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THE NILPOTENT VARIETY OF $W(1;n)_p$ IS IRREDUCIBLE

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Abstract. In the late 1980s, Premet conjectured that the nilpotent variety of any finite dimensional restricted Lie algebra over an algebraically closed field of characteristic $p > 0$ is irreducible. This conjecture remains open, but it is known to hold for a large class of simple restricted Lie algebras, e.g. for Lie algebras of connected reductive algebraic groups, and for Cartan series W, S and H. In this paper, with the assumption that $p > 3$, we confirm this conjecture for the minimal p-envelope $W(1;n)_p$ of the Zassenhaus algebra $W(1;n)$ for all $n \geq 2$.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$, and let $\mathfrak{g}$ be a finite dimensional restricted Lie algebra over k with $[p]$-th power map $x \mapsto x^{[p]}$. The nilpotent variety $\mathcal{N}(\mathfrak{g})$ is the set of all $x \in \mathfrak{g}$ such that $x^{[p]^N} = 0$ for $N \gg 0$. It is well known that $\mathcal{N}(\mathfrak{g})$ is Zariski closed in $\mathfrak{g}$, and it can be presented as a finite union $\mathcal{N}(\mathfrak{g}) = Z_1 \cup Z_2 \cup \cdots \cup Z_t$ of pairwise distinct irreducible components Z_i. In [5], Premet conjectured that the variety $\mathcal{N}(\mathfrak{g})$ is irreducible, i.e. $t = 1$ in this decomposition. The evidence to support this conjecture is that if $\mathfrak{g}$ is the Lie algebra of a connected reductive algebraic group G', and $\mathfrak{n}$ is the set of nilpotent elements in a Borel subalgebra of $\mathfrak{g}$, then $\mathcal{N}(\mathfrak{g}) = \{g.n \mid g \in G', n \in \mathfrak{n}\}$. Since G' is connected and $\mathfrak{n}$ is irreducible, the variety $\mathcal{N}(\mathfrak{g})$ is irreducible [3, p. 64]. Moreover, this conjecture holds for the Jacobson-Witt algebra $W(n;\underline{1})$ [6], for the Special Lie algebras $S(n;\underline{1})$ [13] and for the Hamiltonian Lie algebras $H(2n;\underline{1})$ [14]. In this paper, we are interested in the minimal p-envelope of the Zassenhaus algebra.

Throughout this paper we assume that k is an algebraically closed field of characteristic $p > 3$, and $n \in \mathbb{N}$. The divided power algebra $\mathcal{O}(1;n)$ has a k-basis $\{x^{(a)} \mid 0 \leq a \leq p^n - 1\}$, and the product in $\mathcal{O}(1;n)$ is given by $x^{(a)}x^{(b)} = \binom{a+b}{a}x^{(a+b)}$ if $0 \leq a+b \leq p^n - 1$ and 0 otherwise. We write $x^{(1)}$ as x. Note that $\mathcal{O}(1;n)$ is a local algebra with the unique maximal ideal $\mathfrak{m}$ spanned by all $x^{(a)}$ such that $a \geq 1$. A system of divided powers is defined on $\mathfrak{m}$, $f \mapsto f^{(r)} \in \mathcal{O}(1;n)$ where $r \geq 0$; see [11, Definition 2.1.1]. An automorphism Φ of $\mathcal{O}(1;n)$ is called *admissible* if $\Phi(f^{(r)}) = \Phi(f)^{(r)}$ for all $f \in \mathfrak{m}$ and $r \geq 0$. Let G denote the group of all admissible automorphisms of $\mathcal{O}(1;n)$. It is well known that G is a connected algebraic group of dimension $p^n - n$ [15, Theorem 2].

A derivation $\mathcal{D}$ of $\mathcal{O}(1;n)$ is called *special* if $\mathcal{D}(x^{(a)}) = x^{(a-1)}\mathcal{D}(x)$ for $1 \leq a \leq p^n - 1$ and 0 otherwise. The set of all special derivations of $\mathcal{O}(1;n)$ forms a Lie subalgebra of $\mathrm{Der}(\mathcal{O}(1;n))$ denoted $\mathfrak{L} = W(1;n)$ and called the *Zassenhaus algebra*. It is well known that $\mathfrak{L}$ is a free $\mathcal{O}(1;n)$-module of rank 1 generated by the special derivation ∂ such that $\partial(x^{(a)}) = x^{(a-1)}$ if $1 \leq a \leq p^n - 1$ and 0 otherwise [10, Ch. 4, Proposition 2.2(1)]. When $n = 1$, $\mathfrak{L}$ coincides with the Witt algebra $W(1;1) := \mathrm{Der}(\mathcal{O}(1;1))$, a simple and restricted Lie algebra. When $n \geq 2$, $\mathfrak{L}$ provides the first example of a simple, non-restricted Lie algebra [10, Ch. 4, Theorem 2.4(1)]. From now on we always assume that $n \geq 2$. By [8, Theorem 12.8], any automorphism of $\mathfrak{L}$ is induced by a unique admissible automorphism of $\mathcal{O}(1;n)$ so that $\mathrm{Aut}(\mathfrak{L}) \cong G$ as algebraic groups.

Let $\mathfrak{L}_p = W(1;n)_p$ denote the p-envelope of $\mathfrak{L} \cong \mathrm{ad}\,\mathfrak{L}$ in $\mathrm{Der}(\mathfrak{L})$. This semisimple restricted Lie algebra is referred to as the *minimal p-envelope* of $\mathfrak{L}$. Recent studies have shown that the variety $\mathcal{N}(\mathfrak{L}) := \mathcal{N}(\mathfrak{L}_p) \cap \mathfrak{L}$ is reducible [9]. So investigating the variety $\mathcal{N}(\mathfrak{L}_p)$ becomes critical for verifying Premet's conjecture. Our main result is the following theorem.

Theorem 1.1. *The variety $\mathcal{N}(\mathfrak{L}_p)$ coincides with the Zariski closure of*

$$\mathcal{N}_{reg} := G.(k^* \partial + k\, \partial^p + \cdots + k\, \partial^{p^{n-1}})$$

and hence is irreducible.

Our paper is organized as follows. In Section 2, we recall some basic results on $\mathfrak{L}$ and $\mathfrak{L}_p$. In Section 3, we study some nilpotent elements of $\mathfrak{L}_p$ and then prove the main result. The proof is similar to Premet's proof for the Jacobson-Witt algebra $W(n;\underline{1})$[6]. It relies on the fact that the variety $\mathcal{N}(\mathfrak{L}_p)$ is equidimensional of dimension $p^n - 1$; see [7, Theorem 4.2] and [11, Theorem 7.6.3(2)]. Then we need to prove that $\mathcal{N}_{\mathrm{sing}} := \mathcal{N}(\mathfrak{L}_p) \setminus \mathcal{N}_{\mathrm{reg}}$ is Zariski closed of codimension $\geq n+1$ in $\mathfrak{L}_p$ by constructing an $(n+1)$-dimensional subspace V in $\mathfrak{L}_p$ such that $V \cap \mathcal{N}_{\mathrm{sing}} = \{0\}$. The $(n+1)$-dimensional subspace used in $W(n;\underline{1})$ has no obvious analogue for $\mathfrak{L}_p$. Therefore, a new V is constructed using the original definition of $\mathfrak{L}$ due to H. Zassenhaus. In general, constructing analogues of V for the minimal p-envelopes of $W(n;\underline{m})$, where $\underline{m} = (m_1, \ldots, m_n)$ and $m_i > 1$ for some i, would enable one to check Premet's conjecture for this class of restricted Lie algebras.

2. Preliminaries

2.1. Let k be an algebraically closed field of characteristic $p > 3$ and $n \in \mathbb{N}$. The divided power algebra $\mathcal{O}(1;n)$ has a k-basis $\{x^{(a)} \mid 0 \leq a \leq p^n - 1\}$, and the product in $\mathcal{O}(1;n)$ is given by $x^{(a)}x^{(b)} = \binom{a+b}{a} x^{(a+b)}$ if $0 \leq a+b \leq p^n - 1$ and 0 otherwise. In the following, we write $x^{(1)}$ as x. It is straightforward to see that $\mathcal{O}(1;n)$ is a local algebra with the unique maximal ideal $\mathfrak{m}$ spanned by all $x^{(a)}$ such that $a \geq 1$. A system of divided powers is defined on $\mathfrak{m}$, $f \mapsto f^{(r)} \in \mathcal{O}(1;n)$ where $r \geq 0$; see [11, Definition 2.1.1].

A derivation $\mathcal{D}$ of $\mathcal{O}(1;n)$ is called *special* if $\mathcal{D}(x^{(a)}) = x^{(a-1)}\mathcal{D}(x)$ for $1 \leq a \leq p^n - 1$ and 0 otherwise. The set of all special derivations of $\mathcal{O}(1;n)$ forms a Lie subalgebra of $\mathrm{Der}(\mathcal{O}(1;n))$ denoted $\mathfrak{L} = W(1;n)$ and called the *Zassenhaus algebra*. When $n = 1$, $\mathfrak{L}$ coincides with the Witt algebra $W(1;1) := \mathrm{Der}(\mathcal{O}(1;1))$, a simple and restricted Lie algebra. When $n \geq 2$, $\mathfrak{L}$ provides the first example of a simple, non-restricted Lie algebra [10, Ch. 4, Theorem 2.4(1)]. From now on we always assume that $n \geq 2$.

The Zassenhaus algebra $\mathfrak{L}$ admits an $\mathcal{O}(1;n)$-module structure via $(f\mathcal{D})(x) = f\mathcal{D}(x)$ for all $f \in \mathcal{O}(1;n)$ and $\mathcal{D} \in \mathfrak{L}$. Since each $\mathcal{D} \in \mathfrak{L}$ is uniquely determined by its effect on x, it is easy to see that $\mathfrak{L}$ is a free $\mathcal{O}(1;n)$-module of rank 1 generated by the special derivation ∂ such that $\partial(x^{(a)}) = x^{(a-1)}$ if $1 \leq a \leq p^n - 1$ and 0 otherwise [10, Ch. 4, Proposition 2.2(1)]. Hence the Lie bracket in $\mathfrak{L}$ is given by $[x^{(i)}\,\partial, x^{(j)}\,\partial] = \left(\binom{i+j-1}{i} - \binom{i+j-1}{j}\right)x^{(i+j-1)}\,\partial$ if $1 \leq i+j \leq p^n$ and 0 otherwise.

