



Multilevel ensemble Kalman filtering

DOI:
[10.1137/15M100955X](https://doi.org/10.1137/15M100955X)

Document Version
Accepted author manuscript

[Link to publication record in Manchester Research Explorer](#)

Citation for published version (APA):
Hoel, H., Law, K. J. H., & Tempone, R. (2016). Multilevel ensemble Kalman filtering. *SIAM JOURNAL ON NUMERICAL ANALYSIS*, 54(3), 1813-1839. <https://doi.org/10.1137/15M100955X>

Published in:
SIAM JOURNAL ON NUMERICAL ANALYSIS

Citing this paper
Please note that where the full-text provided on Manchester Research Explorer is the Author Accepted Manuscript or Proof version this may differ from the final Published version. If citing, it is advised that you check and use the publisher's definitive version.

General rights
Copyright and moral rights for the publications made accessible in the Research Explorer are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

Takedown policy
If you believe that this document breaches copyright please refer to the University of Manchester's Takedown Procedures [<http://man.ac.uk/04Y6Bo>] or contact uml.scholarlycommunications@manchester.ac.uk providing relevant details, so we can investigate your claim.



MULTILEVEL ENSEMBLE KALMAN FILTERING

HÅKON HOEL*, KODY J. H. LAW†, AND RAUL TEMPONE‡

Abstract. This work embeds a multilevel Monte Carlo (MLMC) sampling strategy into the Monte Carlo step of the ensemble Kalman filter (EnKF) in the setting of finite dimensional signal evolution and noisy discrete-time observations. The signal dynamics is assumed to be governed by a stochastic differential equation (SDE), and a hierarchy of time grids is introduced for multilevel numerical integration of that SDE. The resulting multilevel ensemble Kalman filter method (MLEnKF) is proved to asymptotically outperform EnKF in terms of computational cost vs. approximation accuracy. The theoretical results are illustrated numerically.

Key words: Monte Carlo, multilevel, filtering, Kalman filter, ensemble Kalman filter.

AMS subject classification: 65C30, 65Y20.

1. Introduction. Filtering refers to the sequential estimation of the state u and/or parameters p of a system through sequential incorporation of online data y . The most complete estimation of the state u_n at time n is given by its probability density conditional on the observations up to the given time $\pi(u_n|y_1, \dots, y_n)$ [28, 3]. For linear Gaussian systems the analytical solution may be given in closed form, via an update formulae for the mean and covariance known as the Kalman filter [29]. However, in general there is no closed form solution. One must therefore resort to either algorithms which approximate the probabilistic solution by leveraging ideas from control theory [30, 28], or Monte Carlo methods to approximate the filtering distribution itself [3, 13, 12]. The ensemble Kalman filter (EnKF) [7, 15] combines elements of both approaches. In the linear Gaussian case, it converges to the Kalman filter solution [38], and even in the nonlinear case, under suitable assumptions it converges [35, 34] to what one may argue is the optimal filter among those which incorporate the data linearly [34, 37, 40]. In the case of spatial models approximated on a numerical grid, the state space itself may become very high-dimensional and even the linear solves may become intractable. Therefore, one may be inclined to use the EnKF filter even for linear Gaussian problems in which the solution is intractable despite being given in closed form on paper by the Kalman filter.

For problems which admit hierarchies of approximations with cost inversely proportional to accuracy, it is natural to leverage solutions to less expensive and less accurate approximations in order to accelerate the convergence of the more expensive and more accurate approximations. This idea originated in the iterative solution of numerical approximation of elliptic PDE as early as [16]. In the context of iterative solution of PDE, the methodology, which may be used both for solution as well as pre-conditioner, has become known as multigrid – see [23] for a general reference. The same idea may be applied in the context of Monte Carlo approximation of random fields as proposed in [24], and later studied in detail in the context of SDE in [18] and PDE in [10]. There has been an explosion of recent activity since [18], including for example [19, 11, 26, 6, 39]. Even more recently, work is beginning to emerge

*Department of Mathematics, University of Oslo, Norway (haakonah@math.uio.no), and Applied Mathematics and Computational Sciences, KAUST, Thuwal, Saudi Arabia

†Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, TN, USA (lawkj@ornl.gov)

‡Applied Mathematics and Computational Sciences, KAUST, Thuwal, Saudi Arabia (raul.tempone@kaust.edu.sa)

extending the same multilevel framework beyond "vanilla" Monte Carlo to its manifestation in the context of Bayesian inference, anywhere that one has a discretization error inversely proportional to cost in the computation of a single sample and the Monte Carlo $\mathcal{O}(M^{-1/2})$ rate of convergence. Examples in the context of Markov Chain Monte Carlo appeared in [31, 25]. To the knowledge of the authors there has yet to be extension of the methodology to the filtering context. As a first step, this work explores the extension of the EnKF to its multilevel implementation, which is naturally referred to as multilevel EnKF (MLEnKF). In the case of linear Gaussian problems, the limiting distribution is the gold-standard Bayesian posterior distribution, while in the non-Gaussian case it is something else (see, e.g. [34, 37, 40] and references therein for further discussion).

The rest of the paper will be organized as follows. In section 2 the filtering problem will be introduced, the Kalman filter and EnKF will be reviewed, and the new multilevel EnKF (MLEnKF) will be introduced for the first time in sub-section 2.4. In section 3 it is proven that indeed the MLEnKF inherits almost the same favorable asymptotic "cost-to- ε " as the standard MLMC for a finite time horizon, and its mean-field limiting distribution is the filtering distribution in the linear and Gaussian case. In section 4 the theory is illustrated with numerical examples of the Ornstein-Uhlenbeck stochastic differential equation, and the geometric Brownian motion. Finally, conclusions and future directions are presented in section 5.

2. Kalman filtering. Here the filtering problem will be introduced in section 2.1, and then the Kalman filter will be introduced for linear Gaussian state-space models in section 2.2. In section 2.3, its Monte Carlo implementation of the Kalman filter will be introduced, which is known as the ensemble Kalman filter (EnKF). In section 2.4, a the multilevel implementation is introduced for the first time.

2.1. General set-up. Let $(\Omega, \mathcal{E}, \mathbb{P})$ be a complete probability space, where Ω is the set of all possible outcomes, \mathcal{E} is the sigma algebra generated by Ω and \mathbb{P} is the associated probability measure. Consider the general stochastic signal evolution for the random variables $u_n : \Omega \rightarrow \mathbb{R}^d$, with $d < \infty$,

$$u_{n+1} = \Psi(u_n), \quad (2.1)$$

for $n = 0, 1, \dots, N - 1$. Given the history of a noisy signal observation

$$y_n = H u_n + \eta_n,$$

where $H \in \mathbb{R}^{m \times d}$ and η_n are i.i.d. with $\eta_1 \sim N(0, \Gamma)$, $\Gamma \in \mathbb{R}^{m \times m}$ symmetric positive definite, the objective is to track the signal u_n given observations $Y_n = Y_n^{\text{obs}}$ with $Y_n = (y_1, y_2, \dots, y_n)$ and $Y_n^{\text{obs}} = (y_1^{\text{obs}}, y_2^{\text{obs}}, \dots, y_n^{\text{obs}})$ the given observed realization of Y_n . In other words, the aim is to approximate the random variable $u_n | (Y_n = Y_n^{\text{obs}})$. Notice that under the given assumptions this is a hidden Markov model and the density of the random variable we seek to approximate admits the following sequential structure

$$\begin{aligned} \pi_{u_n}(u | Y_n = Y_n^{\text{obs}}) &= \frac{\pi_{y_n}(y_n^{\text{obs}} | u_n = u) \pi_{u_n}(u | Y_{n-1} = Y_{n-1}^{\text{obs}})}{\pi_{y_n}(y_n^{\text{obs}} | Y_{n-1} = Y_{n-1}^{\text{obs}})}, \quad (2.2) \\ \pi_{u_n}(u | Y_{n-1} = Y_{n-1}^{\text{obs}}) &= \int_{\mathbb{R}^d} \pi_{u_n}(u | u_{n-1} = v) \pi_{u_{n-1}}(v | Y_{n-1} = Y_{n-1}^{\text{obs}}) dv, \\ \pi_{y_n}(y_n^{\text{obs}} | Y_{n-1} = Y_{n-1}^{\text{obs}}) &= \int_{\mathbb{R}^d} \pi_{y_n}(y_n^{\text{obs}} | u_n = u) \pi_{u_n}(u | Y_{n-1} = Y_{n-1}^{\text{obs}}) du, \end{aligned}$$

Here, $\pi_{X_1}(x_1|X_2 = x_2)$ denotes the marginal probability density of X_1 conditional that $X_2 = x_2$.

It will be assumed that $\Psi(\cdot)$ cannot be evaluated exactly, but rather only approximately, and that there exists a hierarchy of accuracies at which it can be evaluated each with its associated cost. The explicit dependence on ω will be suppressed where confusion is not possible. In particular, we will be concerned herein with the case in which $u_{n+1} = \Psi(u_n) := u(1; u(0) = u_n)$ is given by the evolution at $t = 1$ of the following SDE

$$du(t) = a(u(t))dt + b(u(t))dW(t+n), \quad t \in (0, 1), \quad (2.3)$$

given the prescribed initial condition $u(0) = u_n$, where $a : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $b : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times r}$, and $W : \Omega \times [0, \infty) \rightarrow \mathbb{R}^r$ is an r -dimensional Wiener process. Furthermore, a and b will satisfy the following conditions

$$|a(u) - a(v)| + |b(u) - b(v)| \leq c|u - v| \text{ for all } u, v \in \mathbb{R}^d \text{ and some } c > 0. \quad (2.4)$$

This fits into the framework of (2.1). Notice that the randomness comes from the initial condition and the Wiener process W only, and not the coefficients, however it could in principle come also from the latter. Indeed if the analytical solution is unknown, then the system above must be approximated, leading to the hierarchy of approximations to $\Psi(\cdot)$. In particular, denote by $\{\Psi^\ell\}_{\ell=0}^\infty$ a hierarchy of approximations to the solution $\Psi := \Psi^\infty$ of (2.3). First some assumptions must be made.

ASSUMPTION 1. *For every $p \geq 2$, the solution operators $\{\Psi^\ell\}_{\ell=0}^\infty$ satisfy the following conditions, for some $0 < c_\Psi < \infty$ depending on Ψ :*

- (i) $\|\Psi^\ell(u) - \Psi^\ell(v)\|_p < c_\Psi \|u - v\|_p$,
- (ii) $\|\Psi^\ell(u)\|_p^p \leq c_\Psi (1 + \|u\|_p^p)$,

where the following notation is introduced $\|u\|_p := \mathbb{E}[|u|^p]^{1/p}$.

For many numerical solvers the assumptions can be verified by application of Gronwall's inequality, cf. [20, 9]. For notational simplicity, we consider the particular case in which (2.3) is *autonomous*, such that the coefficients on the right-hand side do not depend explicitly on time. Note that the results easily extend to the non-autonomous case, provided the given assumptions on Ψ are uniform with respect to $\{\Psi_n\}_{n=1}^N$. The specialization is merely for notational convenience.

2.2. Linear Gaussian case. Consider the linear instance of (2.1), in which

$$\Psi(u_n) = Au_n + \xi_n, \quad (2.5)$$

where $A \in \mathbb{R}^{d \times d}$, and ξ_n are i.i.d. normal random variables with $\xi_1 \sim N(0, \Sigma)$, $\Sigma \in \mathbb{R}^{d \times d}$ symmetric positive definite. This case arises when the coefficients of (2.3) are given by

$$a(u) = a_1 u + a_0 \quad \text{and} \quad b(u) = b_0. \quad (2.6)$$

Again we suppress the possible time-dependence of the random maps Ψ and matrices A and Σ just to simplify notation. For this class of problems, given a Gaussian initial condition, the filtering distribution (2.2) is known to be Gaussian, and is therefore defined uniquely by its mean and covariance. Kalman filtering provides a two step

iterative procedure for computing the mean and covariance of $\hat{v}_{n+1} := u_{n+1}|(Y_{n+1} = Y_{n+1}^{\text{obs}})$ given $\hat{v}_n := u_n|(Y_n = Y_n^{\text{obs}}) \sim N(\hat{m}_n, \hat{C}_n)$ where

$$\hat{m}_n := \mathbb{E}[u_n|Y_n = Y_n^{\text{obs}}] \quad \text{and} \quad \hat{C}_n := \mathbb{E}[(u_n - \hat{m}_n)(u_n - \hat{m}_n)^\top | Y_n = Y_n^{\text{obs}}].$$

The classical Kalman filter consists of a two-step formula which maps the distribution of \hat{v}_n to the distribution of \hat{v}_{n+1} . In particular, the map from (\hat{m}_n, \hat{C}_n) to $(\hat{m}_{n+1}, \hat{C}_{n+1})$ is described by the following two prediction equations and two update equations

$$\begin{aligned} m_{n+1} &= A\hat{m}_n, & C_{n+1} &= A\hat{C}_nA^\top + \Sigma \\ \hat{m}_{n+1} &= (I - K_{n+1}H)m_{n+1} + K_{n+1}y_{n+1}^{\text{obs}}, & \hat{C}_{n+1} &= (I - K_{n+1}H)C_{n+1}. \end{aligned}$$

where the *Kalman gain* K_{n+1} is given by

$$K_{n+1} = C_{n+1}H^\top S_{n+1}^{-1}, \quad S_{n+1} = \Gamma + HC_{n+1}H^\top.$$

REMARK 1. *When observations are in a lower dimension than the true signal, the iterations are more efficiently computed by introducing $d_{n+1} = y_{n+1}^{\text{obs}} - Hm_{n+1}$ and updating the mean as follows*

$$\hat{m}_{n+1} = m_{n+1} + K_{n+1}d_{n+1}.$$

It is nonetheless instructive to observe the alternative form presented above, as it is the properties of the operators $I - K_{n+1}H$ which are responsible for stability of the algorithm [33].

