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# CRAMÉR’S ESTIMATE FOR THE REFLECTED PROCESS REVISITED<sup>1</sup>

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The reflected process of a random walk or Lévy process arises in many areas of applied probability, and a question of particular interest is how the tail of the distribution of the heights of the excursions away from zero behaves asymptotically. The Lévy analogue of this is the tail behaviour of the characteristic measure of the height of an excursion. Apparently, the only case where this is known is when Cramér’s condition hold. Here, we establish the asymptotic behaviour for a large class of Lévy processes, which have exponential moments but do not satisfy Cramér’s condition. Our proof also applies in the Cramér case, and corrects a proof of this given in Doney and Maller [*Ann. Appl. Probab.* **15** (2005) 1445–1450].

**1. Introduction.** The reflected process  $R = (R_n, n \geq 0)$  formed from a random walk  $S = (S_n, n \geq 0)$  by setting

$$R_n = S_n - I_n \quad \text{where } I_n = \min_{i \leq n} S_i, n \geq 0,$$

arises in many areas of applied probability, including queuing theory, risk theory and mathematical genetics. In all these areas, the i.i.d. sequence of random variables defined by

$$h_i = \max_{0 \leq n < \widehat{T}_i - \widehat{T}_{i-1}} \{S_{\widehat{T}_{i-1}+n} - S_{\widehat{T}_{i-1}}\}, \quad i = 1, 2, \dots,$$

where  $\widehat{T}_0 = 0$  and  $\widehat{T}_i = \min\{k : S_k < S_{T_{i-1}}\}, i \geq 1$  are the strict descending ladder times, is of central importance. These random variables give the heights of the excursions of  $R$  away from 0, or equivalently the heights of the excursions of  $S$  above its minimum. Our main focus will be on the asymptotic behaviour of  $P(h_1 > x)$ , which among other things is useful in the study of the point process of excursion heights.

In continuous time, we replace the random walk by a Lévy process  $X = (X_t, t \geq 0)$  and study  $R = (R_t, t \geq 0)$  where

$$R_t = X_t - \underline{X}_t \quad \text{and} \quad \underline{X}_t = \inf_{s \leq t} X_s.$$

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In mathematical finance,  $R$  is called the drawup. When indexed by local time at the infimum, the excursions of  $R$  away from 0 form a Poisson point process whose characteristic measure we denote by  $\hat{n}$ . If  $h$  denotes the height of a generic excursion, then  $\hat{n}(h > x)$ , which is the expected number of excursions whose height exceeds  $x$  in unit local time, is the Lévy analogue of  $P(h_1 > x)$ .

Our main interest is in the Lévy process case, but we start by reviewing some discrete time results where we make the following assumption.

**ASSUMPTION S.** Without loss of generality we will treat only the case of nonlattice random walks, or the case that  $S$  takes values on the integers, and is aperiodic, which we refer to as the lattice case.

A classical case where the asymptotic behaviour of  $P(h_1 > x)$  is known is when  $S$  satisfies Cramér’s condition, namely  $E(e^{\gamma S_1}) = 1$  for some  $\gamma \in (0, \infty)$ . Then  $e^{\gamma S_n}$  is a nonnegative martingale, which necessarily converges, and this forces  $S_n \rightarrow -\infty$  a.s. Thus for  $x > 0$ , the first time passage  $\tau_x = \inf\{n : S_n > x\}$  of  $S$  to  $(x, \infty)$  is defective and by Cramér’s estimate satisfies

$$(1.1) \quad \lim_{x \rightarrow \infty} e^{\gamma x} P(\tau_x < \infty) = \Gamma,$$

where  $\Gamma$  is a known nonnegative constant. Here, in the lattice case, the limit is taken through the integers. It then follows immediately from the identity

$$(1.2) \quad P(\tau_x < \infty) = P(h_1 > x) + \int_0^\infty P(h_1 \leq x, \widehat{H}_1 \in dy) P(\tau_{x+y} < \infty),$$

where  $\widehat{H}_1 = |S_{\widehat{T}_1}|$  is the first strict descending ladder height, that

$$(1.3) \quad \lim_{x \rightarrow \infty} e^{\gamma x} P(h_1 > x) = \Gamma \{1 - E(e^{-\gamma \widehat{H}_1})\}.$$

This argument is due to Iglehart [10].

An obvious question is whether one can find a larger class of random walks with exponentially small tails for which something similar to (1.3) holds. To see that this is the case, we make the following definitions. For any nonnegative function  $f$ , let us say that  $f \in \mathcal{L}^{(\alpha)}$ ,  $\alpha \geq 0$ , if

$$(1.4) \quad \lim_{x \rightarrow \infty} \frac{f(x + y)}{f(x)} = e^{-\alpha y} \quad \text{for all } y.$$

In the nonlattice case,  $Z \in \mathcal{L}^{(\alpha)}$  means  $P(Z > x) \in \mathcal{L}^{(\alpha)}$ , and in the lattice case it means that (1.4) holds when  $x$  and  $y$  are restricted to the integers. For  $\alpha > 0$ , the assumption  $Z \in \mathcal{L}^{(\alpha)}$  is equivalent to a local statement, specifically that the measure  $P(Z \in x + dy)/P(Z > x)$  converges weakly to the  $\text{Exp}(\alpha)$  distribution, or the  $\text{Geom}(e^{-\alpha})$  distribution in the lattice case. This does not extend to  $\mathcal{L}^{(0)}$ , the class of long-tailed distributions. For a measure,  $\mu \in \mathcal{L}^{(\alpha)}$  means that  $\bar{\mu}(x) := \mu((x, \infty)) \in \mathcal{L}^{(\alpha)}$ . So for a random walk which satisfies the Cramér condition

above, if  $\Gamma \neq 0$ , then  $P(\tau_x < \infty) \in \mathcal{L}^{(\gamma)}$ ,  $h_1 \in \mathcal{L}^{(\gamma)}$ , and the ratio of  $P(h_1 > x)$  to  $P(\tau_x < \infty)$  converges to the constant  $1 - E(e^{-\gamma \hat{H}_1})$ . Our first result extends this as follows.

**THEOREM 1.1.** *Fix  $\alpha > 0$ . For any random walk  $S$ ,*

$$P(\tau_x < \infty) \in \mathcal{L}^{(\alpha)}$$

*if and only if*

$$P(h_1 > x) \in \mathcal{L}^{(\alpha)},$$

*in which case*

$$\lim_{x \rightarrow \infty} \frac{P(h_1 > x)}{P(\tau_x < \infty)} = 1 - E(e^{-\alpha \hat{H}_1}).$$

In the Cramér case, we know the asymptotic behaviour of  $P(\tau_x < \infty)$ , but in the general case more work is required to obtain a useful asymptotic estimate. The first step is to connect the distributions of  $S$  and  $H = (H_n, n \geq 0)$ , the weak increasing ladder height process, which is defective when  $S_n \rightarrow -\infty$ . This is defined by  $H_n = S_{T_n}$ , where  $T_0 = 0$  and  $T_n = \min\{k : S_k \geq S_{T_{n-1}}\}$ ,  $n \geq 1$ . If  $T_n = \infty$ , we set  $H_n$  equal to some cemetery state, in which case probabilities and expectation are understood to be taken over only noncemetery values. The following result, which is Lemma 1 in [3], gives the required connection between  $S$  and  $H$ . Assume  $\alpha > 0$ : then

$$(1.5) \quad S_1 \in \mathcal{L}^{(\alpha)} \iff H_1 \in \mathcal{L}^{(\alpha)},$$

and when this happens

$$(1.6) \quad \frac{P(S_1 > x)}{P(H_1 > x)} \rightarrow 1 - E(e^{-\alpha \hat{H}_1}).$$

Provided that  $S_n \xrightarrow{a.s.} -\infty$  the connection between  $P(\tau_x < \infty)$  and  $P(H_1 > x)$  is implicit in the relationship

$$P(\tau_x < \infty) = e^{-B} \sum_1^\infty P(H_n > x),$$

where  $B = \sum_1^\infty n^{-1} P(S_n \geq 0) < \infty$ ; see equation (20) of [3]. Since  $H_n$  is the sum of  $n$  i.i.d. copies of  $H_1$  it is clear that to go further we need to study a subclass of  $\mathcal{L}^{(\alpha)}$  which is closed under convolution powers. The appropriate class is  $S^{(\alpha)}$ , the convolution equivalent class of parameter  $\alpha > 0$ . We say a (possibly defective) random variable  $Z \in S^{(\alpha)}$  if  $Z \in \mathcal{L}^{(\alpha)}$  and additionally, when  $Z_1, Z_2$  are independent copies of  $Z$ ,

$$(1.7) \quad \lim_{x \rightarrow \infty} \frac{P(Z_1 + Z_2 > x)}{P(Z_1 > x)} \in (0, \infty).$$

