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INTERMEDIATE $\beta$-SHIFTS OF FINITE TYPE

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Abstract. An aim of this article is to highlight dynamical differences between the greedy, and hence the lazy, $\beta$-shift (transformation) and an intermediate $\beta$-shift (transformation), for a fixed $\beta \in (1, 2)$. Specifically, a classification in terms of the kneading invariants of the linear maps $T_{\beta, \alpha}: x \mapsto \beta x + \alpha \mod 1$ for which the corresponding intermediate $\beta$-shift is of finite type is given. This characterisation is then employed to construct a class of pairs $(\beta, \alpha)$ such that the intermediate $\beta$-shift associated with $T_{\beta, \alpha}$ is a subshift of finite type and where the map $T_{\beta, \alpha}$ is not transitive. This is in contrast to the situation for the corresponding greedy and lazy $\beta$-shifts and $\beta$-transformations, in that these two properties do not hold.

1. Introduction, motivation and the main results

1.1. Introduction and motivation. For a given real number $\beta > 1$ and a real number $x \in [0, 1/(\beta - 1)]$ an infinite word $(\omega_n)_{n \in \mathbb{Z}}$ over the alphabet $\{0, 1\}$ is called a $\beta$-expansion of the point $x$ if

$$x = \sum_{k=1}^{\infty} \omega_k \beta^{-k}.$$ 

If $\beta$ is a natural number, then the $\beta$-expansions of a point $x$ correspond to the $\beta$-adic expansions of $x$. Moreover, in this case, almost all positive real numbers have a unique $\beta$-expansion. On the other hand, in [29] it has been shown that if $\beta$ is not a natural number, then, for Lebesgue almost all $x$, the cardinality of the set of $\beta$-expansions of $x$ is equal to the cardinality of the continuum.

The theory of $\beta$-expansions was initiated by Rényi [27] and Parry [24, 25]. Here an important link to symbolic dynamics is made. Indeed, through iterating the $\beta$-transformation $G_{\beta}: x \mapsto \beta x \mod 1$ and the $\beta$-transformation $L_{\beta}: x \mapsto \beta(x - 1) + 2 \mod 1$ one obtains subsets of $[0, 1]^\mathbb{Z}$ known as the greedy and (normalised) lazy $\beta$-shifts, respectively, where each point $\omega^+$ of the greedy $\beta$-shift is a $\beta$-expansion, and corresponds to a unique point in $[0, 1]$, and each point $\omega^-$ of the lazy $\beta$-shift is a $\beta$-expansion, and corresponds to a unique point in $[(2 - \beta)/(\beta - 1), 1/(\beta - 1)]$. (Note that in the case that $(2 - \beta)/(\beta - 1) \leq 1$, if $\omega^+$ and $\omega^-$ are $\beta$-expansions of the same point, then $\omega^+$ and $\omega^-$ do not necessarily have to be equal, see [19].) Through this connection we observe one of the most appealing features of the theory of $\beta$-expansions, namely that it links symbolic dynamics to number theory. In particular, one can ask questions of the form, for what class of numbers the greedy and lazy $\beta$-shift has given properties and vice versa. In fact, although the arithmetic, Diophantine and ergodic properties of the greedy and lazy $\beta$-shifts have been extensively studied, see [6, 8, 28] and references therein, there are many open problems of this form. Further, applications of this theory to the efficiency of analog-to-digital conversion have also been explored in [12]. Moreover, through understanding $\beta$-expansions of real numbers advances have been made in understanding Bernoulli convolutions, see [9, 10, 11] and reference therein.

There are many ways, other than using the greedy and lazy $\beta$-shift, to generate a $\beta$-expansion of a positive real number. For instance the intermediate $\beta$-shifts $\Omega_{\beta, \alpha}$, which
arise from the intermediate $\beta$-transformations $T_{\beta,\alpha}^\pm : [0, 1] \cup$, where $(\beta, \alpha) \in \Delta$,
\[ \Delta := \{(\beta, \alpha) \in \mathbb{R}^2 : \beta \in (1, 2) \text{ and } 0 \leq \alpha \leq 2 - \beta \} \]
and where the maps $T_{\beta,\alpha}^\pm$ are defined as follows. Letting $p = p_{\beta,\alpha} := (1 - \alpha)/\beta$ we set
\[ T_{\beta,\alpha}^+(p) := 0 \quad \text{and} \quad T_{\beta,\alpha}^+(x) := \beta x + \alpha \text{ mod } 1, \]
for all $x \in [0, 1] \setminus \{p\}$, see Figure 1. Similarly, we define
\[ T_{\beta,\alpha}^- (p) := 1 \quad \text{and} \quad T_{\beta,\alpha}^-(x) := \beta x + \alpha \text{ mod } 1, \]
for all $x \in [0, 1] \setminus \{p\}$. Indeed we have that the maps $T_{\beta,\alpha}^\pm$ are equal everywhere except at the point $p$ and that
\[ T_{\beta,\alpha}^- (x) = 1 - T_{\beta,\alpha}^+ (1 - x), \]
for all $x \in [0, 1]$. Observe that, when $\alpha = 0$, the maps $G_\beta$ and $T_{\beta,\alpha}^+$ coincide, and when $\alpha = 2 - \beta$, the maps $L_\beta$ and $T_{\beta,\alpha}^-$ coincide. We note that the maps defined above are sometimes also called linear Lorenz maps and arise naturally from the Poincaré maps of the geometric model of Lorenz differential equations, see for instance [13, 22, 30, 32]. Here we make the observation that for all $(\beta, \alpha) \in \Delta$, the symbolic space $\Omega_{\beta,\alpha}$ is always a subshift, meaning that it is invariant under the left shift map, see Corollary 2.

\begin{figure}[h]
\centering
\begin{subfigure}{0.3	extwidth}
\centering
\includegraphics[width=\textwidth]{Lbeta.png}
\caption{Plot of $L_\beta$.}
\end{subfigure}
\begin{subfigure}{0.3	extwidth}
\centering
\includegraphics[width=\textwidth]{Tbeta.png}
\caption{Plot of $T_{\beta,\alpha}^+$.}
\end{subfigure}
\begin{subfigure}{0.3	extwidth}
\centering
\includegraphics[width=\textwidth]{GBeta.png}
\caption{Plot of $G_\beta$.}
\end{subfigure}
\caption{Plot of $T_{\beta,\alpha}^+$ for $\beta = (\sqrt{5} + 1)/2$ and $\alpha = 1 - 0.474 \beta$, and the corresponding lazy and greedy $\beta$-transformations. (The height of the filled in circle determines the value of the map at the point of discontinuity.)}
\end{figure}

Here, the intermediate $\beta$-transformations and $\beta$-shifts are the main topic of study, and thus, to illustrate their importance we recall some of the known results in this area. Parry [26] proved that any topological mixing interval map with a single discontinuity is topologically conjugate to a map on the form $T_{\beta,\alpha}^\pm$ where $(\beta, \alpha) \in \Delta$. Moreover, in [15, 17] it is shown that a topologically expansive piecewise continuous map $T$ can be described up to topological conjugacy by the kneading invariants of the points of discontinuity of $T$. (In the case that $T = T_{\beta,\alpha}^\pm$, the kneading invariants of the point of discontinuity $p$ are precisely given by the points in the associated intermediate $\beta$-shift which are a $\beta$-expansion of $p + \alpha/(\beta - 1)$.) For such maps, assuming that there exists a single discontinuity, the authors of [3, 17] gave a simple condition on pairs of infinite words in the alphabet $\{0, 1\}$ which is satisfied if and only if that pair of sequences are the kneading invariants of the point of discontinuity of $T$.

Our main results, Theorems 1.1 and 1.3 and Corollary 1, contribute to the ongoing efforts in determining the dynamical properties of the intermediate $\beta$-shifts and examining whether these properties also hold for the counterpart greedy and lazy $\beta$-shifts. In particular, we demonstrate that it is possible to construct pairs $(\beta, \alpha) \in \Delta$ such that the associated intermediate $\beta$-shift is a subshift of finite type but for which the maps $T_{\beta,\alpha}^\pm$ are not transitive. In contrast to this, the corresponding greedy and lazy $\beta$-shifts are not a subshift of finite type.
(or even a sofic shift, namely a factor of a subshift of finite type) and, moreover, the maps $G_{\beta}$ and $L_{\beta}$ are transitive. Recall that an interval map $T : [0, 1] \cup$ is called transitive if and only if for all open subintervals $J$ of $[0, 1]$ there exists $m \in \mathbb{N}$ such that $\bigcup_{i=0}^{m} T^i(J) = (0, 1).

To prove the transitivity part of our result we will use the results of Palmer [23] and Glendinning [15]. Here they show that for any $1 < \beta < 2$ the maps $G_{\beta}$ and $L_{\beta}$ are transitive. In fact, they give a complete classification of the set of points $(\beta, \alpha) \in \Delta$ for which the maps $T_{\beta, \alpha}^k$ are not transitive. (For completeness we restate their classification in Section 4.)

Before formally stating our main results let us emphasise the importance of subshifts of finite type. These symbolic spaces give a simple representation of dynamical systems with a finite Markov partitions. There are many applications of subshifts of finite type, for instance in coding theory, transmission and storage of data or tilings. We refer the reader to [7, 18, 21, 30] and references therein for more on subshifts of finite type and their applications.

1.2. Main results. Recall, from for instance [8, Example 3.3.4], that the multinacci number of order $n \in \mathbb{N}$ is the real number $\gamma_n \in (1, 2)$ which is the unique positive real solution of the equation $1 = x^{-1} + x^{-2} + \cdots + x^{-n}$. The smallest multinacci number is the multinacci number of order 2 and is equal to the golden mean $(1 + \sqrt{5})/2$, and $\gamma_{n+1} > \gamma_n$, for all $n \in \mathbb{N}$. Further, for $n, k \in \mathbb{N}_0$ with $n \geq 2$, we define the algebraic integers $\beta_{n,k}$ and $\alpha_{n,k}$ by $\beta_{n,k}^2 = \gamma_n$ and $\alpha_{n,k} = 1 - \beta_{n,k}/2$. (Here and in the sequel, we let the symbol $\mathbb{N}_0$ denote the set of non-negative integers.)

**Theorem 1.1.** For all $n, k \in \mathbb{N}$ with $n \geq 2$, we have the following.

(i) The intermediate $\beta_{n,k}$-transformations $T_{\beta_{n,k}, \alpha_{n,k}}^k$ are not transitive.

(ii) The intermediate $\beta_{n,k}$-shift $\Omega_{\beta_{n,k}, \alpha_{n,k}}$ is a subshift of finite type.