2.3. EnKF. EnKF uses an ensemble of particles to estimate means and covariance matrices appearing in the Kalman filter, however the framework can be generalized to non-Gaussian models. Let $v_{n,i}$ and $\hat{v}_{n,i}$ respectively denote the prediction and update of the i -th particle at simulation time $t_n = n$. One EnKF two-step transition consists not of the propagation of a mean and covariance as in the original Kalman filter, but instead the propagation of an ensemble $\{\hat{v}_{n,i}\}_{i=1}^M \mapsto \{\hat{v}_{n+1,i}\}_{i=1}^M$.¹ This procedure consists nonetheless in the prediction and update steps. In the prediction step, M particle paths are computed over one interval, i.e.,

$$v_{n+1}(\omega_i) = \Psi(\hat{v}_n(\omega_i), \omega_i) \tag{2.7}$$

for $i = 1, \dots, M$, where $v_n(\omega_i) := v_{n,i}$ denotes a realization corresponding to the event sample ω_i of the random variable $v_n : \Omega \rightarrow \mathbb{R}^d$, and $\Psi(\cdot, \omega_i)$ signifies the corresponding realization of the map for a given initial condition. Indeed the notation for random variable realizations, e.g. $\xi_{n,i}$ and $\xi_n(\omega_i)$, will be used interchangeably where confusion is not possible. The impetus for introduction of the latter notation will become apparent in the next section. For this presentation it suffices to assume a single infinite precision map, however there indeed may also be numerical approximation errors, i.e. Ψ^L may be used in place of Ψ for some satisfactory resolution L . The

¹Due to the implicit linear and Gaussian assumptions underlying the formulation, one may determine that it is reasonable to summarize the ensemble in its sample mean and covariance and indeed this is often done. In this case, one may construct a Gaussian from the empirical statistics and resample from that.

prediction step is completed by using the particle paths to compute sample mean and covariance:

$$\begin{aligned} m_{n+1}^{\text{MC}} &= E_M[v_{n+1}] \\ C_{n+1}^{\text{MC}} &= \text{Cov}_M[v_{n+1}] \end{aligned}$$

where the following notations are introduced

$$E_M[v] := \frac{1}{M} \sum_{i=1}^M v(\omega_i), \quad (2.8)$$

$$\text{Cov}_M[u, v] := E_M[uv^\top] - E_M[u](E_M[v])^\top, \quad (2.9)$$

as well as the shorthand $\text{Cov}_M[u] := \text{Cov}_M[u, u]$. The update step consists of computing (1) auxiliary matrices

$$S_{n+1}^{\text{MC}} = HC_{n+1}^{\text{MC}}H^\top + \Gamma \text{ and } K_{n+1}^{\text{MC}} = C_{n+1}^{\text{MC}}H^\top(S_{n+1}^{\text{MC}})^{-1},$$

and (2) measurement corrected particle paths for $i = 1, 2, \dots, M$,

$$\begin{aligned} \tilde{y}_{n+1,i} &= y_{n+1}^{\text{obs}} + \eta_{n+1,i}, \\ \hat{v}_{n+1,i} &= (I - K_{n+1}^{\text{MC}}H)v_{n+1,i} + K_{n+1}^{\text{MC}}\tilde{y}_{n+1,i}, \end{aligned}$$

where $\{\eta_{n+1,i}\}_{i=1}^M$ are i.i.d. with $\eta_{n+1,1} \sim N(0, \Gamma)$. This last procedure may appear somewhat ad-hoc. Indeed it was originally introduced in [7] to correct the statistical error induced in its absence in implementations following the original formulation of the ensemble Kalman filter in [14]. It has become known as the perturbed observation implementation. Due to the form of the update, all ensemble members are correlated to one another after the first update. So, the ensemble is no longer Gaussian after the first update. The measurement corrected sample mean and covariance, which need not be computed, would be given by:

$$\begin{aligned} \hat{m}_{n+1}^{\text{MC}} &= E_M[\hat{v}_{n+1}], \\ \hat{C}_{n+1}^{\text{MC}} &= \text{Cov}_M[\hat{v}_{n+1}]. \end{aligned}$$

The sample empirical distribution is defined by

$$\mu_n^{\text{MC}} = \frac{1}{M} \sum_{i=1}^M \delta_{v_n(\omega_i)}, \quad (2.10)$$

and, for $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$, the following shorthand notation is introduced $\mu_n^{\text{MC}}(\varphi) = \int \varphi d\mu_n^{\text{MC}} = E_M[v_n]$. It was shown in [38, 35] that if Ψ is of the form (2.5) and $\mathbb{E}|v_0|^p < \infty$ for all $p \geq 2$, then for all Lipschitz $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ and all $p \geq 2$,

$$(\mathbb{E}|\mu_n^{\text{MC}}(\varphi) - \mu_n(\varphi)|^p)^{1/p} \lesssim M^{-1/2}, \quad (2.11)$$

where μ_n is the filtering distribution. The notation $f(M) \lesssim g(M)$ here is used to denote $f(M) = \mathcal{O}(g(M))$.

2.4. Multilevel EnKF. MLEnKF computes particle paths on a hierarchy of accuracy levels, in this case given by increasing refinement of the temporal discretization. Let $v_n^\ell, \widehat{v}_n^\ell$ respectively denote the prediction and update of a particle on solution level ℓ at simulation time t_n . A solution on level ℓ is computed by the numerical integrator $v_{n+1}^\ell = \Psi^\ell(\widehat{v}_n^\ell)$. Furthermore, let the difference operator for level ℓ be given by

$$\Delta_\ell v_n(\omega) := \begin{cases} v_n^0(\omega), & \text{if } \ell = 0, \\ v_n^\ell(\omega) - v_n^{\ell-1}(\omega), & \text{else if } \ell > 0. \end{cases} \quad (2.12)$$

Then the transition from approximation of the distribution of \widehat{v}_n to the distribution of \widehat{v}_{n+1} in the MLEnKF framework consists of the predict/update step of generating *pairwise coupled* particle realizations on a set of levels $\ell = 0, 1, \dots, L$. However, it is important to note that here one has correlation between pairs and also between levels due to the update, unlike the standard MLMC in which one has i.i.d. pairs. This point will be very important, and we return to it in the following section.

Similarly to the standard EnKF, the MLEnKF transition is between *multilevel* ensembles $\{[\widehat{v}_n^\ell(\omega_{\ell,i}), \widehat{v}_n^{\ell-1}(\omega_{\ell,i})]_{i=1}^{M_\ell}\}_{\ell=0}^L \mapsto \{[\widehat{v}_{n+1}^\ell(\omega_{\ell,i}), \widehat{v}_{n+1}^{\ell-1}(\omega_{\ell,i})]_{i=1}^{M_\ell}\}_{\ell=0}^L$, with the convention that $\widehat{v}_k^{-1} := 0$ for all k for ease of notation. This consists, as for EnKF, of the predict and update steps. In the predict step, particle paths are first computed on a hierarchy of levels. That is, the particle paths are computed one step forward by

$$\begin{aligned} v_{n+1}^{\ell-1}(\omega_{\ell,i}) &= \Psi^{\ell-1}(\widehat{v}_n^{\ell-1}(\omega_{\ell,i}), \omega_{\ell,i}), \\ v_{n+1}^\ell(\omega_{\ell,i}) &= \Psi^\ell(\widehat{v}_n^\ell(\omega_{\ell,i}), \omega_{\ell,i}), \end{aligned} \quad (2.13)$$

for the levels $\ell = 0, 1, \dots, L$ and level particles $i = 1, 2, \dots, M_\ell$ (where for convenience we introduce the convention that $v^{-1} := 0$). Here the introduction of noise in the second argument of the Ψ^ℓ are correlated only within pairs, and are otherwise independent. Thereafter, sample mean and covariance matrices are computed as a sum of sample moments over all levels:

$$\begin{aligned} m_{n+1}^{\text{ML}} &= \sum_{\ell=0}^L E_{M_\ell}[\Delta_\ell v_{n+1}(\omega_{\ell,\cdot})], \\ C_{n+1}^{\text{ML}} &= \sum_{\ell=0}^L \text{Cov}_{M_\ell}[v_{n+1}^\ell(\omega_{\ell,\cdot})] - \text{Cov}_{M_\ell}[v_{n+1}^{\ell-1}(\omega_{\ell,\cdot})], \end{aligned}$$

where we recall the sample moment notation (2.8) and (2.9).

It is necessary for stability of the algorithm that the sample covariance appearing in the denominator of the gain is positive semi-definite, a condition which is *not* guaranteed for multilevel estimators. This will therefore be *imposed* in the algorithm. It would be of independent interest to devise multilevel estimators which preserve positivity without such imposition. Let

$$C_n^{\text{ML}} = \sum_{k=1}^d \lambda_k q_k q_k^\top$$

denote the eigenvalue decomposition of the symmetric multilevel covariance. Notice that the condition $\min_k(\lambda_k) \geq 0$ may not hold. Define

$$\widetilde{C}_n^{\text{ML}} = \sum_{k=1; \lambda_k > 0}^d \lambda_k q_k q_k^\top. \quad (2.14)$$

It is worth noting that this is not the only way to do this, and it may be possible to use a less invasive and/or or less expensive method to guarantee non-negativity of the covariance. For example, banding [5], shrinkage [36], thresholding [4], or localization [1] are some prospective alternatives. In particular, it will be necessary to consider such alternatives as the dimension grows and the cost of factorizing C_n^{ML} becomes a dominant consideration, but this is outside the scope of the present work. In the update step the multilevel Kalman gain is defined as follows

$$K_{n+1}^{\text{ML}} = C_{n+1}^{\text{ML}} H^T (S_{n+1}^{\text{ML}})^{-1}, \text{ where } S_{n+1}^{\text{ML}} = H \tilde{C}_{n+1}^{\text{ML}} H^T + \Gamma. \quad (2.15)$$

Next, all particle paths are corrected according to measurements and perturbed observations are added:

$$\begin{aligned} \tilde{y}_{n+1,i}^\ell &= y_{n+1}^{\text{obs}} + \eta_{n+1,i}^\ell, \\ \hat{v}_{n+1}^{\ell-1}(\omega_{i,\ell}) &= (I - K_{n+1}^{\text{ML}} H) v_{n+1}^{\ell-1}(\omega_{i,\ell}) + K_{n+1}^{\text{ML}} \tilde{y}_{n+1,i}^\ell, \\ \hat{v}_{n+1}^\ell(\omega_{i,\ell}) &= (I - K_{n+1}^{\text{ML}} H) v_{n+1}^\ell(\omega_{i,\ell}) + K_{n+1}^{\text{ML}} \tilde{y}_{n+1,i}^\ell, \end{aligned} \quad (2.16)$$

where $\{\eta_{n+1,i}^\ell\}_{i=1}^{M_\ell}$ are i.i.d. with $\eta_{n+1,1}^{\{0\}} \sim N(0, \Gamma)$. It is in this step precisely that the pairs all become correlated with one another and the situation becomes significantly more complex than the i.i.d. case. After the first update, this correlation propagates forward through (2.13) to the next observation time via this ensemble. This is the conclusion of the update step of the MLEnKF, and this multilevel ensemble is subsequently propagated forward to the next prediction time via (2.13).

The *multilevel* sample mean and covariance (in the case that (2.14) has not modified the covariance, i.e. it has all non-negative eigenvalues without truncation) of this multilevel ensemble are given by:

$$\begin{aligned} \hat{m}_{n+1}^{\text{ML}} &= \sum_{\ell=0}^L E_{M_\ell} [\Delta_\ell \hat{v}_{n+1}(\omega_{\ell,\cdot})] \\ &= (I - K_{n+1}^{\text{ML}} H) m_{n+1}^{\text{ML}} + K_{n+1}^{\text{ML}} [E_{M_0} [\tilde{y}_{n+1,\cdot}^0 - y_{n+1}^{\text{obs}}] + y_{n+1}^{\text{obs}}], \end{aligned} \quad (2.17)$$

$$\begin{aligned} \hat{C}_{n+1}^{\text{ML}} &= (I - K_{n+1}^{\text{ML}} H) C_{n+1}^{\text{ML}} (I - K_{n+1}^{\text{ML}} H)^T + K_{n+1}^{\text{ML}} \text{Cov}_{M_0} [\tilde{y}_{n+1,\cdot}^0 - y_{n+1}^{\text{obs}}] K_{n+1}^{\text{ML}T} \\ &= (I - K_{n+1}^{\text{ML}} H) C_{n+1}^{\text{ML}} + K_{n+1}^{\text{ML}} [\text{Cov}_{M_0} [\tilde{y}_{n+1,\cdot}^0 - y_{n+1}^{\text{obs}}] - \Gamma] K_{n+1}^{\text{ML}T}. \end{aligned} \quad (2.18)$$

The second term appearing in each case is unbiased. For computing general quantities of interest, it is instructive to introduce the empirical measure of the multilevel ensemble $\{[\hat{v}_n^\ell(\omega_{\ell,i}), \hat{v}_n^{\ell-1}(\omega_{\ell,i})]_{i=1}^{M_\ell}\}_{\ell=0}^L$, i.e. ²

$$\mu_n^{\text{ML}} = \frac{1}{M_0} \sum_{i=1}^{M_0} \delta_{\hat{v}_n^0(\omega_{0,i})} + \sum_{\ell=1}^L \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} (\delta_{\hat{v}_n^\ell(\omega_{\ell,i})} - \delta_{\hat{v}_n^{\ell-1}(\omega_{\ell,i})}). \quad (2.19)$$

Then, the following shorthand notation for multilevel sample averages can be introduced. For any $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$, let

$$\mu_n^{\text{ML}}(\varphi) := \int \varphi d\mu_n^{\text{ML}} = \sum_{\ell=0}^L \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \varphi(\hat{v}_n^\ell(\omega_{\ell,i})) - \varphi(\hat{v}_n^{\ell-1}(\omega_{\ell,i})).$$

²Similar may be done for the predicting distributions, but the updated distributions will be our primary interest.