It is easily seen that  $Z \in \mathcal{L}^{(\alpha)}$  implies  $E(e^{\theta Z}) < \infty$  for  $\theta < \alpha$ , and  $E(e^{\theta Z}) = \infty$  for  $\theta > \alpha$ , but when  $\theta = \alpha$  the expectation can be finite or infinite. If in addition  $Z \in \mathcal{S}^{(\alpha)}$  then  $E(e^{\alpha Z}) < \infty$  and the limit in (1.7) is given by  $2E(e^{\alpha Z})$ : see for example the discussion in Section 5 of [14]. Observe that if  $S_n \rightarrow -\infty$  and  $E(e^{\alpha S_1}) \geq 1$  then  $\exists \gamma \in (0, \alpha]$  with  $E(e^{\gamma S_1}) = 1$  and so we are back in the Cramér setting. This explains the second assumption in the following, which is Theorem 1 in [3]; for  $\alpha > 0$  if

$$(1.8) \quad S_1 \in \mathcal{S}^{(\alpha)} \quad \text{and} \quad E(e^{\alpha S_1}) < 1$$

then

$$P(\tau_x < \infty) \in \mathcal{S}^{(\alpha)},$$

and furthermore,

$$\lim_{x \rightarrow \infty} \frac{P(\tau_x < \infty)}{P(S_1 > x)} = \frac{e^{-B}}{(1 - E(e^{-\alpha S_1}))(1 - E(e^{\alpha H_1}))} := C \in (0, \infty).$$

Combining these results, we see that (1.8) is a sufficient condition for  $P(h_1 > x)$ ,  $P(S_1 > x)$ ,  $P(H_1 > x)$  and  $P(\tau_x < \infty)$  to all have the same asymptotic behaviour, modulo constants. However, this leaves open the possibility that this could also happen when  $S_1 \in \mathcal{L}^{(\alpha)} \setminus \mathcal{S}^{(\alpha)}$ : in fact, this is not the case.

**THEOREM 1.2.** *Assume  $\alpha > 0$ ,  $S_1 \in \mathcal{L}^{(\alpha)}$  and  $E(e^{\alpha S_1}) < 1$ . Then*

$$\frac{P(\tau_x < \infty)}{P(S_1 > x)} \rightarrow L \in (0, \infty) \iff S_1 \in \mathcal{S}^{(\alpha)}$$

*in which case, necessarily,  $L = C$ .*

**REMARK 1.1.** This result is actually a statement about the defective renewal process  $H = (H_n, n \geq 1)$ . To see this, we require the Wiener–Hopf factorisation

$$(1.9) \quad 1 - E(e^{-\alpha S_1}) = (1 - E(e^{\alpha H_1}))(1 - E(e^{-\alpha \hat{H}_1})).$$

Then note that the assumptions in Theorem 1.2 are equivalent to  $P(H_1 > x) \in \mathcal{L}^{(\alpha)}$  [by (1.5)] and  $E(e^{\alpha H_1}) < 1$  [by (1.9)]. The conclusion is equivalent to

$$\lim_{x \rightarrow \infty} \frac{P(\tau_x^H < \infty)}{P(H_1 > x)} = L' \in (0, \infty) \iff H_1 \in \mathcal{S}^{(\alpha)},$$

where necessarily

$$L' = \frac{e^{-B}}{(1 - E(e^{\alpha H_1}))^2}.$$

Here,  $\tau_x^H$  is the first passage time of  $H$  to  $(x, \infty)$ , and clearly  $P(\tau_x^H < \infty) = P(\tau_x < \infty)$ . Moreover, since  $\mathcal{S}^{(\alpha)}$  is closed under asymptotic tail-equivalence,  $S_1 \in \mathcal{S}^{(\alpha)} \Leftrightarrow H_1 \in \mathcal{S}^{(\alpha)}$ , by (1.6). Finally, again by (1.6),

$$L' = L(1 - E(e^{-\alpha \hat{H}_1})) = \frac{e^{-B}}{(1 - E(e^{\alpha H_1}))^2}.$$

In fact, our proof shows that this version of the result holds for any defective subordinators.

The Lévy version of Cramér's condition is that  $E(e^{\gamma X_1}) = 1$  for some  $\gamma > 0$ , and assuming this Bertoin and Doney [2] proved the following analogue of (1.1); if  $X_1$  is nonlattice then

$$(1.10) \quad \lim_{x \rightarrow \infty} e^{\gamma x} P(\tau_x < \infty) = \Gamma^*,$$

where  $\tau_x = \inf\{t : X_t > x\}$  is now the first passage time of  $X$  to  $(x, \infty)$  and  $\Gamma^*$  is a known nonnegative constant. The analogue of (1.3) now becomes

$$(1.11) \quad \lim_{x \rightarrow \infty} e^{\gamma x} \hat{n}(h > x) = \hat{\kappa}(\gamma) \Gamma^*,$$

where  $\hat{\kappa}$  is the Laplace exponent of the strictly decreasing ladder height process. A proof of this result was given in [7], but there is a problem with the argument presented there. Specifically, equation (15) therein is not correct, the problem being that the conditions required to employ the compensation formula are not satisfied. So our first aim is to rectify this, and we do so by using a different approach which applies to a much more general situation.

**THEOREM 1.3.** *Fix  $\alpha > 0$ . For any Lévy process  $X$ ,*

$$(1.12) \quad P(\tau_x < \infty) \in \mathcal{L}^{(\alpha)}$$

*if and only if*

$$(1.13) \quad \hat{n}(h > x) \in \mathcal{L}^{(\alpha)},$$

*in which case*

$$(1.14) \quad \lim_{x \rightarrow \infty} \frac{\hat{n}(h > x)}{P(\tau_x < \infty)} = \hat{\kappa}(\alpha).$$

Thus in particular, (1.11) is now proved provided  $\Gamma^* \neq 0$ . (If  $\Gamma^* = 0$ , (1.11) continues to hold; see Remark 4.1.)

As in the random walk case, to deduce useful results about  $\hat{n}(h > x)$  we need to investigate how the asymptotic behaviour of  $P(\tau_x < \infty)$  is related to that of  $\bar{\Pi}_X(x) = \Pi_X((x, \infty))$ , where  $\Pi_X(dy)$  denotes the Lévy measure of our Lévy process  $X$ . We will give a complete answer to this under the natural assumption that  $\bar{\Pi}_X \in \mathcal{L}^{(\alpha)}$ , but first we consider the situation that  $\bar{\Pi}_X \in \mathcal{S}^{(\alpha)}$ , the class

of  $\alpha$ -convolution equivalent functions, for some  $\alpha > 0$ . By this, we mean that the probability distribution defined by  $G(dy) = \Pi_X(dy)\mathbf{1}_{\{y>1\}}/\overline{\Pi}_X(1)$  is in  $\mathcal{S}^{(\alpha)}$ . Thus in particular,

$$(1.15) \quad \lim_{x \rightarrow \infty} \frac{\overline{G * G}(x)}{2\overline{G}(x)} = \int_1^\infty e^{\alpha y} G(dy) < \infty.$$

