OPTION PRICING UNDER EXPONENTIAL JUMP DIFFUSION PROCESSES

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The main contribution of this thesis is to derive the properties and present a closed from solution of the exotic options under some specific types of Lévy processes, such as American put options, American call options, British put options, British call options and American knock-out put options under either double exponential jump-diffusion processes or one-sided exponential jump-diffusion processes. Compared to the geometric Brownian motion, exponential jump-diffusion processes can better incorporate the asymmetric leptokurtic features and the volatility smile observed from the market. Pricing the option with early exercise feature is the optimal stopping problem to determine the optimal stopping time to maximize the expected options payoff. Due to the Markovian structure of the underlying process, the optimal stopping problem is related to the free-boundary problem consisting of an integral differential equation and suitable boundary conditions. By the local time-space formula for semi-martingales [39], the closed form solution for the options value can be derived from the free-boundary problem and we characterize the optimal stopping boundary as the unique solution to a nonlinear integral equation arising from the early exercise premium (EEP) representation.

Chapter 2 and Chapter 3 discuss American put options and American call options respectively. When pricing options with early exercise feature under the double exponential jump-diffusion processes, a non-local integral term will be found in the infinitesimal generator of the underlying process. By the local time-space formula for semi-martingales [39], we show that the value function and the optimal stopping boundary are the unique solution pair to the system of two integral equations. The significant contributions of these two chapters are to prove the uniqueness of the value function and the optimal stopping boundary under less restrictive assumptions compared to previous literatures. In the degenerate case with only one-sided jumps, we find that the results are in line with the geometric Brownian motion models, which extends the analytical tractability of the Black-Scholes analysis to alternative models with jumps.

In Chapter 4 and Chapter 5, we examine the British payoff mechanism under one-sided exponential jump-diffusion processes, which is the first analysis of British options for process with jumps. We show that the optimal stopping boundaries of British put options with only negative jumps or British call options with only positive jumps can also be characterized as the unique solution to a nonlinear integral equation arising from the early exercise premium representation.

Chapter 6 provides the study of American knock-out put options under negative exponential jump-diffusion processes. The conditional memoryless property of the exponential distribution enables us to obtain an analytical form of the arbitrage-free price for American knock-out put options, which is usually more difficult for many other jump-diffusion models.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

According to the theory of modern finance in [57] and [52], valuating the arbitrage-free price of American-type option is the optimal stopping problem with gain function set by the option writer. The optimal stopping time in this problem is the first time when the stock price (geometric Brownian motion) falls into the optimal stopping region shaped by the time-dependent boundary (free boundary).

The optimal stopping problem originated in the Walds sequential analysis [56] representing a method of statistical inference. Snell [53] formulated a general optimal stopping problem for the discrete-time case, and he characterized the value function as the smallest super-martingale (called Snell's envelope) dominating the payoff (gain) process. Dynkin [14] formulated a general optimal stopping problem for Markov process and characterized the solution by means of the smallest superharmonic function dominating the gain function. Dynkin treated the case of discrete time in detail and indicated that the analogous results also hold in the case of discrete time in detail and indicated that the analogous results also hold in the case of continuous time. The research in this direction often called as martingales methods are summarized in [10] (General optimal stopping problems (Chapter 4) and optimal stopping problems in Markov case (Chapter 5)). To price the American-type options, El Karoui and Karatzas [21] established the Riesz decomposition of the value function associated with an optimal stopping time problem. Their result are subsequently applied by Myneni [35] to price an American put option. The Riesz decomposition broke the value function (Snell envelop of the payoff function) into two parts, a martingale and a potential, and the latter corresponds to the EEP component of the option value.

Another way to solve the optimal stopping problem is to convert the optimal stopping problem into the free-boundary problem. The first mathematical analysis of this problem in finance is due to McKean [32] who considered the "discounted" American call option and derived the free-boundary problem of the value function and unknown time-dependent free boundary. He expresses the value function as the solution of a countable system of nonlinear integral equations. Following the work of McKean, Van Moerbeke [34] used a single integral equation to represent the value function. However, the approach of expressing the value function by McKean was in line with the idea coming from earlier work of Kolodner [24] on the free-boundary problem in mathematical physics (such as Stefan's ice melting problem), so it is not obvious to see how the condition of heat equation and the expression of value function related to the economic mechanism behind the arbitrage-free price of American options.

This problem has been solved in the early 1990's, Jacka [18], Carr et al. [8] and Kim [22] independently expressed the value of American put options as a nonlinear Volterra integral equation (EEP representation), which is equal to the value of European put options plus the early exercise premium. Meanwhile, the paper [8] also gave the delay exercise premium representation of American put options. Kallast and Kivinukk [19] illustrated how to solve out the free boundary and the value function numerically. Finally, Peskir [37] using the local time-space formula on curves showed that the free boundary is the unique solution of the nonlinear integral equation. The local time-space formula [36] and [39] gave a convenient way to price the options with early exercise feature such as American Asian options [45], Russian options [38] and others. For this thesis, we also use the local time-space formula to price the option under exponential jump-diffusion processes.

Despite the success of the Black-Scholes model based on geometric Brownian motion and normal distribution, two empirical phenomena have received much attention recently: (1) the asymmetric leptokurtic features. In other words, the return distribution is skewed to the left, and has a higher peak and two heavier tails than those of the normal distribution. (2) The volatility smile. More precisely, if the Black-Scholes model is correct, then the implied volatility should be constant. In reality, it is widely recognized that the implied volatility curve resembles a smile, meaning it is a convex curve of the strike price.

To incorporate the asymmetric leptokurtic features in asset pricing, a variety of models have been proposed, including fractal Brownian motion, generalized hyperbolic models and time-changed Brownian motions; see, for example, Rogers [51], Barndorff-Nielsen and Shephard [3] and Heyde [16]. In a parallel development, There are also several models proposed to incorporate the volatility smile, such as stochastic volatility model, CGMY models and CEV models; see Hull and White [17], Cox and Ross [11] and Carr et al. [7].

Compared with the original Black-Scholes model, the major shortcoming of these alternative models is the analytical tractability. Although many of them can lead to analytic solution for vanilla European options, it is difficult to do so for options involving path-dependent or early exercise. However, the numerical approach for these derivatives are also not efficiency enough, due to the low convergence rates of binomial trees and Monte Carlo simulation; see Boyle et al. [6].

Additionally, most of the alternative models mentioned above employ diffusion-based processes as the underlying dynamic. They can reproduce the profile of implied volatilities at a given maturity fairly well. However, they have more trouble across maturities, they cannot yield a realistic term structure of implied volatilities; see Bates [5]. In particular the "at-the-money" skew. To handle this problem, an effective approach is to allow jump on the trajectories of diffusion processes. Model with jumps not only lead to a variety of smile patterns but also propose a smile explanation in terms of market anticipations: the presence of a skew is attributed to the fear of large negative jumps by market participants.

As one of the best known kind of Lévy processes, the classic Wiener process requires strictly path continuity for every trajectory. By loosing this constrain to stochastic continuity, we can obtained a more general Lévy process which is naturally suitable for considering jumps on the pure diffusion sample path. This category of Léry process is called jump-diffusion models, the "normal" evolution of prices is given by a diffusion process, punctuated by jumps at random intervals. From one extreme to the other, we can even have pure jump models. There are another well developed Léry process category for financial modelling, which are called infinite activity models. In these models, one dose not need to introduce a diffusion component since the dynamics of jumps is already rich enough to generate nontrivial small time behavior. These pure jumps models are out of the scope of this thesis, please see [54] for further introduction.

As a hybrid of pure diffusion and pure jump processes, the evolution of a jump-diffusion model can usually be represented by modelling the (log-) price as a Léry process with a nonzero Gaussian component and a jump part, which is a compound Poisson process with finitely many jumps in every time interval. Such a representation was first proposed by Merton in [33], called the normal jump-diffusion Model. In this model, the distribution of jump size is assume to be Gaussian.

Inspired by Merton's work, Kou proposed the double exponential jump-diffusion model in [25], where the logarithm of the asset price is assumed to follow a Brownian motion plus a compound Poisson process with jump sizes double exponential distributed. Moreover, Kou and Wang demonstrated that the double exponential jump diffusion model can lead to analytical solutions for popular path-dependent options (such as lookback, barrier, and American options) in [27]. For the finite-horizon American options, they constructed a possible format of the value function based on the Barone-Adesi and Whaley method (an approximation for the early exercise premium) [4] and proved that this function can solve the free-boundary problem linked to the option pricing.

In this thesis, we would like to apply the exponential jump-diffusion model to more kinds of exotic options. Using the local time-space formula for semi-martingales [39], we price and analyze the American put options, American call options, British put options, British call options and American knock-out put options. Some proof in this thesis are also inspired by Lamberton and Mikou [29], [30] and Pham [46] which provide several important properties about the American options in exponential jump-diffusion models.

Chapter 2 mainly discusses the American put option under the double exponential jump-diffusion processes. We use stochastic analysis to derive the important properties for American put options and price them by the local time-space formula. Some properties such as the continuity of the optimal stopping boundary, the smooth-fit property and the critical price at maturity are given by Lamberton and Mikou in [29] and [30]. By the local time-space formula, we get the EEP representation of American put options under the double exponential jump-diffusion processes. However, due to the existence of positive jumps on the path, we also find that the infinitesimal generator will introduce a non-local integral term in the EEP representation. Inspired by Pham's work in [46], we still show that the value function and the optimal stopping boundary of the American put option are the unique solution pair to the system of two integral differential equations. The main contribution of this proof is that it relies on less restrictive assumptions compared to [46]. In the degenerate case with only negative jumps, we find that the results for exponential jump-diffusion processes are in line with the geometric Brownian motion models, which extends the analytical tractability of the Black-Scholes analysis to alternative models with jumps.

The American call option with dividends under the double exponential jump-diffusion processes is analyzed in Chapter 3. We uses stochastic analysis to derive the important properties for American call options and price them by the local time-space formula. By following the derivation method in [29] and [30], we give rigorous proof for the properties such as the continuity of the optimal stopping boundary, the smooth-fit property, and the critical price of the American call option at maturity. To show that the stopping set of an American call option with dividends is not empty, we also derive some preliminary results for the perpetual American call options under double exponential jump-diffusion processes. The main results we have for the American call option is parallel with that in Chapter 2 for American put options. In the degenerate case with only positive jumps, we also find that the results for exponential jump-diffusion processes are in line with the geometric Brownian motion models.

In Chapter 4 and Chapter 5, we examine the British payoff mechanism under one-sided exponential jump-diffusion processes, which is the first analysis of British options for process with jumps. In 2011 and 2013, Peskir and Samee ([42] and [43]) presented a new call and put option where the holder enjoyed the early exercise feature of American options where his payoff was the "best prediction" of the European payoff under the hypothesis that the true drift of the stock price equaled to a contract drift. The inherent contract drift for the British call or put option is the tolerant drift to protect the option buyer for the unfavourable movement of the stock price. Considering the analytical tractability and a clear comparison between the return of American and British option, we price and analyze the British put option under the negative exponential jump-diffusion processes in Chapter 4; the British call option under the positive exponential jump-diffusion process in Chapter 5. We tailor the drift term in the underlying process to emphasize the influence of the contract drift. Using the local time-space formula, we get the EEP representation of British put and call options and prove that the free boundary embedded in the representation can be characterized as the unique solution to a nonlinear integral equation. For the financial analysis, we compare the returns between American options and British options and find that with the contract drift properly selected the British option becomes a very attractive alternative to the classic American options.

Chapter 6 mainly studies of American knock-out put options under negative exponential jump-diffusion processes. Using the local time-space formula, we get the EEP representation of American knock-out put options and prove that the free boundary embedded in the representation can be characterized as the unique solution to a nonlinear integral equation. To price such a kind of barrier options, it is crucial to study the first passage times that the process crosses a flat boundary with a given level. However, when a jump-diffusion process crosses the boundary, sometimes it hits the boundary exactly and sometimes it incurs an overshoot, which makes the pricing more difficult for many other jump-diffusion models. One significant advantage of the exponential jump-diffusion processes is that the conditional memoryless property of the exponential distribution will enable us to find the dependence structure between the overshoot and the terminal value. Therefore, with the joint distribution of the overshoot and the terminal value, we derive an analytical form of the EEP representation by the Laplace transform and the change of measure (Girsanov theorem). A financial analysis is also given based on such analytical solutions to the American knock-out put option under negative exponential jump-diffusion processes.

Chapter 2

American Put Option for Double Exponential Jump Diffusion Processes

2.1 Introduction

Despite the success of the Black-Scholes model based on Brownian motion and normal distribution, two empirical phenomena have received much attention recently: (1) the asymmetric leptokurtic features. In other words, the return distribution is skewed to the left, and has a higher peak and two heavier tails than those of the normal distribution. (2) The volatility smile. More precisely, if the Black-Scholes model is correct, then the implied volatility should be constant. In reality, it is widely recognized that the implied volatility curve resembles a smile, meaning it is a convex curve of the strike price.

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Compared with the original Black-Scholes model, the major shortcoming of these alternative models is the analytical tractability. Although many of them can lead to analytic solution for vanilla European options, it is difficult to do so for options involving path-dependent or early exercise. However, the numerical approach for these derivatives are also not efficiency enough, due to the low convergence rates of binomial trees and Monte Carlo simulation; see Boyle et al. [6].

Additionally, most of the alternative models mentioned above employ diffusion-based processes as the underlying dynamic. They can reproduce the profile of implied volatilities at a given maturity fairly well. However, they have more trouble across maturities, they cannot yield a realistic term structure of implied volatilities; see Bates [5]. In particular the "at-the-money" skew. To handle this problem, an effective approach is to allow jump on the trajectories of diffusion processes. Model with jumps not only lead to a variety of smile patterns but also propose a smile explanation in terms of market anticipations: the presence of a skew is attributed to the fear of large negative jumps by market participants.

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As a hybrid of pure diffusion and pure jump processes, the evolution of a jump-diffusion model can usually be represented by modelling the (log-) price as a Léry process with a nonzero Gaussian component and a jump part, which is a compound Poisson process with finitely many jumps in every time interval. Such a representation was first proposed by Merton in [33], called the normal jump-diffusion Model. In this model, the distribution of jump size is assume to be Gaussian.

Inspired by Merton's work, Kou proposed the double exponential jump-diffusion model in [25], where the logarithm of the asset price is assumed to follow a Brownian motion plus a compound Poisson process with jump sizes double exponential distributed. In this chapter, we show that, under the double exponential jump-diffusion processes, the optimal stopping boundary for the American put option with finite horizon can be characterized as the unique solution to a system of nonlinear integral equations arising from the early exercise premium (EEP) representation.

The article is organised as follows. Section 2 introduces the double exponential jump-diffusion model and its basic properties. In Section 3, we derive the system of nonlinear integral equations which leads to the unique solution pair of the American put option's value and its optimal exercise boundary. In Section 4, we introduce a degenerate case of the double exponential jump-diffusion model. Under this degenerate model, we derive a closed form expression for the arbitrage-free price in terms of the optimal exercise boundary and show that the boundary itself can be characterised as the unique solution to a nonlinear integral equation. Using these results, in Section 5 we present a financial analysis of the American put option under the negative exponential jump-diffusion process.

2.2 Double Exponential Jump Diffusion Model

In this section, we will give a brief introduction to this model. And we will call it Kou's model or DEJD model for convenience afterwards.

In Kou's model, we assume a financial market consisting of risky stock Z_t and riskless bond B_t:

$$\frac{dZ_t}{dZ_{t}-} = \mu\, dt + \sigma\, dW_t + d\Big(\sum_{i=1}^{N_t}(V_i - 1)\Big) \qquad (Z_0 = z), \tag{2.1}$$

$$dB_t = rB_t\, dt \qquad (B_0 = 1), \tag{2.2}$$

where μ is the personal appreciation drift of the stock, σ is the volatility, r is the risk-free interest rate, W_t is a standard Brownian motion, N_t is a Poisson process with rate λ, and $\{V_i\}$ is a sequence of independent identically distributed nonnegative random variables such that $Y = \log V$ has an asymmetric double exponential distribution with the density:

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y<0\}} \qquad \eta_1 > 1, \eta_2 > 0, \tag{2.3}$$

where $p, q \geq 0, p+q=1$,represent the probabilities of upward and downward jumps. In other words,

$$\log V = Y \stackrel{d}{=} \begin{cases} \xi^{+} & \text{with probability } p \\ -\xi^{-} & \text{with probability } q \end{cases}, \tag{2.4}$$

where ξ^{+} and ξ^{-} are exponential random variables with means $1/\eta_1$ and $1/\eta_2$, respectively, and the notation $\stackrel{d}{=}$ means equal in distribution. The drift μ and the volatility σ are assumed to be constants. Also N_t, W_t and Y are assumed to be independent. Note that:

$$\begin{aligned} \mathsf{E}[Y] &= \frac{p}{\eta_1} - \frac{q}{\eta_2}, \\ \mathsf{Var}[Y] &= pq(\frac{1}{\eta_1} + \frac{1}{\eta_2})^2 + (\frac{p}{{\eta_1}^2} + \frac{q}{{\eta_2}^2}), \\ \mathsf{E}[V] &= \mathsf{E}[e^Y] \\ &= q\frac{\eta_2}{\eta_2+1} + p\frac{\eta_1}{\eta_1-1}, \qquad \eta_1 > 1, \eta_2 > 0. \end{aligned} \tag{2.5}$$

The requirement $\eta_1 > 1$ is to ensure that $\mathsf{E}[V] < \infty$ and $\mathsf{E}[Z_t] < \infty$. It just means that the average upward jump cannot exceed 100%, which is quite reasonable.

Although the fundamental setting for DEJD model seems to be far more complex than original Black-Scholes, it can indeed provide some interesting and useful properties for further valuation. The first property is that the double exponential distribution, which performs two-sided jumps, has the leptokurtic feature of the jump size that provides the peak and tails of the return distribution found in reality. The second one is that by allowing jumps on the continuous path it can capture the feature of the implied volatility smile when we perform the calibration. The third property is that the exponential distribution has a memoryless feature which makes it possible to obtain analytical expressions for expectations involving first passage times. It is a significant advantage compared Merton's model. This memoryless property helps also to solve the problem of overshoots which is the major obstacle of option pricing for jump-diffusion process. For more details and analysis of this model, see Kou's original papers [25].

With the properties mentioned above, Kou and Wang demonstrated that a double exponential jump diffusion model can lead to analytical solutions for popular path-dependent options (such as lookback, barrier, and American options) in [27]. For the finite-horizon American options, they constructed a possible format of the value function based on the Barone-Adesi and Whaley method [4] and proved that this function can solve the free-boundary problem linked to the option pricing. A limitation of the Barone-Adesi and Whaley method is the use of approximation for the early exercise premium. In this chapter, we would like to derive the value function, as well as the exercise boundary, of a finite-horizon American put option in a more theoretically rigorous way.

We follow the martingale approach, which is also known as risk-neutral pricing method, for derivative pricing in the rest of this chapter. By this approach, we can assume that the value of an American style option is just the expectation of its payoff at a random optimal stopping time under a risk-neutral measure. This expectation form itself can provide us some key properties of the value function. Afterwards, we prove that the optimal stopping time can be illustrated into an optimal exercise boundary dividing the whole plane into two regions. We also prove that both these two regions are not empty for the American put option with no dividend. Thanks to the previous work done by Lamberton and Mikou [29] on American put option under exponential Lévy processes, we could directly obtain some properties of the early exercise boundary. These properties allow us to use the change-of-variable formula with local time on surfaces derived by Peskir in [39] on the value function of American put. This will lead to the EEP representation of the value function. Different from this kind of representation obtained by Peskir [37] for the American put under geometric Brownian motion, the value function we have also involves the early exercise boundary, which means that the pair of our option value and the boundary function is a solution pair to a system of nonlinear integral equations. The uniqueness of this solution is also proved, under some additional conditions coming from free-boundary problems. Since the nonlinear integral equations system seems not feasible enough for analytical calculations, we study a degenerate case of the DEJD model in the last few sections. By restricting the jumps to only one side, we can obtained some results with great analytical tractability, as well as a more generic proof for the uniqueness, which can also be viewed as a main contribution of this chapter. The financial analysis of this degenerate case illustrates the difference and connection between option pricing under jump-diffusion processes and pure diffusion models.

Before moving to the research of any specific style of options, we need to do some general preparation on this model, which can be used in the risk-neutral pricing method. First, note that Z_t in (2.1) is actually a stochastic exponential. Using the relation between stochastic and ordinary exponentials given by [54, Proposition 8.22], we can rewrite Z_t as an ordinary exponential of a real value Lévy process X_t:

$$Z_t = e^{X_t}, \tag{2.6}$$

where

$$\begin{aligned} X_t &= \mu t + \sigma W_t + \sum_{i=1}^{N_t}(V_i - 1) - \frac{\sigma^2}{2}t + \sum_{0\leq s\leq t}^{\Delta Z_s \neq 0}(\ln(1+\Delta Z_s) - \Delta Z_s) \qquad (2.7) \\ &= \mu t + \sigma W_t - \frac{\sigma^2}{2}t + \sum_{i=1}^{N_t}(V_i - 1) + \sum_{i=1}^{N_t}(\ln(V_i) - (V_i - 1)) \\ &= (\mu - \frac{\sigma^2}{2})t + \sigma W_t + \sum_{i=1}^{N_t}(Y_i) \qquad (X_0 = x = \ln z). \end{aligned}$$

We can see that X_t consists of three parts: the combination of the first two parts is a Brownian motion with drift; and last part is a compound Poisson process with the jump size Y_i following a double exponential distribution. The name of this model comes from this fact. Using a Brownian motion with drift $B_t = (\mu - \sigma^2/2)t + \sigma W_t$ to replace X_t in (2.6), then we can get back to the standard Black-Scholes framework.

Following the martingale approach for derivative pricing, the drift parameter μ has to assure that the discounted price process $e^{-rt}Z_t$ becomes a martingale under the physical probability measure P. Here we denote this new risk-neutral dynamic as Z_t^r

$$Z_t^r = e^{X_t^r} = e^{(r-\lambda\zeta-\frac{\sigma^2}{2})t+\sigma W_t+\sum_{i=1}^{N_t}(Y_i)}. \tag{2.8}$$

where

$$\zeta = \mathsf{E}[V] - 1 = q\frac{\eta_2}{\eta_2+1} + p\frac{\eta_1}{\eta_1-1} - 1. \tag{2.9}$$

It will allows us to state that the present value of a derivative is the expectation of a payoff function of (2.8) under the physical measure P. This transform will make the definition of the value function for American options much easier to understand in the next section.

Note that X_t^r is also a strong Markov process, it would be very useful to introduce the infinitesimal generator of X_t^r here:

$$(\mathbb{L}_{X^r}F)(x) = \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(x) + (r - \lambda\zeta - \frac{\sigma^2}{2})\frac{\partial F}{\partial x}(x) + \lambda\int_{-\infty}^{+\infty}[F(x+y) - F(x)]f_Y(y)dy, \tag{2.10}$$

for every $F \in C_b^2(\mathbb{R})$, where $C_b^2(\mathbb{R})$ denotes the set of all bounded, twice continuously differentiable functions with bounded derivatives.

With these fundamental framework and properties, this chapter attempts extend the analytical tractability of Black-Scholes analysis for the classical geometric Brownian motion to alternative models with jumps. In particular, we demonstrate that the double exponential jump diffusion model can lead to the unique pair of solutions to value and boundary functions for finite-horizon American put options, and even analytical solutions under the degenerate one-sided jump case.

2.3 American Put Option for Double Exponential Jump Diffusion Processes

An American option is an option that can be exercised anytime during its life. American options allow option holders to exercise the option at any time prior to and including its maturity date, thus increasing the value of the option to the holder relative to European options, which can only be exercised at maturity. The majority of exchange-traded options are American or American based. The difference between the price of an American option and European option with the same characteristics is called the early exercise premium. The idea behind valuing options with early exercise is to decide when the option should be exercised. To correctly price American options we must place ourselves in the view of the option writer. We assume that the writer is always hedging his option position by trading in the underlying asset. The hedging strategy is dynamic and referred to as delta hedging. By maintaining such a hedge, the writer does not care about the direction in which the underlying moves, he eliminates all asset price risk. However, he does remain exposed to the exercise strategy of the option holder. To reduce this exposure, the writer must price the option under the assumption that the holder exercises at the worst possible time for the writer. This is often referred to as the optimal stopping time, although as far as the writer is concerned it is the last thing he wants to happen. Thus through out this chapter, the optimal stopping time means the optimal exercise strategy only for a delta hedger who does not have a view on the movement of the market.

The early exercise premium formula and the optimal exercise boundary are crucial in our research. The EEP formula of the American option is well studied in the Black-Scholes model. The connection between option pricing with early exercise features and the free-boundary problems in mathematical physics was considered and proved by McKean [32] and van Moerbeke [34]. Based on the exist results from mathematical physics, Kim [22], Jacka [18] and Carr et al. [8] established the EEP formula for the American style option independently, as well as a nonlinear integral equation for the exercise boundary. By introducing a more comprehensive change-of-variable formula with local time on curves derived in [36], G. Peskir characterized the optimal stopping boundary directly from the free-boundary problem for the American put option and also proved the uniqueness and regularity of this boundary in [37].

For the EEP formula of the same case in the jump-diffusion model, Pham first established the result for a specific type of exponential Lévy processes in [46]. Lamberton and Mikou extended this result to some more general cases and provided several important properties of both value function and the optimal exercise boundary, including the boundary behaviour near maturity and the smooth-fit property of the value function when crossing the boundary, in [29], [30] and [31]. The results of our research are in line with the results of [37] for Black-Scholes framework and [31] for exponential Lévy processes.

2.3.1 Assumptions and Notations

We assume that the underlying asset pays no dividend through out this chapter. For an American put option with strike price K, define the payoff function as:

$$\bar{G}(z) = (K - z)^+. \tag{2.11}$$

Following the framework of risk-neutral pricing, the value function of this option should be the expectation of the payoff function (2.11) under the physical probability measure

P:

$$\bar{V}(t,z) = \sup_{0\leq\tau\leq T-t} \mathsf{E}_{t,z}\left(e^{-r\tau}(K - Z^r_{t+\tau})^+\right) \tag{2.12}$$
$$= \sup_{0\leq\tau\leq T-t} \mathsf{E}_{t,z}\left(e^{-r\tau}\bar{G}(Z^r_{t+\tau})\right),$$

where τ is a stopping time of the exponential Lévy process Z^r_t defined in (2.8).

Here we would also like to introduce another form of payoff and value function of American put options with respect to the double exponential jump diffusion process X^r_t:

$$G(x) = (K - e^x)^+, \tag{2.13}$$

$$V(t,x) = \sup_{0\leq\tau\leq T-t} \mathsf{E}_{t,x}\left(e^{-r\tau}(K - e^{X^r_{t+\tau}})^+\right) \tag{2.14}$$
$$= \sup_{0\leq\tau\leq T-t} \mathsf{E}_{t,x}\left(e^{-r\tau}G(X^r_{t+\tau})\right),$$

where $x \in (-\infty, \infty)$, $t \in [0, T]$ and τ is a stopping time of the jump-diffusion process X^r_t also defined in (2.8).

Please note that the value function $\bar{V}(t,z)$ under the exponential Lévy process Z^r_t in (2.12) and the value function $V(t,x)$ under the jump-diffusion process X^r_t in (2.14) will exactly lead to the same result. By using the form $\bar{V}(t,z)$, the properties of the value function can be proved easily and concisely, while $V(t,x)$ is more convenient to be used in the derivation of the EEP formula, the nonlinear integral equations system and the properties of the early exercise boundary. This is the reason for having both of them in this section. The same approach is applied to the boundary function $\bar{b}(t)$ and $b(t)$ in the following content.

2.3.2 Structure of Optimal Stopping Regions

Since the payoff function (2.11) is continuous, it is possible to apply [44, Corollary 2.9] with [44, Remark 2.10]. Then we can have the following theorem with respect to the optimal stopping region and conclude that there exists an optimal stopping time for (2.12).

Theorem 2.1. *The continuation region and stopping region of the American put option described above are:*

$$C = \{(t, z) \in [0, T) \times (0, \infty) | \bar{V}(t, z) > \bar{G}(z)\} \tag{2.15}$$

$$\bar{D} = \{(t, z) \in [0, T] \times (0, \infty) | \bar{V}(t, z) = \bar{G}(z)\} \tag{2.16}$$

It means that the stopping time $\tau_{\bar{D}}$ *defined by:*

$$\tau_{\bar{D}} = \inf\{0 \leq s \leq T - t | Z^r_{t+s} \in \bar{D}\}, \tag{2.17}$$

is optimal in (2.12).

The following property for the structure of the optimal stopping region will also be useful for our further research.

Property 2.1. *The continuation set* C *and the stopping set* $\bar{D}$ *defined in Theorem 2.1 for the American put option are both not empty.*

To prove this property, we need to introduce a result from [27, Theorem 3]. It states that for the American put option with infinite horizon, there exists an optimal stopping point $0 < z^* < K$ such that

$$\bar{V}_\infty(z) = \bar{G}(z) = K - z \qquad \text{if} \qquad z < z^*, \tag{2.18}$$

where $\bar{V}_\infty(z)$ is the value function of the perpetual American put option. Now we can prove the theorem above.

Proof.

1. We claim that all points (t, z) with $z \geq K > 0$ for $0 \leq t < T$ belong to the continuous set C. Indeed, this is easily verified by considering $\tau_\varepsilon = \inf\{0 \leq s \leq T - t | Z^r_{t+s} \leq K - \varepsilon\}$ for $0 < \varepsilon < K$ and noting that $\mathsf{P}_{t,z}(0 < \tau_\varepsilon < T - t) > 0$ if $z \geq K$ with $0 \leq t < T$. The strict inequality implies that $\mathsf{E}_{t,z}\left(e^{-r\tau_\varepsilon}(K - Z^r_{t+\tau_\varepsilon})^+\right) > 0$. Then by the definition of the value function, we have that

$$\bar{V}(t, z) = \sup_{0 \leq \tau \leq T-t} \mathsf{E}_{t,z}\left(e^{-r\tau}(K - Z^r_{t+\tau})^+\right) > 0.$$

On the other hand, $\bar{G}(z) = (K - z)^+ = 0$ for all $z \geq K$ for $0 \leq t < T$. Thus the continuous set C is not empty as claimed.

2. From the definition of the value function (2.12), we can state that $\bar{V}(t,z) \leq \bar{V}_\infty(z)$ for any $0 \leq t \leq T$. Thus for any points (t,z) with $0 < z < z^*$ for $0 \leq t \leq T$, we have $\bar{V}(t,z) \leq \bar{V}_\infty(z) = \bar{G}(z)$. It is clear that such a point belongs to the stopping set $\bar{D}$. The stopping set $\bar{D}$ is not empty as claimed. □

2.3.3 Some Properties of the Value and Boundary Functions

In this part, we will give all the properties of the value and boundary functions which should be used in the derivation of the EEP representation of the American put option. Detailed proofs of them are also provided, except those which have already been introduced by other authors.

Property 2.2.

- *For* $t \in [0,T]$*, the function* $z \mapsto \bar{V}(t,z)$ *is decreasing and convex on* $(0,\infty)$*.*
- *For* $z \in (0,\infty)$*, the function* $t \mapsto \bar{V}(t,z)$ *is continuous and decreasing on* $[0,T]$*.*

This property was introduced and proved in [30, Proposition 2.2]. Now take any given $t^* \in [0,T]$, we have

- The function $z \mapsto \bar{V}(t^*,z)$ is decreasing and convex.
- The payoff function is $\bar{G}(z) = (K-z)^+$.
- Both continuation set C and stopping set $\bar{D}$ are not empty.
- $\bar{V}(t^*,z) = \bar{G}(z)$ for $z < z^*$ since all points (t,z) with $0 < z \leq z^*$ for $0 \leq t \leq T$ belongs to the stopping set $\bar{D}$.
- $\bar{V}(t^*,z) > \bar{G}(z)$ for $z \geq K$ since all points (t,z) with $z \geq K$ for $0 \leq t < T$ belong to the continuous set C

The combination of these facts confines that the shape of the value function and the payoff function can only take the form as illustrated in Figure 2.1. Observe this figure, for each $t \in [0,T]$, we can have a point where the value function joins the payoff function. Denote it as $\bar{b}(t)$, then we have the following theorem.

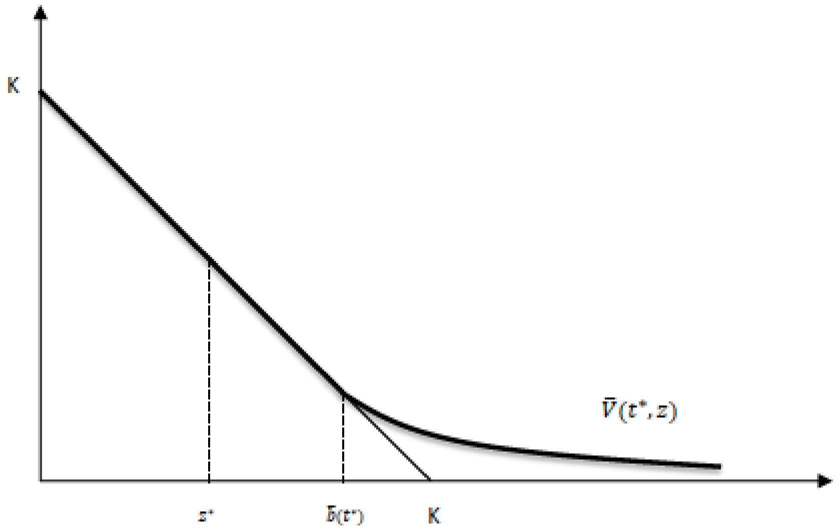

Figure 2.1: A simple illustration of the behaviour of the value function and the payoff function of an American put option at a fixed time $t^* \in [0, T]$.

Theorem 2.2. *There exists a function* $\bar{b} : [0, T] \to \mathbb{R}$ *satisfying* $0 < z^* \leq \bar{b}(t) < K$ *for all* $0 \leq t < T$ *such that the continuation set* C *of (2.12) equals*

$$C = \{(t, z) \in [0, T) \times (0, \infty) | z > \bar{b}(t)\} \tag{2.19}$$

and the stopping set $\bar{D}$ *is the closure of the set*

$$D = \{(t, z) \in [0, T) \times (0, \infty) | z < \bar{b}(t)\} \tag{2.20}$$

joined with the remaining points (T, z) *for* $z \geq \bar{b}(T)$.

This function $\bar{b}(t)$ is just the early exercise boundary which we are interested in. It can also be called as optimal stopping boundary or free-boundary. The name "free-boundary" comes from the connection between the option pricing problem and the ice melting problem in Physics. Before moving to the study of free-boundary $\bar{b}(t)$, we would like to introduce some more properties of the value function, which will be used in further derivations.

Property 2.3. *The value function* $(t, z) \mapsto \bar{V}(t, z)$ *is continuous on* $[0, T] \times (0, \infty)$.

Proof. For this, it is enough to prove that

$$z \mapsto \bar{V}(t, z) \text{ is continuous at } z_0, \tag{2.21}$$

$$t \mapsto \bar{V}(t, z) \text{ is uniformly continuous at } t_0 \text{ for } z \in [z_0 - \delta, z_0 + \delta], \tag{2.22}$$

for each $(t_0, z_0) \in [0, T] \times (0, \infty)$ with some $\delta > 0$ small enough. Indeed, (2.21) follows from Property 2.2 directly. And following the similar proof in Peskir and Shiryaev's book [44, Section 25, p.381] we can easily prove that (2.22) also holds. Let us fix arbitrary $0 \le t_1 < t_2 \le T$ and $z \in (0, \infty)$ and let $\tau_1 = \tau_*(t_1, z)$ denote the optimal stopping time for $\bar{V}(t_1, z)$. Set $\tau_2 = \tau_1 \wedge (T - t_2)$. Thus we have $\tau_2 \le \tau_1$ and $\tau_2 \le T - t_2$. It is also easy to verify that $t \mapsto \bar{V}(t, z)$ is decreasing on $[0, T]$ by the definition of value function (2.12). Then we have

$$\begin{aligned} 0 &\le \bar{V}(t_1, z) - \bar{V}(t_2, z) \\ &\le \mathsf{E}[e^{-r\tau_1}(zZ^r_{\tau_1} - K)^+] - \mathsf{E}[e^{-r\tau_2}(zZ^r_{\tau_2} - K)^+] \\ &\le \mathsf{E}[e^{-r\tau_2}((zZ^r_{\tau_1} - K)^+ + (zZ^r_{\tau_2} - K)^+)] \\ &\le \mathsf{E}[e^{-r\tau_2}(zZ^r_{\tau_1} - zZ^r_{\tau_2})^+] \\ &\le z\mathsf{E}(Z^r_{\tau_1} - Z^r_{\tau_2})^+. \end{aligned} \tag{2.23}$$

Using the fact that X^r_t is a Levy process with stationary independent increments and $\tau_1 - \tau_2 \le t_1 - t_2$, we know that

$$\begin{aligned} \mathsf{E}(Z^r_{\tau_1} - Z^r_{\tau_2})^+ &= \mathsf{E}(\mathsf{E}(Z^r_{\tau_1} - Z^r_{\tau_2})^+ | \mathcal{F}_{\tau_2}) \\ &= \mathsf{E}(Z^r_{\tau_2}\mathsf{E}(Z^r_{\tau_1}/Z^r_{\tau_2} - 1)^+ | \mathcal{F}_{\tau_2}) \\ &= \mathsf{E}(Z^r_{\tau_2})\mathsf{E}(e^{X^r_{\tau_1} - X^r_{\tau_2}} - 1)^+ \\ &= \mathsf{E}(Z^r_{\tau_2})\mathsf{E}(\sup_{0 \le t \le t_2 - t_1} e^{X^r_t} - 1)^+ \\ &=: \mathsf{E}(Z^r_{\tau_2})L(t_2 - t_1). \end{aligned} \tag{2.24}$$

By the property of the jump-diffusion process X^r_t, it can be seen that $L(t_2 - t_1)$ as $t_2 - t_1 \to 0$. And we have

$$0 \le \bar{V}(t_1, z) - \bar{V}(t_2, z) \le z\mathsf{E}(Z^r_{\tau_2})L(t_2 - t_1) \le ze^{rT}L(t_2 - t_1), \tag{2.25}$$

from where (2.22) becomes evident. This complete the proof. □

Property 2.4. *$\bar{V}(t, z)$ is $C^{1,2}$ on C (and $C^{1,2}$ on $\bar{D}$).*

Proof. This follows from the strong Markov property of Z^r_t, Theorem 2.1 and Property 2.3. □

The following property regarding to the free-boundary $\bar{b}(t)$ is needed in proving the smooth-fit property of the value function below.

Property 2.5. *The boundary $\bar{b}(t)$ is increasing on $[0, T]$.*

Proof. Recall that $t \mapsto \bar{V}(t, z)$ is decreasing on $[0, T]$ and the payoff function $\bar{G}(z)$ is not depending on the time t. Hence if $(t, z) \in C$, then take any other $0 < t' < t$, we have $\bar{V}(t', z) - \bar{G}(z) \geq \bar{V}(t, z) - \bar{G}(z) > 0$. This means that $(t', z) \in C$ for all $0 < t' < t$. Combined with the definition of the continuation region C in (2.19), we can state that $t \mapsto \bar{b}(t)$ is increasing on $[0, T]$. □

Now we can introduce an important property for the behaviour of the value function when it crosses the free-boundary. This property will alow us to eliminate the local-time term and simplify the EEP representation in the next section. The proof of the smooth-fit property for general exponential Lévy processes in American put options was given by Lamberton and Mikou in [30, Theorem 4.1].

Property 2.6. *The smooth-fit property holds, i.e. that $z \mapsto \bar{V}(t, z)$ is C^1 at $\bar{b}(t)$:*

$$\bar{V}_z(t, z) = \bar{G}'(z) = -1 \quad \text{for} \quad z = \bar{b}(t). \tag{2.26}$$

Proof. Since X_t^r is a jump-diffusion process which has infinite variation, by [28, Theorem 6.5] we can say that 0 is regular for $(-\infty, 0)$ for X_t^r. Together with Property 2.5, the existence of the smooth-fit property comes from [30, Theorem 4.1] directly. □

The properties of the free-boundary for American put options were fully studied by Lamberton and Mikou in [30] and [29]. We will introduce some of them here without detailed proof.

Property 2.7. *The boundary $\bar{b}(t)$ is continuous on $[0, T]$.*

Proof. See [29, Theorem 4.2] for detailed proof. □

Property 2.8. *At the maturity, the value of the boundary $\bar{b}$ equals to*

$$\bar{b}(T) \overset{def}{=} \lim_{t \to T} \bar{b}(t) = \begin{cases} = K & \text{if} \quad \frac{\lambda p}{\eta_1 - 1} \leq r \\ = \ell & \text{if} \quad \frac{\lambda p}{\eta_1 - 1} > r \end{cases} \tag{2.27}$$

where ℓ is the unique real number in the interval $(0, K)$ such that

$$\bar{\psi}(z) = rK, \tag{2.28}$$

where $\bar{\psi}$ is the function defined $z \in (0, K)$:

$$\bar{\psi}(z) = \lambda \int_{-\infty}^{\infty} (ze^y - K)^+ f_Y(y) dy. \tag{2.29}$$

Proof. Following [30, Theorem 3.2], we only need to calculate the value of $\int(e^y - 1)^+\nu(dy)$:

$$\begin{aligned}\int(e^y-1)^+\nu(dy) &= \int_0^\infty (e^y-1)\lambda f_Y(y)dy \\ &= \lambda p\eta_1\int_0^\infty (e^y-1)e^{-\eta_1 y}dy \\ &= \frac{\lambda p}{\eta_1-1}.\end{aligned}$$

□

From now on, we will focus on the value function of the form $V(t,x)$ defined in (2.14). Similar to Theorem 2.2, we need to introduce the free boundary $b(t)$ w.r.t $V(t,x)$:

Theorem 2.3. *There exists a function* $b:[0,T]\to\mathbb{R}$ *satisfying* $lnz^* \leq b(t) < \ln K$ *for all* $0\leq t<T$ *such that the continuation set* $C=\{(t,x)\in[0,T)\times(-\infty,\infty)|V(t,x)>G(x)\}$ *of the value function (2.14) equals*

$$C=\{(t,x)\in[0,T)\times(-\infty,\infty)|x>b(t)\} \tag{2.30}$$

and the stopping set $\bar{D}=\{(t,x)\in[0,T)\times(-\infty,\infty)|V(t,x)=G(x)\}$ *is the closure of the set*

$$D=\{(t,x)\in[0,T)\times(-\infty,\infty)|x<b(t)\} \tag{2.31}$$

joined with the remaining points (T,x) *for* $x\geq b(T)$.

Since the pair of function $V(t,x)$ and $b(t)$ is essentially equivalent to $\bar{V}(t,z)$ with $\bar{b}(t)$, we can easily convert Property 2.3 - 2.8 introduced above to the following collections.

Property 2.9. *The properties of the value function* $V(t,x)$*:*

- *The value function* $(t,x)\mapsto V(t,x)$ *is continuous on* $[0,T]\times(-\infty,\infty)$.
- *The function* $x\mapsto V(t,x)$ *is decreasing and convex.*
- *The function* $t\mapsto V(t,x)$ *is decreasing and continuous.*
- *The smooth fit holds:* $V_x(t,x)=G'(x)$ *for* $x=b(t)$.

- *$V(t,x)$ is $C^{1,2}$ on C (and $C^{1,2}$ on $\bar{D}$).*

Property 2.10. *The properties of the free-boundary $b(t)$:*

- *$b(t)$ is increasing and continuous on $[0,T]$.*
- $b(T) \overset{def}{=} \lim_{t\to T} b(t) = \begin{cases} = \ln K & \textit{if} \quad \frac{\lambda p}{\eta_1 - 1} \le r \\ = \ln \ell & \textit{if} \quad \frac{\lambda p}{\eta_1 - 1} > r \end{cases}$

2.3.4 EEP Representation of American Put Options

Comparing the American option with European options, the American style option allows holder to exercise before maturity, so the value of an American option should equal to the value of the corresponding European option plus an early exercise premium. Writing the value function of American options into this form, we will have the EEP representation. This section is dedicated to the derivation of the EEP representation of finite horizon American put options.

Standard arguments [44, p.131, Killed version] based on the strong Markov property link the unknown value function $V(t,x)$ and the unknown optimal stopping boundary $b(t)$ to the following free-boundary problem for $(t,x) \in [0,T) \times (-\infty, \infty)$:

$$V_t + \mathbb{L}_{X^r} V = rV \qquad \text{in} \quad C, \tag{2.32}$$

$$V(t,x) = (K - e^x)^+ \qquad \text{for} \quad x = b(t), \tag{2.33}$$

$$V_x(t,x) = -e^x \qquad \text{for} \quad x = b(t), \tag{2.34}$$

$$V(t,x) > (K - e^x)^+ \qquad \text{in} \quad C, \tag{2.35}$$

$$V(t,x) = (K - e^x)^+ \qquad \text{in} \quad D. \tag{2.36}$$

This means that the function $V(t,x)$ and $b(t)$ is a solution pair to the system of equation (2.32) - (2.36). This connection is an important breakthrough in handling option pricing problems with early exercise features. It was widely used in many articles such as [25] and [27]. However, since the uniqueness of the solution is not easy to prove for this free-boundary problem. Most of these articles left the uniqueness to be an open question, which can be seen as a small flaw in preciseness. The uniqueness of the free-boundary problem is also not a major study object for this chapter. But with the results obtained in the previous section, we can transfer the free boundary problem into an equation system which only contains two nonlinear integral equations for the function $V(t,x)$ and $b(t)$. Then we will eventually prove the uniqueness of solutions for this equation system with some additional conditions including the free-boundary equation (2.35) and (2.36), which an be view as a main contribution of this section. The EEP representation acts as the foundation of our research, so we need to derive it first.

Peskir proposed a series of change-of-variable formulas with local time for semi-martingales with jumps in [39]. By Property 2.9 and 2.10, as well as equation (2.32) - (2.36), it is easy to verify that for the jump-diffusion process X_t^r, the conditions of [39, Theorem 3.1] hold. Applying this change-of-variable formula to $V(t, X_t^r)$, we can have,

$$\begin{aligned} V(t,X_t^r) =& V(0,X_0^r) + \int_0^t \frac{1}{2}\Big(\frac{\partial V}{\partial t}(s,X_s^r+) + \frac{\partial V}{\partial t}(s,X_s^r-)\Big)ds \\ &+ \int_0^t \frac{1}{2}\Big(\frac{\partial V}{\partial x}(s,X_s^r+) + \frac{\partial V}{\partial x}(s,X_s^r-)\Big)dX_s^{r,c} \\ &+ \frac{1}{2}\int_0^t \frac{1}{2}\Big(\frac{\partial^2 V}{\partial x^2}(s,X_s^r+) + \frac{\partial^2 V}{\partial x^2}(s,X_s^r-)\Big)d[X^{r,c},X^{r,c}]_s \\ &+ \sum_{0\le s\le t}^{\Delta X_s^r\neq 0}\Big(V(s,X_s^r) - V(s-,X_{s-}^r)\Big). \end{aligned} \tag{2.37}$$

The local time term is eliminated in (2.37) due to the existence of the smooth-fit property for the value function $V(t,x)$ (Property 2.9). Moreover, the following fact holds

$$\mathsf{P}\left(X_s^r = b(s)\right) = 0 \quad \text{for} \quad s \in (0,t] \tag{2.38}$$

since X_s^r has a continuous density on $\mathbb{R}$. This fact (2.38) implies that the integrals in (2.37) can be simplified further to

$$\begin{aligned} V(t,X_t^r) =& V(0,X_0^r) + \int_0^t \frac{\partial V}{\partial t}(s,X_s^r)ds + \int_0^t \frac{\partial V}{\partial x}(s,X_s^r)dX_s^{r,c} \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial x^2}(s,X_s^r+)d[X^{r,c},X^{r,c}]_s + \sum_{0\le s\le t}^{\Delta X_s^r\neq 0}\Big(V(s,X_s^r) - V(s-,X_{s-}^r)\Big). \end{aligned} \tag{2.39}$$

Expand and reorganize (2.39), using the infinitesimal generator introduced in (2.10) we have

$$V(t,X_t^r) = V(0,X_0^r) + \int_0^t (V_t + \mathbb{L}_{X^r}V)(s,X_s^r)ds + M_t^1 + M_t^2, \tag{2.40}$$

where

$$M_t^1 = \sigma \int_0^t \frac{\partial V}{\partial x}(s, X_s^r) dW_s, \tag{2.41}$$

$$M_t^2 = \sum_{0 \le s \le t}^{\Delta X_s^r \neq 0} \Big(V(s, X_s^r) - V(s-, X_{s-}^r) \Big) \tag{2.42}$$

$$- \lambda \int_0^t \int_{-\infty}^{+\infty} [V(s-, X_{s-}^r + y) - V(s-, X_{s-}^r)] f_Y(y) dy ds.$$

Following the similar derivation path for American put options under Black-Scholes framework proposed by Peskir in [37], we would like to prove that M_t^1 and M_t^2 are martingales.

Proposition 2.1. *$M^1 = (M_t^1)_{0 \le t \le T}$ defined in (2.41) is a martingale under* P.