(iii) The greedy and lazy $\beta_{n,k}$-transformations are transitive.

(iv) The greedy and lazy $\beta_{n,k}$-shifts, $\Omega_{\beta_{n,k}, 0}$ and $\Omega_{\beta_{n,k}, 2-\beta_{n,k}}$ respectively, are not sofic, and hence not a subshift of finite type.

Part (iii) is well known and holds for every greedy, and hence lazy, $\beta$-transformations, see for instance [15, 23]. It is included here both for completeness and to emphasise the dynamical differences which can occur between the greedy, and hence the lazy, $\beta$-transformation (shift) and an associated intermediate $\beta$-transformation (shift).

To verify Theorem 1.1(ii) we apply Theorem 1.3 given below. This latter result is a generalisation the following result of Parry [24]. Here and in the sequel, $\tau_{\beta, \alpha}^k(p)$ denotes the points in the associated intermediate $\beta$-shifts which are a $\beta$-expansion of $p + \alpha/\beta - 1,$ where $p = p_{\beta, \alpha}.$

**Theorem 1.2** ([24]). For $\beta \in (1, 2)$, we have that

(i) the greedy $\beta$-shift is a subshift of finite type if and only if $\tau_{\beta, 0}^k(p)$ is periodic and

(ii) the lazy $\beta$-shift is a subshift of finite type if and only if $\tau_{\beta, 2-\beta}^k(p)$ is periodic.

**Theorem 1.3.** Let $\beta \in (1, 2)$ and $\alpha \in (0, 2-\beta).$ The intermediate $\beta$-shift $\Omega_{\beta, \alpha}$ is a subshift of finite type if and only if both $\tau_{\beta, \alpha}^k(p)$ are periodic.

The following example demonstrates that it is possible to have that one of the sequences $\tau_{\beta, \alpha}^k(p)$ is periodic and that the other is not periodic.

**Example 1.** Let $\beta = \gamma_2$, $\alpha = 1/\beta^4$ and $p = (1-\alpha)/\beta = 1/\beta^3$. Clearly $\alpha \in (0, 2-\beta)$. Moreover, an elementary calculation will show that $\tau_{\beta, \alpha}^k(p)$ is the infinite periodic word with period $(0, 1, 1, 0)$ and $\tau_{\beta, 0}^k(p)$ is the concatenation of the finite word $(1, 0, 0)$ with the infinite periodic word with period $(1, 0)$.

For $\beta > 2$ and $\alpha \in [0, 1]$ the forward implications of Theorems 1.2 and 1.3 can be found in, for instance, [31, Theorem 6.3]. Further, for $\beta > 1$, necessary and sufficient conditions of
when the shift space $\Omega_{\beta,\alpha}$ is sofic shift is given in [18, Theorem 2.14]. This latter result in fact gives that if $\beta \in (1, 2)$ is a Pisot number, then along the fiber $\Delta(\beta) := \{ (b, \alpha) \in \Delta : b = \beta \}$ there exists a dense set of points $(b, \alpha)$ such that $\Omega_{b, \alpha}$ is a sofic shift.

This leads us to the final problem of studying how a greedy (or lazy) $\beta$-shift being a subshift of finite type is related to the corresponding intermediate $\beta$-shifts being a subshift of finite type. We already know that it is possible to find intermediate $\beta$-shifts of finite type in the fibres $\Delta(\beta)$ even though the corresponding greedy and lazy $\beta$-shifts are not subshift of finite type. Given this, a natural question to ask is, can one determine when no intermediate $\beta$-shift is a subshift of finite type, for a given fixed $\beta$. This is precisely what we address in the following result which is an almost immediate application of the characterisation provided in Theorems 1.2 and 1.3.

**Corollary 1.** If $\beta \in (1, 2)$ is not the solution of any polynomial of finite degree with coefficients in $[-1, 0, 1]$, then for all $\alpha \in [0, 2 - \beta]$ the intermediate $\beta$-shift $\Omega_{\beta, \alpha}$ is not a subshift of finite type.

We observe that the values $\beta_{\alpha, k}$, which are considered in Theorem 1.1, are indeed a solution of a polynomial with coefficients in the set $[-1, 0, 1]$ and thus do not satisfy the conditions of Corollary 1; this is verified in the proof of Theorem 1.1(iv).

1.3. **Outline.** In Section 2 we present basic definitions, preliminaries and auxiliary results required to prove Theorems 1.1 and 1.3 and Corollary 1. The proofs of Theorem 1.3 and Corollary 1 are presented in Section 3 and the proof of Theorem 1.1 is given in Section 4.

2. **Definitions and Auxiliary Results**

2.1. **Subshifts.** We equipped the space $[0, 1]^\mathbb{N}$ of infinite sequences with the topology induced by the word metric $d : [0, 1]^\mathbb{N} \times [0, 1]^\mathbb{N} \to \mathbb{R}$ which is given by

$$d(\omega, \nu) := \begin{cases} 0 & \text{if } \omega = \nu, \\ 2^{-|\omega \wedge \nu|+1} & \text{otherwise.} \end{cases}$$

Here $|\omega \wedge \nu| := \min\{ n \in \mathbb{N} : \omega_n \neq \nu_n \}$, for all $\omega = (\omega_1, \omega_2, \ldots), \nu = (\nu_1, \nu_2, \ldots) \in [0, 1]^\mathbb{N}$ with $\omega \neq \nu$. We let $\sigma$ denote the left-shift map which is defined on the set

$$[0, 1]^\mathbb{N} \cup \{ \emptyset \} \cup \bigcup_{n=1}^{\infty} [0, 1]^n,$$

of finite and infinite words in the alphabet $\{0, 1\}$ by $\sigma(\omega) := \emptyset$, if $\omega \in [0, 1] \cup \{ \emptyset \}$, and otherwise we set $\sigma(\omega_1, \omega_2, \ldots) := (\omega_2, \omega_3, \ldots)$. A *subshift* is any closed set $\Omega \subseteq [0, 1]^\mathbb{N}$ such that $\sigma(\Omega) \subseteq \Omega$.

Given a subshift $\Omega$ and $n \in \mathbb{N}$ set

$$\Omega_n := \{ (\omega_1, \ldots, \omega_n) \in [0, 1]^n : \text{there exists } (\xi_1, \ldots) \in \Omega \text{ with } (\xi_1, \ldots, \xi_n) = (\omega_1, \ldots, \omega_n) \}$$

and write $\Omega^* := \bigcup_{n=1}^{\infty} \Omega_n$ for the collection of all finite words.

A subshift $\Omega$ is called a *subshift of finite type* (SFT) if there exists an $M \in \mathbb{N}$ such that, for all $\omega = (\omega_1, \omega_2, \ldots, \omega_n), \xi = (\xi_1, \xi_2, \ldots, \xi_m) \in \Omega^*$ with $n, m \in \mathbb{N}$ and $n \geq M$,

$$(\omega_{n-M+1}, \omega_{n-M+2}, \ldots, \omega_n, \xi_1, \xi_2, \ldots, \xi_m) \in \Omega^*$$

if and only if

$$(\omega_1, \omega_2, \ldots, \omega_n, \xi_1, \xi_2, \ldots, \xi_m) \in \Omega^*.$$
Theorem 2.1. A shift space $\Omega$ is a SFT if and only if there exists a finite set $F$ of finite words in the alphabet $[0, 1]$ such that if $\xi \in F$, then $\sigma^n(\omega)|_{|\xi|} \neq \xi$, for all $\omega \in \Omega$.

The set $F$ in Theorem 2.1 is referred to as the set of forbidden words. Further, a subshift $\Omega$ is called sofic if and only if it is a factor of a SFT.

For $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in [0, 1]^n$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_m) \in \{0, 1\}^m$, denote by $\langle \nu, \xi \rangle$ the concatenation $(\nu_1, \nu_2, \ldots, \nu_n, \xi_1, \xi_2, \ldots, \xi_m) \in [0, 1]^{n+m}$, for $n, m \in \mathbb{N}$. We use the same notation when $\xi \in \{0, 1\}^\mathbb{N}$.

An infinite sequence $\omega = (\omega_1, \omega_2, \ldots) \in [0, 1]^\mathbb{N}$ is called periodic with period $n \in \mathbb{N}$, if

$$(\omega_1, \omega_2, \ldots, \omega_n) = (\omega_{(m-1)n+1}, \omega_{(m-1)n+2}, \ldots, \omega_{mn}),$$

for all $m \in \mathbb{N}$, and we write $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$. Similarly, $\omega = (\omega_1, \omega_2, \ldots) \in [0, 1]^\mathbb{N}$ is called eventually periodic if there exists an $n, k \in \mathbb{N}$ such that

$$(\omega_k, \omega_{k+1}, \ldots, \omega_{k+n}) = (\omega_{k+(m-1)n+1}, \omega_{k+(m-1)n+2}, \ldots, \omega_{k+mn}),$$

for all $m \in \mathbb{N}$, and we write $\omega = (\omega_1, \omega_2, \ldots, \omega_k, \omega_{k+1}, \ldots, \omega_{k+mn})$.

2.2. Intermediate $\beta$-shifts and expansions. We now give the formal definition of the intermediate $\beta$-shift. Throughout this section we let $(\beta, \alpha) \in \Delta$ be fixed and let $p$ denote the value $p_{\beta, \alpha} = (1 - \alpha) / \beta$.

For $\tau \in \{+, -\}$, the $T_{\beta, \alpha}^\tau$-expansion $T_{\beta, \alpha}^{\tau}(x)$, of a point $x$ with respect to $T_{\beta, \alpha}$, is defined to be the infinite word $\omega^{\tau} = (\omega_1^{\tau}, \omega_2^{\tau}, \ldots) \in [0, 1]^\mathbb{N}$, where

$$\omega_n^{\tau} := \begin{cases} 0, & \text{if } (T_{\beta, \alpha}^{-1})^{\tau-1}(x) \leq p, \\ 1, & \text{otherwise}, \end{cases}$$

and

$$\omega_n^{\tau} := \begin{cases} 0, & \text{if } (T_{\beta, \alpha}^{\tau})^{\tau-1}(x) < p, \\ 1, & \text{otherwise}. \end{cases}$$

for all $n \in \mathbb{N}$. We will denote the images of the unit interval under $T_{\beta, \alpha}^{\tau}$, by $\Omega_{\beta, \alpha}^{\tau}$, respectively, and write $\Omega_{\beta, \alpha}$ for the union $\Omega_{\beta, \alpha}^{+} \cup \Omega_{\beta, \alpha}^{-}$. The upper and lower kneading invariants of $T_{\beta, \alpha}^{\tau}$ are defined to be the infinite words $T_{\beta, \alpha}^{\tau}(p)$, respectively, and will turn out to be of great importance.

