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DOI:

[10.1090/tran/7514](https://doi.org/10.1090/tran/7514)

Document Version

Accepted author manuscript

[Link to publication record in Manchester Research Explorer](#)

Citation for published version (APA):

Browning, T., & Loughran, D. (2018). SIEVING RATIONAL POINTS ON VARIETIES. *Transactions of the American Mathematical Society*. <https://doi.org/10.1090/tran/7514>

Published in:

Transactions of the American Mathematical Society

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SIEVING RATIONAL POINTS ON VARIETIES

TIM BROWNING AND DANIEL LOUGHRAN

ABSTRACT. An upper bound sieve for rational points on suitable varieties is developed, together with applications to counting rational points in thin sets, local solubility in families, and to the notion of “friable” rational points with respect to divisors. In the special case of quadrics, sharper estimates are obtained by developing a version of the Selberg sieve for rational points.

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1. INTRODUCTION

Sieves are a ubiquitous tool in analytic number theory and have numerous applications. Typically, one is given a subset $\Omega_p \subset \mathbb{Z}/p\mathbb{Z}$ for each prime p and the challenge is to count the number of integers n in an interval for which $n \bmod p \in \Omega_p$ for all p . In favourable situations one can deduce asymptotic formulae from suitable equidistribution statements. In this paper, however, our focus is on upper bound sieves. These can be obtained through a variety of means, the most successful being variants of the large sieve or the Selberg sieve, as explained in [14, Chapters 7–9].

The above set-up can be generalised in many ways, such as in the abstract version of the large sieve developed by Kowalski [16, §2.1], for example. In our investigation we adopt the following approach: one is given a smooth projective variety X over a number field k , together with a height function H and a model \mathcal{X} over the ring of integers \mathfrak{o}_k of k , and for each non-zero prime ideal \mathfrak{p} of \mathfrak{o}_k a subset $\Omega_{\mathfrak{p}} \subset \mathcal{X}(\mathfrak{o}_k/\mathfrak{p})$. The goal is to obtain upper bounds for

$$\#\{x \in X(k) : H(x) \leq B, x \bmod \mathfrak{p} \in \Omega_{\mathfrak{p}} \text{ for all } \mathfrak{p}\}.$$

We adopt two points of view in addressing this counting problem. First we see how much can be achieved by working in as general a set-up as possible. The set-up we take is that of varieties whose rational points are *equidistributed* with respect to a suitable adelic Tamagawa measure, a property that allows us to

Date: 17th January 2018.

2010 Mathematics Subject Classification. 14G05; 11N36, 11P55, 14D10.

sieve by any finite list of local conditions. Given the generality we work in, we are only able to obtain little o -results here, rather than precise upper bounds. Next, by specialising to the case of quadric hypersurfaces, we use the Hardy–Littlewood circle method to develop a version of the Selberg sieve for quadrics, which ultimately gives explicit upper bounds.

1.1. Equidistribution and sieving rational points.

1.1.1. *Manin’s conjecture and equidistribution.* We begin by recalling Manin’s conjecture [9], [1], [28, §3]. We work with the following classes of varieties.

Definition 1.1. A smooth projective geometrically integral variety X over a field k is called *almost Fano* if

- $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$;
- The geometric Picard group $\text{Pic } \bar{X}$ is torsion free;
- The anticanonical divisor $-K_X$ is big.

Let X be an almost Fano variety over a number field k and H an anticanonical height function on X (that is, a height function associated to a choice of adelic metric on the anticanonical bundle of X). If $X(k) \neq \emptyset$, Manin’s original conjecture predicts the existence of Zariski open subset $U \subset X$ such that

$$N(U, H, B) := \#\{x \in U(k) : H(x) \leq B\} \sim c_{X,H} B(\log B)^{\rho(X)-1}, \quad (1.1)$$

where $\rho(X)$ is the rank of the Picard group of X and $c_{X,H} > 0$. The leading constant $c_{X,H}$ in (1.1) has a conjectural interpretation due to Peyre [27], which is expressed in terms of a certain Tamagawa measure on the adelic space $X(\mathbf{A}_k)$.

For our first results we assume that the rational points of bounded height are *equidistributed*. Intuitively, this means that conditions imposed at finitely many different places are asymptotically independent, and alter the leading constant in (1.1) by the Tamagawa measure of the imposed conditions. We recall the relevant definitions in §3.1. This property, first introduced to the subject by Peyre [27, §3], is very natural; Peyre showed that it holds if (1.1) holds with Peyre’s constant with respect to all choices of anticanonical height function.

The equidistribution property is known to hold for the following classes of almost Fano varieties: flag varieties [27, §6.2.4], toric varieties [7, §3.10], equivariant compactifications of additive groups [5, Rem. 0.2], and complete intersections in many variables (proved over \mathbb{Q} in [27, Prop. 5.5.3]; the result over general number fields is obtained by modifying the arguments given in [22, §4.3]).

The equidistribution property trivially allows one to sieve with respect to finitely many primes. One can use it to give upper bounds for sieving with respect to infinitely many primes by taking the limit over the conditions.

1.1.2. *Thin sets.* The original version of Manin’s conjecture (1.1) is false in general, as first shown in [2]. The problem is that the union of the accumulating subvarieties in X can be Zariski dense, so that there is no sufficiently small open set $U \subset X$ on which the expected asymptotic formula holds.

Numerous authors have recently investigated a “thin” version of Manin’s conjecture (see [28, §8], [20], [3] or [19]), where one is allowed to remove a thin subset of $X(k)$, rather than just a Zariski closed set. (We use the term *thin set* in the sense of Serre [33, §3.1]; the various definitions are recalled in §3.2.)

A natural question is whether removing a thin subset could change the asymptotic behaviour of the counting function $N(U, H, B)$. We show that this is not the case when the rational points are equidistributed.

Theorem 1.2. *Let X be an almost Fano variety over k and H an anticanonical height function on X . Assume that the rational points are equidistributed on some dense open subset $U \subset X$. Let $\Upsilon \subset U(k)$ be thin. Then*

$$\lim_{B \rightarrow \infty} \frac{\#\{x \in \Upsilon : H(x) \leq B\}}{N(U, H, B)} = 0.$$

Theorem 1.2 recovers the well-known fact that a thin subset of $\mathbb{P}^n(k)$ contains only 0% of the total number of rational points of $\mathbb{P}^n(k)$, when ordered by height. This special case is due to Cohen [8] and Serre [32, Thm. 13.3].

1.1.3. *Fibrations.* Given a family of varieties $\pi : Y \rightarrow X$, one would like to understand how many varieties in the family have a rational point. To this end, we study the following counting function

$$N(U, H, \pi, B) = \#\{x \in U(k) : H(x) \leq B, x \in \pi(Y(k))\},$$

for suitable open subsets $U \subset X$. As discovered in [23] and [24], the asymptotic behaviour of such counting functions is controlled by the Galois action on the irreducible components of fibres over the codimension 1 points of X . We work with the following types of fibres, first defined in [25].

Definition 1.3. Let $x \in X$ with residue field $\kappa(x)$. We say that a fibre $Y_x = \pi^{-1}(x)$ is *pseudo-split* if every element of $\text{Gal}(\overline{\kappa(x)}/\kappa(x))$ fixes some multiplicity one irreducible component of $Y_x \otimes \overline{\kappa(x)}$.

Note that if Y_x is *split*, i.e. contains a multiplicity one irreducible component which is geometrically irreducible [35, Def. 0.1], then Y_x is pseudo-split.

The large sieve was employed in [24] to give upper bounds for $N(\mathbb{P}^n, H, \pi, B)$. Good upper bounds are not realistic in our generality, but we are able to obtain the following zero density result, which generalises [24, Thm. 1.1].

Theorem 1.4. *Let X be an almost Fano variety over k and H an anticanonical height function on X . Assume that the rational points are equidistributed on some dense open subset $U \subset X$. Let $\pi : Y \rightarrow X$ be a proper dominant morphism with Y geometrically integral and non-singular. Assume that there is a non-pseudo-split fibre over some codimension one point of X . Then*

$$\lim_{B \rightarrow \infty} \frac{N(U, H, \pi, B)}{N(U, H, B)} = 0.$$

1.1.4. *Friable integral points.* Friable numbers are a fundamental tool in analytic number theory. A comprehensive survey on what is known about their distribution can be found in [13]. We introduce the following notion of friable integral points. (Note that, as we are working in a geometric setting, it is preferable to use the term “friable” over “smooth”.)

Definition 1.5. Let X be a finite type scheme over \mathfrak{o}_k and $Z \subset X$ a closed subscheme. For $y > 0$, we say that an integral point $x \in X(\mathfrak{o}_k)$ is *y-friable with respect to Z* if all non-zero prime ideals $\mathfrak{p} \subset \mathfrak{o}_k$ with $x \bmod \mathfrak{p} \in Z$ satisfy $N \mathfrak{p} \leq y$.

One recovers the usual notion of a y -friable number by taking $X = \mathbb{A}_{\mathbb{Z}}^1$ and Z to be the origin. Allowing different subschemes Z is also very natural: given a polynomial $f \in \mathbb{Z}[x]$, a y -friable integral point of $\mathbb{A}_{\mathbb{Z}}^1$ with respect to the subscheme $Z = \{f(x) = 0\}$ is an integer x such that $f(x)$ is y -friable. Lagarias and Soundararajan [17, Thm. 1.4] have investigated the case of the linear equation $X = \{x_1 + x_2 = x_3\} \subset \mathbb{A}_{\mathbb{Z}}^3$, with $Z = \{x_1 x_2 x_3 = 0\}$. Given $c > 8$, they assume GRH and succeed in proving that there are infinitely many (primitive) y -friable integral points with respect to Z , for any $y > (\log \max\{|x_1|, |x_2|, |x_3|\})^c$. (An unconditional version of this result is available in recent work of Harper [11], for c large enough.) We can give the following zero density result in our setting.

Theorem 1.6. *Let X be an almost Fano variety over k and H an anticanonical height function on X . Assume that the rational points are equidistributed on some dense open subset $U \subset X$. Let y be fixed and $Z \subset X$ a divisor. Let \mathcal{X} be a model of X over \mathfrak{o}_k and \mathcal{Z} the closure of Z in \mathcal{X} . Then*

$$\lim_{B \rightarrow \infty} \frac{\#\{x \in U(k) : H(x) \leq B, x \text{ is } y\text{-friable with respect to } \mathcal{Z}\}}{N(U, H, B)} = 0.$$

Here, a *model* is a proper scheme $\mathcal{X} \rightarrow \text{Spec } \mathfrak{o}_k$ whose generic fibre is isomorphic to X . A model can be obtained, for example, by choosing an embedding $X \subset \mathbb{P}_k^n$ and letting \mathcal{X} be the closure of X inside $\mathbb{P}_{\mathfrak{o}_k}^n$. Note that in Theorem 1.6 we view $x \in U(k) \subset \mathcal{X}(\mathfrak{o}_k)$, so that its reduction $x \bmod \mathfrak{p} \in \mathcal{X}(\mathbb{F}_{\mathfrak{p}})$ is well-defined.

It is crucial in Theorem 1.6 that Z be a divisor; the conclusion can fail for higher codimension subvarieties. For an example of this phenomenon in the affine setting, consider $\mathcal{X} = \mathbb{A}_{\mathbb{Z}}^2$ and \mathcal{Z} the origin. Then a y -friable integral point with respect to \mathcal{Z} is a pair of integers (x_1, x_2) whose greatest common divisor is y -friable; clearly a positive proportion of all pairs of integers satisfy this property.

1.2. Sieving on quadrics. In many cases it is possible to get quantitatively stronger versions of the previous results. We pursue this for smooth quadric hypersurfaces, but we expect that results of a similar flavour go through for hypersurfaces of arbitrary degree. The advantage of quadrics is that sharper bounds are available through the smooth δ -function variant of the Hardy–Littlewood circle method. Note that smooth quadric hypersurfaces are flag varieties; hence they are Fano and have equidistributed rational points [27, §6.2.4].

For the remainder of this section $X \subset \mathbb{P}_{\mathbb{Q}}^n$ is a smooth quadric hypersurface of dimension at least 3 over \mathbb{Q} and $H : \mathbb{P}^n(\mathbb{Q}) \rightarrow \mathbb{R}$ is the standard exponential height function associated to the supremum norm. There is a natural choice of model \mathcal{X} given by the closure of X inside $\mathbb{P}_{\mathbb{Z}}^n$; we shall abuse notation and write $X(\mathbb{Z}) = \mathcal{X}(\mathbb{Z})$ and $X(\mathbb{Z}/m\mathbb{Z}) = \mathcal{X}(\mathbb{Z}/m\mathbb{Z})$.

1.2.1. A version of the Selberg sieve. Our fundamental tool will be a version of the Selberg sieve for rational points on quadrics. Let $m \in \mathbb{N}$ be fixed once and for all. For each prime p we suppose that we are given a non-empty set of residue classes $\Omega_{p^m} \subseteq X(\mathbb{Z}/p^m\mathbb{Z})$. Our goal is to measure the density of points $x \in X(\mathbb{Q})$ whose reduction modulo p^m lands in Ω_{p^m} for each prime p . Namely, we are interested in the behaviour of the counting function

$$N(X, H, \Omega, B) = \#\{x \in X(k) : H(x) \leq B, x \bmod p^m \in \Omega_{p^m} \text{ for all } p\}$$

as $B \rightarrow \infty$, where $\Omega = (\Omega_{p^m})_p$. This has order of magnitude B^{n-1} when $\Omega_{p^m} = X(\mathbb{Z}/p^m\mathbb{Z})$ for all p , but we expect it to be significantly smaller when Ω_{p^m} is a proper subset of $X(\mathbb{Z}/p^m\mathbb{Z})$ for many primes p . We define the density function

$$\omega_p = 1 - \frac{\#\Omega_{p^m}}{\#X(\mathbb{Z}/p^m\mathbb{Z})} \in [0, 1], \quad (1.2)$$

for any prime p . The following is our main result for quadrics.

Theorem 1.7. *Assume that $X \subset \mathbb{P}^n$ is a smooth quadric of dimension at least 3 over \mathbb{Q} . Let $m \in \mathbb{N}$ and let $\Omega_{p^m} \subseteq X(\mathbb{Z}/p^m\mathbb{Z})$ for each prime p . Assume that*

$$0 \leq \omega_p < 1, \quad \text{for all } p.$$

Then, for any $\xi \geq 1$ and any $\varepsilon > 0$, we have

$$N(X, H, \Omega, B) \ll_{\varepsilon, X} \frac{B^{n-1}}{G(\xi)} + \xi^{m(n+1)+2+\varepsilon} B^{(n+1)/2+\varepsilon},$$

where $G(\xi) = \sum_{k < \xi} \mu^2(k) \prod_{p|k} \left(\frac{\omega_p}{1-\omega_p} \right)$.