Proof. Recall that $x \mapsto V(t,x)$ is decreasing and convex. And $V(t,x) = G(x)$ for $x \le b(t) \le \ln K$. Thus we have

$$-e^x \le V_x(t,x) \le 0. \tag{2.43}$$

To prove M^1 is a martingale under P, we only have to prove that $\mathsf{E}\langle M^1, M^1 \rangle_T < \infty$. $\langle M^1, M^1 \rangle_T$ stands for the quadratic variation of M_T^1, see [54, Proposition 8.6] for details. Indeed,

$$\begin{aligned} \mathsf{E}\langle M, M \rangle_T &= \sigma^2 \mathsf{E}[\int_0^T (V_x(s, X_s^r))^2 ds] \\ &\le \sigma^2 \int_0^T \mathsf{E}(e^{X_s^r})^2 ds \\ &= \sigma^2 \int_0^T \mathsf{E}(e^{2X_s^r}) ds. \end{aligned} \tag{2.44}$$

Recall the moment-generating function of X_t^r is given by $\mathsf{E}[e^{\theta X_t^r}] = e^{F(\theta)t}$, where the function $F(\theta)$ is defined by

$$F(\theta) = \theta(r - \frac{\sigma^2}{2} - \lambda\zeta) + \frac{1}{2}\theta^2\sigma^2 + \lambda(\frac{p\eta_1}{\eta_1 - \theta} + \frac{q\eta_2}{\eta_2 + \theta} - 1). \tag{2.45}$$

Thus $F(2)$ is a constant denoted as F^*, then we have

$$\sigma^2 \int_0^T \mathsf{E}(e^{2X_s^r}) ds \tag{2.46}$$

$$= \sigma^2 \int_0^T e^{F^* s} ds$$

$$= \frac{\sigma^2}{F^*}(e^{TF^*} - 1) < \infty. \tag{2.47}$$

Thus the proof is completed. □

Proposition 2.2. *$M^2 = (M_t^2)_{0\le t\le T}$ defined in (2.42) is a martingale under* P.

Proof. We can rewrite the first part of M_t^2 as

$$\begin{aligned} &\sum_{0\le s\le t}^{\Delta X_s^r \neq 0} \Big(V(s, X_s^r) - V(s-, X_{s-}^r) \Big) \\ &= \int_0^t \int_{-\infty}^{+\infty} [V(s-, X_{s-}^r + y) - V(s-, X_{s-}^r)] J_{X^r}(dy \times ds), \end{aligned} \tag{2.48}$$

where J_{X^r} is the jump measure of X_t^r. Recall that the compensated jump measure of X_t^r is $\widetilde{J}_{X^r}(dy \times ds) = J_{X^r}(dy \times ds) - \lambda ds f(dy)$, where $f(dy)$ is the density function of Y_i. Thus M_t^2 equals

$$\begin{aligned} M_t^2 = &\int_0^t \int_{-\infty}^{+\infty} [V(s-, X_{s-}^r + y) - V(s-, X_{s-}^r)] J_{X^r}(dy \times ds) \\ &- \lambda \int_0^t \int_{-\infty}^{+\infty} [V(s-, X_{s-}^r + y) - V(s-, X_{s-}^r)] f_Y(y) dy ds \\ = &\int_0^t \int_{-\infty}^{+\infty} [V(s-, X_{s-}^r + y) - V(s-, X_{s-}^r)] \big(J_{X^r}(dy \times ds) - \lambda ds f(dy) \big) \\ = &\int_0^t \int_{-\infty}^{+\infty} [V(s-, X_{s-}^r + y) - V(s-, X_{s-}^r)] \widetilde{J}_{X^r}(dy \times ds). \end{aligned} \tag{2.49}$$

From [28, Corollary 4.6], we know that this integral with respect to the compensated jump measure of a Lévy process is a martingale under P if

$$\mathsf{E}\Big[\int_0^t \int_{-\infty}^{+\infty} |V(s-, X_{s-}^r + y) - V(s-, X_{s-}^r)| \lambda f_Y(y) dy ds \Big] < \infty. \tag{2.50}$$

Indeed, since the value function is bounded with $0 < V(t,x) \le K$ for any $(t,x) \in [0,\infty) \times (-\infty,\infty)$ by its definition, we can easily have

$$\mathsf{E}\Big[\int_0^t \int_{-\infty}^{+\infty} |V(s-, X_{s-}^r + y) - V(s-, X_{s-}^r)| \lambda f_Y(y) dy ds \Big] \le \lambda K < \infty$$

Thus the proof is completed. □

Based on the similar method above, applying [39, Theorem 3.1] to $e^{-rs}V(t + s, X_{t+s}^r)$, taking expectation $\mathsf{E}_{t,x}$ on both sides and set $s = T - t$, we will have

$$V(t,x) = e^{-r(T-t)} \mathsf{E}_{t,x} G(X_T^r) - \mathsf{E}_{t,x} \Bigg(\int_0^{T-t} e^{-ru} (V_t + \mathbb{L}_{X^r} V - rV)(t+u, X_{t+u}^r) du \Bigg). \tag{2.51}$$

Recalling the free boundary problem (2.32), equation (2.51) can be write as

$$\begin{aligned} V(t,x) =& e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r) \\ &- \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(V_t + \mathbb{L}_{X^r}V - rV)(t+u, X_{t+u}^r)I(X_{t+u}^r \le b(t+u))du\right). \end{aligned} \tag{2.52}$$

If we were pricing the option under a pure diffusion model such as classic Black-Scholes, then the existence of the indicator function $I(X_{t+u}^r \le b(t+u))$ will alow us to replace $(V_t + \mathbb{L}_{X^r}V - rV)$ by $(G_t + \mathbb{L}_{X^r}G - rG)$ directly in (2.52). Since the payoff function $G(x)$ is already known and deterministic, then it will be much easier to derive an explicit form for the value function. However, by observing the infinitesimal generator defined in (2.10), we can see it also depends on the global property of the function $V(t,x)$ if the underlying process involves jumps. The main difficulty for option pricing with jumps comes from this non local integral term in the operator $\mathbb{L}_{X^r}$. Replacing $\mathbb{L}_{X^r}V$ by $\mathbb{L}_{X^r}G$ is infeasible in our approach here, thus we can only apply the add-minus trick and leave the remain part as it is in the EEP representation. This is why our value function and free boundary have to be a pair of unique solution to a system of two equations, while Peskir proved that under the Black-Scholes framework the optimal boundary itself can be the unique solution to a single nonlinear integral equation in [37]. Our research here can be seen as an extension of Peskir's method in deriving EEP formula and proving uniqueness.

An interesting fact we should note here is that if only negative jumps are allowed on the path, then it will be reasonable to replace $(V_t+\mathbb{L}_{X^r}V-rV)$ by $(G_t+\mathbb{L}_{X^r}G-rG)$. Moreover, for such a one-sided jump-diffusion process, we can obtain a result which is much more compact and very similar to the conclusion of [37]. This degenerate case will be studied in the next section.

Back to the derivation of the EEP formula, before applying the add-minus technique to equation (2.52), introduce a function

$$g(x) = K - e^x \tag{2.53}$$

It is obvious that $V(t,x) = G(x) = g(x)$ for $x \le b(t) \le \ln K$. Now let's compute the following expression first: for any $x \in \mathbb{R}$

$$
\begin{aligned}
&(g_t + \mathbb{L}_{X^r} g - rg)(x) \qquad (2.54)\\
&= \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial x^2}(x) + (r - \lambda\zeta - \frac{\sigma^2}{2})\frac{\partial g}{\partial x}(x) + \lambda \int_{-\infty}^{+\infty} [g(x+y) - g(x)] f_Y(y) dy - rg(x)\\
&= \frac{1}{2}\sigma^2(-e^x) + (r - \lambda\zeta - \frac{\sigma^2}{2})(-e^x) + \lambda \int_{-\infty}^{+\infty} [e^x - e^{x+y}] f_Y(y) dy - r(K - e^x)\\
&= -rK + \lambda e^x \zeta + \lambda e^x \int_{-\infty}^{+\infty} [1 - e^y] f_Y(y) dy\\
&= -rK + \lambda e^x \big(\zeta + \int_{-\infty}^{+\infty} f_Y(y) dy - \int_{-\infty}^{+\infty} e^y f_Y(y) dy\big)\\
&= -rK + \lambda e^x (\zeta + 1 - \mathsf{E}(e^Y))\\
&= -rK,
\end{aligned}
$$

where the last equation comes from the definition of $\mathsf{E}(e^Y)$ in (2.5) and the definition of ζ in (2.9). Thus by using the add-minus trick, equation (2.52) can be rewrite as

$$
\begin{aligned}
& V(t,x) \qquad (2.55)\\
= \;& e^{-r(T-t)} \mathsf{E}_{t,x} G(X_T^r)\\
& - \mathsf{E}_{t,x}\bigg(\int_0^{T-t} e^{-ru} (V_t + \mathbb{L}_{X^r} V - rV)(t+u, X_{t+u}^r) I(X_{t+u}^r \le b(t+u)) du \bigg)\\
& - \lambda \mathsf{E}_{t,x}\bigg[\int_0^{T-t} e^{-ru} \int_0^{+\infty} \big(g(X_{t+u}^r + y)\\
& \quad - g(X_{t+u}^r + y)\big) f_Y(y) dy I\big(X_{t+u}^r \le b(t+u)\big) du \bigg]\\
= \;& e^{-r(T-t)} \mathsf{E}_{t,x} G(X_T^r)\\
& - \mathsf{E}_{t,x}\bigg(\int_0^{T-t} e^{-ru} (g_t + \mathbb{L}_{X^r} g - rg)(X_{t+u}^r) I(X_{t+u}^r \le b(t+u)) du \bigg)\\
& - \lambda \mathsf{E}_{t,x}\bigg[\int_0^{T-t} e^{-ru} \int_0^{+\infty} \big(V(t+u, X_{t+u}^r + y)\\
& \quad - g(X_{t+u}^r + y)\big) f_Y(y) dy I\big(X_{t+u}^r \le b(t+u)\big) du \bigg].
\end{aligned}
$$

where the last equation also comes from the fact that $V(t,x) = g(x)$ for $x \le b(t) \le \ln K$. Inserting the result from (2.54), we will have the EEP representation for an American put option with no dividends:

Theorem 2.4. *The arbitrage-free price of the American put options without dividends admits the following early exercise premium representation under a double exponential* *jump diffusion process*

$$
\begin{aligned}
& V(t,x) && (2.56)\\
= \; & e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r) + rK\int_0^{T-t} e^{-ru}\mathsf{P}_{t,x}\big(X_{t+u}^r \le b(t+u)\big)du \\
& - \lambda\mathsf{E}_{t,x}\Bigg[\int_0^{T-t} e^{-ru}\int_0^{+\infty}\big(V(t+u, X_{t+u}^r + y) \\
& \quad - g(X_{t+u}^r + y)\big)f_Y(y)dyI\big(X_{t+u}^r \le b(t+u)\big)du\Bigg].
\end{aligned}
$$

The first part of (2.56) is the arbitrage-free price of the corresponding European put option. And the rest is well known as the early exercise premium for the American style options for its special feature.

2.3.5 The Uniqueness of the Value Function and the Free Boundary

The EEP formula (2.56) itself alone is meaningless in calculating the arbitrage free price $V(t,x)$ for options, since it involves another unknown function $b(t)$. To obtain one more nonlinear integral equation for the early exercise boundary $b(t)$, we just need to substitute $x = b(t)$ into the EEP formula (2.56):

$$
\begin{aligned}
& K - e^{b(t)} && (2.57)\\
= \; & e^{-r(T-t)}\mathsf{E}_{t,b(t)}(K - e^{X_T^r})^+ + rK\int_0^{T-t} e^{-ru}\mathsf{P}_{t,b(t)}\big(X_{t+u}^r \le b(t+u)\big)du \\
& - \lambda\mathsf{E}_{t,b(t)}\Bigg[\int_0^{T-t} e^{-ru}\int_0^{+\infty}\big(V(t+u, X_{t+u}^r + y) \\
& \quad - g(X_{t+u}^r + y)\big)f_Y(y)dyI\big(X_{t+u}^r \le b(t+u)\big)du\Bigg].
\end{aligned}
$$

From the previous derivation we know that the value function $V(t,x)$ and free boundary $b(t)$ is a solution pair for the nonlinear integral equation system consisting of (2.56) and (2.57). To make this result more effective and rigorous in practice, the uniqueness of this equation system is essential for our research. Peskir provided a clean and robust proof for the uniqueness problem under the Black-Scholes framework in [37]. But due to the existence of a nonlocal integral term in both equation (2.56) and (2.57), directly employing the main approach in [37] can no longer prove the uniqueness. Inspired by Pham's work on free-boundary problems for jump diffusion problem in

[46], we would like to introduce some additional conditions into the equation system to ensure its uniqueness. Pham proved the uniqueness for a complex equation system based on the free-boundary problems (2.32) - (2.36). His proof also requires several strong assumptions and conditions. By using a technique proposed by Peskir and Shiryaev in [44, p.392, Remark 25.4], we successfully reduce the number of conditions required and make the proof of uniqueness partly independent from the free-boundary problems comparing to that of Pham's. There might be some ways by which we can derive the uniqueness without all these additional conditions for jump diffusion processes. Eliminating the dependence on them will leave to be an open question for future studies. Here is our result for the uniqueness.

Theorem 2.5. *Assume the following condition holds:*

$$r \geq \lambda \frac{p}{\eta_1 - 1}. \tag{2.58}$$

Then the pair of value function and early exercise boundary of the American put options without dividends (V, b) is the unique solution pair (v, b^v) of the following nonlinear integral equation system:

$$\begin{aligned} & v(t,x) \\ = \; & e^{-r(T-t)} \mathsf{E}_{t,x} G(X_T^r) + rK \int_0^{T-t} e^{-ru} \mathsf{P}_{t,x}\big(X_{t+u}^r \leq b^v(t+u)\big) du \\ & - \lambda \mathsf{E}_{t,x} \bigg[\int_0^{T-t} e^{-ru} \int_0^{+\infty} \big(v(t+u, X_{t+u}^r + y) \\ & \quad - g(X_{t+u}^r + y) \big) f_Y(y) dy I\big(X_{t+u}^r \leq b^v(t+u)\big) du \bigg], \end{aligned} \tag{2.59}$$

$$\begin{aligned} & K - e^{b^v(t)} \\ = \; & e^{-r(T-t)} \mathsf{E}_{t,b^v(t)} (K - e^{X_T^r})^+ + rK \int_0^{T-t} e^{-ru} \mathsf{P}_{t,b^v(t)}\big(X_{t+u}^r \leq b^v(t+u)\big) du \\ & - \lambda \mathsf{E}_{t,b^v(t)} \bigg[\int_0^{T-t} e^{-ru} \int_0^{+\infty} \big(v(t+u, X_{t+u}^r + y) \\ & \quad - g(X_{t+u}^r + y) \big) f_Y(y) dy I\big(X_{t+u}^r \leq b^v(t+u)\big) du \bigg], \end{aligned} \tag{2.60}$$

within the class $v : [0,T] \times \mathbb{R} \to \mathbb{R}$, $b^v : [0,T] \to \mathbb{R}$, *and satisfying:*

$$b^v(t) \le \ln K \quad \text{for all} \quad 0 \le t \le T, \tag{2.61}$$

$$x \mapsto v(t,x) \quad \text{is decreasing on} \quad \mathbb{R}, \tag{2.62}$$

$$v(t,x) > G(x) = (K - e^x)^+ \quad \text{if} \quad x > b^v(t), \tag{2.63}$$

$$v(t,x) = G(x) = (K - e^x)^+ \quad \text{if} \quad x \le b^v(t). \tag{2.64}$$

To prove Theorem 2.5, we need to introduce the following lemmas.

Lemma 2.1. *If* $X = (X_t)_{t\ge 0}$ *is a Markov process, set* $F(t,x) = \mathsf{E}_{t,x}H(T,X_T)$ *for a measurable function* H *with* $\mathsf{P}_{t,x}(X_t = x) = 1$, *then the Markov property of* X *implies that* $F(t+s, X_{t+s})$ *is a martingale under* $\mathsf{P}_{t,x}$ *for* $0 \le s \le T-t$.

Proof. We have $F(t+s, X_{t+s}) = \mathsf{E}_{t+s,X_{t+s}}H(T,X_T)$. For any $s' < s$

$$\begin{aligned}
\mathsf{E}_{t,x}[F(t+s,X_{t+s})|\mathcal{F}_{t+s'}] =& \mathsf{E}_{t,x}[\mathsf{E}_{t+s,X_{t+s}}H(T,X_T)|\mathcal{F}_{t+s'}] \\
=& \mathsf{E}_{t,x}[\mathsf{E}_{t,x}H(T,X_T)|\mathcal{F}_{t+s}|\mathcal{F}_{t+s'}] \\
=& \mathsf{E}_{t,x}[H(T,X_T)|\mathcal{F}_{t+s'}] \\
=& \mathsf{E}_{t+s',X_{t+s'}}[H(T,X_T)] \\
=& F(t+s', X_{t+s'})
\end{aligned}$$

□

Lemma 2.2. *If* $X = (X_t)_{t\ge 0}$ *is a Markov process, set* $F(t,x) = \mathsf{E}_{t,x}\int_0^{T-t} H(t+u, X_t + u)du$ *for a measurable function* H *with* $\mathsf{P}_{t,x}(X_t = x) = 1$, *then* $F(t+s, X_{t+s}) + \int_0^s H(t+u, X_{t+u})du$ *is a martingale under* $\mathsf{P}_{t,x}$ *for* $0 \le s \le T-t$.

Proof. For any $s' < s$

$$
\begin{aligned}
&\mathsf{E}_{t,x}\Big[F(t+s,X_{t+s}) + \int_0^s H(t+u,X_{t+u})du|\mathcal{F}_{t+s'}\Big]\\
=&\mathsf{E}_{t,x}\Big[\mathsf{E}_{t+s,X_{t+s}}\int_0^{T-t-s} H(t+s+u,X_{t+s+u})du + \int_0^s H(t+u,X_{t+u})du|\mathcal{F}_{t+s'}\Big]\\
=&\mathsf{E}_{t,x}\Big[\mathsf{E}_{t,x}\Big[\int_s^{T-t} H(t+u,X_{t+u})du|\mathcal{F}_{t+s'}\Big] + \int_0^s H(t+u,X_{t+u})du|\mathcal{F}_{t+s'}\Big]\\
=&\mathsf{E}_{t,x}\Big[\int_s^{T-t} H(t+u,X_{t+u})du|\mathcal{F}_{t+s'}\Big]\\
&\qquad + \mathsf{E}\Big[\int_0^{s'} H(t+u,X_{t+u})du + \int_{s'}^s H(t+u,X_{t+u})du|\mathcal{F}_{t+s'}\Big]\\
=&\mathsf{E}_{t,x}\Big[\int_{s'}^{T-t} H(t+u,X_{t+u})du|\mathcal{F}_{t+s'}\Big] + \int_0^{s'} H(t+u,X_{t+u})du\\
=&\mathsf{E}_{t+s',X_{t+s'}}\Big[\int_{s'}^{T-t} H(t+u,X_{t+u})du\Big] + \int_0^{s'} H(t+u,X_{t+u})du\\
=&\mathsf{E}_{t+s',X_{t+s'}}\Big[\int_0^{T-t-s'} H(t+s'+u,X_{t+s'+u})du\Big] + \int_0^{s'} H(t+u,X_{t+u})du\\
=&F(t+s',X_{t+s'}) + \int_0^{s'} H(t+u,X_{t+u})du
\end{aligned}
$$

□

Proof of Theorem 2.5

Proof. Suppose that there exists such a function pair (U, c) where $U : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $c : [0, T] \to \mathbb{R}$ satisfying the conditions of Theorem 2.5 and solving the equation

(2.59) - (2.60):

$$
\begin{aligned}
&U(t,x) &(2.65)\\
=\;& e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r) + rK\int_0^{T-t} e^{-ru}\mathsf{P}_{t,x}\big(X_{t+u}^r \le c(t+u)\big)du\\
&-\lambda\mathsf{E}_{t,x}\Big[\int_0^{T-t} e^{-ru}\int_0^{+\infty}\big(U(t+u, X_{t+u}^r + y)\\
&\quad - g(X_{t+u}^r + y)\big)f_Y(y)dyI\big(X_{t+u}^r \le c(t+u)\big)du\Big],\\
&K - e^{c(t)} &(2.66)\\
=\;& e^{-r(T-t)}\mathsf{E}_{t,c(t)}(K - e^{X_T^r})^+ + rK\int_0^{T-t} e^{-ru}\mathsf{P}_{t,c(t)}\big(X_{t+u}^r \le c(t+u)\big)du\\
&-\lambda\mathsf{E}_{t,c(t)}\Big[\int_0^{T-t} e^{-ru}\int_0^{+\infty}\big(U(t+u, X_{t+u}^r + y)\\
&\quad - g(X_{t+u}^r + y)\big)f_Y(y)dyI\big(X_{t+u}^r \le c(t+u)\big)du\Big].
\end{aligned}
$$

We just need to prove that this solution pair (U, c) globally coincides with (V, b).

1. Inspired by the method proposed in [44, p.392, Remark 25.4], we would like to introduce an stochastic process $M(s, X_s^r)$ which defined by

$$
\begin{aligned}
&M(s, X_s^r) &(2.67)\\
=\;& e^{-rs}U(t+s, x+X_s^r) + rK\int_0^s e^{-ru}I(x + X_u^r \le c(t+u))du\\
&-\lambda\int_0^s e^{-ru}\int_0^{+\infty}\big(U(t+u, x+X_u^r+y)\\
&\quad - g(x+X_u^r+y)\big)f_Y(y)dyI\big(x+X_u^r \le c(t+u)\big)du,
\end{aligned}
$$

where $X_0^r = 0$ and $(t, x) \in [0, T] \times \mathbb{R}$ is given and fixed. Insert equation (2.65) into this expression and rearrange it, we can have:

$$
M(s, X_s^r) = M_1 + M_2 - M_3, \tag{2.68}
$$

where

$$
\begin{aligned}
M_1 =& \mathsf{E}_{t+s,x+X_s^r}[e^{-r(T-t-s)}G(X_T^r)], && (2.69)\\
M_2 =& rK\int_0^s e^{-ru}I(X_{t+u}^r \le c(t+u))du && (2.70)\\
&+ rK\mathsf{E}_{t+s,x+X_s^r}\Big[\int_0^{T-t-s} e^{-r(s+u)}I(X_{t+s+u}^r \le c(t+s+u))du\Big],\\
M_3 =& \lambda\int_0^s e^{-ru}\int_0^{+\infty}\big(U(t+u,X_{t+u}^r+y)\\
&\quad - g(X_{t+u}^r+y)\big)f_Y(y)dyI\big(X_{t+u}^r\le c(t+u)\big)du && (2.71)\\
&+ \lambda\mathsf{E}_{t+s,x+X_s^r}\Bigg[\int_0^{T-t-s} e^{-r(s+u)}\int_0^{+\infty}\big(U(t+s+u,X_{t+s+u}^r+y)\\
&\quad - g(X_{t+s+u}^r+y)\big)f_Y(y)dyI\big(X_{t+s+u}^r\le c(t+s+u)\big)du\Bigg].
\end{aligned}
$$

Applying Lemma 2.1 to M_1, setting $H(T,X_T^r)=e^{-rT}G(X_T^r)$, we can prove that M_1 is a martingale under $\mathsf{P}_{t,x}$. And applying Lemma 2.2 to M_2, setting $H(t+u,X_{t+u}^r)=rKe^{-ru}I(X_{t+u}^r \le c(t+u))$, we can prove that M_2 is also a martingale under $\mathsf{P}_{t,x}$. Furthermore, appying Lemma 2.2 to M_3, setting $H(t+u,X_{t+u}^r)=\lambda e^{-ru}I\big(X_{t+u}^r \le c(t+u)\big)\int_0^{+\infty}\big(U(t+u,X_{t+u}^r+y)-g(X_{t+u}^r+y)\big)f_Y(y)dy$, we can prove that M_3 is also a martingale under $\mathsf{P}_{t,x}$. Finally, we can say that the stochastic process $M(s,X_s^r)$ is a martingale under $\mathsf{P}_{t,x}$ for $s\in[0,T-t]$.

2. We want to show that $U(t,x)\ge V(t,x)$ for all $t,x\in[0,T]\times(-\infty,\infty)$. For this, observe the martingale process $M(s,X_s^r)$ and applying the reducing method to the last part on the right hand side of (2.67),

$$
\begin{aligned}
&\lambda\int_0^s e^{-ru}\int_0^{+\infty}\big(U(t+u,x+X_u^r+y)\\
&\qquad - g(x+X_u^r+y)\big)f_Y(y)dyI\big(x+X_u^r\le c(t+u)\big)du && (2.72)\\
\le&\lambda\int_0^s e^{-ru}\int_0^{+\infty}\big(U(t+u,x+X_u^r)-g(x+X_u^r+y)\big)f_Y(y)dyI\big(x+X_u^r\le c(t+u)\big)du\\
=&\lambda\int_0^s e^{-ru}\int_0^{+\infty}\big(g(x+X_u^r)-g(x+X_u^r+y)\big)f_Y(y)dyI\big(x+X_u^r\le c(t+u)\big)du\\
=&\lambda\int_0^s e^{-ru}e^{x+X_u^r}\int_0^{+\infty}(e^y-1)f_Y(y)dyI\big(x+X_u^r\le c(t+u)\big)du\\
\le&\lambda\int_0^s e^{-ru}K\int_0^{+\infty}(e^y-1)f_Y(y)dyI\big(x+X_u^r\le c(t+u)\big)du\\
=&\int_0^s e^{-ru}K\lambda\frac{p}{\eta_1-1}I\big(x+X_u^r\le c(t+u)\big)du,
\end{aligned}
$$

where the first inequality is true because $x \mapsto U(t,x)$ is decreasing on $\mathbb{R}$, the first equality holds since $U(t,x) = G(x) = g(x)$ when $x \leq c(t)$, the second inequality comes from the simple fact that $e^x \leq K$ when $x \leq c(t) \leq \ln K$, and the last equation is just the calculation for $\int_0^{+\infty}(e^y - 1) f_Y(y)dy$. Now combine the last two parts of (2.67) with the inequality of (2.72), we can get

$$\begin{aligned} & rK \int_0^s e^{-ru} I(x + X_u^r \leq c(t+u))du \\ & - \lambda \int_0^s e^{-ru} \int_0^{+\infty} \big(U(t+u, x + X_u^r + y) \\ & \qquad - g(x + X_u^r + y)\big) f_Y(y)dy I\big(x + X_u^r \leq c(t+u)\big)du \\ \geq & \int_0^s e^{-ru} rKI(x + X_u^r \leq c(t+u))du - \int_0^s e^{-ru} K\lambda \frac{p}{\eta_1 - 1} I\big(x + X_u^r \leq c(t+u)\big)du \\ = & \int_0^s e^{-ru} K(r - \lambda \frac{p}{\eta_1 - 1}) I\big(x + X_u^r \leq c(t+u)\big)du. \end{aligned} \tag{2.73}$$

If the condition (2.58) holds, we can see that the integrand in the last equation of (2.73) never goes smaller than zero. Since $M(s, X_s^r)$ itself is a martingale as proved previously, we can deduce that $e^{-rs}U(t+s, x + X_s^r)$ could only be a submartingale under P for $s \in [0, T-t]$. The property of submartingales ensures that, for any stopping time $\tau \in [0,T]$

$$\begin{aligned} U(t,x) & \geq \mathsf{E}_{t,x}\left(e^{-r\tau} U(t+\tau, X_{t+\tau}^r)\right) \\ & \geq \mathsf{E}_{t,x}\left(e^{-r\tau} G(X_{t+\tau}^r)\right), \end{aligned} \tag{2.74}$$

where the second inequality is derived from the condition $U(t,x) \geq G(x)$ for all $t, x \in [0,T] \times (-\infty, \infty)$. Then by the definition of the value function $V(t,x)$ in (2.14), the inequality of (2.74) implies that

$$U(t,x) \geq V(t,x), \tag{2.75}$$

for all $t, x \in [0,T] \times (-\infty, \infty)$.

3. Now we would like to show that $U(t,x) = V(t,x)$ for all $t, x \in [0,T] \times (-\infty, \infty)$. Indeed, we just need to show that $U(t,x) \leq V(t,x)$ given that $U(t,x) \geq V(t,x)$ has already been proved previously. For this, let us consider a stopping time:

$$\tau_c = \inf\{s \in [0, T-t] | x + X_s^r \leq c(t+s)\}. \tag{2.76}$$

If $x \le c(t)$, then by the condition (2.64) we know that

$$U(t,x) = G(x) \le V(t,x). \tag{2.77}$$

If $x > c(t)$, then we have $U(t+\tau_c, x+X^r_{\tau_c}) = G(x+X^r_{\tau_c})$ by the definition of τ_c in (2.76). So replace s by τ_c in the martingale process $M(s, X^r_s)$ defined by (2.67) and take expectation E on both sides, we can get

$$\begin{aligned} &U(t,x) \\ =&\mathsf{E}[e^{-r\tau_c}U(t+\tau_c, x+X^r_{\tau_c})] + rK\mathsf{E}[\int_0^{\tau_c} e^{-ru} I(x+X^r_u \le c(t+u))du] \\ &-\lambda\mathsf{E}\Big[\int_0^{\tau_c} e^{-ru}\int_0^{+\infty}\big(U(t+u, x+X^r_u+y) \\ &\quad - g(x+X^r_u+y)\big)f_Y(y)dy I\big(x+X^r_u \le c(t+u)\big)du\Big], \\ =&\mathsf{E}[e^{-r\tau_c}G(x+X^r_{\tau_c})], \end{aligned} \tag{2.78}$$

where the last two part on the right hand side of (2.78) equals to zero by the definition of τ_c. Meanwhile, by the definition of the value function $V(t,x)$ in (2.14), this result implies that:

$$U(t,x) \le V(t,x), \tag{2.79}$$

for all $t,x \in [0,T] \times (-\infty,\infty)$. Thus we have $U(t,x) = V(t,x)$ as claimed.

4. The remaining work is to prove that $c(t) = b(t)$ for all $t \in [0,T]$. Provided by the strong conditions (2.63) and (2.64), this fact is easy to prove with the previous result $U(t,x) = V(t,x)$.

First, suppose there exists a $t \in [0,T)$ such that $b(t) > c(t)$. Then take any x between $b(t)$ and $c(t)$ such that $b(t) > x > c(t)$. By the free-boundary definition Theorem 2.3 of $V(t,x)$ and the condition (2.63) of $U(t,x)$, we know that $V(t,x) = G(x)$ and $U(t,x) > G(x)$, which leads to $V(t,x) > U(t,x)$ for this pair of (t,x). This provides a contradiction to the result of Step **3**.

Second, suppose there exists a $t \in [0,T)$ such that $b(t) < c(t)$. Then take an x between $b(t)$ and $c(t)$ such that $b(t) < x < c(t)$. By the free-boundary definition Theorem 2.3 of $V(t,x)$ and the condition (2.64) of $U(t,x)$, we know that $V(t,x) > G(x)$ and $U(t,x) = G(x)$, which leads to $V(t,x) < U(t,x)$ for this pair of (t,x). This provides a contradiction to the result of Step **3**.

Thus we can have that for any point t, $c(t)$ must equal to $b(t)$. This means that $c(t) = b(t)$ for all $t \in [0, T - t]$ as claimed.

To this end, we have already proved that the solution pair (U, c) satisfying (2.65) and (2.66) globally coincides with (V, b). The proof of Theorem 2.5 is completed. □

Remark 2.1. *Note that the additional condition (2.58) for the uniqueness of the solution pair simply means that the riskless interest rate corrected by the jump risk is nonnegative, which is a reasonable assumption in practice. Another simplification comes with this condition is that we can directly know the value of the boundary function at maturity* $b(T) = \ln K$ *without extra calculation from Property 2.8 and 2.10. Knowing the value of* $b(T)$ *is crucial for the analytical tractability of our research, since we will need to use it to numerically retrieve the whole boundary curve* $t \to b(t)$ *backward on* $[0, T]$ *in practice. We will briefly go through the numerical method which is appropriate for option pricing with the EEP representation in the next section for one-sided jumps case.*

2.4 Degenerate Case: One Sided Negative Jumps

As stated before, the major difficulty of option pricing under jump-diffusion processes is a nonlocal integral term inside the infinitesimal generator defined in (2.10). Comparing Theorem 2.4 in the previous section with the EEP representation derived by Peskir in [37], the value function of American put options will have an additional global integral term involving early exercise boundary in the EEP formula for underlying processes with jumps. This global integral term also makes the uniqueness of the solution can only be held under relatively stronger conditions. Observing the EEP representation in (2.56), we notice that the global integral term only depends on positive jump parts of the underlying process for an American put option. Thus if we put a constraint on the compound Poisson part of the equation (2.8) to allow only negative jumps on the diffusion trajectories, then we will obtain a more compact and clean result, which is very similar in forms to the conclusion of [37].

Suppose that the probability p of positive jumps is zero, whereas the probability $q = 1 - p$ of negative jumps is one in the definition of the jump density (2.3), then we can have a new corresponding process for the underlying asset:

$$Z_t^r = e^{X_t^r} = e^{(r-\lambda\zeta-\frac{\sigma^2}{2})t+\sigma W_t+\sum_{i=1}^{N_t}(Y_i)}, \tag{2.80}$$

where

$$f_Y(y) = \eta_2 e^{\eta_2 y} 1_{\{y<0\}} \qquad \eta_2 > 0, \tag{2.81}$$

$$\zeta = \mathsf{E}[V] - 1 = \frac{\eta_2}{\eta_2 + 1} - 1. \tag{2.82}$$

Here we get a degenerate case of the DEJD model and would like to name it as negative exponential jump-diffusion (NEJD) process for future reference in this section. All the other fundamental settings remain the same, including basic assumptions of the American put option, structures of optimal stopping regions, and forms of the value function $V(t,x)$ and early exercise boundary $b(t)$. It is trivial to verified that under the NEJD process, $V(t,x)$ and $b(t)$ have the same properties as in DEJD models. And the connection to free-boundary problems (2.32) - (2.36) is still hold in this degenerate case. These is no need to provide detailed proof for these properties and theorems since there will be no difference comparing to the derivation in the previous section; restricting the density function by $p = 1$ and $q = 0$ will not change the forms and results of the derivation. The effect of limiting jumps to negative side only emerges after the equation (2.55) in deriving EEP representation. Thus we will not repeat those similar part mentioned above and start directly from equation (2.55). What should be noted is that $\int (e^y - 1)^+ \lambda f_Y(y) dy = 0$ in this degenerated case. This means that, similar to the effect of the additional condition (2.58) for Theorem 2.5, there will be no conditional judgement needed in Property 2.10 for the maturity behaviour of the boundary function b: it will always be $b(T) \overset{def}{=} \lim_{t \to T} b(t) = \ln K$ whatever parameter value we take for this NEJD model.

2.4.1 EEP Representation of American Put Options for NE-JD processes

Since positive jumps are not allowed in this degenerate case, the third part on the right hand side of (2.55) now can be cancelled:

$$
\begin{aligned}
& V(t,x) \qquad (2.83)\\
= \;& e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r)\\
& - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(g_t + \mathbb{L}_{X^r}g - rg)(X_{t+u}^r)I(X_{t+u}^r \le b(t+u))du\right)\\
& - \lambda\mathsf{E}_{t,x}\bigg[\int_0^{T-t} e^{-ru}\int_0^{+\infty}\big(V(t+u, X_{t+u}^r + y)\\
& \quad - g(X_{t+u}^r + y)\big)f_Y(y)dyI\big(X_{t+u}^r \le b(t+u)\big)du\bigg]\\
= \;& e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r) - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(g_t + \mathbb{L}_{X^r}g\right.\\
& \quad \left. - rg)(X_{t+u}^r)I(X_{t+u}^r \le b(t+u))du\right),
\end{aligned}
$$

where the last equality comes from the restriction that $f_Y(y) = 0$ for $y \ge 0$. Again, inserting the result from (2.54), we will have the EEP representation for an American put option with no dividends:

Theorem 2.6. *The arbitrage-free price of the American put options without dividends admits the following early exercise premium representation under a negative exponential jump diffusion process*

$$V(t,x) = e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r) + rK\int_0^{T-t} e^{-ru}\mathsf{P}_{t,x}\big(X_{t+u}^r \le b(t+u)\big)du. \qquad (2.84)$$

The first part of (2.84) is the arbitrage-free price of the corresponding European put option. It should be noted that it will be different from the result calculated by [25, Theorem 2], since only negative jumps are considered now. This form is in line with the EEP formula under Black-Scholes model in [37]. The difference comes from the cumulative distribution of the underlying process X_t^r. We will write this into analytic form in the next section for financial analysis.

2.4.2 The Uniqueness of the Free Boundary for NEJD processes

Observing the EEP formula in Theorem 2.6, we can see that the right hand side of (2.84) contains only the free boundary function $b(t)$ in this degenerate case. This means that if we substitute $x = b(t)$ into the EEP formula (2.84), we will obtain a single nonlinear integral equation which can be solved for the free boundary $b(t)$:

$$K - e^{b(t)} = e^{-r(T-t)}\mathsf{E}_{t,b(t)}(K - e^{X_T^r})^+ + rK\int_0^{T-t} e^{-ru}\mathsf{P}_{t,b(t)}\big(X_{t+u}^r \le b(t+u)\big)du. \tag{2.85}$$

Similar to the previous section, we would like to prove the uniqueness of $b(t)$. Comparing equation (2.85) with (2.56) and (2.57), we can see two advantages brought by restricting jump directions. First, the nonlinear integral equation system is now reduced to one single equation. Second, there is no global integral part on the right hand side. These two simplification in the NEJD model enable us to loose the conditions for uniqueness and follow the main approach for uniqueness proof proposed by Peskir in [37]. Conditions (2.58), (2.62), (2.63) and (2.64) are no longer needed in this case. Just with some weak conditions on the continuity and range of the boundary function $b(t)$, we can have the following theorem about the uniqueness.

Theorem 2.7. *Under the negative exponential jump diffusion process, the early exercise boundary b in the value function (2.14) of the American put option without dividends can be characterized as the unique solution of the following nonlinear integral equation:*

$$K - e^{b^v(t)} = e^{-r(T-t)}\mathsf{E}_{t,b^v(t)}(K - e^{X_T^r})^+ + rK\int_0^{T-t} e^{-ru}\mathsf{P}_{t,b^v(t)}\big(X_{t+u}^r \le b(t+u)\big)du, \tag{2.86}$$

in the class of the continuous functions $b^v : [0,T] \to \mathbb{R}$ satisfying $b^v(t) < \ln K$ for all $0 < t < T$.

Proof. Suppose that there exists such a function $c : [0,T] \to \mathbb{R}$ satisfying the conditions of Theorem 2.7 and solving the equation (2.86). We just need to prove that $c(t)$ coincide with the early exercise boundary $b(t)$.

1. Introduce a corresponding function $U^c : [0, T] \times (-\infty, \infty) \to \mathbb{R}$ defined by

$$U^c(t, x) = e^{-r(T-t)} \mathsf{E}_{t,x}(K - e^{X_T^r})^+ + rK \int_0^{T-t} e^{-ru} \mathsf{P}_{t,x}\big(X_{t+u}^r \le c(t+u)\big) du. \quad (2.87)$$

Simply inserting $x = c(t)$ into (2.87), we can have that

$$U^c(t, c(t)) = K - e^{c(t)} = G(c(t)). \quad (2.88)$$

since c satisfy the condition $c(t) < \ln K$ for all $0 < t < T$.

2. We need to show that $U^c(t, x) = G(x)$ for all $t, x \in [0, T] \times (-\infty, \infty)$ such that $x \le c(t)$. First introduce an stochastic process $M(s, X_s^r)$ which defined by

$$M(s, X_s^r) = e^{-rs} U^c(t + s, x + X_s^r) + rK \int_0^s e^{-ru} I(x + X_u^r \le c(t + u)) du, \quad (2.89)$$

where $X_0^r = 0$ and $(t, x) \in [0, T] \times \mathbb{R}$ is given and fixed. Then follow the similar approach in the previous section, with Lemma 2.1 and 2.2, we can prove that $M(s, X_s^r)$ is a martingale under P for $s \in [0, T - t]$. Now take a pair of (t, x) such that $x \le c(t)$ and consider the stopping time

$$\sigma_c = \inf\{s \in [0, T - t] | x + X_s^r \ge c(t + s)\}. \quad (2.90)$$

Since $M(s, X_s^r)$ is a martingale, we have

$$M(0, X_0^r) = \mathsf{E}[M(\sigma_c, X_{\sigma_c}^r)]. \quad (2.91)$$

For the left hand side of (2.91) we have:

$$M(0, X_0^r) = U^c(t, x), \quad (2.92)$$

by the definition of $M(s, X_s^r)$. Also note that $U^c(t + \sigma_c, x + X_{\sigma_c}^r) = G(x + X_{\sigma_c}^r)$ holds for this degenerated process since we assume that there is no positive jumps. Back to our proof, for the right hand side of (2.91) we have:

$$\begin{aligned} \mathsf{E}[M(\sigma_c, X_{\sigma_c}^r)] =& \mathsf{E}[e^{-r\sigma_c} U^c(t + \sigma_c, x + X_{\sigma_c}^r)] + \mathsf{E}(rK \int_0^{\sigma_c} e^{-ru} du) \quad (2.93) \\ =& \mathsf{E}[e^{-r\sigma_c} G(x + X_{\sigma_c}^r)] + rK\mathsf{E}(\int_0^{\sigma_c} e^{-ru} du). \end{aligned}$$

Then the equation (2.91) can be rewrite as

$$U^c(t, x) = \mathsf{E}[e^{-r\sigma_c} G(x + X_{\sigma_c}^r)] + rK\mathsf{E}(\int_0^{\sigma_c} e^{-ru} du). \quad (2.94)$$

Apply the change-of-variable formula [39, Theorem 3.1] to $e^{-rs}G(x+X_s^r)$, replace s by σ_c and take expectation E on both side, following the similar derivation of Theorem 2.6 we can have

$$\mathsf{E}[e^{-r\sigma_c}G(x+X_{\sigma_c}^r)] = G(x) - rK\mathsf{E}(\int_0^{\sigma_c} e^{-ru}du). \tag{2.95}$$

Thus insert (2.95) into (2.94), we have proved that

$$U^c(t,x) = G(x), \tag{2.96}$$

for all $t, x \in [0,T] \times (-\infty, \infty)$ such that $x \leq c(t)$.

3. We want to show that $U^c(t,x) \leq V(t,x)$ for all $t, x \in [0,T] \times (-\infty, \infty)$. For this, take any such (t,x) and consider the stopping time

$$\tau_c = \inf\{s \in [0, T-t] | x + X_s^r \leq c(t+s)\}. \tag{2.97}$$

If $x \leq c(t)$, then by the result of Step **2** in (2.96) we know that

$$U^c(t,x) = G(x) \leq V(t,x). \tag{2.98}$$

If $x > c(t)$, then we have $U^c(t+\tau_c, x+X_{\tau_c}^r) = G(x+X_{\tau_c}^r)$ by the definition of τ_c in (2.97). So replacing s by τ_c in $M(s, X_s^r)$ and taking expectation E on both sides, we find that

$$\begin{aligned} U^c(t,x) =& \mathsf{E}[e^{-r\tau_c}U^c(t+\tau_c, x+X_{\tau_c}^r)] + rK\mathsf{E}[\int_0^{\tau_c} e^{-ru} I(x+X_u^r \leq c(t+u))du] \quad (2.99) \\ =& \mathsf{E}[e^{-r\tau_c}G(x+X_{\tau_c}^r)], \end{aligned}$$

where the second part on the right hand side of (2.99) equals to zero by the definition of τ_c. Then the definition of the value function $V(t,x)$ in (2.14) implies that

$$U^c(t,x) \leq V(t,x), \tag{2.100}$$

for all $t, x \in [0,T] \times (-\infty, \infty)$.

4. Let us now show that $b(t) \leq c(t)$ for all $t \in [0,T]$. For this, suppose there exists a $t \in [0,T)$ such that $b(t) > c(t)$. Then for this kind of fixed t, take an x such that $(t,x) \in \{[0,T] \times (-\infty, \infty) | x \leq c(t)\}$. So we have $x \leq c(t) < b(t)$. Now consider the stopping time

$$\sigma_b = \inf\{s \in [0, T-t] | x + X_s^r \geq b(t+s)\}. \tag{2.101}$$

Replacing s by σ_b in $M(s, X_s^r)$ and taking expectation E on both sides, we get

$$\mathsf{E}[e^{-r\sigma_b}U^c(t+\sigma_b, x+X_{\sigma_b}^r)] = U^c(t,x) - rK\mathsf{E}[\int_0^{\sigma_b} e^{-ru}I(x+X_u^r \le c(t+u))du]. \tag{2.102}$$

And applying the change-of-formula [39, Theorem 3.1] to $e^{-rs}V(t+s, X_{t+s}^r)$ with the same $(t,x) \in \{[0,T] \times (-\infty, \infty) | x \le c(t)\}$, we have

$$e^{-rs}V(t+s, x+X_s^r) = V(t, X_t^r) - rK\int_0^s e^{-ru}I(x+X_u^r \le b(t+u))du + M_s, \tag{2.103}$$

where M_s is a martingale under P. The proof is similar to the proofs of Proposition 2.1 and 2.2. Also replacing s by σ_b in (2.103) and taking expectation E on both sides, we get

$$\begin{aligned}\mathsf{E}[e^{-r\sigma_b}V(t+\sigma_b, x+X_{\sigma_b}^r)] =& V(t,x) - rK\mathsf{E}[\int_0^{\sigma_b} e^{-ru}I(x+X_u^r \le b(t+u))du] \qquad (2.104)\\ =& V(t,x) - rK\mathsf{E}[\int_0^{\sigma_b} e^{-ru}du]\\ =& V(t,x) - rK\mathsf{E}[\int_0^{\sigma_b} e^{-ru}I(x+X_u^r \le c(t+u))du]\\ &- rK\mathsf{E}[\int_0^{\sigma_b} e^{-ru}I(x+X_u^r > c(t+u))du].\end{aligned}$$

The comparing of (2.102) and (2.104) implies that

$$\mathsf{E}[\int_0^{\sigma_b} e^{-ru}I(x+X_u^r > c(t+u))du] \le 0. \tag{2.105}$$

However, the fact that $c(t) < b(t)$ and the continuity of the functions c and b force the expectation in (2.105) strictly positive and provides a contradiction. Thus $b(t) \le c(t)$ for all $t \in [0,T]$ as claimed.

5. Finally, we show that $b(t) = c(t)$ for all $t \in [0,T]$. For this, suppose there exists a $t \in [0,T)$ such that $b(t) < c(t)$. Then for this kind of fixed t, take any x satisfying $b(t) < x < c(t)$. Now consider the stopping time

$$\tau_b = \inf\{s \in [0, T-t] | x + X_s^r \le b(t+s)\}. \tag{2.106}$$

Replacing s by τ_b in $M(s, X_s^r)$, and taking expectation E on both sides, we get

$$\mathsf{E}[e^{-r\tau_b}U^c(t+\tau_b, x+X_{\tau_b}^r)] = U^c(t,x) - rK\mathsf{E}[\int_0^{\tau_b} e^{-ru}I(x+X_u^r \le c(t+u))du]. \tag{2.107}$$

Again, replacing s by τ_b in (2.103), and taking expectation E on both sides, we find that

$$\mathsf{E}[e^{-r\tau_b}V(t+\tau_b, x+X^r_{\tau_b})] = V(t,x) - rK\mathsf{E}[\int_0^{\tau_b} e^{-ru}I(x+X^r_u \le b(t+u))du]. \tag{2.108}$$

$$= V(t,x),$$

where the last equation follows from the definition of τ_b. Comparing (2.107) and (2.108) implies that

$$\mathsf{E}[\int_0^{\tau_b} e^{-ru}I(x+X^r_u \le c(t+u))du] \le 0. \tag{2.109}$$

Thus similar to Step **4**, the fact that $b(t) < c(t)$ and the continuity of the functions c and b force the expectation in (2.109) strictly positive and provides a contradiction. Thus $b(t) \ge c(t)$ for all $t \in [0,T]$. Combining with the result of Step **4**, we have that $b = c$ for all $t \in [0,T]$ and the proof for Theorem 2.7 is complete. □

2.5 Financial Analysis under NEJD processes

While the EEP representation Theorem 2.4 and the uniqueness Theorem 2.5 for the value function and free boundary under a double exponential jump diffusion process are more theoretically generic and meaningful, the corresponding results Theorem 2.6 and Theorem 2.7 for the degenerate case NEJD model are much more feasible for financial analysis in practice. In this financial analysis section, we will only consider the one sided jumps case. Nevertheless, it still can provide us a glance on how will allowing jumps on the path affect the values and boundaries of American put options.

2.5.1 Analytical Form of the EEP Representation

The first thing we need to do is to write the EEP representation (2.84) of the value function into analytical form. A key component for this is the cumulative distribution function of the underlying NEJD process X^r_t defined in (2.80). Kou derived a generic analytical form of CDF for double exponential jump diffusion processes when calculating the European option price in [25]. Rather than quoting his result directly without any explanation, we prefer to follow his approach here to derive a simplified result for our degenerate case in consideration of rigorous and completeness.

For the simplicity of the derivation below, we would like to define a new generic stochastic process X_t^u by

$$X_t^{\mu} = \mu t + \sigma W_t + \sum_{i=1}^{N_t} (Y_i), \tag{2.110}$$

where the density of jump sizes follows equation (2.81). If we can calculate the CDF of X_T^{μ} and define a generic function Υ based on it, then for any given t, we can express the CDF of X_t^r with Υ by amending some corresponding parameters. Now we will focus on the random variable X_T^{μ} and derive the function Υ step by step.