2.5. Nonlinear Kalman filtering. It will be useful to introduce the limiting process, in the case of nonlinear non-Gaussian forward model (2.1), i.e. nonlinear (2.3). The following nonlinear Markov process defines the mean-field EnKF [34]:

$$\text{Prediction} \begin{cases} v_{n+1} &= \Psi(\hat{v}_n), \\ m_{n+1} &= \mathbb{E}[v_{n+1}], \\ C_{n+1} &= \mathbb{E}[(v_{n+1} - m_{n+1}) \otimes (v_{n+1} - m_{n+1})] \end{cases} \quad (2.20)$$

$$\text{Update} \begin{cases} S_{n+1} &= HC_{n+1}H^\top + \Gamma \\ K_{n+1} &= C_{n+1}H^\top S_{n+1}^{-1} \\ \tilde{y}_{n+1} &= y_{n+1}^{\text{obs}} + \tilde{\eta}_{n+1} \\ \hat{v}_{n+1} &= (I - K_{n+1}H)v_{n+1} + K_{n+1}\tilde{y}_{n+1}. \end{cases} \quad (2.21)$$

Here $\{\tilde{\eta}_n\}_{n=1}^N$ are i.i.d. draws from $N(0, \Gamma)$. The expectations appearing above in (2.20) are with respect to the random variable v_{n+1} , which depends upon the randomness from the initial condition $v_0 = u_0$, the maps Ψ , and $\tilde{\eta}_0, \dots, \tilde{\eta}_n$. The observed value y^{obs} is considered fixed and is not averaged over. It is easy to verify that in the linear Gaussian case of the Section 2.2, the mean and variance of the above process correspond to the mean and variance of the filtering distribution. Furthermore, it was shown in [38, 35] that the single level EnKF converges to the Kalman filtering distribution with the standard rate $\mathcal{O}(M^{-1/2})$ in this case, as stated formally in (2.11). It was furthermore shown in [35] and [34] that for nonlinear Gaussian state-space models and fully non-Gaussian models (2.1), respectively, the *same convergence property holds*, with the measure corresponding to v_n in (2.20) and (2.21) replacing μ_n in (2.11), as long as the model satisfies a Lipschitz criterion as in Assumption 1. In this work, the aim is to show that the MLEnKF converges as well, and with a cost-to- ε which is strictly smaller than its single level EnKF counterpart. The true filtering distribution of $u_n | (Y_n = Y_n^{\text{obs}})$ will not appear in the remainder of this work, and the variable v_n will correspond to the solution of the above system (noting that the two are equivalent in the linear Gaussian case).

3. Theoretical Results. The approximation error and computational cost of approximating the true filtering distribution by MLEnKF when given a sequence of observations y_1, y_2, \dots, y_n will be studied in this section. The notation $|\cdot|$ will be used for standard Euclidean norm (and the induced matrix norm) and the covariance matrix of random variables $Z, X \in \mathbb{R}^d$ will be denoted

$$\text{Cov}[Z, X] := \mathbb{E}[(Z - \mathbb{E}[Z])(X - \mathbb{E}[X])^\top],$$

with the shorthand $\text{Cov}[Z] = \text{Cov}[Z, Z]$. Before stating the main approximation theorem, it will be useful to present the basic assumptions that will be used throughout and the corresponding standard MLMC approximation results for i.i.d. samples, as well as a slight variant which will be useful in what follows.

ASSUMPTION 2. Consider the d -dimensional SDE (2.3) with initial data $u_0 \in \cup_{p \in \mathbb{N}} L^p(\Omega)$. For the hierarchy of solution operators defined in Section 2, let Ψ^ℓ denote a numerical solver using a uniform time step $\Delta t^\ell = 1/N_\ell$ with $N_\ell/N_{\ell-1} \geq \hat{N} > 1$ for $\ell = 0, 1, \dots$. Let \mathcal{F} denote the set of functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ which, for all $\ell \geq 0$ and all $u, v \in \cup_{p \in \mathbb{N}} L^p(\Omega)$, and a given set of constants $\alpha, \beta, \gamma > 0$ with $\alpha \geq \min(\beta, \gamma)/2$, fulfill

- (i) $|\mathbb{E}[\varphi(\Psi^\ell(u)) - \varphi(\Psi(v))]| \lesssim N_\ell^{-\alpha}$, and $|\mathbb{E}[\varphi(u) - \varphi(v)]| \lesssim N_\ell^{-\alpha}$, provided $|\mathbb{E}[u - v]| \lesssim N_\ell^{-\alpha}$;
- (ii) $\|\varphi(\Psi^\ell(v)) - \varphi(\Psi^{\ell-1}(v))\|_p \lesssim N_\ell^{-\beta/2}$, for all $p \geq 2$;
- (iii) $\mathcal{C}_\ell := \text{Cost}(\Psi^\ell(v)) \lesssim N_\ell^\gamma$;

where, as stated above, the notation $f(M) \lesssim g(M)$ here is used to denote $f(M) = \mathcal{O}(g(M))$. Assume further that all monomials of degree less than or equal to 2, are contained in \mathcal{F} .

REMARK 2. An implication of the above condition (ii) is that condition (i) holds with $\alpha = \beta/2$. However, for many numerical schemes, there are settings where it is possible to achieve rates $\alpha > \beta/2$ (implemetationally, this may yield savings in the computational cost). The literature [32, Theorem 14.5.2] and [20, Chapter 7] provide sufficient regularity conditions on the SDE problem and φ for the the Euler–Maruyama method to achieve the rate exponents $\alpha = 1$ and $\beta = 1$, and the Milstein method to achieve $\alpha = 1$ and $\beta = 2$.

We will now state the main theorem of this paper. It gives an upper bound for the computational cost of achieving a sought accuracy in L^p -norm when using the MLEnKF method to approximate the expectation of an observable. The theorem may be considered an extension to the data assimilation setting of earlier “one-step” cost vs. error results in multilevel Monte Carlo, cf. [18, Theorem 3.1] and [10, Theorem 1]. To reduce the number of repetitions in the below proofs we notice once and for all that the process itself is in L^p by Assumption 1, hence the realization giving rise to the observations u_n and the observations themselves y_n are as well, for $n = 1, 2, \dots, N$. It follows from this and the finite norm of K_n [35, 34, 38] that the elements \tilde{v}_n and \bar{v}_n given by (2.20) and (2.21) are also in L^p for $n = 1, 2, \dots, N$. It will be assumed that the update comes at a marginal cost with respect to the prediction. This may be the case for complicated forward solution with small error tolerance, large ensemble, and comparably modest dimension d .

DEFINITION 3.1. A function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be locally Lipschitz continuous with at most polynomial growth at infinity provided that there exist positive scalars $\nu, C_\varphi < \infty$ such that

$$|\varphi(x) - \varphi(y)| \leq C_\varphi |x - y| (1 + |x|^\nu + |y|^\nu), \quad \forall x, y \in \mathbb{R}^d. \quad (3.1)$$

The notation $f(M) \approx g(M)$ will be used to indicate that there exist constants $\tilde{c}_1, \tilde{c}_2 > 0$ such that $\tilde{c}_1 g(M) \leq f(M) \leq \tilde{c}_2 g(M)$.

THEOREM 3.2 (MLEnKF accuracy vs. cost). Suppose Assumptions 1 and 2 hold. For a given $\varepsilon > 0$, let L and $\{M_\ell\}_{\ell=0}^L$ be defined under the constraints $L \approx \log(\varepsilon^{-1})$ and

$$M_\ell \approx \left[N_\ell^{-\frac{\beta+2\gamma}{3}} \begin{cases} N_L^{2\alpha}, & \text{if } \beta > \gamma, \\ L^2 N_L^{2\alpha}, & \text{if } \beta = \gamma, \\ N_L^{2\alpha + \frac{2}{3}(\gamma - \beta)}, & \text{if } \beta < \gamma. \end{cases} \right]. \quad (3.2)$$

Then for all functions $\varphi \in \mathcal{F}$ that are locally Lipschitz continuous with at most polynomial growth at infinity, cf. Definition 3.1, we have that

$$\|\mu_n^{\text{ML}}(\varphi) - \mu_n(\varphi)\|_p \lesssim |\log(\varepsilon)|^n \varepsilon, \quad (3.3)$$

where μ_n^{ML} is the multilevel empirical measure defined in (2.19), where the samples are given by the multilevel predict (2.13) and update (2.16) formulae, approximating

the time $t_n = n$ mean-field EnKF distribution μ_n (the filtering distribution $\mu_n = N(m_n, C_n)$ in the linear Gaussian case). And the computational cost of the MLEnKF estimator over the time sequence satisfies

$$\text{Cost (MLEnKF)} \lesssim \begin{cases} \varepsilon^{-2}, & \text{if } \beta > \gamma, \\ \varepsilon^{-2} |\log(\varepsilon)|^3, & \text{if } \beta = \gamma, \\ \varepsilon^{-(2+\frac{\gamma-\beta}{\alpha})}, & \text{if } \beta < \gamma. \end{cases} \quad (3.4)$$

REMARK 3. The growth in error factor $|\log(\varepsilon)|^n$ in (3.3) is due to a propagation of perturbed observation errors of the MLEnKF estimator that has been conservatively bounded by the triangle inequality in (3.28). In our numerical tests we do however not observe the error growth factor, and therefore we believe it might be possible to eliminate this factor by sharper theoretical bounds.

The proof of Theorem 3.2 follows roughly along the same lines as that of [35], however with more notation and longer calculations due to the multilevel aspect. The proof also has connections to the work [6], in which an MLMC method is developed for estimation of higher order central moments.

It will be convenient to introduce the mean-field limiting multilevel ensemble $\{[\bar{v}_n^\ell(\omega_{\ell,i}), \bar{v}_n^{\ell-1}(\omega_{\ell,i})]_{i=1}^{M_\ell}\}_{\ell=0}^L$, [35, 34, 38], which evolves according to the same equations with the same realizations of noise except the covariance C_n , hence the Kalman gain K_n , are given by limiting formulae in (2.20) and (2.21). That is, the intra-level pairs of ensemble members $(\bar{v}_n^\ell(\omega_{\ell,i}), \bar{v}_n^{\ell-1}(\omega_{\ell,i}))$ are *independent and identically distributed* (i.i.d.) over index i , and they are independent between levels. An ensemble member $(\hat{v}_n^\ell(\omega_{\ell,i}), \hat{v}_n^{\ell-1}(\omega_{\ell,i}))$ maps to $(\bar{v}_{n+1}^\ell(\omega_{\ell,i}), \bar{v}_{n+1}^{\ell-1}(\omega_{\ell,i}))$ as in (2.13). Then $(\hat{v}_{n+1}^\ell(\omega_{\ell,i}), \hat{v}_{n+1}^{\ell-1}(\omega_{\ell,i}))$ is obtained as in (2.16), except with K_{n+1} from (2.21) replacing K_{n+1}^{ML} in (2.16). The noise realizations $\{\omega_{\ell,i}\}$ are assumed to be the same as the EnKF ensemble member $(v_n^\ell(\omega_{\ell,i}), v_n^{\ell-1}(\omega_{\ell,i}))$. The sole difference is that the limiting ensemble is independent between levels and the pairs within a level are i.i.d. This is because the covariance and gain come from the infinite limiting system (2.20) and (2.21). The only correlations are between $\bar{v}_n^\ell(\omega_{\ell,i})$ and $\bar{v}_n^{\ell-1}(\omega_{\ell,i})$, due to the $\omega_{\ell,i}$. Hence there is no multiplicative propagation of correlations within a level or between levels. This crucial fact allows to (a) on the one hand extend standard multilevel theory for i.i.d. draws over multiple updates, and (b) on the other hand, establish the required proximity of the two multilevel ensembles particle-wise, based on convergence of the random gains K_n^{ML} to the deterministic ones K_n . The latter will require the greatest effort and will dominate the proof by means technical lemmas. Note that $\|\bar{v}_n^\ell\|_p, \|\hat{v}_n^\ell\|_p, |K_n| < \infty$, following from Assumptions 1.

The first step is to bound the multilevel predicting covariance in terms of its constituents, the ensemble members. The gain is then bounded in terms of the covariance, and ultimately the updated ensemble in terms of the predicting ensemble and the covariance. The rate appears only by virtue of the convergence of the i.i.d. ensemble covariance, and it is propagated forward by induction. Only the *predicting* covariance will be considered and hats will be omitted to avoid unnecessary notation.

Recall the multilevel Kalman gain is defined as follows

$$K_n^{\text{ML}} = C_n^{\text{ML}} H^\top (H \tilde{C}_n^{\text{ML}} H^\top + \Gamma)^{-1},$$

where

$$\tilde{C}_n^{\text{ML}} = \sum_{k=1; \lambda_k > 0}^d \lambda_k q_k q_k^\top, \quad (3.5)$$

for eigenpairs $\{\lambda_k, q_k\}$ of C_n^{ML} . The following micro-lemma will be necessary to control the error in the gain.

