_i.$$

Appendix B

Background Materials

B.1 NDR

Definition B.1 ([14], page 114). *Let A be a subspace $A \subset X$. Then, we call the pair (X, A) a neighbourhood deformation retract (NDR), if there exists a homotopy $\psi: X \times I \rightarrow X$, and a function $v: X \rightarrow I$ such that:*

- (1) $A = v^{-1}(0)$
- (2) $\psi(x, 0) = x$ for $x \in X$
- (3) $\psi(a, t) = a$ for $(a, t) \in A \times I$
- (4) $\psi(x, 1) \in A$ for $1 > v(x)$.

B.2 Semidirect Products

We recall below some basics of semidirect products. The references include [25, 51, 56].

Definition B.2 ([25]). *A group G is a semidirect product of a subgroup N by a subgroup H if the following conditions are satisfied:*

- $G = NH$
- N is a normal subgroup of G
- $H \cap N = \{1\}$.

Remark B.3 ([25]). *If G is the semidirect product of N by H , then*

$$G/N = NH/N \cong H/(H \cap N) = H/\{1\} \cong H,$$

and we say that G is an extension of N by H . (Although in general when we say G is an extension of A by B , B is not required to be a subgroup of G).

Example B.4. *The direct product of groups N , H is a semidirect product where both H and N are normal subgroups of G . The dihedral group $D(4)$ is a semidirect product, which is not a direct product.*

Notation: A common notation for the semidirect product N by H is $N \rtimes H$ or $H \ltimes N$, however some authors use the symbol pointing in the opposite direction. To avoid ambiguity, it is better to state which subgroup is a normal subgroup when necessary.

Proposition B.5 ([25]). *Let G be a semidirect product of N by H . For each element h of H , the map $\theta_h: N \rightarrow N$ defined by $\theta_h(n) = hnh^{-1}$ is an automorphism of N . The map $\theta: H \rightarrow \text{Aut}(N)$ defined by $\theta(h) = \theta_h$ is a homomorphism.*

An element of a semidirect product N by H can be uniquely written as an ordered pair (n, h) , $n \in N$, $h \in H$, or more simply nh . Then the group multiplication is given by either $(n_1, h_1)(n_2, h_2) = (n_1\theta_{h_1}(n_2), h_1h_2)$, or $n_1h_1n_2h_2 = n_1\theta_{h_1}(n_2)h_1h_2$.

The semidirect product G can be put into a short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1,$$

which is split [51, page 313] where maps are given by $n \mapsto (n, 1)$, $(n, h) \mapsto h$, and the splitting by $h \mapsto (1, h)$.

B.3 Serre Spectral Sequence

There are many references for spectral sequences, which include [39], a very good reference where readers can find full details on the subject.

We list the main properties related to the cohomology Serre or Leray-Serre spectral sequence.

Theorem B.6. (*[39, Theorem 5.2, page 135]*) *Let R be a commutative ring with unit. Suppose $F \rightarrow X \xrightarrow{\pi} M$ is a fibration, where M is path-connected and F is connected. Then there is a first quadrant spectral sequence of algebras, $\{E_r^{*,*}, d_r\}$, converging to $H^*(X_+; R)$ as an algebra, with*

$$E_2^{p,q} \cong H^p(M_+; \mathcal{H}^q(F_+; R)),$$

the cohomology of the space M with local coefficients in the cohomology of the fibre of π . This spectral sequence is natural with respect to fibre-preserving maps of fibrations. Furthermore, the cup product on cohomology with local coefficients and the product \cdot_2 on $E_2^{,*}$ are related by $u \cdot_2 v = (-1)^{p'q} uv$ when $u \in E_2^{p,q}$ and $v \in E_2^{p',q'}$.*

Note that the differential d_r is given by

$$d_r: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1} \quad \text{and} \quad E_{r+1}^{p,q} = \ker d_r / \text{Im } d_r \quad \text{at} \quad E_r^{p,q}$$

where

$$\ker d_r: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1} \quad \text{and} \quad \text{Im } d_r: E_r^{p-r, q+r-1} \rightarrow E_r^{p,q}.$$

The spectral sequence becomes more manageable in some cases.