There is a $\mathbb{Z}$-grading on $\mathfrak{L}$, i.e. $\mathfrak{L} = \bigoplus_{i=-1}^{p^n-2} kd_i$ with $d_i := x^{(i+1)}\partial$. Put $\mathfrak{L}_{(i)} := \bigoplus_{j\geq i}^{p^n-2} kd_j$ for $-1 \leq i \leq p^n - 2$. Then this $\mathbb{Z}$-grading induces a natural filtration

$$\mathfrak{L} = \mathfrak{L}_{(-1)} \supset \mathfrak{L}_{(0)} \supset \mathfrak{L}_{(1)} \supset \cdots \supset \mathfrak{L}_{(p^n-2)} \supset 0$$

on $\mathfrak{L}$. It is known that

$$d_i^p = \begin{cases} d_i, & \text{if } i = 0, \\ d_{pi}, & \text{if } i = p^t - 1 \text{ for some } 1 \leq t \leq n-1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

In particular, $\mathfrak{L}_{(0)}$ is a restricted subalgebra of $\mathrm{Der}(\mathcal{O}(1;n))$, kd_0 is a 1-dimensional torus in $\mathfrak{L}_{(0)}$ and $\mathfrak{L}_{(1)} = \mathrm{nil}\,(\mathfrak{L}_{(0)})$; see [9, p. 3].

It is also useful to mention that the Zassenhaus algebra $\mathfrak{L}$ has another presentation. Let $q = p^n$ and let $\mathbb{F}_q \subset k$ be the set of all roots of $x^q - x = 0$. This is a finite field of q elements. Then $\mathfrak{L}$ has a k-basis $\{e_\alpha \mid \alpha \in \mathbb{F}_q\}$ with the Lie bracket given by $[e_\alpha, e_\beta] = (\beta - \alpha)e_{\alpha+\beta}$ [11, Theorem 7.6.3(1)].

Let $\mathfrak{L}_p = W(1;n)_p$ denote the p-envelope of $\mathfrak{L} \cong \mathrm{ad}\,\mathfrak{L}$ in $\mathrm{Der}(\mathfrak{L})$. This semisimple restricted Lie algebra is referred to as the *minimal p-envelope* of $\mathfrak{L}$. By [11, Theorems 7.1.2(1) and 7.2.2(1)], we see that $\mathfrak{L}_p$ coincides with $\mathrm{Der}(\mathfrak{L}) = \mathfrak{L} + \sum_{i=1}^{n-1} k\partial^{p^i}$. Here we identify $\mathfrak{L}$ with $\mathrm{ad}\,\mathfrak{L} \subset \mathrm{Der}(\mathfrak{L})$ and regard ∂^{p^n} as 0. Then $\dim \mathfrak{L}_p = p^n + (n-1)$.

Let $\mathcal{N}$ denote the variety of nilpotent elements in $\mathfrak{L}_p$. It is well known that $\mathcal{N}$ is Zariski closed in $\mathfrak{L}_p$. One should note that the maximal dimension of toral subalgebras in $\mathfrak{L}_p$ equals n [11, Theorem 7.6.3(2)]. Moreover, $\mathfrak{L}_p$ possesses a toral Cartan subalgebra; see [5, p. 555]. Hence the set of all semisimple elements of $\mathfrak{L}_p$ is Zariski dense in $\mathfrak{L}_p$; see [4, Theorem 2]. It follows from these facts, [5, Corollary 2] and [7, Theorem 4.2] that there exist nonzero homogeneous polynomial functions $\varphi_0, \ldots, \varphi_{n-1}$ on $\mathfrak{L}_p$ such that $\mathcal{N}$ coincides with the set of all common zeros of $\varphi_0, \ldots, \varphi_{n-1}$. The variety $\mathcal{N}$ is equidimensional of dimension $p^n - 1$. Furthermore, any $\mathcal{D} \in \mathcal{N}$ satisfies $\mathcal{D}^{p^n} = 0$.

2.2. An automorphism Φ of $\mathcal{O}(1;n)$ is called *admissible* if $\Phi(f^{(r)}) = \Phi(f)^{(r)}$ for all $f \in \mathfrak{m}$ and $r \geq 0$. Let G denote the group of all admissible automorphisms of $\mathcal{O}(1;n)$. It is well known that G is a connected algebraic group, and each $\Phi \in G$ is uniquely determined by its effect on x. By [15, Lemmas 8, 9 and 10], an assignment $\Phi(x) := y = \sum_{i=1}^{p^n-1} \alpha_i x^{(i)}$ with $\alpha_i \in k$ such that $\alpha_1 \neq 0$ and $\alpha_{p^i} = 0$ for $1 \leq i \leq n-1$ extends to an admissible automorphism of $\mathcal{O}(1;n)$. Conversely, for any $y \in \mathfrak{m}$ as above, there is a unique $\Phi \in G$ such that $\Phi(x) = y$ [15, Corollary 1]. Hence $\dim G = p^n - n$; see also [15, Theorem 2].

Any automorphism of the Zassenhaus algebra $\mathfrak{L}$ is induced by a unique admissible automorphism Φ of $\mathcal{O}(1;n)$ via the rule $\mathcal{D}^\Phi = \Phi\mathcal{D}\Phi^{-1}$, where $\mathcal{D} \in \mathfrak{L}$ [8, Theorem 12.8]. So from now on we shall identify G with the automorphism group $\mathrm{Aut}(\mathfrak{L})$. It is known that G respects the natural filtration of $\mathfrak{L}$. In [12], Tyurin stated explicitly that if $\Phi \in G$ is such that $\Phi(x) = y$, then $\Phi(g(x)\partial) = (y')^{-1}g(y)\partial$ for any $g(x) \in \mathcal{O}(1;n)$. Extend this by defining $\Phi(\partial^{p^i}) = \Phi(\partial)^{p^i}$ for $1 \leq i \leq n-1$ one gets an automorphism of $\mathfrak{L}_p$.

It follows from the above description of G that $\mathrm{Lie}(G) \subseteq \mathfrak{L}_{(0)}$. More precisely,

Lemma 2.1. *The set $\{d_i \mid 0 \leq i \leq p^n - 2$ and $i \neq p^t - 1$ for $1 \leq t \leq n-1\}$ forms a k-basis of* $\mathrm{Lie}(G)$.

Proof. Let $\psi : \mathbb{A}^1 \to G$ be the map defined by $t \mapsto (x \mapsto x + tx^{(i+1)})$, where $0 \leq i \leq p^n - 2$ and $i \neq p^t - 1$ for $1 \leq t \leq n-1$. It is easy to check that ψ is a morphism of algebraic varieties. Then the differential $d_0\psi$ of ψ at 0 is the map $d_0\psi : k \to \mathrm{Lie}(G)$. So $d_0\psi(k) \subseteq \mathrm{Lie}(G)$.

Let us compute $d_0\psi(k)$. The morphism ψ sends $\mathbb{A}^1$ to the set of admissible automorphisms $\{\Phi_t \mid t \in \mathbb{A}^1\}$, where $\Phi_t(x) = x + tx^{(i+1)}$. Since Φ_t is uniquely determined by its effect on x and "admissible" is equivalent to the condition that $\Phi_t(x^{(p^j)}) = \Phi_t(x)^{(p^j)}$ for $1 \leq j \leq n-1$; see [15, Lemma 8]. Then by [11, Definition 2.1.1] we have that

$$\Phi_t(x^{(p^j)}) = (x + tx^{(i+1)})^{(p^j)} = x^{(p^j)} + tx^{(p^j-1)}x^{(i+1)} + \text{terms of higher degree in } t.$$

Passing to $d_0\psi(t)$ we get

$$x \mapsto x^{(i+1)},$$
$$x^{(p^j)} \mapsto x^{(p^j-1)}x^{(i+1)}.$$

These results are the same as $d_i = x^{(i+1)}\partial$ acting on x and $x^{(p^j)}$, respectively. Hence $d_i \in d_0\psi(t) \subseteq d_0\psi(k)$. Note that $\{d_i \mid 0 \leq i \leq p^n - 2 \text{ and } i \neq p^t - 1 \text{ for } 1 \leq t \leq n-1\}$ is a set of $p^n - n$ linearly independent vectors. Since $\dim \mathrm{Lie}(G) = \dim G = p^n - n$, they form a basis of $\mathrm{Lie}(G)$. This completes the proof. □

3. The variety $\mathcal{N}$

3.1. In § 2.1, we observed that any elements of $\mathfrak{L}_{(1)}$ are nilpotent, but they do not tell us much information about $\mathcal{N}$. The interesting nilpotent elements are contained in the complement of $\mathfrak{L}_{(1)}$ in $\mathcal{N}$, denoted $\mathcal{N} \setminus \mathfrak{L}_{(1)}$. They are of the form $\sum_{i=0}^{n-1} \alpha_i \partial^{p^i} + f(x)\partial$ for some $f(x) \in \mathfrak{m}$ and $\alpha_i \in k$ with at least one $\alpha_i \neq 0$. In this subsection, we study elements of this form.

Lemma 3.1. *Let $\mathcal{D} = \partial^{p^{n-1}} + \sum_{i=0}^{n-2} \beta_i \partial^{p^i} + g(x)\partial$ be an element of $\mathfrak{L}_p$, where $\beta_i \in k$ and $g(x) \in \mathfrak{m}$. Then $\mathcal{D}$ is conjugate under G to*

$$\partial^{p^{n-1}} + \sum_{i=0}^{n-2} \beta_i \partial^{p^i} + x^{(p^n - p^{n-1})} h(x)\partial$$

for some $h(x) = \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)}$ with $\mu_i \in k$.