The implied constant in this upper bound is allowed to depend on the choice of ε and the quadric X . In order to prove Theorem 1.7 we shall use Heath-Brown's version [12] of the circle method to study the distribution of zeros of isotropic quadratic forms that are constrained to lie in a fixed set of congruence classes. The main result, Theorem 4.1, is uniform in the modulus and may be of independent interest. Once combined with the Selberg sieve, it easily leads to the statement of Theorem 1.7. In fact, although not the focus of our present investigation, Theorem 4.1 gives an effective strong approximation result which could also be fed into lower bound sieves, in the spirit of work by Nevo and Sarnak [26] on the affine linear sieve for homogeneous spaces. Finally, by appealing to work of Browning and Vishe [4] instead of [12], we remark that it would be possible to obtain a version of Theorem 1.7 over arbitrary number fields, and to extend the results in the next section to a similar level of generality.

1.2.2. Applications. We now give some applications of Theorem 1.7, which serve to strengthen the results in §1.1 for smooth quadrics $X \subset \mathbb{P}^n$ of dimension at least 3 which are defined over \mathbb{Q} .

To begin with, an old result of Cohen [8] and Serre [32, Thm. 13.3] gives a quantitative improvement of Theorem 1.2 when X is projective space. The following result extends this to quadrics.

Theorem 1.8. *Let $\Upsilon \subset X(\mathbb{Q})$ be a thin set. Then there exists $\delta_n > 0$ such that*

$$\#\{x \in \Upsilon : H(x) \leq B\} \ll_{\Upsilon, X} B^{n-1-\delta_n}.$$

We shall see in §5.2 that any $\delta_n < \frac{1}{2} - \frac{7}{2(n+4)}$ is admissible. In particular, δ_n approaches $\frac{1}{2}$ as $n \rightarrow \infty$, which is the saving recorded in [32, Thm. 13.3]. A well-known application of the latter result is that almost all integer polynomials f of degree n have Galois group the symmetric group S_n . (Here we define $\text{Gal}(f)$ to be the Galois group of the splitting field of f over \mathbb{Q} .) Theorem 1.8 yields a similar application, but where the coefficients run over a thinner set.

Example 1.9. Let $n \geq 4$. We claim that

$$\# \left\{ f(x) = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x] : \begin{array}{l} |a_i| \leq B, \text{Gal}(f) \neq S_n, \\ 2a_n^2 = a_{n-1}^2 + \cdots + a_0^2 \end{array} \right\} \ll B^{n-1-\delta_n}.$$

To see this, note that the polynomial $x^n - x^{n-1} - 1$ lives in this family and is irreducible with Galois group S_n , by the remarks at the end of [33, §4.4]. This implies that the generic Galois group in the family is also S_n . Hilbert's irreducibility theorem [33, Thm. 3.3.1] now implies that the Galois group becomes strictly smaller only on some thin subset of the set of rational points on the associated quadric hypersurface. The claim now follows easily from Theorem 1.8.

Our next result concerns fibrations. We extend [24, Thm. 1.2], in which the base is \mathbb{P}^n , to a result involving quadric hypersurfaces. To state the result, we recall the definition of the Δ -invariants from [24]. Let $\pi : Y \rightarrow Z$ be a dominant map of non-singular proper varieties over a number field k with geometrically integral generic fibre. For each codimension 1 point $D \in Z^{(1)}$, the absolute Galois group $\text{Gal}(\overline{\kappa(D)}/\kappa(D))$ of the residue field of D acts on the irreducible components of $\pi^{-1}(D) \otimes \overline{\kappa(D)}$; we choose a finite subgroup $\Gamma_D(\pi)$ through which this action factors. As in [24, Eq. (1.4)], we then define $\delta_D(\pi) = \#\Gamma_D^\circ(\pi)/\#\Gamma_D(\pi)$, where $\Gamma_D^\circ(\pi)$ is the set of $\gamma \in \Gamma_D(\pi)$ which fix some multiplicity 1 irreducible component of $\pi^{-1}(D) \otimes \overline{\kappa(D)}$. Let

$$\Delta(\pi) = \sum_{D \in Z^{(1)}} (1 - \delta_D(\pi)). \quad (1.3)$$

For the next two results we recall our assumption that $X \subset \mathbb{P}^n$ is a smooth quadric of dimension at least 3 which is defined over \mathbb{Q} . We shall deduce the following result from Theorem 1.7.

Theorem 1.10. *Let $\pi : Y \rightarrow X$ be a dominant proper map with geometrically integral generic fibre and Y non-singular. Then*

$$N(X, H, \pi, B) \ll_{X, \pi} \frac{B^{n-1}}{(\log B)^{\Delta(\pi)}}.$$

Note that $\Delta(\pi) > 0$ if and only if there is a non-pseudo-split fibre over some $D \in X^{(1)}$. Thus Theorem 1.10 is a refinement of Theorem 1.4 in the special case that X is a quadric and $k = \mathbb{Q}$. As in [24, Conj. 1.6], we expect Theorem 1.10 to be sharp for the related problem of counting everywhere locally soluble fibres, provided there is an everywhere locally soluble fibre and the fibre over every codimension 1 point contains an irreducible component of multiplicity 1. As outlined in the setting of fibrations over \mathbb{P}^n [24, §5], Theorem 1.10 has several applications. For example, using a variant of the proof of [24, Thm. 5.10], one can obtain a version for quadrics of Serre's result [31, Thm. 2] on zero loci of Brauer group elements.

Our final application of Theorem 1.7 refines Theorem 1.6 for quadrics over \mathbb{Q} .

Theorem 1.11. *Let $y > 0$ and let $Z \subset X$ be a divisor. Then*

$$\#\{x \in X(\mathbb{Q}) : H(x) \leq B, x \text{ is } y\text{-friable with respect to } Z\} \ll_{X, y, Z} \frac{B^{n-1}}{(\log B)^{r(Z)}},$$

where $r(Z)$ is the number of irreducible components of Z .

Layout of the paper. In §2 we collect together some versions of Hensel's lemma which will be required in our proofs. In §3 we prove the results stated in §1.1, and also obtain some general volume estimates which will be required for our results concerning quadrics. Theorem 1.7 will be proved in §4 and §5. Finally, Theorems 1.8, 1.10 and 1.11 will be deduced in §§5.2–5.4.

Notation. For a smooth variety X over a field k , let $\mathrm{Br} X = \mathrm{H}^2(X, \mathbb{G}_m)$ denote its Brauer group and $\mathrm{Br}_0 X = \mathrm{Im}(\mathrm{Br} k \rightarrow \mathrm{Br} X)$. Let \mathfrak{p} be a non-zero prime ideal of the ring of integers \mathfrak{o}_k of a number field k . We let $\mathbb{F}_{\mathfrak{p}}$ be the residue field of \mathfrak{p} , let $N \mathfrak{p} = \#\mathbb{F}_{\mathfrak{p}}$ be its norm, and let $\mathfrak{o}_{\mathfrak{p}}$ be the completion of \mathfrak{o}_k at \mathfrak{p} . For a variety X over a field k and an extension $k \subset L$, we let $X \otimes L = X \times_{\mathrm{Spec} k} \mathrm{Spec} L$.

Acknowledgements. The authors are very grateful to Hung Bui, Christopher Frei, Adam Harper, Roger Heath-Brown, and Damaris Schindler for useful conversations. Thanks are also due to the anonymous referee for simplifying the deduction of Theorem 1.7 from Theorem 4.1. During the preparation of this paper the first author was supported by EPSRC grant EP/P026710/1 and by the NSF under Grant No. DMS-1440140, while in residence at the *Mathematical Sciences Research Institute* in Berkeley, California, during the Spring 2017 semester.

2. HENSEL'S LEMMA AND TRANSVERSALITY

We begin with some versions of Hensel's lemma. Throughout this section k is a number field and \mathfrak{p} is a non-zero prime ideal of \mathfrak{o}_k .

2.1. A quantitative version of Hensel's lemma. Versions of the following lemma have been known for some time.

Lemma 2.1. *Let $X \rightarrow \mathrm{Spec} \mathfrak{o}_{\mathfrak{p}}$ be a smooth finite type morphism of relative dimension n . Let $x_0 \in X(\mathbb{F}_{\mathfrak{p}})$ and $m \in \mathbb{N}$. Then*

$$\#\{x \in X(\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m) : x \bmod \mathfrak{p} = x_0\} = (N \mathfrak{p})^{n(m-1)}.$$

In particular $\#X(\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m) = \#X(\mathbb{F}_{\mathfrak{p}})(N \mathfrak{p})^{n(m-1)}$.

Proof. We want to calculate the number of morphisms $\mathrm{Spec} \mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m \rightarrow X$ whose image is x_0 . This is in bijection with the set of local $\mathfrak{o}_{\mathfrak{p}}$ -algebra homomorphisms $\mathrm{Hom}(\mathcal{O}_{X, x_0}, \mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m)$. Since $\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m$ is Artinian it is complete. Hence by the universal property of the completion we find that

$$\mathrm{Hom}(\mathcal{O}_{X, x_0}, \mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m) \cong \mathrm{Hom}(\widehat{\mathcal{O}}_{X, x_0}, \mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m).$$

However, as $X \rightarrow \mathrm{Spec} \mathfrak{o}_{\mathfrak{p}}$ is smooth, we have $\widehat{\mathcal{O}}_{X, x_0} \cong \mathfrak{o}_{\mathfrak{p}}[[t_1, \dots, t_n]]$ as local $\mathfrak{o}_{\mathfrak{p}}$ -algebras by [21, Ex. 6.2.2.1]. To prove the result, it suffices to note that

$$\#\mathrm{Hom}(\mathfrak{o}_{\mathfrak{p}}[[t_1, \dots, t_n]], \mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m) = (N \mathfrak{p})^{n(m-1)}.$$

Indeed, every element of $\mathrm{Hom}(\mathfrak{o}_{\mathfrak{p}}[[t_1, \dots, t_n]], \mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m)$ has the form

$$t_i \mapsto a_i, \quad i \in \{1, \dots, n\},$$

for non-units $a_1, \dots, a_n \in \mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m$. But $\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m$ has exactly $(N \mathfrak{p})^{m-1}$ non-units. \square

2.2. Transverse intersections. Following Harari [10, §2.4.2], we use the following notion of intersection multiplicity.

Definition 2.2. Let $X \rightarrow \text{Spec } \mathfrak{o}_{\mathfrak{p}}$ be a smooth finite type morphism of relative dimension n and let $D \subset X$ be an irreducible divisor which is flat over $\mathfrak{o}_{\mathfrak{p}}$. Let $x \in X(\mathfrak{o}_{\mathfrak{p}})$ be such that $x \notin D$ and let $t = 0$ be a local equation for $D \subset X$ on some affine patch $U \subset X$ containing x . We define the *intersection multiplicity* of x and D above \mathfrak{p} to be the integer ι which satisfies

$$x^*t = \varpi^\iota,$$

where ϖ denotes a uniformising parameter of $\mathfrak{o}_{\mathfrak{p}}$ and x^*t is the pull-back of t via $x : \text{Spec } \mathfrak{o}_{\mathfrak{p}} \rightarrow U$. We say that x and D meet *transversely* above \mathfrak{p} if $\iota = 1$.

This definition is independent of the choice of t and ϖ . Moreover, whether or not x and D meet transversely above \mathfrak{p} only depends on $x \bmod \mathfrak{p}^2$. In particular, asking whether a point in $X(\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^2)$ meets D transversely above \mathfrak{p} is well-defined.

Proposition 2.3. *Let $X \rightarrow \text{Spec } \mathfrak{o}_{\mathfrak{p}}$ be a smooth finite type morphism of relative dimension n . Let $D \subset X$ be a flat irreducible divisor and let $x_0 \in D(\mathbb{F}_{\mathfrak{p}})$ be a smooth point of D . Then*

$$\left| \# \left\{ x \in X(\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^2) : \begin{array}{l} x \bmod \mathfrak{p} = x_0, \\ x \text{ meets } D \text{ transversely} \end{array} \right\} - (\mathbb{N} \mathfrak{p})^n \right| \leq (\mathbb{N} \mathfrak{p})^{n-1}.$$

Proof. The problem is local around x_0 . Thus without loss of generality, we may assume that X is affine and that D is smooth and has the equation $t = 0$. Let $T(x_0)$ be the cardinality in question. Lemma 2.1 shows that $T(x_0) \leq (\mathbb{N} \mathfrak{p})^n$.

For the reverse inequality, we use an argument inspired by the proof on p. 233 of [10]. Let $\varpi \in \mathfrak{o}_{\mathfrak{p}}$ be a uniformising parameter and let $U_{\mathfrak{p}} \subset \mathfrak{o}_{\mathfrak{p}}^*$ be a collection of $(\mathbb{N} \mathfrak{p} - 1)$ units which are distinct modulo \mathfrak{p} . For $u \in U_{\mathfrak{p}}$ consider the divisors

$$D_u : t = u\varpi \subset X.$$

A simple calculation shows that each D_u is also smooth. Moreover, we have $D_u \otimes \mathbb{F}_{\mathfrak{p}} = D \otimes \mathbb{F}_{\mathfrak{p}}$ and

$$D_u \cap D = D_{u'} \cap D_u = D \otimes \mathbb{F}_{\mathfrak{p}},$$

for $u' \neq u$. Clearly any point $x \in D_u(\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^2)$ with $x \bmod \mathfrak{p} = x_0$ meets D transversely at x_0 . Applying Lemma 2.1 to the D_u , we therefore deduce that

$$\begin{aligned} T(x_0) &\geq \sum_{u \in U_{\mathfrak{p}}} \# \{ x \in D_u(\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^2) : x \bmod \mathfrak{p} = x_0 \} = \sum_{u \in U_{\mathfrak{p}}} (\mathbb{N} \mathfrak{p})^{n-1} \\ &= (\mathbb{N} \mathfrak{p} - 1)(\mathbb{N} \mathfrak{p})^{n-1}. \quad \square \end{aligned}$$

From Proposition 2.3 we easily deduce the following global statement.