In this scenario,  $E(e^{\alpha X_1}) < \infty$  by Theorem 25.3 of Sato [13], and since (1.12) implies  $X_t \rightarrow -\infty$  a.s., we can then assume, without loss of generality, that  $E(e^{\alpha X_1}) < 1$ . This is because if not there exists a  $\gamma \in (0, \alpha]$  such that  $E(e^{\gamma X_1}) = 1$ , so we are back in the Cramér situation. When  $\overline{\Pi}_X \in \mathcal{S}^{(\alpha)}$  and  $E(e^{\alpha X_1}) < 1$ , it has been shown in Klüppelberg, Kyprianou and Maller [11], Lemma 3.5, that

$$(1.16) \quad \lim_{x \rightarrow \infty} \frac{P(\tau_x^H < \infty)}{\overline{\Pi}_H(x)} = \frac{q}{\kappa(-\alpha)^2},$$

where  $\Pi_H$  is the Lévy measure and  $\tau_x^H$  the first passage time for the increasing ladder height process  $H$ , and  $\kappa$  and  $q$  are the Laplace exponent and killing rate of  $H$  respectively. Since  $P(\tau_x^H < \infty) = P(\tau_x < \infty)$  and it was also claimed in Proposition 5.3 of [11] that  $\overline{\Pi}_X \in \mathcal{L}^{(\alpha)}$  if and only if  $\overline{\Pi}_H \in \mathcal{L}^{(\alpha)}$  and then  $\overline{\Pi}_X(x) \sim \widehat{\kappa}(\alpha)\overline{\Pi}_H(x)$ , (1.16) is apparently equivalent to

$$\lim_{x \rightarrow \infty} \frac{P(\tau_x < \infty)}{\overline{\Pi}_X(x)} = \frac{q}{\widehat{\kappa}(\alpha)\kappa(-\alpha)^2}.$$

Together with our Theorem 1.3, this would solve the problem in this convolution equivalent case. However, there is a problem with the proof of the claimed equivalence of  $\overline{\Pi}_X$  and  $\overline{\Pi}_H$ , specifically in display (7.18) of [11], where an unjustified change of limit operations is used. We circumvent this problem in proving

**THEOREM 1.4.** *Fix  $\alpha > 0$ . For any Lévy process  $X$ ,*

$$(1.17) \quad \overline{\Pi}_X \in \mathcal{L}^{(\alpha)}$$

*if and only if*

$$(1.18) \quad \overline{\Pi}_H \in \mathcal{L}^{(\alpha)}$$

*in which case*

$$(1.19) \quad \lim_{x \rightarrow \infty} \frac{\overline{\Pi}_X(x)}{\overline{\Pi}_H(x)} = \widehat{\kappa}(\alpha).$$

**REMARK 1.2.** Note that, unlike Proposition 5.3 of [11], we do not require the assumption that  $X_t \rightarrow -\infty$  a.s. in this result.

Our last main result addresses the possibility that there are situations where  $\overline{\Pi}_X \in \mathcal{L}^{(\alpha)} \setminus \mathcal{S}^{(\alpha)}$  and  $P(\tau_x < \infty)$  [and hence  $\widehat{n}(h > x)$ ] has the same asymptotic behaviour as  $\overline{\Pi}_X(x)$ .

THEOREM 1.5. Assume  $\alpha > 0, \bar{\Pi}_X \in \mathcal{L}^{(\alpha)}$  and  $E(e^{\alpha X_1}) < 1$ . Then

$$(1.20) \quad \lim_{x \rightarrow \infty} \frac{P(\tau_x < \infty)}{\bar{\Pi}_X(x)} = L \in (0, \infty)$$

if and only if  $\bar{\Pi}_X \in \mathcal{S}^{(\alpha)}$ , in which case case  $L = \frac{q}{\hat{\kappa}(\alpha)\kappa(-\alpha)^2}$ .

This result may be reformulated in terms of  $\bar{\Pi}_H$  rather than  $\bar{\Pi}_X$ , and this is the form in which we will prove it. In fact, our proof shows that this version of the result holds for any defective subordinator.

THEOREM 1.6. Assume  $\alpha > 0, \bar{\Pi}_H \in \mathcal{L}^{(\alpha)}$  and  $E(e^{\alpha H_1}) < 1$ . Then

$$(1.21) \quad \lim_{x \rightarrow \infty} \frac{P(\tau_x^H < \infty)}{\bar{\Pi}_H(x)} = L' \in (0, \infty)$$

if and only if  $\bar{\Pi}_H \in \mathcal{S}^{(\alpha)}$ , in which case case  $L' = \frac{q}{\kappa(-\alpha)^2}$ .

REMARK 1.3. To see the equivalence of Theorems 1.5 and 1.6, note that by Theorem 1.4,  $\bar{\Pi}_X \in \mathcal{L}^{(\alpha)} \Leftrightarrow \bar{\Pi}_H \in \mathcal{L}^{(\alpha)}$ ,  $\bar{\Pi}_X \in \mathcal{S}^{(\alpha)} \Leftrightarrow \bar{\Pi}_H \in \mathcal{S}^{(\alpha)}$  and  $L' = \hat{\kappa}(\alpha)L$ . Finally,  $E(e^{\alpha X_1}) < 1 \Leftrightarrow E(e^{\alpha H_1}) < 1$  by the Wiener–Hopf factorisation; see, for example, Proposition 5.1 of [11].

REMARK 1.4. Note that, in particular, our results show that when  $\alpha > 0, \bar{\Pi}_X \in \mathcal{S}^{(\alpha)}$  and  $E(e^{\alpha X_1}) < 1$ , the quantities  $\bar{\Pi}_H(x), P(\tau_x < \infty)$  and  $\hat{n}(h > x)$  all have the same asymptotic behaviour as  $\bar{\Pi}_X(x)$  modulo a constant. This contrasts with the Cramér case, when  $P(\tau_x < \infty)$  and  $\hat{n}(h > x)$  are comparable to each other but not to  $\bar{\Pi}_X(x)$  since then, by Theorem 25.3 of [13],  $\bar{\Pi}_X(x) = o(e^{-\gamma x})$ .

REMARK 1.5. If  $\bar{\Pi}_X \in \mathcal{L}^{(\alpha)}$  the limit in (1.20) cannot be 0 but it can be  $\infty$ . This does not preclude the possibility that the limit may not exist. See the discussion at the end of Section 4.

We conclude this section by remarking that our techniques yield some results for the case  $\alpha = 0$ . These can be found in Section 5.

**2. Proofs in the random walk case.** Condition (1.4) in the definition of  $\mathcal{L}^{(\alpha)}$  can be replaced by the apparently weaker condition

$$(2.1) \quad \lim_{x \rightarrow \infty} \frac{f(x + y)}{f(x)} \quad \text{exists finitely for all } y > 0.$$

This is because (2.1) implies that  $g(x) = f(\ln x)$  is regularly varying at infinity. Exploiting this connection with regularly varying functions further, a very useful bound for the ratio in (2.1) can be obtained from Potter's theorem.



LEMMA 2.1. *Suppose that  $f \in \mathcal{L}^{(\alpha)}$  with  $\alpha > 0$ , and  $f$  is bounded away from 0 and  $\infty$  on compact subsets of  $[1, \infty)$ . Then for every  $\varepsilon > 0$  there exists an  $A = A_\varepsilon$  such that*

$$(2.2) \quad \frac{f(x+y)}{f(x)} \leq A(e^{-(\alpha-\varepsilon)y} \vee e^{-(\alpha+\varepsilon)y}) \quad \text{for all } x \geq 1, y \geq 1-x.$$

PROOF. This follows immediately by applying Potter’s theorem (Theorem 1.5.6(ii) of [4]) to the slowly varying function  $h(x) = (x \vee e)^\alpha f(\ln(x \vee e))$ . □

PROOF OF THEOREM 1.1. In the nonlattice case, on the assumption that  $P(\tau_x < \infty) \in \mathcal{L}^{(\alpha)}$ , we can divide (1.2) by  $P(\tau_x < \infty)$  to see that

$$\begin{aligned} \frac{P(h_1 > x)}{P(\tau_x < \infty)} &= 1 - \int_0^\infty P(h_1 \leq x, \widehat{H}_1 \in dy) \frac{P(\tau_{x+y} < \infty)}{P(\tau_x < \infty)} \\ &\rightarrow 1 - \int_0^\infty P(\widehat{H}_1 \in dy) e^{-\alpha y} = 1 - E(e^{-\alpha \widehat{H}_1}). \end{aligned}$$

To argue the other way, recalling that  $\widehat{T}_r$  is the  $r$ th strict descending ladder time and writing  $S_{\widehat{T}_r} = \widehat{H}_r$ , we note the decomposition

$$P(\tau_x < \infty) = \sum_{k=1}^\infty \int_0^\infty P\left(\sup_{m \leq \widehat{T}_{k-1}} S_m \leq x, \widehat{H}_{k-1} \in dy\right) P(h_k > x+y).$$