Proposition B.7. (*[40, Lemma 2.15, page 111]*) *Given a fibration $F \rightarrow X \rightarrow M$. Suppose that the system of local coefficient rings $\mathcal{H}^*(F_+; R)$ is trivial, M and F are path-connected, and either $H^*(M_+; R)$ or $H^*(F_+; R)$ is a finitely generated free R -module in each dimension. Then the product $E_2^{p,0} \otimes E_2^{0,q} \rightarrow E_2^{p,q}$ induces an isomorphism*

$$H^p(M_+; R) \otimes_R H^q(F_+; R) \cong E_2^{p,q}.$$

The above condition on $\mathcal{H}^*(F_+; R)$ means that the action of $\pi_1(M)$ is trivial.

For example,

$$E_2^{p,0} \cong H^p(M_+; H^0(F_+; R)) \cong H^p(M_+; R),$$

$$E_2^{0,q} \cong H^0(M_+; H^q(F_+; R)) \cong H^q(F_+; R).$$

Appendix C

Examples

C.1 $H^5(SP^2(\mathbb{H}\mathbb{P}^n); \mathbb{Z})$ and $H^9(SP^2(\mathbb{H}\mathbb{P}^n); \mathbb{Z})$

The following is a simple check that $H^5(SP_n^2) = H^9(SP_n^2) = 0$ for any $n \geq 1$ using the commutative ladder

$$\begin{array}{ccccccc}
 \longleftarrow & H^*(\Delta_n) & \xleftarrow{i^*} & H^*(SP_n^2) & \xleftarrow{s^*} & H^*(SP_n^2/\Delta_n) & \xleftarrow{\delta} & H^{*-1}(\Delta_n) & \longleftarrow \\
 & \downarrow & & \downarrow \pi^* & & \cong \downarrow & & \downarrow & \\
 \longleftarrow & H^*(\widehat{\Delta}_n) & \xleftarrow{f^*} & H^*(B_n) & \xleftarrow{\beta^*} & H^*(B_n/\widehat{\Delta}_n) & \xleftarrow{\delta} & H^{*-1}(\widehat{\Delta}_n) & \longleftarrow
 \end{array}$$

where $H^*(-) := H^*(-; \mathbb{Z})$.

Case $*$ = 9. We have the ladder

$$\begin{array}{ccccccc}
 0 \longleftarrow & H^9(SP_n^2) & \xleftarrow{s^*} & H^9(SP_n^2/\Delta_n) & \xleftarrow{\delta} & H^8(\Delta_n) & \xleftarrow{i^*} & H^8(SP_n^2) & \longleftarrow \\
 & \downarrow \pi^* & & \cong \downarrow & & \downarrow & & \downarrow \pi^* & \\
 0 = & H^9(B_n) & \xleftarrow{\beta^*} & H^9(B_n/\widehat{\Delta}_n) & \xleftarrow{\delta} & H^8(\widehat{\Delta}_n) & \xleftarrow{f^*} & H^8(B_n) & \longleftarrow
 \end{array}$$

If $n \geq 2$, then $H^9(SP_n^2) \cong H^9(SP^2) = 0$, since

$$H^8(SP_n^2) = \mathbb{Z}\langle g^2 \rangle \oplus \mathbb{Z}\langle h \rangle, \quad H^8(B_n) = \mathbb{Z}\langle x^2 \rangle \oplus \mathbb{Z}\langle y \rangle \oplus \mathbb{Z}/2\langle w^4 \rangle$$

$$H^8(\Delta_n) = \mathbb{Z}\langle z^2 \rangle, \quad H^8(\widehat{\Delta}_n) = \mathbb{Z}/2\langle w^4 \rangle \oplus \mathbb{Z}/2\langle w^2 z \rangle$$

and the homomorphisms are given by

$$f^*(w^4) = w^4, \quad f^*(x^2) = (2z)^2, \quad f^*(y) = z^2 + w^2 z$$

$$i^*(g^2) = (2z)^2, \quad i^*(h) = 2z^2, \quad (\delta \circ f^*)(y) = \delta(z^2 + w^2 z) = 0$$

so $H^9(B_n/\widehat{\Delta}_n) = \mathbb{Z}/2\langle\delta(z^2)\rangle$ where $\delta(z^2) = \delta(w^2z)$. All the above are the same as for $H^9(SP^2)$.