Corollary 2.4. *Let $X \rightarrow \text{Spec } \mathfrak{o}_k$ be a smooth finite type morphism of relative dimension n . Let $D \subset X$ be a flat irreducible divisor and $Z \subset D$ a closed subscheme which contains the non-smooth locus of D and is of codimension at least 2 in X . Then*

$$\# \left\{ x \in X(\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^2) : \begin{array}{l} x \bmod \mathfrak{p} \in (D \setminus Z)(\mathbb{F}_{\mathfrak{p}}), \\ x \text{ meets } D \text{ transversely} \end{array} \right\} = \#D(\mathbb{F}_{\mathfrak{p}})(\mathbb{N} \mathfrak{p})^n + O((\mathbb{N} \mathfrak{p})^{2(n-1)}),$$

where the implied constant depends on Z and D .

Proof. Applying Proposition 2.3 and the Lang–Weil estimates [18], we find that the cardinality in question equals

$$\#(D \setminus Z)(\mathbb{F}_p) \left((N\mathfrak{p})^n + O((N\mathfrak{p})^{n-1}) \right) = \#D(\mathbb{F}_p)(N\mathfrak{p})^n + O((N\mathfrak{p})^{2(n-1)}). \quad \square$$

Remark 2.5. Proposition 2.3 is a quantitative improvement of the fact, often used in proofs, that any smooth \mathbb{F}_p -point on D lifts to an \mathfrak{o}_p -point of X which meets D transversely above \mathfrak{p} . (cf. the proof of Theorem 2.1.1 on p. 233 of [10] or the proof of [25, Thm. 4.2]).

3. EQUIDISTRIBUTION AND SIEVING

In this section we prove the results stated in §1.1.

3.1. Tamagawa measures and equidistribution. We first recall some notions and results concerning Tamagawa measures and equidistribution of rational points for varieties over a number field k . Our references here are [27] and [6].

3.1.1. Tamagawa measures. We now recall the construction of Peyre’s Tamagawa measure. (In practice we will use Lemma 3.2 for calculations.) Choose Haar measures dx_v on each k_v such that $\text{vol}(\mathfrak{o}_v) = 1$ for all but finitely many v . These give rise to a Haar measure dx on the adèles \mathbf{A}_k of k ; we normalise our Haar measures so that $\text{vol}(\mathbf{A}_k/k) = 1$ with respect to the induced quotient measure.

Now let X be a smooth projective variety over k and let $\|\cdot\| = (\|\cdot\|_v)_{v \in \text{Val}(k)}$ be a choice of adelic metric on the canonical bundle of X as in [6, §§2.1–2.2]. Let ω be a top degree differential form on some dense open subset $U \subset X$. By a classical construction [6, §2.1.7], for any place v of k we obtain a measure $|\omega|_v$ on $U(k_v)$ which depends on the choice of dx_v . The measure $|\omega|_v/\|\omega\|_v$ turns out to be independent of ω . Peyre’s local Tamagawa measure $\tau_{\|\cdot\|_v}$ on $X(k_v)$ is obtained by glueing these measures. The product of the $\tau_{\|\cdot\|_v}$ does not converge in general, to which end convergence factors are introduced. Let M_X be the free part of the geometric Néron–Severi group $\text{NS}(\bar{X})$, with Artin L -function $L(s, M_X) = \prod_v L_v(s, M_X)$. (For an archimedean place v we set $L_v(s, M_X) = 1$.) Under the additional assumption that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, it is proved in [6, Thm. 1.1.1] that $\lambda_v = L_v(1, M_X)^{-1}$ are a collection of convergence factors. In this way

$$\tau_{\|\cdot\|} = \lim_{s \rightarrow 1} (s-1)^{\rho(X)} L(s, M_X) \prod_{v \in \text{Val}(k)} \lambda_v \tau_{\|\cdot\|_v}$$

yields a measure on $X(\mathbf{A}_k)$, called Peyre’s global Tamagawa measure.

The above construction applies when X is almost Fano. In this case we also denote the measure by τ_H , where H is the anticanonical height function associated to the adelic metric $\|\cdot\|$. The conjecture for the leading constant in (1.1) is $c_{X,H} = \alpha(X)\beta(X)\tau_H(X(\mathbf{A}_k)^{\text{Br}})$, where $X(\mathbf{A}_k)^{\text{Br}}$ is the subset of $X(\mathbf{A}_k)$ which is orthogonal to $\text{Br } X$, $\beta(X) = \#\text{H}^1(k, \text{Pic } \bar{X})$, and $\alpha(X)$ is Peyre’s “effective cone constant”. (The precise definition of $\alpha(X)$, which can be found in [27, Def. 2.4], is irrelevant here.)

The following result implies that the Tamagawa measure $\tau_H(X(\mathbf{A}_k)^{\text{Br}})$ is essentially given by a product of local densities.

Lemma 3.1. *Let X be an almost Fano variety over k . Then $\mathrm{Br} X/\mathrm{Br}_0 X$ is finite and there exists a finite set of places S and a compact open subset $A \subset \prod_{v \in S} X(k_v)$ such that $X(\mathbf{A}_k)^{\mathrm{Br}} = A \times \prod_{v \notin S} X(k_v)$.*

Proof. The finiteness of $\mathrm{Br} X/\mathrm{Br}_0 X$ is [29, Lem. 6.10]. For each $b \in \mathrm{Br} X$, the map $X(\mathbf{A}_k) \rightarrow \mathbb{Q}/\mathbb{Z}$ induced by the Brauer pairing is locally constant [29, Cor. 6.7]. Thus the inverse image of 0 is a compact open subset. As the Brauer pairing factorises through the finite group $\mathrm{Br} X/\mathrm{Br}_0 X$, the result follows. \square

We calculate the Tamagawa measure using the following formula, which follows immediately from [29, Thm. 2.14(b)] (cf. [29, Cor. 2.15]).

Lemma 3.2. *Let X be a smooth projective variety of dimension n over k with a choice of adelic metric $\|\cdot\|$ on $-K_X$. Let \mathcal{X} be a model of X over \mathfrak{o}_k . Then there exists a finite set S of prime ideals of \mathfrak{o}_k such that*

$$\tau_{\|\cdot\|_{\mathfrak{p}}}(\{x \in X(k_{\mathfrak{p}}) : x \bmod \mathfrak{p}^m \in \Omega\}) = \frac{\#\Omega}{(\mathbf{N} \mathfrak{p})^{mn}},$$

for any $\mathfrak{p} \notin S$, any $m > 0$ and any $\Omega \subset \mathcal{X}(\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m)$.

3.1.2. *Equidistribution.* We now recall the definition of equidistribution of rational points, as given by Peyre [27, §3] and further developed in [6, §2.5].

Definition 3.3. Let X be an almost Fano variety over a number field k with $X(k) \neq \emptyset$. Let H be an anticanonical height function on X with associated Tamagawa measure τ_H . We say that the rational points on X are *equidistributed* with respect to H and some dense open subset $U \subset X$ if $U(k) \neq \emptyset$ and for any open subset $W \subset X(\mathbf{A}_k)$ with $\tau_H(\partial W) = 0$, we have

$$\lim_{B \rightarrow \infty} \frac{\#\{x \in U(k) \cap W : H(x) \leq B\}}{\#\{x \in U(k) : H(x) \leq B\}} = \frac{\tau_H(W \cap X(\mathbf{A}_k)^{\mathrm{Br}})}{\tau_H(X(\mathbf{A}_k)^{\mathrm{Br}})}. \quad (3.1)$$

As proved in [27, §3], if the equidistribution property holds with respect to *some* choice of anticanonical height, then it holds for *all* choices of anticanonical height. Moreover, the equidistribution property holds if one knows (1.1) with Peyre's constant with respect to all choices of adelic metric on the anticanonical bundle. (In fact, it follows from [6, Prop. 2.5.1] and the Stone–Weierstrass theorem that one need only prove this with respect to all *smooth* adelic metrics.)

Example 3.4. Assume that the rational points on X are equidistributed with respect to H on a dense open subset $U \subset X$. Let \mathcal{X} be a model of X over \mathfrak{o}_k and let S be a finite set of non-zero prime ideals of k . Let $m > 0$ and $\Omega_{\mathfrak{p}^m} \subset \mathcal{X}(\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m)$ for $\mathfrak{p} \in S$. Then Lemma 3.1, Lemma 3.2 and (3.1) imply that

$$\lim_{B \rightarrow \infty} \frac{\#\{x \in U(k) : H(x) \leq B, x \bmod \mathfrak{p}^m \in \Omega_{\mathfrak{p}^m} \forall \mathfrak{p} \in S\}}{N(U, H, B)} \ll \prod_{\mathfrak{p} \in S} \frac{\#\Omega_{\mathfrak{p}^m}}{\#\mathcal{X}(\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^m)},$$

where the implied constant depends on \mathcal{X}, H, m but is *independent* of S and $\Omega_{\mathfrak{p}^m}$.

Remark 3.5. The equidistribution property can be viewed as a quantitative version of weak approximation; indeed, if W is an open neighbourhood of a point $(x_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$ with $\tau_H(\partial W) = 0$, then (3.1) implies that W contains many rational points. In particular $\overline{X(k)} = X(\mathbf{A}_k)^{\mathrm{Br}}$ and so the Brauer–Manin

obstruction is the only obstruction to weak approximation. Moreover, weak approximation holds on X away from the finite set of places S by Lemma 3.1.

Remark 3.6. A natural problem is to formulate a version of Definition 3.3 for the “thin” version of Manin’s conjecture. Here one should replace the condition $x \in U(k)$ from the counting functions in (3.1) by the condition that x lies in the complement of an appropriate thin subset of $X(k)$. It would be interesting to see whether this version holds for the examples considered by Le Rudulier in [20].

3.2. Thin sets. We recall Serre’s definition of thin sets from [33, §3.1].

Definition 3.7. Let X be an integral variety over a field F . A *type I thin subset* is a set of the form $Z(F) \subset X(F)$, where Z is a closed subvariety with $Z \neq X$. A *type II thin subset* is a set of the form $\pi(Y(F))$, where $\pi : Y \rightarrow X$ is a generically finite dominant morphism with $\deg \pi \geq 2$ and Y geometrically integral. A *thin subset* is a subset contained in a finite union of thin subsets of type I and II.

To prove Theorem 1.2, we require information on thin sets modulo \mathfrak{p} .

Lemma 3.8. *Let k be a number field, let $X \rightarrow \text{Spec } \mathfrak{o}_k$ be a smooth integral finite type scheme of relative dimension n and $\Upsilon \subset X(\mathfrak{o}_k)$ be thin in $X(k)$.*

- (1) *If Υ has type I then $\#(\Upsilon \bmod \mathfrak{p}) \ll_{\Upsilon} (\mathbb{N} \mathfrak{p})^{n-1}$.*
- (2) *If Υ has type II, then there exists a finite Galois extension k_{Υ}/k and a constant $c_{\Upsilon} \in (0, 1)$ such that for all primes \mathfrak{p} of \mathfrak{o}_k which split completely in k_{Υ} we have $\#(\Upsilon \bmod \mathfrak{p}) \leq c_{\Upsilon} (\mathbb{N} \mathfrak{p})^n + O_{\Upsilon}((\mathbb{N} \mathfrak{p})^{n-1/2})$.*

Proof. The first part follows from applying the Lang–Weil estimates [18] to each component of the closure of Υ . The second part is [33, Thm. 3.6.2]. \square

3.2.1. Proof of Theorem 1.2. To prove the theorem we may reduce to the case of thin sets of type I or II. The case of type I is easy, so we assume that Υ is a thin set of type II. Let $z > 1$ and let \mathcal{P} be the set of primes \mathfrak{p} in \mathfrak{o}_k which split completely in k_{Υ} . As the rational points on X are equidistributed, it follows from Example 3.4, Lemma 3.8, and the Lang–Weil estimates [18] that

$$\lim_{B \rightarrow \infty} \frac{\#\{x \in U(k) : H(x) \leq B, x \bmod \mathfrak{p} \in (\Upsilon \bmod \mathfrak{p}) \forall \mathbb{N} \mathfrak{p} \leq z\}}{N(U, H, B)} \ll_{X, H} \prod_{\substack{\mathfrak{p} \in \mathcal{P} \\ \mathbb{N} \mathfrak{p} \leq z}} \left(c_{\Upsilon} + O_{\Upsilon} \left(\frac{1}{\sqrt{\mathbb{N} \mathfrak{p}}} \right) \right).$$

The set \mathcal{P} is infinite by the Chebotarev density theorem. Since $0 < c_{\Upsilon} < 1$, the result follows on taking $z \rightarrow \infty$. \square

3.3. Local solubility densities. Let k be a number field. We gather some tools for the proof of Theorem 1.4. This is proved with an analogous strategy to Theorem 1.2, by deriving upper bounds for the size of the set in question modulo \mathfrak{p}^m , for some m . In Lemma 3.8 it was sufficient to take $m = 1$, but as first noticed by Serre [31] (and further developed in [24]), for fibrations one needs to sieve modulo higher powers of \mathfrak{p} . For example, consider the conic bundle

$$x^2 + y^2 = tz^2 \quad \subset \mathbb{P}_{\mathbb{Z}}^1 \times \mathbb{A}_{\mathbb{Z}}^1. \quad (3.2)$$

For any odd prime p , the fibre over *every* \mathbb{F}_p -point of $\mathbb{A}_{\mathbb{Z}}^1$ has an \mathbb{F}_p -point; but there are clearly fibres over \mathbb{Q} which have no \mathbb{Q}_p -point. So sieving modulo p gives no information. One obtains good upper bounds here by sieving modulo p^2 , using the fact that if $p \equiv 3 \pmod{4}$ and the p -adic valuation of t is equal to 1, then the corresponding conic (3.2) has no \mathbb{Q}_p -point.

These observations were greatly generalised by Loughran and Smeets in [24]. The condition that the p -adic valuation of t is 1 can be interpreted geometrically as requiring that a certain intersection is transverse over p (see Definition 2.2). The required generalisation is the following ‘‘sparsity theorem’’ from [24], which gives an explicit criterion for non-solubility at sufficiently large primes.

Proposition 3.9. *Let $\pi : Y \rightarrow X$ be a dominant morphism of finite type \mathfrak{o}_k -schemes with Y_k and X_k smooth geometrically integral k -varieties. Let T be a reduced divisor in X such that the restriction of π to $X \setminus T$ is smooth. Then there exists a finite set of prime ideals S and a closed subset $Z \subset T_{\mathfrak{o}_k, S}$ containing the singular locus of $T_{\mathfrak{o}_k, S}$ and of codimension 2 in $X_{\mathfrak{o}_k, S}$, such that for all non-zero prime ideals $\mathfrak{p} \notin S$ the following holds:*

Let $x \in X(\mathfrak{o}_{\mathfrak{p}})$ be such that the image of $x : \text{Spec } \mathfrak{o}_{\mathfrak{p}} \rightarrow X$ meets $T_{\mathfrak{o}_k, S}$ transversally over \mathfrak{p} outside of Z and such that the fibre above $x \bmod \mathfrak{p} \in T(\mathbb{F}_{\mathfrak{p}})$ is non-split. Then $(Y \times_X x)(\mathfrak{o}_{\mathfrak{p}}) = \emptyset$; i.e. the fibre over x has no $\mathfrak{o}_{\mathfrak{p}}$ -point.