Remark 1. For ease of notation let $\omega^\tau = (\omega_1^\tau, \omega_2^\tau, \ldots)$ denote the infinite words $T_{\beta, \alpha}^{\tau}(p)$, respectively. By definition, $\omega_1^\tau = \omega_2^\tau = 0$ and $\omega_1^\tau = \omega_2^\tau = 1$. Furthermore, we have that $(\omega_k^\tau, \omega_{k+1}^\tau, \ldots) = (0, 0, 0, \ldots)$ if and only if $k = 2$ and $\alpha = 0$; and $(\omega_k^\tau, \omega_{k+1}^\tau, \ldots) = (1, 1, 1, \ldots)$ if and only if $k = 2$ and $\alpha = 2 - \beta$.

The connection between the maps $T_{\beta, \alpha}^\tau$ and the $\beta$-expansions of real numbers is given through the $T_{\beta, \alpha}^\tau$-expansions of a point and the projection map $\pi_{\beta, \alpha} : [0, 1)^\mathbb{N} \to [0, 1]$, which we will shortly define. The projection map $\pi_{\beta, \alpha}$ is linked to the underlying (overlapping) iterated function system (IFS), namely $\{([0, 1]^\mathbb{N}), f_\omega : x \mapsto x/\beta, f_1 : x \mapsto x/\beta + 1 - 1/\beta, \omega \in (1, 2)\}$. (We refer the reader to [14] for the definition of and further details on IFSs.) The projection map $\pi_{\beta, \alpha}$ is defined by

$$\pi_{\beta, \alpha}(\omega_1, \omega_2, \ldots) := (\beta - 1)^{-1} \left( \lim_{n \to \infty} f_{\omega_1} \circ \cdots \circ f_{\omega_n}([0, 1]) - \alpha \right) = \alpha(1 - \beta)^{-1} + \sum_{k=1}^{\infty} \omega_k \beta^{-k}.$$

An important property of the projection map is that the following diagram commutes.

$$
\begin{array}{ccc}
\Omega_{\beta, \alpha}^+ & \xrightarrow{\sigma} & \Omega_{\beta, \alpha}^- \\
\pi_{\beta, \alpha} \downarrow & & \downarrow \pi_{\beta, \alpha} \\
[0, 1] & \xrightarrow{T_{\beta, \alpha}} & [0, 1]
\end{array}
$$
Theorem 2.2 \([1, 2, 4, 17, 18]\)

This result is readily verifiable from the definitions of the maps involved and a sketch of the proof of this result can be found in \([4, \text{Section 5}]\). From this, we conclude that, for each \(x \in \left[\alpha / (\beta - 1), 1 + \alpha / (\beta - 1)\right]\), a \(\beta\)-expansion of \(x\) is the infinite word \(\tau_{\beta, \alpha}^+(x - \alpha (\beta - 1)^{-1})\).

2.3. The extended model. In order to aid us in our proof, we will consider the extended model given as follows. For \((\beta, \alpha) \in \Delta\), we let \(p = p_{\beta, \alpha} = (1 - \alpha) / \beta\) and we define \(\bar{T}_{\beta, \alpha}^\pm : [-\alpha / (\beta - 1), (1 - \alpha) / (\beta - 1)] \cup \beta, \alpha \) by

\[
\bar{T}_{\beta, \alpha}^+(x) = \begin{cases} 
\beta x + \alpha & \text{if } x < p, \\
\beta x + \alpha - 1 & \text{otherwise}
\end{cases}
\]

and
\[
\bar{T}_{\beta, \alpha}^-(x) = \begin{cases} 
\beta x + \alpha & \text{if } x \leq p, \\
\beta x + \alpha - 1 & \text{otherwise}
\end{cases}
\]

(see Figure 2). Note \(\bar{T}_{\beta, \alpha}^\pm \mid_{(0, 1)} = T_{\beta, \alpha}^\pm\) and, for all \(x \in (-\alpha / (\beta - 1), 0) \cup (1, (1 - \alpha) / (1 - \beta))\), there exists a minimal \(n = n(x) \in \mathbb{N}\) such that \(\bar{T}_{\beta, \alpha}^m(x) \in [0, 1]\), for all \(m \geq n\).

![Plot of \(\bar{T}_{\beta, \alpha}^\pm\) for \(\beta = (\sqrt{5} + 1) / 2\) and \(\alpha = 1 - 0.474 \beta\).](image)

Figure 2. Plot of \(\bar{T}_{\beta, \alpha}^\pm\) for \(\beta = (\sqrt{5} + 1) / 2\) and \(\alpha = 1 - 0.474 \beta\). (The height of the filled in circle determines the value of the map at the point of discontinuity.)

The \(\bar{T}_{\beta, \alpha}^\pm\)-expansions \(\bar{T}_{\beta, \alpha}^\pm(x)\) of points \(x \in [-\alpha / (\beta - 1), (1 - \alpha) / (\beta - 1)]\) is defined in the same way as for the maps \(T_{\beta, \alpha}^\pm\). Similarly, we let \(\bar{\Omega}_{\beta, \alpha}^\pm := \bar{T}_{\beta, \alpha}^\pm([-\alpha / (\beta - 1), (1 - \alpha) / (\beta - 1)])\) and we set \(\bar{\Omega}_{\beta, \alpha} = \bar{\Omega}_{\beta, \alpha}^+ \cup \bar{\Omega}_{\beta, \alpha}^-\). Further, the upper and lower kneading invariant of \(\bar{T}_{\beta, \alpha}^\pm\) are the infinite words \(\tau_{\beta, \alpha}^\pm(p)\), respectively.

2.4. Structure of \(\bar{\Omega}_{\beta, \alpha}^\pm\) and \(\bar{\Omega}_{\beta, \alpha}^\pm\). The following theorem, on the structure of the spaces \(\bar{\Omega}_{\beta, \alpha}^\pm\) and \(\bar{\Omega}_{\beta, \alpha}^\pm\), will play a crucial role in the proof of Theorem 1.3. A partial version of this result can be found in \([16, \text{Lemma 1}]\). To the best of our knowledge, this first appeared in \([17, \text{Theorem 1}]\), and later in \([1, 2, 4, 18]\). Moreover, a converse of Theorem 2.2 holds true and was first given in \([17, \text{Theorem 1}]\) and later in \([3, \text{Theorem 1}]\).

Theorem 2.2 \([1, 2, 4, 17, 18]\). For \((\beta, \alpha) \in \Delta\), the shift spaces \(\bar{\Omega}_{\beta, \alpha}^\pm\) and \(\bar{\Omega}_{\beta, \alpha}^\pm\) are completely determined by upper and lower kneading invariants of \(T_{\beta, \alpha}^\pm\) and \(T_{\beta, \alpha}^\pm\), respectively.

\[
\Omega_{\beta, \alpha}^+ = \left\{ \omega \in [0, 1)^3 : \tau_{\beta, \alpha}^+(0) \leq \sigma^n(\omega) < \tau_{\beta, \alpha}^+(p) \text{ or } \tau_{\beta, \alpha}^+(0) \leq \sigma^n(\omega) \leq \tau_{\beta, \alpha}^+(1) \forall n \geq 0 \right\}
\]

\[
\Omega_{\beta, \alpha}^- = \left\{ \omega \in [0, 1)^3 : \tau_{\beta, \alpha}^-(0) \leq \sigma^n(\omega) < \tau_{\beta, \alpha}^-(p) \text{ or } \tau_{\beta, \alpha}^-(0) \leq \sigma^n(\omega) \leq \tau_{\beta, \alpha}^-(1) \forall n \geq 0 \right\}
\]

\[
\tilde{\Omega}_{\beta, \alpha}^+ = \left\{ \omega \in [0, 1)^3 : \tilde{\tau}_{\beta, \alpha}^+(0) \leq \sigma^n(\omega) < \tilde{\tau}_{\beta, \alpha}^+(p) \text{ or } \tilde{\tau}_{\beta, \alpha}^+(0) \leq \sigma^n(\omega) \leq \tilde{\tau}_{\beta, \alpha}^+(1) \forall n \geq 0 \right\}
\]

\[
\tilde{\Omega}_{\beta, \alpha}^- = \left\{ \omega \in [0, 1)^3 : \tilde{\tau}_{\beta, \alpha}^-(0) \leq \sigma^n(\omega) < \tilde{\tau}_{\beta, \alpha}^-(p) \text{ or } \tilde{\tau}_{\beta, \alpha}^-(0) \leq \sigma^n(\omega) \leq \tilde{\tau}_{\beta, \alpha}^-(1) \forall n \geq 0 \right\}
\]
(Here the symbols $<, \leq, >$ and $\geq$ denote the lexicographic orderings on $[0,1]^{\mathbb{N}}$.) Moreover,
\[ \Omega_{\beta, \alpha} = \Omega_{\beta, \alpha}^{+} \cup \Omega_{\beta, \alpha}^{-} \quad \text{and} \quad \widetilde{\Omega}_{\beta, \alpha} = \widetilde{\Omega}_{\beta, \alpha}^{+} \cup \widetilde{\Omega}_{\beta, \alpha}^{-}. \tag{2} \]

**Remark 2.** By the commutativity of the diagram in (1), we have that $\tau_{\beta, \alpha}^{+}(0) = \sigma(\tau_{\beta, \alpha}^{-}(p))$ and $\tau_{\beta, \alpha}^{-}(1) = \sigma(\tau_{\beta, \alpha}^{+}(p))$. Additionally, from the definition of the $\tilde{T}_{\beta, \alpha}^{+}$-expansion of a point
\[ \tilde{T}_{\beta, \alpha}^{+}(\frac{\nu}{p-1}) = \tilde{T}_{\beta, \alpha}(\frac{\nu}{p-1}) = (\tilde{0}) \quad \text{and} \quad \tilde{T}_{\beta, \alpha}^{-}(\frac{\nu}{p-1}) = \tilde{T}_{\beta, \alpha}^{-}(\frac{\nu}{p-1}) = (\tilde{1}). \]
Further, since $\tilde{T}_{\beta, \alpha}^{+}([0,1]) = T_{\beta, \alpha}^{+}$, it follows that $\tau_{\beta, \alpha}^{+}(p) = \tau_{\beta, \alpha}^{-}(p)$ and thus, from now on we will write $\tau_{\beta, \alpha}^{+}(p)$ for the common value $\tau_{\beta, \alpha}^{+}(p) = \tau_{\beta, \alpha}^{-}(p)$ and $\tau_{\beta, \alpha}^{-}(p)$ for the common value $\tau_{\beta, \alpha}^{-}(p) = \tau_{\beta, \alpha}^{-}(p)$.

