

# Modular Representation Theory

# *Modular Representation Theory: Blocks, Defects & Inertial Quotients*

*Elliot McKernon*

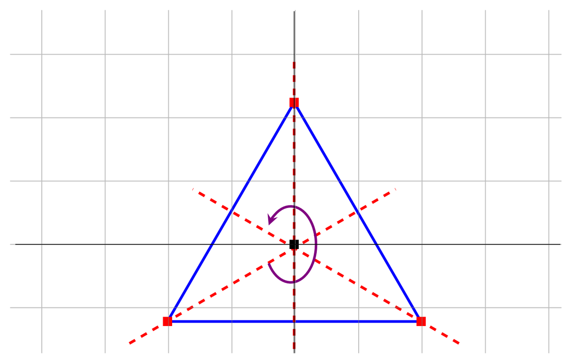
## What is modular representation theory?

**Symmetry** is a reasonably interesting and fundamental property of things, notably in chemistry and physics. For example, the speed of light is invariant in all frames of reference: this is a symmetry.

The symmetries of an object or system are abstractly represented with **groups**: a set with a binary operation, such that there is an inactive 'identity', and that each element has an inverse. In the group of integers $\mathbb{Z} = \{..., -1, 0, 1, 2, ...\}$ with operation '+', the identity is $0$, and the inverse of $n$ is $-n$. *Given an object we can find a group describing its symmetries - given a group we can find an object whose symmetries it describes*. The Poincarè group describes the speed-of-light symmetry.

> ***Representation theory****: a tool to understand groups of symmetries through vector spaces.*

Groups are hard, but we can capitalise on their relation to symmetry by realising the group as the symmetries of a **vector space**, which is defined over a **field**: a set with additive *and* multiplicative identities and inverses: $\mathbb{Z}$ isn't one, but $\mathbb{Q}$ & $\mathbb{R}$ are. Fields, and vector spaces especially, are nicely linear (their study is called linear algebra), and we exploit this linearity to understand the group. This is a **representation** of the group. The symmetries of a triangle, for example, could be studied as symmetries of $\mathbb{R}^2$ instead.



The triangle is embedded into $\mathbb{R}^2$. Its symmetries - $3$ rotations, $3$ reflections - can then be realised as matrices (the strength of vector spaces is really just the manifestation of the power of linear algebra). The reflection in the $y$-axis has matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, while the clockwise rotation of the triangle by $\frac{2\pi}{3}$ has matrix $\begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$.

In $\mathbb{R}$ and $\mathbb{Q}$, summing $1$ never yields $0$, but in other fields it can: in the binary field of two elements, $1 + 1 = 0$. If $1 + ... + 1 = 0$ with $p$ summands, the field has characteristic $p$. As long as the size of a group isn't divisible by $p$, **Mashke's theorem** tells us the representations break apart into simple pieces. Sadly, when $p$ *does* divide the size of the group, trying to prove Mashke's theorem results in a divide-by-$0$-error. **Modular representation theory** studies this more anarchic case.

## How do we do modular representation theory?

Let $G$ denote a group. The information about a representation of $G$, on a vector space over a field $F$, is contained in a composite structure called the **group algebra**, denoted $FG$. Think of this as a coordinate system, with one axis indexed by $F$ and the other by $G$. Understanding the structure of $FG$ is the same as understanding the representation.

To do this, we decompose $FG$ into **blocks**: these are indecomposable ideals (nice, simple substructures). There is a pleasant way to find the blocks: an **idempotent** is an element $e$ such that $e^2 = e$, and we can decompose the identity $1$ into a sum of idempotents (central primitive idempotents, to be exact). This sum of idempotents is in $1 : 1$ correspondence with the blocks above, and multiplying the group algebra by one of the idempotents gives the block.

> *Decomposing the group algebra into blocks corresponds with decomposing the identity into idempotents, with $FGe_i = B_i$:*
> $FG = B_1 + B_2 + ... + B_n,$
> $1 = e_1 + e_2 + ... + e_n.$

So far we've followed the yellow block road from *groups→vector spaces (via representations)→group algebras→blocks*, which we can identify using idempotents. The complexity of these blocks is key to modular representation theory, and we can return full circle to groups by identifying a particular sub-group within $G$, called the **defect group**, which determines that complexity to a large degree. When the defect group is very small, the blocks are very easy to understand: when it's large, the blocks rise in intricacy (in 'defectiveness').

Though their definition is somewhat convoluted, there are a number of conjectures that imbue defect groups with significance, and a lot of work in the area is comprised of progress towards these conjectures.

> **Donovan's conjecture**: Given a group, there are only finitely many classes of blocks with that defect group.

Some cases are known: for example, if the defect group has the same structure as $\mathbb{Z}$ modulo $p$, for $p$ prime, then Donovan's conjecture is proven to hold. There have also been classifications of *what* blocks arise. That is, the structure of each class.

## Reductions, Inertial Quotients and a Conjecture

There are reduction techniques: subgroups of a group which are known to have the *same class* as their parent, but are smaller and lend themselves more easily to study. If $H$ is a subgroup of $G$ that contains the normaliser of $D$ (the elements of $G$ that stabilise $D$), then the **Brauer correspondence** describes a $1 : 1$ pairing:

$$\{\text{Blocks of } G, \text{ defect group } D\} \leftrightarrow \{\text{Blocks of } H, \text{ defect group } D\}.$$

Especially powerful is applying this to **normal subgroups**: subgroups which look the same after any change in perspective (i.e. which are stable under twisting by *any* element of $G$). The first step in proofs, such as the $\mathbb{Z}$ mod $p$-defect group mentioned previously, is reducing $G$ as much as possible, till we are left with some kind of irreducible core.

Looking at the subgroup in $G$ of elements that commute with the defect group $D$ (i.e. $g \in G$ such that $gd = dg, \forall d \in D$), we can form **Brauer pairs**, $(D, e)$, consisting of a defect group $D$ of the block, together with the idempotent $e$. Though the structure of $D$ is important, it is its *role within G's structure* which is the deeper clue to the block's secrets. Looking at those non-trivial elements of $G$ which stabilise the Brauer pair $(D, e)$, much as the way certain elements stabilised the triangle in our first example, defines the **inertial quotient**, labelled $E$.

> The defect group $D$ describes the blocks, and the inertial quotient $E$ describes how $G$ **acts** on $D$, just as the group acted on the $2d$-plane in the triangle example. An action of a group on an object details the structure of both.

Currently I am working on the following conjecture of Charles Eaton:

Let $G$ be a finite group, and $F$ a field of characteristic $p$, where $p$ divides the size of $G$. Suppose that $B$ is a block of $FG$, and that the associated defect group $D$ has the structure of $n$ copies of $\mathbb{Z}$ mod $2$. Suppose further that the inertial quotient $E$ is a (sub)group of $\mathbb{Z}$ mod $2^n - 1$. Then we claim the block $B$ falls into two possible classes, and that the classes have the structure either of $B_0(FSL_2(2^n))$ (a particular block of a particular group), or of the group formed by $E$ acting on $D$.

If we suppose further that $E$ is stricly smaller than $\mathbb{Z}$ mod $2^n - 1$ then the only latter case $(D \rtimes E)$ arises.

elliot.mckernon@manchester.ac.uk