Υ function for NEJD processes

1. First note that for each of the i.i.d. random variable Y_i representing the jump size, we have

$$Y_i \stackrel{d}{=} -\xi_i, \tag{2.111}$$

where ξ_i is an exponential random variable with rate η_2. We know that the sum of n i.i.d. exponential random variables follows the Erlang distribution, thus for any given $n \geq 1$, $\sum_{i=1}^{n}(\xi_i) = -\sum_{i=1}^{n}(Y_i)$ is a random variable with the density

$$f_{\sum_{i=1}^{n}(\xi_i)}(x) = f_{\sum_{i=1}^{n}(\xi_i)}(x; n, \eta_2) = \frac{\eta_2^n x^{n-1} e^{-\eta_2 x}}{(n-1)!}. \tag{2.112}$$

for all $x \geq 0$. The generic process X_t^{μ} can be rewrite as

$$X_t^{\mu} = \mu t + \sigma W_t - \sum_{i=1}^{N_t} (\xi_i). \tag{2.113}$$

2. Introduce a special function from mathematical physics named as Hh function. For every integer $n \geq 0$, the Hh function is a nonincreasing function defined by

$$Hh_n(x) = \int_x^{\infty} Hh_{n-1}(y)dy = \frac{1}{n!}\int_x^{\infty} (t-x)^n e^{-\frac{t^2}{2}} dt, \tag{2.114}$$

$$Hh_{-1}(x) = e^{-\frac{x^2}{2}} = \sqrt{2\pi}\varphi(x), \tag{2.115}$$

$$Hh_0(x) = \sqrt{2\pi}\Phi(-x), \tag{2.116}$$

where $\varphi(x)$ and $\Phi(x)$ are the density function and the CDF for standard normal distribution respectively:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \tag{2.117}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy. \tag{2.118}$$

The Hh function can be viewed as a generalization of the cumulative normal distribution function. A three-term recursion for it is also available:

$$nHh_n(x) = Hh_{n-2} - xHh_{n-1}(x),. \tag{2.119}$$

for all integer $n \geq 1$. This means that is we can compute all $Hh_n(x)$, $n \geq 1$, by using the normal density function and normal distribution function. In this case, it would be a great practical advantage if we could write Υ in terms of Hh functions.

3. Now we can calculate the density of an auxiliary random variable $z - \sum_{i=1}^{n}(\xi_i)$ where z is a normal random variable with distribution $N(0, \sigma^2)$ and n is any integer with $n \geq 1$:

$$\begin{aligned} f_{z-\sum_{i=1}^{n}(\xi_i)}(t) &= \int_{-\infty}^{\infty} f_z(x) f_{\sum_{i=1}^{n}(\xi_i)}(x-t)dx \qquad (2.120)\\ &= e^{t\eta_2}(\eta_2^n)\int_t^{\infty} \frac{e^{-x\eta_2}(x-t)^{n-1}}{(n-1)!}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-x^2}{2\sigma^2}}dx \\ &= e^{t\eta_2}(\eta_2^n)e^{\frac{(\sigma\eta_2)^2}{2}}\int_t^{\infty} \frac{(x-t)^{n-1}}{(n-1)!}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x+\sigma^2\eta_2)^2}{2\sigma^2}}dx. \end{aligned}$$

Letting $y = \frac{x+\sigma^2\eta_2}{\sigma}$ yields

$$\begin{aligned} f_{z-\sum_{i=1}^{n}(\xi_i)}(t) &= e^{t\eta_2}(\eta_2^n)e^{\frac{(\sigma\eta_2)^2}{2}}\sigma^{n-1}\int_{\frac{t}{\sigma}+\sigma\eta_2}^{\infty} \frac{\left(y-(\sigma\eta_2+\frac{t}{\sigma})\right)^{n-1}}{(n-1)!}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \qquad (2.121)\\ &= \frac{e^{\frac{(\sigma\eta_2)^2}{2}}}{\sqrt{2\pi}}(\sigma^{n-1}\eta_2^n)e^{t\eta_2}Hh_{n-1}(\frac{t}{\sigma}+\sigma\eta_2), \end{aligned}$$

where the last equality is from the definition of Hh function in (2.114).

4. For option pricing under exponential jumps, it is important to evaluate the integral $I_n(c, \alpha, \beta, \omega)$ defined by

$$I_n(c, \alpha, \beta, \omega) = \int_c^{\infty} e^{\alpha x} Hh_n(\beta x - \omega)dx \tag{2.122}$$

for all integer $n \geq 0$ and arbitrary constants α, c and β. Here we will omit the detailed calculation but introduce the result from [25, Proposition B.2] directly.

Fact 2.1. *If $\beta > 0$ and $\alpha \neq 0$, then for all $n \geq -1$,*

$$I_n(c, \alpha, \beta, \omega) = -\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}(\frac{\beta}{\alpha})^{n-i}Hh_i(\beta c - \omega) + (\frac{\beta}{\alpha})^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\omega}{\beta}+\frac{\alpha^2}{2\beta^2}}\Phi(-\beta c + \omega + \frac{\alpha}{\beta}). \tag{2.123}$$

If $\beta < 0$ and $\alpha < 0$, then for all $n \geq -1$,

$$I_n(c,\alpha,\beta,\omega) = -\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}(\frac{\beta}{\alpha})^{n-i}Hh_i(\beta c-\omega) - (\frac{\beta}{\alpha})^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\omega}{\beta}+\frac{\alpha^2}{2\beta^2}}\Phi(\beta c-\omega-\frac{\alpha}{\beta}). \tag{2.124}$$

This fact will be very useful in numerically calculating the arbitrage-free price of an American put option.

5. We can write the probability $\mathsf{P}(z-\sum_{i=1}^{n}(\xi_i)\geq x)$ in terms of function I:

$$\mathsf{P}(z-\sum_{i=1}^{n}(\xi_i)\geq x) = \frac{(\sigma\eta_2)^n e^{\frac{(\sigma\eta_2)^2}{2}}}{\sigma\sqrt{2\pi}}\int_x^\infty e^{t\eta_2}Hh_{n-1}(\frac{t}{\sigma}+\sigma\eta_2)dt \tag{2.125}$$

$$= \frac{(\sigma\eta_2)^n e^{\frac{(\sigma\eta_2)^2}{2}}}{\sigma\sqrt{2\pi}}I_{n-1}(x,\eta_2,\frac{1}{\sigma},-\sigma\eta_2).$$

Also recall that for any given t, $\mu t+\sigma W_t$ is a normal random variable with distribution $N(\mu t,\sigma^2 t)$ which is the same for random variable $\mu t+\sqrt{t}z$. And N_t is a random variable following Poisson distribution with rate λt. We can denote as

$$\pi_n(t) := \mathsf{P}(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}. \tag{2.126}$$

6. With all the above results, we can calculate the probability $\mathsf{P}(X_T^u \geq a)$ for a given T and a constant $a \in \mathbb{R}$, and denote it as the function Υ:

$$\begin{aligned}\Upsilon(\mu,\sigma,\lambda,\eta_2;a,T) :=& \mathsf{P}(X_T^\mu \geq a) \qquad (2.127)\\
=& \sum_{n=0}^{\infty}\pi_n(T)\mathsf{P}(\mu T+\sqrt{T}z-\sum_{i=1}^{n}(\xi_i)\geq a)\\
=& \pi_0(T)\mathsf{P}(\mu T+\sqrt{T}z\geq a)+\sum_{n=1}^{\infty}\pi_n(T)\mathsf{P}(\mu T+\sqrt{T}z-\sum_{i=1}^{n}(\xi_i)\geq a)\\
=& \pi_0(T)\Phi(-\frac{a-\mu T}{\sigma\sqrt{T}})\\
&+\frac{e^{\frac{T(\sigma\eta_2)^2}{2}}}{\sigma\sqrt{2\pi T}}\sum_{n=1}^{\infty}\pi_n(T)(\sigma\sqrt{T}\eta_2)^n I_{n-1}(a-\mu T,\eta_2,\frac{1}{\sigma\sqrt{T}},-\sigma\eta_2\sqrt{T}).\end{aligned}$$

Although the form of Υ seems complicated, it can easily calculated by computer by equation (2.123), (2.124) and (2.119).

Analytical representation for European put options

As has been stated in the previous section, the arbitrage-free price of the corresponding European put option in the EEP representation (2.84) of NEJD model will be different from the result calculated by [25, Theorem 2]. However, following the similar method introduced by Kou, we can also derive the analytical form of $e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r)$ in terms of Υ function.

$$\begin{aligned}
e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r) =& e^{-r(T-t)}\mathsf{E}\big[G(x+X_{T-t}^r)\big] \\
=& e^{-r(T-t)}\mathsf{E}\big[(K-e^{x+X_{T-t}^r})^+\big] \\
=& e^{-r(T-t)}\mathsf{E}\big[(K-e^{x+X_{T-t}^r})I(x+X_{T-t}^r \le \ln K)\big] \\
=& Ke^{-r(T-t)}\mathsf{P}(X_{T-t}^r \le \ln K - x) \\
&- e^{-r(T-t)}\mathsf{E}\big[e^{x+X_{T-t}^r}I(X_{T-t}^r \le \ln K - x)\big] \\
=& Ke^{-r(T-t)}\big(1-\Upsilon(r-\lambda\zeta-\frac{1}{2}\sigma^2,\sigma,\lambda,\eta_2;\ln K - x, T-t)\big) \\
&- e^x\mathsf{E}\big[e^{-r(T-t)}e^{X_{T-t}^r}I(X_{T-t}^r \le \ln K - x)\big]
\end{aligned} \tag{2.128}$$

For the second term on the right hand side of the last equality, we can use a change of numerable argument to eliminate the annoying part $e^{-r(T-t)}e^{X_{T-t}^r}$. More precisely, introduce a new probability measure P^* defined as

$$\begin{aligned}
\frac{d\mathsf{P}^*}{d\mathsf{P}} &= e^{-r(T-t)}e^{X_{T-t}^r} \\
&= e^{(\lambda\zeta-\frac{\sigma^2}{2})(T-t)+\sigma W_{T-t}+\sum_{i=1}^{N_{T-t}}(Y_i)}
\end{aligned} \tag{2.129}$$

Note that this is a well-defined probability as $\mathsf{E}[e^{-r(T-t)}e^{X_{T-t}^r}] = 1$. We have, by the Girsanov theorem for jump processes, $W_t^* := W_t - \sigma t$ is a new Brownian motion under P^*, and the original process

$$\begin{aligned}
X_t^* &= (r-\lambda\zeta-\frac{\sigma^2}{2})t+\sigma W_t+\sum_{i=1}^{N_t}(Y_i) \\
&= (r-\lambda\zeta+\frac{\sigma^2}{2})t+\sigma W_t^*+\sum_{i=1}^{N_t^*}(Y_i)
\end{aligned} \tag{2.130}$$

is a new NEJD process with the Poisson process N_t^* having a new rate $\lambda^* = \lambda\mathsf{E}(e^Y) = \lambda(1+\zeta)$, and the jump sizes Y_i being i.i.d. with a new density given by

$$\begin{aligned}
\frac{e^y f_Y(y)}{\mathsf{E}(e^Y)} &= \frac{e^y\eta_2e^{\eta_2 y}1_{\{y<0\}}}{\frac{\eta_2}{\eta_2+1}} \\
&= \eta_2 e^{(\eta_2+1)y}1_{\{y<0\}}\frac{\eta_2+1}{\eta_2} \\
&= (\eta_2+1)e^{(\eta_2+1)y}1_{\{y<0\}}
\end{aligned} \tag{2.131}$$

Thus it is still a negative exponential jump density with $\eta_2^* = \eta_2 + 1$. In summary, we have

$$
\begin{aligned}
&e^x \mathsf{E}\big[e^{-r(T-t)} e^{X^r_{T-t}} I(X^r_{T-t} \le \ln K - x)\big] \qquad (2.132)\\
=&e^x \mathsf{P}^*(X^r_{T-t} \le \ln K - x)\\
=&e^x (1 - \Upsilon(r - \lambda\zeta + \frac{1}{2}\sigma^2, \sigma, \lambda^*, \eta_2^*; \ln K - x, T - t)\big)
\end{aligned}
$$

And the analytical form of the arbitrage-free price of an European put option without dividends at time t is

$$
\begin{aligned}
e^{-r(T-t)}\mathsf{E}_{t,x} G(X^r_T) =&K e^{-r(T-t)}\big(1 - \Upsilon(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; \ln K - x, T - t)\big) \qquad (2.133)\\
&- e^x\big(1 - \Upsilon(r - \lambda\zeta + \frac{1}{2}\sigma^2, \sigma, \lambda^*, \eta_2^*; \ln K - x, T - t)\big)
\end{aligned}
$$

Analytical form of the EEP representation for American put options

With the equation (2.133) and (2.127), we can write the EEP representation of the value function $V(t,x)$ proposed in Theorem 2.6 into the following analytical form:

$$
\begin{aligned}
V(t,x) =&e^{-r(T-t)}\mathsf{E}_{t,x} G(X^r_T) + rK \int_0^{T-t} e^{-ru} \mathsf{P}\big(X^r_u \le b(t+u) - x\big) du \qquad (2.134)\\
=&K e^{-r(T-t)}\big(1 - \Upsilon(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; \ln K - x, T - t)\big)\\
&- e^x (1 - \Upsilon(r - \lambda\zeta + \frac{1}{2}\sigma^2, \sigma, \lambda^*, \eta_2^*; \ln K - x, T - t)\big)\\
&+ rK \int_0^{T-t} e^{-ru} (1 - \Upsilon(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; b(t+u) - x, u)\big) du
\end{aligned}
$$

where $\lambda^* = \lambda(1+\zeta)$, $\eta_2^* = \eta_2 + 1$ and the early exercise boundary $b(t)$ is the unique solution of the following nonlinear integral equation:

$$
\begin{aligned}
K - e^{b(t)} =&K e^{-r(T-t)}\big(1 - \Upsilon(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; \ln K - b(t), T - t)\big) \qquad (2.135)\\
&- e^{b(t)} (1 - \Upsilon(r - \lambda\zeta + \frac{1}{2}\sigma^2, \sigma, \lambda^*, \eta_2^*; \ln K - b(t), T - t)\big)\\
&+ rK \int_0^{T-t} e^{-ru} (1 - \Upsilon(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; b(t+u) - b(t), u)\big) du
\end{aligned}
$$

in the class of the continuous functions $b : [0,T] \to \mathbb{R}$ satisfying $b(t) < \ln K$ for all $0 < t < T$.

2.5.2 Financial Analysis for American Put Options

In this part, we briefly show how will allowing jumps on the path affects the values and boundaries of American put options by some figures. Here we still focus on the negative exponential jump-diffusion processes due to its analytical tractability.

1. With the terminal condition of $b(T)$ given in Property 2.10, we can sequentially solve out the whole early exercise boundary $b(t)$ backward from $t = T$ to $t = 0$ by applying Trapezoid rule and Newton-Raphson method on the nonlinear integral equation (2.135), see [47] for more details of this numerical method. As shown in Figure 2.2, the rational exercise boundary of the American put option varies with different intensity parameter λ. We assume that the strike price $K = 10$, the maturity time $T = 1$, the interest rate $r = 0.1$, the volatility coefficient $\sigma = 0.4$, the rate parameter of the negative exponential random variable $\eta_2 = 1.8$. We can see from Figure 2.2, the value the intensity parameter λ will not influence the shape and the terminal value of the early exercise boundary: $b(t)$ is always increasing on $[0, T]$ and its value at maturity T is always $b(T) = \ln K$, which is consistent with the theoretical results verified in Property 2.10. However, the position of the early exercise boundary before maturity is decided by λ: the boundary $b(t)$ will be lower when the intensity parameter λ we choose is higher. Meanwhile, as λ tends to zero, the early exercise boundary of American put options under negative exponential jump-diffusion processes will converge to that under geometric Brownian motion. This feature can be well explained by the definition of the intensity λ: it represents the frequency of jumps which occurs on the path of the underlying dynamics. Therefore, $\lambda \downarrow 0$ means that the path is fairly similar to that of a pure diffusion process i.e. geometric Brownian motion. Moreover, this convergence can also be viewed as a numerical evidence for the validity of the theoretical results we derived in previous sections.

2. With the early exercise boundary $b(t)$ calculated in Figure 2.2, we can obtain value of the corresponding American put option by the analytical form of the EEP representation (2.134). As shown in Figure 2.3, the value function $x \mapsto V(t, x)$ of the American put option at a given time $t = 0$ varies with different intensity parameter λ. We assume that the strike price $K = 10$, the maturity time $T = 1$, the interest rate $r = 0.1$, the volatility coefficient $\sigma = 0.4$, the rate parameter of the negative exponential random variable $\eta_2 = 1.8$. We can see from Figure 2.3, the value the

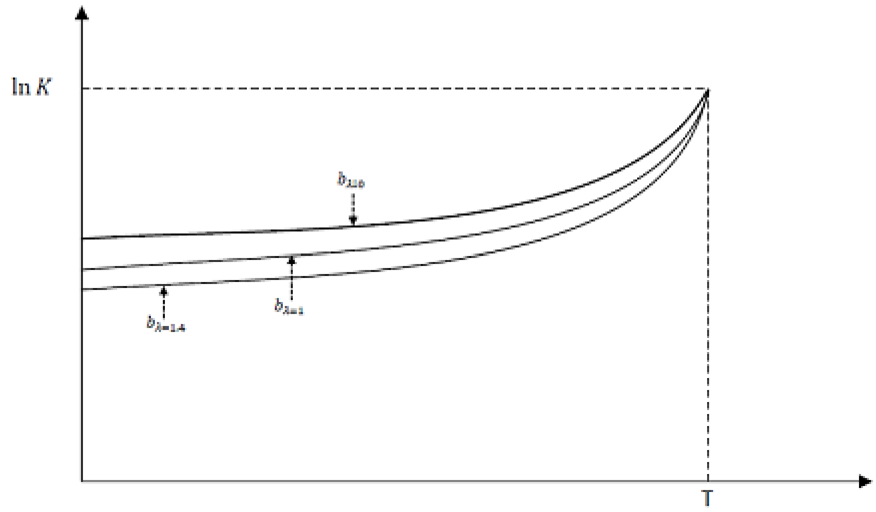

Figure 2.2: A computer drawing comparing the rational exercise boundary of the American put option under the negative exponential jump-diffusion processes with $K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\eta_2 = 1.8$ when the intensity parameter $\lambda = 1.4$ and $\lambda = 1$.

intensity parameter λ will not influence the shape of value function: $V(t, x)$ is always decreasing and convex on $[0, T]$, which is consistent with the theoretical results verified in Property 2.9. However, the position of value function before touching the early exercise point is decided by λ: the value of the American put option will be higher when the intensity parameter λ we choose is higher; and its corresponding early exercise point will be lower. Meanwhile, as λ tends to zero, the value function of American put options under negative exponential jump-diffusion processes will converge to that under geometric Brownian motion. We can see that the results observed from Figure 2.2 and Figure 2.3 are in accordance with each other. Moreover, they are also in line with the financial meaning of pricing American put option with only negative jumps: the higher frequency (represented by intensity λ) of "favourable" jumps (negative jumps for put options) leads to higher value of the option and the lower optimal exercise boundary.

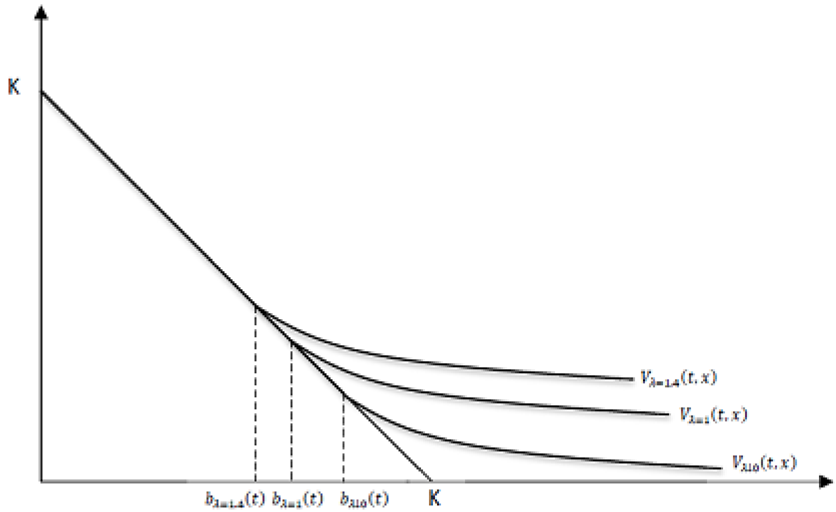

Figure 2.3: A computer drawing comparing the value of the American put option under the negative exponential jump-diffusion processes at a given time $t = 0$ with $K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\eta_2 = 1.8$ when the intensity parameter $\lambda = 1.4$ and $\lambda = 1$.

Chapter 3

American Call Option with Dividends for DEJD Processes

3.1 Introduction

Much research has been conducted to modify the Black-Scholes model based on Brownian motion and normal distribution in order to incorporate two empirical features: (1) the asymmetric leptokurtic features. (2) The volatility smile. Some well-known models dedicated to explain asymmetric leptokurtic features are fractal Brownian motion, generalized hyperbolic models and time-changed Brownian motions; see, for example, Rogers [51], Barndorff-Nielsen and Shephard [3] and Heyde [16]. And to incorporate the volatility smile, a variety of models have been proposed such as stochastic volatility model, CGMY models and CEV models; see Hull and White [17], Cox and Ross [11] and Carr et al. [7]. However, there are still several shortcomings in these alternative models. A more effective and practically approach to improve the performance of option pricing models is introducing jumps on the continuous trajectories of diffusion processes; see more detailed discussion about the motivation of this approach in Tankov and Cont [54] and Chapter 2. Such a Lévy process based model was first proposed by Merton in [33], called the normal jump-diffusion Model. In this model, the distribution of jump size is assume to be Gaussian.

Inspired by Merton's work, Kou proposed the double exponential jump-diffusion model in [25], where the logarithm of the asset price is assumed to follow a Brownian motion plus a compound Poisson process with jump sizes double exponential distributed. We will call it Kou's model or DEJD model for convenience afterwards.

Although the fundamental setting for DEJD model seems to be more complex than original Black-Scholes, it can indeed provide some interesting and useful properties for further valuation such as the leptokurtic feature of the jump size that provides the peak and tails of the return distribution found in reality, the implied volatility smile when we perform the calibration and the memoryless feature coming from the exponential distribution which makes it possible to obtain analytical expressions for expectations involving first passage times. For more details and analysis of this model, see Kou's original papers [25]. With the properties mentioned above, Kou and Wang demonstrated that a double exponential jump diffusion model can lead to analytical solutions for popular path-dependent options (such as lookback, barrier, and American options) in [27]. For more detailed introduction to the development of Kou's model and the limitation of Kou and Wang's results on American options, see Chapter 2.

In this chapter, we would like to derive the value function, as well as the exercise boundary, of a finite-horizon American call option with dividends in a more rigorous way comparing to the approximation method used in [27] for American puts. We will follow the martingale approach, which is also known as risk-neutral pricing method, for derivative pricing in the rest of this section. By this approach, we can assume that the value of an American style option is just the expectation of its payoff at a random optimal stopping time under a risk-neutral measure. This expectation form itself can provide us some key properties of the value function. Afterwards, we prove that the optimal stopping time can be illustrated into a optimal exercise boundary dividing the whole plane into two regions. We also prove that both these two regions are not empty for the American call option with dividends. Inspiring by the approach proposed by Lamberton and Mikou [29] on American put option under exponential Lévy processes, we can derive some similar properties of the early exercise boundary for American calls. These properties allow us to use the change-of-variable formula with local time on surfaces derived by Peskir in [39] on the value function of call options. This will lead to the EEP representation of the value function. Different from this kind of representation obtained by Peskir [37] for the American put under geometric Brownian motion, the value function we have also involves the early exercise boundary, which means that the pair of our option value and the boundary function is a solution to a system of nonlinear integral equations. The uniqueness of this solution is also proved, under some additional conditions coming from free-boundary problems. Since the nonlinear integral equations system seems very unfriendly for analytical calculation, we study a degenerate case of the DEJD model in the last few sections. By restricting the jumps to only one side, we can obtained some results with great analytical tractability, as well as a more generic proof for the uniqueness, which can also be viewed as a main contribution of this chapter. The financial analysis of this degenerate case illustrates the improvement and advantage we have from our work compared to pure diffusion models.

The article is organised as follows. Section 2 introduces the double exponential jump-diffusion model and its basic properties. In Section 3, we derive the system of nonlinear integral equations which leads to the unique solution of the American call option's value and its optimal exercise boundary. In Section 4, we introduce a degenerate case of the double exponential jump-diffusion model. Under this degenerate model, we derive a closed form expression for the arbitrage-free price in terms of the optimal exercise boundary and show that the boundary itself can be characterised as the unique solution to a nonlinear integral equation. Using these results, in Section 5 we present a financial analysis of the American call option under the positive exponential jump-diffusion process.

3.2 Double Exponential Jump Diffusion Model

In this section, we will give a brief introduction to the DEJD model. Here we assume a financial market consisting of risky stock Z_t and riskless bond B_t:

$$\frac{dZ_t}{dZ_{t}-} = \mu\, dt + \sigma\, dW_t + d\Big(\sum_{i=1}^{N_t}(V_i - 1)\Big) \qquad (Z_0 = z), \tag{3.1}$$

$$dB_t = rB_t\, dt \qquad (B_0 = 1), \tag{3.2}$$

where μ is the personal appreciation drift of the stock, σ is the volatility, r is the risk-free interest rate, W_t is a standard Brownian motion, N_t is a Poisson process with rate λ, and $\{V_i\}$ is a sequence of independent identically distributed nonnegative random variables such that $Y = \log V$ has an asymmetric double exponential distribution with the density:

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y<0\}} \qquad \eta_1 > 1, \eta_2 > 0, \tag{3.3}$$

where $p, q \geq 0, p + q = 1$,represent the probabilities of upward and downward jumps. In other words,

$$\log V = Y \stackrel{d}{=} \begin{cases} \xi^+ & \text{with probability } p \\ -\xi^- & \text{with probability } q \end{cases}, \tag{3.4}$$

where ξ^+ and ξ^- are exponential random variables with means $1/\eta_1$ and $1/\eta_2$, respectively, and the notation $\stackrel{d}{=}$ means equal in distribution. The drift μ and the volatility σ are assumed to be constants. Also N_t, W_t and Y are assumed to be independent. Note that:

$$\begin{aligned} \mathsf{E}[Y] &= \frac{p}{\eta_1} - \frac{q}{\eta_2}, \\ \mathsf{Var}[Y] &= pq(\frac{1}{\eta_1} + \frac{1}{\eta_2})^2 + (\frac{p}{\eta_1{}^2} + \frac{q}{\eta_2{}^2}), \\ \mathsf{E}[V] &= \mathsf{E}[e^Y] \\ &= q\frac{\eta_2}{\eta_2 + 1} + p\frac{\eta_1}{\eta_1 - 1}, \qquad \eta_1 > 1, \eta_2 > 0. \end{aligned} \tag{3.5}$$

The requirement $\eta_1 > 1$ is to ensure that $\mathsf{E}[V] < \infty$ and $\mathsf{E}[Z_t] < \infty$. It just means that the average upward jump cannot exceed 100%, which is quite reasonable.

Before moving to the research of any specific style of options, we need to do some general preparation on this model, which can be used in the risk-neutral pricing method. Similar to the approach adopted in Chapter 2, we first rewrite the underlying dynamic Z_t in (3.1) into an ordinary exponential of a real value Lévy process X_t:

$$Z_t = e^{X_t}, \tag{3.6}$$

where

$$\begin{aligned} X_t &= \mu t + \sigma W_t + \sum_{i=1}^{N_t}(V_i - 1) - \frac{\sigma^2}{2}t + \sum_{0 \leq s \leq t}^{\Delta Z_s \neq 0}(\ln(1 + \Delta Z_s) - \Delta Z_s) \\ &= \mu t + \sigma W_t - \frac{\sigma^2}{2}t + \sum_{i=1}^{N_t}(V_i - 1) + \sum_{i=1}^{N_t}(\ln(V_i) - (V_i - 1)) \\ &= (\mu - \frac{\sigma^2}{2})t + \sigma W_t + \sum_{i=1}^{N_t}(Y_i) \qquad (X_0 = x = \ln z). \end{aligned} \tag{3.7}$$

We can see that X_t consists of three parts: the combination of the first two parts is a Brownian motion with drift; and last part is a compound Poisson process with the jump size Y_i following a double exponential distribution. The name of this model comes from this fact. Using a Brownian motion with drift $B_t = (\mu - \sigma^2/2)t + \sigma W_t$ to replace X_t in (3.6), then we can get back to the standard Black-Scholes framework.

Following the martingale approach for derivative pricing, the drift parameter μ has to assure that the discounted price process $e^{-(r-\delta)t}Z_t$ becomes a martingale under the physical probability measure P. Here we denote this new risk-neutral dynamic as Z_t^r

$$Z_t^r = e^{X_t^r} = e^{(r-\delta-\lambda\zeta-\frac{\sigma^2}{2})t+\sigma W_t+\sum_{i=1}^{N_t}(Y_i)}. \tag{3.8}$$

where

$$\zeta = \mathsf{E}[V] - 1 = q\frac{\eta_2}{\eta_2+1} + p\frac{\eta_1}{\eta_1-1} - 1. \tag{3.9}$$

It will allow us to state that the present value of a derivative is the expectation of a payoff function of Z_t^r or X_t^r defined in (3.8) under the physical measure P. This transform will make the definition of the value function for American options much easier to understand in the next section.

Note that X_t^r is also a strong Markov process, it would be very useful to introduce the infinitesimal generator of X_t^r here:

$$\begin{aligned}(\mathbb{L}_{X^r}F)(x) =& \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(x) + (r-\delta-\lambda\zeta-\frac{\sigma^2}{2})\frac{\partial F}{\partial x}(x)\\ &+\lambda\int_{-\infty}^{+\infty}[F(x+y)-F(x)]f_Y(y)dy,\end{aligned} \tag{3.10}$$

for every $F \in C_b^2(\mathbb{R})$, where $C_b^2(\mathbb{R})$ denotes the set of all bounded, twice continuously differentiable functions with bounded derivatives.

With these fundamental framework and properties, this chapter attempts extend the analytical tractability of Black-Scholes analysis for the classical geometric Brownian motion to alternative models with jumps. In particular, we demonstrate that the double exponential jump diffusion model can lead to the unique pair of solutions to value and boundary functions for finite-horizon American call options with dividends, and even analytical solutions under the degenerate one-sided jump case.

3.3 American Call Option with Dividends for DEJD Processes

An American option is an option that can be exercised anytime during its life. The majority of exchange-traded options are American or American based. The difference between the price of an American option and European option with the same characteristics is called the early exercise premium. The idea behind valuing options with early exercise is to decide when the option should be exercised. This particular choice of time is often referred to as the optimal stopping time. For more discussion about the definition of "optimal" for option pricing, see Chapter 2.

The early exercise premium formula and the optimal exercise boundary are the main study objects of our research. The EEP formula of the American option is well studied both in the Black-Scholes model and in the jump-dissuasion models. For the Black-Scholes framework, see Kim [22], Jacka [18], Carr et al. [8] and Peskir[37]. For jump-diffusion models, see Pham [46], Lamberton and Mikou [29], [30] and [31]. Chapter 2 provided a brief introduction to the path of the progression of these studies, so we will not repeat here.

In [29], [30], Lamberton and Mikou proposed several important properties of American put options under exponential Lévy processes, including the boundary behaviour near maturity and the smooth-fit property of the value function when crossing the boundary, which simplifies our derivation for the EEP representation of finite horizon American put options to a great extend in Chapter 2. Due to the difference in payoff functions and the existence of dividends, we can no longer use their results directly in this section. However, we will derive and prove these essential properties for our American calls by following the same method employed by Lamberton and Mikou in [29]. The EEP formula for call options obtained in our research are in line with the results for American puts concluded by [37] within Black-Scholes framework and [31] under exponential Lévy processes.

3.3.1 Assumptions and Notations

We assume that the underlying asset pays dividends as rate δ through out this section. For an American call option with strike price $K > 0$, define the payoff function as:

$$\bar{G}(z) = (z - K)^+. \tag{3.11}$$

Following the framework of risk-neutral pricing, the value function of this option should be the expectation of the payoff function (3.11) the physical probability measure P:

$$\begin{aligned} \bar{V}(t,z) &= \sup_{0 \le \tau \le T-t} \mathsf{E}_{t,z}\left(e^{-r\tau}(Z^r_{t+\tau} - K)^+\right) \\ &= \sup_{0 \le \tau \le T-t} \mathsf{E}_{t,z}\left(e^{-r\tau}\bar{G}(Z^r_{t+\tau})\right), \end{aligned} \tag{3.12}$$

where τ is a stopping time of the exponential Lévy process Z^r_t defined in (3.8).

Here we would like to also introduce another form of payoff and value function of American call options with respect to the double exponential jump diffusion process X^r_t:

$$G(x) = (e^x - K)^+, \tag{3.13}$$

$$\begin{aligned} V(t,x) &= \sup_{0 \le \tau \le T-t} \mathsf{E}_{t,x}\left(e^{-r\tau}(e^{X^r_{t+\tau} - K})^+\right) \\ &= \sup_{0 \le \tau \le T-t} \mathsf{E}_{t,x}\left(e^{-r\tau}G(X^r_{t+\tau})\right), \end{aligned} \tag{3.14}$$

where $x \in (-\infty, \infty)$, $t \in [0, T]$ and τ is a stopping time of the jump-diffusion process X^r_t also defined in (3.8).

Please note that the value function $\bar{V}(t,z)$ under the exponential Lévy process Z^r_t in (3.12) and the value function $V(t,x)$ under the jump-diffusion process X^r_t in (3.14) will exactly lead to the same result. By using the form $\bar{V}(t,z)$, the properties of the value function can be proved easily and concisely, while $V(t,x)$ is more convenient to be used in the derivation of the EEP formula, the nonlinear integral equations system and the properties of the early exercise boundary. This is the reason for having both of them in this section. The same approach is applied to the boundary function $\bar{b}(t)$ and $b(t)$ in the following content.

Assuming that the underlying asset pays positive dividends is essential for American call options. The following theorem will illustrate this phenomenon:

Theorem 3.1. *For the American call option, if the underlying asset has no dividend, then it is always optimal to hold this call option till maturity.*

Proof. With the definition of the value function $V(t,x)$ in (3.14), it is equivalent to prove $e^{-rt}(e^{X_t^r} - K)^+$ is a sub-martingale under P. Indeed, taking any $0 < s < t$, $X_s^r = x$, then by the Jensen's Inequality we have

$$\begin{aligned} \mathsf{E}[e^{-rt}(e^{X_t^r} - K)^+|\mathcal{F}_s] &= \mathsf{E}[(e^{-rt}e^{X_t^r} - e^{-rt}K)^+|\mathcal{F}_s] \\ &\geq \Big(\mathsf{E}[e^{-rt}e^{X_t^r}|\mathcal{F}_s] - e^{-rt}K\Big)^+ \\ &= \Big(e^{-rs}e^x - e^{-rt}K\Big)^+ \\ &> e^{-rs}(e^x - K)^+. \end{aligned} \tag{3.15}$$

where the last equality coming from the fact that the discounted price process should be a martingale under P. Thus $e^{-rt}(e^{X_t^r} - K)^+$ is a sub-martingale under P and the Theorem 3.1 holds as claimed □

Thus American call without dividend is essentially equal to its corresponding European call option. So we will always assume that the dividend rate δ is strictly larger than zero in the following text.

3.3.2 Structure of Optimal Stopping Regions

Since the payoff function (3.11) is continuous, it is possible to apply [44, Corollary 2.9] with [44, Remark 2.10]. Then we can have the following theorem with respect to the optimal stopping region and conclude that there exists an optimal stopping time for (3.12).

Theorem 3.2. *The continuation region and stopping region of the American call option described above are:*

$$C = \{(t,z) \in [0,T) \times (0,\infty) | \bar{V}(t,z) > \bar{G}(z)\} \tag{3.16}$$

$$\bar{D} = \{(t,z) \in [0,T] \times (0,\infty) | \bar{V}(t,z) = \bar{G}(z)\} \tag{3.17}$$

It means that the stopping time $\tau_{\bar{D}}$ defined by:

$$\tau_{\bar{D}} = \inf\{0 \leq s \leq T - t | Z_{t+s}^r \in \bar{D}\}, \tag{3.18}$$

is optimal in (3.12).

The following property for the structure of the optimal stopping region is also essential for our further research.

Property 3.1. *The continuation set C and the stopping set $\bar{D}$ defined in Theorem 3.2 for the American call option are both not empty.*

We introduced a result about perpetual American put options from [27, Theorem 3] when deriving this nonempty property for American puts in Chapter 2. However, here we can only temporarily assume that a similar result still holds for American calls and continue the derivation.

Assumption 3.1. *For the American call option with infinite horizon, there exists an optimal stopping point $z^* > K > 0$ such that*

$$\bar{V}_\infty(z) = \bar{G}(z) = z - K \qquad \text{if} \qquad z > z^*, \tag{3.19}$$

where $\bar{V}_\infty(z)$ is the value function of the perpetual American call option with dividends.

$$\begin{aligned} \bar{V}_\infty(z) &= \sup_\tau \mathsf{E}_z \left(e^{-r\tau} \bar{G}(Z^r_\tau) \right) \\ &= \sup_\tau \mathsf{E}_z \left(e^{-r\tau} (Z^r_\tau - K)^+ \right). \end{aligned} \tag{3.20}$$

We will briefly go through the valuation problem for perpetual American calls with dividends in the next part where a proof for this Assumption 3.1 will also be given. Now we focus on the proof of the nonempty Property 3.1.

Proof.

1. We claim that all points (t, z) with $0 < z \leq K$ for $0 \leq t < T$ belong to the continuous set C. Indeed, this is easily verified by considering $\tau_\varepsilon = \inf\{0 \leq s \leq T - t | Z^r_{t+s} \geq K + \varepsilon\}$ for $0 < \varepsilon$ and noting that $\mathsf{P}_{t,z}(0 < \tau_\varepsilon < T - t) > 0$ if $z \leq K$ with $0 \leq t < T$. The strict inequality implies that $\mathsf{E}_{t,z} \left(e^{-r\tau_\varepsilon} (Z^r_{t+\tau_\varepsilon} - K)^+ \right) > 0$. Then by the definition of the value function, we have that $\bar{V}(t, z) = \sup_{0 \leq \tau \leq T-t} \mathsf{E}_{t,z} \left(e^{-r\tau} (Z^r_{t+\tau} - K)^+ \right) > 0$. On the other hand, $\bar{G}(z) = (z - K)^+ = 0$ for all $0 < z \leq K$ with $0 \leq t < T$. Thus the continuous set C is not empty as claimed.

2. From the definition of the value function (3.12) and (3.20), we can state that $\bar{V}(t, z) \leq \bar{V}_\infty(z)$ for any $0 \leq t \leq T$.Thus for any points (t, z) with $z > z^* > K$ for $0 \leq t \leq T$, we have $\bar{V}(t, z) \leq \bar{V}_\infty(z) = \bar{G}(z)$. It is clear that such a point belongs to the stopping set $\bar{D}$. The stopping set $\bar{D}$ is not empty as claimed. □

3.3.3 A Brief Introduction to Perpetual American Calls

For the completeness of this chapter and also for the rigorous of our derivation in previous sections, here we would like to give a brief introduction to American call options with infinite horizon and derive its value function and optimal stopping point. The conclusion of this part is in line with the results obtained by Kou and Wang in [27, Theorem 3] for perpetual American puts.

First, we need to rewrite the value function of perpetual American calls with dividends $\bar{V}_\infty(z)$ defined in (3.20) into the form $V_\infty(x)$ for the jump-diffusion process X_t^r defined by (3.8), since it is more convenient to derive the analytical solution of value function and optimal stopping point under this representation:

$$\begin{aligned} V_\infty(x) &= \sup_\tau \mathsf{E}_x \left(e^{-r\tau} G(X_\tau^r) \right) \\ &= \sup_\tau \mathsf{E}_x \left(e^{-r\tau} (e^{X_\tau^r} - K)^+ \right). \end{aligned} \tag{3.21}$$

From this definition, we see that all points $x \in [0, \ln K]$ belongs to the continuation set. Thus there should exists an optimal stopping point $x^* > \ln K$ such that the stopping time

$$\tau_{x^*} = \inf\{t \geq 0 | X_t^r \geq x^*\}, \tag{3.22}$$

is optimal in the problem (3.21). Note that here we does not exclude the case where $x^* = \infty$ and thus $\tau_{x^*} = \inf\{\emptyset\} = \infty$. After the following calculation we will see that x^* should be a non infinity constant under the current setting for underlying dynamic that $\eta_1 > 1$ and $\eta_2 > 0$.

In this short section for perpetual calls, we will not explore the EEP formula and the uniqueness of it in detail. Nevertheless, we will directly solve the corresponding free-boundary problem to obtain the analytical representation of the value function $V_\infty(x)$ and the optimal stopping point x^* for simplicity. Similar to the approach adopted by [27, Theorem 3], we conclude that $V_\infty(x)$ and x^* is the unique solution to the following free-boundary problem

$$\mathbb{L}_{X^r} V_\infty = rV_\infty \qquad \text{for} \quad x < x^*, \tag{3.23}$$

$$V_\infty(x) = (e^x - K)^+ \qquad \text{for} \quad x = x^* > \ln K, \tag{3.24}$$

$$V'_\infty(x) = e^x \qquad \text{for} \quad x = x^* > \ln K, \tag{3.25}$$

$$V_\infty(x) = (e^x - K)^+ \qquad \text{for} \quad x > x^* > \ln K, \tag{3.26}$$

$$V_\infty(x) > (e^x - K)^+ \qquad \text{for} \quad x < x^*, \tag{3.27}$$

where $\mathbb{L}_{X^r}$ is the infinitesimal generator defined in (3.10). To solve this free-boundary problem, we need to introduce a special function $F(\theta)$. Recall the moment-generating function of X_t^r is given by $\mathsf{E}[e^{\theta X_t^r}] = e^{F(\theta)t}$, where the function $F(\theta)$ is defined by

$$F(\theta) = \theta(r - \delta - \frac{\sigma^2}{2} - \lambda\zeta) + \frac{1}{2}\theta^2\sigma^2 + \lambda(\frac{p\eta_1}{\eta_1 - \theta} + \frac{q\eta_2}{\eta_2 + \theta} - 1). \tag{3.28}$$

In [26, Lemma 3.1], Kou and Wang showed that for any $\alpha > 0$, $F(\theta) = \alpha$ has exactly four roots $\beta_{1,\alpha}, \beta_{2,\alpha}, -\beta_{3,\alpha}$ and $-\beta_{4,\alpha}$ where

$$0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty,$$

$$0 < \beta_{3,\alpha} < \eta_2 < \beta_{4,\alpha} < \infty.$$

Now we can make a guess for the solution and construct the $V_\infty(x)$ as

$$V_\infty(x) = \begin{cases} e^x - K & x \geq x^* \\ Ae^{-x\beta_3} + Be^{-x\beta^4} & x < x^* \end{cases} \tag{3.29}$$

where A and B are unknown constants, $\beta_3 \equiv \beta_{3,r}$, $\beta_4 \equiv \beta_{4,r}$, and $-\beta_{3,r}$, $-\beta_{4,r}$ are the roots of $F(\theta) = r$. In this case, we have three unknown values A, B and x^* with three equations (3.23), (3.24) and (3.25). This explicit construction of $V_\infty(x)$ enable us to calculate the global integral $\int_{-\infty}^{+\infty} V_\infty(x+y) f_Y(y)dy$ for $x < x^*$ in the infinitesimal generator $\mathbb{L}_{X^r} V_\infty$, which always acts as the main obstacle to evaluate options with jump processes.

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} V_\infty(x+y) f_Y(y) dy \qquad (3.30)\\
=&\int_{x^*-x}^{+\infty} (e^{x+y} - K) p\eta_1 e^{-y\eta_1} dy\\
&+\int_0^{x^*-x} (Ae^{-\beta_3(x+y)} + Be^{-\beta_4(x+y)}) p\eta_1 e^{-y\eta_1} dy\\
&+\int_{-\infty}^{0} (Ae^{-\beta_3(x+y)} + Be^{-\beta_4(x+y)}) q\eta_2 e^{y\eta_2} dy\\
=&pe^{(x-x^*)\eta_1}(\frac{e^{x^*}\eta_1}{\eta_1 - 1} - K)\\
&\frac{Ap\eta_1}{\beta_3+\eta_1}(e^{-\beta_3 x} - e^{-\beta_3 x^*} e^{(x-x^*)\eta_1}) + \frac{Bp\eta_1}{\beta_4+\eta_1}(e^{-\beta_4 x} - e^{-\beta_4 x^*} e^{(x-x^*)\eta_1})\\
&+\frac{Aq\eta_2 e^{-\beta_3 x}}{\eta_2 - \beta_3} + \frac{Bq\eta_2 e^{-\beta_4 x}}{\eta_2 - \beta_4}.
\end{aligned}
$$

Then we can insert (3.30) into $\mathbb{L}_{X^r} V_\infty$ of equation (3.23). After some expansion and reorganising we will have

$$
\begin{aligned}
&\mathbb{L}_{X^r} V_\infty - rV_\infty \qquad (3.31)\\
=&Ae^{-x\beta_3}[F(-\beta_3) - r] + Be^{-x\beta_4}[F(-\beta_4) - r]\\
&+\lambda pe^{(x-x^*)\eta_1}[\frac{\eta_1 e^{x^*}}{\eta_1 - 1} - K - \frac{A\eta_1 e^{-x^*\beta_3}}{\eta_1+\beta_3} - \frac{B\eta_1 e^{-x^*\beta_4}}{\eta_1+\beta_4}]\\
=&\lambda pe^{(x-x^*)\eta_1}[\frac{\eta_1 e^{x^*}}{\eta_1 - 1} - K - \frac{A\eta_1 e^{-x^*\beta_3}}{\eta_1+\beta_3} - \frac{B\eta_1 e^{-x^*\beta_4}}{\eta_1+\beta_4}],
\end{aligned}
$$

for any $x < x^*$. Now rewrite equations (3.23), (3.24) and (3.25) into explicit form:

$$\frac{\eta_1 e^{x^*}}{\eta_1 - 1} - K - \frac{A\eta_1 e^{-x^*\beta_3}}{\eta_1+\beta_3} - \frac{B\eta_1 e^{-x^*\beta_4}}{\eta_1+\beta_4} = 0, \qquad (3.32)$$

$$Ae^{-x^*\beta_3} + Be^{-x^*\beta_4} = e^{x^*} - K, \qquad (3.33)$$

$$A\beta_3 e^{-x^*\beta_3} + B\beta_4 e^{-x^*\beta_4} = -e^{x^*}. \qquad (3.34)$$

Solving this equation system will provide us the explicit solution to the value function

$V_\infty(x)$ and its optimal stopping point x^*:

$$A = \frac{e^{x^*\beta_3}}{\beta_4 - \beta_3}[(\beta_4 + 1)e^{x^*} - \beta_4 K], \tag{3.35}$$

$$B = \frac{-e^{x^*\beta_4}}{\beta_4 - \beta_3}[(\beta_3 + 1)e^{x^*} - \beta_3 K], \tag{3.36}$$

$$V_\infty(x) = \begin{cases} e^x - K & x \geq x^* \\ \frac{e^{x^*\beta_3}}{\beta_4-\beta_3}[(\beta_4 + 1)e^{x^*} - \beta_4 K]e^{-x\beta_3} + \frac{-e^{x^*\beta_4}}{\beta_4-\beta_3}[(\beta_3 + 1)e^{x^*} - \beta_3 K]e^{-x\beta^4} & x < x^* \end{cases}, \tag{3.37}$$

$$x^* = \ln(-\frac{K(\eta_1 - 1)\beta_3\beta_4}{\eta_2(\beta_3 + 1)(\beta_4 + 1)}). \tag{3.38}$$

To this end, we have obtained the analytical representation for perpetual American call options with dividends. The explicit form of x^* can be easily interpreted into the optimal stopping point z^* for $\bar{V}_\infty(z)$

$$z^* = e^{x^*} = -\frac{K(\eta_1 - 1)\beta_3\beta_4}{\eta_2(\beta_3 + 1)(\beta_4 + 1)} > K. \tag{3.39}$$

Thus the validity of Assumption 3.1 can be proved here.

3.3.4 Some Properties of the Value and Boundary Functions

In this part, we will give all the properties of the value and boundary functions which should be used in the derivation of the EEP representation of the American call option. In the similar part of Chapter 2, we introduced several important properties without detailed proofs such as smooth-fit, continuity of exercise boundary, and the boundary behaviour near maturity, since they are directly coming from the result of [29] and [30]. Here we cannot take this advantage again but prove them one by one in details in the following context. Luckily, we have [29] and [44] as references for effective ideas or methods of these proofs.

First, let us introduce some useful properties of the value function $\bar{V}(t, z)$ which can be deduced from its definition (3.12) alone.

Property 3.2.

- *For $t \in [0, T]$, the function $z \mapsto \bar{V}(t, z)$ is increasing on $(0, \infty)$.*
- *For $z \in (0, \infty)$, the function $t \mapsto \bar{V}(t, z)$ is decreasing on $[0, T]$.*

This monotonicity property can be easily concluded directly from the definition of the value function $\bar{V}(t,z)$, thus we will not present this trivial proof. The introductions and proofs of the convexity, continuity and smooth-fit properties will be given below.

Property 3.3. *The function* $z \mapsto \bar{V}(t,z)$ *is convex for* $z \in (0,\infty)$.