**Corollary 2.** For $(\beta, \alpha) \in \Delta$, we have that $\Omega_{\beta, \alpha}^{+}$, $\Omega_{\beta, \alpha}^{-}$, $\tilde{\Omega}_{\beta, \alpha}^{+}$ and $\tilde{\Omega}_{\beta, \alpha}$ are subshifts. Moreover, $\sigma(\Omega_{\beta, \alpha}^{+}) \subseteq \Omega_{\beta, \alpha}^{+}$, $\sigma(\tilde{\Omega}_{\beta, \alpha}^{+}) \subseteq \tilde{\Omega}_{\beta, \alpha}$, $\sigma(\Omega_{\beta, \alpha}^{-}) = \Omega_{\beta, \alpha}$ and $\sigma(\tilde{\Omega}_{\beta, \alpha}) = \tilde{\Omega}_{\beta, \alpha}$.

**Proof.** This is a direct consequence of Theorem 2.2 and the commutativity of the diagram given in (1).

As one can see the code space structure of $\tilde{\Omega}_{\beta, \alpha}^{+}$ is simpler than the code space structure of $\Omega_{\beta, \alpha}^{+}$. In fact, to prove Theorem 1.3, we will show that it is necessary and sufficient to show that $\tilde{\Omega}_{\beta, \alpha}$ is a SFT.

### 2.5. Periodicity and zero/one-full words

We now give a sufficient condition for the kneading invariants of $\tilde{T}_{\beta, \alpha}^{+}$ (and hence, by Remark 2, of $T_{\beta, \alpha}^{+}$) to be periodic. For this, we will require the following notation and definition. For a given subset $E$ of $\mathbb{N}$, we write $|E|$ for the cardinality of $E$.

**Definition 2.3.** Let $(\beta, \alpha) \in \Delta$. An infinite word $\omega = (\omega_1, \omega_2, \ldots) \in [0,1]^{\mathbb{N}}$ is called zero-full with respect to $(\beta, \alpha)$ if
\[ N_0(\omega) := \left| \left\{ k > 2 : (\omega_1, \omega_2, \ldots, \omega_{k-1}, 1) \in \tilde{\Omega}_{\beta, \alpha}^{-} \text{ and } \omega_k = 0 \right\} \right|, \]
is finite, and is called one-full with respect to $(\beta, \alpha)$ if
\[ N_1(\omega) := \left| \left\{ k > 2 : (\omega_1, \omega_2, \ldots, \omega_{k-1}, 0) \in \Omega_{\beta, \alpha}^{+} \text{ and } \omega_k = 1 \right\} \right|, \]
is finite.

Before, stating our next result we remark that $p_{\beta, \alpha} = 1 - 1/\beta$ if and only if $\alpha = 2 - \beta$ and $p_{\beta, \alpha} = 1/\beta$ if and only if $\alpha = 0$.

**Lemma 2.4.** Let $(\beta, \alpha) \in \Delta$ and set $p = p_{\beta, \alpha}$.

(i) If $\tau_{\beta, \alpha}^{+}(p)$ is not periodic and $\alpha \neq 2 - \beta$, then $\sigma(\tau_{\beta, \alpha}^{-}(p))$ is one-full.

(ii) If $\tau_{\beta, \alpha}^{-}(p)$ is not periodic and $\alpha \neq 0$, then $\sigma(\tau_{\beta, \alpha}^{-}(p))$ is zero-full.

**Proof.** As the proofs for (i) and (ii) are essentially the same, we only include a proof of (ii). For ease of notation let $\nu = (\nu_1, \nu_2, \ldots)$ denote the infinite word $\sigma(\tau_{\beta, \alpha}^{-}(p))$ and note that since $\tau_{\beta, \alpha}^{-}(p)$ is not periodic, $\sigma(\tau_{\beta, \alpha}^{-}(p)) \neq \tau_{\beta, \alpha}^{-}(p)$, for all $k \in \mathbb{N}$.

Define the non-constant monotonic sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers as follows. Let $n_1 \in \mathbb{N}$ be the least natural number such that $n_{n_1 - 1} = 1$ and $n_1 = 0$ and for all $k > 1$, let $n_k \in \mathbb{N}$ be such that
\[ \sigma^{n_k}(\nu)_{n_k - n_{k-1} - 1} = \tau_{\beta, \alpha}^{-}(p)_{n_k - n_{k-1}} \quad \text{and} \quad \sigma^{n_k}(\nu)_{n_k - n_{k-1} - 1} = \tau_{\beta, \alpha}^{-}(p)_{n_k - n_{k-1} + 1}. \tag{3} \]
Such a sequence exists by Remark 1 and because $\tau_{\beta, \alpha}^{-}(p)$ is assumed to be not periodic. We will now show that $n_k = 0$ and that
\[ (\nu_1, \nu_2, \ldots, \nu_{n_2 - 1}, 1) \in \tilde{\Omega}_{\beta, \alpha}^{-}. \]
for all integers $k > 1$. By the orderings given in (3) it immediately follows that $ν_n = 0$. In order to conclude the proof, since $ν ∈ Ω_{β,α}^−$, and since $τ_{β,α}^−(p)$ is not periodic, by Theorem 2.2, it is sufficient to show, for all integers $k > 1$ and $m ∈ \{0, 1, 2, \ldots, n_k - 2, n_k - 1\}$, that

$$
σ^m(θ) \begin{cases} ≤ τ_{β,α}^−(p) & \text{if } ν_{m+1} = 0, \\
> τ_{β,α}^−(p) & \text{otherwise},
\end{cases}
$$

where $θ := (ν_1, ν_2, \ldots, ν_{n_k-1}, ν_{n_k-n_k-1+1}, \ldots, ν_{n_k-1})$.

Suppose that $l ∈ \{1, 2, \ldots, k\}$ and $m ∈ \{n_l-1, n_l-1 + 1, \ldots, n_l - 2\}$, where $n_0 := 0$. If $ν_{m+1} = 1$, then

$$
σ^m(θ) = (ν_{m+1}, ν_{m+2}, \ldots, ν_{n_l-1}, ν_{n_l-1}, ν_{n_l-1}, 1, ν_{n_k-n_k-1+1}, ν_{n_k-n_k-1+2}, \ldots)
\begin{cases}
> (ν_{m+1}, ν_{m+2}, \ldots, ν_{n_l-1}, ν_{n_l-1}, ν_{n_l-1}, 0, ν_{n_k+1}, ν_{n_l+2}, \ldots)
\end{cases}
= σ^m(ν)
\begin{cases}
> τ_{β,α}^−(p).
\end{cases}
$$

Now suppose $ν_{m+1} = 0$. By (3), we have that $(ν_{m-1}, ν_{n_l-1}+1, \ldots, ν_{n_l-1}, 1) = τ_{β,α}^−(p)|_{n_l-1+1}$, and so,

$$
σ^{m-n_l+1}(τ_{β,α}^−(p)) = (ν_{m+1}, ν_{m+2}, \ldots, ν_{n_l-1}, ν_{n_l-1}, ν_{n_l-1}+1, ν_{n_k-n_k-1+2}, \ldots).
$$

Let us first consider the case $l = k$. Since by assumption $ν_{m+1} = 0$, and since, by Theorem 2.2, we have that $σ^{m-n_l+1}(τ_{β,α}^−(p)) ≤ τ_{β,α}^−(p)$, it follows that

$$
σ^m(θ) = (ν_{m+1}, ν_{m+2}, \ldots, ν_{n_l-1}, ν_{n_l-1}, ν_{n_l-1}+1, ν_{n_k-n_k-1+2}, \ldots) ≤ τ_{β,α}^−(p).
$$

Therefore, in this case, namely $l = k$, the result follows. Now let us assume that $l < k$. In this case we have that

$$
σ^m(θ) = (ν_{m+1}, ν_{m+2}, \ldots, ν_{n_l-1}, ν_{n_l-1}, ν_{n_l-1}, 1, ν_{n_k-n_k-1+1}, ν_{n_k-n_k-1+2}, \ldots)
\begin{cases}
≤ (ν_{m+1}, ν_{m+2}, \ldots, ν_{n_l-1}, ν_{n_l-1}, ν_{n_l-1}+1, ν_{n_k-n_k-1+2}, \ldots)
\end{cases}
= τ_{β,α}^−(p).
$$

If $l ∈ \{1, 2, \ldots, k-1\}$ and $m = n_l - 1$, then by the orderings given in (3),

$$
σ^m(θ) = (ν_{m-1}, \ldots, ν_{n_l-1}, ν_{n_l-1}, 1, ν_{n_k-n_k-1+1}, ν_{n_k-n_k-1+2}, \ldots) ≤ τ_{β,α}^−(p).
$$

If $m = n_l - 1$, then by the orderings given in (3),

$$
σ^m(θ) = (\underbrace{ν_{n_l-1}, \ldots, ν_{n_l-1}, ν_{n_k-n_k-1+1}, ν_{n_k-n_k-1+2}, \ldots}_{τ_{β,α}^−(p)|_{n_l-1+1}}) = τ_{β,α}^−(p).
$$

Finally, if $m = n_k - 1$, then either $ν_{n_k-n_k-1+1} = 1$, in which case the result follows from Remark 1 or else $ν_{n_k-n_k-1+1} = 0$ in which case

$$
σ^m(θ) = (1, ν_{n_k-n_k-1+1}, ν_{n_k-n_k-1}+2, \ldots) = σ^{n_k-n_k-1}(τ_{β,α}^−(p)) > τ_{β,α}^+(p).
$$

This concludes the proof. □

2.6. Symmetric intermediate $β$-shifts. The following remarks and result on symmetric intermediate $β$-shifts will assist us in shortening the proof of Theorem 1.3 and the proof of Theorem 1.1(ii). However, before doing this, let us recall that two maps $R : X ⊃ Y ⊃ S : Y ⊃ Y$ defined on compact metric spaces are called topologically conjugate if there exists a homeomorphism $h : X → Y$ such that $S ∘ h = h ∘ R$.