LEMMA 3.3 (multilevel covariance approximation error). *Let \tilde{C}_n^{ML} be given by (3.5). Then the following holds*

$$|\tilde{C}_n^{\text{ML}} - C_n^{\text{ML}}| \leq |C_n^{\text{ML}} - C_n|, \quad (3.6)$$

where $|\cdot|$ denotes the induced 2-norm for matrices.

Proof. Notice that

$$|\tilde{C}_n^{\text{ML}} - C_n^{\text{ML}}| = \max_{\{j; \lambda_j < 0\}} \{|\lambda_j|\}. \quad (3.7)$$

Denote the associated eigenvector by u_{\max} (normalized to $|u_{\max}| = 1$). Notice that for any $A = A^\top$,

$$|A| = \sup_u \frac{|u^\top A u|}{|u|^2} = \max_k |\lambda_k|,$$

where λ_k are the eigenvalues of A . Since $C_n \geq 0$, one has that

$$|u_{\max}^\top (C_n^{\text{ML}} - C_n) u_{\max}| = u_{\max}^\top C_n u_{\max} - u_{\max}^\top C_n^{\text{ML}} u_{\max} \geq |\tilde{C}_n^{\text{ML}} - C_n^{\text{ML}}|.$$

□

The next step is to bound the gain error, which is done in the following lemma.

LEMMA 3.4 (Continuity of the gain in the covariance). *There is a constant $c_n < \infty$, depending on $|H|$, γ_{\min} , and $|K_n H|$ such that*

$$|K_n^{\text{ML}} - K_n| \leq c_n |C_n^{\text{ML}} - C_n|, \quad (3.8)$$

where $\gamma_{\min} > 0$ is the smallest eigenvalue of Γ .

Proof. Recall that

$$\begin{aligned} K_n - K_n^{\text{ML}} &= C_n H^\top \left((H C_n H^\top + \Gamma)^{-1} - (H \tilde{C}_n^{\text{ML}} H^\top + \Gamma)^{-1} \right) \\ &\quad + (C_n - C_n^{\text{ML}}) H^\top (H \tilde{C}_n^{\text{ML}} H^\top + \Gamma)^{-1}, \end{aligned} \quad (3.9)$$

where $\tilde{C}_n^{\text{ML}} \geq 0$ is defined in (2.14), and notice that

$$\begin{aligned} (H C_n H^\top + \Gamma)^{-1} - (H \tilde{C}_n^{\text{ML}} H^\top + \Gamma)^{-1} \\ = (H C_n H^\top + \Gamma)^{-1} H (\tilde{C}_n^{\text{ML}} - C_n) H^\top (H \tilde{C}_n^{\text{ML}} H^\top + \Gamma)^{-1}. \end{aligned} \quad (3.10)$$

So

$$\begin{aligned} K_n - K_n^{\text{ML}} &= K_n H (\tilde{C}_n^{\text{ML}} - C_n) H^\top (H \tilde{C}_n^{\text{ML}} H^\top + \Gamma)^{-1} \\ &\quad + (C_n - C_n^{\text{ML}}) H^\top (H \tilde{C}_n^{\text{ML}} H^\top + \Gamma)^{-1}. \end{aligned}$$

Note that $x^\top (\Gamma + B) x \geq x^\top \Gamma x \geq \gamma_{\min}$ for all $x \in \mathbb{R}^d$ whenever $B = B^\top \geq 0$, and this implies that $|(H \tilde{C}_n^{\text{ML}} H^\top + \Gamma)^{-1}| \leq 1/\gamma_{\min}$. It follows by (3.6) that

$$|K_n - K_n^{\text{ML}}| \leq \frac{|H|}{\gamma_{\min}} (1 + 2|K_n H|) |C_n - C_n^{\text{ML}}|.$$

□

It is worth noting that the multilevel gain error is bounded by the *unmodified* multilevel sample covariance error, following from Lemma 3.3, so modification in (3.5) will not affect the ultimate approximation error.

THEOREM 3.5. *Suppose Assumptions 1 and 2 hold. For any $\varepsilon > 0$, let L and $\{M_\ell\}_{\ell=0}^L$ be defined as in Theorem 3.2. Then the following asymptotic inequality holds*

$$\|C_n^{\text{ML}} - C_n\|_p \lesssim \varepsilon + \|C_n^{\text{ML}} - \bar{C}_n^{\text{ML}}\|_p \quad (3.11)$$

with a cost which satisfies (3.4).

Proof. Let C_n^L denote the predicting covariance of the final L^{th} level limiting system at time n , in the sense that the forward map above is replaced by Ψ^L , but the gain comes from the continuum mean-field limiting system. Furthermore, let \bar{C}_n^{ML} denote the covariance associated to the multilevel ensemble $\{(\bar{v}_{n,i}^\ell)_{i=1}^{M_\ell}\}_{\ell=1}^L$. The triangle inequality is used to split

$$|C_n^{\text{ML}} - C_n| \leq |C_n^L - C_n| + |\bar{C}_n^{\text{ML}} - C_n^L| + |C_n^{\text{ML}} - \bar{C}_n^{\text{ML}}|, \quad (3.12)$$

and each term will be dealt with in turn, in the following three lemmas. The proof of the theorem is done after establishing Lemmas 3.6, 3.7 and 3.8, which provide the asymptotic bound on the first two terms. \square

LEMMA 3.6. *Suppose Assumptions 1 and 2 hold. For any $\varepsilon > 0$, let L and $\{M_\ell\}_{\ell=0}^L$ be defined as in Theorem 3.2. Then, for any finite n ,*

$$\max\left(\left|\mathbb{E}[\bar{v}_n^L - v_n]\right|, \left|\mathbb{E}[\bar{\tilde{v}}_n^L - \hat{v}_n]\right|\right) \lesssim \varepsilon, \quad (3.13)$$

and for $\ell = 0, 1, \dots, L$ and all $p \geq 2$,

$$\max\left(\left\|\bar{v}_n^\ell - \bar{v}_n^{\ell-1}\right\|_p, \left\|\bar{\tilde{v}}_n^\ell - \bar{\tilde{v}}_n^{\ell-1}\right\|_p\right) \lesssim N_\ell^{-\beta/2}. \quad (3.14)$$

Proof. Since the initial data $\bar{v}_0^L = \bar{v}_0^L = \hat{v}_0 = v_0$ is the same,

$$\left|\mathbb{E}[\bar{\tilde{v}}_0^L - \hat{v}_0]\right| = 0.$$

Now assume that the following holds

$$\max\left(\left|\mathbb{E}[\bar{\tilde{v}}_{n-1}^L - \hat{v}_{n-1}]\right|, \left|\mathbb{E}[\bar{v}_{n-1}^L - v_{n-1}]\right|\right) \lesssim \varepsilon$$

Assumption 2 (i) directly implies

$$\left|\mathbb{E}[\bar{v}_n^L - v_n]\right| \lesssim \varepsilon.$$

Futhermore, since $|K_n H| < \infty$ for any finite n ,

$$\left|\mathbb{E}[\bar{\tilde{v}}_n^L - \hat{v}_n]\right| \leq |I - K_n H| \left|\mathbb{E}[\bar{v}_n^L - v_n]\right| \lesssim \varepsilon,$$

and inequality (3.13) follows by induction.

To prove inequality (3.14), recall that due to the matching initial data, the inequality holds trivially at $n = 0$. Assume

$$\max\left(\left\|\bar{v}_{n-1}^\ell - \bar{v}_{n-1}^{\ell-1}\right\|_p, \left\|\bar{\tilde{v}}_{n-1}^\ell - \bar{\tilde{v}}_{n-1}^{\ell-1}\right\|_p\right) \lesssim N_\ell^{-\beta/2}.$$

By Assumptions 1 (i) and 2 (ii),

$$\begin{aligned} \|\bar{v}_n^\ell - \bar{v}_n^{\ell-1}\|_p &\leq \left\| \Psi^\ell(\tilde{v}_{n-1}^\ell) - \Psi^{\ell-1}(\tilde{v}_{n-1}^\ell) \right\|_p + \left\| \Psi^{\ell-1}(\tilde{v}_{n-1}^\ell) - \Psi^{\ell-1}(\tilde{v}_{n-1}^{\ell-1}) \right\|_p \\ &\lesssim N_\ell^{-\beta/2}, \\ \left\| \tilde{v}_n^\ell - \tilde{v}_n^{\ell-1} \right\|_p &\lesssim |I - K_n H| \|\bar{v}_n^\ell - \bar{v}_n^{\ell-1}\|_p \lesssim N_\ell^{-\beta/2}, \end{aligned}$$

and inequality (3.14) holds by induction. \square

LEMMA 3.7 (Covariance discretization error). *Suppose Assumptions 1 and 2 hold. For any $\varepsilon > 0$, let L be defined as in Theorem 3.2. Then the following asymptotic inequality holds*

$$|C_n^L - C_n| \lesssim \varepsilon. \quad (3.15)$$

Proof. It is possible to show that for any symmetric matrix A , the following inequality holds

$$|A| \leq \sum_{j,j'=1}^d |A^{jj'}|. \quad (3.16)$$

Furthermore, by adding the terms $\pm \mathbb{E}[(v_n)^j] \mathbb{E}[(\bar{v}_n^L)^{j'}]$,

$$\begin{aligned} |(C_n^L - C_n)^{jj'}| &= \left| \mathbb{E}[(\bar{v}_n^L)^j (\bar{v}_n^L)^{j'}] - \mathbb{E}[(v_n)^j (v_n)^{j'}] \right. \\ &\quad \left. - \mathbb{E}[(\bar{v}_n^L)^j] \mathbb{E}[(\bar{v}_n^L)^{j'}] + \mathbb{E}[(v_n)^j] \mathbb{E}[(v_n)^{j'}] \right| \\ &\leq \left| \mathbb{E}[(\bar{v}_n^L)^j (\bar{v}_n^L)^{j'} - (v_n)^j (v_n)^{j'}] \right| \\ &\quad + \left| \mathbb{E}[(\bar{v}_n^L)^j] \right| \left| \mathbb{E}[(\bar{v}_n^L)^{j'} - (v_n)^{j'}] \right| + \left| \mathbb{E}[(v_n)^j] \right| \left| \mathbb{E}[(\bar{v}_n^L)^{j'} - (v_n)^{j'}] \right| \\ &\lesssim N_L^{-\alpha} \approx \varepsilon. \end{aligned} \quad (3.17)$$

The last inequality follows by Lemma 3.6 and Assumption 2 (i), noting that \mathcal{F} contains all monomials of degree less than or equal to 2.

\square

Notice that

$$C_n^L = \sum_{\ell=0}^L \text{Cov}[\bar{v}_n^\ell] - \text{Cov}[\bar{v}_n^{\ell-1}],$$

and

$$C_n^{\text{ML}} = \sum_{\ell=0}^L \text{Cov}_{M_\ell}[v_n^\ell] - \text{Cov}_{M_\ell}[v_n^{\ell-1}],$$

with the convention that $v^{-1} = \bar{v}^{-1} := 0$. Consider also the partner covariance to the above

$$\bar{C}_n^{\text{ML}} = \sum_{\ell=0}^L \text{Cov}_{M_\ell}[\bar{v}_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}].$$

The next two differences are bounded in terms of the single-level differences, using the triangle inequality to extend to the sum.

LEMMA 3.8 (multilevel i.i.d. sample covariance error). *Suppose Assumptions 1 and 2 hold, and for any $\varepsilon > 0$, let L and $\{M_\ell\}_{\ell=0}^L$ be defined as in Theorem 3.2. Then the following asymptotic inequality holds*

$$\|\bar{C}_n^{\text{ML}} - C_n^L\|_p \lesssim \varepsilon. \quad (3.18)$$

Proof. Notice the following triangle inequality

$$|\bar{C}_n^{\text{ML}} - C_n^L| \leq \sum_{\ell=0}^L |\text{Cov}_{M_\ell}[\bar{v}_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}] - (\text{Cov}_{M_\ell}[\bar{v}_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}])|.$$

To avoid needlessly long terms when bounding the summands of the above equation, we now make the assumption in this proof that $\mathbb{E}[\bar{v}_n^\ell] = 0$, without loss of generality. We may then obtain the rearrangement

$$\text{Cov}_{M_\ell}[\bar{v}_n^\ell] = E_{M_\ell}[\bar{v}_n^\ell (\bar{v}_n^\ell)^\top] - E_{M_\ell}[\bar{v}_n^\ell] (E_{M_\ell}[\bar{v}_n^\ell])^\top,$$

and similarly for the $\ell - 1$ term. Using the identity $aa^\top - bb^\top = \frac{1}{2}[(a+b)(a-b)^\top + (a-b)(a+b)^\top]$ for $a, b \in \mathbb{R}^d$ on each of the outer products with $\ell, \ell - 1$, respectively, and then using (3.16) again for the first term, and Cauchy-Schwartz for the second (and grouping like terms arising from the $(j, j') \rightarrow (j', j)$ symmetry of $\frac{1}{2}(a^j b^{j'} + a^{j'} b^j)$), one has

$$\begin{aligned} & |\text{Cov}_{M_\ell}[\bar{v}_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}] - (\text{Cov}[\bar{v}_n^\ell] - \text{Cov}[\bar{v}_n^{\ell-1}])| \\ & \leq \sum_{j \leq j'=1}^d \left| E_{M_\ell}[(\bar{v}_n^\ell + \bar{v}_n^{\ell-1})^j (\bar{v}_n^\ell - \bar{v}_n^{\ell-1})^{j'}] - \mathbb{E}[(\bar{v}_n^\ell + \bar{v}_n^{\ell-1})^j (\bar{v}_n^\ell - \bar{v}_n^{\ell-1})^{j'}] \right| \\ & + \left| E_{M_\ell}[(\bar{v}_n^\ell + \bar{v}_n^{\ell-1})^{j'} (\bar{v}_n^\ell - \bar{v}_n^{\ell-1})^j] - \mathbb{E}[(\bar{v}_n^\ell + \bar{v}_n^{\ell-1})^{j'} (\bar{v}_n^\ell - \bar{v}_n^{\ell-1})^j] \right| \\ & + |E_{M_\ell}[\bar{v}_n^\ell + \bar{v}_n^{\ell-1}]^j| |E_{M_\ell}[(\bar{v}_n^\ell - \bar{v}_n^{\ell-1})^{j'}]| \\ & + |E_{M_\ell}[(\bar{v}_n^\ell + \bar{v}_n^{\ell-1})^{j'}]| |E_{M_\ell}[(\bar{v}_n^\ell - \bar{v}_n^{\ell-1})^j]|. \end{aligned} \quad (3.19)$$

Almost sure convergence follows by the law of large numbers. The rate in L^p is shown now.