Proof. For rational points this is proved in [24, Thm. 2.8]. The adaptation to integral points is straightforward and omitted. \square

The following is the main result of this section. It is phrased in terms of the invariant $\Delta(\pi)$ that was defined in (1.3).

Proposition 3.10. *Let $\pi : Y \rightarrow X$ be a dominant morphism of finite type \mathfrak{o}_k -schemes with Y_k and X_k smooth geometrically integral k -varieties. Assume that the generic fibre of π is geometrically integral and that $Y(\mathfrak{o}_{\mathfrak{p}}) \neq \emptyset$ for all primes \mathfrak{p} . For any non-zero prime ideal $\mathfrak{p} \subset \mathfrak{o}_k$ let*

$$\Theta_{\mathfrak{p}} = \#\{x \in X(\mathfrak{o}_k/\mathfrak{p}^2) : x \notin \pi(Y(\mathfrak{o}_{\mathfrak{p}})) \bmod \mathfrak{p}^2\}.$$

Then

$$\bullet \quad \frac{\Theta_{\mathfrak{p}}}{\#X(\mathfrak{o}_k/\mathfrak{p}^2)} \ll \frac{1}{N\mathfrak{p}}, \quad (3.3)$$

$$\bullet \quad \sum_{N\mathfrak{p} \leq B} \frac{\Theta_{\mathfrak{p}} \log N\mathfrak{p}}{\#X(\mathfrak{o}_k/\mathfrak{p}^2)} \sim \Delta(\pi) \log B, \text{ and} \quad (3.4)$$

$$\bullet \quad \prod_{N\mathfrak{p} \leq B} \frac{\#(\pi(Y(\mathfrak{o}_{\mathfrak{p}})) \bmod \mathfrak{p}^2)}{\#X(\mathfrak{o}_k/\mathfrak{p}^2)} \asymp \frac{1}{(\log B)^{\Delta(\pi)}}. \quad (3.5)$$

Proof. Let $n = \dim X_k$. To begin with we claim that

$$\Theta_{\mathfrak{p}} = \#\{x \in X(\mathbb{F}_{\mathfrak{p}}) : \pi^{-1}(x) \text{ non-split}\} (N\mathfrak{p})^n + O((N\mathfrak{p})^{2(n-1)}). \quad (3.6)$$

To prove this, let T be a divisor of X which contains the singular locus of π and let $x \in X(\mathfrak{o}_k/\mathfrak{p}^2)$. If $\pi^{-1}(x \bmod \mathfrak{p})$ is split then, by the Lang–Weil estimates [18] and Hensel’s lemma, for large enough \mathfrak{p} we find that the fibre over x has an

$\mathfrak{o}_k/\mathfrak{p}^2$ -point. Thus for large enough \mathfrak{p} we have

$$\Theta_{\mathfrak{p}} = \# \left\{ x \in X(\mathfrak{o}_k/\mathfrak{p}^2) : \begin{array}{l} x \notin \pi(Y(\mathfrak{o}_{\mathfrak{p}})) \bmod \mathfrak{p}^2, x \bmod \mathfrak{p} \in T, \\ \pi^{-1}(x \bmod \mathfrak{p}) \text{ is non-split} \end{array} \right\}. \quad (3.7)$$

However the Lang–Weil estimates and Lemma 2.1 yield

$$\#X(\mathfrak{o}_k/\mathfrak{p}^2) = (\mathbf{N} \mathfrak{p})^n \#X(\mathbb{F}_{\mathfrak{p}}) = (\mathbf{N} \mathfrak{p})^{2n} + O((\mathbf{N} \mathfrak{p})^{2n-1/2}) \quad (3.8)$$

and

$$\#\{x \in X(\mathfrak{o}_k/\mathfrak{p}^2) : x \bmod \mathfrak{p} \in T\} \ll (\mathbf{N} \mathfrak{p})^{2n-1}.$$

These and (3.7) already yield the upper bound (3.3). Moreover, Lemma 2.1 and (3.7) show that $\Theta_{\mathfrak{p}} \leq \#\{x \in X(\mathbb{F}_{\mathfrak{p}}) : \pi^{-1}(x) \text{ non-split}\}(\mathbf{N} \mathfrak{p})^n$, which gives the upper bound in (3.6). For the lower bound, let Z be as in Proposition 3.9. Then Proposition 3.9 and Proposition 2.3 (cf. the proof of Corollary 2.4) give

$$\begin{aligned} \Theta_{\mathfrak{p}} &\geq \# \left\{ x \in X(\mathfrak{o}_k/\mathfrak{p}^2) : \begin{array}{l} x \bmod \mathfrak{p} \in T \setminus Z, x \text{ meets } T \text{ transversely above } \mathfrak{p}, \\ \pi^{-1}(x \bmod \mathfrak{p}) \text{ is non-split} \end{array} \right\} \\ &= \#\{x \in X(\mathbb{F}_{\mathfrak{p}}) : \pi^{-1}(x) \text{ non-split}\}(\mathbf{N} \mathfrak{p})^n + O((\mathbf{N} \mathfrak{p})^{2(n-1)}), \end{aligned}$$

whence (3.6). Here, we have used the fact that

$$\{x \in T(\mathbb{F}_{\mathfrak{p}}) : \pi^{-1}(x) \text{ non-split}\} = \{x \in X(\mathbb{F}_{\mathfrak{p}}) : \pi^{-1}(x) \text{ non-split}\}$$

for large enough \mathfrak{p} . Next, we claim that

$$\sum_{\mathbf{N} \mathfrak{p} \leq B} \#\{x \in X(\mathbb{F}_{\mathfrak{p}}) : \pi^{-1}(x) \text{ non-split}\} = \frac{\Delta(\pi)B^n}{\log(B^n)} + O\left(\frac{B^n}{(\log B)^2}\right). \quad (3.9)$$

Indeed, an easy modification of the proof of [24, Prop. 3.10], which is stated without an explicit error term, shows that

$$\sum_{\mathbf{N} \mathfrak{p} \leq B} \#\{x \in X(\mathbb{F}_{\mathfrak{p}}) : \pi^{-1}(x) \text{ non-split}\} = \Delta(\pi) \sum_{\mathbf{N} \mathfrak{p} \leq B} (\mathbf{N} \mathfrak{p})^{n-1} + O\left(\frac{B^n}{(\log B)^2}\right),$$

on using Serre’s version of the Chebotarev density theorem [33, Thm. 9.11]. The claim (3.9) follows from an application of the prime ideal theorem and partial summation. We obtain (3.4) using (3.6), (3.8), (3.9) and a further application of partial summation. Next, taking logarithms it follows from (3.3) that

$$\begin{aligned} \log \prod_{\mathbf{N} \mathfrak{p} \leq B} \frac{\#(\pi(Y(\mathfrak{o}_{\mathfrak{p}})) \bmod \mathfrak{p}^2)}{\#X(\mathfrak{o}_k/\mathfrak{p}^2)} &= \sum_{\mathbf{N} \mathfrak{p} \leq B} \log \left(1 - \frac{\Theta_{\mathfrak{p}}}{\#X(\mathfrak{o}_k/\mathfrak{p}^2)}\right) \\ &= - \sum_{\mathbf{N} \mathfrak{p} \leq B} \frac{\Theta_{\mathfrak{p}}}{\#X(\mathfrak{o}_k/\mathfrak{p}^2)} + O(1). \end{aligned}$$

On combining this with (3.4) and partial summation, we deduce that

$$\log \prod_{\mathbf{N} \mathfrak{p} \leq B} \frac{\#(\pi(Y(\mathfrak{o}_{\mathfrak{p}})) \bmod \mathfrak{p}^2)}{\#X(\mathfrak{o}_k/\mathfrak{p}^2)} = -\Delta(\pi) \log \log B + O(1).$$

The bounds recorded in (3.5) are now obvious. \square

We give a consequence which is required for the proof of Theorem 1.10. To achieve this we use the following version of Wirsing’s theorem over number fields.

Lemma 3.11. *Let g be a non-negative multiplicative arithmetic function on the non-zero ideals of \mathfrak{o}_k . Assume that there exist $\alpha, \beta > 0$ such that*

$$\sum_{N\mathfrak{p} \leq x} \frac{g(\mathfrak{p}) \log N\mathfrak{p}}{N\mathfrak{p}} \sim \alpha \log x \quad (3.10)$$

as $x \rightarrow \infty$ and $g(\mathfrak{p}^v) \leq \beta^v$ for all non-zero prime ideals \mathfrak{p} and all $v \in \mathbb{N}$. Then there exists $c_g > 0$ such that

$$\sum_{N\mathfrak{a} \leq x} g(\mathfrak{a}) \sim c_g \frac{x}{\log x} \prod_{N\mathfrak{p} \leq x} \left(1 + \frac{g(\mathfrak{p})}{N\mathfrak{p}} + \frac{g(\mathfrak{p}^2)}{N\mathfrak{p}^2} + \dots \right).$$

Proof. Over \mathbb{Q} this is a special case of [36, Satz 1.1]. We deduce the case of a general number field from this as follows. Let g be as in the lemma and let $d = [k : \mathbb{Q}]$. Define the arithmetic function over \mathbb{Q} via

$$h(n) = \sum_{N\mathfrak{a}=n} g(\mathfrak{a}).$$

Note that as ideals of prime norm are prime we have

$$h(p) = \sum_{N\mathfrak{p}=p} g(\mathfrak{p}).$$

Using unique factorisation of ideals, one easily verifies that h is a non-negative multiplicative function. We have $h(p^v) \leq (d\beta)^v$ for all primes p and all $v \in \mathbb{N}$. Moreover,

$$\sum_{N\mathfrak{p} \leq x} \frac{g(\mathfrak{p}) \log N\mathfrak{p}}{N\mathfrak{p}} = \sum_{p \leq x} \frac{h(p) \log p}{p} + \sum_{\substack{p, v \geq 2 \\ p^v \leq x}} \frac{h(p^v) \log p^v}{p^v} = \sum_{p \leq x} \frac{h(p) \log p}{p} + O(1).$$

Thus h also satisfies the hypotheses of the lemma and it follows that

$$\sum_{N\mathfrak{a} \leq x} g(\mathfrak{a}) = \sum_{n \leq x} h(n) \sim c_h \frac{x}{\log x} \prod_{p \leq x} \left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \dots \right).$$

The asymptotic behaviour of the above product is determined by the term $h(p)/p$. We deduce that there is a constant $c'_g > 0$ such that

$$\prod_{p \leq x} \left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \dots \right) \sim c'_g \prod_{N\mathfrak{p} \leq x} \left(1 + \frac{g(\mathfrak{p})}{N\mathfrak{p}} + \frac{g(\mathfrak{p}^2)}{N\mathfrak{p}^2} + \dots \right)$$

as $x \rightarrow \infty$, since higher order terms and prime ideals of non-prime norm do not affect the asymptotic behaviour. This completes the proof. \square

Combining Wirsing's result with Proposition 3.10, we can deduce the following.

Corollary 3.12. *Assume that $\Delta(\pi) > 0$ and that the assumptions of Proposition 3.10 hold. Let*

$$\omega_{\mathfrak{p}} = 1 - \frac{\#\pi(Y(\mathfrak{o}_{\mathfrak{p}}) \bmod \mathfrak{p}^2)}{\#X(\mathfrak{o}_k/\mathfrak{p}^2)}, \quad G(B) = \sum_{N\mathfrak{a} \leq B} \mu_k^2(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} \left(\frac{\omega_{\mathfrak{p}}}{1 - \omega_{\mathfrak{p}}} \right).$$

where μ_k is the Möbius function on the ideals of \mathfrak{o}_k . Then

$$G(B) \asymp (\log B)^{\Delta(\pi)}.$$

Proof. We shall show that the conditions of Lemma 3.11 are satisfied with

$$g(\mathfrak{a}) = (N \mathfrak{a}) \mu_k^2(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} \frac{\omega_{\mathfrak{p}}}{1 - \omega_{\mathfrak{p}}}.$$

This function is non-negative, multiplicative and supported on square-free ideals of \mathfrak{o}_k . Since $\omega_{\mathfrak{p}} = O(1/N \mathfrak{p})$, by (3.3), we also have $g(\mathfrak{p}) = O(1)$. Next, it follows from (3.4) that (3.10) holds with $\alpha = \Delta(\pi)$. Hence Lemma 3.11 yields

$$\sum_{N \mathfrak{a} \leq B} g(\mathfrak{a}) \sim c \frac{B}{\log B} \prod_{N \mathfrak{p} \leq B} \left(1 + \frac{g(\mathfrak{p})}{N \mathfrak{p}}\right),$$

for a suitable constant $c > 0$. But, in view of (3.5) we have

$$\prod_{N \mathfrak{p} \leq B} \left(1 + \frac{g(\mathfrak{p})}{N \mathfrak{p}}\right) = \prod_{N \mathfrak{p} \leq B} (1 - \omega_{\mathfrak{p}})^{-1} \asymp (\log B)^{\Delta(\pi)},$$

Thus

$$\sum_{N \mathfrak{a} \leq B} g(\mathfrak{a}) \asymp B (\log B)^{\Delta(\pi)-1}.$$

The desired bounds for $G(B)$ now follow on using partial summation to remove the factor $N \mathfrak{a}$ in $g(\mathfrak{a})$. \square

3.4. Proof of Theorem 1.4. Let $\pi : Y \rightarrow X$ be as in Theorem 1.4. First assume that the generic fibre of π is not geometrically integral. Then, as Y is smooth over k , the generic fibre is smooth thus not geometrically connected. Hence we may consider the Stein factorisation [15, Cor. III.11.5]

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ & \searrow f & \nearrow g \\ & & Z \end{array}$$

of π , where g is now finite of degree at least 2. It follows that $\pi(X(k))$ is a thin set. The result in this case thus follows from Theorem 1.2.