The maps $T_{β,α}$ and $T_{β,2-α-β}$ are topologically conjugate and the maps $T_{β,α}^−$ and $T_{β,2-α-β}^+$ are topologically conjugate, where the conjugating map is given by $h : x → 1 - x$. This is immediate from the definition of the maps involved. Similarly, the maps $T_{β,α}^+$ and $T_{β,2-α-β}^−$ are topologically conjugate and the maps $T_{β,α}^−$ and $T_{β,2-α-β}^+$ are topologically conjugate,
where the conjugating map is also \( h \). When lifted to \([0,1]^{\mathbb{N}}\) the mapping \( h \) becomes the symmetric map \( *: [0,1]^{\mathbb{N}} \cup \) defined by,
\[
*(\omega_1, \omega_2, \ldots) = (\omega_1 + 1 \mod 2, \omega_2 + 1 \mod 2, \ldots),
\]
for all \( \omega \in [0,1]^{\mathbb{N}} \). Thus, \( \sigma|_{\Omega^h_{\beta,\alpha}} \) and \( \sigma|_{\Omega^{*\beta}_{\beta,\alpha} - \beta} \) are topologically conjugate, \( \sigma|_{\Omega^{*\beta}_{\beta,\alpha}} \) and \( \sigma|_{\Omega^{\beta}_{\beta,\alpha} - \beta} \) are topologically conjugate, \( \sigma|_{\Omega^{+\beta}_{\beta,\alpha}} \) and \( \sigma|_{\Omega^{*\beta}_{\beta,\alpha} + \beta} \) are topologically conjugate and \( \sigma|_{\Omega^{-\beta}_{\beta,\alpha}} \) and \( \sigma|_{\Omega^{-\beta}_{\beta,\alpha}} \) are topologically conjugate. In all these four cases the conjugating map is the symmetric map \( * \).

**Theorem 2.5.** For all \((\beta, \alpha) \in \Delta\) we have that \( \Omega^{\pm}_{\beta,\alpha} = \Omega^{\mp}_{\beta^2 - \alpha} \) and \( \Omega^{\pm}_{\beta,\alpha} = \Omega^{\mp}_{\beta^2 - \alpha} \). Moreover, given \( \beta \in (1,2) \), there exists a unique point \( \alpha \in [0,2 - \beta] \), namely \( \alpha = 1 - \beta/2 \), such that \( \tau_{\beta,\alpha}^\pm(p) = *\left(\tau_{\beta,\alpha}^\pm(p)\right) \), where \( p = p_{\beta,\alpha} = (1-\alpha)/\beta \).

**Proof.** The first part of the result follows immediately from the above comments and Theorem 2.2. The second part of the result follows from an application of [2, Theorem 2] together with the fact that the maps \( T_{\beta,\alpha}^\pm \) are topologically conjugate to a sub-class of maps considered in [2], where the conjugating map \( h_{\beta,\alpha}: [0,1] \to [-\alpha/(\beta - 1), (1-\alpha)/(\beta - 1)] \) is given by \( h_{\beta,\alpha}(x) = (x - \alpha)/(\beta - 1) \).

\[\square\]

3. PROOFS OF THEOREM 1.3 AND COROLLARY 1

Let us first prove an analogous result for the extended model \( \hat{\Omega}_{\beta,\alpha} \). Also, throughout this section, for a given \((\beta, \alpha) \in \Delta\), we set \( p = p_{\beta,\alpha} \).

**Proposition 1.** Let \( \beta \in (1,2) \).

(i) If \( \alpha \in (0,2 - \beta) \), then \( \Omega^-_{\beta,\alpha} \) is a SFT if and only if both \( \tau_{\beta,\alpha}^\pm(p) \) are periodic.

(ii) If \( \alpha = 2 - \beta \), then \( \Omega^0_{\beta,\alpha} \) is a SFT if and only if \( \tau_{\beta,\alpha}^\pm(p) \) is periodic.

(iii) If \( \alpha = 0 \), then \( \Omega^0_{\beta,\alpha} \) is a SFT if and only if \( \tau_{\beta,\alpha}^\pm(p) \) is periodic.

**Proof of Proposition 1(i).** Assume by way of contradiction that \( \Omega^-_{\beta,\alpha} \) is a SFT, but that \( \tau_{\beta,\alpha}^\pm(p) \) is not periodic. Then by Lemma 2.4 the infinite word \( v = \sigma(\tau_{\beta,\alpha}^\pm(p)) \) is zero-full, namely
\[
N_0(v) = \left\lvert \left\{ k > 2 : (v_1, v_2, \ldots, v_{k-1}, 1) \in \hat{\Omega}^-_{\beta,\alpha}; k \text{ and } v_k = 0 \right\} \right\rvert
\]
is not finite. Thus, there exists a non-constant monotonic sequence \((n_k)_{k \in \mathbb{N}}\) of natural numbers such that
\[
(v_1, v_2, \ldots, v_{n_k-1}) \in \hat{\Omega}^-_{\beta,\alpha}^{n_k} \quad \text{and} \quad (0, v_1, v_2, \ldots, v_{n_k-1}, 0) = \tau_{\beta,\alpha}^\pm(p)_{n_k+1}.
\]
This together with Corollary 2 implies that \( \hat{\Omega}_{\beta,\alpha} \) is not a SFT, contradicting our assumption.

The statement that if \( \Omega^0_{\beta,\alpha} \) is a SFT, then \( \tau_{\beta,\alpha}^\pm(p) \) is periodic, follows in an identical manner to above, where we use one-fullness instead of zero-fullness.

We will now prove the converse statement: for \((\beta, \alpha) \in \Delta\), if both the kneading invariants of \( T_{\beta,\alpha}^\pm \) are periodic, then \( \Omega^0_{\beta,\alpha} \) is a SFT. Without loss of generality we may assume that both the kneading invariants of \( T_{\beta,\alpha}^\pm \) have the same period length \( n \). This together with Corollary 2 implies it is sufficient to prove that for all \( \omega = (\omega_1, \omega_2, \ldots, \omega_k) \), \( \xi = (\xi_1, \xi_2, \ldots, \xi_m) \in \hat{\Omega}_{\beta,\alpha} \) with \( k > n \), if
\[
(\omega_{k-n+1}, \omega_{k-n+2}, \ldots, \omega_k, \xi_1, \xi_2, \ldots, \xi_m) \in \hat{\Omega}_{\beta,\alpha},
\]
then
\[
(\omega_1, \omega_2, \ldots, \omega_k, \xi_1, \xi_2, \ldots, \xi_m) \in \hat{\Omega}_{\beta,\alpha}.
\]
If the inclusion in (4) holds, then there exists at least one \( \eta = (\eta_1, \eta_2, \eta_1 \ldots) \in \{0, 1\}^\mathbb{N} \) such that
\[
(\omega_{k-n+1}, \omega_{k-n+2}, \ldots, \omega_k, \xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \ldots) \in \Omega_{\beta, \alpha}.
\]
Fix such an \( \eta \) and consider the infinite word \( \theta := (\omega_1, \omega_2, \ldots, \omega_k, \xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \ldots) \).

We claim that \( \theta \) belongs to the space \( \Omega_{\beta, \alpha} \). For suppose not, then, by Theorem 2.2, there exists \( l \) belonging to the set \( \{0, 1, \ldots, k - n - 1\} \) such that
\[
\tau^-_{\beta, \alpha}(p) < \alpha^m(\theta) < \tau^+_{\beta, \alpha}(p).
\]
By the definition of the lexicographic ordering and Corollary 2, since both \( \tau^\pm_{\beta, \alpha}(p) \) are periodic with period length \( n \) and \( \omega = (\omega_1, \omega_2, \ldots, \omega_k) \in \Omega_{\beta, \alpha}^\mathbb{N} \), there exists \( t \in \mathbb{N} \) with \( l + (t - 1)n < k \leq l + tn \) and if \( m < l + tn - k \), then
\[
\tau^-_{\beta, \alpha}(p)|_n < (\omega_{t+(t-1)n+1}, \omega_{t+(t-1)n+2}, \ldots, \omega_k, \xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \ldots, \eta_{l+tn-k}) < \tau^+_{\beta, \alpha}(p)|_n.
\]
In particular, as \( \beta, \alpha \) is a SFT if and only if there exists a set of forbidden word \( F \) such that if \( \xi \in F \), then \( \alpha^m(\omega)|_n \not\prec \xi \), for all \( \omega \in \Omega \).

**Proof of Proposition 1(ii).** The forward direction follows in the same way as in the proof of the forward direction of Proposition 1(i). Noting that, \( p = 1 - 1/\beta \), since \( \beta = 2 - \beta \), and thus \( \tau^\pm_{\beta, \alpha}(p) = (1, 0, 0, \ldots) \). The reverse direction follows analogously to the proof of the reverse direction of Proposition 1(i), except we set \( n \) to be the length of the period of \( \tau^-_{\beta, \alpha}(p) \). \( \square \)

**Proof of Proposition 1(iii).** This follows from Proposition 1(ii) and Theorem 2.5. \( \square \)

We now show in the following proposition, Proposition 2, that \( \Omega_{\beta, \alpha}^\mathbb{N} \) is a SFT if and only if \( \Omega_{\beta, \alpha}^\mathbb{N} \) is a SFT. This result together with Proposition 1 completes the proof of Theorem 1.3. In order to prove the backwards implication of Proposition 2, we will use Theorem 2.1. Namely that a subshift \( \Omega \) is a SFT if and only if there exists a set of forbidden word \( F \) such that if \( \xi \in F \), then \( \alpha^m(\omega)|_n \not\prec \xi \), for all \( \omega \in \Omega \).

**Proposition 2.** For \( (\beta, \alpha) \in \Delta \) the subshift \( \Omega_{\beta, \alpha} \) is a SFT if and only if the subshift \( \Omega_{\beta, \alpha} \) is a SFT.

**Proof.** We first prove the forward direction: if the shift space \( \Omega_{\beta, \alpha} \) is a SFT, then \( \Omega_{\beta, \alpha} \) is a SFT. We proceed by way of contradiction. Suppose that \( \Omega_{\beta, \alpha} \) is a SFT, but \( \Omega_{\beta, \alpha} \) is not a SFT. Proposition 1 then implies that \( \tau^\pm_{\beta, \alpha}(p) \) are both periodic and we may assume, without loss of generality, that they have the same period length \( m \). Moreover, by Corollary 2, since, by assumption, \( \Omega_{\beta, \alpha} \) is not a SFT, there exist \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \), \( \xi = (\xi_1, \xi_2, \ldots, \xi_k) \in \Omega_{\beta, \alpha}^\mathbb{N} \), with \( n, k \in \mathbb{N} \) and \( n \geq m \) such that
\[
(\omega_{n-m+1}, \omega_{n-m+2}, \ldots, \omega_n, \xi_1, \xi_2, \ldots, \xi_k) \in \Omega_{\beta, \alpha}^\mathbb{N}
\]
but \( (\omega_1, \omega_2, \ldots, \omega_n, \xi_1, \xi_2, \ldots, \xi_k) \not\in \Omega_{\beta, \alpha}^\mathbb{N} \).