Case $n = 1$. Again, $H^9(SP_1^2) = 0$, which follows from

$$\begin{array}{ccccc} 0 & \longleftarrow & H^9(SP_1^2) & \xleftarrow{s^*} & H^9(SP_1^2/\Delta_1) & \longleftarrow & 0 \\ & & \downarrow \pi^* & & \cong \downarrow & & \\ 0 & = & H^9(B_1) & \xleftarrow{\beta^*} & H^9(B_1/\widehat{\Delta}_1) & \longleftarrow & 0 \end{array}$$

Case $* = 5$. This is similar to $H^9(SP_n^2)$, and $H^5(SP^2) \cong H^5(SP_n^2)$

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & H^5(SP_n^2) & \xleftarrow{s^*} & H^5(SP_n^2/\Delta_n) & \xleftarrow{\delta} & H^4(\Delta_n) & \xleftarrow{i^*} & H^4(SP_n^2) & \longleftarrow & 0 \\ & & \downarrow \pi^* & & \cong \downarrow & & \downarrow & & \downarrow & & \\ 0 & = & H^5(B_n) & \xleftarrow{\beta^*} & H^5(B_n/\widehat{\Delta}_n) & \xleftarrow{\delta} & H^4(\widehat{\Delta}_n) & \xleftarrow{f^*} & H^4(B_n) & \longleftarrow & 0 \end{array}$$

where $H^4(SP_n^2) = \mathbb{Z}\langle g \rangle$, $H^4(B_n) = \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}/2\langle w^2 \rangle$, $H^4(\Delta_n) = \mathbb{Z}\langle z \rangle$ and $H^4(\widehat{\Delta}_n) = \mathbb{Z}/2\langle w^2 \rangle \oplus \mathbb{Z}\langle z \rangle$. The homomorphisms are given by $f^*(x) = 2z$, $f^*(w^2) = w^2$, $i^*(g) = 2z$, $(\delta \circ f^*)(x) = \delta(2z) = 0$. Thus $H^5(B_n/\widehat{\Delta}_n) = \mathbb{Z}/2\langle\delta(z)\rangle$, and $(s^* \circ \delta)(z) = 0$. Therefore, $H^5(SP_n^2) = 0$, which confirms Nakaoka (8.2).

C.2 $\theta_{m,1}$ and $\theta_{m,2}$

For the reader's convenience and future use, we describe the polynomials $\theta_{m,1}$ and $\theta_{m,2}$ for $1 \leq m \leq 4$.

For $m = 1$, we have

$$\theta_{1,1} = g^2 - h, \quad \theta_{1,2} = \frac{1}{2}gh,$$

and for any $m \geq 2$ we have expressions

$$\theta_{m,1} = g \theta_{m-1,1} - \theta_{m-1,2}, \quad 2 \theta_{m,2} = h \theta_{m-1,1};$$

for example,

$$\begin{aligned} \theta_{2,1} &= g^3 - 3 \cdot \frac{1}{2}gh \\ \theta_{2,2} &= \frac{1}{2}g^2h - \frac{1}{2}h^2 \\ \theta_{3,1} &= g^4 - 4 \cdot \frac{1}{2}g^2h + \frac{1}{2}h^2 \\ \theta_{3,2} &= \frac{1}{2}g^3h - 3 \cdot \frac{1}{4}gh^2 \\ \theta_{4,1} &= g^5 - 5 \cdot \frac{1}{2}g^3h + 5 \cdot \frac{1}{4}gh^2 \\ \theta_{4,2} &= \frac{1}{2}g^4h - 4 \cdot \frac{1}{4}g^2h^2 + \frac{1}{4}h^3. \end{aligned}$$