We may therefore assume that the generic fibre of π is geometrically integral. Choose models \mathcal{X} and \mathcal{Y} for X and Y over \mathfrak{o}_k , together with a map $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ which restricts to the original map π on X and Y . Then we clearly have

$$\begin{aligned} N(U, H, \pi, B) &\leq \#\{x \in U(k) : H(x) \leq B, x \in \pi(Y(k_{\mathfrak{p}})) \forall \mathfrak{p}\} \\ &\leq \#\{x \in U(k) : H(x) \leq B, x \bmod \mathfrak{p}^2 \in \pi(\mathcal{Y}(\mathfrak{o}_{\mathfrak{p}})) \bmod \mathfrak{p}^2 \forall \mathfrak{p}\}. \end{aligned}$$

Let $z > 0$. Imposing the above local conditions for all \mathfrak{p} with $N \mathfrak{p} \leq z$, we may use equidistribution, Example 3.4, and (3.5), to obtain

$$\lim_{B \rightarrow \infty} \frac{N(U, H, \pi, B)}{N(U, H, B)} \ll_{X, H} \prod_{N \mathfrak{p} \leq z} \frac{\#\pi(\mathcal{Y}(\mathfrak{o}_{\mathfrak{p}})) \bmod \mathfrak{p}^2}{\#\mathcal{X}(\mathfrak{o}_k/\mathfrak{p}^2)} \ll \frac{1}{(\log z)^{\Delta(\pi)}},$$

where the implied constant is independent of z . Our assumption that there is a non-pseudo-split fibre over some codimension 1 point implies that $\Delta(\pi) > 0$. Taking $z \rightarrow \infty$ completes the proof of Theorem 1.4. \square

3.5. Proof of Theorem 1.6. We begin with the following result.

Lemma 3.13. *Let X be a finite type scheme over \mathfrak{o}_k whose generic fibre X_k is geometrically integral. Assume that $X(\mathbb{F}_{\mathfrak{p}}) \neq \emptyset$ for all primes \mathfrak{p} and let $Z \subset X$ a divisor which is flat over \mathfrak{o}_k . Then*

$$\prod_{N\mathfrak{p} < z} \left(1 - \frac{\#Z(\mathbb{F}_{\mathfrak{p}})}{\#X(\mathbb{F}_{\mathfrak{p}})} \right) \asymp (\log z)^{-r(Z)}, \quad z \rightarrow \infty,$$

where $r(Z)$ denotes the number of irreducible components of Z_k .

Proof. Let $n = \dim X_k$. By [34, Cor. 7.13] we have

$$\sum_{N\mathfrak{p} < z} \#Z(\mathbb{F}_{\mathfrak{p}}) = r(Z) \frac{z^{n-1}}{\log(z^{n-1})} + O\left(\frac{z^{n-1}}{(\log z)^2}\right).$$

Note that $\#X(\mathbb{F}_{\mathfrak{p}}) = (N\mathfrak{p})^n + O((N\mathfrak{p})^{n-1/2})$ by Lang–Weil [18]. Hence, on taking logarithms and combining this with partial summation, we obtain

$$\begin{aligned} \log \prod_{N\mathfrak{p} < z} \left(1 - \frac{\#Z(\mathbb{F}_{\mathfrak{p}})}{\#X(\mathbb{F}_{\mathfrak{p}})} \right) &= \sum_{N\mathfrak{p} < z} \log \left(1 - \frac{\#Z(\mathbb{F}_{\mathfrak{p}})}{\#X(\mathbb{F}_{\mathfrak{p}})} \right) \\ &= - \sum_{N\mathfrak{p} < z} \frac{\#Z(\mathbb{F}_{\mathfrak{p}})}{\#X(\mathbb{F}_{\mathfrak{p}})} + O(1) \\ &= -r(Z) \log \log z + O(1). \end{aligned} \tag{3.11}$$

Exponentiating yields the result. \square

Let now $X, \mathcal{X}, Z, \mathcal{Z}$ be as in Theorem 1.6 and let $z > y > 0$. Example 3.4 and Lemma 3.13 yield

$$\begin{aligned} \lim_{B \rightarrow \infty} \frac{\#\{x \in U(k) : H(x) \leq B, x \text{ is } y\text{-friable with respect to } \mathcal{Z}\}}{N(U, H, B)} \\ \ll \prod_{y < N\mathfrak{p} < z} \left(1 - \frac{\#\mathcal{Z}(\mathbb{F}_{\mathfrak{p}})}{\#\mathcal{X}(\mathbb{F}_{\mathfrak{p}})} \right) \ll \left(\frac{\log y}{\log z} \right)^{r(Z)}. \end{aligned}$$

Taking $z \rightarrow \infty$ completes the proof. \square

4. ZEROS OF QUADRATIC FORMS IN FIXED RESIDUE CLASSES

Let $F \in \mathbb{Z}[x_1, \dots, x_n]$ be an isotropic quadratic form with non-zero discriminant $\Delta_F \in \mathbb{Z}$. For any positive integer M and each prime power factor $p^m \parallel M$ suppose that we are given a non-empty subset

$$\Omega_{p^m} \subseteq \{\mathbf{x} \in (\mathbb{Z}/p^m\mathbb{Z})^n : p \nmid \mathbf{x}, F(\mathbf{x}) \equiv 0 \pmod{p^m}\}. \tag{4.1}$$

Put $\Omega_M = \prod_{p^m \parallel M} \Omega_{p^m}$. For $\mathbf{x} \in \mathbb{Z}^n$, we write $[\mathbf{x}]_M$ for its reduction modulo M . In this section we shall use the Hardy–Littlewood circle method to produce an asymptotic formula for the counting function

$$\widehat{N}(B, \Omega_M) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n, F(\mathbf{x})=0 \\ [\mathbf{x}]_M \in \Omega_M}} w(\mathbf{x}/B),$$

where $w : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is an infinitely differentiable function with compact support.

Associated to F and w is the weighted real density $\sigma_\infty(w)$, as defined in [12, Thm. 3]. It satisfies $1 \ll_{F,w} \sigma_\infty(w) \ll_{F,w} 1$. Moreover, we have the associated p -adic density

$$\sigma_p = \lim_{k \rightarrow \infty} p^{-(n-1)k} \#\{\mathbf{x} \in (\mathbb{Z}/p^k\mathbb{Z})^n : F(\mathbf{x}) \equiv 0 \pmod{p^k}\}, \quad (4.2)$$

for each prime p . The goal of this section is to prove the following result.

Theorem 4.1. *Assume that $n \geq 5$ and that $\nabla F(\mathbf{x}) \gg 1$ for all $\mathbf{x} \in \text{supp}(w)$. Assume that M is coprime to $2\Delta_F$ and let Ω_M be as in (4.1). Then*

$$\widehat{N}(B, \Omega_M) = \sigma_\infty(w) B^{n-2} \prod_{p \nmid M} \sigma_p \prod_{p^m \parallel M} \frac{\#\Omega_{p^m}}{p^{m(n-1)}} + O_{\varepsilon, F, w}(B^{n/2+\varepsilon} M^{n/2+\varepsilon}), \quad \forall \varepsilon > 0.$$

In this result and henceforth in this section, the implied constant is allowed to depend on the choice of ε , the form F and the weight function w , but not on the modulus M . To ease notation we shall suppress this dependence in what follows.

Some comments are in order about the statement of this result. The condition that $\nabla F(\mathbf{x}) \gg 1$ for any \mathbf{x} in the support of w is required to simplify the analysis of certain oscillatory integrals in the argument. The assumptions $(M, 2\Delta_F) = 1$ and $(\mathbf{x}, M) = 1$ for any $\mathbf{x} \in \Omega_M$ are made purely to simplify the expression for the leading constant in the asymptotic formula for $\widehat{N}(B, \Omega_M)$.

It is possible to obtain a version of Theorem 4.1 by exploiting existing work in the literature, such as using work of Sardari [30, Thm. 1.8] to handle the contribution from $\mathbf{x} \equiv \mathbf{a} \pmod{M}$, for each $\mathbf{a} \in \Omega_M$. However, this leads to weaker results than our approach. Nonetheless, several facets of Theorem 4.1 could still be improved. Firstly, one can do better in the B -aspect of the error term when n is odd. Secondly, it would not be hard to deal with the cases $n = 3$ or 4 . Finally, when M is square-free it is possible to improve the error term to $O(B^{n/2+\varepsilon} \#\Omega_M^{1/2})$. In order to simplify our exposition we have not pursued these improvements here. In our application Ω_M will be comparable in size to the set of $\mathbf{x} \in (\mathbb{Z}/M\mathbb{Z})^n$ for which $F(\mathbf{x}) \equiv 0 \pmod{M}$, leading us to relax the dependence on $\#\Omega_M$, often to the extent that we employ the trivial inequality $\#\Omega_M \leq M^n$.

4.1. First steps. We begin the proof of Theorem 4.1 by invoking the version of the circle method developed by Heath-Brown [12, Thm. 1]. This implies that

$$\widehat{N}(B, \Omega_M) = \frac{c_Q}{Q^2} \sum_{q=1}^{\infty} \sum_{a \pmod{q}}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ [\mathbf{x}]_M \in \Omega_M}} w(\mathbf{x}/B) e_q(aF(\mathbf{x})) h\left(\frac{q}{Q}, \frac{F(\mathbf{x})}{Q^2}\right),$$

for any $Q > 1$. Here c_Q is a positive constant satisfying $c_Q = 1 + O_A(Q^{-A})$ for any $A > 0$ and, moreover, $h(x, y)$ is a smooth function defined on the set $(0, \infty) \times \mathbb{R}$ such that $h(x, y) \ll x^{-1}$ for all y , with $h(x, y)$ non-zero only for $x \leq \max\{1, 2|y|\}$. In particular, we are only interested in $q \ll Q$ in this sum.

We will henceforth take $Q = B$. It is natural to break the sum into residue classes modulo the least common multiple $[q, M]$ and then apply Poisson summation, as in the proof of [12, Thm. 2]. This leads to the expression

$$\widehat{N}(B, \Omega_M) = \frac{c_B}{B^2} \sum_{q \ll B} \sum_{\mathbf{c} \in \mathbb{Z}^n} [q, M]^{-n} S_{q, M}(\mathbf{c}) J_{q, M}(\mathbf{c}),$$

where

$$S_{q,M}(\mathbf{c}) = \sum_{a \bmod q}^* \sum_{\substack{\mathbf{y} \bmod [q,M] \\ [\mathbf{y}]_M \in \Omega_M}} e_q(aF(\mathbf{y})) e_{[q,M]}(\mathbf{c} \cdot \mathbf{y}) \quad (4.3)$$

and

$$\begin{aligned} J_{q,M}(\mathbf{c}) &= \int_{\mathbb{R}^n} w(\mathbf{x}/B) h\left(\frac{q}{B}, \frac{F(\mathbf{x})}{B^2}\right) e_{[q,M]}(-\mathbf{c} \cdot \mathbf{x}) d\mathbf{x} \\ &= B^n \int_{\mathbb{R}^n} w(\mathbf{x}) h\left(\frac{q}{B}, F(\mathbf{x})\right) e_{[q,M]}(-B\mathbf{c} \cdot \mathbf{x}) d\mathbf{x}. \end{aligned}$$

For any $r > 0$ and $\mathbf{v} \in \mathbb{R}^n$ it will be convenient to set

$$I_r^*(\mathbf{v}) = \int_{\mathbb{R}^n} w(\mathbf{x}) h(r, F(\mathbf{x})) e_r(-\mathbf{v} \cdot \mathbf{x}) d\mathbf{x}. \quad (4.4)$$

In this notation, which coincides with that of [12, §7], we may clearly write $J_{q,M}(\mathbf{c}) = B^n I_r^*(M'^{-1}\mathbf{c})$, where $r = q/B$ and $M' = [q, M]/q = M/(M, q)$. Thus

$$\widehat{N}(B, \Omega_M) = c_B B^{n-2} \sum_{q \ll B} \sum_{\mathbf{c} \in \mathbb{Z}^n} [q, M]^{-n} S_{q,M}(\mathbf{c}) I_r^*(M'^{-1}\mathbf{c}). \quad (4.5)$$

4.2. The exponential sum. In this section we analyse the sum $S_{q,M}(\mathbf{c})$ in (4.3) for $q, M \in \mathbb{N}$ with $(M, 2\Delta_F) = 1$. We begin by establishing the following.

Lemma 4.2. *Let $M = M_1 M_2$. Suppose that $(q_1 M_1, q_2 M_2) = 1$ and choose integers s, t such that $[q_1, M_1]s + [q_2, M_2]t = 1$. Then*

$$S_{q_1 q_2, M}(\mathbf{c}) = S_{q_1, M_1}(t\mathbf{c}) S_{q_2, M_2}(s\mathbf{c}).$$

Proof. Note that $[q_1 q_2, M] = [q_1, M_1][q_2, M_2]$. As \mathbf{y}_1 runs modulo $[q_1, M_1]$ and \mathbf{y}_2 runs modulo $[q_2, M_2]$, so $\mathbf{y} = \mathbf{y}_1[q_2, M_2]t + \mathbf{y}_2[q_1, M_1]s$ runs over a full set of residue classes modulo $[q_1 q_2, M]$. Now let $\bar{q}_1, \bar{q}_2 \in \mathbb{Z}$ be such that $q_1 \bar{q}_1 + q_2 \bar{q}_2 = 1$. Then $a = a_1 q_2 \bar{q}_2 + a_2 q_1 \bar{q}_1$ runs over $(\mathbb{Z}/q_1 q_2 \mathbb{Z})^*$ as a_1 (resp. a_2) runs over $(\mathbb{Z}/q_1 \mathbb{Z})^*$ (resp. $(\mathbb{Z}/q_2 \mathbb{Z})^*$). Under these transformations $[\mathbf{y}]_M \in \Omega_M \Leftrightarrow [\mathbf{y}_i]_{M_i} \in \Omega_{M_i}$ for $i = 1, 2$, since $[q_1, M_1]s + [q_2, M_2]t = 1$. Furthermore,

$$e_{[q_1 q_2, M]}(\mathbf{c} \cdot \mathbf{y}) = e_{[q_1, M_1]}(t\mathbf{c} \cdot \mathbf{y}_1) e_{[q_2, M_2]}(s\mathbf{c} \cdot \mathbf{y}_2)$$

and

$$\begin{aligned} e_{q_1 q_2}(aF(\mathbf{y})) &= e_{q_1}(a_1 \bar{q}_2 F(\mathbf{y})) e_{q_2}(a_2 \bar{q}_1 F(\mathbf{y})) \\ &= e_{q_1}(a_1 \bar{q}_2 ([q_2, M_2]t)^2 F(\mathbf{y}_1)) e_{q_2}(a_2 \bar{q}_1 ([q_1, M_1]s)^2 F(\mathbf{y}_2)). \end{aligned}$$

Note that $(\bar{q}_2 ([q_2, M_2]t)^2, q_1) = (\bar{q}_1 ([q_1, M_1]s)^2, q_2) = 1$. A further change of variables in the a_1 and a_2 summations therefore proves the lemma. \square

For any divisor $L \mid M$, we henceforth set

$$K_L(\mathbf{c}) = S_{1,L}(\mathbf{c}) = \sum_{\mathbf{y} \in \Omega_L} e_L(\mathbf{c} \cdot \mathbf{y}).$$

While it is clear that $K_L(\mathbf{0}) = \#\Omega_L$, we expect $K_L(\mathbf{c})$ to be rather smaller than $\#\Omega_L$ for typical values of $\mathbf{c} \in \mathbb{Z}^n$. This will be established in §4.4.