Proof. Take any $z_1, z_2 \in (0,\infty)$ satisfying $z_1 < z_2$, then for any $\lambda \in [0,1]$

$$\begin{aligned}
&\bar{V}(t, \lambda z_1 + (1-\lambda)z_2) \\
=& \sup_{0\le\tau\le T-t} \mathsf{E}[e^{-r\tau}((\lambda z_1 + (1-\lambda)z_2)Z^r_\tau - K)^+] \\
=& \sup_{0\le\tau\le T-t} \mathsf{E}[e^{-r\tau}(\lambda(z_1 Z^r_\tau - K) + (1-\lambda)(z_2 Z^r_\tau - K))^+] \\
\le& \sup_{0\le\tau\le T-t} \mathsf{E}[e^{-r\tau}(\lambda(z_1 Z^r_\tau - K)^+ + (1-\lambda)(z_2 Z^r_\tau - K)^+)] \\
\le& \sup_{0\le\tau\le T-t} \mathsf{E}[e^{-r\tau}\lambda(z_1 Z^r_\tau - K)^+] + \sup_{0\le\tau\le T-t} \mathsf{E}[e^{-r\tau}(1-\lambda)(z_2 Z^r_\tau - K)^+] \\
=& \lambda\bar{V}(t,z_1) + (1-\lambda)\bar{V}(t,z_2),
\end{aligned} \tag{3.40}$$

where the first inequality comes from the fact that $(x+y)^+ \le x^+ + y^+$ for $x, y \in \mathbb{R}$, and the second inequality is from the property of supremum. Thus $z \mapsto \bar{V}(t,z)$ is convex as claimed. □

Property 3.4. *The value function* $(t,z) \mapsto \bar{V}(t,z)$ *is continuous on* $[0,T] \times (0,\infty)$.

Proof. For this, it is enough to prove that

$$z \mapsto \bar{V}(t,z) \text{ is continuous at } z_0, \tag{3.41}$$

$$t \mapsto \bar{V}(t,z) \text{ is uniformly continuous at } t_0 \text{ for } z \in [z_0 - \delta, z_0 + \delta] \tag{3.42}$$

for each $(t_0, z_0) \in [0,T] \times (0,\infty)$ with some $\delta > 0$ small enough.

For (3.41), it follows directly from Property 3.3 about the convexity of $z \mapsto \bar{V}(t,z)$. For (3.42), let us fix arbitrary $0 \le t_1 < t_2 \le T$ and $z \in (0,\infty)$ and let $\tau_1 = \tau_*(t_1, z)$ denote the optimal stopping time for $\bar{V}(t_1, z)$. Set $\tau_2 = \tau_1 \wedge (T - t_2)$. Thus we have $\tau_2 \le \tau_1$ and $\tau_2 \le T - t_2$. By the monotonicity Property 3.2 that $t \mapsto \bar{V}(t,z)$ is decreasing on $[0, T]$, we can have

$$
\begin{aligned}
0 &\leq \bar{V}(t_1, z) - \bar{V}(t_2, z) \qquad (3.43)\\
&\leq \mathsf{E}[e^{-r\tau_1}(zZ^r_{\tau_1} - K)^+] - \mathsf{E}[e^{-r\tau_2}(zZ^r_{\tau_2} - K)^+]\\
&\leq \mathsf{E}[e^{-r\tau_2}((zZ^r_{\tau_1} - K)^+ - (zZ^r_{\tau_2} - K)^+)]\\
&\leq \mathsf{E}[e^{-r\tau_2}(zZ^r_{\tau_1} - zZ^r_{\tau_2})^+]\\
&\leq z\mathsf{E}(Z^r_{\tau_1} - Z^r_{\tau_2})^+,
\end{aligned}
$$

where we also use an inequality that $(x - K)^+ - (y - K)^+ \leq (x - y)^+$ for $x, y \in \mathbb{R}$. Further, recall the fact that X^r_t is a Lévy process with stationary independent increments and $\tau_1 - \tau_2 \leq t_1 - t_2$, we know that

$$
\begin{aligned}
\mathsf{E}(Z^r_{\tau_1} - Z^r_{\tau_2})^+ &= \mathsf{E}(\mathsf{E}(Z^r_{\tau_1} - Z^r_{\tau_2})^+|\mathcal{F}_{\tau_2}) \qquad (3.44)\\
&= \mathsf{E}(Z^r_{\tau_2}\mathsf{E}(Z^r_{\tau_1}/Z^r_{\tau_2} - 1)^+|\mathcal{F}_{\tau_2})\\
&= \mathsf{E}(Z^r_{\tau_2})\mathsf{E}(e^{X^r_{\tau_1} - X^r_{\tau_2}} - 1)^+\\
&= \mathsf{E}(Z^r_{\tau_2})\mathsf{E}(\sup_{0 \leq t \leq t_2 - t_1} e^{X^r_t} - 1)^+\\
&=: \mathsf{E}(Z^r_{\tau_2})L(t_2 - t_1). \qquad (3.45)
\end{aligned}
$$

By the property of the jump-diffusion process X^r_t, it can be seen that $L(t_2 - t_1)$ as $t_2 - t_1 \to 0$. And we have

$$
0 \leq \bar{V}(t_1, z) - \bar{V}(t_2, z) \leq z\mathsf{E}(Z^r_{\tau_2})L(t_2 - t_1) \leq ze^{rT}L(t_2 - t_1), \qquad (3.46)
$$

from where (3.42) becomes evident. This complete the proof. □

Property 3.5. *$\bar{V}(t, z)$ is $C^{1,2}$ on C (and $C^{1,2}$ on $\bar{D}$).*

Proof. This follows from the strong Markov property of Z^r_t, Theorem 3.2 and Property 3.4. □

Now, we can provide a more concrete description about the structure of optimal stopping regions using these properties concluded so far: Take any given $t^* \in [0, T]$, we have

- The function $z \mapsto \bar{V}(t^*, z)$ is increasing and convex.
- The payoff function is $\bar{G}(z) = (z - K)^+$.

- Both continuation set C and stopping set $\bar{D}$ are not empty.

- $\bar{V}(t^*, z) = \bar{G}(z)$ for $z > z^* > K$ since all points (t, z) with $z > z^*$ for $0 \leq t \leq T$ belongs to the stopping set $\bar{D}$.

- $\bar{V}(t^*, z) > \bar{G}(z)$ for $0 < z \leq K$ since all points (t, z) with $0 < z \leq K$ for $0 \leq t < T$ belong to the continuous set C

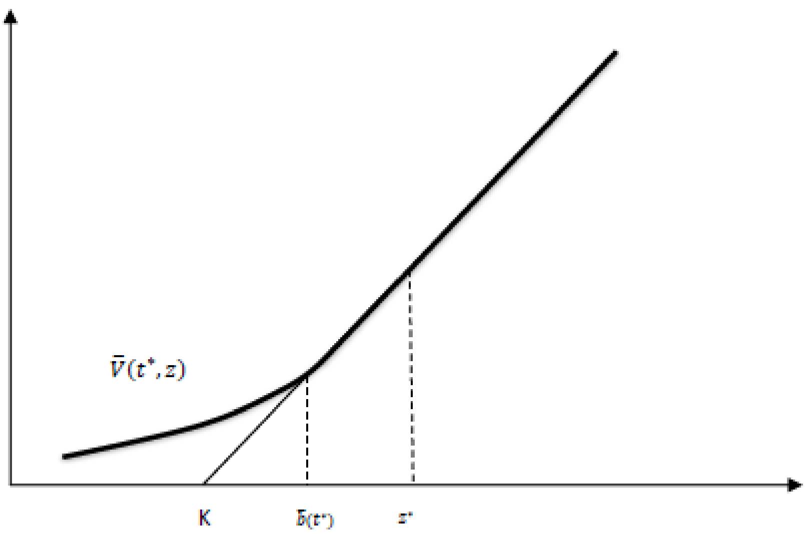

Figure 3.1: A simple illustration of the behaviour of the value function and the payoff function of an American call option at a fixed time $t^* \in [0, T]$.

The combination of these fact confines that the shape of the value function and the payoff function can only take the form as illustrated in Figure 3.1. Observe this figure, for each $t \in [0, T]$, we can have a point where the value function joins the payoff function. Denote it as $\bar{b}(t)$, then we have the following theorem.

Theorem 3.3. *There exists a function* $\bar{b} : [0, T] \to \mathbb{R}$ *satisfying* $0 < K < \bar{b}(t) \leq z^*$ *for all* $0 \leq t < T$ *such that the continuation set* C *of (3.12) equals*

$$C = \{(t, z) \in [0, T) \times (0, \infty) | z < \bar{b}(t)\} \tag{3.47}$$

and the stopping set $\bar{D}$ *is the closure of the set*

$$D = \{(t, z) \in [0, T) \times (0, \infty) | z > \bar{b}(t)\} \tag{3.48}$$

joined with the remaining points (T, z) *for* $z \leq \bar{b}(T)$*.*

This function $\bar{b}(t)$ is just the early exercise boundary which we are interested in. It can also be called as optimal stopping boundary or free-boundary. The name "free-boundary" comes from the connection between the option pricing problem and the ice melting problem in Physics. Before moving to the smooth-fit property of value function $\bar{V}(t,z)$, we would like to introduce the following monotonicity property of the early exercise boundary $\bar{b}(t)$, which is essential in the proof of smooth-fit.

Property 3.6. *The early exercise boundary $\bar{b}(t)$ is decreasing on $[0,T]$.*

Proof. Recall that $t \mapsto \bar{V}(t,z)$ is decreasing on $[0,T]$ in Property 3.2 and the payoff function $\bar{G}(z)$ is not depending on the time t. Hence if $(t,z) \in C$, then take any other $0 < t' < t$, we have $\bar{V}(t',z) - \bar{G}(z) \geq \bar{V}(t,z) - \bar{G}(z) > 0$. This means that $(t',z) \in C$ for all $0 < t' < t$. Combined with the definition of the continuation region C in (3.47), we can state that $t \mapsto \bar{b}(t)$ is decreasing on $[0,T]$. □

Now we can introduce an important property for the behaviour of the value function when crossing the free-boundary. This property will alow us to eliminate the local-time term and simplify the EEP representation in the next section. The proof of the smooth-fit property follows [30, Theorem 4.1].

Property 3.7. *The smooth-fit property holds, i.e. that $z \mapsto \bar{V}(t,z)$ is C^1 at $\bar{b}(t)$:*

$$\bar{V}_z(t,z) = \bar{G}'(z) = 1 \quad \textit{for} \quad z = \bar{b}(t) \tag{3.49}$$

Proof. Recall that X_t^r has infinite variation. Then by [28, Theorem 6.5] we can say that 0 is regular for $(0,\infty)$ for X_t^r. This means that

$$\mathsf{P}(\tau_0^+ = 0) = 1 \tag{3.50}$$

where $\tau_0^+ = \inf\{t > 0 | X_t^r > 0\}$ and $X_0^r = 0$.

Now we want to show that for any $t \in [0,T)$, $z \mapsto \bar{V}(t,z)$ is differentiable at $\bar{b}(t)$ and $\bar{V}_z(t,\bar{b}(t)) = \bar{G}'(\bar{b}(t))$. To simplify the proof, we only consider the case $t = 0$, and it is easy to extend to all other cases. First note that for any $h > 0$

$$\frac{\bar{V}(0,\bar{b}(0)) - \bar{V}(0,\bar{b}(0)-h)}{h} \leq \frac{\bar{G}(\bar{b}(0)) - \bar{G}(\bar{b}(0)-h)}{h}, \tag{3.51}$$

since $\bar{V}(t,z) \geq \bar{G}(z)$ and $\bar{V}(0,\bar{b}(0)) = \bar{G}(\bar{b}(0))$. So we have

$$\limsup_{h\to 0^+}\left(\frac{\bar{V}(0,\bar{b}(0)) - \bar{V}(0,\bar{b}(0)-h)}{h}\right) \leq \bar{G}'(\bar{b}(0)). \tag{3.52}$$

And since $\bar{b}(t) > K$ for $t \in [0, T)$, we know that

$$\bar{G}(\bar{b}(t)) = \bar{b}(t) - K \tag{3.53}$$

$$\bar{G}'(\bar{b}(t)) = 1, \tag{3.54}$$

and $\bar{G}$ is continuously differentiable in a neighborhood of $\bar{b}(t)$ for all $t \in [0, T)$. Next, we consider the optimal stopping time for $\bar{V}(0, \bar{b}(0) - h)$

$$\begin{aligned}\tau_h &= \inf\{t \in [0,T) | (\bar{b}(0) - h) Z_t^r \geq \bar{b}(t)\} \\ &= \inf\{t \in [0,T) | (X_t^r \geq \ln(\frac{\bar{b}(t)}{\bar{b}(0) - h})\} \\ &\leq \inf\{t \in [0,T) | (X_t^r \geq \ln(\frac{\bar{b}(0)}{\bar{b}(0) - h})\} \\ &=: \tau_h^*.\end{aligned} \tag{3.55}$$

where the inequality follows Property 3.6 that $\bar{b}(t)$ is decreasing. Recall that $\mathsf{P}(\tau_0^+ = 0) = 1$. Thus on the set $\{\tau_0^+ = 0\}$, given a fixed $t \in (0, T)$, there exists $s \in [0, t)$ such that $X_s^r > 0$. For small enough h, we have that $X_s^r > \ln(\bar{b}(0)/\bar{b}(0) - h)$, so that $\tau_h^* \leq s$. Therefore, $\lim_{h \to 0^*} \tau_h^* \leq t$. Since t is arbitrary, we deduce that $\tau_h^* \to 0$ a.s. when h goes to 0. Hence,

$$\lim_{h \to 0^+} \tau_h = 0 \quad \text{a.s.} \tag{3.56}$$

Moreover, since

$$\bar{V}(0, \bar{b}(0)) \geq \mathsf{E}[e^{-r\tau_h} \bar{G}(\bar{b}(0) e^{X_{\tau_h}^r})], \tag{3.57}$$

by the definition of $\bar{V}(t, z)$ in (3.12), we have

$$\begin{aligned}&\frac{\bar{V}(0, \bar{b}(0)) - \bar{V}(0, \bar{b}(0) - h)}{h} \\ =&\frac{\bar{V}(0, \bar{b}(0)) - \mathsf{E}[e^{-r\tau_h} \bar{G}((\bar{b}(0) - h) e^{X_{\tau_h}^r})]}{h} \\ \geq&\mathsf{E}[e^{-r\tau_h} \frac{\bar{G}(\bar{b}(0) e^{X_{\tau_h}^r}) - \bar{G}((\bar{b}(0) - h) e^{X_{\tau_h}^r})}{h}].\end{aligned} \tag{3.58}$$

Since $\bar{G}$ is continuously differentiable in a neighborhood of $\bar{b}(0)$ as stated before, we have

$$\lim_{h \to 0} \frac{\bar{G}(\bar{b}(0) e^{X_{\tau_h}^r}) - \bar{G}((\bar{b}(0) - h) e^{X_{\tau_h}^r})}{h} = \bar{G}'(\bar{b}(0)). \tag{3.59}$$

Then using the Lipschitz continuity of $\bar{G}$, by dominated convergence we obtain that

$$\liminf_{h\to 0^+}\left(\frac{\bar{V}(0,\bar{b}(0)) - \bar{V}(0,\bar{b}(0)-h)}{h}\right) \geq \bar{G}'(\bar{b}(0)). \tag{3.60}$$

Combine (3.52) and (3.60) we can conclude that

$$\bar{V}_z(0,\bar{b}(0)) = \bar{G}'(\bar{b}(0)) = 1. \tag{3.61}$$

Moreover, we have

$$\bar{V}_z(t,z) = \bar{G}'(z) = 1 \quad \text{at} \quad z = \bar{b}(t), \tag{3.62}$$

and the smooth-fit property holds as claimed. □

For the derivation of the EEP formula and the proof of uniqueness in following sections, we still need some more useful properties about the early exercise boundary. However, we prefer to introduce these free boundary properties with the jump-diffusion process X_t^r and its corresponding form of the value function $V(t,x)$ defined in (3.14), since it is more convenient to prove them by using the infinitesimal generator $\mathbb{L}_{X^r}$ of X_t^r defined in (3.10). So it is essential to introduce the optimal stopping boundary $b(t)$ with respect to $V(t,x)$ here:

Theorem 3.4. *There exists a function* $b : [0,T] \to \mathbb{R}$ *satisfying* $0 < \ln K < b(t) \leq \ln z^*$ *for all* $0 \leq t < T$ *such that the continuation set* $C = \{(t,x) \in [0,T) \times (-\infty,\infty) | V(t,x) > G(x)\}$ *of the value function (3.14) equals*

$$C = \{(t,x) \in [0,T) \times (-\infty,\infty) | x < b(t)\} \tag{3.63}$$

and the stopping set $\bar{D} = \{(t,x) \in [0,T) \times (-\infty,\infty) | V(t,x) = G(x)\}$ *is the closure of the set*

$$D = \{(t,x) \in [0,T) \times (-\infty,\infty) | x > b(t)\} \tag{3.64}$$

joined with the remaining points (T,x) *for* $x \leq b(T)$*.*

This theorem directly comes from Theorem 3.3. Since the pair of function $V(t,x)$ and $b(t)$ is essentially equivalent to $\bar{V}(t,z)$ with $\bar{b}(t)$, we can easily convert Property 3.2 - 3.7 proposed above into the following collections.

Property 3.8. *The properties of the value function* $V(t,x)$*:*

- *The value function* $(t, x) \mapsto V(t, x)$ *is continuous on* $[0, T] \times (-\infty, \infty)$.
- *The function* $x \mapsto V(t, x)$ *is increasing and convex.*
- *The function* $t \mapsto V(t, x)$ *is decreasing and continuous.*
- *The smooth fit holds:* $V_x(t, x) = G'(x)$ *for* $x = b(t)$.
- $V(t, x)$ *is* $C^{1,2}$ *on* C *(and* $C^{1,2}$ *on* $\bar{D}$*).*

Proof. Here we only present the proof of convexity $x \mapsto V(t, x)$ for clarity. Take any $x_1, x_2 \in (-\infty, \infty)$ satisfying $x_1 < x_2$, then for any $\lambda \in [0, 1]$

$$\begin{aligned}
&V(t, \lambda x_1 + (1-\lambda)x_2) \qquad (3.65)\\
=&\sup_{0\le\tau\le T-t} \mathsf{E}[e^{-r\tau}(e^{\lambda x_1+(1-\lambda)x_2} e^{X_\tau^r} - K)^+]\\
\le&\sup_{0\le\tau\le T-t} \mathsf{E}[e^{-r\tau}((\lambda e^{x_1} + (1-\lambda)e^{x_2})e^{X_\tau^r} - K)^+]\\
=&\sup_{0\le\tau\le T-t} \mathsf{E}[e^{-r\tau}(\lambda(e^{x_1}e^{X_\tau^r} - K) + (1-\lambda)(e^{x_2}e^{X_\tau^r} - K))^+]\\
\le&\sup_{0\le\tau\le T-t} \mathsf{E}[e^{-r\tau}(\lambda(e^{x_1}e^{X_\tau^r} - K)^+ + (1-\lambda)(e^{x_2}e^{X_\tau^r} - K)^+)]\\
\le&\sup_{0\le\tau\le T-t} \mathsf{E}[e^{-r\tau}\lambda(e^{x_1}e^{X_\tau^r} - K)^+] + \sup_{0\le\tau\le T-t} \mathsf{E}[e^{-r\tau}(1-\lambda)(e^{x_2}e^{X_\tau^r} - K)^+]\\
=&\lambda V(t, x_1) + (1-\lambda)V(t, x_2),
\end{aligned}$$

where the first inequality comes from the convexity of the exponential function, the second inequality comes from the fact that $(x + y)^+ \le x^+ + y^+$ for $x, y \in \mathbb{R}$, and the third inequality is from the property of supremum. Thus $x \mapsto V(t, x)$ is convex as claimed. □

Property 3.9. *The early exercise boundary* $b(t)$ *is decreasing on* $[0, T]$.

The properties of the free-boundary for American put options were fully studied by Lamberton and Mikou in [30] and [29]. However, there is no proved result for American calls that can be directly referred to. Thus here we will introduce two important properties for the early exercise boundary of American call options, and prove them in details. One is for the continuity of the boundary function $b(t)$, and the other is for the boundary behaviour at maturity.

Property 3.10. *The early exercise boundary* $b(t)$ *is continuous on* $[0, T]$

Proof. Standard arguments [44, p.131, Killed version] based on the strong Markov property and the definition of the early exercise boundary $b(t)$ in Theorem 3.4 implies that the value function $V(t,x)$ and its infinitesimal generator for X_t^* admits the following relationship:

$$(V_t + \mathbb{L}_{X^r} V - rV)(t,x) \begin{cases} = 0 & \text{if} \quad x < b(t) \\ < 0 & \text{if} \quad x \geq b(t) \geq \ln K \end{cases} \tag{3.66}$$

for any fixed $t \in [0,T)$. The inequality is what makes the American option different from European ones. It is the result of potential "suboptimal" choice of stopping time for delta hedgers. See Chapter 2 for more detailed illustration of the concept of "optimal" in American style derivatives.

Now we would like to calculate $(\mathbb{L}_{X^r} V - rV)(t,x)$ for $x \geq b(t)$, which is essential for our proof below. As stated in Chapter 2, the non local integral term in the operator $\mathbb{L}_{X^r}$ defined in (3.10) act as a main difficulty for option pricing with jumps. Here this non local integral term also make it impossible for us to obtain an explicit result of $(\mathbb{L}_{X^r} V - rV)(t,x)$. However, by applying some appropriate techniques, we can still get a simplified form of it which is enough for further derivations. First, let's introduce a corresponding function to $G(x)$:

$$g(x) = e^x - K \tag{3.67}$$

It is obvious that $V(t,x) = G(x) = g(x)$ for $x \geq b(t) \geq \ln K$. Since the form of $g(x)$ is already known globally, it is possible to calculate $(\mathbb{L}_{X^r} g - rg)(x)$ into explicit form:

$$\begin{aligned}
&(\mathbb{L}_{X^r} g - rg)(x) \qquad\qquad (3.68)\\
&= \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial x^2}(x) + (r - \delta - \lambda\zeta - \frac{\sigma^2}{2})\frac{\partial g}{\partial x}(x) + \lambda \int_{-\infty}^{+\infty} [g(x+y) - g(x)] f_Y(y)dy - rg(x) \\
&= \frac{1}{2}\sigma^2 (e^x) + (r - \delta - \lambda\zeta - \frac{\sigma^2}{2})(e^x) + \lambda \int_{-\infty}^{+\infty} [e^{x+y} - e^x] f_Y(y)dy - r(e^x - K) \\
&= rK - \delta e^x - \lambda e^x \zeta + \lambda e^x \int_{-\infty}^{+\infty} [e^y - 1] f_Y(y)dy \\
&= rK - \delta e^x - \lambda e^x \big(\zeta + \int_{-\infty}^{+\infty} f_Y(y)dy - \int_{-\infty}^{+\infty} e^y f_Y(y)dy\big) \\
&= rK - \delta e^x - \lambda e^x (\zeta + 1 - \mathsf{E}(e^Y)) \\
&= rK - \delta e^x,
\end{aligned}$$

for any $x \in \mathbb{R}$. The last equation comes from the definition of $\mathsf{E}(e^Y)$ in (3.5) and the definition of ζ in (3.9). With the result of (3.68), we can rewrite $(\mathbb{L}_{X^r}V - rV)(t,x)$ as below:

$$\begin{aligned}
&(\mathbb{L}_{X^r}V - rV)(t,x) \qquad (3.69)\\
&= (\mathbb{L}_{X^r}g - rg)(x) + \lambda \int_{-\infty}^{\infty} [V(t,x+y) - g(x+y)] f_Y(y)dy \\
&= rK - \delta e^x + \psi(t,x),
\end{aligned}$$

for $(t,x) \in [0,T) \times [b(t),\infty)$. The first equality is the result of an add-minus trick applied inside the non local integral term of $\mathbb{L}_{X^r}$ and the fact that $V(t,x) = g(x)$ for $x \geq b(t) \geq \ln K$. The function ψ is denote as

$$\psi(t,x) = \lambda \int_{-\infty}^{\infty} [V(t,x+y) - g(x+y)] f_Y(y)dy, \qquad (3.70)$$

for $(t,x) \in [0,T) \times [b(t),\infty)$. We can say that for any fixed $t \in [0,T)$, the function $x \mapsto \psi(t,x)$ is nonnegative, nonincreasing and continuous. The nonnegative property is from the fact that $V(t,x) \geq G(x) \geq g(x)$ globally. The continuity follows from the continuity of $V(t,x)$ by dominated convergence. And the nonincreasing property is the consequence of both the convexity of $x \mapsto V(t,x)$ and the fact that $V(t,x) = G(x) = g(x)$ for $x \geq b(t) \geq \ln K$. Additionally, the continuity of $\psi(t,x)$ will also leads to the continuity of $(\mathbb{L}_{X^r}V - rV)(t,x)$. We will not illustrate these trivial proof here.

Now we can focus on the continuity of $b(t)$. First let's show that $b(t)$ is right continuous. For this, take any fixed $t \in [0,T)$ and let $(t_n)_{n\geq 1}$ be a decreasing sequence such that $\lim_{n\to\infty} t_n = t$. Since the function b is decreasing, the sequence $(b(t_n))_{n\geq 1}$ is increasing and $\lim_{n\to\infty} b(t_n) \leq b(t)$. On the other hand, we have

$$V(t_n, b(t_n)) = G(b(t_n)), \qquad (3.71)$$

by the definition of the early exercise boundary $b(t)$ in Theorem 3.4. Then from the continuity of V and G, we have

$$V(t, \lim_{n\to\infty} b(t_n)) = G(\lim_{n\to\infty} b(t_n)). \qquad (3.72)$$

Hence the point $(t, \lim_{n\to\infty} b(t_n))$ lies on $\bar{D}$, and we have $\lim_{n\to\infty} b(t_n) \geq b(t)$. Therefore $\lim_{n\to\infty} b(t_n) = b(t)$ and the right-continuity is proved.

Secondly, we want to prove that b is also left continuous. Take any fixed $t \in (0, T)$ and denote $b(t^-)$ as the left limit. Since the function b is decreasing, the left limit exists and $b(t^-) \geq b(t)$. Take any point $(s, x) \in (0, t) \times (b(t), b(t^-))$ then we know that $x < b(t^-) < b(s)$. Thus on the open set $(0, t) \times (b(t), b(t^-))$, we have $(V_t + \mathbb{L}_{X^r} V - rV)(s, x) = 0$ from relation (3.66). This leads to

$$(\mathbb{L}_{X^r} V - rV)(s, x) = -V_t(s, x) > 0 \quad \text{on} \quad (0, t) \times (b(t), b(t^-)), \tag{3.73}$$

since $t \mapsto V(t, x)$ is decreasing as stated in Property 3.2.then by the continuity of $(\mathbb{L}_{X^r} V - rV)(s, x)$, we have:

$$(\mathbb{L}_{X^r} V - rV)(t, x) \geq 0 \quad \text{for} \quad x \in (b(t), b(t^-)). \tag{3.74}$$

Note that $x \in (b(t), b(t^-))$ which means that we can apply the result of (3.69) here:

$$(\mathbb{L}_{X^r} V - rV)(t, x) = rK - \delta e^x + \psi(t, x) \geq 0 \quad \text{for} \quad x \in (b(t), b(t^-)). \tag{3.75}$$

On the other hand, on the set $(s, x) \in (t, T) \times (b(t), \infty)$ we have $x > b(t) > b(s)$ since the function b is decreasing. This implies that $(V_t + \mathbb{L}_{X^r} V - rV)(s, x) < 0$ on this set from the equation (3.66). Recall that $V(s, x) = G(x) = g(x)$ for $x \geq b(s) \geq \ln K$, thus the time derivative of $V(s, x)$ is zero on $(s, x) \in (t, T) \times (b(t), \infty)$. So we have

$$(V_t + \mathbb{L}_{X^r} V - rV)(s, x) = (\mathbb{L}_{X^r} V - rV)(s, x) < 0 \quad \text{on} \quad (t, T) \times (b(t), \infty). \tag{3.76}$$

Again, by the continuity of $(\mathbb{L}_{X^r} V - rV)(s, x)$, we have:

$$(\mathbb{L}_{X^r} V - rV)(t, x) \leq 0 \quad \text{for} \quad x \in (b(t), \infty). \tag{3.77}$$

It is also possible to apply the result of (3.69) to (3.77) since $x > b(t)$:

$$(\mathbb{L}_{X^r} V - rV)(t, x) = rK - \delta e^x + \psi(t, x) \leq 0 \quad \text{for} \quad x \in (b(t), \infty). \tag{3.78}$$

The combination of equation (3.75) and (3.78) leads to:

$$\psi(t, x) - \delta e^x = -rK \quad \text{for} \quad x \in (b(t), b(t^-)). \tag{3.79}$$

Now let

$$\hat{\psi}(t, x) = \delta e^x + \psi(t, x), \tag{3.80}$$

then from the property of $\psi(t, x)$ we know that $x \mapsto \hat{\psi}(t, x)$ is continuous and strictly decreasing on $[b(t), \infty)$. This contradicts equation (3.79) which indicates $\hat{\psi}(t, x)$ remains a constant over an interval. Thus the function b is continuous as claimed. $\square$

Property 3.11. *At the maturity, the value of the early exercise boundary b equals to*

$$b(T) \stackrel{def}{=} \lim_{t \to T} b(t) = \begin{cases} = \ln K & \text{if } \frac{\lambda q}{\eta_2+1} \leq \delta - r \\ = \ell & \text{if } \frac{\lambda q}{\eta_2+1} > \delta - r \end{cases} \tag{3.81}$$

where ℓ is the unique real number in the interval $(\ln K, \infty)$ such that

$$\hat{\psi}_T(x) = -rK, \tag{3.82}$$

where $\hat{\psi}_T(x)$ is the function defined on $x \in (\ln K, \infty)$

$$\hat{\psi}_T(x) = \lambda \int_{-\infty}^{\infty} (K - e^{x+y})^+ f_Y(y) dy - \delta e^x. \tag{3.83}$$

Proof. Note that

$$\hat{\psi}_T(x) = \lim_{t \to T} \hat{\psi}(t, x), \tag{3.84}$$

on $(\ln K, \infty)$ by the fact that $\lim_{t \to T} V(t, x) = V(T, x) = G(x) = (e^x - K)^+$ globally. And the function $\hat{\psi}(t, x)$ is introduced by (3.80) in the proof of Property 3.10. Thus from the property of $\hat{\psi}(t, x)$ we can easily see that $x \mapsto \hat{\psi}_T(x)$ is continuous and strictly decreasing on $(\ln K, \infty)$. The connection between function $\hat{\psi}_T$ and $\hat{\psi}$ also indicates that the derivation of this particular function carries on the same idea of the previous proof.

Now let us focus on the maturity behaviour of the boundary function. $b(T)$ is define by $b(T) = \lim_{t \to T} b(t)$. It is clear that $b(T) \geq \ln K$ by the continuity of b and the fact that $b(t) > \ln K$ for $t \in [0, T)$. Recall that some results we already had in the proof of the previous property, for any (t, x) on the set $(0, T) \times (b(t), \infty)$, the inequality $(V_t + \mathbb{L}_{X^r} V - rV)(t, x) \leq 0$ reads

$$\hat{\psi}(t, x) + rK \leq 0. \tag{3.85}$$

Hence, by the continuity of $t \mapsto \hat{\psi}(t, x)$, we can say that

$$\hat{\psi}_T(x) \leq -rK, \tag{3.86}$$

for any $x \in (b(T), \infty)$. Note that the continuity of $t \mapsto \hat{\psi}(t, x)$ follows from the continuity of $V(t, x)$ by dominated convergence.

On the other hand, on the set $(0, T) \times (-\infty, b(t))$, we have

$$(V_t + \mathbb{L}_{X^r} V - rV)(t, x) = 0. \tag{3.87}$$

Therefore, for $t \in (0,T)$, we have $(\mathbb{L}_{X^r}V - rV)(t,x) \geq 0$ on the interval $(-\infty, b(t))$ since $t \mapsto V(t,x)$ is decreasing. Note that $\lim_{t\to T}(\mathbb{L}_{X^r}V - rV)(t,x) = (\mathbb{L}_{X^r}G - rG)(x)$ in the sense of distributions. Thus we have $(\mathbb{L}_{X^r}G - rG)(x) \geq 0$ on the interval $(-\infty, b(T))$. It is easy to check that on the interval $(\ln K, \infty)$, we have $(\mathbb{L}_{X^r}G - rG)(x) = \hat{\psi}_T(x) + rK$. Hence,

$$\hat{\psi}_T(x) \geq -rK, \tag{3.88}$$

for any $x \in (-\infty, b(T)) \cap (\ln K, \infty)$, or it can also be written as $x \in (\ln K, b(T))$. Note that this interval should be empty if $b(T) = \ln K$.

Recall that $\hat{\psi}_T(x)$ is strictly decreasing on $(\ln K, \infty)$. If $\lambda \int_{-\infty}^{\infty}(1 - e^y)^+ f_Y(y)dy \leq \delta - r$, then $\lambda \int_{-\infty}^{\infty}(K - e^{\ln K + y})^+ f_Y(y)dy \leq \delta e^{\ln K} - rK$, so that $\lim_{x\to\ln K}\hat{\psi}_T(x) \leq -rK$. Therefore, the strictly decreasing property leads to that $\hat{\psi}_T(x) \leq -rK$ on the whole interval of $(\ln K, \infty)$. We then deduce from previous statements of equation (3.88) that $b(T) = \ln K$.

On the other hand, suppose that $\lambda \int_{-\infty}^{\infty}(1 - e^y)^+ f_Y(y)dy > \delta - r$, then $\lambda \int_{-\infty}^{\infty}(K - e^{\ln K + y})^+ f_Y(y)dy > \delta e^{\ln K} - rK$, so that $\lim_{x\to\ln K}\hat{\psi}_T(x) > -rK$. Also note that $\hat{\psi}_T(x) \to -\infty$ when $x \to \infty$. Together with the fact that $\hat{\psi}_T(x)$ is strictly decreasing on $(\ln K, \infty)$, we can say that the equation $\hat{\psi}_T(x) = -rK$ has a unique solution ℓ in $(\ln K, \infty)$. Moreover, we have $\hat{\psi}_T(x) > -rK$ for $x < \ell$ and $\hat{\psi}_T(x) < -rK$ for $x > \ell$. Therefore, from equation (3.86) and (3.88), we deduce that $b(T) = \ell$.

Finally, we just need to calculate the value of $\lambda \int_{-\infty}^{\infty}(1 - e^y)^+ f_Y(y)dy$ and complete the proof of Property 3.11:

$$\begin{aligned} \lambda \int_{-\infty}^{\infty}(1 - e^y)^+ f_Y(y)dy &= \lambda \int_{-\infty}^{0}(1 - e^y) f_Y(y)dy \\ &= \lambda q \eta_2 \int_{-\infty}^{0}(1 - e^y)e^{\eta_2 y}dy \\ &= \frac{\lambda q}{\eta_2 + 1}. \end{aligned} \tag{3.89}$$

□

To this end, we can move on to the derivation of the EEP formula for American calls and the uniqueness solution theorem with all the properties of the value function $V(t,x)$ and the boundary function $b(t)$ we have obtained so far.

3.3.5 EEP Representation of American Call Options with dividends

Comparing the American option with European options, the American style option allows holder to exercise before maturity, so the value of an American option should equal to the value of the corresponding European option plus an early exercise premium. Writing the value function of American options into this form, we will have the EEP representation. This section is dedicated to the derivation of the EEP representation of finite horizon American call options with dividends.

Standard arguments [44, p.131, Killed version] based on the strong Markov property link the unknown value function $V(t,x)$ and the unknown optimal stopping boundary $b(t)$ to the following free-boundary problem for $(t,x) \in [0,T) \times (-\infty,\infty)$:

$$V_t + \mathbb{L}_{X^r} V = rV \qquad \text{in} \quad C, \tag{3.90}$$

$$V(t,x) = (e^x - K)^+ \qquad \text{for} \quad x = b(t), \tag{3.91}$$

$$V_x(t,x) = e^x \qquad \text{for} \quad x = b(t), \tag{3.92}$$

$$V(t,x) > (e^x - K)^+ \qquad \text{in} \quad C, \tag{3.93}$$

$$V(t,x) = (e^x - K)^+ \qquad \text{in} \quad D. \tag{3.94}$$

This means that the pair of function $V(t,x)$ and $b(t)$ is a solution to the system of equation (3.90) - (3.94). This connection is an important breakthrough in handling option pricing problems with early exercise feature. See Chapter 2 for a brief introduction to the advantages and disadvantages of using this connection to price options. As what we have achieved in Chapter 2, with the results obtained in the previous section, we can transfer the free-boundary problems into an equation system which only contains two nonlinear integral equations for the function $V(t,x)$ and $b(t)$. Then we will eventually prove the uniqueness of solutions for this equation system with some additional conditions including the free-boundary equation (3.93) and (3.94), which an be view as a main contribution of this section. The EEP representation acts as the foundation of our research, so we need to derive it first.

Peskir proposed a series of change-of-variable formulas with local time for semimartingales with jumps in [39]. By Property 3.8 to 3.11, as well as equation (3.90) - (3.94), it is easy to verify that for the jump-diffusion process X_t^r, the conditions of

[39, Theorem 3.1] hold. Then for fixed t and $X_t^r = x$, applying this change-of-variable formula to $e^{-rs}V(t+s, X_{t+s}^r)$ in terms of s, we can have,

$$e^{-rs}V(t+s, X_{t+s}^r) = V(t, X_t^r) + \int_0^s e^{-ru}(V_t + \mathbb{L}_{X^r}V - rV)(t+u, X_{t+u}^r)du + M_s^1 + M_s^2, \tag{3.95}$$

where

$$M_s^1 = \sigma \int_0^s e^{-ru}\frac{\partial V}{\partial x}(t+u, X_{t+u}^r)dW_u, \tag{3.96}$$

$$\begin{aligned} M_s^2 = & \sum_{0\le u\le s}^{\Delta X_{t+u}^r \neq 0} \Big(e^{-ru}V(t+u, X_{t+u}^r) - e^{-ru}V(t+u, X_{t+u-}^r) \Big) \\ & - \lambda \int_0^s e^{-ru} \int_{-\infty}^{+\infty} [V(t+u, X_{t+u-}^r + y) - V(t+u, X_{t+u-}^r)] f_Y(y)dydu. \end{aligned} \tag{3.97}$$

Note that there is no local time term in (3.95) due to the smooth-fit property of the value function $V(t,x)$ (Property 3.8). Following the similar derivation path for American put options employed in Chapter 2, we would like to prove that M_s^1 and M_s^2 are martingales.

Proposition 3.1. *$M^1 = (M_s^1)_{0\le s\le T-t}$ defined in (3.96) is a martingale under* P.

Proof. Recall that $x \mapsto V(t,x)$ is increasing and convex. And $V(t,x) = G(x)$ for $x \ge b(t) \ge \ln K$. Thus we have

$$0 \le V_x(t,x) \le e^x. \tag{3.98}$$

To prove M^1 is a martingale under P, we only have to prove that $\mathsf{E}_{t,x}\langle M, M\rangle_{T-t} < \infty$. $\langle M^1, M^1\rangle_T$ stands for the quadratic variation of M_T^1, see [54, Proposition 8.6] for details. Indeed,

$$\begin{aligned} \mathsf{E}_{t,x}\langle M, M\rangle_{T-t} &= \sigma^2 \mathsf{E}[\int_0^{T-t} e^{-2ru}(V_x(t+u, x+X_u^r))^2 du] \\ &\le \sigma^2 \int_0^{T-t} \mathsf{E}(e^{x+X_u^r})^2 du \\ &= \sigma^2 e^{2x} \int_0^{T-t} \mathsf{E}(e^{2X_u^r})du. \end{aligned} \tag{3.99}$$

Recall that $\mathsf{E}[e^{\theta X_t^r}] = e^{F(\theta)t}$ is just the moment generating function of X_t^r where the function $F(\theta)$ is introduced by (3.28) in previous sections. Thus $F(2)$ is a constant denoted as F^*, then we have

$$
\begin{aligned}
&\sigma^2 e^{2x} \int_0^{T-t} \mathsf{E}(e^{2X_u^r}) du && (3.100)\\
&= \sigma^2 e^{2x} \int_0^{T-t} e^{F^* u} du \\
&= \frac{\sigma^2 e^{2x}}{F^*} (e^{(T-t)F^*} - 1) < \infty. && (3.101)
\end{aligned}
$$

Thus the proof is completed. □

Proposition 3.2. *$M^2 = (M_s^2)_{0 \le s \le T-t}$ defined in (3.97) is a martingale under* P.

Proof. We can rewrite the first part of M_s^2 as

$$
\begin{aligned}
&\sum_{0 \le u \le s}^{\Delta X_{t+u}^r \neq 0} \Big(e^{-ru} V(t+u, X_{t+u}^r) - e^{-ru} V(t+u, X_{t+u-}^r) \Big) && (3.102)\\
&= \int_0^s \int_{-\infty}^{+\infty} e^{-ru} [V(t+u, X_{t+u-}^r + y) - V(t+u, X_{t+u-}^r)] J_{X^r}(dy \times du),
\end{aligned}
$$

where J_{X^r} is the jump measure of X_t^r. Recall that the compensated jump measure of X_t^r is $\widetilde{J}_{X^r}(dy \times du) = J_{X^r}(dy \times du) - \lambda du f(dy)$, where $f(dy)$ is the density function of Y_i. Thus M_s^2 equals

$$
\begin{aligned}
M_s^2 &= \int_0^s \int_{-\infty}^{+\infty} e^{-ru} [V(t+u, X_{t+u-}^r + y) - V(t+u, X_{t+u-}^r)] J_{X^r}(dy \times du) && (3.103)\\
&\quad - \lambda \int_0^s \int_{-\infty}^{+\infty} e^{-ru} [V(t+u, X_{t+u-}^r + y) - V(t+u, X_{t+u-}^r)] f_Y(y) dy du \\
&= \int_0^s \int_{-\infty}^{+\infty} e^{-ru} [V(t+u, X_{t+u-}^r + y) - V(t+u, X_{t+u-}^r)] \big(J_{X^r}(dy \times du) \\
&\quad - \lambda du f(dy) \big) \\
&= \int_0^s \int_{-\infty}^{+\infty} e^{-ru} [V(t+u, X_{t+u-}^r + y) - V(t+u, X_{t+u-}^r)] \widetilde{J}_{X^r}(dy \times du).
\end{aligned}
$$

From [28, Corollary 4.6], we know that this integral with respect to the compensated jump measure of a Lévy process is a martingale under P if

$$
\mathsf{E}\Big[\int_0^s \int_{-\infty}^{+\infty} |V(t+u, X_{t+u-}^r + y) - V(t+u, X_{t+u-}^r)| \lambda f_Y(y) dy du \Big] < \infty. \quad (3.104)
$$

Indeed, since $x \mapsto V(t,x)$ is increasing and convex. And $V(t,x) = G(x)$ for $x \ge b(t) \ge$

$\ln K$, we can have

$$
\begin{aligned}
&\mathsf{E}\Big[\int_0^s \int_{-\infty}^{+\infty} |V(t+u, X^r_{t+u-}+y) - V(t+u, X^r_{t+u-})|\lambda f_Y(y)dydu\Big] \\
\leq &\lambda \mathsf{E}\Big[\int_0^s \int_{-\infty}^{+\infty} |e^{X^r_{t+u-}+y} - e^{X^r_{t+u-}}|f_Y(y)dydu\Big] \\
= &\lambda \mathsf{E}\Big[\int_0^s e^{X^r_{t+u-}} \int_{-\infty}^{+\infty} |e^y - 1|f_Y(y)dydu\Big] \\
= &\lambda\left[\frac{q}{\eta_2+1} + \frac{p}{\eta_1-1}\right]\mathsf{E}\Big[\int_0^s e^{X^r_{t+u-}}du\Big] \\
= &\lambda\left[\frac{q}{\eta_2+1} + \frac{p}{\eta_1-1}\right]\int_0^s \mathsf{E}[e^{X^r_{t+u-}}]du < \infty
\end{aligned}
$$

where the last inequality comes from the fact that $e^{-(r-\delta)t+X^r_t}$ is a martingale under the physical probability measure P. Thus the proof is completed. □

Now take expectation $\mathsf{E}_{t,x}$ on both sides of equation (3.95) and set $s = T - t$, we will have

$$
V(t,x) = e^{-r(T-t)}\mathsf{E}_{t,x}G(X^r_T) - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(V_t + \mathbb{L}_{X^r}V - rV)(t+u, X^r_{t+u})du\right). \tag{3.105}
$$

Recalling the free-boundary problem (3.90), equation (3.105) can be write as

$$
\begin{aligned}
V(t,x) = &e^{-r(T-t)}\mathsf{E}_{t,x}G(X^r_T) \qquad (3.106)\\
&- \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(V_t + \mathbb{L}_{X^r}V - rV)(t+u, X^r_{t+u})I(X^r_{t+u} \geq b(t+u))du\right).
\end{aligned}
$$

As stated in Chapter 2, the main difficulty for option pricing with jumps comes from the non local integral term in the operator $\mathbb{L}_{X^r}$ of equation (3.106). We can only apply the add-minus trick and leave the remain part as it is in the EEP representation. This is why our value function and free boundary have to be a pair of unique solution to a system of two equations, while Peskir proved that under the Black-Scholes framework the optimal boundary itself can be the unique solution to a single nonlinear integral equation in [37]. Our research here can be seen as an extension of Peskir's method in deriving EEP formula and proving uniqueness.

Not like the degenerated case introduced in Chapter 2 assuming that only negative jumps exists, here for American call options, the degenerated case assumes that positive jumps are allowed on the path. In this special case, the existence of the indicator function $I(X^r_{t+u} \geq b(t+u))$ in equation (3.106) will alow us to replace $(V_t + \mathbb{L}_{X^r}V - rV)$ by $(g_t + \mathbb{L}_{X^r}g - rg)$, where the function g is introduced by (3.67) in the proof of continuity of boundary function b. Moreover, for such a one-sided jump-diffusion process, we can obtain a result which is much more compact and very similar to the conclusion of [37]. This degenerate case will be studied in the next section.

Recall the calculating result of $(\mathbb{L}_{X^r}g - rg)(x)$ in (3.68), we can have:

$$(g_t + \mathbb{L}_{X^r}g - rg)(x) = rK - \delta e^x, \tag{3.107}$$

for any $x \in \mathbb{R}$. Thus by applying the add-minus trick, equation (3.106) can be rewrite as:

$$\begin{aligned}
& V(t,x) \qquad\qquad (3.108)\\
= \; & e^{-r(T-t)}\mathsf{E}_{t,x}G(X^r_T) \\
& - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(V_t + \mathbb{L}_{X^r}V - rV)(t+u, X^r_{t+u})I(X^r_{t+u} \geq b(t+u))du\right) \\
& - \lambda\mathsf{E}_{t,x}\Big[\int_0^{T-t} e^{-ru}\int_{-\infty}^0 \big(g(X^r_{t+u}+y) \\
& \quad - g(X^r_{t+u}+y)\big)f_Y(y)dyI\big(X^r_{t+u} \geq b(t+u)\big)du\Big] \\
= \; & e^{-r(T-t)}\mathsf{E}_{t,x}G(X^r_T) \\
& - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(g_t + \mathbb{L}_{X^r}g - rg)(X^r_{t+u})I(X^r_{t+u} \geq b(t+u))du\right) \\
& - \lambda\mathsf{E}_{t,x}\Big[\int_0^{T-t} e^{-ru}\int_{-\infty}^0 \big(V(t+u, X^r_{t+u}+y) \\
& \quad - g(X^r_{t+u}+y)\big)f_Y(y)dyI\big(X^r_{t+u} \geq b(t+u)\big)du\Big].
\end{aligned}$$

where the last equation also comes from the fact that $V(t,x) = g(x)$ for $x \geq b(t) \geq \ln K$. Inserting the result from (3.107), we will have the EEP representation for an American call option with dividends:

Theorem 3.5. *The arbitrage-free price of the American call options with dividends admits the following early exercise premium representation under a double exponential* *jump diffusion process*

$$
\begin{aligned}
& V(t,x) && (3.109)\\
= \;& e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r) - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(rK-\delta e^{X_{t+u}^r})I(X_{t+u}^r \geq b(t+u))du\right)\\
& - \lambda\mathsf{E}_{t,x}\left[\int_0^{T-t} e^{-ru}\int_{-\infty}^0 \big(V(t+u,X_{t+u}^r+y)\right.\\
& \left. - g(X_{t+u}^r+y)\big)f_Y(y)dyI\big(X_{t+u}^r \geq b(t+u)\big)du\right].
\end{aligned}
$$

The first part of (3.109) is just the arbitrage-free price of the corresponding European call option which is calculated by [25, Theorem 2]. And the rest is well known as the early exercise premium for the American style options for its special feature.

3.3.6 The Uniqueness of the Value Function and the Free Boundary

The EEP formula (3.109) itself alone is meaningless in calculating the arbitrage free price $V(t,x)$ for options, since it involves another unknown function $b(t)$. To obtain one more nonlinear integral equation for the early exercise boundary $b(t)$, we just need to substitute $x = b(t)$ into the EEP formula (3.109):

$$
\begin{aligned}
& e^{b(t)} - K && (3.110)\\
= \;& e^{-r(T-t)}\mathsf{E}_{t,b(t)}(e^{X_T^r}-K)^+ - \mathsf{E}_{t,b(t)}\left(\int_0^{T-t} e^{-ru}(rK-\delta e^{X_{t+u}^r})I(X_{t+u}^r \geq b(t+u))du\right)\\
& - \lambda\mathsf{E}_{t,b(t)}\left[\int_0^{T-t} e^{-ru}\int_{-\infty}^0 \big(V(t+u,X_{t+u}^r+y)\right.\\
& \left. - g(X_{t+u}^r+y)\big)f_Y(y)dyI\big(X_{t+u}^r \geq b(t+u)\big)du\right].
\end{aligned}
$$

From the previous derivation we know that the value function $V(t,x)$ and free boundary $b(t)$ is a solution pair for the nonlinear integral equation system consisting of (3.109) and (3.110). To make this result more effective and rigorous in practice, the uniqueness of this equation system is essential for our research. Similar to the approach we employed in Chapter 2, here we would like to introduce some additional conditions into the equation system to ensure the uniqueness. There might be some ways by which we can derive the uniqueness without all these additional conditions for jump diffusion processes. Eliminating the dependence on them will leave to be an open question for future studies. Here is our result for the uniqueness.