Assuming that  $P(h_1 > x) \in \mathcal{L}^{(\alpha)}$ , dividing through by  $P(h_1 > x)$  and using (2.2) to bound the ratio  $P(h_1 > x+y)/P(h_1 > x)$ , dominated convergence gives

$$\begin{aligned} \frac{P(\tau_x < \infty)}{P(h_1 > x)} &= \sum_{k=1}^\infty \int_0^\infty P\left(\sup_{m \leq \widehat{T}_{k-1}} S_m \leq x, \widehat{H}_{k-1} \in dy\right) \frac{P(h_1 > x+y)}{P(h_1 > x)} \\ &\rightarrow \sum_{k=1}^\infty \int_0^\infty P(\widehat{H}_{k-1} \in dy) e^{-\alpha y} \\ &= \sum_{k=1}^\infty E(e^{-\alpha \widehat{H}_1})^{k-1} = (1 - E(e^{-\alpha \widehat{H}_1}))^{-1}, \end{aligned}$$

as required. In the lattice case, the same argument works by restricting  $x$  to integer values. □

PROOF OF THEOREM 1.2. We prove the equivalent version given in Remark 1.1, so assume we are in the nonlattice case,  $P(H_1 > x) \in \mathcal{L}^{(\alpha)}$ ,  $E(e^{\alpha H_1}) < 1$  and

$$(2.3) \quad \lim_{x \rightarrow \infty} \frac{P(\tau_x^H < \infty)}{P(H_1 > x)} = L' \in (0, \infty).$$

Then for  $x > 0$ ,

$$\frac{P(\tau_x^H < \infty)}{P(H_1 > x)} = 1 + \int_0^x P(H_1 \in dy) \frac{P(\tau_{x-y}^H < \infty)}{P(H_1 > x)},$$

and so

$$\lim_{x \rightarrow \infty} \int_0^x P(H_1 \in dy) \frac{P(\tau_{x-y}^H < \infty)}{P(H_1 > x)} \text{ exists.}$$

By bounded convergence followed by monotone convergence,

$$\begin{aligned} \lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \int_0^K P(H_1 \in dy) \frac{P(\tau_{x-y}^H < \infty)}{P(H_1 > x)} &= \lim_{K \rightarrow \infty} \int_0^K P(H_1 \in dy) L' e^{\alpha y} \\ &= L' E(e^{\alpha H_1}). \end{aligned}$$

Next, let  $\bar{H}_\infty = \sup_n H_n$ . Then  $P(\tau_x^H < \infty) = P(\bar{H}_\infty > x)$ , and so by (2.3) and  $E(e^{\alpha H_1}) < \infty$ , we have  $E(e^{\alpha \bar{H}_\infty}) < \infty$ . Thus

$$\begin{aligned} \lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \int_{x-K}^x P(H_1 \in dy) \frac{P(\tau_{x-y}^H < \infty)}{P(H_1 > x)} \\ &= \lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \int_0^K \frac{P(H_1 \in x - dy)}{P(H_1 > x)} P(\tau_y^H < \infty) \\ &= \lim_{K \rightarrow \infty} \int_0^K \alpha e^{\alpha y} P(\bar{H}_\infty > y) dy \\ &= E(e^{\alpha \bar{H}_\infty}) < \infty. \end{aligned}$$

Hence

$$\lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \int_K^{x-K} P(H_1 \in dy) \frac{P(\tau_{x-y}^H < \infty)}{P(H_1 > x)} \text{ exists,}$$

and so by (2.3)

$$(2.4) \quad \lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \int_K^{x-K} P(H_1 \in dy) \frac{P(H_1 > x - y)}{P(H_1 > x)} \text{ exists.}$$

Now let  $Z_1$  and  $Z_2$  be independent copies of  $H_1$ . Display (1) of [8] gives

$$\begin{aligned} &\int_K^{x-K} P(Z_1 \in dy) \frac{P(Z_1 > x - y)}{P(Z_1 > x)} \\ &= \frac{P(Z_1 + Z_2 > x)}{P(Z_1 > x)} - 2 \int_0^K \frac{P(Z_1 > x - y)}{P(Z_1 > x)} P(Z_1 \in dy) \\ &\quad - \frac{P(Z_1 > x - K) P(Z_1 > K)}{P(Z_1 > x)} \\ &\sim \frac{P(Z_1 + Z_2 > x)}{P(Z_1 > x)} - 2 \int_0^K e^{\alpha y} P(Z_1 \in dy) - e^{\alpha K} P(Z_1 > K) \end{aligned}$$

as  $x \rightarrow \infty$ . Thus letting  $K \rightarrow \infty$ , we obtain

$$\lim_{x \rightarrow \infty} \frac{P(Z_1 + Z_2 > x)}{P(Z_1 > x)} = \lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \int_K^{x-K} P(Z_1 \in dy) \frac{P(Z_1 > x - y)}{P(Z_1 > x)} + 2Ee^{\alpha Z_1}$$

which exists by (2.4). This means that  $H_1 \in \mathcal{S}^{(\alpha)}$ . The converse holds by Lemma 1 and Theorem 1 of [3], where the value of  $L'$  is also calculated.

The proof in the lattice case consists of restricting  $x$  to integer values and replacing integrals by sums. This shows condition (1.7) holds when  $x$  is restricted to the integers. But this trivially implies (1.7) without this restriction, in the lattice case.  $\square$

REMARK 2.1. A compound Poisson process with zero drift is a time-change of a random walk, so we have essentially proved the Lévy process results in this case. The only thing that needs checking is the value of the constants. So in the remainder of the paper we need only deal with Lévy processes which are either compound Poisson with nonzero drift or have  $\Pi_X(\mathbb{R}) = \infty$ . Note that such processes cannot take values on a lattice.

**3. Preliminaries in the Lévy case.** We briefly collect the pertinent properties of a Lévy process to be used in this paper. Further details can be found, for example, in [1, 6, 12] and [13]. Let  $(L_s^{-1}, H_s)_{s \geq 0}$  denote the *weakly* ascending bivariate ladder process of  $X$ . When  $X_t \rightarrow -\infty$  a.s.,  $(L^{-1}, H)$  is defective and may be obtained from a nondefective process by exponential killing at some appropriate rate  $q > 0$ . When the process is killed, it is sent to some cemetery state, in which case probabilities and expectation are understood to be taken over only noncemetery values. The renewal function of  $H$  is

$$V(x) = \int_0^\infty P(H_s \leq x) ds.$$

Note that  $V(\infty) := \lim_{x \rightarrow \infty} V(x) = q^{-1}$ . The Laplace exponent  $\kappa$  of  $H$ , defined by  $e^{-\kappa(\lambda)} = Ee^{-\lambda H_1}$  for values of  $\lambda \in \mathbb{R}$  for which the expectation is finite, satisfies

$$\kappa(\lambda) = q + d\lambda + \int_0^\infty (1 - e^{-\lambda x}) \Pi_H(dx).$$

Observe that

$$\int_{y \geq 0} e^{-\lambda y} V(dy) = \frac{1}{\kappa(\lambda)}$$

for all  $\lambda \in \mathbb{R}$  with  $\kappa(\lambda) > 0$ .

Let  $\widehat{X}_t = -X_t, t \geq 0$ , denote the dual process, and  $(\widehat{L}^{-1}, \widehat{H})$  the corresponding *strictly* ascending bivariate ladder processes of  $\widehat{X}$ . All quantities relating to  $\widehat{X}$  will be denoted in the obvious way, for example,  $\widehat{\kappa}, \widehat{d}, \Pi_{\widehat{H}}$  and  $\widehat{V}$ . We may assume

the normalisations of  $L$  and  $\widehat{L}$  are chosen so that the constant in the Wiener–Hopf factorisation is 1; see (4) in Section VI.2 of [1].  $\widehat{L}$  is a local time at 0 for the reflected process  $R$ , and the excursion  $e_t$  of  $R$  at local time  $t$  is given by

$$e_t(s) = X_{(\widehat{L}_{t-}^{-1}+s)\wedge\widehat{L}_t^{-1}} - \underline{X}_{\widehat{L}_t^{-1}}.$$

If  $e_t \neq 0$ , that is  $\Delta\widehat{L}_t^{-1} > 0$ , then  $e_t$  takes values in the space of excursions

$$\mathcal{E} = \{\varepsilon \in D : \varepsilon(s) \geq 0 \text{ for all } 0 \leq s < \zeta, \zeta > 0\},$$

where  $D$  is the Skorohod space of cadlag functions and  $\zeta = \zeta(\varepsilon) = \inf\{s : \varepsilon(u) = \varepsilon(v) \text{ all } u, v \geq s\}$  is the lifetime of the excursion. Furthermore,  $\{(t, e_t) : e_t \in \mathcal{E}\}$  is a Poisson point process with intensity (excursion) measure  $\widehat{n}$ .