In particular, as \( \Omega_{\beta, \alpha} \) is a SFT, Theorem 2.2 implies that there exists an \( l \in \{0, 1, \ldots, n-1\} \) such that either
\[
(\omega_{l+1}, \omega_{l+2}, \ldots, \omega_n, \xi_1, \xi_2, \ldots, \xi_k) < \tau^+_{\beta, \alpha}(0)|(m+n-l) = \alpha(\tau^+_{\beta, \alpha}(p))|_{k+n-l}
\]
or
\[
(\omega_{l+1}, \omega_{l+2}, \ldots, \omega_n, \xi_1, \xi_2, \ldots, \xi_k) > \tau^-_{\beta, \alpha}(1)|(m+n-l) = \alpha(\tau^-_{\beta, \alpha}(p))|_{k+n-l}
\]
However, since \( \omega, \xi \in \Omega_{\beta, \alpha}^\mathbb{N} \) and since \( \tau^\pm_{\beta, \alpha}(p) \) are both periodic with period length \( m \), it follows that \( l + 1 \geq n - m \) contradicting the fact that \( (\omega_{n-m+1}, \omega_{n-m+2}, \ldots, \omega_n, \xi_1, \xi_2, \ldots, \xi_k) \in \Omega_{\beta, \alpha}^\mathbb{N} \).
We will now show the converse: if the shift space $\Omega_{\beta,\alpha}$ is a SFT, then $\Omega_{\beta,\alpha}$ is a SFT. Let $F$ denote the set of forbidden words of $\Omega_{\beta,\alpha}$, and assume that $F$ is the smallest such set. Then for each $\xi \in F$ either
\[ \xi < \tau_{\beta,\alpha}(0)_{|\xi|}, \quad \xi > \tau_{\beta,\alpha}(1)_{|\xi|}, \quad \text{or} \quad \tau_{\beta,\alpha}(p)_{|\xi|} < \xi \leq \tau_{\beta,\alpha}(p)_{|\xi|}. \]
We claim that the set
\[ Y := \{ \xi \in F : \tau_{\beta,\alpha}(p)_{|\xi|} < \xi \leq \tau_{\beta,\alpha}(p)_{|\xi|} \} \]
is a finite set of forbidden words for $\Omega_{\beta,\alpha}$. For if not, this would contradict Theorem 2.2. \hfill \Box

**Proof of Theorem 1.3.** This is an immediate consequence of Propositions 1 and 2. \hfill \Box

Let us conclude this section with the proof of Corollary 1.

**Proof of Corollary 1.** By construction $\pi_{\beta,\alpha} \circ \tau_{\beta,\alpha}^+$ is the identity function on $[0, 1]$ and so $\beta$ is a root of the polynomial
\[ \pi_{\beta,\alpha}(\tau_{\beta,\alpha}^+(p)) - \pi_{\beta,\alpha}(\tau_{\beta,\alpha}^-(p)) - \pi_{\beta,\alpha}(\tau_{\beta,\alpha}(p)) + \pi_{\beta,\alpha}(\tau_{\beta,\alpha}(p)). \]
By the definition of the projection map $\pi_{\beta,\alpha}$, this latter polynomial is independent of $\alpha$ and all of its coefficients belong to the set $\{-1, 0, 1\}$. An application of Theorem 1.2 and 1.3 completes the proof. \hfill \Box

4. **Proof of Theorem 1.1**

Having proved Theorem 1.3, we are now equipped to prove Theorem 1.1. Before setting out to prove Theorem 1.1, let us recall the result of Palmer and Glendinning on the classification of the point $(\beta, \alpha) \in \Delta$ such that $T_{\beta,\alpha}^+$, and hence $T_{\beta,\alpha}^+$, is transitive.

**Definition 4.1.** Suppose that $1 < k < n$ are natural numbers such that $\gcd(k, n) = 1$. Let $s, m \in \mathbb{N}$ be such that $0 \leq s < k$ and $n = mk + s$. For $1 \leq j \leq s$ define $V_j$ and $r_j$ by $jk = V_j s + r_j$, where $r_j, V_j \in \mathbb{N}$ and $0 \leq r_j < s$. Also, define $h_j$ inductively via the formula $V_j = h_1 + h_2 + \cdots + h_j$. We define the set $D_{k,n}$ to be the set of points $(\beta, \alpha) \in \Delta$, such that $1 < \beta^n \leq 2$ and
\[ \frac{1 + \beta(\sum_{j=1}^{s} W_j - 1)}{\beta(\beta^n + \cdots + 1)} \leq \alpha \leq \frac{\beta(\sum_{j=1}^{s} W_j) - \beta^{s+1} + \beta^n + \beta - 1}{\beta(\beta^n + \cdots + 1)}. \]
Here,
\[ W_1 := \sum_{i=1}^{V_1} \beta^{(i-1)(m+s-1)} \quad \text{and} \quad W_j := \sum_{i=1}^{h_j} \beta^{(V_j-1)(m+s-1)}, \]
for $2 \leq j \leq s$. Further, for each natural number $n > 1$, we define the set $D_{k,n}$ to be the set of points $(\beta, \alpha) \in \Delta$ such that $1 < \beta^n \leq 2$ and
\[ \frac{1}{\beta(\beta^n + \cdots + 1)} \leq \alpha \leq \frac{-\beta^{s+1} + \beta^n + 2\beta - 1}{\beta(\beta^n + \cdots + 1)}. \]
(See Figure 3 for an illustration of the regions $D_{k,n}$.)

**Theorem 4.2 ([123] and [15, Proposition 2]).** If $(\beta, \alpha) \in D_{k,n}$ for some $1 \leq k < n$ with $\gcd(k, n) = 1$, then $T_{\beta,\alpha}^+$, and hence $T_{\beta,\alpha}^+$, is not transitive.

In order to prove Theorem 1.1 we will also use the following notation. For $n \in \mathbb{N}$ set
\[ \xi_{-\beta,\alpha} = (0, 1, 1, \ldots, 1) \]
and, for $k \geq 1$, set
\[ \xi_{\alpha, k} = (\xi_{\alpha, k-1} \ast (\xi_{\alpha, k-1} \ast (\xi_{\alpha, k-1} \ast (\xi_{\alpha, k-1} \ast (\xi_{\alpha, k-1} \ast \cdots \ast (\xi_{\alpha, k-1} \ast \xi_{\alpha, k-1})_{|\xi_{\alpha, k-1}|})_{|\xi_{\alpha, k-1}|})_{|\xi_{\alpha, k-1}|})_{|\xi_{\alpha, k-1}|})_{|\xi_{\alpha, k-1}|})_{|\xi_{\alpha, k-1}|})_{|\xi_{\alpha, k-1}|})_{|\xi_{\alpha, k-1}|})_{|\xi_{\alpha, k-1}|}). \]
Further, for \( n, k \in \mathbb{N} \), we set \( \xi_{n,k}^\alpha := \ast(\xi_{n,k}^\beta) \). Note that, \( \xi_{n,k}^\alpha |_{2^l} < \xi_{n,k}^\beta |_{2^l} \), for all \( n, k \in \mathbb{N} \). We also make the observation that if \( k \geq 2 \) and if \( l \in \{2,3,\ldots,k\} \), then we have the following equalities.

\[
\begin{align*}
\xi_{n,k}^\alpha |_{2^{l-1,2}} &= (\xi_{n,k}^\alpha |_{2^{l-1}}, \ast(\xi_{n,k}^\alpha |_{2^{l-1}}), \ast(\xi_{n,k}^\alpha |_{2^{l-1}}, \xi_{n,k}^\alpha |_{2^{l-1}})) \\
\xi_{n,k}^\beta |_{2^{l-1,2}} &= (\ast(\xi_{n,k}^\beta |_{2^{l-1}}, \xi_{n,k}^\beta |_{2^{l-1}}, \xi_{n,k}^\beta |_{2^{l-1}}))
\end{align*}
\]  

(5)

This observation will become important in the proof of Theorem 1.1(ii).

\textbf{Proof of Theorem 1.1(i).} Fix \( n, k \in \mathbb{N} \) with \( n \geq 2 \) and recall that \( \alpha_{n,k} = (2 - \beta_{n,k})/2 \). Observe that

\[
\frac{2 - \beta_{n,k}^3}{2} \geq \frac{1}{\beta_{n,k}(\beta_{n,k} + 1)} \quad \text{if and only if} \quad (2 - \beta_{n,k}^2)(\beta_{n,k} - 1) \geq 0
\]

(6)

and that

\[
\frac{2 - \beta_{n,k}}{2} \leq \frac{-\beta_{n,k}^3 + \beta_{n,k}^2 + 2\beta_{n,k} - 1}{\beta_{n,k}(\beta_{n,k} + 1)} \quad \text{if and only if} \quad (\beta_{n,k}^2 - 2)(\beta_{n,k} - 1) \leq 0.
\]

(7)

The later inequalities give in (6) and (7) both hold true since \( 1 < \beta_{n,k} < (\beta_{n,k})^2 \leq (\beta_{n,k})^2 = \gamma_n \) and since \( \gamma_n \in (1, 2) \). Therefore, we have that \( (\beta_{n,k}, \alpha_{n,k}) \in D_{1,2} \) and so the results follows from an application of Theorem 4.2. \( \square \)

Using the results of [3] together with Theorems 1.3 and 2.5 and the following three propositions (Proposition 3, 4, 5), we obtain Theorem 1.1(ii).

\textbf{Proof of Theorem 1.1(ii).} Let \( n, k \in \mathbb{N}_0 \) with \( n \geq 2 \) be fixed. By combining Theorems 1.3 and 2.5 together with the results of [3] it suffices to show the following, for all \( n, k \in \mathbb{N}_0 \) with \( n \geq 2 \).

(i) The pair \( (\xi_{n,k}^-, \xi_{n,k}^+) \) are admissible, namely that, for all \( m \in \mathbb{N} \),

\[
(\sigma^m(\xi_{n,k}^-) \leq \xi_{n,k}^+ \text{ or } \sigma^m(\xi_{n,k}^-) > \xi_{n,k}^+) \quad \text{and} \quad (\sigma^m(\xi_{n,k}^+) < \xi_{n,k}^- \text{ or } \sigma^m(\xi_{n,k}^+) \geq \xi_{n,k}^-).
\]

(ii) The value \( \beta_{n,k} \) is the maximal real root of the polynomial

\[
x \mapsto \sum_{m=1}^{\infty} (\xi_{n,k,m}^- - \xi_{n,k,m}^+)x^{-m},
\]

where \( \xi_{n,k,m}^\pm \) denotes the \( m \)-th letter of the word \( \xi_{n,k}^\pm \), respectively.