Theorem 3.6. *Assume the following condition holds:*

$$\delta - r \geq \lambda \frac{q}{\eta_2 + 1}. \tag{3.111}$$

Then the pair of value function and early exercise boundary of the American call options with dividends (V, b) *is the unique solution pair* (v, b^v) *of the following nonlinear integral equation system:*

$$\begin{aligned} & v(t,x) \qquad (3.112) \\ = \;& e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r) - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(rK - \delta e^{X_{t+u}^r})I(X_{t+u}^r \geq b^v(t+u))du\right) \\ & - \lambda \mathsf{E}_{t,x}\bigg[\int_0^{T-t} e^{-ru}\int_{-\infty}^0 \big(v(t+u, X_{t+u}^r + y) \\ & \quad - g(X_{t+u}^r + y)\big) f_Y(y)dy I\big(X_{t+u}^r \geq b^v(t+u)\big)du\bigg], \end{aligned}$$

$$\begin{aligned} & e^{b^v(t)} - K \qquad (3.113) \\ = \;& e^{-r(T-t)}\mathsf{E}_{t,b^v(t)}(e^{X_T^r} - K)^+ \\ & - \mathsf{E}_{t,b^v(t)}\left(\int_0^{T-t} e^{-ru}(rK - \delta e^{X_{t+u}^r})I(X_{t+u}^r \geq b^v(t+u))du\right) \\ & - \lambda \mathsf{E}_{t,b^v(t)}\bigg[\int_0^{T-t} e^{-ru}\int_{-\infty}^0 \big(v(t+u, X_{t+u}^r + y) \\ & \quad - g(X_{t+u}^r + y)\big) f_Y(y)dy I\big(X_{t+u}^r \geq b^v(t+u)\big)du\bigg], \end{aligned}$$

within the class $v : [0,T] \times \mathbb{R} \to \mathbb{R}$, $b^v : [0,T] \to \mathbb{R}$, *and satisfying:*

$$b^v(t) \geq \ln K \quad \textit{for all} \quad 0 \leq t \leq T, \tag{3.114}$$

$$x \mapsto v(t,x) \quad \textit{is increasing on} \quad \mathbb{R}, \tag{3.115}$$

$$v(t,x) > G(x) = (e^x - K)^+ \quad \textit{if} \quad x < b^v(t), \tag{3.116}$$

$$v(t,x) = G(x) = (e^x - K)^+ \quad \textit{if} \quad x \geq b^v(t). \tag{3.117}$$

Proof. Suppose that there exists such a function pair (U, c) where $U : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $c : [0,T] \to \mathbb{R}$ satisfying the conditions of Theorem 3.6 and solving the equation

(3.112) - (3.113):

$$
\begin{aligned}
& U(t,x) \qquad (3.118)\\
= \;& e^{-r(T-t)}\mathsf{E}_{t,x}G(X^r_T) - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(rK - \delta e^{X^r_{t+u}})I(X^r_{t+u} \geq c(t+u))du\right)\\
& - \lambda \mathsf{E}_{t,x}\Big[\int_0^{T-t} e^{-ru}\int_{-\infty}^0 \big(U(t+u, X^r_{t+u}+y)\\
& \quad - g(X^r_{t+u}+y)\big) f_Y(y)dy I\big(X^r_{t+u} \geq c(t+u)\big)du\Big],
\end{aligned}
$$

$$
\begin{aligned}
& e^{c(t)} - K \qquad (3.119)\\
= \;& e^{-r(T-t)}\mathsf{E}_{t,c(t)}(e^{X^r_T} - K)^+\\
& - \mathsf{E}_{t,c(t)}\left(\int_0^{T-t} e^{-ru}(rK - \delta e^{X^r_{t+u}})I(X^r_{t+u} \geq c(t+u))du\right)\\
& - \lambda \mathsf{E}_{t,c(t)}\Big[\int_0^{T-t} e^{-ru}\int_{-\infty}^0 \big(U(t+u, X^r_{t+u}+y)\\
& \quad - g(X^r_{t+u}+y)\big) f_Y(y)dy I\big(X^r_{t+u} \geq c(t+u)\big)du\Big].
\end{aligned}
$$

We just need to prove that this solution pair (U, c) globally coincides with (V, b).

1. Inspired by the method proposed in [44, p.392, Remark 25.4], we would like to introduce an stochastic process $M(s, X^r_s)$ which defined by

$$
\begin{aligned}
& M(s, X^r_s) \qquad (3.120)\\
= \;& e^{-rs}U(t+s, x+X^r_s)) - \int_0^s e^{-ru}(rK - \delta e^{x+X^r_u})I(x+X^r_u \geq c(t+u))du\\
& - \lambda \int_0^s e^{-ru}\int_{-\infty}^0 \big(U(t+u, x+X^r_u+y)\\
& \quad - g(x+X^r_u+y)\big) f_Y(y)dy I\big(x+X^r_u \geq c(t+u)\big)du,
\end{aligned}
$$

where $X^r_0 = 0$ and $(t,x) \in [0,T] \times \mathbb{R}$ is given and fixed. Following the similar approach employed in Chapter 2, the Markov property of the underlying dynamic X^r implies that the stochastic process $M(s, X^r_s)$ is a martingale under $\mathsf{P}_{t,x}$ for $s \in [0, T-t]$.

2. We want to show that $U(t,x) \geq V(t,x)$ for all $t, x \in [0,T] \times (-\infty, \infty)$. For this, observe the martingale process $M(s, X^r_s)$ and applying the reducing method to the last part on the right hand side of (3.120),

$$\begin{aligned}
&\lambda \int_0^s e^{-ru} \int_{-\infty}^0 \big(U(t+u, x+X_u^r+y) \\
&\qquad - g(x+X_u^r+y)\big) f_Y(y) dy I\big(x+X_u^r \geq c(t+u)\big) du \qquad (3.121)\\
\leq &\lambda \int_0^s e^{-ru} \int_{-\infty}^0 \big(U(t+u, x+X_u^r) - g(x+X_u^r+y)\big) f_Y(y) dy I\big(x+X_u^r \geq c(t+u)\big) du \\
= &\lambda \int_0^s e^{-ru} \int_{-\infty}^0 \big(g(x+X_u^r) - g(x+X_u^r+y)\big) f_Y(y) dy I\big(x+X_u^r \geq c(t+u)\big) du \\
= &\lambda \int_0^s e^{-ru} e^{x+X_u^r} \int_{-\infty}^0 (1-e^y) f_Y(y) dy I\big(x+X_u^r \geq c(t+u)\big) du \\
= &\int_0^s e^{-ru} \lambda \frac{q}{\eta_2+1} e^{x+X_u^r} I\big(x+X_u^r \geq c(t+u)\big) du,
\end{aligned}$$

where the first inequality is true because $x \mapsto U(t,x)$ is increasing on $\mathbb{R}$, the first equality holds since $U(t,x) = G(x) = g(x)$ when $x \geq c(t)$, and the last equation is just the calculation for $\int_{-\infty}^0 (1-e^y) f_Y(y) dy$. Now combine the last two parts of (3.120) with the inequality of (3.121), we can get

$$\begin{aligned}
&- \int_0^s e^{-ru}(rK - \delta e^{x+X_u^r}) I(x+X_u^r \geq c(t+u)) du \qquad (3.122)\\
&- \lambda \int_0^s e^{-ru} \int_{-\infty}^0 \big(U(t+u, x+X_u^r+y) \\
&\qquad - g(x+X_u^r+y)\big) f_Y(y) dy I\big(x+X_u^r \geq c(t+u)\big) du \\
\geq &\int_0^s e^{-ru}(\delta e^{x+X_u^r} - rK) I(x+X_u^r \geq c(t+u)) du \\
&- \int_0^s e^{-ru} \lambda \frac{q}{\eta_2+1} e^{x+X_u^r} I\big(x+X_u^r \geq c(t+u)\big) du \\
\geq &\int_0^s e^{-ru}(\delta e^{x+X_u^r} - r e^{x+X_u^r} - \lambda \frac{q}{\eta_2+1} e^{x+X_u^r}) I\big(x+X_u^r \geq c(t+u)\big) du \\
= &\int_0^s e^{-ru} e^{x+X_u^r} (\delta - r - \lambda \frac{q}{\eta_2+1}) I\big(x+X_u^r \geq c(t+u)\big) du,
\end{aligned}$$

where the last inequality comes from the simple fact that $e^x \geq K$ when $x \geq c(t) \geq \ln K$. If the condition (3.111) holds, we can see that the integrand in the last equation of (3.122) never goes smaller than zero. Since $M(s, X_s^r)$ itself is a martingale as proved previously, we can deduce that $e^{-rs}U(t+s, x+X_s^r)$ could only be a submartingale under P for $s \in [0, T-t]$. The property of submartingales ensures that, for any stopping time $\tau \in [0,T]$

$$\begin{aligned}
U(t,x) &\geq \mathsf{E}_{t,x}\left(e^{-r\tau} U(t+\tau, X_{t+\tau}^r)\right) \qquad (3.123)\\
&\geq \mathsf{E}_{t,x}\left(e^{-r\tau} G(X_{t+\tau}^r)\right),
\end{aligned}$$

where the second inequality is derived from the condition $U(t,x) \geq G(x)$ for all $t, x \in [0,T] \times (-\infty, \infty)$. Then by the definition of the value function $V(t,x)$ in (3.14), the inequality of (3.123) implies that

$$U(t,x) \geq V(t,x), \tag{3.124}$$

for all $t, x \in [0,T] \times (-\infty, \infty)$.

3. Now we would like to show that $U(t,x) = V(t,x)$ for all $t, x \in [0,T] \times (-\infty, \infty)$. Indeed, we just need to show that $U(t,x) \leq V(t,x)$ given that $U(t,x) \geq V(t,x)$ has already been proved previously. For this, let us consider a stopping time:

$$\tau_c = \inf\{s \in [0, T-t] | x + X_s^r \geq c(t+s)\}. \tag{3.125}$$

If $x \geq c(t)$, then by the condition (3.117) we know that

$$U(t,x) = G(x) \leq V(t,x). \tag{3.126}$$

If $x < c(t)$, then we have $U(t+\tau_c, x + X_{\tau_c}^r) = G(x + X_{\tau_c}^r)$ by the definition of τ_c in (3.125). So replace s by τ_c in the martingale process $M(s, X_s^r)$ defined by (3.120) and take expectation E on both sides, we can get

$$\begin{aligned} & U(t,x) \qquad\qquad (3.127) \\ =& \mathsf{E}[e^{-r\tau_c} U(t+\tau_c, x + X_{\tau_c}^r)] - \mathsf{E}[\int_0^{\tau_c} e^{-ru}(rK - \delta e^{x+X_u^r}) I(x + X_u^r \geq c(t+u)) du] \\ & - \lambda \mathsf{E}\bigg[\int_0^{\tau_c} e^{-ru} \int_0^{+\infty} \big(U(t+u, x + X_u^r + y) \\ & \quad - g(x + X_u^r + y)\big) f_Y(y) dy I\big(x + X_u^r \geq c(t+u)\big) du\bigg], \\ =& \mathsf{E}[e^{-r\tau_c} G(x + X_{\tau_c}^r)], \end{aligned}$$

where the last two part on the right hand side of (3.127) equals to zero by the definition of τ_c. Meanwhile, by the definition of the value function $V(t,x)$ in (3.14), this result implies that:

$$U(t,x) \leq V(t,x), \tag{3.128}$$

for all $t, x \in [0,T] \times (-\infty, \infty)$. Thus we have $U(t,x) = V(t,x)$ as claimed.

4. The remaining work is to prove that $c(t) = b(t)$ for all $t \in [0,T]$. Provided by the strong conditions (3.116) and (3.117), this fact is easy to prove with the previous result $U(t,x) = V(t,x)$.

First, suppose there exists a $t \in [0, T)$ such that $b(t) < c(t)$. Then take an x between $b(t)$ and $c(t)$ such that $b(t) < x < c(t)$. By the free-boundary definition Theorem 3.4 of $V(t, x)$ and the condition (3.116) of $U(t, x)$, we know that $V(t, x) = G(x)$ and $U(t, x) > G(x)$, which leads to $V(t, x) < U(t, x)$ for this pair of (t, x). This provides a contradiction to the result of Step **3**.

Second, suppose there exists a $t \in [0, T)$ such that $b(t) > c(t)$. Then take an x between $b(t)$ and $c(t)$ such that $b(t) > x > c(t)$. By the free-boundary definition Theorem 3.4 of $V(t, x)$ and the condition (3.117) of $U(t, x)$, we know that $V(t, x) > G(x)$ and $U(t, x) = G(x)$, which leads to $V(t, x) > U(t, x)$ for this pair of (t, x). This provides a contradiction to the result of Step **3**.

Thus we can have that for any point t, $c(t)$ must equal to $b(t)$. This means that $c(t) = b(t)$ for all $t \in [0, T - t]$ as claimed.

To this end, we have already proved that the solution pair (U, c) satisfying (3.118) and (3.119) globally coincides with (V, b). The proof of Theorem 3.6 is completed. □

Remark 3.1. *Similar to what we have discussed in Chapter 2, the additional condition (3.111) also means that we can directly know the value of the boundary function at maturity* $b(T) = \ln K$ *without extra calculation from Property 3.11. Knowing the value of* $b(T)$ *is crucial for the analytical tractability of our research, since we will need to use it to numerically retrieve the whole boundary curve* $t \to b(t)$ *backward on* $[0, T]$ *in practice.*

3.4 Degenerate Case: One Sided Positive Jumps

As stated before, the major difficulty of option pricing under jump-diffusion processes is a nonlocal integral term inside the infinitesimal generator defined in (3.10). Comparing Theorem 3.5 in the previous section with the EEP representation derived by Peskir in [37] for American puts, the value function of American call options will have an additional global integral term involving early exercise boundary in the EEP formula for underlying processes with jumps. This global integral term also makes the uniqueness of the solution can only be held under relatively stronger conditions. Observing the EEP representation in (3.109), we notice that the global integral term only depends on negative jump parts of the underlying process for an American call option. Thus inspired by the assumption we made in Chapter 2, if we put a constraint on the compound Poisson part of the equation (3.8) to allow only positive jumps on the diffusion trajectories, then we will obtain a more compact and clean result, which is very similar in forms to the conclusion of [37].

Suppose that the probability q of negative jumps is zero, whereas the probability $p = 1 - q$ of positive jumps is one in the definition of the jump density (3.3), then we can have a new corresponding process for the underlying asset:

$$Z_t^r = e^{X_t^r} = e^{(r-\delta-\lambda\zeta-\frac{\sigma^2}{2})t+\sigma W_t+\sum_{i=1}^{N_t}(Y_i)}, \tag{3.129}$$

where

$$f_Y(y) = \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} \qquad \eta_1 > 1, \tag{3.130}$$

$$\zeta = \mathsf{E}[V] - 1 = \frac{\eta_1}{\eta_1 - 1} - 1. \tag{3.131}$$

Here we get a degenerate case of the DEJD model and would like to name it as Positive Exponential Jump Diffusion (PEJD) process for future reference in this section. All the other fundamental settings remain the same, including basic assumptions of the American call option, structures of optimal stopping regions, and forms of the value function $V(t, x)$ and early exercise boundary $b(t)$. It is trivial to verified that under the PEJD process, $V(t, x)$ and $b(t)$ have the same properties as in DEJD models. And the connection to free-boundary problems (3.90) - (3.94) is still hold in this degenerate case. These is no need to provide detailed proof for these properties and theorems since there will be no difference comparing to the derivation in the previous section; restricting the density function by $q = 1$ and $p = 0$ will not change the forms and results of the derivation. The effect of limiting jumps to positive side only emerges after the equation (3.108) in deriving EEP representation. Thus we will not repeat those similar part mentioned above and start directly from equation (3.108).

What should be noted is that $\int (1 - e^y)^+ \lambda f_Y(y) dy = 0$ in this degenerated case. This means that the boundary behaviour of function b near maturity introduced by Property 3.11 can be simplified by: $b(T) \stackrel{def}{=} \lim_{t \to T} b(t) = \max(\ln K, \ln \frac{rK}{\delta})$ as in the standard Black-Scholes model.

3.4.1 EEP Representation of American Call Options with Dividends for PEJD processes

Since the negative jumps is not allowed in this degenerate case, the third part on the right hand side of (3.108) now can be cancelled:

$$\begin{aligned}
& V(t,x) \qquad (3.132)\\
= \; & e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r)\\
& - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(g_t + \mathbb{L}_{X^r}g - rg)(X_{t+u}^r)I(X_{t+u}^r \geq b(t+u))du\right)\\
& - \lambda\mathsf{E}_{t,x}\left[\int_0^{T-t} e^{-ru}\int_{-\infty}^0 \big(V(t+u, X_{t+u}^r + y)\right.\\
& \qquad \left. - g(X_{t+u}^r + y)\big) f_Y(y)dy I\big(X_{t+u}^r \geq b(t+u)\big)du\right]\\
= \; & e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r) - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(\mathbb{L}_{X^r}g - rg)(X_{t+u}^r)I(X_{t+u}^r \geq b(t+u))du\right),
\end{aligned}$$

where the last equality comes from the restriction that $f_Y(y) = 0$ for $y \leq 0$. Again, inserting the result from (3.68), we will have the EEP representation for an American call option:

Theorem 3.7. *The arbitrage-free price of the American call option without dividends admits the following early exercise premium representation under a positive exponential jump diffusion process*

$$V(t,x) = e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r) - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(rK - \delta e^{X_{t+u}^r})I(X_{t+u}^r \geq b(t+u))du\right). \tag{3.133}$$

The first part of (3.133) is the arbitrage-free price of the corresponding European call option. It has already been calculated by Kou in [25, Theorem 2]. This form is in line with the EEP formula under Black-Scholes model in [37]. The difference comes from the additional term $\delta e^{X_{t+u}^r}$ inside the second expectation and the cumulative distribution of the underlying process X_t^r. We will write this into analytic form in the next section for financial analysis.

3.4.2 The Uniqueness of the Free Boundary for PEJD processes

Observing the EEP formula in Theorem 3.7, we can see that the right hand side of (3.133) contains only the free boundary function $b(t)$ in this degenerate case. This means that if we substitute $x = b(t)$ into the EEP formula (3.133), we will obtain a single nonlinear integral equation which can be solved for the free boundary $b(t)$:

$$\begin{aligned} e^{b(t)} - K =& e^{-r(T-t)}\mathsf{E}_{t,b(t)}(e^{X_T^r} - K)^+ \\ &- \mathsf{E}_{t,b(t)}\left(\int_0^{T-t} e^{-ru}(rK - \delta e^{X_{t+u}^r})I(X_{t+u}^r \geq b^v(t+u))du\right). \end{aligned} \tag{3.134}$$

Similar to the previous section, we would like to prove the uniqueness of $b(t)$. Comparing equation (3.134) with (3.109) and (3.110), we can see two advantages brought by restricting jump directions. First, the nonlinear integral equation system is now reduced to one single equation. Second, there is no global integral part on the right hand side. These two simplification in the PEJD model enable us to loose the conditions for uniqueness and follow the main approach for uniqueness proof proposed by Peskir in [37]. Conditions (3.111), (3.115), (3.116) and (3.117) are no longer needed in this case. Just with some weak conditions on the continuity and range of the boundary function $b(t)$, we can have the following theorem about the uniqueness.

Theorem 3.8. *Under the positive exponential jump diffusion process, the early exercise boundary b in the value function (3.14) of the American call option with dividends can be characterized as the unique solution of the following nonlinear integral equation:*

$$\begin{aligned} e^{b^v(t)} - K =& e^{-r(T-t)}\mathsf{E}_{t,b^v(t)}(e^{X_T^r} - K)^+ \\ &- \mathsf{E}_{t,b^v(t)}\left(\int_0^{T-t} e^{-ru}(rK - \delta e^{X_{t+u}^r})I(X_{t+u}^r \geq b(t+u))du\right), \end{aligned} \tag{3.135}$$

in the class of the continuous functions $b^v : [0,T] \to \mathbb{R}$ satisfying that $b^v(t) > \max(\ln K, \ln \frac{rK}{\delta})$ for all $0 < t < T$.

Proof. Suppose that there exists such a function $c : [0,T] \to \mathbb{R}$ satisfying the conditions of Theorem 3.8 and solving the equation (3.135). We just need to prove that $c(t)$ coincide with the early exercise boundary $b(t)$.

1. Introduce a corresponding function $U^c : [0,T] \times (-\infty, \infty) \to \mathbb{R}$ defined by

$$U^c(t,x) = e^{-r(T-t)}\mathsf{E}_{t,x}(e^{X^r_T} - K)^+ - \mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(rK - \delta e^{X^r_{t+u}})I(X^r_{t+u} \geq c(t+u))du\right). \tag{3.136}$$

Simply inserting $x = c(t)$ into (3.136), we can have that

$$U^c(t, c(t)) = e^{c(t)} - K = G(c(t)). \tag{3.137}$$

since c satisfy the condition $c(t) > \ln K$ for all $0 < t < T$.

2. We need to show that $U^c(t,x) = G(x)$ for all $t, x \in [0,T] \times (-\infty, \infty)$ such that $x \geq c(t)$. First introduce an stochastic process $M(s, X^r_s)$ which defined by

$$M(s, X^r_s) = e^{-rs}U^c(t+s, x+X^r_s) - \int_0^s e^{-ru}(rK - \delta e^{x+X^r_u})I(x + X^r_u \geq c(t+u))du, \tag{3.138}$$

where $X^r_0 = 0$ and $(t,x) \in [0,T] \times \mathbb{R}$ is given and fixed. Then follow the similar approach in Chapter 2, the Markov property of X^r_t implies that $M(s, X^r_s)$ is a martingale under P for $s \in [0, T-t]$. Now take a pair of (t,x) such that $x \geq c(t)$ and consider the stopping time

$$\sigma_c = \inf\{s \in [0, T-t] | x + X^r_s \leq c(t+s)\}. \tag{3.139}$$

Since $M(s, X^r_s)$ is a martingale, we have

$$M(0, X^r_0) = \mathsf{E}[M(\sigma_c, X^r_{\sigma_c})]. \tag{3.140}$$

For the left hand side of (3.140) we have:

$$M(0, X^r_0) = U^c(t,x), \tag{3.141}$$

by the definition of $M(s, X^r_s)$. Also note that $U^c(t+\sigma_c, x+X^r_{\sigma_c}) = G(x + X^r_{\sigma_c})$ holds for this degenerated process since we assume that there is no negative jumps. Back to our proof, for the right hand side of (3.140) we have:

$$\begin{aligned}\mathsf{E}[M(\sigma_c, X^r_{\sigma_c})] =& \mathsf{E}[e^{-r\sigma_c}U^c(t+\sigma_c, x+X^r_{\sigma_c})] \\ &- \mathsf{E}\left(\int_0^{\sigma_c} e^{-ru}(rK - \delta e^{x+X^r_u})I(x+X^r_u \geq c(t+u))du\right) \qquad (3.142) \\ =& \mathsf{E}[e^{-r\sigma_c}G(x+X^r_{\sigma_c})] - \mathsf{E}\left(\int_0^{\sigma_c} e^{-ru}(rK - \delta e^{x+X^r_u})du\right),\end{aligned}$$

where the last equation follows from the definition of the stopping time σ_c. Then the equation (3.140) can be rewrite as

$$U^c(t,x) = \mathsf{E}[e^{-r\sigma_c}G(x+X^r_{\sigma_c})] - \mathsf{E}\left(\int_0^{\sigma_c} e^{-ru}(rK - \delta e^{x+X^r_u})du\right). \tag{3.143}$$

If we apply the change-of-variable formula [39, Theorem 3.1] to $e^{-rs}G(x+X^r_s)$ in terms of s, then we can have

$$\begin{aligned} e^{-rs}G(x+X^r_s) =& G(x) + \int_0^s e^{-ru}(rK - \delta e^{x+X^r_u})I(x+X^r_u \geq \ln K)du \\ &+ M_s + \frac{K}{2}\int_0^s dl^{\ln K}_u(X^r), \end{aligned} \tag{3.144}$$

where M_s is the martingale part under P, and $dl^{\ln K}_u(X^r)$ refers to the integration w.r.t. the continuous increasing function $u \mapsto l^{\ln K}_u(X^r)$, $l^{\ln K}_u(X^r) = \mathsf{P} - \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^u I(\ln K - \varepsilon < X^r_v < \ln K + \varepsilon)d\langle X^r, X^r\rangle_v$. Note that this equation (3.144) only holds for PEJD processes. There will also be an additional non-local integral term if the underlying dynamic allows negative jumps. Now replace s by σ_c and take expectation E on both sides of (3.144), following the similar derivation of Theorem 3.7 we can obtain that

$$\mathsf{E}[e^{-r\sigma_c}G(x+X^r_{\sigma_c})] = G(x) + \mathsf{E}\left(\int_0^{\sigma_c} e^{-ru}(rK - \delta e^{x+X^r_u})du\right), \tag{3.145}$$

Since $c(t) > \ln K$. Thus insert (3.145) into (3.143), we have proved that

$$U^c(t,x) = G(x), \tag{3.146}$$

for all $t,x \in [0,T]\times(-\infty,\infty)$ such that $x \geq c(t)$.

3. We want to show that $U^c(t,x) \leq V(t,x)$ for all $t,x \in [0,T]\times(-\infty,\infty)$. For this, take any such (t,x) and consider the stopping time

$$\tau_c = \inf\{s \in [0,T-t] | x + X^r_s \geq c(t+s)\}. \tag{3.147}$$

If $x \geq c(t)$, then by the result of Step **2** in (3.146) we know that

$$U^c(t,x) = G(x) \leq V(t,x). \tag{3.148}$$

If $x < c(t)$, then we have $U^c(t+\tau_c, x+X^r_{\tau_c}) = G(x+X^r_{\tau_c})$ by the definition of τ_c in (3.147). So replacing s by τ_c in $M(s, X^r_s)$ and taking expectation E on both sides, we find that

$$
\begin{aligned}
U^c(t,x) =& \mathsf{E}[e^{-r\tau_c}U^c(t+\tau_c, x+X^r_{\tau_c})] \\
&- \mathsf{E}[\int_0^{\tau_c} e^{-ru}(rK-\delta e^{x+X^r_u})I(x+X^r_u \geq c(t+u))du] \\
=& \mathsf{E}[e^{-r\tau_c}G(x+X^r_{\tau_c})],
\end{aligned} \tag{3.149}
$$

where the second part on the right hand side of (3.149) equals to zero by the definition of τ_c. Then the definition of the value function $V(t,x)$ in (3.14) implies that

$$
U^c(t,x) \leq V(t,x), \tag{3.150}
$$

for all $t,x \in [0,T] \times (-\infty,\infty)$.

4. Let us now show that $b(t) \geq c(t)$ for all $t \in [0,T]$. For this, suppose there exists a $t \in [0,T)$ such that $b(t) < c(t)$. Then for this kind of fixed t, take an x such that $(t,x) \in \{[0,T] \times (-\infty,\infty) | x \geq c(t)\}$. So we have $x \geq c(t) > b(t)$. Now consider the stopping time

$$
\sigma_b = \inf\{s \in [0,T-t] | x + X^r_s \leq b(t+s)\}. \tag{3.151}
$$

Replacing s by σ_b in $M(s,X^r_s)$ and taking expectation E on both sides, we get

$$
\begin{aligned}
\mathsf{E}[e^{-r\sigma_b}U^c(t+\sigma_b, x+X^r_{\sigma_b})] =& U^c(t,x) \\
&+ \mathsf{E}[\int_0^{\sigma_b} e^{-ru}(rK-\delta e^{x+X^r_u})I(x+X^r_u \geq c(t+u))du].
\end{aligned} \tag{3.152}
$$

Recall equation (3.95) in previous section under DEJD processes

$$
\begin{aligned}
e^{-rs}V(t+s, X^r_{t+s}) =& V(t, X^r_t) + \int_0^s e^{-ru}(V_t + \mathbb{L}_{X^r}V - rV)(t+u, X^r_{t+u})du \\
&+ M^1_s + M^2_s.
\end{aligned}
$$

Here in our degenerated case, it can be rewritten as

$$
\begin{aligned}
e^{-rs}V(t+s, X^r_{t+s}) =& V(t, X^r_t) + \int_0^s e^{-ru}(\mathbb{L}_{X^r}g - rg)(X^r_{t+u})I(X^r_{t+u} \geq b(t+u))du \\
&+ M^1_s + M^2_s \\
=& V(t, X^r_t) + \int_0^s e^{-ru}(rK - \delta e^{X^r_{t+u}})I(X^r_{t+u} \geq b(t+u))du \\
&+ M^1_s + M^2_s.
\end{aligned} \tag{3.153}
$$

Note that this derivation also implies that $(rK - \delta e^{X^r_{t+u}})I(X^r_{t+u} \geq b(t+u)) < 0$ by the fact that $V_t + \mathbb{L}_{X^r}V - rV < 0$ in the stopping region $\bar{D}$. Now replacing s by σ_b in (3.153) and taking expectation E on both sides, we get

$$
\begin{aligned}
\mathsf{E}[e^{-r\sigma_b}V(t+\sigma_b, x+X^r_{\sigma_b})] =& V(t,x) \\
&+ \mathsf{E}[\int_0^{\sigma_b} e^{-ru}(rK - \delta e^{x+X^r_u})I(x+X^r_u \geq b(t+u))du] \qquad (3.154) \\
=& V(t,x) + \mathsf{E}[\int_0^{\sigma_b} e^{-ru}(rK - \delta e^{x+X^r_u})du] \\
=& V(t,x) \\
&+ \mathsf{E}[\int_0^{\sigma_b} e^{-ru}(rK - \delta e^{x+X^r_u})I(x+X^r_u \geq c(t+u))du] \\
&+ \mathsf{E}[\int_0^{\sigma_b} e^{-ru}(rK - \delta e^{x+X^r_u})I(x+X^r_u < c(t+u))du].
\end{aligned}
$$

The comparing of (3.152) and (3.154) implies that

$$
\mathsf{E}[\int_0^{\sigma_b} e^{-ru}(rK - \delta e^{x+X^r_u})I(x+X^r_u < c(t+u))du] \geq 0. \qquad (3.155)
$$

However, the fact that $c(t) > b(t)$ and the continuity of the functions c and b force the expectation in (3.155) strictly negative and provides a contradiction. Thus $b(t) \geq c(t)$ for all $t \in [0,T]$ as claimed.

5. Finally, we show that $b(t) = c(t)$ for all $t \in [0,T]$. For this, suppose there exists a $t \in [0,T)$ such that $b(t) > c(t)$. Then for this kind of fixed t, take any x satisfying $c(t) < x < b(t)$. Now consider the stopping time

$$
\tau_b = \inf\{s \in [0, T-t] | x + X^r_s \geq b(t+s)\}. \qquad (3.156)
$$

Replacing s by τ_b in $M(s, X^r_s)$, and taking expectation E on both sides, we get

$$
\begin{aligned}
\mathsf{E}[e^{-r\tau_b}U^c(t+\tau_b, x+X^r_{\tau_b})] =& U^c(t,x) \\
&+ \mathsf{E}[\int_0^{\tau_b} e^{-ru}(rK - \delta e^{x+X^r_u})I(x+X^r_u \geq c(t+u))du]. \qquad (3.157)
\end{aligned}
$$

Again, replacing s by τ_b in (3.153), and taking expectation E on both sides, we find that

$$\begin{aligned}\mathsf{E}[e^{-r\tau_b}V(t+\tau_b, x+X^r_{\tau_b})] =& V(t,x) \\ &+ rK\mathsf{E}[\int_0^{\tau_b} e^{-ru}(rK-\delta e^{x+X^r_u})I(x+X^r_u \geq b(t+u))du]. \\ =& V(t,x),\end{aligned} \tag{3.158}$$

where the last equation follows from the definition of τ_b. Comparing (3.157) and (3.158) implies that

$$\mathsf{E}[\int_0^{\tau_b} e^{-ru}(rK-\delta e^{x+X^r_u})I(x+X^r_u \geq c(t+u))du] \geq 0. \tag{3.159}$$

Thus similar to Step **4**, the fact that $b(t) > c(t)$, the continuity of the functions c and b and the range condition that $c(t) > \max(\ln K, \ln \frac{rK}{\delta})$ force the expectation in (3.159) strictly negative and provides a contradiction. Thus $b(t) \leq c(t)$ for all $t \in [0,T]$. Combining with the result of Step **4**, we have that $b = c$ for all $t \in [0,T]$ and the proof for Theorem 3.8 is complete. □

3.5 Financial Analysis under PEJD processes

While the EEP representation Theorem 3.5 and the uniqueness Theorem 3.6 for the value function and free boundary under a double exponential jump diffusion process are more theoretically generic and meaningful, the corresponding results Theorem 3.7 and Theorem 3.8 for the degenerate case PEJD model are much more feasible for financial analysis in practice. In this financial analysis section, we will only consider the one sided jumps case. Nevertheless, it still can provide us a glance on how will allowing jumps on the path affect the values and boundaries of American call options.

3.5.1 Analytical Form of the EEP Representation

The first thing we need to do is to write the EEP representation (3.133) of the value function into analytical form. A key component for this is the cumulative distribution function of the underlying PEJD process X^r_t defined in (3.129). Using the same approach adopted by Chapter 2, we would like to define a new generic stochastic process

X_t^u by

$$X_t^\mu = \mu t + \sigma W_t + \sum_{i=1}^{N_t}(Y_i), \tag{3.160}$$

where the density of jump sizes follows equation (3.130). If we can calculate the CDF of X_T^μ and define a generic function Υ based on it, then for any given t, we can express the CDF of X_t^r with Υ by amending some corresponding parameters.Now we will focus on the random variable X_T^μ and derive the function Υ.

Υ function for PEJD processes

we will omit some repeated parts of the derivation of the function Υ since it has been illustrated clearly for the NEJD processes in Chapter 2. The only difference will be changing the jump direction and density parameters.

1. First note that for each of the i.i.d. random variable Y_i representing the jump size, we have

$$Y_i \stackrel{d}{=} \xi_i, \tag{3.161}$$

where ξ_i is an exponential random variable with rate η_1. We know that the sum of n i.i.d. exponential random variables follows the Erlang distribution, thus for any given $n \geq 1$, $\sum_{i=1}^n(\xi_i) = \sum_{i=1}^n(Y_i)$ is a random variable with the density

$$f_{\sum_{i=1}^n(\xi_i)}(x) = f_{\sum_{i=1}^n(\xi_i)}(x; n, \eta_1) = \frac{\eta_1^n x^{n-1} e^{-\eta_1 x}}{(n-1)!}. \tag{3.162}$$

for all $x \geq 0$. The generic process X_t^μ can be rewrite as

$$X_t^\mu = \mu t + \sigma W_t + \sum_{i=1}^{N_t}(\xi_i). \tag{3.163}$$

2. Introduce a special function from mathematical physics named as Hh function. For every integer $n \geq 0$, the Hh function is a nonincreasing function defined by

$$Hh_n(x) = \int_x^\infty Hh_{n-1}(y)dy = \frac{1}{n!}\int_x^\infty (t-x)^n e^{-\frac{t^2}{2}} dt, \tag{3.164}$$

$$Hh_{-1}(x) = e^{-\frac{x^2}{2}} = \sqrt{2\pi}\varphi(x), \tag{3.165}$$

$$Hh_0(x) = \sqrt{2\pi}\Phi(-x), \tag{3.166}$$

where $\varphi(x)$ and $\Phi(x)$ are the density function and the CDF for standard normal distribution respectively:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \tag{3.167}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy. \tag{3.168}$$

The Hh function can be viewed as a generalization of the cumulative normal distribution function. A three-term recursion for it is also available:

$$nHh_n(x) = Hh_{n-2} - xHh_{n-1}(x), . \tag{3.169}$$

for all integer $n \geq 1$. This means that is we can compute all $Hh_n(x)$, $n \geq 1$, by using the normal density function and normal distribution function. In this case, it would be a great practical advantage if we could write Υ in terms of Hh functions.

3. For option pricing under exponential jumps, it is important to evaluate the integral $I_n(c, \alpha, \beta, \omega)$ defined by

$$I_n(c, \alpha, \beta, \omega) = \int_c^{\infty} e^{\alpha x} Hh_n(\beta x - \omega) dx \tag{3.170}$$

for all integer $n \geq 0$ and arbitrary constants α, c and β. Here we will omit the detailed calculation but introduce the result from [25, Proposition B.2] directly.

Fact 3.1. *If $\beta > 0$ and $\alpha \neq 0$, then for all $n \geq -1$,*

$$I_n(c, \alpha, \beta, \omega) = -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^{n} (\frac{\beta}{\alpha})^{n-i} Hh_i(\beta c - \omega) + (\frac{\beta}{\alpha})^{n+1} \frac{\sqrt{2\pi}}{\beta} e^{\frac{\alpha\omega}{\beta} + \frac{\alpha^2}{2\beta^2}} \Phi(-\beta c + \omega + \frac{\alpha}{\beta}). \tag{3.171}$$

If $\beta < 0$ and $\alpha < 0$, then for all $n \geq -1$,

$$I_n(c, \alpha, \beta, \omega) = -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^{n} (\frac{\beta}{\alpha})^{n-i} Hh_i(\beta c - \omega) - (\frac{\beta}{\alpha})^{n+1} \frac{\sqrt{2\pi}}{\beta} e^{\frac{\alpha\omega}{\beta} + \frac{\alpha^2}{2\beta^2}} \Phi(\beta c - \omega - \frac{\alpha}{\beta}). \tag{3.172}$$

This fact will be very useful in numerically calculating the arbitrage-free price of an American call option.

4. With all the above results, we can calculate the probability $\mathsf{P}(X_T^u \geq a)$ for a given T and a constant $a \in \mathbb{R}$, and denote it as the function Υ:

$$\begin{aligned}\Upsilon(\mu,\sigma,\lambda,\eta_1;a,T) :=& \mathsf{P}(X_T^\mu \geq a) \\ =& \sum_{n=0}^{\infty} \pi_n(T)\mathsf{P}(\mu T + \sqrt{T}z + \sum_{i=1}^{n}(\xi_i) \geq a) \\ =& \pi_0(T)\mathsf{P}(\mu T + \sqrt{T}z \geq a) + \sum_{n=1}^{\infty} \pi_n(T)\mathsf{P}(\mu T + \sqrt{T}z + \sum_{i=1}^{n}(\xi_i) \geq a) \\ =& \pi_0(T)\Phi(-\frac{a-\mu T}{\sigma\sqrt{T}}) \\ &+ \frac{e^{\frac{T(\sigma\eta_1)^2}{2}}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{\infty} \pi_n(T)(\sigma\sqrt{T}\eta_1)^n \\ &\times I_{n-1}(a-\mu T, -\eta_1, -\frac{1}{\sigma\sqrt{T}}, -\sigma\eta_1\sqrt{T}),\end{aligned} \tag{3.173}$$

where $\pi_n(t)$ is the probability density function of a random variable N_t following Poisson distribution with rate λt:

$$\pi_n(t) := \mathsf{P}(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}. \tag{3.174}$$

Although the form of Υ seems complicated, it can easily calculated by computer by equation (3.171), (3.172) and (3.169).

Analytical representation for European call options

As has been stated in the previous section, the arbitrage-free price of the corresponding European call option in the EEP representation (3.133) of PEJD model has been calculated by Kou in [25, Theorem 2]. Here we will introduce the result of $e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r)$ in terms of Υ function without detailed proof.

$$\begin{aligned}e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T^r) =& e^x\Upsilon(r-\delta-\lambda\zeta+\frac{1}{2}\sigma^2,\sigma,\lambda^*,\eta_1^*;\ln K - x, T-t) \\ &- Ke^{-r(T-t)}\Upsilon(r-\delta-\lambda\zeta-\frac{1}{2}\sigma^2,\sigma,\lambda,\eta_1;\ln K - x, T-t),\end{aligned} \tag{3.175}$$

where $\eta_1^* = \eta_1 - 1$, $\lambda^* = \lambda(\zeta+1)$ and ζ is defined by (3.9).

Analytical form of the EEP representation for American call options

We also need to rewrite the second part of EEP representation (3.133) of the value function $V(t,x)$ proposed in Theorem 3.7 into analytical form:

$$
\begin{aligned}
& -\mathsf{E}_{t,x}\left(\int_0^{T-t} e^{-ru}(rK-\delta e^{X^r_{t+u}})I(X^r_{t+u}\geq b(t+u))du\right) \qquad (3.176)\\
&= -\int_0^{T-t} e^{-ru}\mathsf{E}[(rK-\delta e^{x+X^r_u})I(x+X^r_u\geq b(t+u))]du,
\end{aligned}
$$

where equality follows by Fubini's Theorem. Now let us set

$$
H(x) = rK - \delta e^x \qquad (3.177)
$$

as a function. Then we have

$$
\begin{aligned}
& \int_0^{T-t} e^{-ru}\mathsf{E}[(rK-\delta e^{x+X^r_u})I(x+X^r_u\geq b(t+u))]du \qquad (3.178)\\
&= \int_0^{T-t} e^{-ru}\mathsf{E}[H(x+X^r_u)I(x+X^r_u\geq b(t+u))]du\\
&= \int_t^{T} e^{-r(v-t)}\mathsf{E}[H(x+X^r_{v-t})I(x+X^r_{v-t}\geq b(v))]dv,
\end{aligned}
$$

by setting $v = u + t$. Moreover

$$
\begin{aligned}
& \mathsf{E}[H(x+X^r_{v-t})I(x+X^r_{v-t}>b(v))] \qquad (3.179)\\
&= \int_{b(v)}^{+\infty} H(y)f(v-t,x,y)dy,
\end{aligned}
$$

where $f(v-t,x,y)$ is the density function of $x+X^r_{v-t}$. To obtain the analytical expression of it, see that

$$
\begin{aligned}
\mathsf{P}(x+X^r_{v-t}<y) &= \mathsf{P}(X^r_{v-t}<y-x) \qquad (3.180)\\
&= 1-\mathsf{P}(X^r_{v-t}\geq y-x)\\
&= 1-\Upsilon(r-\delta-\frac{\sigma^2}{2}-\lambda\zeta,\sigma,\lambda,\eta_1;y-x,v-t) \qquad (3.181)
\end{aligned}
$$

where the function Υ is introduced before in (3.173). Therefore, we can have the analytical expression of $f(v-t,x,y)$

$$f(v-t,x,y) = \frac{d(\mathsf{P}(x+X^r_{v-t}<y))}{dy} \tag{3.182}$$
$$= -\frac{d\Upsilon(r-\delta-\frac{\sigma^2}{2}-\lambda\zeta,\sigma,\lambda,\eta_1;y-x,v-t)}{dy}$$
$$= -\frac{e^{(\sigma\eta_1)^2(v-t)/2}}{\sigma\sqrt{2\pi(v-t)}}\sum_{n=1}^{\infty}\pi_n(\sigma\sqrt{v-t}\eta_1)^n$$
$$\times\Big[-e^{-\eta_1(y-x-(r-\delta-\frac{\sigma^2}{2}-\lambda\zeta)(v-t))}$$
$$Hh_{n-1}(-\frac{1}{\sigma\sqrt{v-t}}[y-x-(r-\delta-\frac{\sigma^2}{2}-\lambda\zeta)(v-t)]+\sigma\eta_1\sqrt{v-t})\Big]$$
$$-\pi_0[\frac{1}{\sigma\sqrt{v-t}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(-\frac{y-x-(r-\delta-\frac{\sigma^2}{2}-\lambda\zeta)(v-t)}{\sigma\sqrt{v-t}})^2}{2}}].$$

Define a function J as

$$J(t,x,v,w) = -e^{r(v-t)}\int_w^\infty H(y)f(v-t,x,y)dy, \tag{3.183}$$

With the equation (3.175) and (3.183), we can write the EEP representation of the value function $V(t,x)$ proposed in Theorem 3.7 into the following analytical form:

$$V(t,x) = e^{-r(T-t)}\mathsf{E}_{t,x}G(X^r_T) - \mathsf{E}_{t,x}\left(\int_0^{T-t}e^{-ru}(rK-\delta e^{X^r_{t+u}})I(X^r_{t+u}\geq b(t+u))du\right) \tag{3.184}$$
$$= e^x\Upsilon(r-\delta-\lambda\zeta+\frac{1}{2}\sigma^2,\sigma,\lambda^*,\eta_1^*;\ln K-x,T-t)$$
$$-Ke^{-r(T-t)}\Upsilon(r-\delta-\lambda\zeta-\frac{1}{2}\sigma^2,\sigma,\lambda,\eta_1;\ln K-x,T-t)$$
$$+\int_t^T J(t,x,v,b(v))dv$$

where $\lambda^*=\lambda(1+\zeta)$, $\eta_1^*=\eta_1-1$. And the early exercise boundary $b(t)$ is the unique solution of the following nonlinear integral equation:

$$e^{b(t)}-K = e^{b(t)}\Upsilon(r-\delta-\lambda\zeta+\frac{1}{2}\sigma^2,\sigma,\lambda^*,\eta_1^*;\ln K-b(t),T-t) \tag{3.185}$$
$$-Ke^{-r(T-t)}\Upsilon(r-\delta-\lambda\zeta-\frac{1}{2}\sigma^2,\sigma,\lambda,\eta_1;\ln K-b(t),T-t)$$
$$+\int_t^T J(t,b(t),v,b(v))dv$$

in the class of the continuous functions $b:[0,T]\to\mathbb{R}$ satisfying $b(t)>\max(\ln K,\ln\frac{rK}{\delta})$ for all $0<t<T$.

3.5.2 Financial Analysis for American Call Options

In this part, we briefly show how will allowing jumps on the path affects the values and boundaries of American call options by some figures. Here we still focus on the positive exponential jump-diffusion processes due to its analytical tractability.

1. With the terminal condition of $b(T)$ given in Property 3.11, we can sequentially solve out the whole early exercise boundary $b(t)$ backward from $t = T$ to $t = 0$ by applying Trapezoid rule and Newton-Raphson method on the nonlinear integral equation (3.185), see [47] for more details of this numerical method. As shown in Figure 3.2, the rational exercise boundary of the American call option varies with different intensity parameter λ. We assume that the strike price $K = 10$, the maturity time $T = 1$, the interest rate $r = 0.1$, the dividend rate $\delta = 0.03$, the volatility coefficient $\sigma = 0.4$, the rate parameter of the positive exponential random variable $\eta_1 = 2.3$. We can see from Figure 3.2, the value the intensity parameter λ will not influence the shape and the terminal value of the early exercise boundary: $b(t)$ is always decreasing on $[0, T]$ and its value at maturity T is always $b(T) = \ln \frac{rK}{\delta}$ (since we assume $\delta < r$), which is consistent with the theoretical results verified in Property 3.9 and Property 3.11. However, the position of the early exercise boundary before maturity is decided by λ: the boundary $b(t)$ will be higher when the intensity parameter λ we choose is higher. Meanwhile, as λ tends to zero, the early exercise boundary of American call options under positive exponential jump-diffusion processes will converge to that under geometric Brownian motion. This feature can be well explained by the definition of the intensity λ: it represents the frequency of jumps which occurs on the path of the underlying dynamics. Therefore, $\lambda \downarrow 0$ means that the path is fairly similar to that of a pure diffusion process i.e. geometric Brownian motion. Moreover, this convergence can also be viewed as a numerical evidence for the validity of the theoretical results we derived in previous sections.

2. With the early exercise boundary $b(t)$ calculated in Figure 3.2, we can obtain value of the corresponding American call option by the analytical form of the EEP representation (2.134). As shown in Figure 3.3, the value function $x \mapsto V(t, x)$ of the American call option at a given time $t = 0$ varies with different intensity parameter λ. We assume that the strike price $K = 10$, the maturity time $T = 1$, the interest rate $r = 0.1$, the dividend rate $\delta = 0.03$, the volatility coefficient $\sigma = 0.4$, the rate

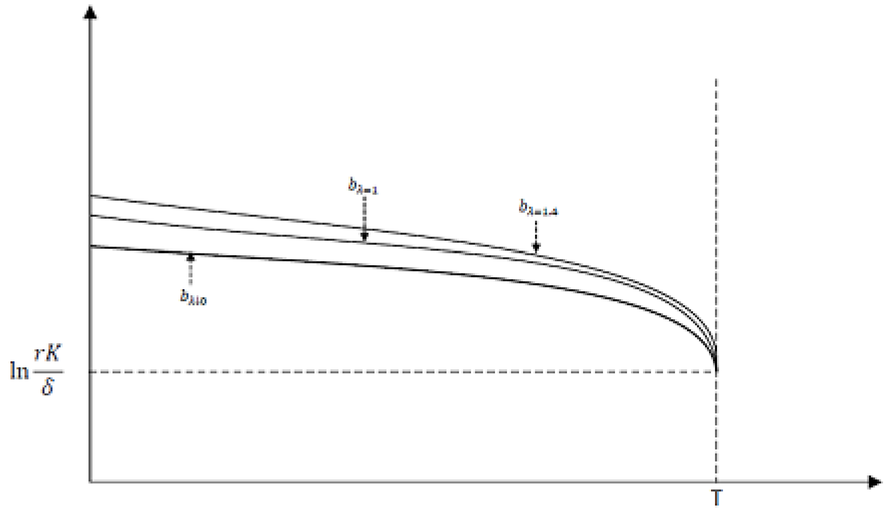

Figure 3.2: A computer drawing comparing the rational exercise boundary of the American call option under the positive exponential jump-diffusion processes with $K = 10$, $T = 1$, $r = 0.1$, $\delta = 0.03$, $\sigma = 0.4$, $\eta_1 = 2.3$ when the intensity parameter $\lambda = 1.4$ and $\lambda = 1$.

parameter of the negative exponential random variable $\eta_1 = 2.3$. We can see from Figure 3.3, the value the intensity parameter λ will not influence the shape of value function: $V(t, x)$ is always increasing and convex on $[0, T]$, which is consistent with the theoretical results verified in Property 3.8. However, the position of value function before touching the early exercise point is decided by λ: the value of the American call option will be higher when the intensity parameter λ we choose is higher; and its corresponding early exercise point will also be higher. Meanwhile, as λ tends to zero, the value function of American call options under positive exponential jump-diffusion processes will converge to that under geometric Brownian motion. We can see that the results observed from Figure 3.2 and Figure 3.3 are in accordance with each other. Moreover, they are also in line with the financial meaning of pricing American call option with only positive jumps: the higher frequency (represented by intensity λ) of "favourable" jumps (positive jumps for call options) leads to higher value of the option and the higher optimal exercise boundary.