For  $\varepsilon \in \mathcal{E}$ , let  $h = h(\varepsilon) = \sup_{s \geq 0} \varepsilon(s)$  be the height of the excursion  $\varepsilon$ . Note that  $\widehat{n}(h = 0) > 0$  if and only if  $X$  is compound Poisson. Set  $|\widehat{n}| = \widehat{n}(\mathcal{E}) = \widehat{n}(h \geq 0)$ . It might seem that the major technical problems will be in the case  $|\widehat{n}| = \infty$ , when there is no first excursion. But when  $|\widehat{n}| < \infty$ , so that there is a first excursion, the fact that it may not start at 0 also creates problems. It is therefore important to know when  $\widehat{n}$  is finite, and since we could not find this in the literature we include the following result.

**PROPOSITION 3.1.** *The excursion measure  $\widehat{n}$  is finite if and only if one of the following two conditions hold:*

- 0 is irregular for  $[0, \infty)$  and  $\Pi_X(0, \infty) < \infty$ ;
- 0 is irregular for  $(-\infty, 0)$ .

**PROOF.** Excursion intervals are precisely the nonempty intervals of the form  $(\widehat{L}_{t-}^{-1}, \widehat{L}_t^{-1})$ . Let

$$T = \inf\{t : \Delta\widehat{L}_t^{-1} > 0\}.$$

Then  $|\widehat{n}| = \infty$  iff  $T = 0$  a.s. We consider the three possible cases:

Case I: 0 is regular for both  $[0, \infty)$  and  $(-\infty, 0)$ :

Then there are excursion intervals with end points arbitrarily close to 0, that is, there exist  $t_n \downarrow$  such that  $\Delta\widehat{L}_{t_n}^{-1} > 0$  and  $\widehat{L}_{t_n}^{-1} \rightarrow 0$ . If  $t_n \downarrow s > 0$  then by right continuity,  $\widehat{L}_s^{-1} = 0$ . This implies  $\widehat{L}^{-1}$  is compound Poisson which is impossible when 0 is regular for  $(-\infty, 0)$ . Thus  $T = 0$  and so  $|\widehat{n}| = \infty$ .

In the two remaining cases, 0 is irregular for exactly one of  $[0, \infty)$  or  $(-\infty, 0)$ . In particular, this implies  $X$  has bounded variation and so  $X_t = Y_t - Z_t + ct$  where  $Y$  and  $Z$  are pure jump subordinators.

Case II: 0 is irregular for  $[0, \infty)$ :

In this case,  $c \leq 0$  and  $\widehat{L}^{-1}$  is not compound Poisson. Let

$$S = \inf\{s : \Delta X_s > 0\}.$$

Then  $S = \widehat{L}_{T-}^{-1}$  where  $\widehat{L}_{0-}^{-1} = 0$ . By right continuity of  $\widehat{L}^{-1}$ ,  $|\widehat{n}| < \infty$  precisely when  $S > 0$  a.s. which in turn is equivalent to  $\Pi_X(0, \infty) < \infty$ .

Case III: 0 is irregular for  $(-\infty, 0)$ :

In this case,  $\widehat{L}^{-1}$  is compound Poisson by construction (see page 24 of [6]), and so  $T > 0$ . Thus  $|\widehat{n}| < \infty$ .  $\square$

**4. Proofs in the Lévy case.** Applying Corollary 4.1 of [9] to the dual process  $\widehat{X}$ , the Lévy measure of  $\widehat{H}$  is related to  $\widehat{n}$  by the formula

$$(4.1) \quad \Pi_{\widehat{H}}(dx) = \widehat{n}(|\varepsilon(\zeta)| \in dx) + d_{\widehat{L}^{-1}} \Pi_X^-(dx), \quad x > 0,$$

where  $\Pi_X^-(x, \infty) = \Pi_X((-\infty, -x))$  for  $x > 0$  and  $d_{\widehat{L}^{-1}}$  is the drift of  $\widehat{L}^{-1}$ . The final term on the right-hand side allows for the possibility of  $X$  jumping down from a strict current minimum. It is only present when  $d_{\widehat{L}^{-1}} > 0$ , which in turn implies  $X$  has bounded variation. The Poisson point process of excursions can be extended to include these downward jumps from strict minima as follows. Let  $\mathbf{x}$  denote the path  $\mathbf{x}(t) = x$  for all  $t \geq 0$  and let

$$\widetilde{\mathcal{E}} = \mathcal{E} \cup \{\mathbf{x} : x < 0\}.$$

Define

$$\widetilde{e}_t = \begin{cases} e_t & \text{if } e_t \in \mathcal{E}, \\ \mathbf{x} & \text{if } \Delta \widehat{L}_t^{-1} = 0 \text{ and } \Delta X_{\widehat{L}_t^{-1}} = x < 0. \end{cases}$$

Then  $\{(t, \widetilde{e}_t) : \widetilde{e}_t \in \widetilde{\mathcal{E}}\}$  is a Poisson point process with characteristic measure  $\widetilde{n}$  given by

$$\widetilde{n}(A) = \widehat{n}(A \cap \mathcal{E}) + d_{\widehat{L}^{-1}} \Pi_X^-(\{x : \mathbf{x} \in A\}).$$

The properties of Poisson point processes used below can be found in Proposition 0.2 of [1]. For  $\delta \geq 0$ , let

$$A_\delta = \{\varepsilon \in \mathcal{E} : h(\varepsilon) > \delta\}$$

and  $A_\delta^c = \widetilde{\mathcal{E}} \setminus A_\delta$ . Set

$$T_\delta = \inf\{t : e_t \in A_\delta\},$$

and

$$(4.2) \quad h^{(\delta)} = h(e_{T_\delta}), \quad Z^{(\delta)} = \widehat{H}_{T_\delta-} \quad \text{and} \quad D^{(\delta)} = |e_{T_\delta}(\zeta)|.$$

The case  $\delta = 0$  will only be considered when  $|\widehat{n}| < \infty$ . Since the Poisson point processes  $\{(t, e_t) : e_t \in A_\delta\}$  and  $\{(t, \widetilde{e}_t) : \widetilde{e}_t \in A_\delta^c\}$  are independent, we can write  $\widehat{H}$  as the sum of two independent subordinators  $\widehat{H} = J^{(\delta)} + K^{(\delta)}$  where

$$K_t^{(\delta)} = \sum_{s \leq t} |e_s(\zeta)| I(e_s \in A_\delta)$$

is the sum of the jumps of  $\widehat{H}$  that correspond to the ends of excursions for which  $h > \delta$ , and  $J^{(\delta)} = \widehat{H} - K^{(\delta)}$ . Using (4.1), their Laplace exponents are given by

$$(4.3) \quad \begin{aligned} \kappa^{J^{(\delta)}}(\lambda) &= \widehat{d}\lambda + \int_0^\infty (1 - e^{-\lambda x}) \{ \widehat{n}(h \leq \delta, |\varepsilon(\zeta)| \in dx) + d_{\widehat{L}^{-1}} \Pi_{\overline{X}}(dx) \}, \\ \kappa^{K^{(\delta)}}(\lambda) &= \int_0^\infty (1 - e^{-\lambda x}) \widehat{n}(h > \delta, |\varepsilon(\zeta)| \in dx), \end{aligned}$$

respectively. Here, we are assuming  $\widehat{q} = 0$  which will be the case below. Clearly,  $\widehat{H}_t = J_t^{(\delta)}$  for  $t < T_\delta$  and  $J^{(\delta)}$  does not jump at time  $T_\delta$ , so  $Z^{(\delta)} = J_{T_\delta}^{(\delta)}$ . Further,  $J^{(\delta)}$  is independent of  $(T_\delta, e_{T_\delta})$  and  $e_{T_\delta}$  is independent of  $T_\delta$ , thus both  $h^{(\delta)}$  and  $D^{(\delta)}$  are independent of  $Z^{(\delta)}$ . Additionally,  $T_\delta$  has an exponential distribution with parameter  $\widehat{n}(h > \delta)$ , hence

$$(4.4) \quad E e^{-\lambda Z^{(\delta)}} = \int_0^\infty \widehat{n}(h > \delta) e^{-\widehat{n}(h > \delta)t} e^{-\kappa^{J^{(\delta)}}(\lambda)t} dt = \frac{\widehat{n}(h > \delta)}{\widehat{n}(h > \delta) + \kappa^{J^{(\delta)}}(\lambda)}.$$