First, it will be necessary to recall the Marcinkiewicz-Zygmund inequality: for i.i.d. random variables $X_1, \dots, X_N \sim X$ with $\|X\|_p < \infty$ for $p \geq 2$, and $\mathbb{E}[X] = 0$,

$$\|E_N[X]\|_p \leq c_p N^{-1/2} \|X\|_p, \quad (3.20)$$

where the constant depends only on p , cf. [8, 22]; in fact, $c_p \leq 3\sqrt{2p}$, cf. [41].

Using the Marcinkiewicz-Zygmund inequality then Hölder's inequality on each of the first two terms on the right-hand side of (3.19), then the reverse order on the last two, and finally the Assumptions 2, Lemma 3.6, and the fact $\bar{v}_n^\ell \in L^p(\Omega)$ for all $p \geq 2$,

together yield

$$\begin{aligned}
& \|\text{Cov}_{M_\ell}[\bar{v}_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}] - (\text{Cov}[\bar{v}_n^\ell] - \text{Cov}[\bar{v}_n^{\ell-1}])\|_p \leq \sum_{j \leq j'=1}^d M_\ell^{-1/2} \left[\right. \\
& c_p \left(\|(\bar{v}_n^\ell + \bar{v}_n^{\ell-1})^j\|_{2p} \|(\bar{v}_n^\ell - \bar{v}_n^{\ell-1})^{j'}\|_{2p} + \|(\bar{v}_n^\ell + \bar{v}_n^{\ell-1})^{j'}\|_{2p} \|(\bar{v}_n^\ell - \bar{v}_n^{\ell-1})^j\|_{2p} \right) \\
& \left. + c_{2p}^2 M_\ell^{-1/2} \left(\|(\bar{v}_n^\ell + \bar{v}_n^{\ell-1})^j\|_{2p} \|(\bar{v}_n^\ell - \bar{v}_n^{\ell-1})^{j'}\|_{2p} + \|(\bar{v}_n^\ell + \bar{v}_n^{\ell-1})^{j'}\|_{2p} \|(\bar{v}_n^\ell - \bar{v}_n^{\ell-1})^j\|_{2p} \right) \right] \\
& \lesssim M_\ell^{-1/2} N_\ell^{-\beta/2}.
\end{aligned} \tag{3.21}$$

Finally, by the triangle inequality, the following bound holds for (3.18) for all $p \geq 2$,

$$\|\bar{C}_n^{\text{ML}} - C_n^L\|_p \lesssim \sum_{\ell=0}^L M_\ell^{-1/2} N_\ell^{-\beta/2} \lesssim \varepsilon.$$

□

The previous two lemmas complete the proof of Theorem 3.5. Now we turn to the next term in (3.11), the difference between multilevel ensemble covariances, which is continuous in the individual ensemble members. First it will be necessary to recall (see e.g. Lemma 4.3 of [35]) that for identically distributed random variables $x_1, \dots, x_N \in \mathbb{R}^d$,

$$\left(\mathbb{E} \left[\left| E_N \left[|x_n|^p \right]^{1/p} \right|^q \right] \right)^{1/q} \leq \|x_n\|_r, \tag{3.22}$$

where $r = \max\{q, p\}$.

LEMMA 3.9 (Continuity of multilevel sample covariances in particles). *Suppose Assumptions 1 and 2 hold, and for any $\varepsilon > 0$, let L and $\{M_\ell\}_{\ell=0}^L$ be defined as in Theorem 3.2. Then the following asymptotic inequality holds for all $p \geq 2$,*

$$\begin{aligned}
\|C_n^{\text{ML}} - \bar{C}_n^{\text{ML}}\|_p & \leq \sum_{l=0}^L \left(\|v_n^l - \bar{v}_n^l\|_p + 4\|v_n^l - \bar{v}_n^l\|_{2p} \|\bar{v}_n^l\|_{2p} \right. \\
& \left. + \|v_n^{l-1} - \bar{v}_n^{l-1}\|_p + 4\|v_n^{l-1} - \bar{v}_n^{l-1}\|_{2p} \|\bar{v}_n^{l-1}\|_{2p} \right).
\end{aligned} \tag{3.23}$$

Proof. Recall first that

$$|C_n^{\text{ML}} - \bar{C}_n^{\text{ML}}| \leq \sum_{\ell=1}^L |\text{Cov}_{M_\ell}[v_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}] - (\text{Cov}_{M_\ell}[\bar{v}_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}])|.$$

Now the individual terms will be bounded. Note that

$$\text{Cov}_{M_\ell}[v_n^\ell] = E_{M_\ell}[v_n^\ell (v_n^\ell)^\top] - E_{M_\ell}[v_n^\ell] (E_{M_\ell}[v_n^\ell])^\top,$$

and similar for \bar{v}_n^ℓ . Using $a^2 - b^2 = (a-b)^2 + 2b(a-b)$ with $a = u^\top v_n^\ell$ and $b = u^\top \bar{v}_n^\ell$ and again with $a = (1/M_\ell) \sum_{i=1}^{M_\ell} u^\top v_{n,i}^\ell$ and $b = (1/M_\ell) \sum_{i=1}^{M_\ell} u^\top \bar{v}_{n,i}^\ell$ for arbitrary $u \in \mathbb{R}^d$,

these terms are rearranged as follows

$$\begin{aligned} & u^\top (\text{Cov}_{M_\ell}[v_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^\ell])u \\ &= E_{M_\ell}[|u^\top(v_n^\ell - \bar{v}_n^\ell)|^2] + 2E_{M_\ell}[(u^\top \bar{v}_n^\ell)(u^\top(v_n^\ell - \bar{v}_n^\ell))] \\ & - |E_{M_\ell}[u^\top(v_n^\ell - \bar{v}_n^\ell)]|^2 - 2E_{M_\ell}[u^\top \bar{v}_n^\ell]E_{M_\ell}[u^\top(v_n^\ell - \bar{v}_n^\ell)]. \end{aligned}$$

Then, using the Cauchy-Schwartz inequality, the first term of (3.12) is bounded as follows

$$\begin{aligned} & |\text{Cov}_{M_\ell}[v_n^\ell] - \text{Cov}_{M_\ell}[v_n^{\ell-1}] - (\text{Cov}_{M_\ell}[\bar{v}_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}])| \\ &= |\text{Cov}_{M_\ell}[v_n^\ell] - \text{Cov}_{M_\ell}[\bar{v}_n^\ell] + \text{Cov}_{M_\ell}[v_n^{\ell-1}] - \text{Cov}_{M_\ell}[\bar{v}_n^{\ell-1}]| \\ &\leq E_{M_\ell}[|v_n^\ell - \bar{v}_n^\ell|^2] + 4\sqrt{E_{M_\ell}[|\bar{v}_n^\ell|^2]E_{M_\ell}[|v_n^\ell - \bar{v}_n^\ell|^2]} \\ &+ E_{M_\ell}[|v_n^{\ell-1} - \bar{v}_n^{\ell-1}|^2] + 4\sqrt{E_{M_\ell}[|\bar{v}_n^{\ell-1}|^2]E_{M_\ell}[|v_n^{\ell-1} - \bar{v}_n^{\ell-1}|^2]}. \end{aligned} \quad (3.24)$$

After rearrangement, the triangle inequality, (3.22) with $p = \max\{p, 2\}$, and Hölder's inequality complete the proof.

□

It has just been shown that the second term of (3.11) is “close in the prediction ensembles”. Therefore, the error level of the first term will carry over between observation times by induction. This is made rigorous by the next lemma.

LEMMA 3.10 (Distance between ensembles.). *Suppose Assumptions 1 and 2 hold, and for any $\varepsilon > 0$, let L and $\{M_\ell\}_{\ell=0}^L$ be defined as in Theorem 3.2. Then the following asymptotic inequality holds for all $p \geq 2$,*

$$\sum_{\ell=0}^L \left\| \widehat{v}_n^\ell - \widetilde{v}_n^\ell \right\|_p \lesssim |\log(\varepsilon)|^n \varepsilon. \quad (3.25)$$

Proof. First recall that the assertion holds trivially for $n = 0$. Proceeding by induction, assume for $p \geq 2$,

$$\sum_{\ell=0}^L \left\| \widehat{v}_{n-1}^\ell - \widetilde{v}_{n-1}^\ell \right\|_p \lesssim |\log(\varepsilon)|^{n-1} \varepsilon,$$

Then Assumption 1(i) implies the following inequality holds for the prediction

$$\sum_{\ell=0}^L \left\| v_n^\ell - \bar{v}_n^\ell \right\|_p \leq c_\Psi \sum_{\ell=0}^L \left\| \widehat{v}_{n-1}^\ell - \widetilde{v}_{n-1}^\ell \right\|_p \lesssim |\log(\varepsilon)|^{n-1} \varepsilon. \quad (3.26)$$

Using Lemma 3.4, the following inequalities hold for $\ell = 0, \dots, L$,

$$|\widehat{v}_n^\ell - \widetilde{v}_n^\ell| \leq |I - K_n H| |v_n^\ell - \bar{v}_n^\ell| + c_n |C_n^{\text{ML}} - C_n| \left(|v_n^\ell - \bar{v}_n^\ell| + |y_n^\ell - H\bar{v}_n^\ell| \right). \quad (3.27)$$

By Hölder's inequality and since $y_n^\ell, \bar{v}_n^\ell \in L^p(\Omega)$ for all $p \geq 2$,

$$\begin{aligned} \left\| \widehat{v}_n^\ell - \widetilde{v}_n^\ell \right\|_p &\leq |I - K_n H| \left\| v_n^\ell - \bar{v}_n^\ell \right\|_p \\ &+ c_n \|C_n^{\text{ML}} - C_n\|_{2p} \left(\left\| v_n^\ell - \bar{v}_n^\ell \right\|_{2p} + \left\| y_n^\ell - H\bar{v}_n^\ell \right\|_{2p} \right) \\ &\lesssim \left\| v_n^\ell - \bar{v}_n^\ell \right\|_p + \|C_n^{\text{ML}} - C_n\|_{2p} \left(\left\| v_n^\ell - \bar{v}_n^\ell \right\|_{2p} + 1 \right). \end{aligned}$$

Plugging the moment bound (3.26) into the right-hand side of the inequality (3.23) yields that $\|C_n^{\text{ML}} - \bar{C}_n^{\text{ML}}\|_{2p} \lesssim |\log(\varepsilon)|^{n-1} \varepsilon$, which in combination Theorem 3.5 further leads to $\|C_n^{\text{ML}} - C_n\|_{2p} \lesssim |\log(\varepsilon)|^{n-1} \varepsilon$. Therefore, summing the above and using (3.26) again for $p, 2p$

$$\begin{aligned} \sum_{\ell=0}^L \|\hat{v}_n^\ell - \tilde{v}_n^\ell\|_p &\lesssim \sum_{\ell=0}^L \|v_n^\ell - \bar{v}_n^\ell\|_p + \varepsilon (\|v_n^\ell - \bar{v}_n^\ell\|_{2p} + \|y_n^\ell - \bar{v}_n^\ell\|_{2p}) \\ &\lesssim |\log(\varepsilon)|^{n-1} \varepsilon \left(1 + \sum_{\ell=0}^L \|y_n^\ell - \bar{v}_n^\ell\|_{2p}\right) \\ &\lesssim |\log(\varepsilon)|^n \varepsilon, \end{aligned} \quad (3.28)$$

where the last inequality of the proof uses that $\|y_n^\ell - H\bar{v}_n^\ell\|_{2p} \lesssim 1$ and $L \approx |\log(\varepsilon)|$. \square

Induction is complete on the distance between the multilevel ensemble and its i.i.d. shadow in L^p , and it remains only to close the argument, which is done next. Note that the induction actually holds for all n , but we are able to neglect the n -dependence of the constant c_n appearing in (3.27) by considering only a finite number N of steps.

Proof. [Proof of Theorem 3.2]

What remains is to verify that provided L and M_ℓ are defined under the constraints in Theorem 3.2, the error bound (3.3) will be obtained for all the functions $\varphi \in \mathcal{F}$ which are locally Lipschitz continuous with at most polynomial growth at infinity, cf. Definition 3.1. Notice that the triangle inequality gives

$$\begin{aligned} \|\mu_n^{\text{ML}}(\varphi) - \mu_n(\varphi)\|_p &\leq \|\mu_n^{\text{ML}}(\varphi) - \bar{\mu}_n^{\text{ML}}(\varphi)\|_p \\ &\quad + \|\bar{\mu}_n^{\text{ML}}(\varphi) - \bar{\mu}_n^L(\varphi)\|_p + \|\bar{\mu}_n^L(\varphi) - \mu_n(\varphi)\|_p, \end{aligned} \quad (3.29)$$

where $\bar{\mu}_n^{\text{ML}}$ denotes the empirical measure associated to the i.i.d. ensemble, and $\bar{\mu}_n^L$ denotes the probability measure associated to \bar{v}^L . Before treating each term separately, we notice that the two first summands of the right-hand side of the inequality relates to the statistical error, whereas the last relates to the bias.