Next, let $S_q(\mathbf{c}) = S_{q,1}(\mathbf{c})$. This is precisely the exponential sum appearing in [12, Thm. 2]. Recall that the dual form $F^* \in \mathbb{Z}[\mathbf{x}]$ has underlying matrix $\Delta_F \mathbf{A}^{-1}$, where \mathbf{A} is the symmetric matrix of determinant Δ_F that is associated to F . Our next result is a variant of [12, Lem. 28] and concerns the mean square.

Lemma 4.3. *Let $\varepsilon > 0$. Then*

$$\sum_{q \leq R} |S_q(\mathbf{c})|^2 \ll \begin{cases} R^{n+3} & \text{if } n \text{ is even and } F^*(\mathbf{c}) = 0, \\ R^{n+5/2+\varepsilon}(1 + |\mathbf{c}|)^\varepsilon & \text{otherwise.} \end{cases}$$

Proof. The first bound follows directly from [12, Lem. 25]. As in the proof of [12, Lem. 28], we split q into a square-free part u and a square-full part v , finding that $|S_q(\mathbf{c})|^2 \ll u^{n+1+\varepsilon}(u, F^*(\mathbf{c}))v^{n+2}$, where the factor $(u, F^*(\mathbf{c}))$ can be dropped if n is odd. Assuming that n is odd or $F^*(\mathbf{c}) \neq 0$, it therefore follows that

$$\sum_{q \leq R} |S_q(\mathbf{c})|^2 \ll \sum_{v \leq R} v^{n+2} \left(\frac{R}{v}\right)^{n+2+\varepsilon} (1 + |\mathbf{c}|)^\varepsilon \ll R^{n+5/2+\varepsilon}(1 + |\mathbf{c}|)^\varepsilon,$$

since there are $O(R^{1/2})$ square-full values of $v \leq R$. \square

Before returning to the exponential sum $S_{q,M}(\mathbf{c})$ in (4.3), we first record the following estimate.

Lemma 4.4. *Let $a \in (\mathbb{Z}/q\mathbb{Z})^*$, let $\mathbf{c} \in \mathbb{Z}^n$ and let $M \mid q$. Then*

$$\left| \sum_{\substack{\mathbf{y} \bmod q \\ [\mathbf{y}]_M \in \Omega_M}} e_q(aF(\mathbf{y}) + \mathbf{c} \cdot \mathbf{y}) \right| \ll \frac{(qM)^{n/2}}{(q/M, M)^{n/2}}.$$

Proof. Let $T_{q,M}(\mathbf{c})$ denote the sum whose modulus is to be estimated. Then

$$T_{q,M}(\mathbf{c}) = \sum_{\mathbf{u} \in \Omega_M} \sum_{\substack{\mathbf{y} \bmod q \\ \mathbf{y} \equiv \mathbf{u} \bmod M}} e_q(aF(\mathbf{y}) + \mathbf{c} \cdot \mathbf{y}).$$

Let $q' = q/M$. We make the change of variables $\mathbf{y} = \mathbf{u} + M\mathbf{z}$ for $\mathbf{z} \bmod q'$, giving

$$\begin{aligned} |T_{q,M}(\mathbf{c})| &\leq \sum_{\mathbf{u} \in \Omega_M} \left| \sum_{\mathbf{z} \bmod q'} e_{Mq'}(a\{M^2F(\mathbf{z}) + M\mathbf{z} \cdot \nabla F(\mathbf{u})\} + M\mathbf{c} \cdot \mathbf{z}) \right| \\ &= \sum_{\mathbf{u} \in \Omega_M} \left| \sum_{\mathbf{z} \bmod q'} e_{q'}(aMF(\mathbf{z}) + \mathbf{z} \cdot \{\nabla F(\mathbf{u}) + \mathbf{c}\}) \right|. \end{aligned}$$

Let $q'' = q'/h$, where $h = (q', M)$. We write $\mathbf{j} = \nabla F(\mathbf{u}) + \mathbf{c}$ for convenience. The next step is to make the change of variables $\mathbf{z} = \mathbf{z}_1 + q''\mathbf{z}_2$ for $\mathbf{z}_1 \bmod q''$ and $\mathbf{z}_2 \bmod h$. Noting that $q' \mid Mq''$, the inner sum is

$$\begin{aligned} &\sum_{\mathbf{z}_1 \bmod q''} \sum_{\mathbf{z}_2 \bmod h} e_{q'}(aMF(\mathbf{z}_1) + (\mathbf{z}_1 + q''\mathbf{z}_2) \cdot \mathbf{j}) \\ &= \begin{cases} h^n \sum_{\mathbf{z}_1 \bmod q''} e_{q''}(ah^{-1}MF(\mathbf{z}_1) + h^{-1}\mathbf{z}_1 \cdot \mathbf{j}) & \text{if } h \mid \mathbf{j}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

When $h \mid \mathbf{j}$ the sum over \mathbf{z}_1 is $O(q^{m/2})$ by the proof of [12, Lem. 25], since $(q'', h^{-1}M) = 1$. Thus

$$\begin{aligned} |T_{q,M}(\mathbf{c})| &\ll h^n q^{m/2} \# \{ \mathbf{u} \in \Omega_M : 2\mathbf{A}\mathbf{u} \equiv -\mathbf{c} \pmod{h} \} \\ &\leq h^n q^{m/2} \# \{ \mathbf{u} \in (\mathbb{Z}/M\mathbb{Z})^n : 2\mathbf{A}\mathbf{u} \equiv -\mathbf{c} \pmod{h} \}, \end{aligned}$$

where \mathbf{A} is the matrix associated to F . As \mathbf{A} is non-singular, the inner cardinality is $O((M/h)^n)$. We conclude the proof on recalling that $q'' = q/(hM)$. \square

We now return to the exponential sum $S_{q,M}(\mathbf{c})$ in (4.3). There is a unique factorisation into pairwise coprime positive integers u, v_1, v_2 , with v_1 square-free and v_2 square-full, such that

$$q = uv_1v_2, \quad \text{with } (u, M) = 1 \text{ and } v_1v_2 \mid M^\infty. \quad (4.6)$$

Likewise there is a unique factorisation $M = M_{11}M_{12}M_2$, where

$$M_{1i} = (M, v_i) \quad \text{and} \quad M_2 = \frac{M}{M_{11}M_{12}}. \quad (4.7)$$

It follows that $M_{11} = v_1$, since v_1 is square-free and $v_1 \mid M^\infty$. Moreover, we have $(M_2, uv_1v_2) = 1$ and

$$[q, M] = \frac{uv_1v_2M_{11}M_{12}M_2}{(v_1v_2, M_{11}M_{12})} = uv_1v_2M_2.$$

We may now establish the following factorisation of $S_{q,M}(\mathbf{c})$.

Lemma 4.5. *We have*

$$S_{q,M}(\mathbf{c}) = \phi(v_1)S_u(\mathbf{c})S_{v_2, M_{12}}(\overline{uv_1M_2}\mathbf{c})K_{v_1}(\overline{uv_2M_2}\mathbf{c})K_{M_2}(\overline{uv_1v_2}\mathbf{c}),$$

where $\overline{uv_1M_2} \in \mathbb{Z}$ (respectively, $\overline{uv_2M_2} \in \mathbb{Z}$, $\overline{uv_1v_2} \in \mathbb{Z}$) is a multiplicative inverse of uv_1M_2 (respectively, uv_2M_2 , uv_1v_2) modulo v_2 (respectively, v_1 , M_2).

Proof. We write $v = v_1v_2$ and $M_1 = M_{11}M_{12}$ for convenience. Let $\overline{uM_2} \in \mathbb{Z}$ (respectively, $\overline{uv} \in \mathbb{Z}$) be such that $uM_2\overline{uM_2} \equiv 1 \pmod{v}$ (respectively, $uv\overline{uv} \equiv 1 \pmod{M_2}$). Then the factorisation

$$S_{q,M}(\mathbf{c}) = S_u(\mathbf{c})S_{v, M_1}(\overline{uM_2}\mathbf{c})K_{M_2}(\overline{uv}\mathbf{c})$$

is a direct consequence of Lemma 4.2 and the obvious fact that $S_q(t\mathbf{c}) = S_q(\mathbf{c})$, for any $(t, q) = 1$. Note that since $v_i \mid M_{1i}$, we have $[v_i, M_{1i}] = v_i$, for $i = 1, 2$. A further application of Lemma 4.2 now yields

$$S_{v_1v_2, M_{11}M_{12}}(\mathbf{d}) = S_{v_1, v_1}(t\mathbf{d})S_{v_2, M_{12}}(s\mathbf{d}),$$

where $s, t \in \mathbb{Z}$ are such that $v_1s + v_2t = 1$. Finally, we note that

$$S_{v_1, v_1}(t\mathbf{d}) = \sum_{a \pmod{v_1}}^* \sum_{\mathbf{y} \in \Omega_{v_1}} e_{v_1}(aF(\mathbf{y}) + t\mathbf{d}\cdot\mathbf{y}) = \phi(v_1)K_{v_1}(t\mathbf{d}),$$

since $v_1 \mid F(\mathbf{y})$ for any $\mathbf{y} \in \Omega_{v_1}$. This completes the proof of the lemma. \square

The following simple upper bound will suffice to handle the third factor in the factorisation of Lemma 4.5.

Lemma 4.6. *For any $\mathbf{c} \in \mathbb{Z}^n$, we have $S_{v_2, M_{12}}(\mathbf{c}) \ll v_2^{n/2+1}M_{12}^{n/2}$.*

Proof. Recall the definition (4.3) of $S_{v_2, M_{12}}(\mathbf{c})$. We shall sum trivially over a . Noting that $M_{12} \mid v_2$, the statement follows on applying Lemma 4.4 to estimate the inner sum over \mathbf{y} and noting that $(v_2, M_{12}^2) \geq M_{12}$. \square

4.3. Contribution from the trivial character. Returning to (4.5), the contribution $T(B)$, say, from the term $\mathbf{c} = \mathbf{0}$ is found to satisfy

$$T(B) = (1 + O_A(B^{-A})) B^{n-2} \sum_{q \ll B} [q, M]^{-n} S_{q, M}(\mathbf{0}) I_{q/B}^*(\mathbf{0}), \quad (4.8)$$

for any $A > 0$, in the notation of (4.3) and (4.4).

Recall that $n \geq 5$. We shall start by analysing an unweighted version of the sum over q in (4.8). For any prime p , recall the definition (4.2) of the p -adic density σ_p . In particular the product $\prod_p \sigma_p$ is absolutely convergent for $n \geq 5$.

Lemma 4.7. *Let $R \leq B$. Then*

$$\sum_{q \leq R} [q, M]^{-n} S_{q, M}(\mathbf{0}) = \prod_{p \nmid M} \sigma_p \prod_{p^m \parallel M} \frac{\#\Omega_{p^m}}{p^{m(n-1)}} + O(R^{(3+\kappa-n)/2} B^\varepsilon M^{(n-1-\kappa)/2+\varepsilon}),$$

for any $\varepsilon > 0$, where

$$\kappa = \begin{cases} 1 & \text{if } 2 \mid n, \\ 0 & \text{if } 2 \nmid n. \end{cases} \quad (4.9)$$

Proof. We make the change of variables $q = uv_1v_2$ and $M = M_{11}M_{12}M_2$, in the notation of (4.6) and (4.7). Recalling that $[q, M] = uv_1v_2M_2$, an application of Lemma 4.5 implies that

$$\sum_{q \leq R} [q, M]^{-n} S_{q, M}(\mathbf{0}) = \sum_{\substack{v_1v_2 \leq R \\ v_1v_2 \mid M^\infty}} \sum_{\substack{u \leq R/(v_1v_2) \\ (u, M)=1}} \frac{\phi(v_1) S_u(\mathbf{0}) S_{v_2, M_{12}}(\mathbf{0}) \#\Omega_{v_1} \#\Omega_{M_2}}{(uv_1v_2M_2)^n}.$$

Since $R \leq B$, an inspection of the proof of [12, Lemmas 28 and 31] reveals that

$$\sum_{\substack{u \leq R/(v_1v_2) \\ (u, M)=1}} u^{-n} S_u(\mathbf{0}) = \prod_{p \nmid M} \sigma_p + O((R/v_1v_2)^{(3+\kappa-n)/2} B^\varepsilon M^{\varepsilon/2}).$$

Lemma 4.6 yields $S_{v_2, M_{12}}(\mathbf{0}) \ll v_2^{n/2+1} M_{12}^{n/2}$. Thus the overall contribution from the error term is

$$\begin{aligned} &\ll R^{(3+\kappa-n)/2} B^\varepsilon M^{\varepsilon/2} \sum_{\substack{v_1v_2 \leq R \\ v_1v_2 \mid M^\infty}} \frac{(v_1v_2)^{(n-3-\kappa)/2} v_1 v_2^{n/2+1} M_{12}^{n/2} \#\Omega_{v_1} \#\Omega_{M_2}}{(v_1v_2M_2)^n} \\ &\ll R^{(3+\kappa-n)/2} B^\varepsilon M^{\varepsilon/2} \sum_{\substack{v_1v_2 \leq R \\ v_1v_2 \mid M^\infty}} \frac{M_{12}^{n/2} v_1^{(n-1-\kappa)/2}}{v_2^{(1+\kappa)/2}}, \end{aligned}$$

since $\#\Omega_{v_1} \leq v_1^n$ and $\#\Omega_{M_2} \leq M_2^n$. Recalling that $M_{12} \mid v_2$, we next observe that $M_{12}^{n/2}/v_2^{(1+\kappa)/2} \leq M_{12}^{(n-1-\kappa)/2}$. Since

$$\#\{v \leq R : v \mid M^\infty\} \leq \sum_{v \mid M^\infty} \left(\frac{R}{v}\right)^\varepsilon = R^\varepsilon \prod_{p \mid M} (1 - p^{-\varepsilon})^{-1} \ll R^\varepsilon M^{\varepsilon/4},$$

the error term gives the overall contribution $O(R^{(3+\kappa-n)/2} B^{3\varepsilon} M^{(n-1-\kappa)/2+\varepsilon})$.