Since, by (4.3) and dominated convergence,

$$(4.5) \quad \begin{aligned} \lim_{\delta \rightarrow 0} \kappa^{J^{(\delta)}}(\lambda) &= \widehat{d}\lambda \\ &+ \int_0^\infty (1 - e^{-\lambda x}) \{ \widehat{n}(h = 0, |\varepsilon(\zeta)| \in dx) + d_{\widehat{L}^{-1}} \Pi_{\overline{X}}(dx) \}, \end{aligned}$$

it follows from (4.4) that  $Z^{(\delta)} \xrightarrow{P} 0$  if either  $\widehat{n}(h > 0) = \infty$ , or  $\widehat{d} = 0$ ,  $\widehat{n}(h = 0) = 0$  and  $d_{\widehat{L}^{-1}} = 0$ . Recall the condition  $\widehat{n}(h = 0) = 0$  is equivalent to  $X$  not being compound Poisson.

**PROOF OF THEOREM 1.3.** Assume (1.12). Since we have dealt with the compound Poisson case, we need to consider two cases.

Case I:  $\widehat{n}(h > 0) = \infty$ , or  $\widehat{d} = d_{\widehat{L}^{-1}} = 0$  and  $X$  is not compound Poisson.

Recalling (4.2), for any  $x > \delta > 0$  we have

$$\begin{aligned} P(\tau_x < \infty) &= P(h^{(\delta)} > x + Z^{(\delta)}) \\ &+ \int_0^\infty P(h^{(\delta)} \leq x + Z^{(\delta)}, Z^{(\delta)} + D^{(\delta)} \in dy) P(\tau_{x+y} < \infty). \end{aligned}$$

Dividing by  $P(\tau_x < \infty)$  and taking limits gives

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{P(h^{(\delta)} > x + Z^{(\delta)})}{P(\tau_x < \infty)} &= 1 - \int_0^\infty e^{-\alpha y} P(Z^{(\delta)} + D^{(\delta)} \in dy) \\ &= E(1 - e^{-\alpha(Z^{(\delta)} + D^{(\delta)})}). \end{aligned}$$

Since  $h^{(\delta)}$  and  $Z^{(\delta)}$  are independent and  $h^{(\delta)}$  has distribution given by

$$P(h^{(\delta)} \in \cdot) = \frac{\widehat{n}(h \in \cdot, h > \delta)}{\widehat{n}(h > \delta)},$$

it then follows that

$$\lim_{x \rightarrow \infty} \frac{E[\widehat{n}(h > x + Z^{(\delta)})]}{\widehat{n}(h > \delta)P(\tau_x < \infty)} = E(1 - e^{-\alpha(Z^{(\delta)} + D^{(\delta)})}).$$

Now for any  $c > 0$

$$\widehat{n}(h > x + c)P(Z^{(\delta)} \leq c) \leq E[\widehat{n}(h > x + Z^{(\delta)})] \leq \widehat{n}(h > x),$$

hence

$$\begin{aligned} \widehat{n}(h > \delta)E(1 - e^{-\alpha(Z^{(\delta)} + D^{(\delta)})}) &\leq \liminf_{x \rightarrow \infty} \frac{\widehat{n}(h > x)}{P(\tau_x < \infty)} \\ &\leq \limsup_{x \rightarrow \infty} \frac{\widehat{n}(h > x)}{P(\tau_x < \infty)} \\ &\leq \frac{e^{\alpha c} \widehat{n}(h > \delta)}{P(Z^{(\delta)} \leq c)} E(1 - e^{-\alpha(Z^{(\delta)} + D^{(\delta)})}), \end{aligned}$$

where the last inequality again uses (1.12). Let  $\delta \rightarrow 0$  then  $c \rightarrow 0$  to obtain

$$\begin{aligned} \limsup_{\delta \rightarrow 0} \widehat{n}(h > \delta)E(1 - e^{-\alpha(Z^{(\delta)} + D^{(\delta)})}) \\ &\leq \liminf_{x \rightarrow \infty} \frac{\widehat{n}(h > x)}{P(\tau_x < \infty)} \\ &\leq \limsup_{x \rightarrow \infty} \frac{\widehat{n}(h > x)}{P(\tau_x < \infty)} \\ &\leq \liminf_{\delta \rightarrow 0} \widehat{n}(h > \delta)E(1 - e^{-\alpha(Z^{(\delta)} + D^{(\delta)})}). \end{aligned}$$

Thus both limits exist, though possibly infinite, and

$$(4.6) \quad \lim_{x \rightarrow \infty} \frac{\widehat{n}(h > x)}{P(\tau_x < \infty)} = \lim_{\delta \rightarrow 0} \widehat{n}(h > \delta)E(1 - e^{-\alpha(Z^{(\delta)} + D^{(\delta)})}).$$

To evaluate the limit observe that since  $Z^{(\delta)}$  and  $D^{(\delta)}$  are independent

$$\begin{aligned} E(1 - e^{-\alpha(Z^{(\delta)} + D^{(\delta)})}) &= E(1 - e^{-\alpha Z^{(\delta)}}) + E(1 - e^{-\alpha D^{(\delta)}}) \\ &\quad - E(1 - e^{-\alpha Z^{(\delta)}})E(1 - e^{-\alpha D^{(\delta)}}). \end{aligned}$$

By (4.4) and (4.5),

$$\begin{aligned} \lim_{\delta \rightarrow 0} \widehat{n}(h > \delta)E(1 - e^{-\alpha Z^{(\delta)}}) &= \lim_{\delta \rightarrow 0} \frac{\widehat{n}(h > \delta)\kappa^{J^{(\delta)}}(\alpha)}{\widehat{n}(h > \delta) + \kappa^{J^{(\delta)}}(\alpha)} \\ &= \widehat{d}\alpha + \int_0^\infty (1 - e^{-\alpha z})d_{\widehat{L}^{-1}\Pi_{\widehat{X}}}(dz). \end{aligned}$$

Next, since

$$P(D^{(\delta)} \in dz) = \frac{\widehat{n}(h > \delta, |\varepsilon(\zeta)| \in dz)}{\widehat{n}(h > \delta)},$$

we have by monotone convergence

$$\begin{aligned} \widehat{n}(h > \delta)E(1 - e^{-\alpha D^{(\delta)}}) &= \widehat{n}(h > \delta) \left( 1 - \int_0^\infty e^{-\alpha z} \frac{\widehat{n}(h > \delta, |\varepsilon(\zeta)| \in dz)}{\widehat{n}(h > \delta)} \right) \\ (4.7) \qquad \qquad \qquad &= \int_0^\infty (1 - e^{-\alpha z}) \widehat{n}(h > \delta, |\varepsilon(\zeta)| \in dz) \\ &\rightarrow \int_0^\infty (1 - e^{-\alpha z}) \widehat{n}(|\varepsilon(\zeta)| \in dz). \end{aligned}$$

Finally, by (4.7) and  $Z^{(\delta)} \xrightarrow{P} 0$ ,

$$\widehat{n}(h > \delta)E(1 - e^{-\alpha Z^{(\delta)}})E(1 - e^{-\alpha D^{(\delta)}}) \rightarrow 0.$$

Since (1.12) implies  $X_t \rightarrow -\infty$  a.s., this means  $\widehat{q} = 0$  and so by (4.1), the limit in (4.6) is  $\widehat{\kappa}(\alpha)$ . This proves (1.14) which in turn implies (1.13).

Case II:  $\widehat{n}(h > 0) < \infty$  and  $\widehat{d} > 0$  or  $d_{\widehat{L}^{-1}} > 0$ .