The first summand of (3.29) satisfies the following bound

$$\begin{aligned} \|\mu_n^{\text{ML}}(\varphi) - \bar{\mu}_n^{\text{ML}}(\varphi)\|_p &= \left\| \sum_{\ell=0}^L E_{M_\ell} \left[\varphi(\hat{v}_n^\ell) - \varphi(\bar{v}_n^{\ell-1}) - (\varphi(\tilde{v}_n^\ell) - \varphi(\tilde{v}_n^{\ell-1})) \right] \right\|_p \\ &\leq \sum_{\ell=0}^L \|\varphi(\hat{v}_n^\ell) - \varphi(\bar{v}_n^{\ell-1})\|_p + \|\varphi(\tilde{v}_n^{\ell-1}) - \varphi(\bar{v}_n^{\ell-1})\|_p \\ &\leq C \sum_{\ell=0}^L \left[\|\hat{v}_n^\ell - \bar{v}_n^{\ell-1}\|_p \left(1 + |\bar{v}_n^{\ell-1}|^\nu\right) + |\hat{v}_n^\ell - \bar{v}_n^{\ell-1}|^{\nu+1} \right]_p \\ &\quad + \|\tilde{v}_n^{\ell-1} - \bar{v}_n^{\ell-1}\|_p \left(1 + |\bar{v}_n^{\ell-1}|^\nu\right) + |\tilde{v}_n^{\ell-1} - \bar{v}_n^{\ell-1}|^{\nu+1} \Big]_p \\ &\leq C \sum_{\ell=0}^L \left[\|\hat{v}_n^\ell - \bar{v}_n^{\ell-1}\|_{2p} \left(1 + \|\bar{v}_n^{\ell-1}\|_{2p}^\nu\right) + \|\hat{v}_n^\ell - \bar{v}_n^{\ell-1}\|_{p(\nu+1)}^{\nu+1} \right] \\ &\quad + \|\tilde{v}_n^{\ell-1} - \bar{v}_n^{\ell-1}\|_{2p} \left(1 + \|\bar{v}_n^{\ell-1}\|_{2p}^\nu\right) + \|\tilde{v}_n^{\ell-1} - \bar{v}_n^{\ell-1}\|_{p(\nu+1)}^{\nu+1} \Big] \\ &\lesssim |\log(\varepsilon)|^n \varepsilon. \end{aligned} \quad (3.30)$$

The second inequality follows directly from the expression (3.1). The third inequality comes from the triangle inequality and Hölder's inequality, and the fourth inequality arises directly from Lemma 3.10 and the boundedness of \tilde{v}_n in L^p for $p \geq 2$.

For the second summand of (3.29), notice that we can write $\bar{\mu}_n^L = \sum_{\ell=0}^L \bar{\mu}_n^\ell - \bar{\mu}_n^{\ell-1}$, where $\bar{\mu}_n^\ell$ is the measure associated to the level ℓ limiting process \tilde{v}_n^ℓ . Then, by virtue of (3.20) and condition (ii) of Assumption 2,

$$\begin{aligned} \|\bar{\mu}_n^{\text{ML}}(\varphi) - \bar{\mu}_n^L(\varphi)\|_p &\leq \sum_{\ell=0}^L \left\| E_{M_\ell} \left[\varphi(\tilde{v}_n^\ell) - \varphi(\tilde{v}_n^{\ell-1}) - \mathbb{E}[\varphi(\tilde{v}_n^\ell) - \varphi(\tilde{v}_n^{\ell-1})] \right] \right\|_p \\ &\leq C \sum_{\ell=0}^L M_\ell^{-1/2} \|\varphi(\tilde{v}_n^\ell) - \varphi(\tilde{v}_n^{\ell-1})\|_p \\ &\leq C \sum_{\ell=0}^L M_\ell^{-1/2} \|\tilde{v}_n^\ell - \tilde{v}_n^{\ell-1}\|_p \\ &\lesssim \sum_{\ell=0}^L M_\ell^{-1/2} N_\ell^{-\beta/2} \lesssim \varepsilon. \end{aligned} \tag{3.31}$$

Finally, for the bias term,

$$\|\bar{\mu}_n^L(\varphi) - \mu_n(\varphi)\|_p = |\bar{\mu}_n^L(\varphi) - \mu_n(\varphi)| = \left| \mathbb{E} \left[\varphi(\tilde{v}_n^L) - \varphi(\hat{v}_n) \right] \right| \lesssim \varepsilon, \tag{3.32}$$

where the last inequality follows from Lemma 3.6 and Assumption 2 (i).

Putting together (3.30), (3.31), and (3.32) in (3.29) yields the sought bound in (3.3). \square

Theorem 3.2 shows the cost-to- ε performance of MLEnKF, and to verify that it generally outperforms EnKF in this performance measure, we end this section with a comparable result on the cost-to- ε performance of EnKF.

THEOREM 3.11 (EnKF accuracy vs. cost). *Suppose Assumption 1, Assumption 2 (i), and Assumption 2 (iii) hold. For a given $\varepsilon > 0$, let L and M be defined under the constraints $L \approx \log(\varepsilon^{-1})/\alpha$ and $M \approx \varepsilon^{-2}$. Then for all functions $\varphi \in \mathcal{F}$ that are locally Lipschitz continuous with at most polynomial growth at infinity, cf. Definition 3.1, we have for any $p \geq 2$,*

$$\|\mu_n^{\text{MC}}(\varphi) - \mu_n(\varphi)\|_p \lesssim \varepsilon. \tag{3.33}$$

Here μ_n^{MC} denotes the EnKF empirical measure defined in (2.10), where the samples are given by the EnKF predict formulae at resolution level L (i.e., with the numerical integrator Ψ^L), approximating the time $t_n = n$ mean-field EnKF distribution μ_n . The computational cost of the EnKF estimator over the time sequence satisfies

$$\text{Cost (EnKF)} \lesssim \varepsilon^{-(2+\gamma/\alpha)}. \tag{3.34}$$

Proof. [Sketch of proof] By the triangle inequality

$$\begin{aligned} \|\mu_n(\varphi) - \mu_n^{\text{MC}}(\varphi)\|_p &\leq \|\mu_n(\varphi) - \bar{\mu}_n^L(\varphi)\|_p + \|\bar{\mu}_n^L(\varphi) - \bar{\mu}_n^{\text{MC}}(\varphi)\|_p \\ &\quad + \|\bar{\mu}_n^{\text{MC}}(\varphi) - \mu_n^{\text{MC}}(\varphi)\|_p =: I + II + III, \end{aligned}$$

where $\bar{\mu}_n^{\text{MC}}$ denotes the empirical measure associated to an EnKF ensemble $\{\tilde{v}_n^L(\omega_i)\}_{i=1}^M$ and $\bar{\mu}_n^L$ denotes the empirical measure associated to \tilde{v}_n^L . We bound the terms I , II , and III individually.

For the first term, we have

$$I = |\mu_n(\varphi) - \bar{\mu}_n^L(\varphi)| \leq \left| \mathbb{E} \left[\varphi(\hat{v}_n) - \varphi(\tilde{v}_n^L) \right] \right| \lesssim \varepsilon,$$

where the last inequality is implied by inequality (3.13) of Lemma 3.6, which it is straightforward to verify holds under Assumption 2 (i).

For the second term, we first note that we may assume without loss of generality that $\varphi(0) = 0$. Since φ is locally Lipschitz continuous with at most polynomial growth at infinity, there then exists positive scalars ν, C_φ such that

$$|\varphi(x)| \leq C_\varphi(1 + |x|^\nu).$$

By inequality (3.20) and since $\tilde{v}_n^L \in L^p(\Omega)$ for any $p > 1$,

$$II \leq \left\| E_M[\varphi(\tilde{v}_n^L)] - \mathbb{E}[\varphi(\tilde{v}_n^L)] \right\|_p \leq M^{-1/2} \left\| \varphi(\tilde{v}_n^L) \right\|_p \leq C\varepsilon \left\| 1 + |\tilde{v}_n^L|^\nu \right\|_p \lesssim \varepsilon.$$

For the last term, let us first assume that for any $p \geq 2$ and finite n ,

$$\left\| \hat{v}_n^L - \tilde{v}_n^L \right\| \lesssim \varepsilon, \quad (3.35)$$

for the particle dynamics \hat{v}_n^L and \tilde{v}_n^L respectively associated to the EnKF ensemble $\{\hat{v}_{n,i}^L\}_{i=1}^M$ and the mean-field EnKF ensemble $\{\tilde{v}_{n,i}^L\}_{i=1}^M$. Then the assumed regularity of φ , that $\hat{v}_n^L, \tilde{v}_n^L \in L^p(\Omega)$ for all $p \geq 2$, and Hölder's inequality yield that

$$\begin{aligned} III &= \left\| E_M[\varphi(\hat{v}_n^L) - \varphi(\tilde{v}_n^L)] \right\| \leq C_\varphi \left\| \hat{v}_n^L - \tilde{v}_n^L \right\| \left(1 + |\hat{v}_n^L|^\nu + |\tilde{v}_n^L|^\nu \right) \Big\|_p \\ &\lesssim \left\| \hat{v}_n^L - \tilde{v}_n^L \right\|_p \lesssim \varepsilon. \end{aligned}$$

All that remains is to verify (3.35). Since this can be done by very similar steps as in the proof of inequality (3.25), we omit this verification.

□

REMARK 4. Notice that for a given n one can obtain an error $\mathcal{O}(\varepsilon)$ for *MLEnKF* in (3.3) for an additional cost which is given by replacing ε by $\varepsilon |\log \varepsilon|^{-n}$ in (3.4). Furthermore, it is worth noting that, for any m and for any $\delta > 0$, $|\log \varepsilon|^m = \mathcal{O}(\varepsilon^{-\delta})$. Hence one can obtain a cost-of-error rate in (3.4) which is uniform in time and asymptotically superior to *EnKF* (3.34).

4. Numerical Examples. In this section the performance of EnKF and MLEnKF are compared on some very simple numerical examples in terms of computational cost vs. approximation error. First, in section 4.1, underlying dynamics from an Ornstein–Uhlenbeck SDE is considered. Next, in section 4.2, the underlying dynamics geometric Brownian motion is considered. Both of these examples are indeed analytically tractable, however they are approximated as though they were not. This provides a solid benchmark to compute errors and allows the theory to be illustrated.

4.1. An Ornstein-Uhlenbeck SDE. We first consider the simple Ornstein-Uhlenbeck SDE problem

$$du = -udt + \sigma dW_t, \quad u(0) = 1. \quad (4.1)$$

It has the exact solution

$$u(t) = u(0)e^{-t} + \int_0^t \sigma e^{(s-t)} dW_s,$$

and since

$$\int_0^1 \sigma e^{(s-1)} dW_s \sim N\left(0, \underbrace{\frac{\sigma^2}{2}(1 - e^{-2})}_{=:\Sigma}\right),$$

one SDE realization sampled at the observation times $t_n = n$ is generated by the linear solution operator

$$u_{n+1} = e^{-1}u_n + \xi_n =: \Psi(u_n)$$

where $\xi_n \sim N(0, \Sigma)$ i.i.d. The corresponding noisy observations are given by

$$y_n^{\text{obs}} = u_n + \eta_n,$$

with $\eta_n \sim N(0, \Gamma)$ i.i.d.

For the MLEnKF algorithm, a hierarchy of Milstein solution operators $\{\Psi^\ell\}_{\ell=0}^\infty$ are introduced, where the ℓ^{th} level solution operator uses a uniform time-step of size $\Delta t^\ell = 2^{-(\ell+1)}$. A numerical integration step takes the form

$$u_{n,m+1}^\ell = u_{n,m}^\ell(1 - \Delta t^\ell) + \sigma \Delta W_{n,m}^\ell, \quad m = 0, 1, \dots, 2^{\ell+1} - 1, \quad (4.2)$$

where the initial condition is given by $v_{n,0}^\ell = \widehat{v}_{n-1}^\ell$,

$$\Delta W_{n,m}^\ell = W(t_n + (m+1)\Delta t^\ell) - W(t_n + m\Delta t^\ell) \sim N(0, \Delta t^\ell),$$

and $u_n^\ell = u_{n,2^{\ell+1}}^\ell$.

Moreover, since the solution operator for (4.1) is linear, the gold standard becomes the conventional Kalman filter update

$$(\widehat{m}_n^\dagger, \widehat{C}_n^\dagger) = \left((I - K_n H)m_n + K_n y_n^{\text{obs}}, (I - K_n H)C_n \right), \quad n = 1, 2, \dots$$

Problem parameters. In the numerical experiments, $N = 100, 200$ and 400 observation times $\{t_n = n\}_{n=1}^N$ are used, and the covariance parameters are set to $\Gamma = 0.04$ and $\sigma = 0.5$. For a prescribed computational cost $\mathcal{O}(J)$, an EnKF ensemble of size $M = \mathcal{O}(J^{2/3})$ is solved by the Milstein method on a mesh $\Delta t = \mathcal{O}(J^{1/3})$, and for the MLEnKF method, we set L and M_ℓ according to the constraint in Theorem 3.2.