Redefining the choice of $\varepsilon > 0$, we have therefore proved that

$$\begin{aligned} \sum_{q \leq R} [q, M]^{-n} S_{q, M}(\mathbf{0}) &= \prod_{p \nmid M} \sigma_p \sum_{\substack{v_1 v_2 \leq R \\ v_1 v_2 | M^\infty}} \frac{\phi(v_1) S_{v_2, M_{12}}(\mathbf{0}) \# \Omega_{v_1} \# \Omega_{M_2}}{(v_1 v_2 M_2)^n} \\ &\quad + O(R^{(3+\kappa-n)/2} B^\varepsilon M^{(n-1-\kappa)/2+\varepsilon}). \end{aligned}$$

Employing Lemma 4.6 once more, we obtain

$$\begin{aligned} \sum_{\substack{v_1 v_2 > R \\ v_1 v_2 | M^\infty}} \frac{\phi(v_1) \# \Omega_{v_1} \# \Omega_{M_2} S_{v_2, M_{12}}(\mathbf{0})}{(v_1 v_2 M_2)^n} &\ll \sum_{\substack{v_1 v_2 > R \\ v_1 v_2 | M^\infty}} \frac{M_{12}^{n/2}}{v_2^{n/2-1}} \left(\frac{v_1 v_2}{R} \right)^{(n-3-\kappa)/2-\varepsilon} \\ &\ll R^{(3+\kappa-n)/2} B^\varepsilon M^{(n-1-\kappa)/2+\varepsilon}, \end{aligned}$$

on recalling that $v_1 = M_{11}$ and arguing as before. In view of the fact that $\prod_{p \nmid M} \sigma_p \ll 1$ for $n \geq 5$, it follows that

$$\sum_{q \leq R} [q, M]^{-n} S_{q, M}(\mathbf{0}) = C_M \prod_{p \nmid M} \sigma_p + O(R^{(3+\kappa-n)/2} B^\varepsilon M^{(n-1-\kappa)/2+\varepsilon}).$$

Here

$$C_M = \sum_{v_1 v_2 | M^\infty} \frac{\phi(v_1) \# \Omega_{v_1} \# \Omega_{M_2} S_{v_2, M_{12}}(\mathbf{0})}{(v_1 v_2 M_2)^n},$$

where the sum is constrained to satisfy $(v_1, v_2) = \mu^2(v_1) = 1$, with v_2 is square-full. It remains to calculate this quantity. We have

$$\sum_{\substack{v_1 | M^\infty \\ (v_1, v_2) = 1}} \phi(v_1) \mu^2(v_1) = \frac{M^b}{(M, v_2)^b},$$

where $k^b = \prod_{p \nmid k} p$ is the square-free kernel. Since $\# \Omega_{v_1} \# \Omega_{M_2} = \# \Omega_M / \# \Omega_{M_{12}}$, we see that

$$\begin{aligned} C_M &= \frac{\# \Omega_M M^b}{M^n} \sum_{\substack{v_2 | M^\infty \\ v_2 \text{ square-full}}} \frac{S_{v_2, M_{12}}(\mathbf{0}) M_{12}^n}{(M, v_2)^b v_2^n \# \Omega_{M_{12}}} \\ &= \frac{\# \Omega_M M^b}{M^n} \prod_{p^m \parallel M} \left(1 + \frac{1}{p} \sum_{\ell \geq 2} \frac{S_{p^\ell, p^{\min\{\ell, m\}}}(\mathbf{0}) p^{\min\{\ell, m\}n}}{p^{\ell n} \# \Omega_{p^{\min\{\ell, m\}}}} \right). \end{aligned}$$

Let Σ_p denote the sum over ℓ . The contribution to Σ_p from $\ell \in [2, m]$ is $\sum_{2 \leq \ell \leq m} \phi(p^\ell) = p^m - p$. On the other hand, on evaluating the Ramanujan sum, the contribution to Σ_p from $\ell > m$ is

$$\begin{aligned} \frac{p^{mn}}{\# \Omega_{p^m}} \sum_{\ell > m} \frac{S_{p^\ell, p^m}(\mathbf{0})}{p^{\ell n}} &= \frac{p^{mn}}{\# \Omega_{p^m}} \sum_{\ell > m} \frac{p^\ell N(\ell) - p^{n+\ell-1} N(\ell-1)}{p^{\ell n}} \\ &= \frac{p^{mn}}{\# \Omega_{p^m}} \left(\lim_{\ell \rightarrow \infty} p^{-\ell(n-1)} N(\ell) - p^{-m(n-1)} \# \Omega_{p^m} \right), \end{aligned}$$

where $N(k) = \#\{\mathbf{y} \bmod p^k : p^k \mid F(\mathbf{y}), [\mathbf{y}]_{p^m} \in \Omega_{p^m}\}$, for any $k \geq m$. Note that $p \nmid \mathbf{y}$ in $N(k)$ by our assumption on Ω_M in (4.1). Putting everything together, we deduce that

$$C_M = \frac{\#\Omega_M M^b}{M^n} \prod_{p^m \parallel M} \left(\frac{p^{mn-1}}{\#\Omega_{p^m}} \lim_{\ell \rightarrow \infty} p^{-\ell(n-1)} N(\ell) \right) = \prod_{p^m \parallel M} \lim_{\ell \rightarrow \infty} p^{-\ell(n-1)} N(\ell).$$

Finally, since $(M, 2\Delta_F) = 1$, a straightforward application of Lemma 2.1 shows that $N(k+1) = p^{n-1}N(k)$ for all $k \geq m$. Thus we can replace the limit by $p^{-m(n-1)}N(m) = p^{-m(n-1)}\#\Omega_{p^m}$. This completes the proof of the lemma. \square

For convenience we put

$$\mathfrak{S}(M) = \prod_{p \nmid M} \sigma_p \prod_{p^m \parallel M} \frac{\#\Omega_{p^m}}{p^{m(n-1)}}.$$

Next, let $\sigma_\infty(w)$ be the weighted real density associated to F , as defined in [12, Thm. 3]. Since $\nabla F(\mathbf{x}) \gg 1$ throughout the support of w , it follows from [12, Lem. 13] that $I_{q/B}^*(\mathbf{0}) = \sigma_\infty(w) + O_A((q/B)^A)$, for any $A > 0$. Hence

$$\sum_{q \leq B^{1-\varepsilon}} \frac{S_{q,M}(\mathbf{0}) I_{q/B}^*(\mathbf{0})}{[q, M]^n} = \sigma_\infty(w) \sum_{q \leq B^{1-\varepsilon}} [q, M]^{-n} S_{q,M}(\mathbf{0}) + O(B^{-n}),$$

on taking A sufficiently large. We apply Lemma 4.7 with $R = B^{1-\varepsilon}$ to estimate the inner sum, finding that

$$\sum_{q \leq B^{1-\varepsilon}} \frac{S_{q,M}(\mathbf{0}) I_{q/B}^*(\mathbf{0})}{[q, M]^n} = \sigma_\infty(w) \mathfrak{S}(M) + O(B^{(3+\kappa-n)/2+(n+1)\varepsilon} M^{(n-1-\kappa)/2+\varepsilon}).$$

We have the bounds $\frac{\partial^i}{\partial q^i} I_{q/B}^*(\mathbf{0}) \ll q^{-i}$, for $i \in \{0, 1\}$, which are a direct consequence of [12, Lemmas 14 and 15]. Hence we may combine Lemma 4.7 with partial summation to conclude that

$$\sum_{B^{1-\varepsilon} < q \leq B} \frac{S_{q,M}(\mathbf{0}) I_{q/B}^*(\mathbf{0})}{[q, M]^n} \ll B^{(3+\kappa-n)/2+(n+1)\varepsilon} M^{(n-1-\kappa)/2+\varepsilon}.$$

Bringing everything together in (4.8), and redefining the choice of ε , we finally arrive at the estimate

$$T(B) = \sigma_\infty(w) \mathfrak{S}(M) B^{n-2} + O(B^{(n-1+\kappa)/2+\varepsilon} M^{(n-1-\kappa)/2+\varepsilon}).$$

This shows that the contribution from the trivial character is satisfactory for Theorem 4.1.

4.4. Contribution from the non-trivial characters. It remains to consider the contribution $E(B)$, say, from $\mathbf{c} \neq \mathbf{0}$ in (4.5). Thus

$$E(B) \ll B^{n-2} \sum_{q \leq B} \sum_{\mathbf{0} \neq \mathbf{c} \in \mathbb{Z}^n} [q, M]^{-n} |S_{q,M}(\mathbf{c})| |I_r^*(M'^{-1}\mathbf{c})|,$$

where $r = q/B$, $M' = [q, M]/q = M/(q, M)$ and $I_r^*(\mathbf{v})$ is defined in (4.4). It follows from [12, Lemmas 14 and 18] that $I_r^*(\mathbf{v}) \ll_A r^{-1} |\mathbf{v}|^{-A}$ for any $A > 0$. Hence there is a negligible contribution to $E(B)$ from vectors $\mathbf{c} \in \mathbb{Z}^n$ for which

$|\mathbf{c}| \geq M'B^\varepsilon$ for any fixed value of $\varepsilon > 0$. We now apply [12, Lemmas 14 and 22] to deduce that

$$I_r^*(\mathbf{v}) \ll (r^{-2}|\mathbf{v}|)^{\varepsilon/10} (r^{-1}|\mathbf{v}|)^{1-n/2}.$$

Hence

$$E(B) \ll B^{n/2-1+\varepsilon} \sum_{q \ll B} \sum_{\substack{\mathbf{c} \in \mathbb{Z}^n \\ 0 < |\mathbf{c}| \leq M'B^\varepsilon}} |\mathbf{c}|^{1-n/2} \frac{|S_{q,M}(\mathbf{c})|}{[q, M]^{n/2+1}},$$

since $qM' = [q, M]$ and $r^{-1}|\mathbf{v}| = B|\mathbf{c}|/[q, M]$. We carry out the change of variables recorded in (4.6) and (4.7) and recall that $\#\Omega_{v_1} \leq v_1^n$. In this notation, it follows from Lemmas 4.5 and 4.6 that

$$\begin{aligned} \frac{|S_{q,M}(\mathbf{c})|}{[q, M]^{n/2+1}} &= \frac{|\phi(v_1)S_u(\mathbf{c})S_{v_2, M_{12}}(\overline{uv_1 M_2 \mathbf{c}})K_{v_1}(\overline{uv_2 M_2 \mathbf{c}})K_{M_2}(\overline{uv_1 v_2 \mathbf{c}})|}{(uv_1 v_2 M_2)^{n/2+1}} \\ &\ll \frac{|S_u(\mathbf{c})v_1^{n/2}K_{M_2}(\overline{uv_1 v_2 \mathbf{c}})M_{12}^{n/2}|}{(uM_2)^{n/2+1}}, \end{aligned}$$

where $M_2 = M/(M, v_1 v_2)$.

Let \mathcal{V} denote the set of vectors $(v_1, v_2) \in \mathbb{N}^2$ such that $v_1 v_2 \ll B$ and $v_1 v_2 \mid M^\infty$, with $(v_1, v_2) = \mu^2(v_1) = 1$ and v_2 square-full. Noting that $M' = M/(q, M) = M_2$, we deduce that

$$E(B) \ll B^{n/2-1+\varepsilon} \sum_{(v_1, v_2) \in \mathcal{V}} \frac{v_1^{n/2} M_{12}^{n/2}}{M_2^{n/2+1}} E_{v_1, v_2}(B), \quad (4.10)$$

where

$$E_{v_1, v_2}(B) = \sum_{\substack{u \ll B/(v_1 v_2) \\ (u, M) = 1}} \sum_{\substack{\mathbf{c} \in \mathbb{Z}^n \\ 0 < |\mathbf{c}| \leq M_2 B^\varepsilon}} |\mathbf{c}|^{1-n/2} \frac{|S_u(\mathbf{c})K_{M_2}(\overline{uv_1 v_2 \mathbf{c}})|}{u^{n/2+1}}.$$

The presence of \bar{u} in $K_{M_2}(\overline{uv_1 v_2 \mathbf{c}})$ prevents us from executing the sum over u directly. To separate $S_u(\mathbf{c})$ and $K_{M_2}(\overline{uv_1 v_2 \mathbf{c}})$, we shall apply Cauchy's inequality. This gives $E_{v_1, v_2}(B)^2 \leq \Sigma_1 \Sigma_2$, where

$$\begin{aligned} \Sigma_1 &= \sum_{\substack{u \ll B/(v_1 v_2) \\ (u, M) = 1}} \sum_{\substack{\mathbf{c} \in \mathbb{Z}^n \\ 0 < |\mathbf{c}| \leq M_2 B^\varepsilon}} |\mathbf{c}|^{2-n} \frac{|S_u(\mathbf{c})|^2}{u^{n+2}}, \\ \Sigma_2 &= \sum_{\substack{u \ll B/(v_1 v_2) \\ (u, M) = 1}} \sum_{\substack{\mathbf{c} \in \mathbb{Z}^n \\ 0 < |\mathbf{c}| \leq M_2 B^\varepsilon}} |K_{M_2}(\overline{uv_1 v_2 \mathbf{c}})|^2. \end{aligned}$$

The following results are concerned with estimating these quantities.

Lemma 4.8. *We have*

$$\Sigma_1 \ll (M_2 B)^\varepsilon \left(M_2^2 (B/(v_1 v_2))^{1/2} + (B/(v_1 v_2))^{(1+\kappa)/2} \right),$$

where κ is given by (4.9).