If 0 is irregular for  $(-\infty, 0)$ , then  $(\widehat{L}^{-1}, \widehat{H})$  is bivariate compound Poisson, so  $\widehat{d} = d_{\widehat{L}^{-1}} = 0$ . Thus by Proposition 3.1, it is necessarily the case that 0 is irregular for  $[0, \infty)$  and  $\Pi_X(0, \infty) < \infty$ . In particular,  $X$  is of bounded variation but not compound Poisson. Hence  $X_t = Y_t - U_t$  where  $Y$  is a spectrally positive compound Poisson process and  $U$  is an independent subordinator which is not compound Poisson. If  $d_U$  is the drift of  $U$ , then its Laplace exponent is

$$\kappa^U(\lambda) = d_U \lambda + \int_0^\infty (1 - e^{-\lambda x}) \Pi_X^-(dx).$$

Since 0 is irregular for  $[0, \infty)$ , it suffices to prove the result when  $\widehat{L}$  is given by

$$\widehat{L}_t = \int_0^t I(X_s = \underline{X}_s) ds.$$

In this case, we have  $\widehat{L}_t^{-1} = t$  until the time of the first jump of  $Y$ , at which time  $\widehat{L}^{-1}$  also jumps. Thus  $d_{\widehat{L}^{-1}} = 1$ ,  $\widehat{d} = d_U$  and

$$T = \inf\{t : \Delta Y_t > 0\} = \inf\{t : e_t \in \mathcal{E}\}$$

has an exponential distribution with parameter  $\Pi_X(0, \infty) = |\widehat{n}|$ . Setting  $\delta = 0$  in the discussion preceding the proof of Theorem 1.3, we can write  $\widehat{H} = J + K$  where, since  $d_{\widehat{L}^{-1}} = 1$ ,

$$\begin{aligned} \kappa^J(\lambda) &= \widehat{d}\lambda + \int_0^\infty (1 - e^{-\lambda x}) \Pi_X^-(dx), \\ \kappa^K(\lambda) &= \int_0^\infty (1 - e^{-\lambda x}) \widehat{n}(|\varepsilon(\zeta)| \in dx). \end{aligned}$$



Thus  $J$  has the same distribution as  $U$ , but  $J \neq U$ . However,  $J_s = U_s$  for  $s \leq T$ . Let  $h_1 = h(e_T)$  be the height and  $D_1 = |e_T(\zeta)|$  the overshoot of the first excursion. Then, since  $P(D_1 \in dz) = |\widehat{n}|^{-1} \widehat{n}(|\varepsilon(\zeta)| \in dz)$ ,

$$(4.8) \quad E(1 - e^{-\lambda D_1}) = \frac{1}{|\widehat{n}|} \int_0^\infty (1 - e^{-\lambda z}) \widehat{n}(|\varepsilon(\zeta)| \in dz) = \frac{\kappa^K(\lambda)}{|\widehat{n}|}.$$

Also, as noted previously,  $J$  is independent of  $(T, e_T)$  (this would not be true if  $J$  were replaced by  $U$ ), and  $e_T$  is independent of  $T$ . In what follows, it will sometimes be convenient to write  $P(\tau_x < \infty)$  as  $P(\overline{X}_\infty > x)$  where  $\overline{X}_t = \sup_{s \leq t} X_s$ . We also write  $S$  for the right-hand endpoint  $\widehat{L}_T^{-1}$  of the first excursion interval. Then for any  $t > 0$ , since  $J_s = U_s$  for  $s \leq T$ ,

$$\begin{aligned} P(\tau_x < \infty) &= P(T \leq t, h_1 > x + J_T) \\ &\quad + P\left(T \leq t, h_1 \leq x + J_T, \sup_{r \geq 0} (X_{S+r} - X_S) > x + J_T + D_1\right) \\ &\quad + P\left(T > t, \sup_{r \geq 0} (X_{t+r} - X_t) > x + J_t\right) \\ &= \int_0^t P(T \in ds) P(h_1 > x + J_s) + \int_0^t P(T \in ds) E f_x(h_1, D_1, J_s) \\ &\quad + P(T > t) E g_x(J_t), \end{aligned}$$

where

$$\begin{aligned} f_x(y, z, w) &= I(y \leq x + w) P(\overline{X}_\infty > x + w + z) \quad \text{and} \\ g_x(w) &= P(\overline{X}_\infty > x + w). \end{aligned}$$

Thus dividing by  $P(\tau_x < \infty)$  and letting  $x \rightarrow \infty$ , we obtain

$$\begin{aligned} \lim_{x \rightarrow \infty} \int_0^t P(T \in ds) \frac{P(h_1 > x + J_s)}{P(\tau_x < \infty)} &= 1 - P(T > t) E e^{-\alpha J_t} \\ &\quad - \int_0^t P(T \in ds) E e^{-\alpha(J_s + D_1)} \\ &= 1 - e^{-(|\widehat{n}| + \kappa^J(\alpha))t} \\ &\quad - |\widehat{n}| E e^{-\alpha D_1} \int_0^t e^{-(|\widehat{n}| + \kappa^J(\alpha))s} ds. \end{aligned}$$

Now divide by  $t$ , let  $t \rightarrow 0$ , and use (4.8) to get

$$\begin{aligned} \lim_{t \rightarrow 0} \lim_{x \rightarrow \infty} \int_0^t P(T \in ds) \frac{P(h_1 > x + J_s)}{t P(\tau_x < \infty)} \\ (4.9) \quad &= |\widehat{n}| + \kappa^J(\alpha) - |\widehat{n}| \left(1 - \frac{\kappa^K(\alpha)}{|\widehat{n}|}\right) \\ &= \widehat{\kappa}(\alpha). \end{aligned}$$

Since

$$P(h_1 > x + J_t) \leq P(h_1 > x + J_s) \leq P(h_1 > x)$$

for  $0 \leq s \leq t$  and  $J_t \xrightarrow{P} 0$ , it then easily follows from (4.9) that

$$\lim_{x \rightarrow \infty} \frac{|\hat{n}|P(h_1 > x)}{P(\tau_x < \infty)} = \hat{\kappa}(\alpha).$$

This is equivalent to (1.14), which then implies (1.13).

In the converse direction, assume (1.13). By the compensation formula,

$$\begin{aligned} P(\tau_x < \infty) &= E \sum_t I(\bar{X}_{\hat{L}_{t-}^-} \leq x, h(e_t) > x + |\underline{X}_{\hat{L}_{t-}^-}|) \\ (4.10) \quad &= E \int_0^\infty dt I(\bar{X}_{\hat{L}_{t-}^-} \leq x) \hat{n}(h > x + |\underline{X}_{\hat{L}_{t-}^-}|) \\ &= \int_0^\infty dt \int_{y \geq 0} P(\bar{X}_{\hat{L}_{t-}^-} \leq x, |\underline{X}_{\hat{L}_{t-}^-}| \in dy) \hat{n}(h > x + y). \end{aligned}$$

By (2.2), for any  $\varepsilon \in (0, \alpha)$  there exists a constant  $A$  such that

$$\frac{\hat{n}(h > x + y)}{\hat{n}(h > x)} \leq A e^{-(\alpha - \varepsilon)y} \quad \text{for all } x \geq 1, y \geq 0.$$

Thus for  $x \geq 1$

$$\begin{aligned} &\int_0^\infty dt \int_{y \geq 0} P(\bar{X}_{\hat{L}_{t-}^-} \leq x, |\underline{X}_{\hat{L}_{t-}^-}| \in dy) \frac{\hat{n}(h > x + y)}{\hat{n}(h > x)} \\ &\leq A \int_0^\infty dt \int_{y \geq 0} P(|\underline{X}_{\hat{L}_{t-}^-}| \in dy) e^{-(\alpha - \varepsilon)y} \\ &\leq A \int_0^\infty dt E e^{-(\alpha - \varepsilon)\hat{H}_t} \\ &= \frac{A}{\hat{\kappa}(\alpha - \varepsilon)} < \infty. \end{aligned}$$

Hence, dividing (4.10) by  $\hat{n}(h > x)$  and applying dominated convergence we obtain

$$\lim_{x \rightarrow \infty} \frac{P(\tau_x < \infty)}{\hat{n}(h > x)} = \int_0^\infty dt \int_{y \geq 0} P(|\underline{X}_{\hat{L}_{t-}^-}| \in dy) e^{-\alpha y} = \frac{1}{\hat{\kappa}(\alpha)}.$$

Thus (1.14) holds, and hence also (1.12).  $\square$

REMARK 4.1. If  $\Gamma^* = 0$  in (1.10), then a simpler version of the above proof where dividing by  $P(\tau_x < \infty)$  is replaced by dividing by  $e^{-\gamma x}$  shows that the limit in (1.11) is also 0.