Proof. Take $\mathcal{D} = \partial^{p^{n-1}} + \sum_{i=0}^{n-2} \beta_i \partial^{p^i} + g(x)\partial$ as in the lemma. By the proof of [12, Theorem 1], if $\Phi(x) = y$ is any admissible automorphism of $\mathcal{O}(1;n)$ with identical linear part, then

$$\Phi(\mathcal{D}) = \partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + (y')^{-1}\big(\beta_0 + \Phi(g(x)) - \sum_{i=1}^{n-2} \beta_i \partial^{p^i} y - \partial^{p^{n-1}} y\big)\partial.$$

If $g(x)\partial \equiv \gamma_1 x\partial \,(\mathrm{mod}\, \mathfrak{L}_{(1)})$ for some $\gamma_1 \in k$ and $\Phi(x) = y = x + \gamma_1 x^{(p^{n-1}+1)}$, then we can show that

$$\Phi(\mathcal{D}) \equiv \partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + \beta_0\, \partial \,(\mathrm{mod}\, \mathfrak{L}_{(1)}).$$

For $\gamma_1 = 0$, i.e. Φ is the identity automorphism, the result is clear. For $\gamma_1 \in k^*$, let us show this congruence by proving that

$$(y')^{-1}\big(\beta_0 + \Phi(g(x)) - \sum_{i=1}^{n-2} \beta_i \partial^{p^i} y - \partial^{p^{n-1}} y\big)\,\partial - \beta_0\,\partial \in \mathfrak{L}_{(1)}. \tag{3.1}$$

Note that $y' = 1 + \gamma_1 x^{(p^{n-1})}$ which is invertible in $\mathcal{O}(1;n)$. Since $\mathfrak{L}_{(1)}$ is invariant under multiplication of invertible elements of $\mathcal{O}(1;n)$, we can multiply both sides of (3.1) by y' and show that

$$\big(\beta_0 + \Phi(g(x)) - \sum_{i=1}^{n-2} \beta_i \partial^{p^i} y - \partial^{p^{n-1}} y\big)\,\partial - \beta_0 y'\,\partial \in \mathfrak{L}_{(1)}.$$

Since $g(x)\partial \equiv \gamma_1 x \partial \,(\mathrm{mod}\, \mathfrak{L}_{(1)})$ and Φ preserves the natural filtration of $\mathfrak{L}$, in particular, it preserves $\mathfrak{L}_{(1)}$, hence

$$\begin{aligned}
&\big(\beta_0 + \Phi(g(x)) - \sum_{i=1}^{n-2} \beta_i \partial^{p^i} y - \partial^{p^{n-1}} y\big)\,\partial - \beta_0 y'\,\partial \\
\equiv&\big(\beta_0 + \gamma_1(x + \gamma_1 x^{(p^{n-1}+1)}) - \sum_{i=1}^{n-2} \beta_i \gamma_1 x^{(p^{n-1}-p^i+1)} - \gamma_1 x\big)\,\partial - \beta_0(1 + \gamma_1 x^{(p^{n-1})})\,\partial \\
\equiv&0 \,(\mathrm{mod}\, \mathfrak{L}_{(1)}).
\end{aligned}$$

Therefore, $\mathcal{D}$ is conjugate to $\partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + \beta_0 \partial \,(\mathrm{mod}\, \mathfrak{L}_{(1)})$. If $\mathcal{D} \equiv \partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + \beta_0 \partial + \gamma_2 x^{(2)}\,\partial \,(\mathrm{mod}\, \mathfrak{L}_{(2)})$ for some $\gamma_2 \in k$, i.e. $\mathcal{D} = \partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + \beta_0 \partial + g_2(x)\,\partial$ with $g_2(x)\,\partial \equiv \gamma_2 x^{(2)}\,\partial \,(\mathrm{mod}\, \mathfrak{L}_{(2)})$, then repeat the above process by applying the automorphism $\Phi_2(x) = y_2 = x + \gamma_2 x^{(p^{n-1}+2)}$ to $\mathcal{D}$ we have that

$$\Phi_2(\mathcal{D}) = \partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + (y_2')^{-1}\big(\beta_0 + \Phi_2(g_2(x)) - \sum_{i=1}^{n-2} \beta_i \partial^{p^i} y_2 - \partial^{p^{n-1}} y_2\big)\partial.$$

Then we can show that

$$\Phi_2(\mathcal{D}) \equiv \partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + \beta_0\,\partial \,(\mathrm{mod}\, \mathfrak{L}_{(2)}).$$

For $\gamma_2 = 0$, the result is clear. For $\gamma_2 \in k^*$, let us prove that

$$(y_2')^{-1}\big(\beta_0 + \Phi_2(g_2(x)) - \sum_{i=1}^{n-2} \beta_i \partial^{p^i} y_2 - \partial^{p^{n-1}} y_2\big)\partial - \beta_0\,\partial \in \mathfrak{L}_{(2)}. \tag{3.2}$$

By the same reasons as before, we can multiply both sides of (3.2) by y_2' and show that

$$(\beta_0 + \Phi_2(g_2(x)) - \sum_{i=1}^{n-2} \beta_i \partial^{p^i} y_2 - \partial^{p^{n-1}} y_2)\,\partial - \beta_0 y_2'\,\partial \in \mathfrak{L}_{(2)}.$$

Indeed, since $\mathfrak{L}_{(2)}$ is invariant under Φ_2 we have that

$$
\begin{aligned}
&(\beta_0 + \Phi_2(g_2(x)) - \sum_{i=1}^{n-2} \beta_i \partial^{p^i} y_2 - \partial^{p^{n-1}} y_2)\,\partial - \beta_0 y_2'\,\partial \\
\equiv&\big(\beta_0 + \gamma_2(x + \gamma_2 x^{(p^{n-1}+2)})^{(2)} - \sum_{i=1}^{n-2} \beta_i \gamma_2 x^{(p^{n-1}-p^i+2)} - \gamma_2 x^{(2)}\big)\,\partial - \beta_0(1 + \gamma_2 x^{(p^{n-1}+1)})\,\partial \\
\equiv&0 \,(\mathrm{mod}\, \mathfrak{L}_{(2)}).
\end{aligned}
$$

Hence $\mathcal{D}$ is conjugate to $\partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + \beta_0 \partial \,(\mathrm{mod}\, \mathfrak{L}_{(2)})$. Then

$$
\mathcal{D} \equiv \partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + \beta_0\,\partial + \gamma_3 x^{(3)} \partial \,(\mathrm{mod}\, \mathfrak{L}_{(3)})
$$

for some $\gamma_3 \in k$. Apply the automorphism $\Phi_3(x) = y_3 = x + \gamma_3 x^{(p^{n-1}+3)}$ to $\mathcal{D}$ we can show that $\mathcal{D}$ is conjugate to $\partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + \beta_0 \partial \,(\mathrm{mod}\, \mathfrak{L}_{(3)})$. Continue doing this until we get $\mathcal{D}$ is conjugate to $\partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + \beta_0 \partial \,(\mathrm{mod}\, \mathfrak{L}_{(p^n - p^{n-1}-1)})$. Then

$$
\mathcal{D} \equiv \partial^{p^{n-1}} + \sum_{i=1}^{n-2} \beta_i \partial^{p^i} + \beta_0 \partial + \gamma_{p^n - p^{n-1}} x^{(p^n - p^{n-1})}\,\partial \,(\mathrm{mod}\, \mathfrak{L}_{(p^n - p^{n-1})})
$$

for some $\gamma_{p^n - p^{n-1}} \in k$. Next, we were supposed to apply the automorphism $\Phi_{p^n - p^{n-1}}(x) = x + \gamma_{p^n - p^{n-1}} x^{(p^n)}$ to $\mathcal{D}$. But since $x^{(j)} = 0$ for $j \geq p^n$ in $\mathcal{O}(1;n)$, the automorphism $\Phi_{p^n - p^{n-1}}$ is the identity automorphism and we stop here. Therefore, $\mathcal{D}$ is conjugate under G to

$$
\partial^{p^{n-1}} + \sum_{i=0}^{n-2} \beta_i \partial^{p^i} + x^{(p^n - p^{n-1})} h(x) \partial
$$

for some $h(x) = \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)}$ with $\mu_i \in k$. This completes the proof. $\square$

Corollary 3.1. *Let $\mathcal{D} = \partial^{p^m} + \sum_{i=0}^{m-1} \beta_i \partial^{p^i} + g(x)\partial$ be an element of $\mathfrak{L}_p$, where $1 \leq m \leq n-2, \beta_i \in k$ and $g(x) \in \mathfrak{m}$. Then $\mathcal{D}$ is conjugate under G to*

$$
\partial^{p^m} + \sum_{i=0}^{m-1} \beta_i \partial^{p^i} + x^{(p^n - p^m)} h(x) \partial
$$

for some $h(x) = \sum_{i=0}^{p^m - 1} \mu_i x^{(i)}$ with $\mu_i \in k$.

Proof. If $g(x)\,\partial \equiv \gamma_i x^{(i)}\,\partial \,(\mathrm{mod}\, \mathfrak{L}_{(i)})$ for some $\gamma_i \in k$, then the automorphism $\Phi(x) = x + \gamma_i x^{(p^m + i)}$ reduces $\mathcal{D}$ to the form $\mathcal{D} \equiv \partial^{p^m} + \sum_{i=1}^{m-1} \beta_i\,\partial^{p^i} + \beta_0\,\partial \,(\mathrm{mod}\, \mathfrak{L}_{(i)})$. Continue doing this until we get $\mathcal{D} \equiv \partial^{p^m} + \sum_{i=1}^{m-1} \beta_i\,\partial^{p^i} + \beta_0\,\partial \,(\mathrm{mod}\, \mathfrak{L}_{(p^n - p^m - 1)})$. This completes the proof. $\square$

Lemma 3.2. *Let $\mathcal{D} = \partial^{p^{n-1}} + \sum_{i=0}^{n-2} \beta_i \partial^{p^i} + x^{(p^n - p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)} \partial$ be a nilpotent element of $\mathfrak{L}_p$.*

(i) *If $\beta_i = 0$ for all i, then $\mu_0 = \mu_1 = 0$ and $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)}$.*

(ii) (a) *If $j \geq 0$ is the smallest index such that $\beta_j \neq 0$, then $\mu_0 = 0$ and $\mathcal{D}^{p^{n-1-j}}$ is conjugate under G to*

$$\partial^{p^{n-1}} + x^{(p^n - p^{n-1})} \sum_{i=2}^{p^{n-1}-1} \nu_i x^{(i)}\, \partial$$

for some $\nu_i \in k$. Hence $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)}$ for all $j \geq 1$.