Proof. Let $v = v_1 v_2$. To begin with, it follows from Lemma 4.3 and partial summation that

$$\sum_{u \ll B/v} \frac{|S_u(\mathbf{c})|^2}{u^{n+2}} \ll \begin{cases} (B/v)^{1/2+\varepsilon} |\mathbf{c}|^\varepsilon, & \text{if } F^*(\mathbf{c}) \neq 0, \\ (B/v)^{(1+\kappa)/2+\varepsilon} (1+|\mathbf{c}|)^\varepsilon, & \text{if } F^*(\mathbf{c}) = 0. \end{cases}$$

A standard estimate shows that there are $O(C^{n-2})$ vectors $\mathbf{c} \in \mathbb{Z}^n$, such that $|\mathbf{c}| \leq C$ and $F^*(\mathbf{c}) = 0$. Hence

$$\Sigma_1 \ll (M_2 B^\varepsilon)^{2+\varepsilon} (B/v)^{1/2+\varepsilon} + (M_2 B^\varepsilon)^{2\varepsilon} (B/v)^{(1+\kappa)/2+\varepsilon},$$

on breaking the \mathbf{c} -sum into dyadic intervals. This therefore completes the proof of the lemma, on redefining the choice of $\varepsilon > 0$. \square

Lemma 4.9. *We have $\Sigma_2 \ll B^{1+\varepsilon} M_2^{2n} / (v_1 v_2)$.*

Proof. Since $K_{M_2}(\overline{v_1 v_2} \mathbf{c})$ only depends on the value of \mathbf{c} modulo M_2 , we may break into residue classes modulo M_2 , concluding that

$$\begin{aligned} \Sigma_2 &\leq \sum_{u \ll B/(v_1 v_2)} \sum_{\mathbf{a} \bmod M_2} |K_{M_2}(\mathbf{a})|^2 \# \{ \mathbf{c} \in \mathbb{Z}^n : |\mathbf{c}| \leq M_2 B^\varepsilon, \mathbf{c} \equiv \mathbf{a} \bmod M_2 \} \\ &\ll \frac{B^{1+n\varepsilon}}{v_1 v_2} \sum_{\mathbf{a} \bmod M_2} |K_{M_2}(\mathbf{a})|^2. \end{aligned}$$

But the inner sum over \mathbf{a} is $M_2^n \# \Omega_{M_2} \leq M_2^{2n}$, by orthogonality of characters. The lemma follows on redefining ε . \square

Combining Lemma 4.8 and 4.9 in (4.10), we deduce that

$$\begin{aligned} E(B) &\ll B^{(n-1)/2+2\varepsilon} \sum_{(v_1, v_2) \in \mathcal{V}} \frac{v_1^{(n-1)/2} M_{12}^{n/2} M_2^{n/2+\varepsilon}}{v_2^{1/2}} \left(\frac{B}{v_1 v_2} \right)^{(1+\kappa)/4} \\ &\ll B^{n/2+(\kappa-1)/4+2\varepsilon} M^{n/2+\varepsilon} \sum_{(v_1, v_2) \in \mathcal{V}} \frac{1}{(v_1 v_2)^{(3+\kappa)/4}} \\ &\ll B^{n/2+(\kappa-1)/4+3\varepsilon} M^{n/2+2\varepsilon}, \end{aligned}$$

since $v_1 M_{12} M_2 = M$. This completes the proof of Theorem 4.1 on redefining the choice of $\varepsilon > 0$. \square

5. THE SELBERG SIEVE ON QUADRICS

In this section we prove Theorem 1.7 and its applications.

5.1. Proof of Theorem 1.7. Points of $X(\mathbb{Q})$ are represented by vectors $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}_{\text{prim}}^{n+1}$ such that $F(\mathbf{x}) = 0$, where $F \in \mathbb{Z}[x_0, \dots, x_n]$ is the quadratic form defining X . As in the previous section, for any $d \in \mathbb{N}$ we write $[\mathbf{x}]_d$ for the reduction of \mathbf{x} modulo d . Passing to the affine cone, we have

$$N(X, H, \Omega, B) \leq \# \left\{ \mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} : |\mathbf{x}| \leq B, F(\mathbf{x}) = 0, [\mathbf{x}]_{p^m} \in \widehat{\Omega}_{p^m} \text{ for all } p \right\}.$$

where $\widehat{\Omega}_{p^m} = \{ \mathbf{x} \in (\mathbb{Z}/p^m \mathbb{Z})^{n+1} : p \nmid \mathbf{x}, (x_0 : \dots : x_n) \in \Omega_{p^m} \}$ and $|\cdot|$ is the supremum norm on \mathbb{R}^{n+1} . We apply the Selberg sieve to estimate this.

Consider the function $\omega_0 : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$, given by

$$\omega_0(x) = \begin{cases} e^{-(1-x^2)^{-1}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Then ω_0 is infinitely differentiable and compactly supported on $[-1, 1]$. We work with the weight function $w : \mathbb{R}^{n+1} \rightarrow \mathbb{R}_{\geq 0}$, given by

$$w(\mathbf{x}) = \omega_0\left(\frac{5}{2}|\mathbf{A}\mathbf{x}| - 2\right),$$

where \mathbf{A} is the non-singular matrix defining F , with determinant Δ_F . It is clear that $w(\mathbf{x}) = 0$ unless $\frac{2}{5} \leq |\mathbf{A}\mathbf{x}| \leq \frac{6}{5}$. In particular w is supported on a region $\mathbf{x} \ll 1$, where we adhere to the convention that the implied constant in any estimate is allowed to depend on F . Moreover, $\nabla F(\mathbf{x}) \geq \frac{1}{5}$ throughout the support of w . It therefore follows that w belongs to the class of weight functions $\mathcal{C}_0^+(S)$ introduced in [12, §2 and §6], for an appropriate set of parameters S including n and the coefficients of the quadratic form F .

We have $|\mathbf{A}\mathbf{x}| \leq (n+1)\|\mathbf{A}\|B$ for any $\mathbf{x} \in \mathbb{Z}^{n+1}$ such that $|\mathbf{x}| \leq B$, where $\|\mathbf{A}\|$ is the maximum modulus of the coefficients of \mathbf{A} . Let $c = (n+1)\|\mathbf{A}\|$. We break the sum into dyadic intervals for $|\mathbf{A}\mathbf{x}|$, finding that

$$\begin{aligned} N(X, H, \Omega, B) &\leq \sum_{j=0}^{\infty} \# \left\{ \mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} : \begin{array}{l} 2^{-j-1}cB < |\mathbf{A}\mathbf{x}| \leq 2^{-j}cB \\ F(\mathbf{x}) = 0, [\mathbf{x}]_{p^m} \in \widehat{\Omega}_{p^m} \text{ for all } p \end{array} \right\} \\ &\ll \sum_{j=0}^{\infty} \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1}, F(\mathbf{x})=0 \\ [\mathbf{x}]_{p^m} \in \widehat{\Omega}_{p^m} \text{ for all } p}} w(2^j \mathbf{x} / (cB)). \end{aligned}$$

It will clearly suffice to show that

$$\sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1}, F(\mathbf{x})=0 \\ [\mathbf{x}]_{p^m} \in \widehat{\Omega}_{p^m} \text{ for all } p}} w(\mathbf{x}/B) \ll_{\varepsilon, X} \frac{B^{n-1}}{G(\xi)} + \xi^{m(n+1)+2+\varepsilon} B^{(n+1)/2+\varepsilon}, \quad (5.1)$$

for any $B, \xi \geq 1$, with $G(\xi)$ as in the statement of Theorem 1.7.

Let P denote the produce over distinct primes $p < \xi$ for which $\omega_p > 0$. For $n \in \mathbb{N}$ we define the finite sequence of non-negative numbers

$$a_n = \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^{n+1} \\ F(\mathbf{x})=0 \\ n(\mathbf{x})=n}} w(\mathbf{x}/B), \quad \text{where } n(\mathbf{x}) = \prod_{\substack{p|P \\ [\mathbf{x}]_{p^m} \in \widehat{\Omega}_{p^m}^c}} p.$$

The left hand side of (5.1) can be written $\sum_{(n,P)=1} a_n$. We seek to apply the Selberg sieve, in the form [14, Thm. 7.1], to estimate this quantity.

Let $d | P$. First note that $d | n(\mathbf{x})$ if and only if $[\mathbf{x}]_{p^m} \in \widehat{\Omega}_{p^m}^c$ for all $p | d$, for any \mathbf{x} appearing in the definition of a_n . Hence Theorem 4.1 implies that

$$\sum_{d|n} a_n = g(d)B^{n-1}\sigma_{\infty}(w) \prod_p \sigma_p + O_{\varepsilon, X}(d^{m(n+1)/2+\varepsilon/4} B^{(n+1)/2+\varepsilon}),$$

where

$$g(d) = \prod_{p|d} \left(1 - \frac{\#\widehat{\Omega}_{p^m}}{\#\widehat{X}(\mathbb{Z}/p^m\mathbb{Z})} \right) = \prod_{p|d} \omega_p,$$

in the notation of (1.2). In deriving this expression for $g(d)$, we have used Lemma 2.1 for $p \nmid 2\Delta_F$ to usher in the appearance of $\#\widehat{X}(\mathbb{Z}/p^m\mathbb{Z})$ in the denominator. Clearly $g(p) = \omega_p$ satisfies $0 < g(p) < 1$ for every $p \mid P$. It now follows from [14, Thm. 7.1 and Eq. (7.32)] that

$$\sum_{(n,P)=1} a_n \ll_{\varepsilon, X} \frac{B^{n-1}}{G(\xi)} + \sum_{d \leq \xi^2} \tau_3(d) d^{m(n+1)/2+\varepsilon/4} B^{(n+1)/2+\varepsilon},$$

Taking the trivial bound $\tau_3(d) \ll d^{\varepsilon/4}$ and summing over $d \leq \xi^2$, this therefore concludes the proof of (5.1) and so the proof of Theorem 1.7. \square

5.2. Proof of Theorem 1.8. Let $X \subset \mathbb{P}^n$ be a non-singular quadric hypersurface defined over \mathbb{Q} , with dimension $n-1 \geq 3$. Let $\Upsilon \subset X(\mathbb{Q})$ be a thin subset. To prove Theorem 1.8, it suffices to consider thin sets of type I and II (see §3.2).

We begin with the more difficult case of type II . By Lemma 3.8 there is a set of primes \mathcal{P} of positive natural density δ and a constant $c \in (0, 1)$, such that for each $p \in \mathcal{P}$ we have

$$\#\{\Upsilon \bmod p\} \leq cp^{n-1} + O_{\Upsilon}(p^{n-3/2}).$$

Taking $m = 1$ in (1.2), for such p we therefore have $\omega_p \geq 1 - c + O_{\Upsilon}(p^{-1/2})$. It follows that there exists $\eta < (1-c)/c$ such that

$$\frac{\omega_p}{1 - \omega_p} \geq \eta$$

for large enough $p \in \mathcal{P}$. Let \mathcal{P}° denote the set of such $p \in \mathcal{P}$. An application of Lemma 3.11 now yields

$$G(\xi) \geq \sum_{\substack{a \leq \xi \\ p|a \Rightarrow p \in \mathcal{P}^\circ}} \mu^2(a) \eta^{\omega(a)} \gg_{\Upsilon, X} \xi (\log \xi)^{\eta\delta-1} \gg_{\varepsilon, \Upsilon, X} \xi^{1-\varepsilon},$$

for any $\varepsilon > 0$. It therefore follows from Theorem 1.7 that

$$\#\{x \in \Upsilon(\mathbb{Q}) : H(x) \leq B\} \ll_{\varepsilon, \Upsilon, X} \xi^{-1+\varepsilon} B^{n-1} + \xi^{n+3+\varepsilon} B^{(n+1)/2+\varepsilon},$$

Balancing the terms by choosing $\xi = B^{\theta_n}$, with $\theta_n = \frac{n-3}{2(n+4)} = \frac{1}{2} - \frac{7}{2(n+4)}$, this is plainly satisfactory for Theorem 1.8.

Turning to thin sets of type I , we let $Z \subset X$ be a Zariski closed subset with $Z \neq X$. For any prime p , Lemma 3.8 implies that $\#Z(\mathbb{F}_p) \leq cp^{n-2}$, for some $c = c(Z) > 0$. Then $\omega_p \geq 1 - cp^{-1}$ and it follows that

$$\frac{\omega_p}{1 - \omega_p} \geq \frac{1 - cp^{-1}}{cp^{-1}} = \frac{p}{c} - 1.$$

A further application of Lemma 3.11 now implies that $G(\xi) \gg_{\varepsilon, \Upsilon} \xi^{2-\varepsilon}$ for all $\varepsilon > 0$. We complete the proof of Theorem 1.8 by arguing as above. \square

5.3. Proof of Theorem 1.10. Let $\pi : Y \rightarrow X$ be a dominant map with $X \subset \mathbb{P}^n$ a smooth quadric hypersurface of dimension at least 3, as in the statement of Theorem 1.10. Then

$$\begin{aligned} N(X, H, \pi, B) &\leq \#\{x \in X(\mathbb{Q}) : H(x) \leq B, x \in \pi(Y(\mathbb{Q}_p)) \forall p\} \\ &\leq \#\{x \in X(\mathbb{Q}) : H(x) \leq B, x \bmod p^2 \in \pi(Y(\mathbb{Z}_p)) \bmod p^2 \forall p\}. \end{aligned}$$

We now apply Theorem 1.7 with $\Omega_{p^2} = (\pi(Y(\mathbb{Z}_p)) \bmod p^2)$ to find that

$$N(X, H, \pi, B) \ll_{\varepsilon, X} \frac{B^{n-1}}{G(\xi)} + \xi^{2n+4+\varepsilon} B^{(n+1)/2+\varepsilon},$$

where

$$G(\xi) = \sum_{a \leq \xi} \mu^2(a) \prod_{p|a} \left(\frac{\omega_p}{1 - \omega_p} \right), \quad \omega_p = 1 - \frac{\#\pi(Y(\mathbb{Z}_p) \bmod p^2)}{\#X(\mathbb{Z}/p^2\mathbb{Z})}.$$

Taking ξ to be a small power of B , the result follows from Corollary 3.12. \square

5.4. Proof of Theorem 1.11. Let $Z \subset X$ be a divisor and let $r(Z)$ be the number of irreducible components of Z . We are led to apply Theorem 1.7 with $m = 1$ and $\Omega_p = X(\mathbb{F}_p) \setminus Z(\mathbb{F}_p)$ for $p > y$. Thus

$$\omega_p = 1 - \frac{\#X(\mathbb{F}_p) - \#Z(\mathbb{F}_p)}{\#X(\mathbb{F}_p)} = \frac{\#Z(\mathbb{F}_p)}{\#X(\mathbb{F}_p)}. \quad (5.2)$$

As $1 - \omega_p \leq 1$, we obtain

$$G(\xi) \geq \sum_{\substack{k < \xi \\ p|k \Rightarrow p > y}} \mu^2(k) \prod_{p|k} \omega_p.$$

Applying (3.11) and Lemma 3.11, we see that

$$\sum_{\substack{k < \xi \\ p|k \Rightarrow p > y}} k \mu^2(k) \prod_{p|k} \omega_p \asymp_y \frac{\xi}{\log \xi} \prod_{p < \xi} (1 + \omega_p) = \frac{\xi}{\log \xi} \prod_{p < \xi} \frac{1 + \omega_p^2}{1 - \omega_p}.$$

As ω_p is given by (5.2), an application of Lemma 3.13 shows that the product over primes is $\asymp (\log \xi)^{r(Z)}$. Using partial summation to remove k , we have $G(\xi) \gg (\log \xi)^{r(Z)}$. The result follows on taking ξ to be a small power of B . \square

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