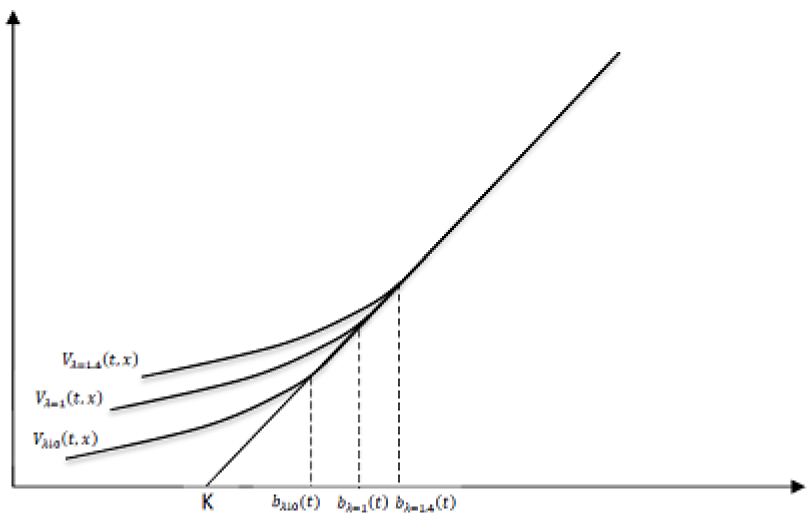

Figure 3.3: A computer drawing comparing the value of the American call option under the positive exponential jump-diffusion processes at a given time $t = 0$ with $K = 10$, $T = 1$, $r = 0.1$, $\delta = 0.03$, $\sigma = 0.4$, $\eta_1 = 2.3$ when the intensity parameter $\lambda = 1.4$ and $\lambda = 1$.

Chapter 4

British Put Option for Negative Exponential Jump Diffusion Processes

4.1 Introduction

The purpose of the present section is to introduce and examine the British options with put payoff under the negative exponential jump-diffusion processes. The British payoff mechanism is intrinsically built into the option contract using the concept of optimal prediction in Du Toit and Peskir [55]. We refer to such contracts as “British” for the reasons outlined by Peskir and Samee [42]. In their article, Peskir and Samee proved that the British put option not only provides a unique protection against unfavourable stock price movement but also enables the option holder to obtain higher returns when the stock price movements are favourable in both liquid and illiquid markets. This remarkable advantage of the British payoff mechanism is reaffirmed by Al-Fagih [2], Peskir et al. [40][41], Kitapbayev [23] and Qiu [48] for the value of British barrier, British Asian, British Russian, British lookback, and British strangle options. These combined features are especially appealing as the problem of liquidity and return are addressed completely endogenously.

Up to now, most researches of option pricing with British payoff mechanism are based on Brownian motion and normal distribution as the classic Black-Scholes model. As an extension to Peskir and Samee [42], we will study the British put option pricing model under a negative exponential jump-diffusion process in this section, which is a degenerate case of the double exponential jump-diffusion processes. The DEJD model was propose by Kou in [25] to incorporate the asymmetric leptokurtic features and the volatility smile which cannot be well explained by the Wiener process based model. For more advantages and useful properties of the DEJD model, please see the detailed introduction in Chapter 2. Also in Chapter 2, we showed that the two sided jump-diffusion processes will lead to an equation system containing additional non-local integral terms, which cannot provide analytical solutions for the price of a corresponding American put option. For a better financial comparison between the British and the American put option, here we assume there are only negative jumps on the path of the underlying asset. Based on the negative jump-diffusion processes, we will show that: (1) the optimal stopping boundary for the British put option with finite horizon can also be characterized as the unique solution to a nonlinear integral equation arising from the early exercise premium representation, where the proof of EEP representation is based on the change-of-variable formula with local time for semi-martingales proposed by Peskir in [36]; (2) the closed-from solution for the price of the British put option is attainable.

The section is organised as follows. In Section 2 we present a basic motivation for the British put option under the negative exponential jump-diffusion processes. In Section 3 we formally define the British put option and present some of its basic properties. We continue in Section 4 to derive a closed-form expression of the arbitrage-free price in terms of the rational exercise boundary and show that the rational exercise boundary can be characterised as the unique solution to a non-linear integral equation. In Section 5 we provide a financial analysis using the results above, making a comparison with American put option under the NEJD processes.

4.2 Basic Motivation for the British Put Option

The economic motivation for the British put option under the geometric Brownian motion was well explained by Peskir and Samee in [42]. Here we would like extend the underlying dynamic to the negative exponential jump-diffusion processes.

1. Consider the financial market consisting a risky stock Z_t and riskless bond B_t:

$$\frac{dZ_t}{dZ_{t}-} = (\mu - \lambda\zeta)\,dt + \sigma\,dW_t + d\Big(\sum_{i=1}^{N_t}(V_i - 1)\Big) \qquad (Z_0 = z), \tag{4.1}$$

$$dB_t = rB_t\,dt \qquad (B_0 = 1), \tag{4.2}$$

where μ is the personal appreciation drift of the stock, σ is the volatility, r is the risk-free interest rate, W_t is a standard Wiener process, N_t is a Poisson process with rate λ, and $\{V_i\}$ is a sequence of independent identically distributed nonnegative random variables such that $Y = \log V$ has a negative exponential distribution with the density:

$$f_Y(y) = \eta_2 e^{\eta_2 y} 1_{\{y<0\}} \qquad \eta_2 > 0, \tag{4.3}$$

The drift μ and the volatility σ are assumed to be constants; N_t, W_t and Y are assumed to be independent. Also note that the drift of the stock price is adjusted by $\lambda\zeta$, where

$$\zeta = \mathsf{E}[V] - 1 = \mathsf{E}[e^Y] - 1 = \frac{\eta_2}{\eta_2 + 1} - 1. \tag{4.4}$$

It is well known that this correction term will have no effect on option prices within the framework of risk-neutral pricing. Meanwhile, by introducing this correction term into the underlying dynamic we can keep the further derivation consistent with Peskir and Samee [42] in the format and focus on the personal appreciation drift μ which acts as the core parameter in the British payoff mechanism.

Here we will directly introduce the following facts of negative exponential jump-diffusion processes. For more detailed derivation, see Chapter 2. We can rewrite the underlying dynamic Z_t in the stochastic exponential (4.1) into an ordinary exponential of a real value Lévy process:

$$Z_t = e^{(\mu - \lambda\zeta - \frac{\sigma^2}{2})t + \sigma W_t + \sum_{i=1}^{N_t}(Y_i)}. \tag{4.5}$$

And its corresponding dynamic is given by:

$$Z_t^r = e^{(r - \lambda\zeta - \frac{\sigma^2}{2})t + \sigma W_t + \sum_{i=1}^{N_t}(Y_i)}. \tag{4.6}$$

The relation between Z_t and Z_t^r is that $\text{Law}(Z|\tilde{\mathsf{P}}) = \text{Law}(Z^r|\mathsf{P})$, where $\tilde{\mathsf{P}}$ is the risk-neutral measure and P refers to the physical measure.

Note that Z_t^r is also a strong Markov process, thus we can have the infinitesimal generator of Z_t^r based on Lamberton and Mikou's paper in [30]:

$$(\mathbb{L}_{Z^r}F)(z) = \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 F}{\partial z^2}(z) + (r - \lambda\zeta)\frac{\partial F}{\partial z}(z) + \lambda \int_{-\infty}^{0} [F(ze^y) - F(z)] f_Y(y)dy, \tag{4.7}$$

for every $F \in C_b^2(\mathbb{R})$, where $C_b^2(\mathbb{R})$ denotes the set of all bounded, twice continuously differentiable functions with bounded derivatives.

2. Recall that an European put option is a financial contract between a seller/hedger and a buyer/holder entitling the latter to sell the underlying stock at a specified strike price $K > 0$ at a specified maturity time $T > 0$. Standard hedging arguments based on self-financing portfolios imply that the arbitrage-free price of the option is given by

$$V = \tilde{\mathsf{E}} e^{-rT}(K - Z_T)^+ \tag{4.8}$$

where the expectation $\tilde{\mathsf{E}}$ is taken with respect to the risk-neutral measure $\tilde{\mathsf{P}}$. Note that the equivalent martingale measure for a jump-diffusion process is not unique, which means that the market is incomplete. In an incomplete market, perfect hedges do not exist and option hedging is actually a risky affair. There are various approaches to hedge options in incomplete market, such as super-hedging, utility maximization, mean-variance hedging and minimal entropy martingale measure. The key difference between these approaches lies in how they measure the risk premia of jumps; i.e. how they define the optimal equivalent martingale measure. For a comprehensive discussion and comparison of these methods, see Tankov and Cont [54]. Nevertheless, the absence of market completeness will not influence the arbitrage-free pricing, which could choose any equivalent martingale measure as a self-consistent pricing rule. Therefore, in this article we will simply follow Merton's hedge approach in [33] which ignores the risk premia for jumps. Assuming the risk premia to be zero also allow us focus more on the core parameter μ of the British option rather than jump parameters λ and η. Thus we (randomly) specify an equivalent martingale measure and set it as the risk-neutral measure $\tilde{\mathsf{P}}$ in all of the following context.

In the sense of Merton's approach, after receiving the amount V from the buyer, the seller can perfectly hedge his position at time T through trading in the underlying stock and bond, and this enables him to meet his obligation without any risk. On the other hand, since the holder can also trade in the underlying stock and bond, he can perfectly hedge his position in the opposite direction and completely eliminate any risk too. Thus the rational performance is risk free, at least from this theoretical standpoint.

3. In this section we will analyse the rational performance from the standpoint of a true buyer. Peskir and Samee [42] define this term as a buyer who has no ability or desire to sell the option nor to hedge his own position in. Thus, every true buyer will exercise the option at time T in accordance with the rational performance. For more details on the motivation and interest for considering a true buyer in this context see [42].

With this in mind we now return to the European put holder and recall that he has right to sell the stock at the strike price K at the maturity time T. Thus his payoff can be expressed as

$$e^{-rT}(K - Z_T(\mu))^+ \tag{4.9}$$

where $Z_T = Z_T(\mu)$ represents the stock price at time T under the physical probability measure P. From equation (4.5) we know that $\mu \mapsto Z_T(\mu)$ is strictly increasing so that $\mu \mapsto e^{-rT}(K - Z_T(\mu))^+$ is decreasing on $\mathbb{R}$. Moreover, it is well known that $\text{Law}(Z(\mu)|\tilde{\mathsf{P}}) = \text{Law}(Z^r|\mathsf{P})$, where Z^r is defined in (4.6). Combining this with (4.8) above we see that if $\mu = r$ then the return is "fair" for the buyer, in the sense that

$$V = \mathsf{E}e^{-rT}(K - Z_T(\mu))^+ \tag{4.10}$$

where the left-hand side represents the value of his investment and the right-hand side represents the expected value of his payoff. On the other hand, if $\mu < r$ then the return is "favourable" for the buyer, in the sense that

$$V < \mathsf{E}e^{-rT}(K - Z_T(\mu))^+ \tag{4.11}$$

and if $\mu > r$ then the return is "unfavourable" for the buyer, in the sense that

$$V > \mathsf{E}e^{-rT}(K - Z_T(\mu))^+ \tag{4.12}$$

with the same interpretations as above. Note that the actual drift μ is unknown at time $t = 0$ and also difficult to estimate at later times $t \in (0, T]$ unless T is unrealistically large.

4. The brief analysis above shows that whilst the actual drift μ of the underlying stock price is irrelevant in determining the arbitrage-free price of the option, to a true buyer it is crucial, and he will buy the option if he believes that $\mu < r$. If this turns out to be the case then on average he will make a profit. Thus, after purchasing the option, the put holder will be happy if the observed stock price movements reaffirm his belief that $\mu < r$.

The British put option seeks to address the opposite scenario: What if the call holder observes stock price movements which change his belief regarding the actual drift and cause him to believe that $\mu > r$ instead? In this contingency the British put holder is effectively able to substitute this unfavourable drift with a contract drift and minimise his losses. In this way he is endogenously protected from any stock price drift greater than the contract drift. The value of the contract drift is therefore selected to present the buyer's expected Level of tolerance for the deviation of the actual drift from his original belief. It will shown below (similar to Peskir and Samee [42]) that the practical implications of this protection feature are most remarkable as not only can the British put holder exercise at or above the strike price to a substantial reimbursement of the original option price (covering the ability to sell in a liquid option market completely endogenously) but also when the stock price movements are favourable he will generally receive higher returns at a lesser price (see Section 5 for further details).

5. Releasing now the true buyer's perspective observe that a put holder who believes that $\mu > r$ may attempt to sell his contract. However, in a real financial market the price will be determined by the market and this may also involve additional transaction costs and taxes. Moreover, it will be increasingly difficult to sell an out-of-the-money option. The latter therefore strongly correlates the buyer's risk exposure to the liquidity of the option market. We remark that the liquidity of the option market can change during the term of the contract. The protection afforded to the British put option holder, on the other hand, is endogenous, i.e. it is always in place regardless of the whether the option market is liquid or not.

4.3 The British Put Option: Definition and Basic Properties

We begin this section by presenting a formal definition of the British put option under the negative exponential jump-diffusion processes. This is then follower by a brief analysis of the optimal stopping problem and the free-boundary problem characterising the arbitrage-free price and the rational exercise strategy.

1. Consider the financial market consisting of a risky stock Z and riskless bond B whose prices evolves as (4.1) and (4.2) respectively. Let a strike price $K > 0$ and a maturity time $T > 0$ be given and fixed.

Definition 4.1. *The British put option is a financial contract between a seller and a buyer entitling the latter to exercise at any (stopping) time τ prior to T whereupon his payoff is the "best prediction" of the European payoff $(K - Z_T)^+$ given all the information up to time τ under the hypothesis that the true drift of the stock price equals to μ_c.*

The quantity μ_c is defined in the option contract and we refer to it as the "contract drift". As we mentioned in the previous section, by tailoring the underlying dynamic to (4.1), μ_c remains to be the only extra quantity given by the British contract, which is in line with British put option under the geometric Brownian motion in [42]. We believe that keeping λ and η out of contract parameters is very essential from a practical and tractable view. Recalling our discussion before, it is natural that the contract drift satisfies

$$\mu_c > r \tag{4.13}$$

since otherwise the British put holder could beat the interest rate r by simply exercising immediately (a formal argument confirming this economic reasoning will be given shortly below). Recall that the value of the contract drift is selected to represent the buyer's expected level of tolerance for the deviation of the true drift μ from his original belief. It will be shown in Section 5 below that this protection feature has remarkable implications both in terms of liquidity and return.

2. Denoting by $(\mathcal{F}_t)_{0 \le t \le T}$ the natural filtration generated by Z, the payoff of the British put option at a given stopping time τ can be formally written as

$$\mathsf{E}^{\mu_c}\big((K - Z_T)^+ | \mathcal{F}_\tau\big) \tag{4.14}$$

where the conditional expectation is taken with respect to a new probability measure P^{μ_c} under which the stock price Z evolves as

$$\frac{dZ_t}{dZ_{t}-} = (\mu_c - \lambda\zeta)\, dt + \sigma\, dW_t + d\Big(\sum_{i=1}^{N_t}(V_i - 1)\Big) \tag{4.15}$$

with $Z_0 = z$ in $(0, \infty)$. Comparing (4.1) and (4.15) we see that the effect of exercising the British put option is to substitute the true but unknown drift of the stock price with the contract drift for the remaining term of the contract.

3. Stationary and independent increments of W governing Z imply that

$$\mathsf{E}^{\mu_c}\big((K - Z_T)^+ | \mathcal{F}_\tau\big) = G^{\mu_c}(t, Z_t) \tag{4.16}$$

where the payoff function G^{μ_c} can be expressed as

$$G^{\mu_c}(t, z) = \mathsf{E}(K - z Z^{\mu_c}_{T-t})^+, \tag{4.17}$$

and $Z^{\mu_c}_t$ is the given by

$$Z^{\mu_c}_t = e^{(\mu_c - \lambda\zeta - \frac{\sigma^2}{2})t + \sigma W_t + \sum_{i=1}^{N_t}(Y_i)}, \tag{4.18}$$

with $\text{Law}(Z|\mathsf{P}^{\mu_c}) = \text{Law}(Z^{\mu_c}|\mathsf{P})$, for $t \in [0, T]$ and $z \in (0, \infty)$. Note that (4.17) is very similar to the arbitrage-free price of an European put option under the negative exponential jump-diffusion processes introduced in Chapter 2. Thus (4.17) can be rewritten as follows

$$\begin{aligned} G^{\mu_c}(t, z) =& K\big(1 - \Upsilon(\mu_c - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; \ln\frac{K}{z}, T - t)\big) \\ & - z e^{r(T-t)}\big(1 - \Upsilon(\mu_c - \lambda\zeta + \frac{1}{2}\sigma^2, \sigma, \lambda^*, \eta_2^*; \ln\frac{K}{z}, T - t)\big), \end{aligned} \tag{4.19}$$

where $\lambda^* = \lambda(1 + \zeta)$, $\eta_2^* = \eta_2 + 1$ and the function Υ is introduced in Chapter 2 as follows

$$\begin{aligned} \Upsilon(\mu, \sigma, \lambda, \eta_2; a, T) =& \mathsf{P}^\mu(Z_T \geq \ln a) \\ =& \pi_0(T)\Phi(-\frac{a - \mu T}{\sigma\sqrt{T}}) \\ & + \frac{e^{\frac{T(\sigma\eta_2)^2}{2}}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{\infty} \pi_n(T)(\sigma\sqrt{T}\eta_2)^n I_{n-1}(a - \mu T, \eta_2, \frac{1}{\sigma\sqrt{T}}, -\sigma\eta_2\sqrt{T}), \end{aligned} \tag{4.20}$$

The following function are needed in Υ

$$\pi_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \tag{4.21}$$

$$I_n(c, \alpha, \beta, \omega) = \int_c^\infty e^{\alpha x} Hh_n(\beta x - \omega)dx, \tag{4.22}$$

$$Hh_n(x) = \int_x^\infty Hh_{n-1}(y)dy = \frac{1}{n!}\int_x^\infty (t - x)^n e^{-\frac{t^2}{2}} dt, \tag{4.23}$$

$$Hh_{-1}(x) = e^{-\frac{x^2}{2}} = \sqrt{2\pi}\varphi(x), \tag{4.24}$$

$$Hh_0(x) = \sqrt{2\pi}\Phi(-x). \tag{4.25}$$

where $\varphi(x)$ and $\Phi(x)$ are the density function and the CDF for standard normal distribution given by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \tag{4.26}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy. \tag{4.27}$$

4. Standard hedging arguments based on self-financing portfolios imply that the arbitrage-free price of the British put option is given by

$$V = \sup_{0 \le \tau \le T} \tilde{\mathsf{E}} \left[e^{-r\tau} \mathsf{E}^{\mu_c} \big((K - Z_T)^+ | \mathcal{F}_\tau \big) \right], \tag{4.28}$$

where the supremum is taken over all stopping times τ of Z with values in $[0, T]$ and $\tilde{\mathsf{E}}$ is taken with respect to the risk-neutral probability measure $\tilde{\mathsf{P}}$ specified before. Making use of the payoff function G^{μ_c} defined in (4.17) and the optional sampling theorem, upon enabling the process Z to start at any point z in $(0, \infty)$ at any time $t \in [0, T]$, we see that the problem (4.28) extends as follows

$$V(t, z) = \sup_{0 \le \tau \le T-t} \tilde{\mathsf{E}}_{t,z} \left[e^{-r\tau} G^{\mu_c}(t + \tau, Z_{t+\tau}) \right], \tag{4.29}$$

where the supremum is taken over all stopping times τ of Z with values in $[0, T - t]$ and $\tilde{\mathsf{E}}_{t,z}$ is taken with respect to the equivalent martingale measure $\tilde{\mathsf{P}}_{t,z}$ under which $Z_t = z$. Since the supremum in (4.29) is attained at the first entry time of Z to the closed set where V equals G^{μ_c}, and the Law$(Z|\tilde{\mathsf{P}})$ is the same as Law$(Z^r|\mathsf{P})$, it follows that

$$V(t, z) = \sup_{0 \le \tau \le T-t} \mathsf{E} \left[e^{-r\tau} G^{\mu_c}(t + \tau, z Z^r_\tau) \right], \tag{4.30}$$

for $t \in [0, T]$ and $z \in (0, \infty)$ where the supremum is taken as in (4.29) above and the process Z^r is defined in (4.6) with $Z^r_0 = 0$.

5. We see from (4.17) that

$$z \mapsto G^{\mu_c}(t, z) \quad \text{is convex} \tag{4.31}$$

and strictly decreasing on $(0, \infty)$ with $G^{\mu_c}(t, 0) = K$ and $G^{\mu_c}(t, \infty) = 0$ for any $t \in [0, T]$ given and fixed. One also sees that $G^{\mu_c}(T, z) = (K - z)^+$ for $z \in (0, \infty)$ showing that the British put payoff coincides with the European put payoff at the time of maturity. Moreover, from the definition of function Υ in (4.20), it can be verified that $G_z^{\mu_c}(t,0+) < -1$ so that the British put payoff function $z \mapsto G^{\mu_c}(t,z)$ goes strictly below the European put payoff function $z \mapsto (K-z)^+$ after starting at the same value K, then cross each other at a point strictly smaller than K. Finally, from (4.30) and (4.31) we easily find that

$$z \mapsto V(t,z) \quad \text{is convex} \tag{4.32}$$

and decreasing on $(0,\infty)$ with $V(t,0) = K$ and $V(t,\infty) = 0$ for any $t \in [0,T)$ given and fixed, and $V(T,z) = (K-z)^+$ for $z \in (0,\infty)$. In this sense the value function of the British put option is similar to the value function of the American put option. The most important technical difference is that whilst the American put boundary is increasing as a function of time, this is not necessarily the case for the British boundary b.

6. Although $G^{\mu_c}(t,z)$ is more complicated than the European or American put payoff function $z \mapsto (K-z)^+$, it is actually a $C^{1,2}$ function which means that we can directly apply Itô's formula to it. Thus to gain a deeper insight into the solution to the optimal stopping problem (4.30), the Itô's formula yields

$$e^{-rs}G^{\mu_c}(t+s, Z^r_{t+s}) = G^{\mu_c}(t,z) + \int_0^s e^{-ru} H^{\mu_c}(t+u, Z^r_{t+u})du + M^1_s + M^2_s, \tag{4.33}$$

where

$$M^1_s = \sigma \int_0^s e^{-ru} Z^r_{t+u} G^{\mu_c}_z(t+u, X^r_{t+u})dW_u, \tag{4.34}$$

$$\begin{aligned} M^2_s = &\sum_{0\le u\le s}^{\Delta Z^r_{t+u}\neq 1} \Big(e^{-ru}G^{\mu_c}(t+u, Z^r_{t+u}) - e^{-ru}G^{\mu_c}(t+u, Z^r_{t+u-}) \Big) \\ &- \lambda \int_0^s e^{-ru} \int_{-\infty}^0 [G^{\mu_c}(t+u, Z^r_{t+u-}e^y) - G^{\mu_c}(t+u, Z^r_{t+u-})] f_Y(y)dydu, \end{aligned} \tag{4.35}$$

and the function $H^{\mu_c} = H^{\mu_c}(t,z)$ is given by

$$H^{\mu_c} = G^{\mu_c}_t + (r-\lambda\zeta)zG^{\mu_c}_z + \frac{\sigma^2}{2}z^2 G^{\mu_c}_{zz} - rG^{\mu_c} + \lambda \int_{-\infty}^0 [G^{\mu_c}(t, ze^y) - G^{\mu_c}(t,z)] f_Y(y)dy. \tag{4.36}$$

Similar to the derivation in Chapter 2, we can see that M^1_s and M^2_s are two martingales for $s \in [0, T-t]$ with $t \in [0,T)$. By the optional sampling theorem we therefore find

$$\mathsf{E}[e^{-r\tau}G^{\mu_c}(t+\tau, zZ^r_\tau)] = G^{\mu_c}(t,z) + \mathsf{E}[\int_0^\tau e^{-r\tau} H^{\mu_c}(t+u, zZ^r_u)du] \tag{4.37}$$

for all stopping times τ of Z^r given by (4.6) with values in $[0, T-t]$ with $t \in [0, T)$ and $z \in (0, \infty)$ given and fixed. On the other hand, it is clear from (4.17) that the payoff function G^{μ_c} satisfies the Kolomogorov backward equation

$$G_t^{\mu_c} + \mathbb{L}_{Z^{\mu_c}} G^{\mu_c} = 0, \tag{4.38}$$

where $\mathbb{L}_{Z^{\mu_c}}$ is the infinitesimal generator similarly defined in (4.7). Thus from (4.36) we see that

$$\begin{aligned} H^{\mu_c} =& G_t^{\mu_c} + \mathbb{L}_{Z^{\mu_c}} G^{\mu_c} + (r-\mu_c) z G_z^{\mu_c} - r G^{\mu_c} \\ =& (r-\mu_c) z G_z^{\mu_c} - r G^{\mu_c}. \end{aligned} \tag{4.39}$$

This representation shows in particular that if $\mu_c \le r$ then $H^{\mu_c} < 0$ so that from (4.37) we see that it is always optimal to exercise immediately as pointed out following (4.13) above. Similar to the derivation in Peskir and Samee [42], by inserting the expression of G^{μ_c} from (4.19) into (4.39), it can be shown that there exists a continuous function $h : [0, T] \to \mathbb{R}$ such that

$$H^{\mu_c}(t, h(t)) = 0 \tag{4.40}$$

for all $t \in [0, T)$ with $H^{\mu_c}(t, z) > 0$ for $z > h(t)$ and $H^{\mu_c}(t, z) < 0$ for $z < h(t)$ when $t \in [0, T)$ is given and fixed. In view of (4.37) this implies that no point (t, z) in $[0, T) \times (0, \infty)$ with $z > h(t)$ is a stopping point. Likewise, it is also clear and can be verified that if $z < h(t)$ and $t < T$ is sufficiently close to T then it is optimal to stop immediately. This shows that the optimal stopping boundary b separating the continuation set from the stopping set satisfies $b(T) = h(T)$ and this value equals rK/μ_c as can be seen from analysis of the boundary behaviour at the maturity in Chapter 2. Also recall from that the optimal stopping boundary in the American put option under the negative exponential jump-diffusion processes takes value K at T.

7. Standard Markovian arguments lead to the following free-boundary problem (see Peskir and Shiryaev [44] for more detailed classification) for the value function

$V = V(t,z)$ and the optimal stopping boundary $b = b(t)$ to be determined:

$$V_t + \mathbb{L}_{Z^r} V = rV \qquad \text{for} \quad z > b(t) \quad \text{and} \quad t \in [0,T), \tag{4.41}$$

$$V(t,z) = G^{\mu_c}(t,z) \qquad \text{for} \quad z = b(t) \quad \text{and} \quad t \in [0,T], \tag{4.42}$$

$$V_z(t,x) = G^{\mu_c}_z(t,z) \qquad \text{for} \quad z = b(t) \quad \text{and} \quad t \in [0,T], \tag{4.43}$$

$$V(T,z) = (K-z)^+ \qquad \text{for} \quad z \geq b(T) = \frac{rK}{\mu_c}, \tag{4.44}$$

$$V(t,\infty) = 0 \qquad \text{for} \quad t \in [0,T], \tag{4.45}$$

where we also set $V(t,z) = G^{\mu_c}(t,z)$ for $z \in (0, b(t))$ and $t \in [0,T]$. It can be shown that this free-boundary problem has a unique solution V and b which coincide with the value function (4.30) and the optimal stopping boundary respectively (see Peskir and Shiryaev [44]). This means that the continuation set is given by $C = \{V > G^{\mu_c}\} = \{(t,z) \in [0,T) \times (0,\infty) | z > b(t)\}$ and the stopping set is given by $D = \{V = G^{\mu_c}\} = \{(t,z) \in [0,T) \times (0,\infty) | z \leq b(t)\} \cup \{(T,z) | z > b(T)\}$ so that the optimal stopping time in (4.30) is given by

$$\tau_b = \inf\{t \in [0,T] | Z^r_t \leq b(t)\}. \tag{4.46}$$

This stopping time represents the rational exercise strategy for the British put option and plays a key role in financial analysis of the option.

Depending on the size of the contract drift μ_c satisfying (4.13) we distinguish three different regimes for the position and shape of the optimal stopping boundary b. Firstly, when μ_c is large than b is an increasing function of time, for which $b(0)$ tends to 0 as $T \to \infty$. Secondly, if $\mu_c > r$ is close to r then b is a skewed U-shaped function of time, for which $b(0)$ tends to ∞ as $T \to \infty$. Third, there is an intermediate case where b can take either of the two shapes depending on the size of T.

The connection between these three regimes is similar to the case of British put option under geometric Brownian motion which was fully discussed in Peskir and Samee [42]. It also should be note that b is not necessarily a monotone function of time, which makes this analysis more complicated in comparison with the American put option.

4.4 The Arbitrage-Free Price and the Rational Exercise Boundary

In this section we derive a closed form expression for the arbitrage-free price V in terms of the rational exercise boundary b and show that the rational exercise boundary b itself can be characterised as the unique solution to a nonlinear integral equation.

Peskir proposed a series of change-of-variable formulas with local time for semi-martingales with jumps in [39]. By the properties of the value function V mentioned in the last section, it is easy to verify that the jump-diffusion process Z_t^r, the condition of [39, Theorem 3.1] hold. Applying this change-of-variable formula to $e^{-rs}V(t+s, Z_{t+s}^r)$ in terms of s with t and $Z_t^r = z$ given and fixed, we can have

$$e^{-rs}V(t+s, Z_{t+s}^r) = V(t,z) + \int_0^s e^{-ru}(V_t + \mathbb{L}_{Z^r}V - rV)(t+u, Z_{t+u}^r)du + M_s^1 + M_s^2, \tag{4.47}$$

where

$$M_s^1 = \sigma \int_0^s e^{-ru} z Z_u^r V_z(t+u, zZ_u^r)dW_u, \tag{4.48}$$

$$\begin{aligned} M_s^2 = & \sum_{0\le u\le s}^{\Delta Z_{t+u}^r \neq 1} \Big(e^{-ru}V(t+u, Z_{t+u}^r) - e^{-ru}V(t+u, Z_{t+u-}^r) \Big) \\ & - \lambda \int_0^s e^{-ru} \int_{-\infty}^0 [V(t+u, Z_{t+u-}^r e^y) - V(t+u, Z_{t+u-}^r)] f_Y(y)dydu, \end{aligned} \tag{4.49}$$

Note that there is no local time term in (4.47) due to the smooth-fit property (4.43) of the value function $V(t,z)$. Following the similar derivation path for American put options employed in Chapter 2, it can be easily proved that M_s^1 and M_s^2 are martingales under P. Using the relation given by (4.41), we can simplify the equation (4.47) as follows

$$\begin{aligned} e^{-rs}V(t+s, Z_{t+s}^r) = & V(t,z) \\ & + \int_0^s e^{-ru}(V_t + \mathbb{L}_{Z^r}V - rV)(t+u, Z_{t+u}^r) I(Z_{t+u}^r < b(t+u))du \\ & + M_s^1 + M_s^2. \end{aligned} \tag{4.50}$$

Recall that only negative jumps are allowed in the NEJD processes, then the existence of the indicator function $I(Z_{t+u}^r < b(t+u))$ will allow us to replace $(V_t + \mathbb{L}_{Z^r}V - rV)$ directly by $(G_t^{\mu_c} + \mathbb{L}_{Z^r}G^{\mu_c} - rG^{\mu_c})$, since no global integral term will be introduced by the operator $\mathbb{L}_{Z^r}$. Now replacing s by $T-t$ in (4.50), using that $V(T,z) = G^{\mu_c}(T,z) = (K-z)^+$, taking $\mathsf{E}_{t,z}$ on both sides and applying the optional sampling theorem, we get

$$V(t,z) = e^{-r(T-t)}G^r(t,z) - \int_0^{T-t} e^{-ru}\mathsf{E}_{t,z}[H^{\mu_c}(t+u, Z^r_{t+u})I(Z^r_{t+u} < b(t+u))]du, \tag{4.51}$$

where $G^r(t,z) = \mathsf{E}_{t,z}(K - Z^r_T)^+$ is defined similar to $G^{\mu_c}(t,z)$ in (4.17). And we will make use of the following function below

$$J(t,z,v,w) = -e^{-r(v-t)}\int_0^w H^{\mu_c}(v,y)f(v-t,z,y)dy, \tag{4.52}$$

for $t \in [0,T)$, $z > 0$, $v \in (t,T)$ and $y > 0$. And the function $y \mapsto f(v-t,z,y)$ is the probability density function of zZ^r_{v-t} given by

$$f(v-t,z,y) = -\frac{d\Upsilon(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; \ln\frac{y}{z}, v-t)}{dy}, \tag{4.53}$$

where the function Υ is defined in (4.20).

Now the main result for the EEP representation and the optimal stopping boundary for an British put option under the NEJD processes can be stated as follows.

Theorem 4.1. *The arbitrage-free price of the British put options admits the following early exercise premium representation under a negative exponential jump diffusion process*

$$V(t,z) = e^{-r(T-t)}G^r(t,z) + \int_t^T J(t,z,v,b(v))dv \tag{4.54}$$

for all $(t,z) \in [0,T] \times (0,\infty)$*, where the first term is the arbitrage-free price of the European put option and the second term is the early-exercise premium.*

The rational exercise boundary of the British put option can be characterised as the unique continuous solution $b : [0,T] \to \mathbb{R}_+$ *to the nonlinear integral equation*

$$G^{\mu_c}(t,b(t)) = e^{-r(T-t)}G^r(t,b(t)) + \int_t^T J(t,b(t),v,b(v))dv \tag{4.55}$$

satisfying $0 \le b(t) \le h(t)$ *for all* $t \in [0,T]$ *where* h *is defined by (4.40) above.*

The analysis from (4.47) to (4.51) establishes the existence of the solution to (4.54), thus only the uniqueness of the optimal stopping boundary b remains to be proved.

Proof. Take any continuous function $c : [0,T] \to \mathbb{R}$ which solves (4.55) and satisfies $0 \le c(t) \le h(t)$ for all $t \in [0,T]$. Motivated by the representation (4.51) above define the function $U^c : [0,T] \times (0,\infty) \to \mathbb{R}$ by setting

$$\begin{aligned} U^c(t,x) =& e^{-r(T-t)}\mathsf{E}[G^{\mu_c}(T, zZ^r_{T-t})] \\ &- \int_0^{T-t} e^{-ru}\mathsf{E}[H^{\mu_c}(t+u, zZ^r_u)I(zZ^r_u < c(t+u))]du, \end{aligned} \tag{4.56}$$

for $(t,z) \in [0,T]\times(0,\infty)$. Observe that c solving (4.51) means exactly that $U^c(t,c(t)) = G^{\mu_c}(t,c(t))$ for all $t \in [0,T]$.

1. We show that $U^c(t,z) = G^{\mu_c}(t,z)$ for all $(t,z) \in [0,T] \times (0,\infty)$ such that $z \le c(t)$. For this, take any such (t,z) and note that the Markov property of Z implies that the stochastic process $M(s, Z^r_s)$ which defined by

$$M(s,Z^r_s) = e^{-rs}U^c(t+s, zZ^r_s) - \int_0^s e^{-ru}\mathsf{E}[H^{\mu_c}(t+u, zZ^r_u)I(zZ^r_u < c(t+u))]du, \tag{4.57}$$

is a continuous martingale under P for $s \in [0, T-t]$. Consider the stopping time

$$\sigma_c = \inf\{s \in [0, T-t] | zZ^r_s \ge c(t+s)\}, \tag{4.58}$$

under P. Since $U^c(t,c(t)) = G^{\mu_c}(t,c(t))$ for all $t \in [0,T]$ and $U^c(T,z) = G^{\mu_c}(T,z)$ for all $z > 0$ we see that $U^c(t+\sigma_c, zZ_{\sigma_c}) = G^{\mu_c}(t+\sigma_c, zZ_{\sigma_c})$. Replacing s by σ_c in (4.57), taking E on both sides and applying the optional sampling theorem, we find that

$$\begin{aligned} U^c(t,z) =& \mathsf{E}[e^{-r\sigma_c}U^c(t+\sigma_c, zZ_{\sigma_c})] \\ &- \mathsf{E}\left(\int_0^{\sigma_c} e^{-ru}H^{\mu_c}(t+u, zZ^r_u)I(zZ^r_u < c(t+u))du\right) \\ =& \mathsf{E}[e^{-r\sigma_c}G^{\mu_c}(t+\sigma_c, zZ_{\sigma_c})] - \mathsf{E}\left(\int_0^{\sigma_c} e^{-ru}H^{\mu_c}(t+u, zZ^r_u)du\right) = G^{\mu_c}(t,z), \end{aligned} \tag{4.59}$$

where the last equality we use (4.37). This shows that U^c equals G^{μ_c} below c as claimed.

2. We show that $U^c(t,z) \le V(t,z)$ for all $(t,z) \in [0,T] \times (0,\infty)$. For this, take any such (t,z) and consider the stopping time

$$\tau_c = \inf\{s \in [0, T-t] | zZ^r_s \le c(t+s)\}, \tag{4.60}$$

under P. We claim that $U^c(t+\tau_c, zZ_{\tau_c}) = G^{\mu_c}(t+\tau_c, zZ_{\tau_c})$. Indeed, if $z \le c(t)$ then $\tau_c = 0$ so that $U^c(t,z) = V(t,z)$ by previous step. On the other hand, if $z > c(t)$ then the claim follows since $U^c(t,c(t)) = G^{\mu_c}(t,c(t))$ for all $t \in [0,T]$ and $U^c(T,z) = G^{\mu_c}(T,z)$ for all $z > 0$. Replacing s by τ_c in (4.57), taking E on both sides and applying the optional sampling theorem, we find that

$$U^c(t,z) = \mathsf{E}[e^{-r\tau_c}U^c(t+\tau_c, zZ_{\tau_c})] - \mathsf{E}\bigg(\int_0^{\tau_c} e^{-ru}H^{\mu_c}(t+u, zZ_u^r)I(zZ_u^r < c(t+u))du\bigg) \tag{4.61}$$

$$= \mathsf{E}[e^{-r\tau_c}G^{\mu_c}(t+\tau_c, zZ_{\tau_c})] \le V(t,z),$$

where in the second equality we used the definition of τ_c. This shows that $U^c \le V$ as claimed.

3. We show that $b(t) \le c(t)$ for all $t \in [0,T]$. For this, suppose this there exists $t \in [0,T)$ such that $c(t) < b(t)$. Take any $z \le c(t)$ and consider the stopping time

$$\sigma_b = \inf\{s \in [0,T-t] | zZ_s^r \ge b(t+s)\}, \tag{4.62}$$

under P. Replacing s with σ_b in (4.50) and (4.57), taking E on both sides of these identities and applying the optional sampling theorem, we find

$$\mathsf{E}[e^{-r\sigma_b}V(t+\sigma_b, zZ_{\sigma_b}^r)] = V(t,z) + \mathsf{E}\bigg(\int_0^{\sigma_b} e^{-ru}H^{\mu_c}(t+u, zZ_u^r)du\bigg) \tag{4.63}$$

$$\mathsf{E}[e^{-r\sigma_b}U^c(t+\sigma_b, zZ_{\sigma_b}^r)] = U^c(t,z) + \mathsf{E}\bigg(\int_0^{\sigma_b} e^{-ru}H^{\mu_c}(t+u, zZ_u^r)I(zZ_u^r < c(t+u))du\bigg). \tag{4.64}$$

Since $z \le c(t)$ we see by step **1** above that $U^c(t,z) = G^{\mu_c}(t,z) = V(t,z)$ where the last equality follows since z lies below $b(t)$. Moreover, by Step **2** above we know that $U^c(t+\sigma_b, zZ_{\sigma_b}) \le V(t+\sigma_b, zZ_{\sigma_b})$ so that (4.63) and (4.64) imply that

$$\mathsf{E}\bigg(\int_0^{\sigma_b} e^{-ru}H^{\mu_c}(t+u, zZ_u^r)I(zZ_u^r \ge c(t+u))du\bigg) \ge 0. \tag{4.65}$$

The fact that $c(t) < b(t)$ and the continuity of the functions c and b imply that there exists $\epsilon > 0$ sufficiently small such that $c(t+u) < b(t+u)$ for all $u \in [0,\epsilon]$. Consequently the P-probability of $(zZ_u^r)_{0\le u\le\epsilon}$ spending a strictly positive amount of time in this set before hitting b is strictly positive. Combine with the fact that b lies below h this forces the expectation in (4.65) to be strictly negative and provides a contradiction. Hence $b \leq c$ as claimed.

4. We show that $b(t) = c(t)$ for all $t \in [0, T]$. For this, suppose that there exists $t \in [0, T)$ such that $b(t) < c(t)$. Take any $z \in (b(t), c(t))$ and consider the stopping time

$$\tau_b = \inf\{s \in [0, T-t] | zZ_s^r \leq b(t+s)\}, \tag{4.66}$$

under P. Replacing s with τ_b in (4.50) and (4.57), taking E on both sides of these identities and applying the optimal sampling theorem, we find

$$\mathsf{E}[e^{-r\tau_b} V(t+\tau_b, zZ_{\tau_b}^r)] = V(t, z) \tag{4.67}$$

$$\mathsf{E}[e^{-r\tau_b} U^c(t+\tau_b, zZ_{\tau_b}^r)] = U^c(t, z) + \mathsf{E}\left(\int_0^{\tau_b} e^{-ru} H^{\mu_c}(t+u, zZ_u^r) I(zZ_u^r < c(t+u)) du \right). \tag{4.68}$$

Since $b \leq c$ by step **3** above and U^c equals G^{μ_c} below c by step **1** above, we see that $U^c(t+\tau_b, zZ_{\tau_b}) = G^{\mu_c}(t+\tau_b, zZ_{\tau_b}) = V(t+\tau_b, zZ_{\tau_b})$ where the last equality follows since V equals G^{μ_c} below b. Moreover, by step **2** we know that $U^c \leq V$ so that (4.67) and (4.68) imply that

$$\mathsf{E}\left(\int_0^{\tau_b} e^{-ru} H^{\mu_c}(t+u, zZ_u^r) I(zZ_u^r < c(t+u)) du \right) \geq 0. \tag{4.69}$$

But then as in step **3** above the continuity of the functions c and b combined with the fact that c lies below h forces the expectation in (4.69) to be strictly negative and provides a contradiction. Thus $c = b$ as claimed and the proof is complete. □

4.5 Financial Analysis for British Put Options

In this section, we focus on the practical features of the British put option under negative exponential jump-diffusion processes. We compare it with the American put option that is universally traded. We address the problem as to what the return would be if the price of the underlying asset enters the given region as a given time. We use "skeletal" approach of Peskir and Samee [42], with a similar parameter set so that we can make reference to the results by Peskir and Samee [42].

1. As shown in Figure 4.1, the rational exercise strategy of the British put option under negative exponential jump-diffusion processes varies with different contract drift

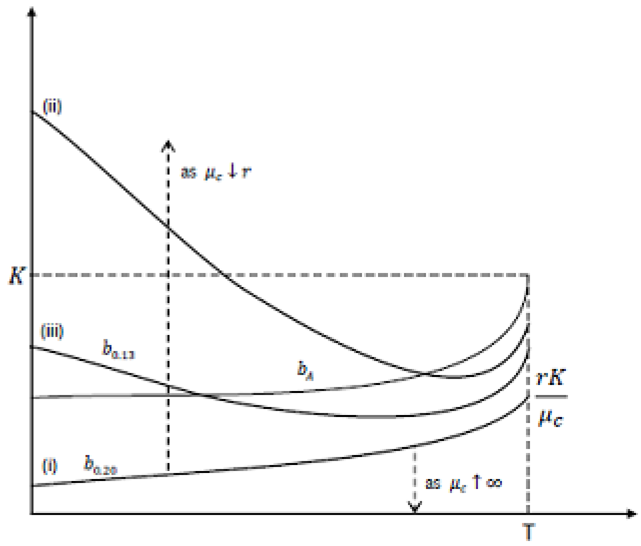

Figure 4.1: A computer drawing showing the rational exercise boundaries of the British put option under the negative exponential jump-diffusion processes with $K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\lambda = 1.0$, $\eta_2 = 1.8$ when the contract drift μ_c equals 0.13 and 0.20 (the thin curved line b_A represents the rational exercise boundary of the American put option).

μ_c. Where depending on the value of μ_c, the boundary b is either an increasing function of time, a skewed U-shaped function of time, or an intermediate case where b can take either of the two shapes depending on the size of T.

In Figure 4.1, we witness the same three regimes for the optimal stopping boundary b as Peskir and Samee [42]. In particular, as μ_c gets closer to r, we see that $b(0) \uparrow \infty$ making the stock price always below $b(0)$. It would be optimal to stop immediately, making the buyer overprotected. On the other hand, when $\mu_c \uparrow \infty$, it is not optional to exercise the option before time T, reducing the British put option to the European put option. Therefore, the contract drift μ_c should not be too close to r (since in this case the buyer is overprotected) and should not be too large (since in this case the British put option effectively reduces to the European put option).

2. In the numerical example below (see Table 4.1, 4.2 and 4.3), the parameter values have been chosen to present the practical features of the British put option. We assume that the initial stock price equals the strike price $K = 10$, the maturity time $T = 1$, the interest rate $r = 0.1$, the volatility coefficient $\sigma = 0.4$, the intensity parameter $\lambda = 1.0$, and the rate parameter of the negative exponential random variable

Time (months)	0	2	4	6	8	10	12
Exercise at K with $\mu_c = 0.13$	99%	92%	85%	75%	64%	48%	0%
Exercise at K with $\mu_c = 0.20$	80%	75%	71%	64%	56%	43%	0%
Exercise at 11 with $\mu_c = 0.13$	70%	63%	54%	45%	34%	18%	0%
Exercise at 11 with $\mu_c = 0.20$	55%	50%	44%	38%	29%	16%	0%
Exercise at 12 with $\mu_c = 0.13$	50%	43%	35%	26%	16%	04%	0%
Exercise at 12 with $\mu_c = 0.20$	38%	33%	27%	21%	14%	03%	0%
Exercise at 13 with $\mu_c = 0.13$	36%	29%	22%	14%	07%	01%	0%
Exercise at 13 with $\mu_c = 0.20$	26%	21%	16%	09%	05%	0.9%	0%
Exercise at 14 with $\mu_c = 0.13$	25%	19%	14%	07%	02%	0.7%	0%
Exercise at 14 with $\mu_c = 0.20$	18%	14%	09%	05%	01%	0%	0%
Exercise at 15 with $\mu_c = 0.13$	18%	11%	08%	03%	0.2%	0%	0%
Exercise at 15 with $\mu_c = 0.20$	12%	08%	05%	01%	0%	0%	0%

Table 4.1: Returns observed upon exercising the British put option at and above the strike price K. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,z)/100 = G^{\mu_c}(t,z)/V(0,K)$. The parameter set is the same as in Figure 4.1 above ($K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\lambda = 1.0$, $\eta_2 = 1.8$) and the initial stock price equals K.

$\eta_2 = 1.8$. By considering the set of parameters above, the arbitrage-free price of the British put option using the EEP representation (4.54) is 1.096 if $\mu_c = 0.2$ and 1.113 if $\mu_c = 0.13$. The price of the European put option under exponential jump-diffusion processes is 1.091 that can be evaluated using the formula in [25]. Using the relevant results proposed in Chapter 2, we find the price of the American put option is 1.203. In terms of the price sizes it can be seen that this example is quite typical since the price of the British put option lies between the prices of the European option and the American option. We get that the closer the contract drift gets to r, the protection feature is stronger and the option price is more expensive as stated above. If the contract drift μ_c is too close to r, the protection feature works and makes the price of the British option extremely close to the American option. If $\mu_c \uparrow \infty$ the British put option will be reduced to the European put option.