(iii) The following equalities hold:

\[
\pi_{n,k,\alpha_{n,k}}(\xi_{n,k}^-) = 1/2 = \pi_{n,k,\alpha_{n,k}}(\xi_{n,k}^+).
\]
However, (i) is precisely the result given in Proposition 3, (ii) is the result presented in Proposition 4 and (iii) is the result given in Proposition 5; these propositions, together with their proofs, are presented directly below.

**Proposition 3.** For \( n, k \in \mathbb{N}_0 \) with \( n \geq 2 \), we have that \((\xi_{n,k}^-)^*\) is admissible.

**Proof.** We first prove the statement for the cases \( k = 0, k = 1 \) and \( k = 2 \). We then show the result for a given \( k > 2 \) using an inductive argument.

Using the observations that

\[
\begin{align*}
\xi_{n,0} &= (0, 1, 1, \ldots, 1), & \xi_{n,0}^* &= (1, 0, 0, \ldots, 0), \\
\xi_{n,1}^- &= (0, 1, 1, \ldots, 1, 0), & \xi_{n,1}^- &= (1, 0, 0, \ldots, 0, 1), \\
\xi_{n,2}^- &= (0, 1, 1, 0, 1, 0, 1, \ldots, 0, 0, 1), & \xi_{n,2}^- &= (1, 0, 0, 1, 1, 0, \ldots, 0, 1, 1, 0),
\end{align*}
\]

it is easy to verify, using the repeating structure of the words \( \xi_{n,k}^* \), that the pairs \((\xi_{n,0}^-, \xi_{n,0}^*), (\xi_{n,1}^-, \xi_{n,2}^*), (\xi_{n,1}^-, \xi_{n,1}^*), (\xi_{n,2}^-, \xi_{n,2}^*)\) are admissible.

Assume that \( k > 2 \) and let \( a \) denote the finite word \( \xi_{n,2}^-|k = (0, 1, 1, 0) \). It is an elementary to show that,

(i) \( \sigma^m(a, a) < (a, \ast(a))|_{\mathbb{N}_m} \) or \( \sigma^m(a, a) > (a, \ast(a))|_{\mathbb{N}_m} \),
(ii) \( \sigma^m(a, \ast(a)) < (a, \ast(a))|_{\mathbb{N}_m} \) or \( \sigma^m(a, \ast(a)) > (a, \ast(a))|_{\mathbb{N}_m} \),
(iii) \( \sigma^m(\ast(a), \ast(a)) < (a, \ast(a))|_{\mathbb{N}_m} \) or \( \sigma^m(\ast(a), \ast(a)) > (a, \ast(a))|_{\mathbb{N}_m} \) and
(iv) \( \sigma^m(\ast(a), a) < (a, \ast(a))|_{\mathbb{N}_m} \) or \( \sigma^m(\ast(a), a) > (a, \ast(a))|_{\mathbb{N}_m} \).

for \( m \in \{1, 2, 3\} \). Note that by construction, for \( k > 2 \), the sequence \( \xi_{n,k}^- \) is an infinite concatenation of the finite words \( a \) and \( \ast(a) \) and, by (5), that \( \xi_{n,k}^-|k = (a, \ast(a)) \). Thus, for any \( m = 4p + l \), where \( p \in \mathbb{N}_0 \) and \( l \in \{1, 2, 3\} \), we have that

\[
(\sigma^m(\xi_{n,k}^-) \leq \xi_{n,k}^- \text{ or } \sigma^m(\xi_{n,k}^-) > \xi_{n,k}^+) \quad \text{and} \quad (\sigma^m(\xi_{n,k}^+) \leq \xi_{n,k}^- \text{ or } \sigma^m(\xi_{n,k}^+) > \xi_{n,k}^+).
\]

Hence, all that remains is to prove, for \( k > 2 \) and for \( m = 4p \), where \( p \in \mathbb{N} \), that

\[
(\sigma^m(\xi_{n,k}^-) \leq \xi_{n,k}^- \text{ or } \sigma^m(\xi_{n,k}^-) > \xi_{n,k}^+) \quad \text{and} \quad (\sigma^m(\xi_{n,k}^+) \leq \xi_{n,k}^- \text{ or } \sigma^m(\xi_{n,k}^+) > \xi_{n,k}^+).
\]

Indeed, for \( p \in \mathbb{N} \) and \( k > 2 \),

\[
\sigma^p(\xi_{n,k}^-) = \sigma^p(\xi_{n,k}^-|k)^2, \ast(\xi_{n,k}^-|k)^2, \ldots, \ast(\xi_{n,k}^-|k)^2
\]

\[
\begin{align*}
\sigma^p(\xi_{n,k}^-|k)^2, \ast(\xi_{n,k}^-|k)^2, \ldots, \ast(\xi_{n,k}^-|k)^2) & \text{ if } p = (n+1)m \text{ and } m \in \mathbb{N}, \\
\sigma^p(\xi_{n,k}^-|k)^2, \ast(\xi_{n,k}^-|k)^2, \ldots, \ast(\xi_{n,k}^-|k)^2) & \text{ if } p = (n+1)m - 1 \text{ and } m \in \mathbb{N}, \\
\ast(\xi_{n,k}^-|k)^2, \ldots, \ast(\xi_{n,k}^-|k)^2, (\xi_{n,k}^-|k)^2, \ast(\xi_{n,k}^-|k)^2) & \text{ otherwise}.
\end{align*}
\]

Thus by the equality given in (5) it follows that

\[
\sigma^p(\xi_{n,k}^-) = \xi_{n,k}^- \text{ if } p = (n+1)m \text{ for some } m \in \mathbb{N}, \quad \xi_{n,k}^+ \text{ otherwise}.
\]
Further, by a symmetrical argument we obtain that

$$\sigma^{k-l}(\xi_{n+k}^+) = \xi_{n+k}^+ \quad \text{if } p = (n + 1)m \text{ for some } m \in \mathbb{N},$$

otherwise.

Now assume that for some $l \in \{1, 2, \ldots, k - 2\}$ and for all $p \in \mathbb{N}$ that

$$\sigma^{k-l}(\xi_{n+k}^+) \leq \xi_{n+k}^+ \quad \text{or} \quad \sigma^{k-l}(\xi_{n+k}^+) > \xi_{n+k}^+,$$

and that

$$\sigma^{k-l}(\xi_{n+k}^+) < \xi_{n+k}^+ \quad \text{or} \quad \sigma^{k-l}(\xi_{n+k}^+) \geq \xi_{n+k}^+.$$

To complete the proof we will show, for all $p \in \mathbb{N}$ with $p = 1 \mod 2$, that

$$\sigma^{k-l}(\xi_{n+k}^+) \leq \xi_{n+k}^+ \quad \text{or} \quad \sigma^{k-l}(\xi_{n+k}^+) > \xi_{n+k}^+,$$

and

$$\sigma^{k-l}(\xi_{n+k}^+) < \xi_{n+k}^+ \quad \text{or} \quad \sigma^{k-l}(\xi_{n+k}^+) \geq \xi_{n+k}^+.$$

To this end observe that by construction, the words $\xi_{n+k}^+$ and $\xi_{n+k}^-$ are made up of the finite words $\xi_{n,k-(i-1)}^{2k-l-1}$ and $*(\xi_{n,k-{(i-1)}}^{2k-l-1})$ and moreover, that

$$\xi_{n,k-(i-1)}^{2k-l-1} = (\xi_{n,k-i}^{2k-1}, *((\xi_{n,k-i}^{2k-1})).$$

Hence, we have one of the following cases.

\begin{itemize}
  \item[(v)] $\sigma^{k-l}(\xi_{n+k}^+)_{3:2k-l} = (\xi_{n,k-i}^{2k-1}, *((\xi_{n,k-i}^{2k-1}))$
  \item[(vi)] $\sigma^{k-l}(\xi_{n+k}^+)_{3:2k-l} = (\xi_{n,k-i}^{2k-1}, \xi_{n,k-i}^{2k-1}, *((\xi_{n,k-i}^{2k-1}))$
  \item[(vii)] $\sigma^{k-l}(\xi_{n+k}^+)_{3:2k-l} = (\xi_{n,k-i}^{2k-1}, \xi_{n,k-i}^{2k-1}, *((\xi_{n,k-i}^{2k-1}))$
  \item[(viii)] $\sigma^{k-l}(\xi_{n+k}^+)_{3:2k-l} = (\xi_{n,k-i}^{2k-1}, \xi_{n,k-i}^{2k-1}, *((\xi_{n,k-i}^{2k-1}))$
\end{itemize}

By the equality in (5), if (v) or (vi) occurs, then $\sigma^{k-l}(\xi_{n+k}^+) > \xi_{n+k}^+$ and $\sigma^{k-l}(\xi_{n+k}^+) < \xi_{n+k}^-$. Further, if (vii) or (viii) occurs, then $\sigma^{k-l}(\xi_{n+k}^-) < \xi_{n+k}^-$ and $\sigma^{k-l}(\xi_{n+k}^+) > \xi_{n+k}^-$. This completes the proof. \[\square\]

**Proposition 4.** For $n, k \in \mathbb{N}_0$ with $n \geq 2$, the value $\beta_{n,k}$ is the maximal real root of the polynomial

$$x \mapsto \sum_{m=1}^{\infty} (\xi_{n,k,m}^- - \xi_{n,k,m}^+) x^{-m}.$$ 

**Proof.** Observe, by the recursive definition of the infinite words $\xi_{n,k}^\pm$, that

$$\sum_{m=1}^{\infty} (\xi_{n,k,m}^- - \xi_{n,k,m}^+) x^{-m} = \frac{x^{n+1}2k}{x^{n+1}2k - 1} \sum_{m=1}^{\infty} (\xi_{n,k,m}^- - \xi_{n,k,m}^+) x^{-m} = \frac{x^{n}2k - (n-1)2k - \cdots - 2k}{x^{n+1}2k - 1} \sum_{m=1}^{\infty} (\xi_{n,k,m}^- - *((\xi_{n,k,m}^+))) x^{-m}.$$