Approximations of the mean and covariance. In our first numerical experiment we approximate the gold standard mean and covariance for a single observation realization using the respective ensemble Kalman filtering methods, and measure the approximation error in terms of the root mean square error (RMSE):

$$\sqrt{\sum_{n=1}^N \frac{|\widehat{m}_n^\dagger - \widetilde{m}_n|^2}{N}}, \quad \sqrt{\sum_{n=1}^N \frac{|\widehat{C}_n^\dagger - \widetilde{C}_n|^2}{N}}, \quad (4.3)$$

with $(\tilde{m}_n, \tilde{C}_n)$ denoting a single realization of either the EnKF or the MLEnKF updates approximating the gold standard moments. These observables are sufficiently smooth to reach the rates $\alpha = 1$ and $\beta = 2$ with the Milstein method, cf. [20]. The respective decay rates are numerically verified over a sequence of times in Figure 4.1. Figure 4.2 presents a numerical performance study measuring RMSE (4.3) vs. computational cost for the respective methods. As is to be expected from Theorem 3.2 the decay of RMSE for the MLEnKF method as a function of the cost J is roughly $\mathcal{O}(J^{-1/2})$, orders of magnitude faster than the observed and expected EnKF decay rate $\mathcal{O}(J^{-1/3})$. Note that the error growth factor $|\log(\varepsilon)|^n$ from the theoretical bound (3.3) is not visible in the experiments. In fact, the constant is even stable (the shift in cost as measured by runtime is simply due to computation of additional updates), indicating that with a more careful analysis the present results may be extended to an infinite time horizon.

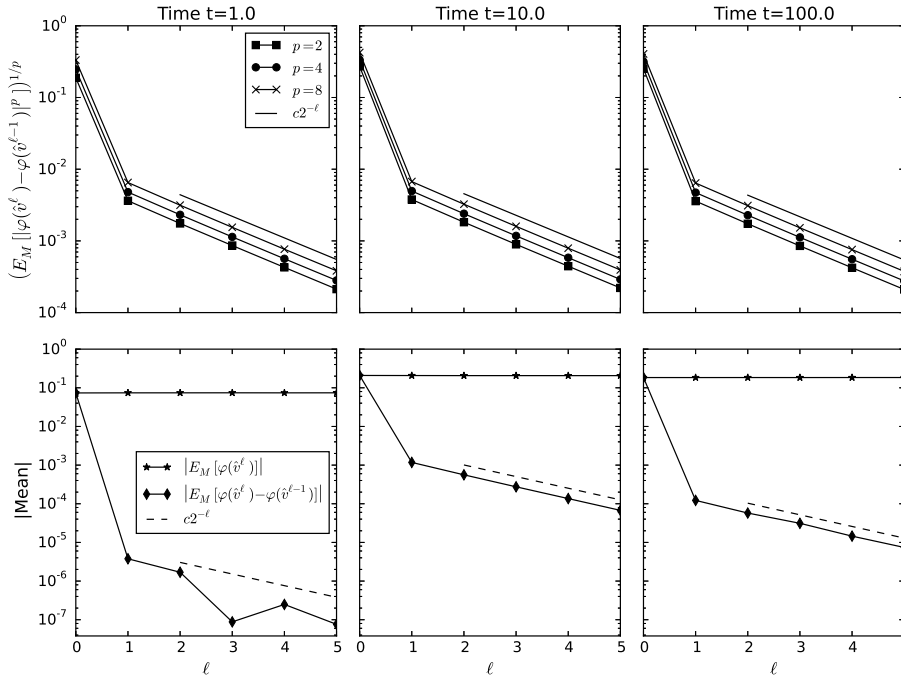


FIGURE 4.1. Numerical estimates of the decay rates over a sequence of times for the problem presented in Section 4.1 with $\varphi(v) = v$. The computations use $M = 10^6$ particles on every level.

Approximations of the exceedance probability. In our second numerical test, we approximate the mean of the observable $\varphi(\hat{u}_n) := \mathbf{1}\{\hat{u}_n > 0.1\}$, which corresponds to the exceedance probability $\mathbb{P}(\hat{u}_n > 0.1) = 1 - \Phi((0.1 - \hat{m}_n^\dagger)/\sqrt{\hat{C}_n^\dagger})$. The Milstein method achieves the weak rate $\alpha = 1$, but while one may show for $p = 2$ and any $\delta > 0$, $\|\varphi(\Psi^\ell(v)) - \varphi(\Psi^{\ell-1}(v))\|_p \lesssim N_\ell^{(1-\delta)/2}$, cf. [17, 2], the low regularity of the observable implies that there does not exist a $\beta > 0$ fulfilling condition (ii) of Assumption 2 for all $p > 2$. A numerical inference of $\beta = 0$ can be made from the numerical estimates of the decay rates in Figure 4.3, where we see that the decay rate of $\|\varphi(\Psi^\ell(v)) - \varphi(\Psi^{\ell-1}(v))\|_p$ consistently decreases towards 0 as p increases over a sequence of times

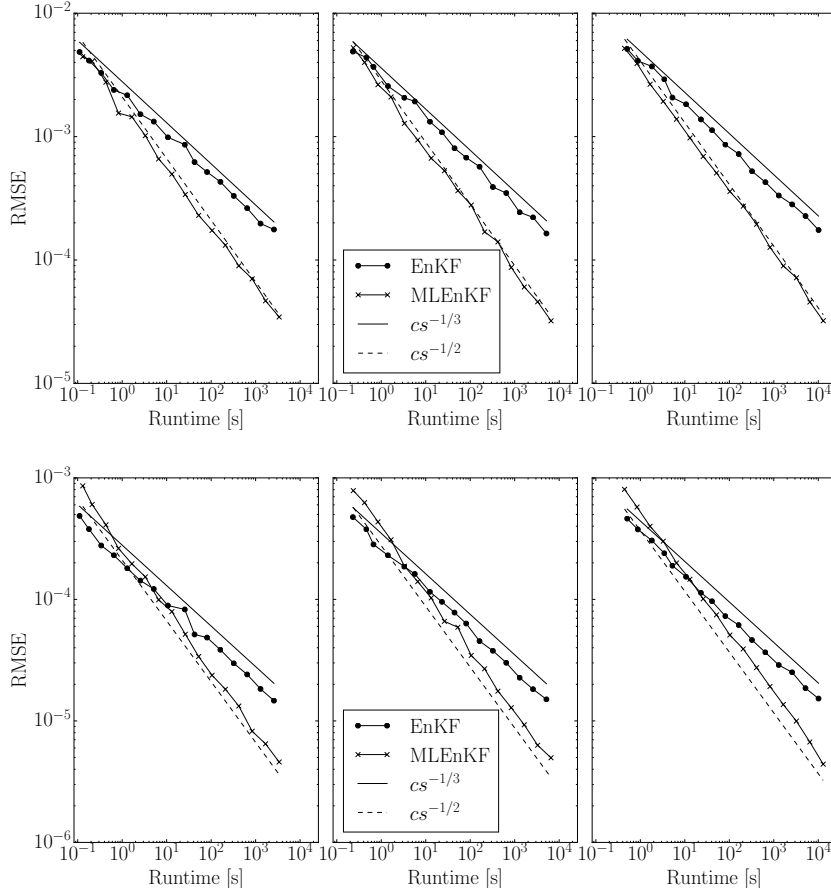


FIGURE 4.2. Comparison of the accuracy vs. computational cost when using the EnKF and MLEnKF methods on the filtering problem presented in Section 4.1. The error is measured in terms of the RMSE (4.3) for the mean (top row) and covariance (bottom row), computed with $N = 100, 200$ and 400 observation times in the first, second and third column, respectively. The computational cost is measured in computer runtime.

(while $\alpha \approx 1$). Theorem 3.2 does therefore not cover the given approximation problem. Nonetheless, implementing with the rates $\beta = 1$ and $\alpha = 1$, a numerical comparison of the performance of EnKF and MLEnKF approximating the exceedance probability is presented in Figure 4.6. A near optimal RMSE decay rate, slightly slower than $\mathcal{O}(J^{-1/2})$, is again achieved for the MLEnKF method.

4.2. Drift-alternating Geometric Brownian Motion. We next consider the SDE

$$du(t+n) = \begin{cases} \sigma^2 u(t+n)dt + \sigma u(t+n)dW(t+n), & \text{if } n \text{ is even,} \\ \sigma u(t+n)dW(t+n), & \text{else,} \end{cases} \quad \text{for } t \in (0, 1), \quad (4.4)$$

and with the initial condition $u(0) = 1$. This equation is analytically tractable as well, and the solution of the transformed equation $z = \log u$ is given via Itô's formula

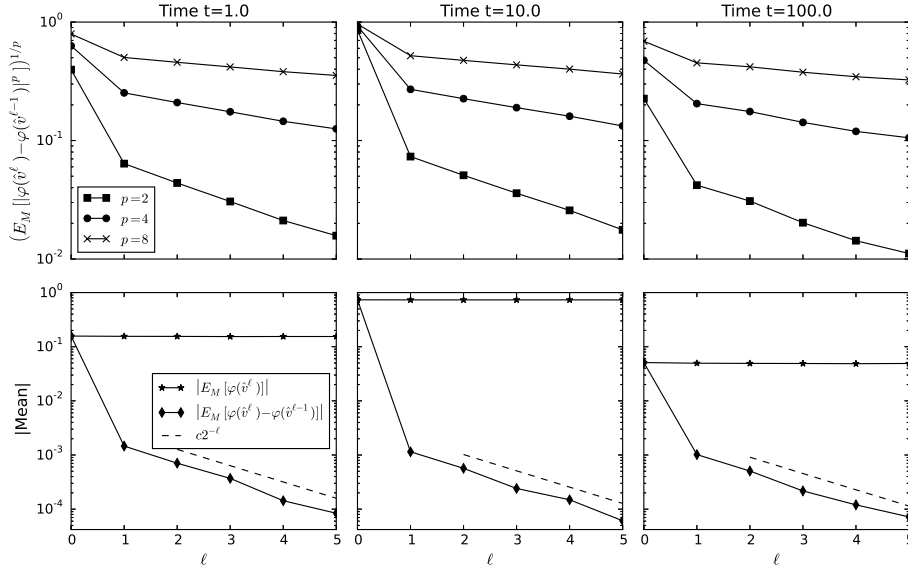


FIGURE 4.3. Numerical estimates of the decay rates over a sequence of times for the problem presented in Section 4.1 when approximating the exceedance probability $\mathbb{P}(\hat{u}_n > 0)$, i.e., with $\varphi(v) = \mathbf{1}\{v > 0.1\}$. The computations use $M = 10^6$ particles on every level.

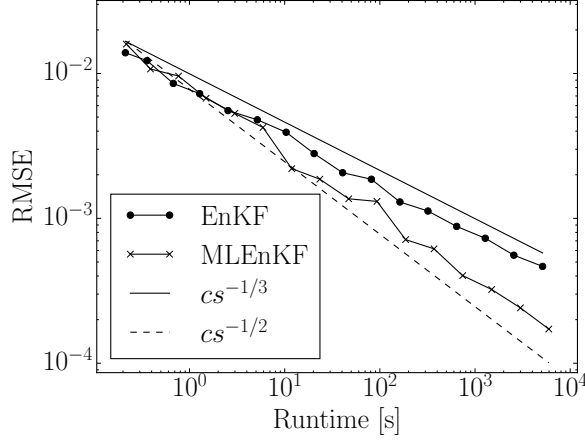


FIGURE 4.4. Accuracy vs. computational cost comparison of the EnKF and MLEnKF methods on the filtering problem presented in Section 4.1 when approximating the exceedance probability $\mathbb{P}(\hat{u}_n > 0.1)$ over $N = 200$ observation times. The error is measured in terms of the RMSE and the computational cost in computer runtime.

by

$$dz(t+n) = (-1)^n \frac{\sigma^2}{2} dt + \sigma dW(t+n).$$

Defining $\xi_n \sim N(0, \sigma^2)$ i.i.d., one has that

$$z_{n+1} = z_n + (-1)^n \frac{\sigma^2}{2} + \xi_n =: \Psi_n(z_n), \quad \text{with } z_0 = \log u_0 = 0,$$

and the solution of (4.4) can be obtained via exponentiation: $u_n = e^{z_n}$. Moreover, noisy observations for u_n are introduced on the form

$$\tilde{y}_n = u_n e^{\eta_n},$$

and $\eta_n \sim N(0, \Gamma)$ i.i.d. Which, upon defining $y_n^{\text{obs}} = \log \tilde{y}_n$, yields the following relation to noisy observations of z_n :

$$y_n^{\text{obs}} = z_n + \eta_n.$$

As the SDE (4.4) does not fulfill the linear Gaussian constraints (2.6) but $z = \log u$ does, we will here update the ensemble of $z = \log u$ processes. However, to add some artificial difficulty to the problem, the numerical integration is done on the u ensemble:

- (i) Numerically integrate a (multilevel or single level) ensemble $u_{n-1} \rightarrow u_n$.
- (ii) Compute sample mean and covariance of $z_n | Y_{n-1}^{\text{obs}}$ using the $z_n = \log u_n$ ensemble.
- (iii) Update the ensemble z_n by the new information provided by the observation y_n^{obs} .
- (iv) Compute the initial condition for the ensemble $u_n = e^{z_n}$ and return to (i).