3. Table 4.1 illustrates the power of the protection feature in practice. For instance, if the stock price is stock price is at K halfway to maturity (clearly representative of unfavourable price movements) then the British put holder can exercise immediately to a payoff with a reimbursement of $64\% - 75\%$ of his original investment. However, in the case of the American put option, the holder is out-of-the-money and wold receive zero payoff. We see that the size of the reimbursement also varies with

Time (months)	0	2	4	6	8	10	12
Exercise at 8 (British put)	180%	177%	174%	172%	170%	169%	178%
Exercise at 8 (American put)	164%	164%	164%	164%	164%	164%	164%
Exercise at b (British put)	183%	202%	221%	240%	257%	262%	206%
Exercise at b (American put)	279%	269%	254%	238%	215%	177%	0%
Exercise at 6 (British put)	314%	317%	322%	329%	337%	348%	360%
Exercise at 6 (American put)	332%	332%	332%	332%	332%	332%	332%
Exercise at 4 (British put)	496%	506%	510%	518%	526%	535%	543%
Exercise at 4 (American put)	489%	489%	489%	489%	489%	489%	489%
Exercise at 2 (British put)	701%	705%	708%	712%	717%	721%	725%
Exercise at 2 (American put)	655%	655%	655%	655%	655%	655%	655%

Table 4.2: Returns observed upon exercising the British put option (with μ_c=0.13) and the American put option below the strike price K. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,z)/100 = G^{\mu_c}(t,z)/V(0,K)$ and $R_A(t,z)/100 = (K-z)^+/V_A(0,K)$ respectively. The parameter set is the same as in Figure 4.1 above ($K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\lambda = 1.0$, $\eta_2 = 1.8$) and the initial stock price equals K.

the contract drift as analysed in the former paragraph. When the contract drift gets closer to r, the option holder gets more protection as well as greater reimbursement.

4. Now we only focus on the British put option with $\mu_c = 0.13$ since in this case the rational exercise strategy most closely resembles that of the American put option and the comparison is therefore more revealing. In Table 4.2, we compare the returns of the British put option and the American version option as the stock price is moving favourably. The result generally indicates that the British option outperforms the American option except a few points. In Table 4.3, we compare the protection feature of the British option with the reimbursement of American option if the holder of the latter can choose to sell his option freely without friction. We see in Table 4.3 that the protection feature of the British put option is remarkably similar to the protection afforded to the American put holder by his ability to sell. However, in the real market the option holder's ability to sell the option is affected by many exogenous factors such as the friction costs, taxes, the liquidity of the market and so on. Therefore, from this point of view it makes the British style option more attractive since the protection feature to the British option is intrinsic and endogenous.

Time (months)	0	2	4	6	8	10	12
Exercise at 8 (British put)	180%	177%	174%	172%	170%	169%	178%
Selling at 8 (American put)	165%	163%	160%	158%	154%	154%	163%
Exercise at 9 (British put)	133%	128%	122%	115%	108%	97%	87%
Selling at 9 (American put)	122%	118%	112%	107%	99%	90%	80%
Exercise at K (British put)	97%	90%	83%	74%	63%	46%	0%
Selling at K (American put)	90%	84%	77%	69%	59%	43%	0%
Exercise at 11 (British put)	70%	63%	54%	45%	34%	18%	0%
Selling at 11 (American put)	65%	59%	51%	43%	32%	17%	0%
Exercise at 12 (British put)	50%	43%	36%	26%	16%	04%	0%
Selling at 12 (American put)	47%	41%	33%	25%	16%	04%	0%
Exercise at 13 (British put)	36%	28%	22%	14%	7%	0%	0%
Selling at 13 (American put)	34%	28%	21%	14%	06%	0%	0%

Table 4.3: Returns observed upon exercising the British put option (with μ_c=0.13) above the rational exercise boundary compared with returns received upon selling the American put option in the same contingency. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,z)/100 = G^{\mu_c}(t,z)/V(0,K)$ and $R_A(t,z)/100 = V_A(t,z)/V_A(0,K)$ respectively. The parameter set is the same as in Figure 4.1 above ($K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\lambda = 1.0$, $\eta_2 = 1.8$) and the initial stock price equals K.

Chapter 5

British Call Option for Positive Exponential Jump Diffusion Processes

5.1 Introduction

The purpose of the present section is to introduce and examine the British options with call payoff under the positive exponential jump-diffusion processes. The British payoff mechanism is intrinsically built into the option contract using the concept of optimal prediction in Du Toit and Peskir [55]. We refer to such contracts as "British" for the reasons outlined by Peskir and Samee [42], [43]. In their articles, Peskir and Samee proved that the British put or call option not only provides a unique protection against unfavourable stock price movement but also enables the option holder to obtain higher returns when the stock price movements are favourable in both liquid and illiquid markets. This remarkable advantage of the British payoff mechanism is reaffirmed by Al-Fagih [2], Peskir et al. [40][41], Kitapbayev [23] and Qiu [48] for the value of British barrier, British Asian, British Russian, British lookback, and British strangle options. These combined features are especially appealing as the problem of liquidity and return are addressed completely endogenously.

Up to now, most researches of option pricing with British payoff mechanism are based on Brownian motion and normal distribution as the classic Black-Scholes model. As an extension to Peskir and Samee [42], we will study the British put option pricing model under a positive exponential jump-diffusion process in this section, which is a degenerate case of the double exponential jump-diffusion processes. The DEJD model was propose by Kou in [25] to incorporate the asymmetric leptokurtic features and the volatility smile which cannot be well explained by the Wiener process based model. For more advantages and useful properties of the DEJD model, please see the detailed introduction in Chapter 2. And in Chapter 3, we showed that the two sided jump-diffusion processes will lead to an equation system containing additional non-local integral terms, which cannot provide analytical solutions for the price of a corresponding American call option. For a better financial comparison between the British and the American call option, here we assume there are only positive jumps on the path of the underlying asset. Based on the positive jump-diffusion processes, we will show that: (1) the optimal stopping boundary for the British call option with finite horizon can also be characterized as the unique solution to a nonlinear integral equation arising from the early exercise premium representation, where the proof of EEP representation is based on the change-of-variable formula with local time for semi-martingales proposed by Peskir in [36]; (2) the closed-from solution for the price of the British call option is attainable.

The section is organised as follows. In Section 2 we present a basic motivation for the British call option under the negative exponential jump-diffusion processes. In Section 3 we formally define the British call option and present some of its basic properties. We continue in Section 4 to derive a closed-form expression of the arbitrage-free price in terms of the rational exercise boundary and show that the rational exercise boundary can be characterised as the unique solution to a non-linear integral equation. In Section 5 we provide a financial analysis using the results above, making a comparison with American call option under the PEJD processes.

5.2 Basic Motivation for the British Call Option

The economic motivation for the British call option under the geometric Brownian motion was well explained by Peskir and Samee in [43]. Here we would like extend the underlying dynamic to the positive exponential jump-diffusion processes.

1. Consider the financial market consisting a risky stock Z_t and riskless bond B_t:

$$\frac{dZ_t}{dZ_t-} = (\mu - \lambda\zeta)\, dt + \sigma\, dW_t + d\Big(\sum_{i=1}^{N_t}(V_i - 1)\Big) \qquad (Z_0 = z), \tag{5.1}$$

$$dB_t = rB_t\, dt \qquad (B_0 = 1), \tag{5.2}$$

where μ is the personal appreciation drift of the stock, σ is the volatility, r is the risk-free interest rate, W_t is a standard Wiener process, N_t is a Poisson process with rate λ, and $\{V_i\}$ is a sequence of independent identically distributed nonnegative random variables such that $Y = \log V$ has a positive exponential distribution with the density:

$$f_Y(y) = \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} \qquad \eta_1 > 1, \tag{5.3}$$

The drift μ and the volatility σ are assumed to be constants; N_t, W_t and Y are assumed to be independent. Also note that the drift of the stock price is adjusted by $\lambda\zeta$, where

$$\zeta = \mathsf{E}[V] - 1 = \mathsf{E}[e^Y] - 1 = \frac{\eta_1}{\eta_1 - 1} - 1. \tag{5.4}$$

It is well known that this correction term will have no effect on option prices within the framework of risk-neutral pricing. Meanwhile, by introducing this correction term into the underlying dynamic we can keep the further derivation consistent with Peskir and Samee [43] in the format and focus on the personal appreciation drift μ which acts as the core parameter in the British payoff mechanism.

Here we will directly introduce the following facts of positive exponential jump-diffusion processes. For more detailed derivation, see Chapter 3. We can rewrite the underlying dynamic Z_t in the stochastic exponential (5.1) into an ordinary exponential of a real value Lévy process:

$$Z_t = e^{(\mu - \lambda\zeta - \frac{\sigma^2}{2})t + \sigma W_t + \sum_{i=1}^{N_t}(Y_i)}. \tag{5.5}$$

And its corresponding dynamic is given by:

$$Z_t^r = e^{(r - \lambda\zeta - \frac{\sigma^2}{2})t + \sigma W_t + \sum_{i=1}^{N_t}(Y_i)}. \tag{5.6}$$

The relation between Z_t and Z_t^r is that $\text{Law}(Z|\tilde{\mathsf{P}}) = \text{Law}(Z^r|\mathsf{P})$, where $\tilde{\mathsf{P}}$ is the risk-neutral measure and P refers to the physical measure.

Note that Z_t^r is also a strong Markov process, thus we can have the infinitesimal generator of Z_t^r based on Lamberton and Mikou's paper in [30]:

$$(\mathbb{L}_{Z^r}F)(z) = \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 F}{\partial z^2}(z) + (r - \lambda\zeta)\frac{\partial F}{\partial z}(z) + \lambda \int_{-\infty}^{0} [F(ze^y) - F(z)] f_Y(y)dy, \tag{5.7}$$

for every $F \in C_b^2(\mathbb{R})$, where $C_b^2(\mathbb{R})$ denotes the set of all bounded, twice continuously differentiable functions with bounded derivatives.

2. Recall that an European call option is a financial contract between a seller/hedger and a buyer/holder entitling the latter to buy the underlying stock at a specified strike price $K > 0$ at a specified maturity time $T > 0$. Standard hedging arguments based on self-financing portfolios imply that the arbitrage-free price of the option is given by

$$V = \tilde{\mathsf{E}} e^{-rT}(Z_T - K)^+ \tag{5.8}$$

where the expectation $\tilde{\mathsf{E}}$ is taken with respect to the risk-neutral measure $\tilde{\mathsf{P}}$. Note that the equivalent martingale measure for a jump-diffusion process is not unique, which means that the market is incomplete. In an incomplete market, perfect hedges do not exist and option hedging is actually a risky affair. There are various approaches to hedge options in incomplete market, such as super-hedging, utility maximization, mean-variance hedging and minimal entropy martingale measure. The key difference between these approaches lies in how they measure the risk premia of jumps; i.e. how they define the optimal equivalent martingale measure. For a comprehensive discussion and comparison of these methods, see Tankov and Cont [54]. Nevertheless, the absence of market completeness will not influence the arbitrage-free pricing, which could choose any equivalent martingale measure as a self-consistent pricing rule. Therefore, in this article we will simply follow Merton's hedge approach in [33] which ignores the risk premia for jumps. Assuming the risk premia to be zero also allow us focus more on the core parameter μ of the British option rather than jump parameters λ and η. Thus we (randomly) specify an equivalent martingale measure and set it as the risk-neutral measure $\tilde{\mathsf{P}}$ in all of the following context.

In the sense of Merton's approach, after receiving the amount V from the buyer, the seller can perfectly hedge his position at time T through trading in the underlying stock and bond, and this enables him to meet his obligation without any risk. On the other hand, since the holder can also trade in the underlying stock and bond, he can perfectly hedge his position in the opposite direction and completely eliminate any risk too. Thus the rational performance is risk free, at least from this theoretical standpoint.

3. In this section we will analyse the rational performance from the standpoint of a true buyer. Peskir and Samee [42] define this term as a buyer who has no ability or desire to sell the option nor to hedge his own position in. Thus, every true buyer will exercise the option at time T in accordance with the rational performance. For more details on the motivation and interest for considering a true buyer in this context see [42].

With this in mind we now return to the European call holder and recall that he has right to sell the stock at the strike price K at the maturity time T. Thus his payoff can be expressed as

$$e^{-rT}(Z_T(\mu) - K)^+ \tag{5.9}$$

where $Z_T = Z_T(\mu)$ represents the stock price at time T under the physical probability measure P. From equation (5.5) we know that $\mu \mapsto Z_T(\mu)$ is strictly increasing so that $\mu \mapsto e^{-rT}(K - Z_T(\mu))^+$ is increasing on $\mathbb{R}$. Moreover, it is well known that $\text{Law}(Z(\mu)|\tilde{\mathsf{P}}) = \text{Law}(Z^r|\mathsf{P})$, where Z^r is defined in (5.6). Combining this with (5.8) above we see that if $\mu = r$ then the return is "fair" for the buyer, in the sense that

$$V = \mathsf{E}e^{-rT}(Z_T(\mu) - K)^+ \tag{5.10}$$

where the left-hand side represents the value of his investment and the right-hand side represents the expected value of his payoff. On the other hand, if $\mu > r$ then the return is "favourable" for the buyer, in the sense that

$$V < \mathsf{E}e^{-rT}(Z_T(\mu) - K)^+ \tag{5.11}$$

and if $\mu < r$ then the return is "unfavourable" for the buyer, in the sense that

$$V > \mathsf{E}e^{-rT}(Z_T(\mu) - K)^+ \tag{5.12}$$

with the same interpretations as above. Note that the actual drift μ is unknown at time $t = 0$ and also difficult to estimate at later times $t \in (0, T]$ unless T is unrealistically large.

4. The brief analysis above shows that whilst the actual drift μ of the underlying stock price is irrelevant in determining the arbitrage-free price of the option, to a true buyer it is crucial, and he will buy the option if he believes that $\mu > r$. If this turns out to be the case then on average he will make a profit. Thus, after purchasing the option, the put holder will be happy if the observed stock price movements reaffirm his belief that $\mu > r$.

The British put option seeks to address the opposite scenario: What if the call holder observes stock price movements which change his belief regarding the actual drift and cause him to believe that $\mu < r$ instead? In this contingency the British call holder is effectively able to substitute this unfavourable drift with a contract drift and minimise his losses. In this way he is endogenously protected from any stock price drift smaller than the contract drift. The value of the contract drift is therefore selected to present the buyer's expected Level of tolerance for the deviation of the actual drift from his original belief. It will shown below (similar to Peskir and Samee [43]) that the practical implications of this protection feature are most remarkable as not only can the British call holder exercise at or below the strike price to a substantial reimbursement of the original option price (covering the ability to sell in a liquid option market completely endogenously) but also when the stock price movements are favourable he will generally receive higher returns (see Section 5 for further details).

5. Releasing now the true buyer's perspective observe that a call holder who believes that $\mu < r$ may attempt to sell his contract. However, in a real financial market the price will be determined by the market and this may also involve additional transaction costs and taxes. Moreover, it will be increasingly difficult to sell an out-of-the-money option. The latter therefore strongly correlates the buyer's risk exposure to the liquidity of the option market. We remark that the liquidity of the option market can change during the term of the contract. The protection afforded to the British call option holder, on the other hand, is endogenous, i.e. it is always in place regardless of the whether the option market is liquid or not.

5.3 The British Call Option: Definition and Basic Properties

We begin this section by presenting a formal definition of the British call option under the positive exponential jump-diffusion processes. This is then follower by a brief analysis of the optimal stopping problem and the free-boundary problem characterising the arbitrage-free price and the rational exercise strategy.

1. Consider the financial market consisting of a risky stock Z and riskless bond

B whose prices evolves as (5.1) and (5.2) respectively. Let a strike price $K > 0$ and a maturity time $T > 0$ be given and fixed.

Definition 5.1. *The British call option is a financial contract between a seller and a buyer entitling the latter to exercise at any (stopping) time τ prior to T whereupon his payoff is the "best prediction" of the European payoff $(Z_T - K)^+$ given all the information up to time τ under the hypothesis that the true drift of the stock price equals to μ_c.*

The quantity μ_c is defined in the option contract and we refer to it as the "contract drift". As we mentioned in the previous section, by tailoring the underlying dynamic to (5.1), μ_c remains to be the only extra quantity given by the British contract, which is in line with British call option under the geometric Brownian motion in [43]. We believe that keeping λ and η out of contract parameters is very essential from a practical and tractable view. Recalling our discussion before, it is natural that the contract drift satisfies the right-hand inequality in

$$0 < \mu_c < r \tag{5.13}$$

since otherwise the British call holder could beat the interest rate r by simply exercising immediately (a formal argument confirming this economic reasoning will be given shortly below). We will also show below that the left-hand inequality in (5.13) must be satisfied as well since otherwise it would be optimal to exercise at time T and the British call option would reduce to the European call option. Recall that the value of the contract drift is selected to represent the buyer's expected level of tolerance for the deviation of the true drift μ from his original belief. It will be shown in Section 5 below that this protection feature has remarkable implications both in terms of liquidity and return.

2. Denoting by $(\mathcal{F}_t)_{0 \le t \le T}$ the natural filtration generated by Z, the payoff of the British call option at a given stopping time τ can be formally written as

$$\mathsf{E}^{\mu_c}\big((Z_T - K)^+ | \mathcal{F}_\tau\big) \tag{5.14}$$

where the conditional expectation is taken with respect to a new probability measure P^{μ_c} under which the stock price Z evolves as

$$\frac{dZ_t}{dZ_{t}-} = (\mu_c - \lambda\zeta)\, dt + \sigma\, dW_t + d\Big(\sum_{i=1}^{N_t}(V_i - 1)\Big) \tag{5.15}$$

with $Z_0 = z$ in $(0,\infty)$. Comparing (5.1) and (5.15) we see that the effect of exercising the British call option is to substitute the true but unknown drift of the stock price with the contract drift for the remaining term of the contract.

3. Stationary and independent increments of W governing Z imply that

$$\mathsf{E}^{\mu_c}\big((Z_T - K)^+ | \mathcal{F}_\tau\big) = G^{\mu_c}(t, Z_t) \tag{5.16}$$

where the payoff function G^{μ_c} can be expressed as

$$G^{\mu_c}(t,z) = \mathsf{E}(z Z^{\mu_c}_{T-t} - K)^+, \tag{5.17}$$

and $Z^{\mu_c}_t$ is the given by

$$Z^{\mu_c}_t = e^{(\mu_c - \lambda\zeta - \frac{\sigma^2}{2})t + \sigma W_t + \sum_{i=1}^{N_t}(Y_i)}, \tag{5.18}$$

with $\text{Law}(Z|\mathsf{P}^{\mu_c}) = \text{Law}(Z^{\mu_c}|\mathsf{P})$, for $t \in [0,T]$ and $z \in (0,\infty)$. Note that (5.17) is very similar to the arbitrage-free price of an European call option under the positive exponential jump-diffusion processes introduced in Chapter 3. Thus (5.17) can be rewritten as follows

$$\begin{aligned} G^{\mu_c}(t,z) =& -K\Upsilon(\mu_c - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_1; \ln\frac{K}{z}, T-t) \\ & + z e^{r(T-t)}\Upsilon(\mu_c - \lambda\zeta + \frac{1}{2}\sigma^2, \sigma, \lambda^*, \eta_1^*; \ln\frac{K}{z}, T-t), \end{aligned} \tag{5.19}$$

where $\lambda^* = \lambda(1+\zeta)$, $\eta_1^* = \eta_1 - 1$ and the function Υ is introduced in Chapter 3 as follows

$$\begin{aligned} \Upsilon(\mu,\sigma,\lambda,\eta_1;a,T) =& \mathsf{P}^\mu(Z_T \geq \ln a) \\ =& \pi_0(T)\Phi(-\frac{a-\mu T}{\sigma\sqrt{T}}) \\ & + \frac{e^{\frac{T(\sigma\eta_1)^2}{2}}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{\infty} \pi_n(T)(\sigma\sqrt{T}\eta_1)^n I_{n-1}(a-\mu T, -\eta_1, -\frac{1}{\sigma\sqrt{T}}, -\sigma\eta_1\sqrt{T}), \end{aligned} \tag{5.20}$$

The following function are needed in Υ

$$\pi_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \tag{5.21}$$

$$I_n(c,\alpha,\beta,\omega) = \int_c^\infty e^{\alpha x} Hh_n(\beta x - \omega)dx, \tag{5.22}$$

$$Hh_n(x) = \int_x^\infty Hh_{n-1}(y)dy = \frac{1}{n!}\int_x^\infty (t-x)^n e^{-\frac{t^2}{2}}dt, \tag{5.23}$$

$$Hh_{-1}(x) = e^{-\frac{x^2}{2}} = \sqrt{2\pi}\varphi(x), \tag{5.24}$$

$$Hh_0(x) = \sqrt{2\pi}\Phi(-x). \tag{5.25}$$

where $\varphi(x)$ and $\Phi(x)$ are the density function and the CDF for standard normal distribution given by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \tag{5.26}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy. \tag{5.27}$$

4. Standard hedging arguments based on self-financing portfolios imply that the arbitrage-free price of the British call option is given by

$$V = \sup_{0 \le \tau \le T} \tilde{\mathsf{E}} \left[e^{-r\tau} \mathsf{E}^{\mu_c} \big((Z_T - K)^+ | \mathcal{F}_\tau \big) \right], \tag{5.28}$$

where the supremum is taken over all stopping times τ of Z with values in $[0, T]$ and $\tilde{\mathsf{E}}$ is taken with respect to the risk-neutral probability measure $\tilde{\mathsf{P}}$ specified before. Making use of the payoff function G^{μ_c} defined in (5.17) and the optional sampling theorem, upon enabling the process Z to start at any point z in $(0, \infty)$ at any time $t \in [0, T]$, we see that the problem (5.28) extends as follows

$$V(t, z) = \sup_{0 \le \tau \le T-t} \tilde{\mathsf{E}}_{t,z} \left[e^{-r\tau} G^{\mu_c}(t + \tau, Z_{t+\tau}) \right], \tag{5.29}$$

where the supremum is taken over all stopping times τ of Z with values in $[0, T - t]$ and $\tilde{\mathsf{E}}_{t,z}$ is taken with respect to the equivalent martingale measure $\tilde{\mathsf{P}}_{t,z}$ under which $Z_t = z$. Since the supremum in (5.29) is attained at the first entry time of Z to the closed set where V equals G^{μ_c}, and the Law$(Z|\tilde{\mathsf{P}})$ is the same as Law$(Z^r|\mathsf{P})$, it follows that

$$V(t, z) = \sup_{0 \le \tau \le T-t} \mathsf{E} \left[e^{-r\tau} G^{\mu_c}(t + \tau, z Z^r_\tau) \right], \tag{5.30}$$

for $t \in [0, T]$ and $z \in (0, \infty)$ where the supremum is taken as in (5.29) above and the process Z^r is defined in (5.6) with $Z^r_0 = 0$.

5. We see from (5.17) that

$$z \mapsto G^{\mu_c}(t, z) \quad \text{is convex} \tag{5.31}$$

and strictly increasing on $(0, \infty)$ with $G^{\mu_c}(t, 0) = 0$ and $G^{\mu_c}(t, \infty) = \infty$ for any $t \in [0, T]$ given and fixed. One also sees that $G^{\mu_c}(T, z) = (z - K)^+$ for $z \in (0, \infty)$ showing that the British call payoff coincides with the European call payoff at the time of maturity. Moreover, if $\mu_c \le 0$ then from (5.17) and (5.18) we see that

$$G^{\mu_c}(t, z) < e^{-r(T-t)} \mathsf{E}(z Z^r_{T-t} - K)^+ \tag{5.32}$$

for $t \in [0, T)$ and $z \in (0, \infty)$ where the term on the right-hand side can be recognised as the payoff obtained by choosing $\tau = T - t$ in (5.30). From the strict inequality in (5.32), and the fact that the supremum in (5.32) is attained at the first entry time of X to the set where V equals G^{μ_c}, we therefore see that it is not optimal to stop in (5.30) before the time of maturity when $\mu_c \leq 0$ (i.e. the optimal stopping time τ in (5.30) equals $T-t$). This establishes the claim about the left-hand inequality following (5.13) above. Denoting by $V_E(t, z)$ the arbitrage-free price of the European call option (given by the term on the right-hand side of (5.32) above) it follows therefore using (5.17) and (5.18) that $V(t, z) = V_E(t, z)$ if $\mu_c \leq 0$ and $V(t, z) > V_E(t, z)$ if $\mu_c > 0$ for all $t \in [0, T)$ and $z \in (0, \infty)$. In particular, this confirms that the British call option is more expensive than the European call option whenever (5.13) holds. Finally, from (5.30) and (5.31) we easily find that

$$z \mapsto V(t, z) \quad \text{is convex} \tag{5.33}$$

and increasing on $(0, \infty)$ with $V(t, 0) = 0$ and $V(t, \infty) = \infty$ for any $t \in [0, T)$ given and fixed, and $V(T, z) = (z - K)^+$ for $z \in (0, \infty)$. In this sense the value function of the British call option is similar to the value function of the European call option. The most important technical difference is that whilst the American call boundary without dividend is trivial (non-existing), this is not necessarily the case for the British call boundary b.

6. Although $G^{\mu_c}(t, z)$ is more complicated than the European or American put payoff function $z \mapsto (K - z)^+$, it is actually a $C^{1,2}$ function which means that we can directly apply Itô's formula to it. Thus to gain a deeper insight into the solution to the optimal stopping problem (5.30), the Itô's formula yields

$$e^{-rs} G^{\mu_c}(t + s, Z^r_{t+s}) = G^{\mu_c}(t, z) + \int_0^s e^{-ru} H^{\mu_c}(t + u, Z^r_{t+u}) du + M^1_s + M^2_s, \tag{5.34}$$

where

$$M^1_s = \sigma \int_0^s e^{-ru} Z^r_{t+u} G^{\mu_c}_z(t + u, X^r_{t+u}) dW_u, \tag{5.35}$$

$$\begin{aligned} M^2_s = & \sum_{0 \leq u \leq s}^{\Delta Z^r_{t+u} \neq 1} \Big(e^{-ru} G^{\mu_c}(t + u, Z^r_{t+u}) - e^{-ru} G^{\mu_c}(t + u, Z^r_{t+u-}) \Big) \\ & - \lambda \int_0^s e^{-ru} \int_0^\infty [G^{\mu_c}(t + u, Z^r_{t+u-} e^y) - G^{\mu_c}(t + u, Z^r_{t+u-})] f_Y(y) dy du, \end{aligned} \tag{5.36}$$

and the function $H^{\mu_c} = H^{\mu_c}(t,z)$ is given by

$$H^{\mu_c} = G_t^{\mu_c} + (r-\lambda\zeta)zG_z^{\mu_c} + \frac{\sigma^2}{2}z^2G_{zz}^{\mu_c} - rG^{\mu_c} + \lambda\int_0^\infty [G^{\mu_c}(t,ze^y) - G^{\mu_c}(t,z)]f_Y(y)dy. \tag{5.37}$$

Similar to the derivation in Chapter 3, we can see that M_s^1 and M_s^2 are two martingales for $s \in [0, T-t]$ with $t \in [0,T)$. By the optional sampling theorem we therefore find

$$\mathsf{E}[e^{-r\tau}G^{\mu_c}(t+\tau, zZ_\tau^r)] = G^{\mu_c}(t,z) + \mathsf{E}[\int_0^\tau e^{-r\tau}H^{\mu_c}(t+u, zZ_u^r)du] \tag{5.38}$$

for all stopping times τ of Z^r given by (5.6) with values in $[0, T-t]$ with $t \in [0,T)$ and $z \in (0,\infty)$ given and fixed. On the other hand, it is clear from (5.17) that the payoff function G^{μ_c} satisfies the Kolomogorov backward equation

$$G_t^{\mu_c} + \mathbb{L}_{Z^{\mu_c}}G^{\mu_c} = 0, \tag{5.39}$$

where $\mathbb{L}_{Z^{\mu_c}}$ is the infinitesimal generator similarly defined in (5.7). Thus from (5.37) we see that

$$\begin{aligned} H^{\mu_c} =& G_t^{\mu_c} + \mathbb{L}_{Z^{\mu_c}}G^{\mu_c} + (r-\mu_c)zG_z^{\mu_c} - rG^{\mu_c} \\ =& (r-\mu_c)zG_z^{\mu_c} - rG^{\mu_c}. \end{aligned} \tag{5.40}$$

This representation shows in particular that if $\mu_c \geq r$ then $H^{\mu_c} < 0$ so that from (5.38) we see that it is always optimal to exercise immediately as pointed out following (5.13) above. Similar to the derivation in Peskir and Samee [43], by inserting the expression of G^{μ_c} from (5.19) into (5.40), it can be shown that there exists a continuous function $h : [0,T] \to \mathbb{R}$ such that

$$H^{\mu_c}(t, h(t)) = 0 \tag{5.41}$$

for all $t \in [0,T)$ with $H^{\mu_c}(t,z) < 0$ for $z > h(t)$ and $H^{\mu_c}(t,z) > 0$ for $z < h(t)$ when $t \in [0,T)$ is given and fixed. In view of (5.38) this implies that no point (t,z) in $[0,T) \times (0,\infty)$ with $z < h(t)$ is a stopping point. Likewise, it is also clear and can be verified that if $z > h(t)$ and $t < T$ is sufficiently close to T then it is optimal to stop immediately. This shows that the optimal stopping boundary b separating the continuation set from the stopping set satisfies $b(T) = h(T)$ and this value equals rK/μ_c as can be seen from analysis of the boundary behaviour at the maturity in Chapter 3.

7. Standard Markovian arguments lead to the following free-boundary problem (see Peskir and Shiryaev [44] for more detailed classification) for the value function $V = V(t,z)$ and the optimal stopping boundary $b = b(t)$ to be determined:

$$V_t + \mathbb{L}_{Z^r} V = rV \qquad \text{for} \quad z > b(t) \quad \text{and} \quad t \in [0,T), \tag{5.42}$$

$$V(t,z) = G^{\mu_c}(t,z) \qquad \text{for} \quad z = b(t) \quad \text{and} \quad t \in [0,T], \tag{5.43}$$

$$V_z(t,x) = G^{\mu_c}_z(t,z) \qquad \text{for} \quad z = b(t) \quad \text{and} \quad t \in [0,T], \tag{5.44}$$

$$V(T,z) = (z-K)^+ \qquad \text{for} \quad z \geq b(T) = \frac{rK}{\mu_c}, \tag{5.45}$$

$$V(t,\infty) = 0 \qquad \text{for} \quad t \in [0,T], \tag{5.46}$$

where we also set $V(t,z) = G^{\mu_c}(t,z)$ for $z > b(t)$ and $t \in [0,T]$. It can be shown that this free-boundary problem has a unique solution V and b which coincide with the value function (5.30) and the optimal stopping boundary respectively (see Peskir and Shiryaev [44]). This means that the continuation set is given by $C = \{V > G^{\mu_c}\} = \{(t,z) \in [0,T) \times (0,\infty) | z < b(t)\}$ and the stopping set is given by $D = \{V = G^{\mu_c}\} = \{(t,z) \in [0,T) \times (0,\infty) | z \geq b(t)\} \cup \{(T,z) | z \in (0, b(T))\}$ so that the optimal stopping time in (5.30) is given by

$$\tau_b = \inf\{t \in [0,T] | Z^r_t \geq b(t)\}. \tag{5.47}$$

This stopping time represents the rational exercise strategy for the British call option and plays a key role in financial analysis of the option.

Depending on the size of the contract drift μ_c satisfying (5.13) we distinguish three different regimes for the position and shape of the optimal stopping boundary b. Firstly, when $\mu_c \in (0,r)$ is close to 0 then b is an decreasing function of time. Secondly, if $\mu_c \in (0,r)$ is close to r then b is a skewed S-shaped function of time. Third, there is an intermediate case where b can take either of the two shapes depending on the size of T. Moreover, the second regime becomes dominant when T is large in the sense that $b(0)$ tends to 0 as $T \to \infty$.

The connection between these three regimes is similar to the case of British put option under geometric Brownian motion which was fully discussed in Peskir and Samee [43]. It also should be note that b is not necessarily a monotone function of time, which makes this analysis more complicated in comparison with the American put option.

5.4 The Arbitrage-Free Price and the Rational Exercise Boundary

In this section we derive a closed form expression for the arbitrage-free price V in terms of the rational exercise boundary b and show that the rational exercise boundary b itself can be characterised as the unique solution to a nonlinear integral equation.

Peskir proposed a series of change-of-variable formulas with local time for semimartingales with jumps in [39]. By the properties of the value function V mentioned in the last section, it is easy to verify that the jump-diffusion process Z_t^r, the condition of [39, Theorem 3.1] hold. Applying this change-of-variable formula to $e^{-rs}V(t+s, Z_{t+s}^r)$ in terms of s with t and $Z_t^r = z$ given and fixed, we can have

$$e^{-rs}V(t+s, Z_{t+s}^r) = V(t,z) + \int_0^s e^{-ru}(V_t + \mathbb{L}_{Z^r}V - rV)(t+u, Z_{t+u}^r)du + M_s^1 + M_s^2, \tag{5.48}$$

where

$$M_s^1 = \sigma \int_0^s e^{-ru} z Z_u^r V_z(t+u, zZ_u^r) dW_u, \tag{5.49}$$

$$\begin{aligned} M_s^2 = & \sum_{0\le u\le s}^{\Delta Z_{t+u}^r \neq 1} \Big(e^{-ru}V(t+u, Z_{t+u}^r) - e^{-ru}V(t+u, Z_{t+u-}^r) \Big) \\ & - \lambda \int_0^s e^{-ru} \int_0^\infty [V(t+u, Z_{t+u-}^r e^y) - V(t+u, Z_{t+u-}^r)] f_Y(y) dy du, \end{aligned} \tag{5.50}$$

Note that there is no local time term in (5.48) due to the smooth-fit property (5.44) of the value function $V(t,z)$. Following the similar derivation path for American call options employed in Chapter 3, it can be easily proved that M_s^1 and M_s^2 are martingales under P. Using the relation given by (5.42), we can simplify the equation (5.48) as follows

$$\begin{aligned} e^{-rs}V(t+s, Z_{t+s}^r) = & V(t,z) \\ & + \int_0^s e^{-ru}(V_t + \mathbb{L}_{Z^r}V - rV)(t+u, Z_{t+u}^r) I(Z_{t+u}^r > b(t+u)) du \\ & + M_s^1 + M_s^2. \end{aligned} \tag{5.51}$$

Recall that only positive jumps are allowed in the PEJD processes, then the existence of the indicator function $I(Z_{t+u}^r > b(t+u))$ will allow us to replace $(V_t + \mathbb{L}_{Z^r}V - rV)$ directly by $(G_t^{\mu_c} + \mathbb{L}_{Z^r}G^{\mu_c} - rG^{\mu_c})$, since no global integral term will be introduced by the operator $\mathbb{L}_{Z^r}$. Now replacing s by $T-t$ in (5.51), using that $V(T,z) = G^{\mu_c}(T,z) = (z-K)^+$, taking $\mathsf{E}_{t,z}$ on both sides and applying the optional sampling theorem, we get

$$V(t,z) = e^{-r(T-t)}G^r(t,z) - \int_0^{T-t} e^{-ru}\mathsf{E}_{t,z}[H^{\mu_c}(t+u, Z^r_{t+u})I(Z^r_{t+u} > b(t+u))]du, \tag{5.52}$$

where $G^r(t,z) = \mathsf{E}_{t,z}(Z^r_T - K)^+$ is defined similar to $G^{\mu_c}(t,z)$ in (5.17). And we will make use of the following function below

$$J(t,z,v,w) = -e^{-r(v-t)}\int_0^w H^{\mu_c}(v,y)f(v-t,z,y)dy, \tag{5.53}$$

for $t \in [0,T)$, $z > 0$, $v \in (t,T)$ and $y > 0$. And the function $y \mapsto f(v-t,z,y)$ is the probability density function of zZ^r_{v-t} given by

$$f(v-t,z,y) = -\frac{d\Upsilon(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_1; \ln\frac{y}{z}, v-t)}{dy}, \tag{5.54}$$

where the function Υ is defined in (5.20).

Now the main result for the EEP representation and the optimal stopping boundary for an British call option under the PEJD processes can be stated as follows.

Theorem 5.1. *The arbitrage-free price of the British call options admits the following early exercise premium representation under a positive exponential jump diffusion process*

$$V(t,z) = e^{-r(T-t)}G^r(t,z) + \int_t^T J(t,z,v,b(v))dv \tag{5.55}$$

for all $(t,z) \in [0,T] \times (0,\infty)$, where the first term is the arbitrage-free price of the European call option and the second term is the early-exercise premium.

The rational exercise boundary of the British call option can be characterised as the unique continuous solution $b : [0,T] \to \mathbb{R}_+$ to the nonlinear integral equation

$$G^{\mu_c}(t,b(t)) = e^{-r(T-t)}G^r(t,b(t)) + \int_t^T J(t,b(t),v,b(v))dv \tag{5.56}$$

satisfying $b(t) \geq h(t)$ for all $t \in [0,T]$ where h is defined by (5.41) above.

The analysis from (5.48) to (5.52) establishes the existence of the solution to (5.55), thus only the uniqueness of the optimal stopping boundary b remains to be proved.

Proof. Take any continuous function $c : [0,T] \to \mathbb{R}$ which solves (5.56) and satisfies $c(t) \geq h(t)$ for all $t \in [0,T]$. Motivated by the representation (5.52) above define the function $U^c : [0,T] \times (0,\infty) \to \mathbb{R}$ by setting

$$\begin{aligned} U^c(t,x) =& e^{-r(T-t)}\mathsf{E}[G^{\mu_c}(T, zZ^r_{T-t})] \\ &- \int_0^{T-t} e^{-ru}\mathsf{E}[H^{\mu_c}(t+u, zZ^r_u)I(zZ^r_u > c(t+u))]du, \end{aligned} \tag{5.57}$$

for $(t,z) \in [0,T]\times(0,\infty)$. Observe that c solving (5.52) means exactly that $U^c(t,c(t)) = G^{\mu_c}(t,c(t))$ for all $t \in [0,T]$.

1. We show that $U^c(t,z) = G^{\mu_c}(t,z)$ for all $(t,z) \in [0,T] \times (0,\infty)$ such that $z \geq c(t)$. For this, take any such (t,z) and note that the Markov property of Z implies that the stochastic process $M(s, Z^r_s)$ which defined by

$$M(s, Z^r_s) = e^{-rs}U^c(t+s, zZ^r_s) - \int_0^s e^{-ru}\mathsf{E}[H^{\mu_c}(t+u, zZ^r_u)I(zZ^r_u > c(t+u))]du, \tag{5.58}$$

is a continuous martingale under P for $s \in [0, T-t]$. Consider the stopping time

$$\sigma_c = \inf\{s \in [0, T-t] | zZ^r_s \leq c(t+s)\}, \tag{5.59}$$

under P. Since $U^c(t,c(t)) = G^{\mu_c}(t,c(t))$ for all $t \in [0,T]$ and $U^c(T,z) = G^{\mu_c}(T,z)$ for all $z > 0$ we see that $U^c(t+\sigma_c, zZ_{\sigma_c}) = G^{\mu_c}(t+\sigma_c, zZ_{\sigma_c})$. Replacing s by σ_c in (5.58), taking E on both sides and applying the optional sampling theorem, we find that

$$\begin{aligned} U^c(t,z) =& \mathsf{E}[e^{-r\sigma_c}U^c(t+\sigma_c, zZ_{\sigma_c})] \\ &- \mathsf{E}\left(\int_0^{\sigma_c} e^{-ru}H^{\mu_c}(t+u, zZ^r_u)I(zZ^r_u > c(t+u))du\right) \\ =& \mathsf{E}[e^{-r\sigma_c}G^{\mu_c}(t+\sigma_c, zZ_{\sigma_c})] - \mathsf{E}\left(\int_0^{\sigma_c} e^{-ru}H^{\mu_c}(t+u, zZ^r_u)du\right) = G^{\mu_c}(t,z), \end{aligned} \tag{5.60}$$

where the last equality we use (5.38). This shows that U^c equals G^{μ_c} below c as claimed.

2. We show that $U^c(t,z) \leq V(t,z)$ for all $(t,z) \in [0,T] \times (0,\infty)$. For this, take any such (t,z) and consider the stopping time

$$\tau_c = \inf\{s \in [0, T-t] | zZ^r_s \geq c(t+s)\}, \tag{5.61}$$

under P. We claim that $U^c(t+\tau_c, zZ_{\tau_c}) = G^{\mu_c}(t+\tau_c, zZ_{\tau_c})$. Indeed, if $z \ge c(t)$ then $\tau_c = 0$ so that $U^c(t,z) = V(t,z)$ by previous step. On the other hand, if $z < c(t)$ then the claim follows since $U^c(t, c(t)) = G^{\mu_c}(t, c(t))$ for all $t \in [0,T]$ and $U^c(T,z) = G^{\mu_c}(T,z)$ for all $z > 0$. Replacing s by τ_c in (5.58), taking E on both sides and applying the optional sampling theorem, we find that

$$U^c(t,z) = \mathsf{E}[e^{-r\tau_c} U^c(t+\tau_c, zZ_{\tau_c})] - \mathsf{E}\left(\int_0^{\tau_c} e^{-ru} H^{\mu_c}(t+u, zZ_u^r) I(zZ_u^r > c(t+u)) du \right) \tag{5.62}$$

$$= \mathsf{E}[e^{-r\tau_c} G^{\mu_c}(t+\tau_c, zZ_{\tau_c})] \le V(t,z),$$

where in the second equality we used the definition of τ_c. This shows that $U^c \le V$ as claimed.

3. We show that $c(t) \le b(t)$ for all $t \in [0,T]$. For this, suppose this there exists $t \in [0,T)$ such that $c(t) > b(t)$. Take any $z \le c(t)$ and consider the stopping time

$$\sigma_b = \inf\{ s \in [0, T-t] | zZ_s^r \le b(t+s) \}, \tag{5.63}$$

under P. Replacing s with σ_b in (5.51) and (5.58), taking E on both sides of these identities and applying the optional sampling theorem, we find

$$\mathsf{E}[e^{-r\sigma_b} V(t+\sigma_b, zZ_{\sigma_b}^r)] = V(t,z) + \mathsf{E}\left(\int_0^{\sigma_b} e^{-ru} H^{\mu_c}(t+u, zZ_u^r) du \right) \tag{5.64}$$

$$\mathsf{E}[e^{-r\sigma_b} U^c(t+\sigma_b, zZ_{\sigma_b}^r)] = U^c(t,z) + \mathsf{E}\left(\int_0^{\sigma_b} e^{-ru} H^{\mu_c}(t+u, zZ_u^r) I(zZ_u^r > c(t+u)) du \right). \tag{5.65}$$

Since $z \ge c(t)$ we see by step **1** above that $U^c(t,z) = G^{\mu_c}(t,z) = V(t,z)$ where the last equality follows since z lies above $b(t)$. Moreover, by Step **2** above we know that $U^c(t+\sigma_b, zZ_{\sigma_b}) \le V(t+\sigma_b, zZ_{\sigma_b})$ so that (5.64) and (5.65) imply that

$$\mathsf{E}\left(\int_0^{\sigma_b} e^{-ru} H^{\mu_c}(t+u, zZ_u^r) I(zZ_u^r \le c(t+u)) du \right) \ge 0. \tag{5.66}$$

The fact that $c(t) > b(t)$ and the continuity of the functions c and b imply that there exists $\epsilon > 0$ sufficiently small such that $c(t+u) > b(t+u)$ for all $u \in [0,\epsilon]$. Consequently the P-probability of $(zZ_u^r)_{0 \le u \le \epsilon}$ spending a strictly positive amount of time in this set before hitting b is strictly positive. Combine with the fact that b lies above h this forces the expectation in (5.66) to be strictly negative and provides a contradiction. Hence $c \le b$ as claimed.

4. We show that $b(t) = c(t)$ for all $\tilde{\in}[0,T]$. For this, suppose that there exists $t \in [0,T)$ such that $c(t) < b(t)$. Take any $z \in (c(t), b(t))$ and consider the stopping time

$$\tau_b = \inf\{s \in [0, T-t] | zZ^r_s \ge b(t+s)\}, \tag{5.67}$$

under P. Replacing s with τ_b in (5.51) and (5.58), taking E on both sides of these identities and applying the optimal sampling theorem, we find

$$\mathsf{E}[e^{-r\tau_b}V(t+\tau_b, zZ^r_{\tau_b})] = V(t,z) \tag{5.68}$$

$$\mathsf{E}[e^{-r\tau_b}U^c(t+\tau_b, zZ^r_{\tau_b})] = U^c(t,z) + \mathsf{E}\bigg(\int_0^{\tau_b} e^{-ru}H^{\mu_c}(t+u, zZ^r_u)I(zZ^r_u > c(t+u))du\bigg). \tag{5.69}$$

Since $c \le b$ by step **3** above and U^c equals G^{μ_c} above c by step **1** above, we see that $U^c(t+\tau_b, zZ_{\tau_b}) = G^{\mu_c}(t+\tau_b, zZ_{\tau_b}) = V(t+\tau_b, zZ_{\tau_b})$ where the last equality follows since V equals G^{μ_c} above b. Moreover, by step **2** we know that $U^c \le V$ so that (5.68) and (5.69) imply that

$$\mathsf{E}\bigg(\int_0^{\tau_b} e^{-ru}H^{\mu_c}(t+u, zZ^r_u)I(zZ^r_u > c(t+u))du\bigg) \ge 0. \tag{5.70}$$

But then as in step **3** above the continuity of the functions c and b combined with the fact that c lies below h forces the expectation in (5.70) to be strictly negative and provides a contradiction. Thus $c = b$ as claimed and the proof is complete. □

5.5 Financial Analysis for British Call Options

In this section, we focus on the practical features of the British call option under positive exponential jump-diffusion processes. We compare it with the American call option that is universally traded. We address the problem as to what the return would be if the price of the underlying asset enters the given region as a given time. We use "skeletal" approach of Peskir and Samee [43], with a similar parameter set so that we can make reference to the results by Peskir and Samee [43].

1. As shown in Figure 5.1, the rational exercise strategy of the British call option under positive exponential jump-diffusion processes varies with different contract drift

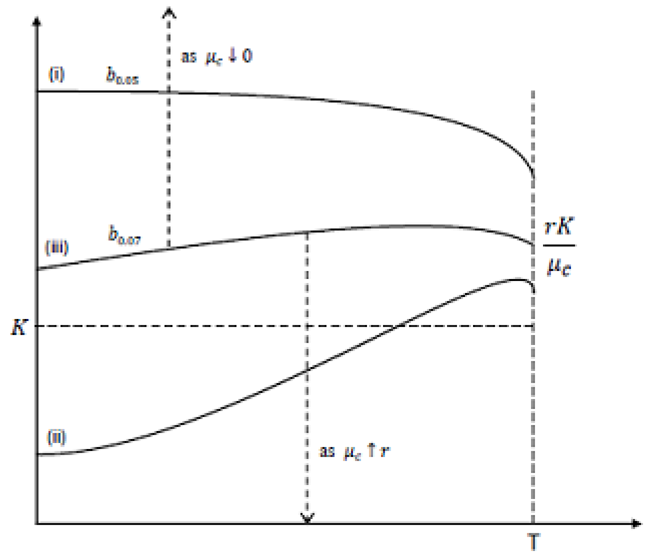

Figure 5.1: A computer drawing showing the rational exercise boundaries of the British call option under the positive exponential jump-diffusion processes with $K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\lambda = 1.0$, $\eta_1 = 2.3$ when the contract drift μ_c equals 0.05 and 0.07.

μ_c. Where depending on the value of μ_c, the boundary b is either an decreasing function of time, a skewed S-shaped function of time, or an intermediate case where b can take either of the two shapes depending on the size of T.

In Figure 5.1, we witness the same three regimes for the optimal stopping boundary b as Peskir and Samee [43]. In particular, as μ_c gets closer to r, we see that $b(0) \downarrow 0$ making the stock price always above $b(0)$. It would be optimal to stop immediately, making the buyer overprotected. On the other hand, when $\mu_c \uparrow r$, it is not optional to exercise the option before time T, reducing the British call option to the European call option. Therefore, the contract drift μ_c should not be too close to r (since in this case the buyer is overprotected) and should not be too close to zero (since in this case the British call option effectively reduces to the European call option).

2. In the numerical example below (see Table 5.1, 5.2 and 5.3), the parameter values have been chosen to present the practical features of the British call option. We assume that the initial stock price equals the strike price $K = 10$, the maturity time $T = 1$, the interest rate $r = 0.1$, the volatility coefficient $\sigma = 0.4$, the intensity parameter $\lambda = 1.0$, and the rate parameter of the positive exponential random variable $\eta_1 = 2.3$. By considering the set of parameters above, the arbitrage-free price of the

Time (months)	0	2	4	6	8	10	12
Exercise at K with $\mu_c = 0.07$	99%	86%	74%	61%	47%	31%	0%
Exercise at K with $\mu_c = 0.05$	92%	81%	70%	58%	46%	30%	0%
Exercise at 9 with $\mu_c = 0.07$	66%	55%	45%	34%	23%	09%	0%
Exercise at 9 with $\mu_c = 0.05$	61%	52%	42%	32%	22%	09%	0%
Exercise at 8 with $\mu_c = 0.07$	40%	33%	23%	15%	07%	0.4%	0%
Exercise at 8 with $\mu_c = 0.05$	37%	29%	21%	14%	06%	0%	0%
Exercise at 7 with $\mu_c = 0.07$	21%	15%	08%	03%	0.3%	0%	0%
Exercise at 7 with $\mu_c = 0.05$	19%	13%	08%	03%	0.2%	0%	0%
Exercise at 6 with $\mu_c = 0.07$	11%	07%	04%	02%	0.1%	0%	0%
Exercise at 6 with $\mu_c = 0.05$	10%	07%	04%	02%	0.1%	0%	0%
Exercise at 5 with $\mu_c = 0.07$	04%	02%	01%	0.3%	0%	0%	0%
Exercise at 5 with $\mu_c = 0.05$	03%	02%	01%	0.3%	0%	0%	0%

Table 5.1: Returns observed upon exercising the British put option at and below the strike price K. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,z)/100 = G^{\mu_c}(t,z)/V(0,K)$. The parameter set is the same as in Figure 5.1 above ($K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\lambda = 1.0$, $\eta_1 = 2.3$) and the initial stock price equals K.