(b) *In particular, if $\beta_0 \neq 0$, then $\mathcal{D}^{p^{n-1}}$ is conjugate under G to $\partial^{p^{n-1}}$. Hence $\mathcal{D} = \partial^{p^{n-1}} + \sum_{i=0}^{n-2} \gamma_i\, \partial^{p^i}$ for some $\gamma_i \in k$ with $\gamma_0 \neq 0$.*

Proof. Let $\mathcal{D} = \partial^{p^{n-1}} + \sum_{i=0}^{n-2} \beta_i \partial^{p^i} + x^{(p^n - p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)} \partial$ be a nilpotent element of $\mathfrak{L}_p$. Then $\mathcal{D}^{p^n} = 0$; see § 2.1. Let us first calculate $\mathcal{D}^p$. Recall Jacobson's formula,

$$(\mathcal{D}_1 + \mathcal{D}_2)^p = \mathcal{D}_1^p + \mathcal{D}_2^p + \sum_{i=1}^{p-1} s_i(\mathcal{D}_1, \mathcal{D}_2) \tag{3.3}$$

for all $\mathcal{D}_1, \mathcal{D}_2 \in \mathfrak{L}_p$, and $s_i(\mathcal{D}_1, \mathcal{D}_2)$ can be computed by the formula

$$\operatorname{ad}(t\mathcal{D}_1 + \mathcal{D}_2)^{p-1}(\mathcal{D}_1) = \sum_{i=1}^{p-1} i s_i(\mathcal{D}_1, \mathcal{D}_2) t^{i-1},$$

where t is a parameter. Set $\mathcal{D}_1 = x^{(p^n - p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)} \partial$ and $\mathcal{D}_2 = \partial^{p^{n-1}} + \sum_{i=0}^{n-2} \beta_i \partial^{p^i}$. Then $\mathcal{D}_1^p = 0$ by (2.1) and $\mathcal{D}_2^p = \sum_{i=0}^{n-2} \beta_i^p \partial^{p^{i+1}}$. By the natural filtration of $\mathfrak{L}$, we have that for any $1 \leq s \leq p-2$,

$$\begin{aligned} [\mathcal{D}_1, (\operatorname{ad} \mathcal{D}_2)^s(\mathcal{D}_1)] &\in [\mathfrak{L}_{(p^n - p^{n-1} - 1)}, \mathfrak{L}_{(p^n - (s+1)p^{n-1} - 1)}] \\ &\subseteq [\mathfrak{L}_{(p^n - p^{n-1} - 1)}, \mathfrak{L}_{(p^{n-1} - 1)}] \\ &\subseteq \mathfrak{L}_{(p^n - 2)} = \operatorname{span}\{x^{(p^n - 1)} \partial\}. \end{aligned}$$

This last term will appear if and only if $s = p-2$. So

$$\mathcal{D}^p = \sum_{i=0}^{n-2} \beta_i^p \partial^{p^{i+1}} + (\operatorname{ad} \mathcal{D}_2)^{p-1}(\mathcal{D}_1) + \mu_{(1)} x^{(p^n - 1)} \partial \tag{3.4}$$

for some $\mu_{(1)} \in k$.

(i) If $\beta_i = 0$ for all i, then $\mathcal{D}^p = (\operatorname{ad} \partial^{p^{n-1}})^{p-1}(\mathcal{D}_1) + \mu_{(1)} x^{(p^n - 1)} \partial$. Since $\partial^{p^{n-1}}$ is a derivation of $\mathfrak{L}$ and $\partial^{p^{n-1}}(\sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)}) = 0$, we have that

$$\mathcal{D}^p = \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)}\, \partial + \mu_{(1)} x^{(p^n - 1)} \partial.$$

If $\mu_0 \neq 0$, then $\mathcal{D}^p \equiv \mu_0 \partial \pmod{\mathfrak{L}_{(0)}}$. By Jacobson's formula,

$$\mathcal{D}^{p^n} \equiv \mu_0^{p^{n-1}} \partial^{p^{n-1}} + \sum_{i=0}^{n-2} \mu_i'\, \partial^{p^i} \pmod{\mathfrak{L}_{(0)}}$$

for some $\mu_i' \in k$. As $\mu_0 \neq 0$, this implies that $\mathcal{D}^{p^n} \not\equiv 0 \,(\mathrm{mod}\, \mathfrak{L}_{(0)})$ and so is not equal to 0. This is a contradiction. Hence $\mu_0 = 0$. Similarly, if $\mu_1 \neq 0$ then $\mathcal{D}^p \equiv \mu_1 x\partial \,(\mathrm{mod}\, \mathfrak{L}_{(1)})$. But $\mathcal{D}^{p^n} \equiv \mu_1^{p^{n-1}} x\partial \not\equiv 0 \,(\mathrm{mod}\, \mathfrak{L}_{(1)})$, a contradiction. Thus $\mu_1 = 0$. Therefore, $\mathcal{D}^p = \sum_{i=2}^{p^{n-1}-1} \mu_i x^{(i)}\, \partial + \mu_{(1)} x^{(p^n-1)}\partial$, which is an element of $\mathfrak{L}_{(1)}$. Since $\mathfrak{L}_{(1)}$ is restricted we have that $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)}$. This proves statement (i).

(ii)(a) Let $j \geq 0$ be the smallest index such that $\beta_j \neq 0$, and let l be the largest index such that $\beta_l \neq 0$, i.e. $0 \leq j \leq l \leq n-2$. Let us consider the special case $j = l$, i.e.

$$\mathcal{D} = \partial^{p^{n-1}} + \beta_j \partial^{p^j} + x^{(p^n - p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)} \partial.$$

We prove by induction that for any $1 \leq r \leq n-1-j$, $\mathcal{D}^{p^r}$ is conjugate under G to

$$\partial^{p^{j+r}} + \beta_{0,(1)}^{p^{r-1}}\, \partial^{p^{r-1}} + x^{(p^n - p^{j+r})} \sum_{i=0}^{p^{j+r}-1} \mu_{i,(r)} x^{(i)}\, \partial$$

for some $\beta_{0,(1)} \in k^*\mu_0$ and $\mu_{i,(r)} \in k$. For $r = 1$, the previous calculation (3.4) gives

$$\mathcal{D}^p = \beta_j^p \partial^{p^{j+1}} + \mathrm{ad}\,\big(\partial^{p^{n-1}} + \beta_j \partial^{p^j}\big)^{p-1}\big(x^{(p^n-p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)}\partial\big) + \mu_{(1)} x^{(p^n-1)}\partial.$$

Note that

$$\begin{aligned}
&\mathrm{ad}\,\big(\partial^{p^{n-1}} + \beta_j \partial^{p^j}\big)^{p-1}\big(x^{(p^n-p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)}\partial\big)\\
&= \mathrm{ad}\,\Big(\sum_{m=0}^{p-1} (-1)^m \beta_j^{p-1-m} \partial^{mp^{n-1}+(p-1-m)p^j}\Big)\Big(\sum_{i=0}^{p^{n-1}-1} \mu_i x^{(p^n-p^{n-1}+i)}\partial\Big)\\
&= \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)}\partial - \beta_j \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(p^{n-1}-p^j+i)}\partial + \cdots + \beta_j^{p-1} \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(p^n-p^{n-1}-(p-1)p^j+i)}\partial.
\end{aligned}$$

The above result can be rewritten as $\mu_0\, \partial + g(x)\, \partial$ for some $g(x) \in \mathfrak{m}$. Hence

$$\mathcal{D}^p = \beta_j^p \partial^{p^{j+1}} + \mu_0\, \partial + g(x)\, \partial + \mu_{(1)} x^{(p^n-1)}\partial.$$

Then the automorphism $\Phi(x) = \alpha x$ with $\alpha^{p^{j+1}} = \beta_j^p$ reduces $\mathcal{D}^p$ to the form

$$\mathcal{D}^p = \partial^{p^{j+1}} + \beta_{0,(1)}\, \partial + f_1(x)\, \partial,$$

where $\beta_{0,(1)} \in k^*\mu_0$ and $f_1(x) \in \mathfrak{m}$. It follows from Lemma 3.1 and Corollary 3.1 that $\mathcal{D}^p$ is conjugate under G to

$$\partial^{p^{j+1}} + \beta_{0,(1)}\, \partial + x^{(p^n-p^{j+1})} \sum_{i=0}^{p^{j+1}-1} \mu_{i,(1)} x^{(i)}\, \partial$$

for some $\mu_{i,(1)} \in k$. Thus the result is true for $r = 1$. Suppose the result is true for $r = k-1$, i.e. $\mathcal{D}^{p^{k-1}}$ is conjugate under G to

$$\partial^{p^{j+k-1}} + \beta_{0,(1)}^{p^{k-2}} \partial^{p^{k-2}} + x^{(p^n - p^{j+k-1})} \sum_{i=0}^{p^{j+k-1}-1} \mu_{i,(k-1)} x^{(i)} \partial$$

for some $\mu_{i,(k-1)} \in k$. Let us calculate $\mathcal{D}^{p^k}$. Set $\mathcal{D}_1 = x^{(p^n-p^{j+k-1})} \sum_{i=0}^{p^{j+k-1}-1} \mu_{i,(k-1)} x^{(i)} \partial$ and $\mathcal{D}_2 = \partial^{p^{j+k-1}} + \beta_{0,(1)}^{p^{k-2}} \partial^{p^{k-2}}$ in the Jacobson's formula (3.3). Then $\mathcal{D}_1^p \in \mathfrak{L}_{(1)}$ and $\mathcal{D}_2^p = \partial^{p^{j+k}} + \beta_{0,(1)}^{p^{k-1}} \partial^{p^{k-1}}$. By the natural filtration of $\mathfrak{L}$, we have that

$$(\operatorname{ad} \mathcal{D}_2)^{p-1}(\mathcal{D}_1) \in \mathfrak{L}_{(p^n - p^{j+k}-1)} \subseteq \mathfrak{L}_{(1)}.$$

Similarly, for any $1 \le s \le p-2$,

$$\begin{aligned}
[\mathcal{D}_1, (\operatorname{ad} \mathcal{D}_2)^s(\mathcal{D}_1)] &\in [\mathfrak{L}_{(p^n-p^{j+k-1}-1)}, \mathfrak{L}_{(p^n-(s+1)p^{j+k-1}-1)}] \\
&\subseteq [\mathfrak{L}_{(1)}, \mathfrak{L}_{(p^n-(s+1)p^{j+k-1}-1)}] \\
&\subseteq \mathfrak{L}_{(p^n-(s+1)p^{j+k-1})}, \\
&\subseteq \mathfrak{L}_{(p^n-(p-1)p^{j+k-1})}, \\
&\subseteq \mathfrak{L}_{(p^n-(p-1)p^{n-2})} \text{ (since } 1 \le j+k-1 \le n-2) \\
&\subseteq \mathfrak{L}_{(1)}.
\end{aligned}$$

Hence $\mathcal{D}^{p^k} = \partial^{p^{j+k}} + \beta_{0,(1)}^{p^{k-1}} \partial^{p^{k-1}} + f_k(x)\partial$ for some $f_k(x) \in \mathfrak{m}$. By Lemma 3.1 and Corollary 3.1, $\mathcal{D}^{p^k}$ is conjugate under G to

$$\partial^{p^{j+k}} + \beta_{0,(1)}^{p^{k-1}} \partial^{p^{k-1}} + x^{(p^n-p^{j+k})} \sum_{i=0}^{p^{j+k}-1} \mu_{i,(k)} x^{(i)} \partial$$

for some $\mu_{i,(k)} \in k$, i.e. the result is true for $r = k$. Therefore, we proved by induction that for any $1 \le r \le n-1-j$, $\mathcal{D}^{p^r}$ is conjugate under G to

$$\partial^{p^{j+r}} + \beta_{0,(1)}^{p^{r-1}} \partial^{p^{r-1}} + x^{(p^n-p^{j+r})} \sum_{i=0}^{p^{j+r}-1} \mu_{i,(r)} x^{(i)} \partial$$

for some $\beta_{0,(1)} \in k^*\mu_0$ and $\mu_{i,(r)} \in k$. In particular, $\mathcal{D}^{p^{n-1-j}}$ is conjugate under G to

$$\partial^{p^{n-1}} + \beta_{0,(1)}^{p^{n-2-j}} \partial^{p^{n-2-j}} + x^{(p^n-p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \mu_{i,(n-1-j)} x^{(i)} \partial . \tag{3.5}$$