This latter term is equal to zero if and only if either

$$x^{n}2k = 0, \quad x^{n}2k - (n-1)2k - \cdots - 2k = 0 \quad \text{or} \quad \sum_{m=1}^{\infty} (\xi_{n,k,m}^- - *((\xi_{n,k,m}^+))) x^{-m} = 0.$$
However, $x^{2^k} > 0$ for all $x > 1$, and, by the definition of a multinacci number, the maximal real solution of $x^{n^2} = x^{2^{k+1}} - x^{2^k} - \cdots - x^2 - 1 = x = \beta_{n,k}$. Further, we claim that

$$
\sum_{m=1}^{2^k} (\xi_{n,k,m} - \ast(\xi_{n,k,m}))x^{-m} < 0,
$$

for all $x > 1$. Given this claim, which we will shortly prove, the result follows. To prove the claim we proceed by induction on $k$. If $k = 0$, then

$$
\sum_{m=1}^{2^k} (\xi_{n,k,m} - \ast(\xi_{n,k,m}))x^{-m} = -x^{-1}.
$$

Suppose that the statement is true for some $k \in \mathbb{N}_0$, then

$$
\sum_{m=1}^{2^{k+1}} (\xi_{n,k+1,m} - \ast(\xi_{n,k+1,m}))x^{-m} = \sum_{m=1}^{2^k} (\xi_{n,k,m} - \ast(\xi_{n,k,m}))x^{-m} + x^{-2^k} \sum_{m=1}^{2^k} (\ast(\xi_{n,k,m}) - \xi_{n,k,m})x^{-m} = \left(1 - x^{-2^k}\right) \sum_{m=1}^{2^k} (\xi_{n,k,m} - \ast(\xi_{n,k,m}))x^{-m}.
$$

Since $x > 1$ the value of the term $1 - x^{-2^k}$ is positive and finite and by our inductive hypothesis

$$
\sum_{m=1}^{2^k} (\xi_{n,k,m} - \ast(\xi_{n,k,m}))x^{-m} < 0.
$$

This completes the proof of the claim. □

**Proposition 5.** For all integers $n, k \in \mathbb{N}$ with $n \geq 2$, we have that

$$
\pi_{\beta_n,\alpha_{n,k}}(\xi_{n,k}^-) = 1/2 = \pi_{\beta_n,\alpha_{n,k}}(\xi_{n,k}^+).
$$

**Proof.** For the proof of this result we require the following. Define $\kappa : [0, 1] \to \{(0, 1), (1, 0)\}$ by $\kappa(0) := (0, 1)$ and $\kappa(1) := (1, 0)$. Let $\overline{\kappa}$ be the substitution map defined on the set of all finite words in the alphabet $[0, 1]$ by

$$
\overline{\kappa}(\omega_1, \omega_2, \ldots, \omega_m) := (\kappa(\omega_1), \kappa(\omega_2), \ldots, \kappa(\omega_m)).
$$

We claim that

$$
\overline{\kappa}(\xi_{n,k-1}^\pm|_{2^{k-1}}) = \xi_{n,k}^\pm|_{2^k},
$$

(9)

for all $n, k \in \mathbb{N}$. We will shortly prove the claim, then using (9) we prove via an inductive argument the statement of the proposition.

Fixed $n \in \mathbb{N}$. We now prove (9) by induction on $k$. By definition we have that

$$
\xi_{n,0}^- = (0, 1, 1, \ldots, 1), \quad \xi_{n,0}^+ = (1, 0, 0, \ldots, 0),
$$

$$
\xi_{n,1}^- = (0, 1, 0, 1, \ldots, 0, 1), \quad \xi_{n,1}^+ = (1, 0, 1, 0, 1, \ldots, 0, 1).
$$

This completes the proof of the base case of the induction. So suppose that (9) holds true for all integers $m < k$, for some integer $k > 1$. Then, by construction of $\xi_{n,k}$, definition of $\overline{\kappa}$ and
the inductive hypothesis, we have that
\[ \tilde{\kappa}(\xi_{n,k-1}^+|_{\Sigma^{2^k}}) = (\tilde{\kappa}(\xi_{n,k-1}^+|_{\Sigma^{2^k}}), \tilde{\kappa}(\xi_{n,k-1}^+|_{\Sigma^{2^k}}), \tilde{\kappa}(\xi_{n,k-1}^+|_{\Sigma^{2^k}})) \]

\[ = (\tilde{\kappa}(\xi_{n,k-1}^+|_{\Sigma^{2^k}}), \tilde{\kappa}(\xi_{n,k-1}^+|_{\Sigma^{2^k}}), \tilde{\kappa}(\xi_{n,k-1}^+|_{\Sigma^{2^k}})) \]

This completes the induction.

For a fixed integer \( n \geq 2 \), we now prove, via an inductive argument on \( k \), the statement of the proposition. Recall that \( \alpha_{n,k} = 1 - \beta_{n,k}/2 \) and observe, for \( k = 0 \), by definition of the projection map \( \pi_{\beta_{n,0}, \alpha_{n,0}} \), that
\[ \pi_{\beta_{n,0}, \alpha_{n,0}}(\xi_{n,0}^+|_{\Sigma^0}) = \frac{1 - \beta_{n,0}/2}{1 - \beta_{n,0}} + \frac{1}{\beta_{n,0}} \sum_{m=1}^{\infty} \frac{1}{\beta_{n,0}^{m+1}} = \frac{1 - \beta_{n,0}/2}{1 - \beta_{n,0}} + \frac{\beta_{n,0}^{-m}}{\beta_{n,0}^{m+1}} = \frac{1}{2^n}. \]

where the last equality follows by using the fact that \( \beta_{n,0} \) is the multinacci number of order \( n \) and so \( 1 = \beta_{n,0}^{-1} + \beta_{n,0}^{-2} + \cdots + \beta_{n,0}^{-n} \). Further, we note that
\[ \pi_{\beta_{n,0}, \alpha_{n,0}}(\xi_{n,0}^+|_{\Sigma^0}) = \frac{1 - \beta_{n,0}/2}{1 - \beta_{n,0}} + \frac{1}{\beta_{n,0}} \left( \frac{1}{\beta_{n,0}^{1}} + \cdots + \frac{1}{\beta_{n,0}^{n}} \right) \sum_{m=1}^{\infty} \frac{1}{\beta_{n,0}^{m+1}} = \pi_{\beta_{n,0}, \alpha_{n,0}}(\xi_{n,0}^+). \]

This completes the proof of the base case of the induction. So suppose the statement of the proposition holds true for all integers \( m < k \), for some \( k \in \mathbb{N} \). By the definition of the projection map \( \pi_{\beta_{n,k}, \alpha_{n,k}} \), that \( \beta_{n,k}^2 = \beta_{n,k-1} \) and \( (9) \), we have that
\[ \pi_{\beta_{n,k}, \alpha_{n,k}}(\xi_{n,k}^+|_{\Sigma^k}) = \frac{\beta_{n,k} + 1}{2} - \frac{1 - \beta_{n,k}^2/2}{1 - \beta_{n,k}} = \frac{(1 - \beta_{n,k}^2/2 + \beta_{n,k})}{2(1 - \beta_{n,k})} = \frac{1}{2^n}. \]

This completes the induction. \( \square \)

**Proof of Theorem 1.1(iii).** This is a direct consequence of Theorem 4.2. \( \square \)

For the proof of Theorem 1.1(iv) we will require the following additional preliminaries.

(i) A Pisot number is a positive real number \( \beta \) whose Galois conjugates all have modulus strictly less than 1. A Perron number is a positive real number \( \beta \) whose Galois conjugates all have modulus strictly less than \( \beta \).

(ii) For \( n \geq 2 \), the \( n \)-th multinacci number is a Pisot number, see [8, Example 3.3.4].

(iii) If the shift space \( \Omega_{\beta,0} \) is sofic, then \( \beta \) is a Perron number, see [28, Theorem 2.2(3)]; here the result is attributed to [5, 20, 24].

**Proof of Theorem 1.1(iv).** Since, by definition, any SFT is sofic, it is sufficient to show that \( \beta_{n,k} \) is not a Perron number, for all \( n,k \in \mathbb{N} \) with \( n \geq 2 \). To this end let \( P_{n,k} \) denote the polynomial given by
\[ P_{n,k}(x) := x^{2^n} - x^{2^k(n-1)} - \cdots - x^{2^k} - 1. \]

Suppose, by way of contradiction, that \( \beta_{n,k} \) was a Perron number. By the definition of \( \beta_{n,k} \) and \( P_{n,k} \), we have that \( P_{n,k}(\beta_{n,k}) = 0 \), and so the minimal polynomial \( Q_{n,k} \) of \( \beta_{n,k} \) divides \( P_{n,k} \). Further, if \( \beta \) was a root of \( Q_{n,k} \) not equal to \( \beta_{n,k} \), then it would also be a root of \( P_{n,k} \). Moreover, \( |\beta| < 1 \). Since if not, then \( P_{n,k}(\beta) = 0 \), since \( \beta \) is a root of \( Q_{n,k} \), and, by our assumptions, \( 1 < |\beta^{2^k}| < \gamma_n \). Hence, \( \beta^{2^k} \) would be a Galois conjugate of \( \gamma_n \) with modules greater than 1, contradicting the fact that \( \gamma_n \) is a Pisot number. Thus, all of the
roots of $Q_n$ with the exception of $\beta_{n,k}$ would be of absolute value strictly less than 1, and hence $\beta_{n,k}$ would be a Pisot number. However, it is well known that there are only two Pisot numbers $\theta_0$ and $\theta_1$ less than $2^{1/2}$, and moreover, $\theta_0 > \theta_1 > 2^{1/3}$. Thus, since $\gamma_n \in (1, 2)$, if $k \geq 2$ we obtain a contradiction to our assumption that $\beta_{n,k}$ is a Perron number. Furthermore, by numerical calculations we know that $1.8 > \theta_0^2$, $1.925 > \theta_1^2$, $P_{k,0}(1.8) < 0$ and $P_{k,0}(1.925) < 0$. Therefore, for all $n \geq 4$, we have that $\theta_0$ nor $\theta_1$ is a root of $P_{n,1}$. Furthermore, for $n \in [2, 3]$, it is a simple calculation to show that the minimal polynomials of $\theta_0$ and $\theta_1$, namely $x \mapsto x^3 - x - 1$ and $x \mapsto x^4 - x^3 - 1$ respectively, do not divide $P_{n,1}$, for $n \in [2, 3]$. Hence, $\theta_0, \theta_1 \notin \{\beta_{2,1}, \beta_{3,1}\}$. This yields a contradiction to the assumption that $\beta_{n,1}$ is a Perron number.

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