REMARK 5. *The numerical integration of the GBM process in step (i) above introduces an artificial difficulty in the filtering problem since the integration may by other means be solved exactly. In practice, this does of course not make sense, but our purpose here is simply to numerically validate the performance of the MLEnKF method on a set of simple filtering problems for which reference solutions exist.*

Numerical integration of u_n is done by the hierarchy of Euler–Maruyama schemes introduced in (4.2) (applied to the GBM problem, the schemes are Euler–Maruyama, while applied to problems with additive noise, the schemes are Milstein), here with the slightly finer mesh hierarchy $\Delta t^\ell = 2^{-3+\ell}$, since the problem less stable. The covariance parameters are set to $\sigma = 1/4$ and $\Gamma = 1/16$, $N = 200$ and the numerical method yields the rates $\alpha = 1$, $\beta = 1$ (and $\gamma = 1$). See Figure 4.5 for a numerical verification of these decay rates over a sequence of times. In Figure 4.6, the gold standard mean and covariance of z_n has been approximated by the filtering methods. We observe an RMSE decay rate slightly slower than $\mathcal{O}(s^{-1/2})$ for MLEnKF and $\mathcal{O}(s^{-1/3})$ for EnKF, where s denotes runtime in seconds.

5. Conclusion. A first attempt, to the knowledge of the authors, at filtering using a multilevel Monte Carlo approach is considered in the present work. A proof based on induction of the optimality of the cost as a function of the error, or equivalently the error as a function of the cost, is given. This shows that an optimality result, which is slightly penalized with respect to the vanilla Monte Carlo result, can extend to the case of sequential inference. There is either a logarithmic term which grows with the number of steps, or a slightly higher rate $\mathcal{O}(\varepsilon^{-2-\delta})$, for $\delta > 0$. The ensemble Kalman filter is considered, which is consistent only in the case of a linear Gaussian model. However, the mean-field limiting equation may be viewed as a one-step optimal linear (in the observation) filter, and the convergence to this limiting distribution has the desired rate for a more general class of models. Since this work was submitted, two papers have appeared which consider consistent nonlinear filters for similar models. The paper [27] considered multilevel particle filters with optimally coupled multinomial resampling and found the rate of strong convergence β is effectively reduced by a factor of 2 as a result of the resampling. The paper [21] considered

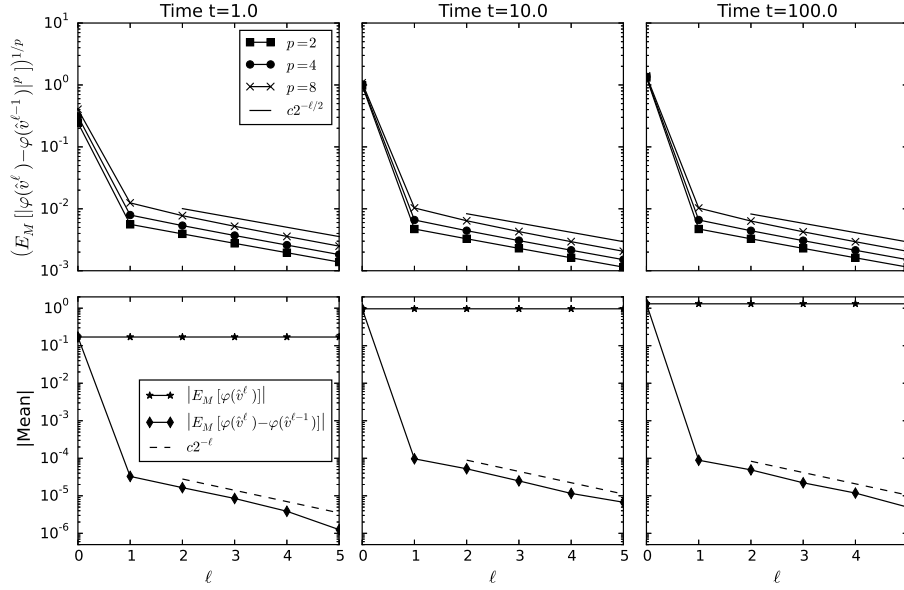


FIGURE 4.5. Numerical estimates of the decay rates over a sequence of times for the problem presented in Section 4.2 with $\varphi(v) = v$. The computations use $M = 10^6$ particles on every level.

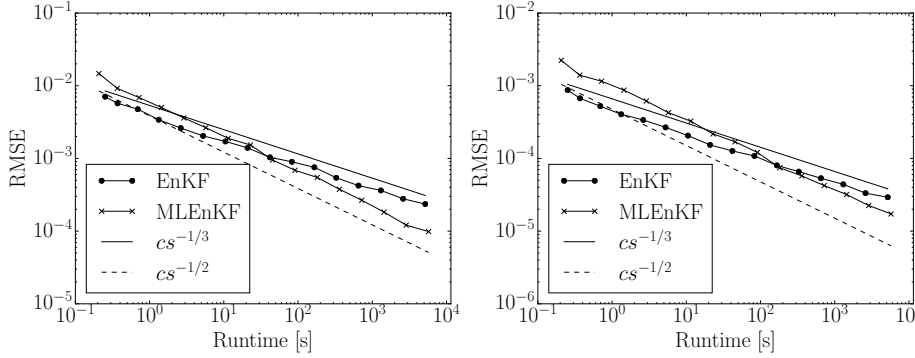


FIGURE 4.6. Accuracy vs. computational cost comparison of the EnKF and MLEnKF methods on the filtering problem presented in Section 4.2. The error is measured in terms of the RMSE (4.3) for the mean (left plot) and covariance (right plot), and the computational cost is measured in computer runtime.

multilevel ensemble transform particle filters, which use an optimally coupled deterministic transformation in place of the standard random resampling mechanism, and numerical results indicated the rate may be reduced in some cases and the same in others.

Acknowledgements Research reported in this publication was supported by the King Abdullah University of Science and Technology (KAUST). HH, KJHL, and RT were members of the SRI Center for Uncertainty Quantification at KAUST for much of the research reported. KJHL was additionally supported by an ORNL LDRD Strategic Hire grant.

REFERENCES

- [1] JEFFREY L ANDERSON, *Localization and sampling error correction in ensemble kalman filter data assimilation*, Monthly Weather Review, 140 (2012), pp. 2359–2371.
- [2] RAINER AVIKAINEN, *On irregular functionals of SDEs and the Euler scheme*, Finance Stoch., 13 (2009), pp. 381–401.
- [3] A. BAIN AND D. CRISAN, *Fundamentals of Stochastic Filtering*, Springer, 2009.
- [4] PETER J BICKEL AND ELIZAVETA LEVINA, *Covariance regularization by thresholding*, The Annals of Statistics, (2008), pp. 2577–2604.
- [5] ———, *Regularized estimation of large covariance matrices*, The Annals of Statistics, (2008), pp. 199–227.
- [6] CLAUDIO BIERIG AND ALEXEY CHERNOV, *Convergence analysis of multilevel Monte Carlo variance estimators and application for random obstacle problems*, Numerische Mathematik, (2014), pp. 1–35.
- [7] GERRIT BURGERS, PETER JAN VAN LEEUWEN, AND GEIR EVENSEN, *Analysis scheme in the ensemble Kalman filter*, Monthly weather review, 126 (1998), pp. 1719–1724.
- [8] OLIVIER CAPPÉ, ERIC MOULINES, AND TOBIAS RYDÉN, *Inference in hidden Markov models*, Springer, 2005.
- [9] J. CARLSSON, MOON K.S., SZEPESSY A., TEMPONE R., AND ZOURARIS G., *Stochastic differential equations: Models and numerics*. Lecture notes, 2010.
- [10] K. A. CLIFFE, M. B. GILES, R. SCHEICHL, AND A. L. TECKENTRUP, *Multilevel Monte Carlo methods and applications to elliptic PDEs with random coefficients*, Comput. Vis. Sci., 14 (2011), pp. 3–15.
- [11] NATHAN COLLIER, ABDUL-LATEEF HAJI-ALI, FABIO NOBILE, ERIK VON SCHWERIN, AND RAL TEMPONE, *A continuation multilevel Monte Carlo algorithm*, BIT Numerical Mathematics, (2014), pp. 1–34.
- [12] PIERRE DEL MORAL, *Feynman-Kac Formulae: Genealogical and Interacting Particle Systems with Applications*, Springer, 2004.
- [13] ARNAUD DOUCET, SIMON GODSILL, AND CHRISTOPHE ANDRIEU, *On sequential Monte Carlo sampling methods for Bayesian filtering*, Statistics and computing, 10 (2000), pp. 197–208.
- [14] GEIR EVENSEN, *Sequential data assimilation with a nonlinear quasi-geostrophic model using Monte Carlo methods to forecast error statistics*, Journal of Geophysical Research: Oceans (1978–2012), 99 (1994), pp. 10143–10162.
- [15] ———, *The ensemble Kalman filter: Theoretical formulation and practical implementation*, Ocean dynamics, 53 (2003), pp. 343–367.
- [16] RADII PETROVICH FEDORENKO, *A relaxation method for solving elliptic difference equations*, Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 1 (1961), pp. 922–927.
- [17] MIKE GILES, *Improved multilevel Monte Carlo convergence using the Milstein scheme*, in Monte Carlo and Quasi-Monte Carlo Methods 2006, Alexander Keller, Stefan Heinrich, and Harald Niederreiter, eds., Springer Berlin Heidelberg, 2008, pp. 343–358.
- [18] M. B. GILES, *Multilevel Monte Carlo path simulation*, Oper. Res., 56 (2008), pp. 607–617.
- [19] M. B. GILES AND L. SZPRUCH, *Antithetic multilevel Monte Carlo estimation for multi-dimensional SDEs without Lévy area simulation*, Ann. Appl. Probab., 24 (2014), pp. 1585–1620.
- [20] CARL GRAHAM AND DENIS TALAY, *Stochastic simulation and Monte Carlo methods*, vol. 68 of Stochastic Modelling and Applied Probability, Springer, Heidelberg, 2013. Mathematical foundations of stochastic simulation.
- [21] ALASTAIR GREGORY, COLIN COTTER, AND SEBASTIAN REICH, *Multilevel ensemble transform particle filtering*, arXiv preprint arXiv:1509.00325, (2015).
- [22] ALLAN GUT, *Probability: a graduate course*, vol. 200, Springer, 2005.
- [23] WOLFGANG HACKBUSCH, *Multi-grid methods and applications*, vol. 4, Springer-Verlag Berlin, 1985.
- [24] STEFAN HEINRICH, *Multilevel Monte Carlo methods*, in Large-scale scientific computing, Springer, 2001, pp. 58–67.
- [25] VIET HA HOANG, CHRISTOPH SCHWAB, AND ANDREW M STUART, *Complexity analysis of accelerated MCMC methods for Bayesian inversion*, Inverse Problems, 29 (2013), p. 085010.
- [26] HÅKON HOEL, ERIK VON SCHWERIN, ANDERS SZEPESSY, AND RAÚL TEMPONE, *Implementation and analysis of an adaptive multilevel Monte Carlo algorithm*, Monte Carlo Methods Appl., 20 (2014), pp. 1–41.
- [27] AJAY JASRA, KENGO KAMATANI, KODY JH LAW, AND YAN ZHOU, *Multilevel particle filter*, arXiv preprint arXiv:1510.04977, (2015).
- [28] A.H. JAZWINSKI, *Stochastic processes and filtering theory*, vol. 63, Academic Pr, 1970.

- [29] RUDOLPH EMIL KALMAN ET AL., *A new approach to linear filtering and prediction problems*, Journal of basic Engineering, 82 (1960), pp. 35–45.
- [30] E. KALNAY, *Atmospheric Modeling, Data Assimilation and Predictability*, Cambridge, 2003.
- [31] C KETELSEN, R SCHEICHL, AND AL TECKENTRUP, *A hierarchical multilevel Markov chain Monte Carlo algorithm with applications to uncertainty quantification in subsurface flow*, arXiv preprint arXiv:1303.7343, (2013).
- [32] P.E. KLOEDEN AND E. PLATEN, *Numerical solution of stochastic differential equations*, vol. 23 of Applications of Mathematics (New York), Springer-Verlag, Berlin, 1992.
- [33] KODY LAW, ANDREW STUART, AND KOSTAS ZYGALAKIS, *Data Assimilation: A Mathematical Introduction*, Springer-Verlag Berlin, 2015.
- [34] KODY JH LAW, HAMIDOU TEMBINE, AND RAUL TEMPONE, *Deterministic mean-field ensemble Kalman filtering*, arXiv preprint arXiv:1409.0628v4, (2014).
- [35] FRANÇOIS LE GLAND, VALÉRIE MONBET, VU-DUC TRAN, ET AL., *Large sample asymptotics for the ensemble Kalman filter*, The Oxford Handbook of Nonlinear Filtering, (2011), pp. 598–631.
- [36] OLIVIER LEDOIT AND MICHAEL WOLF, *A well-conditioned estimator for large-dimensional covariance matrices*, Journal of multivariate analysis, 88 (2004), pp. 365–411.
- [37] DAVID G LUENBERGER, *Optimization by vector space methods*, John Wiley & Sons, 1968.
- [38] JAN MANDEL, LOREN COBB, AND JONATHAN D BEEZLEY, *On the convergence of the ensemble Kalman filter*, Applications of Mathematics, 56 (2011), pp. 533–541.
- [39] S. MISHRA AND C. SCHWAB, *Sparse tensor multi-level Monte Carlo finite volume methods for hyperbolic conservation laws with random initial data*, Math. Comp., 81 (2012), pp. 1979–2018.
- [40] OLIVER PAJONK, BOJANA V ROSIĆ, ALEXANDER LITVINENKO, AND HERMANN G MATTHIES, *A deterministic filter for non-Gaussian Bayesian estimation Applications to dynamical system estimation with noisy measurements*, Physica D: Nonlinear Phenomena, 241 (2012), pp. 775–788.
- [41] YAO-FENG REN AND HAN-YING LIANG, *On the best constant in Marcinkiewicz-Zygmund inequality*, Statist. Probab. Lett., 53 (2001), pp. 227–233.