By Jacobson's formula,

$$\mathcal{D}^{p^{n-j}} = \beta_{0,(1)}^{p^{n-1-j}} \partial^{p^{n-1-j}} + \sum_{i=0}^{p^{n-1}-1} \mu_{i,(n-1-j)} x^{(i)} \partial + f_{n-j}(x)\partial + \mu_{(n-j)} x^{(p^n-1)} \partial \tag{3.6}$$

for some $f_{n-j}(x)\,\partial \in \mathfrak{L}_{(1)}$ and $\mu_{(n-j)} \in k$. Then

$$\mathcal{D}^{p^n} \equiv \beta_{0,(1)}^{p^{n-1}}\,\partial^{p^{n-1}} + \mu_{0,(n-1-j)}^{p^j}\,\partial^{p^j} + \sum_{i=0}^{j-1} \mu_i'\,\partial^{p^i}\ (\operatorname{mod} \mathfrak{L}_{(0)})$$

for some $\mu_i' \in k$. But $\mathcal{D}^{p^n} = 0$, this implies that $\beta_{0,(1)} = 0$ and so $\mu_0 = 0$. We must also have that $\mu_{0,(n-1-j)} = 0$ and $\mu_i' = 0$ for all i. Substitute these into (3.6), we get

$$\begin{aligned}\mathcal{D}^{p^{n-j}} &= \sum_{i=1}^{p^{n-1}-1} \mu_{i,(n-1-j)} x^{(i)}\,\partial + f_{n-j}(x)\,\partial + \mu_{(n-j)} x^{(p^n-1)}\,\partial \\ &\equiv \mu_{1,(n-1-j)} x\,\partial\ (\operatorname{mod} \mathfrak{L}_{(1)}).\end{aligned}$$

Then one can show similarly that $\mu_{1,(n-1-j)} = 0$. Hence $\mathcal{D}^{p^{n-1-j}}$ (3.5) is conjugate under G to

$$\partial^{p^{n-1}} + x^{(p^n-p^{n-1})} \sum_{i=2}^{p^{n-1}-1} \mu_{i,(n-1-j)} x^{(i)}\,\partial\,.$$

If $j < l$, i.e. $\mathcal{D} = \partial^{p^{n-1}} + \sum_{i=j}^{l} \beta_i \partial^{p^i} + x^{(p^n-p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \mu_i x^{(i)}\partial$, then one can show similarly that $\mathcal{D}^{p^{n-1-j}}$ is conjugate under G to

$$\partial^{p^{n-1}} + \lambda\,\partial^{p^{n-2-j}} + \sum_{i=0}^{n-3-j} \lambda_i\,\partial^{p^i} + x^{(p^n-p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \nu_i x^{(i)}\,\partial$$

for some $\lambda \in k^*\mu_0$ and $\lambda_i, \nu_i \in k$. Then by the same arguments as above, one can show that $\mu_0 = 0$ and $\mathcal{D}^{p^{n-1-j}}$ is conjugate under G to

$$\partial^{p^{n-1}} + x^{(p^n-p^{n-1})} \sum_{i=2}^{p^{n-1}-1} \nu_i x^{(i)}\partial.$$

Suppose now $j \geq 1$. By Jacobson's formula,

$$\mathcal{D}^{p^{n-j}} = \sum_{i=2}^{p^{n-1}-1} \nu_i x^{(i)}\partial + \mu_{(n-j)} x^{(p^n-1)}\,\partial$$

for some $\mu_{(n-j)} \in k$. This is an element of $\mathfrak{L}_{(1)}$. Since G preserves $\mathfrak{L}_{(1)}$ we have that $\mathcal{D}^{p^{n-j}} \in \mathfrak{L}_{(1)}$. As $\mathfrak{L}_{(1)}$ is restricted, we have that $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)}$. This proves statement (ii)(a).

(b) If $\beta_0 \neq 0$, then (ii)(a) implies that $\mathcal{D}^{p^{n-1}}$ is conjugate under G to

$$\partial^{p^{n-1}} + x^{(p^n-p^{n-1})} \sum_{i=2}^{p^{n-1}-1} \nu_i x^{(i)}\,\partial$$

for some $\nu_i \in k$. If q is the smallest index such that $\nu_q \neq 0$, then

$$\mathcal{D}^{p^n} = \sum_{i=q}^{p^{n-1}-1} \nu_i x^{(i)}\,\partial + \mu_{(n)} x^{(p^n-1)}\,\partial$$

for some $\mu_{(n)} \in k$. As $\nu_q \neq 0$, this implies that $\mathcal{D}^{p^n} \neq 0$, a contradiction. Hence $\nu_i = 0$ for all i. Therefore, we are interested in the set

$$\mathcal{S} := \{\mathcal{D} \in \big(\partial^{p^{n-1}} + \sum_{i=1}^{n-2} k\, \partial^{p^i} + \mathfrak{L}\big) \cap \mathcal{N} \mid \mathcal{D}^{p^{n-1}} \text{ is conjugate under } G \text{ to } \partial^{p^{n-1}}\}.$$

Since $[\mathcal{D}, \mathcal{D}^{p^{n-1}}] = 0$, the above set $\mathcal{S}$ is a subset of the centraliser $\mathfrak{z}_{\mathfrak{L}_p}(\partial^{p^{n-1}})$ of $\partial^{p^{n-1}}$ in $\mathfrak{L}_p$. It is easy to verify that $\mathfrak{z}_{\mathfrak{L}_p}(\partial^{p^{n-1}})$ is spanned by $\partial^{p^{n-1}}$ and $W(1, n-1)_p$. Since $W(1, n-1)_p$ is a restricted Lie subalgebra of $\mathfrak{L}_p$, we may regard the automorphism group of $W(1, n-1)_p$ as a subgroup of G. Let $\mathcal{D} = \partial^{p^{n-1}} + \sum_{i=1}^{n-2} \gamma_i\, \partial^{p^i} + n$ be an element of $\mathfrak{z}_{\mathfrak{L}_p}(\partial^{p^{n-1}})$, where $\gamma_i \in k$ and $n \in W(1, n-1)$. If $n = 0$, then $\mathcal{D}^{p^{n-1}} = 0$ which is not conjugate to $\partial^{p^{n-1}}$. So $n \neq 0$. If $n \notin W(1, n-1)_{(0)}$, then $n = \gamma_0\, \partial$ for some $\gamma_0 \neq 0$. It is easy to see that $\mathcal{D}^{p^{n-1}}$ is conjugate under G to $\partial^{p^{n-1}}$. If $n \in W(1, n-1)_{(0)}$, then $n = \sum_{i=1}^{p^{n-1}-1} \lambda_i x^{(i)}\, \partial$ with $\lambda_i \neq 0$ for some i. It follows from Lemma 3.1 that $\mathcal{D}$ is conjugate under G to

$$\partial^{p^{n-1}} + \sum_{i=1}^{n-2} \gamma_i\, \partial^{p^i} + x^{(p^n - p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \lambda_i' x^{(i)}\, \partial$$

for some $\lambda_i' \in k$. If $\gamma_i = 0$ for all i, then (i) of this lemma implies that $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)}$ which is not conjugate to $\partial^{p^{n-1}}$. Similarly, if $j \geq 1$ is the smallest index such that $\gamma_j \neq 0$, then (ii)(a) of this lemma implies that $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)}$ which is again not conjugate to $\partial^{p^{n-1}}$. Therefore, the set $\mathcal{S}$ consists of elements of the form $\mathcal{D} = \partial^{p^{n-1}} + \sum_{i=0}^{n-2} \gamma_i\, \partial^{p^i}$ with $\gamma_i \in k$ such that $\gamma_0 \neq 0$. This proves statement (ii)(b). □

Corollary 3.2. *Let $\mathcal{D} = \partial^{p^m} + \sum_{i=0}^{m-1} \alpha_i \partial^{p^i} + x^{(p^n - p^m)} \sum_{i=0}^{p^m-1} \mu_i x^{(i)}\, \partial$ with $1 \leq m \leq n-2$ be a nilpotent element of $\mathfrak{L}_p$.*

(i) *If $\alpha_i = 0$ for all i, then $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)}$.*

(ii) (a) *If $q \geq 0$ is the smallest index such that $\alpha_q \neq 0$, then $\mathcal{D}^{p^{n-1-q}}$ is conjugate under G to*

$$\partial^{p^{n-1}} + x^{(p^n - p^{n-1})} \sum_{i=2}^{p^{n-1}-1} \nu_i x^{(i)}\, \partial$$

for some $\nu_i \in k$. Hence $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)}$ for all $q \geq 1$.

(b) *In particular, if $\alpha_0 \neq 0$, then $\mathcal{D}^{p^{n-1}}$ is conjugate under G to $\partial^{p^{n-1}}$. Hence $\mathcal{D} = \partial^{p^m} + \sum_{i=0}^{m-1} \gamma_i\, \partial^{p^i}$ for some $\gamma_i \in k$ with $\gamma_0 \neq 0$.*

Proof. Take $\mathcal{D}$ as in the corollary. By Corollary 3.1, one can prove by induction that for any $1 \leq r \leq n-1-m$, $\mathcal{D}^{p^r}$ is conjugate under G to

$$\partial^{p^{m+r}} + \sum_{i=0}^{m-1} \alpha_i^{p^r}\, \partial^{p^{i+r}} + x^{(p^n - p^{m+r})} \sum_{i=0}^{p^{m+r}-1} \mu_{i,(r)} x^{(i)}\, \partial$$

for some $\mu_{i,(r)} \in k$. In particular, $\mathcal{D}^{p^{n-1-m}}$ is conjugate under G to

$$\partial^{p^{n-1}} + \sum_{i=0}^{m-1} \alpha_i^{p^{n-1-m}} \partial^{p^{i+n-1-m}} + x^{(p^n - p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \mu_{i,(n-1-m)} x^{(i)} \partial .$$

(i) If $\alpha_i = 0$ for all i, then Lemma 3.2(i) implies that $\mu_{0,(n-1-m)} = \mu_{1,(n-1-m)} = 0$. By Jacobson's formula,

$$\mathcal{D}^{p^{n-m}} = \sum_{i=2}^{p^{n-1}-1} \mu_{i,(n-1-m)} x^{(i)} \partial + \mu_{(n-m)} x^{(p^n-1)} \partial$$

for some $\mu_{(n-m)} \in k$. This is an element of $\mathfrak{L}_{(1)}$. Since G preserves $\mathfrak{L}_{(1)}$, this implies that $\mathcal{D}^{p^{n-m}} \in \mathfrak{L}_{(1)}$. As $\mathfrak{L}_{(1)}$ is restricted, we have that $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)}$. This proves statement (i).