British call option using the EEP representation (5.55) is 2.140 if $\mu_c = 0.05$ and 2.172 if $\mu_c = 0.07$. The price of the European call option under exponential jump-diffusion processes is 2.124 that can be evaluated using the formula in [25]. We get that the closer the contract drift gets to r, the protection feature is stronger and the option price is more expensive as stated above. Moreover, in terms of the price sizes it can be seen that this example is not isolated since the price of the British call stays very close to the price of the European call option unless the contract drift μ_c is unrealistically close to the interest rate r.

3. Table 5.1 illustrates the power of the protection feature in practice. For instance, if the stock price is stock price is at K halfway to maturity (clearly representative of unfavourable price movements) then the British call holder can exercise immediately to a payoff with a reimbursement of $58\% - 61\%$ of his original investment. However, compare this with a "formal American call" holder who in this contingency is out-of-the-money and wold receive zero payoff upon exercise. We see that the size of the reimbursement also varies with the contract drift as analysed in the former paragraph. When the contract drift gets closer to r, the option holder gets more protection as well as greater reimbursement.

4. Now we only focus on the British call option with $\mu_c = 0.07$ since in this

Time (months)	0	2	4	6	8	10	12
Exercise at 20 (British call)	551%	538%	525%	512%	498%	489%	477%
Exercise at 20 (European call)	531%	523%	515%	506%	495%	485%	478%
Exercise at 18 (British call)	449%	436%	424%	412%	401%	390%	380%
Exercise at 18 (European call)	435%	426%	417%	409%	400%	392%	384%
Exercise at 16 (British call)	349%	337%	325%	313%	302%	292%	282%
Exercise at 16 (European call)	340%	331%	321%	312%	302%	294%	285%
Exercise at b (British call)	228%	247%	264%	277%	282%	268%	199%
Exercise at b (European call)	225%	244%	262%	277%	283%	270%	201%
Exercise at 12 (British call)	165%	153%	141%	128%	114%	99%	88%
Exercise at 12 (European call)	164%	153%	142%	130%	116%	101%	88%
Exercise at 11 (British call)	125%	114%	102%	89%	74%	58%	39%
Exercise at 11 (European call)	125%	115%	103%	91%	77%	60%	39%

Table 5.2: Returns observed upon exercising the British call option (with μ_c=0.07) above the strike price K compared with returns received upon selling the European call in the same contingency. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,z)/100 = G^{\mu_c}(t,z)/V(0,K)$ and $R_E(t,z)/100 = V_E(t,z)/V_E(0,K)$ respectively. The parameter set is the same as in Figure 5.1 above ($K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\lambda = 1.0$, $\eta_1 = 2.3$) and the initial stock price equals K.

case the rational exercise strategy is closer to the strike price K and this makes the comparison more interesting economically. In Table 5.2, we compare the returns of the British call option and the European version option as the stock price is moving favourably. The result generally indicates that the British option outperforms the European option except a few points. In Table 5.3, we compare the protection feature of the British option with the reimbursement of European option if the holder of the latter can choose to sell his option freely without friction. We see in Table 5.3 that the protection feature of the British call option is remarkably similar to the protection afforded to the European call holder by his ability to sell. However, in the real market the option holder's ability to sell the option is affected by many exogenous factors such as the friction costs, taxes, the liquidity of the market and so on. Therefore, from this point of view it makes the British style option more attractive since the protection feature to the British option is intrinsic and endogenous.

Time (months)	0	2	4	6	8	10	12
Exercise at K (British put)	97%	85%	73%	61%	47%	31%	0%
Selling at K (American put)	98%	87%	75%	63%	49%	32%	0%
Exercise at 9 (British put)	66%	55%	45%	34%	23%	09%	0%
Selling at 9 (American put)	67%	57%	46%	36%	24%	10%	0%
Exercise at 8 (British put)	40%	32%	23%	15%	07%	0.4%	0%
Selling at 8 (American put)	42%	33%	24%	16%	08%	0.6%	0%
Exercise at 7 (British put)	21%	15%	08%	03%	0.4%	0%	0%
Selling at 7 (American put)	22%	16%	08%	04%	0.4%	0%	0%
Exercise at 6 (British put)	09%	04%	02%	01%	0%	0%	0%
Selling at 6 (American put)	09%	05%	03%	01%	0%	0%	0%
Exercise at 5 (British put)	03%	02%	01%	0.3%	0%	0%	0%
Selling at 5 (American put)	03%	02%	01%	0.4%	0.1%	0%	0%

Table 5.3: Returns observed upon exercising the British call option (with μ_c=0.07) at and below the strike price K compared with returns received upon selling the European call option in the same contingency. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,z)/100 = G^{\mu_c}(t,z)/V(0,K)$ and $R_E(t,z)/100 = V_E(t,z)/V_E(0,K)$ respectively. The parameter set is the same as in Figure 5.1 above ($K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\lambda = 1.0$, $\eta_1 = 2.3$) and the initial stock price equals K.

Chapter 6

The American Knock-Out Put Option for Negative Exponential Jump Diffusion Processes

6.1 Introduction

Barrier options on stocks have been traded in the OTC market for over four decades. Although barrier options are developed in comparison to other exotic options in terms of pricing and other features, they are not easily accessible to the general public. The relatively inexpensive price of barrier options has led to their extensive use by investors in managing risks related to commodities, FX and interest rate exposures. Despite this seemingly attractive feature, an investor buying a barrier option is in fact taking a risk much greater than that of a vanilla option since the option can be terminated or "knock-out" at anytime prior to maturity or, conversely, inactive for the whole lifetime of the option as a result of not being "knock-in".

There are two types of barrier options: knock-in and knock-out, both are exactly like vanilla options except they are activated or terminated respectively once the price of the underlying asset hits a certain level. Within these two categories, we can have: up barriers, where the barrier level is above the spot initial price of the asset and down barriers, where the barrier level is below the spot price. If an investor expects the price of the underlying asset to fluctuate strongly, knock-in barriers can lead to a higher profit as a result of the lower cost in buying the option. Similarly, the investor can reduce costs with a knock-out barrier if he expects the price of the underlying asset to remain within stable price range. Barrier options are path-dependent and can take either American or European form. Barrier options sometimes come with a rebate, which is cash paid out to the option holder if the option expires worthless as a result of the option being knocked-out or failing to knock-in. Up barriers can be both above or below the strick price of the option, although the latter is better known as a reverse barrier option.

There are a number of established methods for pricing European barrier options under the geometric Brownian motions and these have been discussed extensively in the literature (see for example [12], [13], [50] and [49]). Also under the geometric Brownian motions, Katatzas and Wang [20] obtained a closed-form expression for the price and optimal boundary of the perpetual American put option in the presence of an up-and-out barrier using the method of variational inequalities to solve the problem explicitly. Chang et al.[9] provided a numerical approximation scheme for a finite horizon American up-and-out barrier option. Gao, Huang and Subrahmanyam [15] proposed an early exercise premium representation for the American knock-out call and put options in terms of the optimal exercise boundary. Of particular interest to us is the work of Al-Fagih [1] who provided a rigorous derivation of the early exercise premium representation which does not require the assumption that the value function V is regular on the optimal exercise boundary b. In this section, we show that, under the negative exponential jump-diffusion processes, the optimal stopping boundary for the American knock-out put option with finite horizon can also be characterized as the unique solution to a nonlinear integral equation arising from the early exercise premium (EEP) representation. The proof of EEP representation uses the change-of-variable formula with local time on curves derived by Peskir [39]. The analytical result of the arbitrage-free price is based on the pricing formula for European barrier options under the double exponential jump-diffusion processes proposed by Kou and Wang in [27]. We also look briefly at a reverse up and out put option and prove that it is always optimal for the investor to exercise immediately.

The chapter is organised as follows. In Section 2, we formulate the American knock-out put option problem and show the effect of varying the barrier level above or below the strike on the optimal stopping problem. In Section 3, we present the main result and proof. The results are in line with results on the American put options under the negative exponential jump-diffusion processes in Chapter 2. Finally, using these results in Section 4 we conduct a financial analysis of the American knock-out put option making comparisons with the European knock-out and American put options under the NEJD processes.

6.2 Problem Formulation of the American Knock-Out Put Option

1. Here is a brief summary of the negative exponential jump-diffusion processes which was introduced in Chapter 2. Consider the financial market consisting a risky stock Z_t and riskless bond B_t:

$$\frac{dZ_t}{dZ_t-} = \mu\, dt + \sigma\, dW_t + d\Big(\sum_{i=1}^{N_t}(V_i - 1)\Big) \qquad (Z_0 = z), \tag{6.1}$$

$$dB_t = rB_t\, dt \qquad (B_0 = 1), \tag{6.2}$$

where μ is the personal appreciation drift of the stock, σ is the volatility, r is the risk-free interest rate, W_t is a standard Wiener process, N_t is a Poisson process with rate λ, and $\{V_i\}$ is a sequence of independent identically distributed nonnegative random variables such that $Y = \log V$ has a negative exponential distribution with the density:

$$f_Y(y) = \eta_2 e^{\eta_2 y} 1_{\{y<0\}} \qquad \eta_2 > 0, \tag{6.3}$$

The drift μ and the volatility σ are assumed to be constants; N_t, W_t and Y are assumed to be independent. And we defined another useful parameter ζ as follows

$$\zeta = \mathsf{E}[V] - 1 = \mathsf{E}[e^Y] - 1 = \frac{\eta_2}{\eta_2 + 1} - 1. \tag{6.4}$$

The negative exponential jump-diffusion process in a degenerate case of the double exponential jump-diffusion processes. The DEJD model was propose by Kou in [25] to incorporate the asymmetric leptokurtic features and the volatility smile which cannot be well explained by the Wiener process based model. For more advantages and useful properties of the DEJD model, please see the detailed introduction in Chapter 2. Also in Chapter 2, we showed that the two sided jump-diffusion processes will lead to an equation system containing additional non-local integral terms, which cannot provide analytical solutions for the price of a corresponding American put option. For a better financial comparison between the European and the American knock-out put option, here we assume there are only negative jumps on the path of the underlying asset.

2. For any interest rate $r > 0$, maturity $T > 0$, strike price K and barrier level ℓ, the arbitrage-free price of the American knock-out put option is given by

$$V = \sup_{0 \le \tau \le T} \mathsf{E}\left[e^{-r\tau}(K - Z_\tau)^+ I(M_\tau < \ell)\right], \tag{6.5}$$

where the supremum is taken over all stopping times τ of the corresponding risk-neutral process $Z = (Z_t)_{0 \le t \le T}$ solving

$$\frac{dZ_t}{dZ_{t-}} = (r - \lambda\zeta)\, dt + \sigma\, dW_t + d\,\Big(\sum_{i=1}^{N_t}(V_i - 1)\Big) \qquad (Z_0 = 1), \tag{6.6}$$

and $M = \max_{0 \le t \le T} Z_t$ denotes the maximum of the corresponding risk-neutral process Z. Recall the unique strong solution for (6.6) is given by

$$Z_t = e^{(r - \lambda\zeta - \frac{\sigma^2}{2})t + \sigma W_t + \sum_{i=1}^{N_t}(Y_i)}, \tag{6.7}$$

under the physical probability measure P for $t \in [0, T]$. The process Z is strong Markov with infinitesimal generator given by

$$(\mathbb{L}_Z F)(z) = \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 F}{\partial z^2}(z) + (r - \lambda\zeta)\frac{\partial F}{\partial z}(z) + \lambda \int_{-\infty}^{0} [F(ze^y) - F(z)] f_Y(y) dy, \tag{6.8}$$

for every $F \in C_b^2(\mathbb{R})$, where $C_b^2(\mathbb{R})$ denotes the set of all bounded, twice continuously differentiable functions with bounded derivatives.

The option knocking-out is financially equivalent to the price process being "stopped" as soon as it hits the barrier. We make use of the Markov structure of Z and introduce is a new process $Z^\ell = (Z^\ell_t)_{t \ge 0}$ which represents the process Z stopped once it hits the barrier level ℓ. We define this process by $Z^\ell_t = (Z_{t \wedge \tau_\ell})_{t \ge 0}$ where τ_ℓ is the first hitting time of the level ℓ, given by

$$\tau_\ell = \inf\{t \ge 0 | Z_t \ge \ell\}. \tag{6.9}$$

This is particular helpful since it means that we do not need to monitor the maximum process $M = \max_{0 \le t \le T} Z_t$. Moreover, the process Z^ℓ behaves exactly like the process Z for all times t before τ_ℓ which means most of the properties of Z follow naturally for Z^ℓ.

3. Summarising the preceding facts we see that the American knock-out put option problem (6.5) under the negative exponential jump-diffusion process, where $\ell > K$, reduces solve the following optimal stopping problem

$$V(t,z) = \sup_{0 \leq \tau \leq T-t} \mathsf{E}_{t,z}\left[e^{-r\tau} G(Z^{\ell}_{t+\tau})\right], \tag{6.10}$$

where the supremum is taken over all stopping times τ of Z^{ℓ} with values in $[0, T-t]$ and the expectation $E_{t,z}$ is taken with respect to the equivalent martingale measure $\mathsf{P}_{t,z}$ under which $Z_t = z < \ell$. The gain function G is given by

$$G(z) = (K - z)^{+}. \tag{6.11}$$

Using the same arguments as in [44] applied to the process Z^{ℓ} we may infer from general theory of optimal stopping for Markov processes that the continuation set equals $C = \{(t,z) \in [0,T) \times (0,\infty) | V(t,z) > G(z)\}$ and the stopping set is given by $D = \{(t,z) \in [0,T] \times (0,\infty) | V(t,z) = G(z)\}$. Also from [44] we can conclude that the optimal stopping time in (6.10) given by

$$\tau_D = \inf\{s \in [0, T-t] | Z^{\ell}_{t+s} \in D\}, \tag{6.12}$$

is optimal in our problem. It follows that there exists a function $b : [0,T] \to \mathbb{R}$ such that the continuation set equals

$$C = \{(t,z) \in [0,T) \times (0,\infty) | z > b(t)\} \tag{6.13}$$

and the stopping set $\bar{D}$ is the closure of the set

$$D = \{(t,z) \in [0,T) \times (0,\infty) | z < b(t)\} \tag{6.14}$$

joined with the remaining points (T, z) for $z \geq b(T)$. Thus the stopping time

$$\tau_b = \inf\{s \in [0, T-t] | Z^{\ell}_{t+s} \leq b(t+s)\}, \tag{6.15}$$

is optimal in (6.10). Since the gain function $G(z) = (K-z)^{+}$ is a function of space only, it follows that (see [44]) that the map $t \mapsto V(t,z)$ is decreasing on $[0,T]$ for each $z \in (0,\infty)$ and that the boundary $t \mapsto b(t)$ is increasing on $[0,T]$. Recalling from Chapter 2 that $z \mapsto V(t,z)$ is convex on $(0,\infty)$ and $(t,z) \mapsto V(t,z)$ is continuous on $[0,T] \times (0,\infty)$ and the boundary b satisfies the properties that $b : [0,T] \to (0,K]$

is continuous and increasing with $b(T) = K$ under the negative exponential jump-diffusion processes.

4. Standard arguments based on the strong Markov property lead to the following free-boundary problem for the unknown value function V and the unknown boundary b:

$$V_t + \mathbb{L}_Z V = rV \quad \text{for} \quad z > b(t) \quad \text{and} \quad t \in [0,T), \tag{6.16}$$

$$V(t,z) = (K-z)^+ \quad \text{for} \quad z = b(t) \quad \text{and} \quad t \in [0,T], \tag{6.17}$$

$$V_z(t,x) = -1 \quad \text{for} \quad z = b(t) \quad \text{and} \quad t \in [0,T], \tag{6.18}$$

$$V(t,z) > (K-z)^+ \quad \text{for} \quad z > b(t) \quad \text{and} \quad t \in [0,T), \tag{6.19}$$

$$V(t,z) = (K-z)^+ \quad \text{for} \quad z < b(t) \quad \text{and} \quad t \in [0,T), \tag{6.20}$$

$$V(t,z) = 0 \quad \text{for} \quad z \in [\ell, \infty]. \tag{6.21}$$

The relation (6.16)-(6.20) are independently verified in Chapter 2 in the case of the American put option under the NEJD processes; the results can be translated fully into the present setting. Similarly, the following properties of V and b also holds under the negative exponential jump-diffusion processes.

Property 6.1.

- *V is continuous on $[0,T] \times \mathbb{R}_+$.*
- *V is $C^{1,2}$ on C.*
- *$z \mapsto V(t,z)$ is decreasing and convex.*
- *$t \mapsto V(t,z)$ is decreasing with $V(T,z) = (K-z)^+$*
- *$t \mapsto b(t)$ is increasing and continuous with $0 < b(t) \le K$ and $b(T) = K$.*
- *The smooth-fit property holds, i.e. that $V_z(t,z) = G_z(z) = -1$ at $z = b(t)$.*

5. We briefly turn to the American up-and-out put with a reverse barrier, i.e. the case where $\ell < K$. The payoff here is discontinuous since the option is knocked-out while in the money. The arbitrage-free price is thus given by

$$V(t,z) = \sup_{0 \le \tau \le T-t} \mathsf{E}_{t,z} \left[e^{-r\tau} (K - Z^{\ell}_{t+\tau})^+ I(Z^{\ell}_{t+\tau} < \ell) \right]. \tag{6.22}$$

This is the same writing

$$V(t,z) = \sup_{0\le\tau\le T-t} \mathsf{E}_{t,z}\left[e^{-r\tau}(K - Z^{\ell}_{t+\tau})^{+} I(Z^{\ell}_{t+\tau} < \ell)\right]\left[I(\tau_* < \tau_\ell) + I(\tau_* \ge \tau_\ell)\right]. \quad (6.23)$$

where τ_* is an optimal stopping time. If $\tau_* < \tau_\ell$, then we have

$$\begin{aligned} V(t,z) =& \mathsf{E}_{t,z}\left[e^{-r\tau_*}(K - Z^{\ell}_{t+\tau_*})^{+} I(Z^{\ell}_{t+\tau_*} < \ell)\right] \\ =& \mathsf{E}_{t,z}\left[e^{-r\tau_*}(K - Z_{t+\tau_*})\right] \\ =& K\mathsf{E}e^{-r\tau_*} - z \le K - z \end{aligned}$$

where the third equality follows from the optional sampling theorem (since the jump-diffusion process Z_t is right-continuous and the stopping time τ_* is bounded in between $[0, T-t]$) and using the fact that $e^{-rt}Z_t$ is a martingale under the measure P. On the other hand, if $\tau_* > \tau_\ell$, then the option would have already been knocked-out and $V(t,z) = 0$. We conclude from this brief analysis that one can make profit by exercising at any time and thus it is always optimal to stop immediately in the case of a reverse up-and-out put option with no rebates.

6.3 The Arbitrage-Free Price of the Rational Exercise Boundary

In this section we derive a closed form expression for the arbitrage-free price V in terms of the rational exercise boundary b and show that the rational exercise boundary b itself can be characterised as the unique solution to a nonlinear integral equation.

Peskir proposed a series of change-of-variable formulas with local time for semi-martingales with jumps in [39]. By the Property 6.1 for the value function V and the exercise boundary b mentioned in the last section, it is easy to verify that the jump-diffusion process Z^{ℓ}_t, the condition of [39, Theorem 3.1] hold. Applying this change-of-variable formula to $e^{-rs}V(t+s, Z^{\ell}_{t+s})$ in terms of s with t and $Z^{\ell}_t = z$ given and fixed, we can have

$$e^{-rs}V(t+s, Z^{\ell}_{t+s}) = V(t,z) + \int_0^s e^{-ru}(V_t + \mathbb{L}_Z V - rV)(t+u, Z^{\ell}_{t+u})du + M^1_s + M^2_s, \quad (6.24)$$

where

$$M_s^1 = \sigma \int_0^s e^{-ru} z Z_u^\ell V_z(t+u, zZ_u^\ell) dW_u, \tag{6.25}$$

$$M_s^2 = \sum_{0\le u\le s}^{\Delta Z_{t+u}^\ell \neq 1} \left(e^{-ru} V(t+u, Z_{t+u}^\ell) - e^{-ru} V(t+u, Z_{t+u-}^\ell) \right) \tag{6.26}$$

$$- \lambda \int_0^s e^{-ru} \int_{-\infty}^0 [V(t+u, Z_{t+u-}^\ell e^y) - V(t+u, Z_{t+u-}^\ell)] f_Y(y) dy du,$$

Note that there is no local time term in (6.24) due to the smooth-fit property. Following the similar derivation path for American put options employed in Chapter 2, it can be easily proved that M_s^1 and M_s^2 are martingales under P. Using the relation given by (6.16), we can simplify the equation (6.24) as follows

$$e^{-rs} V(t+s, Z_{t+s}^\ell) = V(t,z) \tag{6.27}$$

$$+ \int_0^s e^{-ru} (V_t + \mathbb{L}_Z V - rV)(t+u, Z_{t+u}^\ell) I(Z_{t+u}^\ell \le b(t+u)) du$$

$$+ M_s^1 + M_s^2.$$

Recall that only negative jumps are allowed in the NEJD processes, then the existence of the indicator function $I(Z_{t+u}^\ell \le b(t+u))$ will allow us to replace $(V_t + \mathbb{L}_Z V - rV)$ directly by $(G_t + \mathbb{L}_Z G - rG)$, since no global integral term will be introduced by the operator $\mathbb{L}_Z$. Also recall from Chapter 2 that $(G_t + \mathbb{L}_Z G - rG)(t+u, Z_{t+u}^\ell) I(Z_{t+u}^\ell \le b(t+u)) = -rKI(Z_{t+u}^\ell \le b(t+u))$. Now replacing s by $T-t$ in (6.27), using that $V(T,z) = G(z) = (K-z)^+$, taking $\mathsf{E}_{t,z}$ on both sides and applying the optional sampling theorem, we get

$$V(t,z) = e^{-r(T-t)} \mathsf{E}_{t,z} G(Z_T^\ell) + rK \int_0^{T-t} e^{-ru} \mathsf{P}_{t,z}(Z_{t+u}^\ell \le b(t+u)) du, \tag{6.28}$$

Now the main result for the EEP representation and the optimal stopping boundary for an American knock-out put option under the NEJD processes can be stated as follows.

Theorem 6.1. *The arbitrage-free price of the up and out American put option admits the following early exercise premium representation under a negative exponential jump diffusion process*

$$V(t,z) = e^{-r(T-t)} \mathsf{E}_{t,z} G(Z_T^\ell) + rK \int_0^{T-t} e^{-ru} \mathsf{P}_{t,z}(Z_{t+u}^\ell \le b(t+u)) du, \tag{6.29}$$

for all $(t,z) \in [0,T] \times (0,\infty)$*, where the first term is the arbitrage-free price of the up and out European put option and the second term is the early-exercise premium.*

The rational exercise boundary of the up and out American put option can be characterised as the unique continuous solution $b : [0,T] \to \mathbb{R}_+$ *to the nonlinear integral equation*

$$K - b(t) = e^{-r(T-t)} \mathsf{E}_{t,b(t)}(K - Z_T^\ell)^+ + \int_0^{T-t} e^{-ru} \mathsf{P}_{t,b(t)}\big(Z_{t+u}^\ell \le b(t+u)\big) du, \quad (6.30)$$

satisfying $0 \le b(t) \le K$ *for all* $t \in [0,T]$

The analysis from (6.24) to (6.28) establishes the existence of the solution to (6.29), thus only the uniqueness of the optimal stopping boundary b remains to be proved.

Proof. Suppose that there exists such a function $c : [0,T] \to \mathbb{R}$ satisfying the conditions of Theorem 6.1 and solving the equation (6.30). We just need to prove that $c(t)$ coincides with the early exercise boundary $b(t)$.

1. Introduce a corresponding function $U^c : [0,T] \times (0,\infty) \to \mathbb{R}$ defined by

$$U^c(t,z) = e^{-r(T-t)} \mathsf{E}_{t,z}(K - Z_T^\ell)^+ + rK \int_0^{T-t} e^{-ru} \mathsf{P}_{t,z}\big(Z_{t+u}^\ell \le c(t+u)\big) du. \quad (6.31)$$

Simply inserting $z = c(t)$ into (6.31), we can have that

$$U^c(t, c(t)) = K - c(t) = G(c(t)). \quad (6.32)$$

since c satisfy the condition $c(t) < K$ for all $0 < t < T$.

2. We need to show that $U^c(t,z) = G(z)$ for all $t, z \in [0,T] \times (0,\infty)$ such that $z \le c(t)$. First introduce an stochastic process $M(s, Z_s^\ell)$ which defined by

$$M(s, Z_s^\ell) = e^{-rs} U^c(t+s, zZ_s^\ell) + rK \int_0^s e^{-ru} I(zZ_u^\ell \le c(t+u)) du, \quad (6.33)$$

where $Z_0^\ell = 1$ and $(t,z) \in [0,T] \times (0,\infty)$ is given and fixed. Then follow the similar approach in Chapter 2, we can prove that $M(s, Z_s^\ell)$ is a martingale under P for $s \in [0, T-t]$. Now take a pair of (t,z) such that $z \le c(t)$ and consider the stopping time

$$\sigma_c = \inf\{s \in [0, T-t] | zZ_s^\ell \ge c(t+s)\}. \quad (6.34)$$

Since $M(s, Z_s^\ell)$ is a martingale, we have

$$M(0, Z_0^\ell) = \mathsf{E}[M(\sigma_c, Z_{\sigma_c}^\ell)]. \quad (6.35)$$

For the left hand side of (6.35) we have:

$$M(0, Z_0^\ell) = U^c(t, z), \tag{6.36}$$

by the definition of $M(s, Z_s^r)$. Also note that $U^c(t+\sigma_c, zZ_{\sigma_c}^\ell) = G(zZ_{\sigma_c}^\ell)$ holds for the NEJD process since we assume that there is no positive jumps. Back to our proof, for the right hand side of (6.35) we have:

$$\begin{aligned}\mathsf{E}[M(\sigma_c, Z_{\sigma_c}^\ell)] =& \mathsf{E}[e^{-r\sigma_c}U^c(t+\sigma_c, zZ_{\sigma_c}^\ell)] + \mathsf{E}(rK\int_0^{\sigma_c} e^{-ru}du) \\ =& \mathsf{E}[e^{-r\sigma_c}G(zZ_{\sigma_c}^\ell)] + rK\mathsf{E}(\int_0^{\sigma_c} e^{-ru}du).\end{aligned} \tag{6.37}$$

Then the equation (6.35) can be rewrite as

$$U^c(t, z) = \mathsf{E}[e^{-r\sigma_c}G(zZ_{\sigma_c}^\ell)] + rK\mathsf{E}(\int_0^{\sigma_c} e^{-ru}du). \tag{6.38}$$

Apply the change-of-variable formula [39, Theorem 3.1] to $e^{-rs}G(zZ_s^r)$, replace s by σ_c and take expectation E on both side, following the similar derivation of first part of Theorem 6.1 we can have

$$\mathsf{E}[e^{-r\sigma_c}G(zZ_{\sigma_c}^r)] = G(z) - rK\mathsf{E}(\int_0^{\sigma_c} e^{-ru}du). \tag{6.39}$$

Thus insert (6.39) into (6.38), we have proved that

$$U^c(t, z) = G(z), \tag{6.40}$$

for all $t, z \in [0, T] \times (0, \infty)$ such that $z \le c(t)$.

3. We want to show that $U^c(t, z) \le V(t, z)$ for all $t, z \in [0, T] \times (0, \infty)$. For this, take any such (t, z) and consider the stopping time

$$\tau_c = \inf\{s \in [0, T-t] | zZ_s^\ell \le c(t+s)\}. \tag{6.41}$$

If $z \le c(t)$, then by the result of Step **2** in (6.40) we know that

$$U^c(t, z) = G(z) \le V(t, z). \tag{6.42}$$

If $z > c(t)$, then we have $U^c(t+\tau_c, zZ_{\tau_c}^\ell) = G(zZ_{\tau_c}^\ell)$ by the definition of τ_c in (6.41). So replacing s by τ_c in $M(s, Z_s^\ell)$ and taking expectation E on both sides, we find that

$$\begin{aligned}U^c(t, z) =& \mathsf{E}[e^{-r\tau_c}U^c(t+\tau_c, zZ_{\tau_c}^\ell)] + rK\mathsf{E}[\int_0^{\tau_c} e^{-ru}I(zZ_u^\ell \le c(t+u))du] \\ =& \mathsf{E}[e^{-r\tau_c}G(zZ_{\tau_c}^\ell)],\end{aligned} \tag{6.43}$$

where the second part on the right hand side of (6.43) equals to zero by the definition of τ_c. Then the definition of the value function $V(t,x)$ in (6.10) implies that

$$U^c(t,z) \leq V(t,z), \tag{6.44}$$

for all $t, z \in [0,T] \times (0,\infty)$.

4. Let us now show that $b(t) \leq c(t)$ for all $t \in [0,T]$. For this, suppose there exists a $t \in [0,T)$ such that $b(t) > c(t)$. Then for this kind of fixed t, take an z such that $(t,z) \in \{[0,T] \times (0,\infty) | z \leq c(t)\}$. So we have $z \leq c(t) < b(t)$. Now consider the stopping time

$$\sigma_b = \inf\{s \in [0, T-t] | zZ^\ell_s \geq b(t+s)\}. \tag{6.45}$$

Replacing s by σ_b in $M(s, Z^\ell_s)$ and taking expectation E on both sides, we get

$$\mathsf{E}[e^{-r\sigma_b} U^c(t+\sigma_b, zZ^\ell_{\sigma_b})] = U^c(t,z) - rK\mathsf{E}[\int_0^{\sigma_b} e^{-ru} I(zZ^\ell_u \leq c(t+u))du]. \tag{6.46}$$

And applying the change-of-formula [39, Theorem 3.1] to $e^{-rs}V(t+s, Z^\ell_{t+s})$ with the same $(t,z) \in \{[0,T] \times (0,\infty) | z \leq c(t)\}$, we have

$$e^{-rs} V(t+s, zZ^\ell_s) = V(t, Z^\ell_t) - rK \int_0^s e^{-ru} I(zZ^\ell_u \leq b(t+u))du + M_s, \tag{6.47}$$

where M_s is a martingale under P. Also replacing s by σ_b in (6.47) and taking expectation E on both sides, we get

$$\begin{aligned} \mathsf{E}[e^{-r\sigma_b} V(t+\sigma_b, zZ^\ell_{\sigma_b})] =& V(t,z) - rK\mathsf{E}[\int_0^{\sigma_b} e^{-ru} I(zZ^\ell_u \leq b(t+u))du] \\ =& V(t,z) - rK\mathsf{E}[\int_0^{\sigma_b} e^{-ru} du] \\ =& V(t,z) - rK\mathsf{E}[\int_0^{\sigma_b} e^{-ru} I(zZ^\ell_u \leq c(t+u))du] \\ &- rK\mathsf{E}[\int_0^{\sigma_b} e^{-ru} I(zZ^\ell_u > c(t+u))du]. \end{aligned} \tag{6.48}$$

The comparing of (6.46) and (6.48) implies that

$$\mathsf{E}[\int_0^{\sigma_b} e^{-ru} I(zZ^\ell_u > c(t+u))du] \leq 0. \tag{6.49}$$

However, the fact that $c(t) < b(t)$ and the continuity of the functions c and b force the expectation in (6.49) strictly positive and provides a contradiction. Thus $b(t) \leq c(t)$ for all $t \in [0,T]$ as claimed.

5. Finally, we show that $b(t) = c(t)$ for all $t \in [0, T]$. For this, suppose there exists a $t \in [0, T)$ such that $b(t) < c(t)$. Then for this kind of fixed t, take any x satisfying $b(t) < z < c(t)$. Now consider the stopping time

$$\tau_b = \inf\{s \in [0, T-t] | zZ^\ell_s \leq b(t+s)\}. \tag{6.50}$$

Replacing s by τ_b in $M(s, Z^\ell_s)$, and taking expectation E on both sides, we get

$$\mathsf{E}[e^{-r\tau_b}U^c(t+\tau_b, zZ^\ell_{\tau_b})] = U^c(t,z) - rK\mathsf{E}[\int_0^{\tau_b} e^{-ru}I(zZ^\ell_u \leq c(t+u))du]. \tag{6.51}$$

Again, replacing s by τ_b in (6.47), and taking expectation E on both sides, we find that

$$\begin{aligned}\mathsf{E}[e^{-r\tau_b}V(t+\tau_b, zZ^\ell_{\tau_b})] &= V(t,z) - rK\mathsf{E}[\int_0^{\tau_b} e^{-ru}I(zZ^\ell_u \leq b(t+u))du]. \\ &= V(t,z),\end{aligned} \tag{6.52}$$

where the last equation follows from the definition of τ_b. Comparing (6.51) and (6.52) implies that

$$\mathsf{E}[\int_0^{\tau_b} e^{-ru}I(zZ^\ell_u \leq c(t+u))du] \leq 0. \tag{6.53}$$

Thus similar to Step **4**, the fact that $b(t) < c(t)$ and the continuity of the functions c and b force the expectation in (6.53) strictly positive and provides a contradiction. Thus $b(t) \geq c(t)$ for all $t \in [0, T]$. Combining with the result of Step **4**, we have that $b = c$ for all $t \in [0, T]$ and the proof for Theorem 6.1 is complete. □

In the remaining part of this section, we would like to derive the analytical form of the EEP representation (6.29) and the boundary equation (6.30) which are essential for the following financial analysis. A key component of such analytical form is the joint distribution between first hitting time and the terminal value

$$\begin{aligned}\mathsf{P}(Z^\ell_t \leq a) =& \mathsf{P}\big(Z_t \leq a, \max_{0\leq s\leq t} Z_s < \ell\big) \\ =& \mathsf{P}\big(Z_t \leq a, \tau_\ell > t\big),\end{aligned} \tag{6.54}$$

where the first hitting time of the level ℓ is introduced in (6.9). Based the conditional memoryless property of the exponential jump-diffusion processes, Kou and Wang [26] derived the Laplace transform of the joint distribution between first hitting time and the terminal, and provided the Gaver-Stehfest algorithm for this Laplace inversion. Thus we denote this joint distribution as a function below

$$\Psi(r-\lambda\zeta-\frac{\sigma^2}{2},\sigma,\lambda,\eta_2;a,\ell,t) := \mathsf{P}\big(Z_t \le a,\ \max_{0\le s\le t} Z_s < \ell\big) \tag{6.55}$$

Now we can rewrite the first part of (6.29) in terms of Ψ function.

$$\begin{aligned} e^{-r(T-t)}\mathsf{E}_{t,z}G(Z_T^\ell) =& e^{-r(T-t)}\mathsf{E}\big[G(zZ_{T-t}^\ell)\big] \\ =& e^{-r(T-t)}\mathsf{E}\big[(zZ_{T-t}-K)^+ I(z\max_{0\le s\le T-t} Z_s < \ell)\big] \\ =& e^{-r(T-t)}\mathsf{E}\big[(zZ_{T-t}-K) I(z\max_{0\le s\le T-t} Z_s < \ell, zZ_{T-t}\le K)\big] \\ =& e^{-r(T-t)}\mathsf{E}\big[zZ_{T-t} I(z\max_{0\le s\le T-t} Z_s < \ell, zZ_{T-t}\le K)\big] \\ &- Ke^{-r(T-t)}\mathsf{P}(z\max_{0\le s\le T-t} Z_s < \ell, zZ_{T-t}\le K)) \\ =& z\mathsf{E}\big[e^{-r(T-t)} Z_{T-t} I(z\max_{0\le s\le T-t} Z_s < \ell, zZ_{T-t}\le K)\big] \\ &- Ke^{-r(T-t)}\Psi(r-\lambda\zeta-\frac{1}{2}\sigma^2,\sigma,\lambda,\eta_2;\frac{K}{z},\frac{\ell}{z},T-t) \end{aligned} \tag{6.56}$$

For the first term on the right hand side of the last equality, we can use a change of numerable argument to eliminate the annoying part $e^{-r(T-t)}Z_{T-t}$. More precisely, introduce a new probability measure P^* defined as

$$\begin{aligned} \frac{d\mathsf{P}^*}{d\mathsf{P}} &= e^{-r(T-t)}Z_{T-t} \\ &= e^{(\lambda\zeta-\frac{\sigma^2}{2})(T-t)+\sigma W_{T-t}+\sum_{i=1}^{N_{T-t}}(Y_i)} \end{aligned} \tag{6.57}$$

Note that this is a well-defined probability as $\mathsf{E}[e^{-r(T-t)}Z_{T-t}] = 1$. We have, by the Girsanov theorem for jump processes, $W_t^* := W_t - \sigma t$ is a new Brownian motion under P^*, and the original process

$$\begin{aligned} Z_t^* &= e^{(r-\lambda\zeta-\frac{\sigma^2}{2})t+\sigma W_t+\sum_{i=1}^{N_t}(Y_i)} \\ &= e^{(r-\lambda\zeta+\frac{\sigma^2}{2})t+\sigma W_t^*+\sum_{i=1}^{N_t^*}(Y_i)} \end{aligned} \tag{6.58}$$

is a new NEJD process with the Poisson process N_t^* having a new rate $\lambda^* = \lambda\mathsf{E}(e^Y) = \lambda(1+\zeta)$, and the jump sizes Y_i being i.i.d. with a new density given by

$$\begin{aligned} \frac{e^y f_Y(y)}{\mathsf{E}(e^Y)} &= \frac{e^y\eta_2 e^{\eta_2 y}1_{\{y<0\}}}{\frac{\eta_2}{\eta_2+1}} \\ &= \eta_2 e^{(\eta_2+1)y}1_{\{y<0\}}\frac{\eta_2+1}{\eta_2} \\ &= (\eta_2+1)e^{(\eta_2+1)y}1_{\{y<0\}} \end{aligned} \tag{6.59}$$

Thus it is still a negative exponential jump density with $\eta_2^* = \eta_2 + 1$. In summary, we have

$$\begin{aligned} & z\mathsf{E}\big[e^{-r(T-t)} Z_{T-t} I(z \max_{0\le s\le T-t} Z_s < \ell, zZ_{T-t} \le K)\big] \\ =& z\mathsf{P}^*(z \max_{0\le s\le T-t} Z_s < \ell, zZ_{T-t} \le K) \\ =& z\Psi(r - \lambda\zeta + \frac{1}{2}\sigma^2, \sigma, \lambda^*, \eta_2^*; \frac{K}{z}, \frac{\ell}{z}, T-t) \end{aligned} \tag{6.60}$$

And the analytical form of the arbitrage-free price of an up and out European put option at time t is

$$\begin{aligned} e^{-r(T-t)}\mathsf{E}_{t,z} G(Z_T^\ell) =& z\Psi(r - \lambda\zeta + \frac{1}{2}\sigma^2, \sigma, \lambda^*, \eta_2^*; \frac{K}{z}, \frac{\ell}{z}, T-t) \\ & - Ke^{-r(T-t)}\Psi(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; \frac{K}{z}, \frac{\ell}{z}, T-t) \end{aligned} \tag{6.61}$$

With the equation (6.61) and (6.55), we can write the EEP representation of the value function $V(t,z)$ proposed in Theorem 6.1 into the following analytical form:

$$\begin{aligned} V(t,z) =& e^{-r(T-t)}\mathsf{E}_{t,z} G(Z_T^\ell) + rK\int_0^{T-t} e^{-ru}\mathsf{P}_{t,z}(Z_{t+u}^\ell \le b(t+u))du \\ =& z\Psi(r - \lambda\zeta + \frac{1}{2}\sigma^2, \sigma, \lambda^*, \eta_2^*; \frac{K}{z}, \frac{\ell}{z}, T-t) \\ & - Ke^{-r(T-t)}\Psi(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; \frac{K}{z}, \frac{\ell}{z}, T-t) \\ & + rK\int_0^{T-t} e^{-ru}\Psi(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; \frac{b(t+u)}{z}, \frac{\ell}{z}, u)du \end{aligned} \tag{6.62}$$

where $\lambda^* = \lambda(1+\zeta)$, $\eta_2^* = \eta_2 + 1$ and the early exercise boundary $b(t)$ is the unique solution of the following nonlinear integral equation:

$$\begin{aligned} K - b(t) =& z\Psi(r - \lambda\zeta + \frac{1}{2}\sigma^2, \sigma, \lambda^*, \eta_2^*; \frac{K}{b(t)}, \frac{\ell}{b(t)}, T-t) \\ & - Ke^{-r(T-t)}\Psi(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; \frac{K}{b(t)}, \frac{\ell}{b(t)}, T-t) \\ & + rK\int_0^{T-t} e^{-ru}\Psi(r - \lambda\zeta - \frac{1}{2}\sigma^2, \sigma, \lambda, \eta_2; \frac{b(t+u)}{b(t)}, \frac{\ell}{b(t)}, u)du \end{aligned} \tag{6.63}$$

in the class of the continuous functions $b : [0,T] \to \mathbb{R}$ satisfying $0 \le b(t) \le K$ for all $t \in [0,T]$.

6.4 Financial Analysis for American Knock-Out Put Options

In this section, we present a numerical example to draw a comparison between the American knock-out put option under negative exponential jump-diffusion processes and its European counterpart with the American put option. We mainly address the question as to what the return would be if the underlying process enters the given region at a given time. Such an analysis, although skeletal, highlights the features and drawbacks of the option irrespective of whether the stock price model is correct or not.

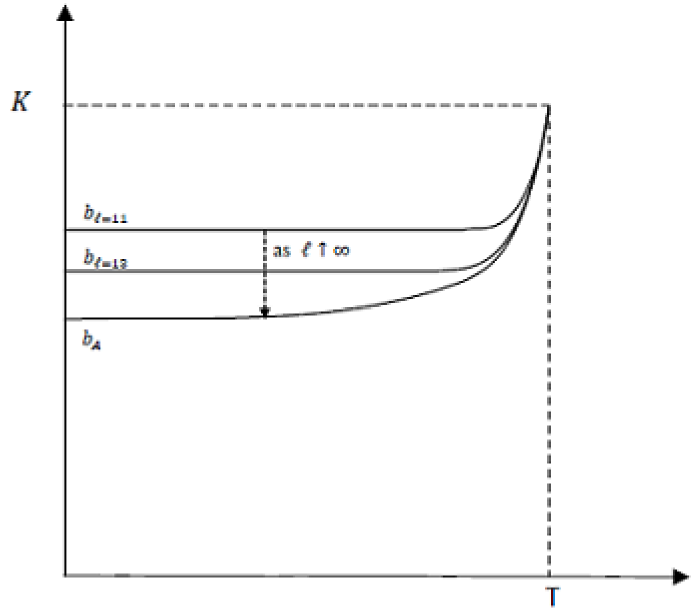

Figure 6.1: A computer drawing showing the rational exercise boundaries of the American knock-out option under the negative exponential jump-diffusion processes with $K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\lambda = 1.0$, $\eta_2 = 1.8$ when the barrier level ℓ equals 11 and 13 (the thin curved line b_A represents the rational exercise boundary of the American put option).

1. Figure 6.1 illustrates how the optimal boundary of the American knock-out put option changes as one varies the barrier level ℓ. We can see that the introduction of a barrier raises the optimal boundary, this is made more apparent as the barrier is lowered further towards the strike. Closer to the end of the lifetime of the option, we note that the optimal boundary for all barrier levels is almost identical; the reason behind this could be the decreasing likeliness of the stock price hitting the barrier before maturity, given that it has not hit it well into the contract. Hence, one can say that the optimal boundary is not much affected by the presence of a barrier as the time draws nearer to maturity. In addition, when the initial point is fixed at $K = 10$, we see that choosing a barrier level of 15, while keeping the remaining parameter fixed, the American knock-out put option behaves comparably to the American put option. This can be also seen in Table 6.1 when the returns of the American put in comparison with

Time (months)	0	2	4	6	8	10	12
Exercise at 8 (Knock-out $\ell = 11$)	343%	343%	343%	343%	343%	343%	343%
Exercise at 8 (Knock-out $\ell = 13$)	189%	189%	189%	189%	189%	189%	189%
Exercise at 8 (Knock-out $\ell = 15$)	171%	171%	171%	171%	171%	171%	171%
Exercise at 8 (American put)	167%	167%	167%	167%	167%	167%	167%
Exercise at b_{11} (Knock-out $\ell = 11$)	453%	452%	448%	440%	420%	366%	0%
Exercise at b_{13} (Knock-out $\ell = 13$)	309%	301%	290%	273%	248%	205%	0%
Exercise at b_{15} (Knock-out $\ell = 15$)	286%	276%	264%	247%	224%	186%	0%
Exercise at b_A (American put)	281%	271%	258%	242%	219%	182%	0%
Exercise at 6 (Knock-out $\ell = 11$)	687%	687%	687%	687%	687%	687%	687%
Exercise at 6 (Knock-out $\ell = 13$)	378%	378%	378%	378%	378%	378%	378%
Exercise at 6 (Knock-out $\ell = 15$)	342%	342%	342%	342%	342%	342%	342%
Exercise at 6 (American put)	335%	335%	335%	335%	335%	335%	335%
Exercise at 4 (Knock-out $\ell = 11$)	1030%	1030%	1030%	1030%	1030%	1030%	1030%
Exercise at 4 (Knock-out $\ell = 13$)	568%	568%	568%	568%	568%	568%	568%
Exercise at 4 (Knock-out $\ell = 15$)	513%	513%	513%	513%	513%	513%	513%
Exercise at 4 (American put)	502%	502%	502%	502%	502%	502%	502%
Exercise at 2 (Knock-out $\ell = 11$)	1373%	1373%	1373%	1373%	1373%	1373%	1373%
Exercise at 2 (Knock-out $\ell = 13$)	757%	757%	757%	757%	757%	757%	757%
Exercise at 2 (Knock-out $\ell = 15$)	683%	683%	683%	683%	683%	683%	683%
Exercise at 2 (American put)	669%	669%	669%	669%	669%	669%	669%

Table 6.1: Returns observed upon exercising the American knock-out put option below the strike price K with barrier levels $\ell = 11, 13, 15$. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t, z)/100 = (K - z)^+/V(0, K)$. The parameter set is the same as in Figure 6.1 above ($K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\lambda = 1.0$, $\eta_2 = 1.8$) and the initial stock price equals K.

the American knock-out show a difference of a maximum of 12%. This is logical since it becomes highly unlikely for the stock price to hit the barrier. In fact, as the barrier rises to infinity, the American knock-out put converges to the American put option. In Table 6.1, we can see that the returns observed on exercising below the strike for a lower barrier are more than double those of a higher barrier. This is however obtained at a much greater risk since the option is more likely to be knocked-out.

2. Table 6.2 shows the returns an investor can extract upon selling the American and European knock-out options and the American put options at and above the strike price K. In practice, the option holder may choose to sell his option at any time during the term of the contract, and thus one may view his payoff as the price he receives upon selling. We choose $\ell = 13$ since this practically illustrated the influence of the presence of a barrier on the returns (not too close to K allowing us to analyse at a greater range of price; and not too far from K so that we are not left with the case of the American put option). Viewing the percentage returns as a measure of

Time (months)	0	2	4	6	8	10	12
Sell at 12 (American put)	53%	46%	39%	30%	20%	8%	0%
Sell at 12 (American knock-out put)	29%	27%	25%	22%	17%	8%	0%
Sell at 12 (European knock-out put)	29%	28%	26%	23%	18%	9%	0%
Sell at 11 (American put)	73%	66%	58%	49%	37%	21%	0%
Sell at 11 (American knock-out put)	61%	59%	55%	49%	37%	21%	0%
Sell at 11 (European knock-out put)	62%	60%	57%	52%	43%	26%	0%
Sell at K (American put)	100%	94%	86%	77%	65%	49%	0%
Sell at K (American knock-out put)	100%	97%	92%	84%	73%	55%	0%
Sell at K (European knock-out put)	100%	98%	95%	89%	78%	60%	0%
Sell at 9 (American put)	137%	132%	125%	118%	109%	97%	84%
Sell at 9 (American knock-out put)	148%	144%	139%	133%	123%	110%	95%
Sell at 9 (European knock-out put)	146%	145%	142%	138%	130%	118%	105%

Table 6.2: Returns observed upon selling the American knock-out, European knock-out and American put option at and above the strike K. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,z)/100 = V(t,z)/V(0,K)$, $R_A(t,z)/100 = V_E(t,z)/V_E(0,K)$ and $R_A(t,z)/100 = (K-z)^+/V_A(0,K)$ respectively. The parameter set is the same as in Figure 6.1 above ($K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$, $\lambda = 1.0$, $\eta_2 = 1.8$, $\ell = 13$) and the initial stock price equals K.

option performance we observe the following in the region directly below K (i.e. when option is in-the-money). The American put naturally produces lower returns since the investor would have paid a higher premium, while selling the European knock-out option produces better returns, then followed by the American knock-out option. On the other hand, in the region above K, selling the American put option produces returns higher than the American and European knock-out options. This essentially reflects the option holder's uneasiness as the stock price approaches the barrier. In reality, the price at which the option holder is able to sell will depend upon a number of exogenous factors such as his inability of access the option market due to the friction costs in the form of transaction costs and taxes involved in selling, and the liquidity of the option market itself and thus it maybe increasingly difficult for the holder to sell the option. The issue of liquidity can be better addressed in the context of barrier options based on the economic rationale of the British option introduced by Peskir and Samee [42]. We will leave this question to future researches on option pricing with jump-diffusion processes.

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