(ii) If $q \geq 0$ is the smallest index such that $\alpha_q \neq 0$, then $\mathcal{D}^{p^{n-1-m}}$ is conjugate under G to

$$\partial^{p^{n-1}} + \sum_{i=q}^{m-1} \alpha_i^{p^{n-1-m}} \partial^{p^{i+n-1-m}} + x^{(p^n - p^{n-1})} \sum_{i=0}^{p^{n-1}-1} \mu_{i,(n-1-m)} x^{(i)} \partial .$$

It follows from Lemma 3.2(ii)(a) that $\mathcal{D}^{p^{n-1-q}}$ is conjugate under G to

$$\partial^{p^{n-1}} + x^{(p^n - p^{n-1})} \sum_{i=2}^{p^{n-1}-1} \nu_i x^{(i)} \partial$$

for some $\nu_i \in k$. Suppose now $q \geq 1$, then it is easy to see that $\mathcal{D}^{p^{n-q}}$ is an element of $\mathfrak{L}_{(1)}$. Since $\mathfrak{L}_{(1)}$ is restricted, we have that $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)}$. This proves statement (ii)(a).

If $\alpha_0 \neq 0$, then the result follows from above and Lemma 3.2(ii)(b). This proves statement (ii)(b). □

3.2. The calculations in the last subsection enable us to identify an irreducible component of $\mathcal{N}$.

Lemma 3.3. *Define* $\mathcal{N}_{reg} := \{\mathcal{D} \in \mathcal{N} \mid \mathcal{D}^{p^{n-1}} \notin \mathfrak{L}_{(0)}\}$. *Then*

$$\mathcal{N}_{reg} = G.(k^* \partial + k\, \partial^p + \cdots + k\, \partial^{p^{n-1}}).$$

Proof. Since $(\partial + \sum_{i=1}^{n-1} \alpha_i \partial^{p^i})^{p^n} = 0$ and $(\partial + \sum_{i=1}^{n-1} \alpha_i \partial^{p^i})^{p^{n-1}} = \partial^{p^{n-1}}$, this shows that any elements which are conjugate under G to $\partial + \sum_{i=1}^{n-1} \alpha_i \partial^{p^i}$ are contained in $\mathcal{N}_{\text{reg}}$. So $G.(k^* \partial + k\, \partial^p + \cdots + k\, \partial^{p^{n-1}}) \subseteq \mathcal{N}_{\text{reg}}$. To show that $\mathcal{N}_{\text{reg}} \subseteq G.(k^* \partial + k\, \partial^p + \cdots + k\, \partial^{p^{n-1}})$, we observe that if $\mathcal{D} \in \mathfrak{L}_{(1)}$, then $\mathcal{D}$ is nilpotent and $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)} \subset \mathfrak{L}_{(0)}$. Hence $\mathcal{D} \notin \mathcal{N}_{\text{reg}}$. Therefore, $\mathcal{N}_{\text{reg}} \subseteq \mathcal{N} \setminus \mathfrak{L}_{(1)}$.

Note that elements of $\mathcal{N} \setminus \mathfrak{L}_{(1)}$ have the form $\mathcal{D} = \sum_{i=0}^{n-1} \alpha_i \partial^{p^i} + f(x)\partial$ for some $f(x) \in \mathfrak{m}$ and $\alpha_i \in k$ with at least one $\alpha_i \neq 0$. If $\alpha_0 \neq 0$ and $\alpha_i = 0$ for all $i \geq 1$, then $\mathcal{D} = \alpha_0 \partial + f(x)\partial$. Hence

$$\mathcal{D}^{p^{n-1}} = \alpha_0^{p^{n-1}} \partial^{p^{n-1}} + \sum_{i=0}^{n-2} \alpha_i' \partial^{p^i} + w$$

for some $\alpha_i' \in k$ and $w \in \mathfrak{L}_{(0)}$. As $\alpha_0 \neq 0$, this implies that $\mathcal{D}^{p^{n-1}} \notin \mathfrak{L}_{(0)}$ and so $\mathcal{D} \in \mathcal{N}_{\mathrm{reg}}$. Apply the automorphism $\Phi(x) = \alpha_0 x$ to $\mathcal{D}$, we may assume that $\mathcal{D}$ has the form

$$\partial + g(x)\partial$$

for some $g(x) \in \mathfrak{m}$. By [1, Lemma 1], $\mathcal{D}$ is conjugate under G to

$$\partial + \sum_{i=1}^{n} \beta_i x^{(p^i-1)}\partial$$

for some $\beta_i \in k$. Then it follows from [9, Proposition 4.3] that $\mathcal{D}$ is nilpotent if and only if $\beta_i = 0$ for all i. Consequently, $\mathcal{D}$ is conjugate under G to ∂. Thus $\mathcal{N}_{\mathrm{reg}} \subseteq G.(k^*\,\partial + k\,\partial^p + \cdots + k\,\partial^{p^{n-1}})$ in this case.

For the other elements of $\mathcal{N} \setminus \mathfrak{L}_{(1)}$, let $1 \leq t \leq n-1$ be the largest index such that $\alpha_t \neq 0$, i.e. $\mathcal{D} = \sum_{i=0}^{t} \alpha_i \partial^{p^i} + f(x)\partial$. Then the automorphism $\Phi(x) = \alpha x$ with $\alpha^{p^t} = \alpha_t$ reduces $\mathcal{D}$ to the form

$$\mathcal{D} = \partial^{p^t} + \sum_{i=0}^{t-1} \beta_i \partial^{p^i} + g(x)\,\partial$$

for some $\beta_i \in k^*\alpha_i$ and $g(x) \in \mathfrak{m}$. It follows from Lemma 3.1 and Corollary 3.1 that $\mathcal{D}$ is conjugate under G to

$$\partial^{p^t} + \sum_{i=0}^{t-1} \beta_i \partial^{p^i} + x^{(p^n-p^t)} \sum_{i=0}^{p^t-1} \mu_i x^{(i)}\,\partial$$

for some $\mu_i \in k$. If $\beta_i = 0$ for all i, then Lemma 3.2 and Corollary 3.2(i) imply that $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)} \subset \mathfrak{L}_{(0)}$. If $j \geq 1$ is the smallest index such that $\beta_j \neq 0$, then Lemma 3.2 and Corollary 3.2(ii)(a) imply that $\mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(1)} \subset \mathfrak{L}_{(0)}$. Hence in both cases $\mathcal{D}$ is not in $\mathcal{N}_{\mathrm{reg}}$. But if $\beta_0 \neq 0$, then it is easy to see that $\mathcal{D}^{p^{n-1}} \notin \mathfrak{L}_{(0)}$. So $\mathcal{D} \in \mathcal{N}_{\mathrm{reg}}$. Moreover, it follows from Lemma 3.2 and Corollary 3.2(ii)(b) that $\mathcal{D}$ is conjugate under G to $\partial^{p^t} + \sum_{i=0}^{t-1} \gamma_i\,\partial^{p^i}$ for some $\gamma_i \in k$ with $\gamma_0 \neq 0$. Hence $\mathcal{N}_{\mathrm{reg}} \subseteq G.(k^*\,\partial + k\,\partial^p + \cdots + k\,\partial^{p^{n-1}})$ in this case. Since we have exhausted all elements of $\mathcal{N}_{\mathrm{reg}}$, this completes the proof. □

Before we proceed to show that the Zariski closure of $\mathcal{N}_{\mathrm{reg}}$ is an irreducible component of $\mathcal{N}$, we need the following results.

Lemma 3.4. *Let $\mathcal{D} = \partial + \sum_{i=1}^{n-1} \lambda_i\,\partial^{p^i}$ with $\lambda_i \in k$ and denote by $\mathfrak{z}_{\mathfrak{L}}(\mathcal{D})$ (respectively $\mathfrak{z}_{\mathfrak{L}_p}(\mathcal{D})$) the centraliser of $\mathcal{D}$ in $\mathfrak{L}$ (respectively $\mathfrak{L}_p$). Then*

(i) $\mathfrak{z}_{\mathfrak{L}}(\mathcal{D}) = \mathrm{span}\{\partial\}$.

(ii) $\mathfrak{z}_{\mathfrak{L}_p}(\mathcal{D}) = \mathrm{span}\{\partial, \partial^p, \ldots, \partial^{p^{n-1}}\}$.

(iii) $\mathfrak{z}_{\mathfrak{L}}(\mathcal{D}) \cap \mathrm{Lie}(G) = \{0\}$.

Proof. (i) Clearly, $\mathrm{span}\{\partial\} \subseteq \mathfrak{z}_{\mathfrak{L}}(\mathcal{D})$. Since $(\mathrm{ad}\,\mathcal{D})^{p^n-1} \neq 0$ and $(\mathrm{ad}\,\mathcal{D})^{p^n} = 0$, the theory of canonical Jordan normal form says that there exists a basis $\mathcal{B}$ of $\mathfrak{L}$ such that the matrix of $\mathrm{ad}\,\mathcal{D}$ with respect to $\mathcal{B}$ is a single Jordan block of size p^n with zeros on the main diagonal. Hence the matrix of $\mathrm{ad}\,\mathcal{D}$ has rank $p^n - 1$. This implies that $\ker(\mathrm{ad}\,\mathcal{D})$ has dimension 1. By definition, $\ker(\mathrm{ad}\,\mathcal{D}) = \mathfrak{z}_{\mathfrak{L}}(\mathcal{D})$. Hence $\mathfrak{z}_{\mathfrak{L}}(\mathcal{D}) = \mathrm{span}\{\partial\}$.

(ii) It is clear that $\operatorname{span}\{\partial, \partial^p, \ldots, \partial^{p^{n-1}}\} \subseteq \mathfrak{z}_{\mathfrak{L}_p}(\mathcal{D})$. Suppose $v \in \mathfrak{z}_{\mathfrak{L}_p}(\mathcal{D})$. Then we can write $v = \sum_{i=0}^{n-1} \alpha_i \, \partial^{p^i} + v_1$ for some $v_1 \in \mathfrak{L}_{(0)}$. Since $\sum_{i=0}^{n-1} \alpha_i \, \partial^{p^i} \in \mathfrak{z}_{\mathfrak{L}_p}(\mathcal{D})$, we must have that $v_1 \in \mathfrak{z}_{\mathfrak{L}_p}(\mathcal{D})$. By (i), the centraliser of $\mathcal{D}$ in $\mathfrak{L}$ is $k\,\partial$ which is not in $\mathfrak{L}_{(0)}$. Hence $v_1 = 0$ and $\mathfrak{z}_{\mathfrak{L}_p}(\mathcal{D}) = \operatorname{span}\{\partial, \partial^p, \ldots, \partial^{p^{n-1}}\}$.

(iii) It follows from (i) and Lemma 2.1. This completes the proof of the lemma. □

Lemma 3.5. *The Zariski closure of $\mathcal{N}_{reg}$ is an irreducible component of $\mathcal{N}$.*

Proof. By Lemma 3.3, it suffices to show that the Zariski closure of $G.(k^* \, \partial + k \, \partial^p + \cdots + k \, \partial^{p^{n-1}})$ is an irreducible component of $\mathcal{N}$. Put $X := k^* \, \partial + k \, \partial^p + \cdots + k \, \partial^{p^{n-1}}$. Then $X \cong \mathbb{A}^{n-1}$ which is irreducible. Moreover, G is a connected algebraic group so that $\overline{G.X}$ is an irreducible variety contained in $\mathcal{N}$. Then $\dim \overline{G.X} \leq \dim \mathcal{N}$. If $\dim \overline{G.X} \geq \dim \mathcal{N}$, then we get the desired result.

Define Ψ to be the morphism

$$\Psi : G \times X \to \overline{G.X}$$
$$(g, \mathcal{D}) \mapsto g.\mathcal{D}$$

Since $G.X$ is dense in $\overline{G.X}$, it contains smooth points of $\overline{G.X}$. As the set of smooth points is G-invariant, there exists $\mathcal{D} \in X$ such that $\Psi(1, \mathcal{D}) = \mathcal{D}$ is a smooth point in $\overline{G.X}$. We may assume without loss of generality that $\mathcal{D} = \partial + \sum_{i=1}^{n-1} \lambda_i \, \partial^{p^i}$ for some $\lambda_i \in k$. Then the differential of Ψ at the smooth point $(1, \mathcal{D})$ is the map

$$d_{(1,\mathcal{D})}\Psi : \operatorname{Lie}(G) \oplus X \to T_{\mathcal{D}}(\overline{G.X}).$$

Since $\dim T_{\mathcal{D}}(\overline{G.X}) = \dim \overline{G.X}$, it is enough to show that $\dim T_{\mathcal{D}}(\overline{G.X}) \geq \dim \mathcal{N} = p^n - 1$. It is easy to see that $T_{\mathcal{D}}(\overline{G.X})$ contains $T_{\mathcal{D}}(X) = X$ which has dimension $n-1$. Since $\mathcal{D} \in X$, $T_{\mathcal{D}}(\overline{G.X})$ also contains $T_{\mathcal{D}}(G.\mathcal{D})$, the image of $\operatorname{Lie}(G) \oplus \mathcal{D}$ under $d_{(1,\mathcal{D})}\Psi$, i.e. $T_{\mathcal{D}}(G.\mathcal{D}) = d_{(1,\mathcal{D})}\Psi(\operatorname{Lie}(G) \oplus \mathcal{D})$. By Lemma 3.4(iii), $\mathfrak{z}_{\mathfrak{L}}(\mathcal{D}) \cap \operatorname{Lie}(G) = \{0\}$, this implies that the restriction of the linear operator $\operatorname{ad} \mathcal{D}$ to $\operatorname{Lie}(G)$ has trivial kernel and so the image $[\mathcal{D}, \operatorname{Lie}(G)]$ is isomorphic to $\operatorname{Lie}(G)$. Hence

$$T_{\mathcal{D}}(G.\mathcal{D}) = d_{(1,\mathcal{D})}\Psi(\operatorname{Lie}(G) \oplus \mathcal{D}) = [\mathcal{D}, \operatorname{Lie}(G)] \cong \operatorname{Lie}(G).$$

Therefore, $T_{\mathcal{D}}(\overline{G.X})$ contains $\operatorname{Lie}(G)$ which has dimension $p^n - n$. It follows from Lemma 2.1 that $X \cap \operatorname{Lie}(G) = \{0\}$. Hence $T_{\mathcal{D}}(\overline{G.X})$ contains the direct sum $X \oplus \operatorname{Lie}(G)$ of X and $\operatorname{Lie}(G)$. Therefore,

$$\dim T_{\mathcal{D}}(\overline{G.X}) \geq \dim \big(X \oplus \operatorname{Lie}(G)\big) = \dim X + \dim \operatorname{Lie}(G) = p^n - 1 = \dim \mathcal{N}.$$

This completes the proof. □

3.3. Our goal is to prove the irreducibility of the variety $\mathcal{N}$. To achieve this, we need the following result.

Proposition 3.1. *Define $\mathcal{N}_{sing} := \mathcal{N} \setminus \mathcal{N}_{reg} = \{\mathcal{D} \in \mathcal{N} \mid \mathcal{D}^{p^{n-1}} \in \mathfrak{L}_{(0)}\}$. Then*

$$\dim \mathcal{N}_{sing} < \dim \mathcal{N}.$$

Clearly, $\mathcal{N}_{\text{sing}}$ is Zariski closed in $\mathcal{N}$. To prove this proposition, we need to construct an $(n+1)$-dimensional subspace V in $\mathfrak{L}_p$ such that $V \cap \mathcal{N}_{\text{sing}} = \{0\}$. Then the result follows from [2, Ch. I, Proposition 7.1]; see a similar proof in [6]. The way V is constructed relies on the original definition of $\mathfrak{L}$ due to H. Zassenhaus and the following lemmas. Recall that $\mathfrak{L}$ has a k-basis $\{e_\alpha \,|\, \alpha \in \mathbb{F}_q\}$ with the Lie bracket given by $[e_\alpha, e_\beta] = (\beta - \alpha)e_{\alpha+\beta}$. Here $\mathbb{F}_q \subset k$ is a finite field of $q = p^n$ elements. The multiplicative group $\mathbb{F}_q^\times$ of $\mathbb{F}_q$ is cyclic of order $p^n - 1$ with generator ξ; see § 2.1 for detail.

Lemma 3.6. *Let $\sigma \in \mathrm{GL}(\mathfrak{L})$ be such that $\sigma(e_\alpha) := \xi^{-1}e_{\xi\alpha}$ for any $\alpha \in \mathbb{F}_q$. Then σ is a diagonalizable automorphism of $\mathfrak{L}$.*

Proof. By definition,

$$[\sigma(e_\alpha), \sigma(e_\beta)] = [\xi^{-1}e_{\xi\alpha}, \xi^{-1}e_{\xi\beta}] = \xi^{-2}(\xi\beta - \xi\alpha)e_{\xi\alpha+\xi\beta} = \xi^{-1}(\beta-\alpha)e_{\xi(\alpha+\beta)} = \sigma([e_\alpha, e_\beta])$$

for any $\alpha, \beta \in \mathbb{F}_q$. So the endomorphism σ is an automorphism of $\mathfrak{L}$. Since $\xi^{p^n-1} = 1$, we have that $\sigma^{p^n-1} = \mathrm{id}$. As k is an algebraically closed field, the automorphism σ is diagonalizable. □

Since σ is an automorphism of $\mathfrak{L}$, it respects the natural filtration $\{\mathfrak{L}_{(i)}\}$ $(i \geq -1)$ of $\mathfrak{L}$.

Lemma 3.7. *The automorphism σ acts as a scalar on each 1-dimensional vector space $\mathfrak{L}_{(i)}/\mathfrak{L}_{(i+1)}$.*

Proof. Let T denote the torus of the p-envelope $\langle e_0 \rangle_p$ in $\mathfrak{L}_p$ generated by e_0. Let $\mathbb{F}_p \subset k$ denote the finite field with p elements. By [11, Theorem 1.3.11(1)], $\dim_k T$ is the $\mathbb{F}_p$-dimension of the $\mathbb{F}_p$-vector space spanned by the T-roots; see also the proof of [11, Theorem 7.6.3(2)]. Since $[e_0, e_\beta] = \beta e_\beta$ for any $\beta \in \mathbb{F}_q$, the endomorphism $\mathrm{ad}(e_0)$ has $q = p^n$ distinct eigenvalues. Therefore, $\dim_k T = n$. As $\sigma(e_0) = \xi^{-1}e_0$ and $\sigma(e_0^{p^j}) = \xi^{-p^j}e_0^{p^j}$ for all $j \geq 1$, we see that $\xi^{-1}, \xi^{-p}, \xi^{-p^2}, \ldots, \xi^{-p^{n-1}}$ are the eigenvalues of σ on T. Note that $e_0 \notin \mathfrak{L}_{(0)}$. Indeed, if $e_0 \in \mathfrak{L}_{(0)}$, then T is contained in $\mathfrak{L}_{(0)}$ as $\mathfrak{L}_{(0)}$ is restricted. But this contradicts the fact that any torus of $\mathfrak{L}_{(0)}$ has dimension 1 [12, p. 67]. Therefore, $e_0 \notin \mathfrak{L}_{(0)}$. As a result, $T \cap \mathfrak{L} = ke_0$.

Consider the surjective map $\pi : \mathfrak{L}_{(-1)} \twoheadrightarrow \mathfrak{L}_{(-1)}/\mathfrak{L}_{(0)}$. Since $e_0 \notin \mathfrak{L}_{(0)}$, the vector space $\mathfrak{L}_{(-1)}/\mathfrak{L}_{(0)} = k\,\partial$ is spanned by $\pi(e_0)$. This implies that e_0 has weight -1 and σ acts on $\mathfrak{L}_{(-1)}/\mathfrak{L}_{(0)}$ as $\xi^{-1}\,\mathrm{id}$. Similarly, we can show that σ acts on $\mathfrak{L}_{(k)}/\mathfrak{L}_{(k+1)}$ as $\xi^k\,\mathrm{id}$ for $0 \leq k \leq p^n - 2$. Indeed, elements of $\mathfrak{L}_{(k)}/\mathfrak{L}_{(k+1)}$ have the form $x + \mathfrak{L}_{(k+1)}$ for some $x \in \mathfrak{L}_{(k)}$. Since $e_0 \in \mathfrak{L}_{(-1)} \setminus \mathfrak{L}_{(0)}$, we have that $[e_0, \mathfrak{L}_{(k+1)}] \subseteq [\mathfrak{L}_{(-1)}, \mathfrak{L}_{(k+1)}] \subseteq \mathfrak{L}_{(k)}$. Hence $[e_0, \mathfrak{L}_{(k+1)}] + \mathfrak{L}_{(k+1)} = \mathfrak{L}_{(k)}$. In particular, $[e_0, \mathfrak{L}_{(0)}] + \mathfrak{L}_{(0)} = \mathfrak{L}_{(-1)}$. If $u \in \mathfrak{L}_{(0)}$ is such that $[e_0, u] \notin \mathfrak{L}_{(0)}$, i.e. $[e_0, u] \neq 0$ on $\mathfrak{L}_{(-1)}/\mathfrak{L}_{(0)}$, then u is an eigenvector of σ corresponding to an eigenvalue, say λ. Then

$$\sigma[e_0, u] = [\sigma(e_0), \sigma(u)] = [\xi^{-1}e_0, \lambda u] = \xi^{-1}\lambda[e_0, u].$$

So $\xi^{-1}\lambda$ is the eigenvalue of σ on $\mathfrak{L}_{(-1)}/\mathfrak{L}_{(0)}$. But σ acts on $\mathfrak{L}_{(-1)}/\mathfrak{L}_{(0)}$ as $\xi^{-1}\,\mathrm{id}$, we must have that $\xi^{-1}\lambda = \xi^{-1}\,\mathrm{id}$. Thus $\lambda = 1$, i.e. σ acts on $\mathfrak{L}_{(0)}/\mathfrak{L}_{(1)}$ as id. Continue in this way, one can show that σ acts on $\mathfrak{L}_{(1)}/\mathfrak{L}_{(2)}$ as $\xi\,\mathrm{id}$, on $\mathfrak{L}_{(k)}/\mathfrak{L}_{(k+1)}$ as $\xi^k\,\mathrm{id}$ and on $\mathfrak{L}_{(p^n-2)}$ as $\xi^{p^n-2}\,\mathrm{id} = \xi^{-1}\,\mathrm{id}$. This completes the proof. □

Remark 3.1. The last lemma shows that

(i) the eigenvalues of σ on $\mathfrak{L}$ are $\xi^{-1}, \xi^0 = 1, \xi, \dots, \xi^{p^n-3}$ and $\xi^{p^n-2} = \xi^{-1}$. All have multiplicity 1 except ξ^{-1} which has multiplicity 2;

(ii) the eigenvalues of σ on $\mathfrak{L}_{(0)}$ are $\xi^0 = 1, \xi, \dots, \xi^{p^n-3}$ and $\xi^{p^n-2} = \xi^{-1}$. All have multiplicity 1;

(iii) the eigenspace $\mathfrak{L}[k] := \{\mathcal{D} \in \mathfrak{L} \,|\, \sigma(\mathcal{D}) = \xi^k \mathcal{D}\}$ corresponding to the eigenvalue ξ^k, where $0 \leq k \leq p^n - 3$, has dimension 1. In particular, the eigenspace $\mathfrak{L}[0] = ku$, which is a torus in $\mathfrak{L}_{(0)}$. Since any torus has a basis consisting of toral elements, we may assume that u is toral, i.e. $u^p = u$;

(iv) the eigenspace $\mathfrak{L}[-1] = \text{span}\,\{e_0, v \,|\, v \in \mathfrak{L}_{(p^n-2)}\}$ and has dimension 2.

Proof of Proposition 3.1. Recall that the n-dimensional torus $T = \langle e_0 \rangle_p$ in $\mathfrak{L}_p$ and the toral element $u \in \mathfrak{L}_{(0)} \setminus \mathfrak{L}_{(1)}$; see Lemma 3.7 and Remark 3.1(iii). Put $V := T \oplus ku = \sum_{i=0}^{n-1} k e_0^{p^i} \oplus ku$. We want to show that $V \cap \mathcal{N}_{\text{sing}} = \{0\}$.

Suppose for contradiction that $V \cap \mathcal{N}_{\text{sing}} \neq \{0\}$. Then take a nonzero element y in $V \cap \mathcal{N}_{\text{sing}}$, we can write

$$y = \sum_{i=0}^{n-1} \lambda_i e_0^{p^i} + \mu u$$

for some $\lambda_i, \mu \in k$ with at least one $\lambda_i \neq 0$. Suppose $\lambda_0 \neq 0$. Since $e_0 \in \mathfrak{L}_{(-1)} \setminus \mathfrak{L}_{(0)}$, we may assume without loss of generality that $e_0 = \partial + \sum_{i=1}^{p^n-1} \alpha_i x^{(i)}\,\partial$ for some $\alpha_i \in k$. As the p-th power map on T is periodic, i.e. $(e_0^{p^k})^{p^{n-1}} = e_0^{p^{k-1}}$ for all $k \geq 1$, this implies that

$$y^{p^{n-1}} = \lambda_0^{p^{n-1}}\,\partial^{p^{n-1}} + \sum_{i=0}^{n-2} \lambda_i'\,\partial^{p^i} + w$$

for some $\lambda_i' \in k$ and $w \in \mathfrak{L}_{(0)}$. Since $\lambda_0 \neq 0$, this shows that $y^{p^{n-1}}$ is not in $\mathfrak{L}_{(0)}$, a contradiction.

Suppose now $\lambda_0 = 0$ and let $1 \leq s \leq n-1$ be the largest index such that $\lambda_s \neq 0$. Then

$$y = \sum_{i=1}^{s} \lambda_i e_0^{p^i} + \mu u.$$

By [11, Lemma 1.1.1] and the fact that $e_0^{p^n} = e_0$, we have that

$$y^{p^{n-s}} = \lambda_s^{p^{n-s}} e_0 + \lambda_{s-1}^{p^{n-s}}\, e_0^{p^{n-1}} + \cdots + \lambda_1^{p^{n-s}}\, e_0^{p^{n-s+1}} + \mu^{p^{n-s}} u + \sum_{l=0}^{n-s-1} v_l^{p^l}, \tag{3.7}$$

where v_l is a linear combination of commutators in $e_0^{p^j}$ $(1 \leq j \leq s)$ and u. By the Jacobi identity, we can rearrange each v_l so that

$$v_l \in \text{span}\{[e_0^{a_0}[u^{b_1}[e_0^{a_1}[u^{b_2}[e_0^{a_2}[\dots[e_0^{a_{t-1}}[u^{b_t}[e_0^{a_t}, u]\dots]\},$$

where $[e_0^{a_0}[u^{b_1}[e_0^{a_1}[u^{b_2}[e_0^{a_2}[\dots[e_0^{a_{t-1}}[u^{b_t}[e_0^{a_t}, u]\dots]$ is a left normed commutator of length p^{n-s-l} with u at the right end, and $p \leq a_i \leq p^s, b_i$ are arbitrary constants. Since e_0 and u are eigenvectors of σ corresponding to eigenvalues ξ^{-1} and $\xi^0 = 1$, respectively, the commutator $[e_0^{a_0}[u^{b_1}[e_0^{a_1}[u^{b_2}[e_0^{a_2}[\dots[e_0^{a_{t-1}}[u^{b_t}[e_0^{a_t},u]\dots]$ is an eigenvector of σ corresponding to the eigenvalue $\xi^{-(a_0+a_1+\cdots+a_t)}$. As

$$a_0+a_1+\cdots+a_t \le (p^{n-s-l}-1)p^s \le (p^{n-s}-1)p^s = p^n-p^s \le p^n-p \ne 1,$$

the eigenvalue $\xi^{-(a_0+a_1+\cdots+a_t)}$ is not equal to ξ^{-1}. Hence $v_l \in \mathfrak{L}_{(0)} \setminus \mathfrak{L}_{(p^n-2)} \subset \mathfrak{L}_{(0)}$. Since $\mathfrak{L}_{(0)}$ is restricted, we have that $v_l^{p^l} \in \mathfrak{L}_{(0)}$ and so $\sum_{l=0}^{n-s-1} v_l^{p^l} \in \mathfrak{L}_{(0)}$. As e_0 is not in $\mathfrak{L}_{(0)}$, this shows that the term $\sum_{l=0}^{n-s-1} v_l^{p^l}$ does not cancel with the first term $\lambda_s^{p^{n-s}} e_0$ in $y^{p^{n-s}}$ (3.7). Therefore,

$$y^{p^{n-s}} = \lambda_s^{p^{n-s}} e_0 + \lambda'_{n-1} e_0^{p^{n-1}} + \cdots + \lambda'_{n-s+1} e_0^{p^{n-s+1}} + w_1$$

for some $\lambda'_i \in k$ and $w_1 \in \mathfrak{L}_{(0)}$. Then we know that $(y^{p^{n-s}})^{p^{n-1}} = y^{p^{2n-s-1}}$ belongs to $\lambda_s^{p^{2n-s-1}} e_0^{p^{n-1}} + \sum_{i=0}^{n-2} \mathfrak{L}^{p^i}$. As $\lambda_s \ne 0$, this implies that $y^{p^{2n-s-1}}$ is not equal to 0. But $2n-s-1 \ge n$, this contradicts that y is nilpotent. Therefore, we proved by contradiction that $V \cap \mathcal{N}_{\text{sing}} = \{0\}$. The result then follows from [2, Ch. I, Proposition 7.1]. This completes the proof. □

Theorem 3.1. *The variety $\mathcal{N}$ is irreducible.*

Proof. The variety $\mathcal{N}$ is equidimensional of dimension p^n-1. The ideal defining $\mathcal{N}$ is homogeneous, hence any irreducible component of $\mathcal{N}$ contains 0 [7, Theorem 4.2]. It follows from Lemma 3.5 that the Zariski closure of $\mathcal{N}_{\text{reg}}$ is an irreducible component of $\mathcal{N}$. Let $Z_1,\dots,Z_t$ be pairwise distinct irreducible components of $\mathcal{N}$, and set $Z_1 = \overline{\mathcal{N}_{\text{reg}}}$. Suppose $t \ge 2$. Then $Z_2 \setminus Z_1$ is contained in $\mathcal{N}_{\text{sing}}$, which is Zariski closed in $\mathcal{N}$ with $\dim \mathcal{N}_{\text{sing}} < \dim \mathcal{N}$ by Proposition 3.1. Since $Z_2 \setminus Z_1 = Z_2 \setminus (Z_1 \cap Z_2)$, this set is Zariski dense in Z_2. Then its closure Z_2 is also contained in $\mathcal{N}_{\text{sing}}$, i.e. $\dim Z_2 = \dim \mathcal{N} \le \dim \mathcal{N}_{\text{sing}}$. This is a contradiction. Hence $t=1$ and the variety $\mathcal{N}$ is irreducible. This completes the proof of Theorem 1.1. □

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