PROOF OF THEOREM 1.4. Assume  $\Pi_H \in \mathcal{L}^{(\alpha)}$ . By Vigon’s *équation amicale* (see (5.3.3) of [6]), for any  $t > 0$ ,

$$\overline{\Pi}_X(t) = \int_0^\infty \Pi_H(t + dy) \overline{\Pi}_{\widehat{H}}(y) + \widehat{d}\Pi'_H(t) + \widehat{q}\overline{\Pi}_H(t),$$

where  $\Pi'_H$  denotes the cadlag version of the density of  $\Pi_H$ , which exists when  $\widehat{d} > 0$ . By Fubini’s theorem,

$$\overline{\Pi}_X(t) = \int_0^\infty (\overline{\Pi}_H(t) - \overline{\Pi}_H(t + y))\Pi_{\widehat{H}}(dy) + \widehat{d}\Pi'_H(t) + \widehat{q}\overline{\Pi}_H(t),$$

thus

$$(4.11) \quad \frac{1}{\overline{\Pi}_H(x)} \int_x^\infty \overline{\Pi}_X(t) dt = \int_0^\infty \Pi_{\widehat{H}}(dy) \int_0^y \frac{\overline{\Pi}_H(x + t)}{\overline{\Pi}_H(x)} dt + \widehat{d} + \frac{\widehat{q}}{\overline{\Pi}_H(x)} \int_x^\infty \overline{\Pi}_H(t) dt.$$

Fix  $\varepsilon \in (0, \alpha)$ . By (2.2), for some  $A$  and all  $x \geq 1, y \geq 0$ ,

$$\int_0^y \frac{\overline{\Pi}_H(x + t)}{\overline{\Pi}_H(x)} dt \leq A \int_0^y e^{-(\alpha-\varepsilon)t} dt = \frac{A(1 - e^{-(\alpha-\varepsilon)y})}{\alpha - \varepsilon}.$$

This final expression is integrable over  $(0, \infty)$  with respect to  $\Pi_{\widehat{H}}(dy)$ , hence we may apply dominated convergence to conclude

$$(4.12) \quad \int_0^\infty \Pi_{\widehat{H}}(dy) \int_0^y \frac{\overline{\Pi}_H(x + t)}{\overline{\Pi}_H(x)} dt \rightarrow \int_0^\infty \Pi_{\widehat{H}}(dy) \frac{(1 - e^{-\alpha y})}{\alpha}.$$

Similarly, another appeal to (2.2) together with dominated convergence gives

$$(4.13) \quad \frac{\widehat{q}}{\overline{\Pi}_H(x)} \int_x^\infty \overline{\Pi}_H(t) dt = \widehat{q} \int_0^\infty \frac{\overline{\Pi}_H(x + t)}{\overline{\Pi}_H(x)} dt \rightarrow \frac{\widehat{q}}{\alpha}.$$

Thus by (4.11), (4.12) and (4.13)

$$\frac{1}{\overline{\Pi}_H(x)} \int_x^\infty \overline{\Pi}_X(t) dt \rightarrow \frac{\widehat{\kappa}(\alpha)}{\alpha}.$$

Now fix  $a > 0$ . Then

$$\begin{aligned} \frac{a\overline{\Pi}_X(x)}{\overline{\Pi}_H(x)} &\leq \frac{1}{\overline{\Pi}_H(x)} \int_{x-a}^x \overline{\Pi}_X(t) dt \\ &= \frac{\overline{\Pi}_H(x - a)}{\overline{\Pi}_H(x)} \frac{1}{\overline{\Pi}_H(x - a)} \int_{x-a}^\infty \overline{\Pi}_X(t) dt - \frac{1}{\overline{\Pi}_H(x)} \int_x^\infty \overline{\Pi}_X(t) dt \\ &\rightarrow \frac{\widehat{\kappa}(\alpha)}{\alpha} (e^{\alpha a} - 1). \end{aligned}$$

Divide by  $a$  and let  $a \rightarrow 0$  to obtain

$$\limsup_{x \rightarrow \infty} \frac{\overline{\Pi}_X(x)}{\overline{\Pi}_H(x)} \leq \widehat{\kappa}(\alpha).$$

Integrating over  $[x, x + a]$  gives the corresponding lower bound. Hence (1.19) holds and consequently  $\overline{\Pi}_X \in \mathcal{L}^{(\alpha)}$ .

The opposite direction is straightforward. Assume  $\overline{\Pi}_X \in \mathcal{L}^{(\alpha)}$ . By Vigon's equation amicale inversée (see (5.3.4) of [6]), for  $x > 0$ ,

$$(4.14) \quad \frac{\overline{\Pi}_H(x)}{\overline{\Pi}_X(x)} = \int_0^\infty \widehat{V}(dy) \frac{\overline{\Pi}_X(x+y)}{\overline{\Pi}_X(x)}.$$

To take the limit inside the integral, we again we use (2.2) and observe

$$\int_0^\infty \widehat{V}(dy) A e^{-(\alpha-\varepsilon)y} = \int_0^\infty \widehat{V}(y) A(\alpha - \varepsilon) e^{-(\alpha-\varepsilon)y} dy < \infty$$

since  $\widehat{V}(y) \leq Cy$  for  $y \geq 1$  by Proposition III.1 of [1]. Thus by dominated convergence,

$$\frac{\overline{\Pi}_H(x)}{\overline{\Pi}_X(x)} \rightarrow \int_0^\infty \widehat{V}(dy) e^{-\alpha y} = \frac{1}{\widehat{\kappa}(\alpha)}.$$

Hence (1.19) holds, and consequently,  $\overline{\Pi}_H \in \mathcal{L}^{(\alpha)}$ .  $\square$

**PROOF OF THEOREM 1.6.** Assume  $\overline{\Pi}_H \in \mathcal{L}^{(\alpha)}$ ,  $E(e^{\alpha H_1}) < 1$  and (1.21) hold. Let  $Z = H_{\tau_1}$  if  $\tau_1 < \infty$  and set  $Z$  equal to some cemetery state otherwise. Then by Proposition III.2 of [1],

$$(4.15) \quad \lim_{x \rightarrow \infty} \frac{P(Z > x)}{\overline{\Pi}_H(x)} = \lim_{x \rightarrow \infty} \int_0^1 V(dz) \frac{\overline{\Pi}_H(x-z)}{\overline{\Pi}_H(x)} = \int_0^1 e^{\alpha z} V(dz).$$

Hence  $P(Z > x) \in \mathcal{L}^{(\alpha)}$ . Further, since  $E e^{\alpha H_1} < \infty$  implies  $\int_1^\infty e^{\alpha y} \Pi_H(dy) < \infty$  by Theorem 25.3 of [13], which in turn is equivalent to

$$(4.16) \quad \int_1^\infty \overline{\Pi}_H(y) \alpha e^{\alpha y} dy < \infty,$$

we have

$$(4.17) \quad E e^{\alpha Z} = \int_0^\infty P(Z > y) \alpha e^{\alpha y} dy < \infty.$$

Now for  $x > 1$ ,

$$(4.18) \quad \frac{P(\tau_x^H < \infty)}{\overline{\Pi}_H(x)} = \frac{P(Z > x)}{\overline{\Pi}_H(x)} + \int_0^x P(Z \in dy) \frac{P(\tau_{x-y}^H < \infty)}{\overline{\Pi}_H(x)}.$$

By (1.21), bounded convergence, then monotone convergence

$$\begin{aligned}
 (4.19) \quad \lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \int_0^K P(Z \in dy) \frac{P(\tau_{x-y}^H < \infty)}{\overline{\Pi}_H(x)} &= \lim_{K \rightarrow \infty} \int_0^K P(Z \in dy) L' e^{\alpha y} \\
 &= \int_0^\infty P(Z \in dy) L' e^{\alpha y} < \infty
 \end{aligned}$$

by (4.17), while

$$\begin{aligned}
 (4.20) \quad &\lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \int_{x-K}^x P(Z \in dy) \frac{P(\tau_{x-y}^H < \infty)}{\overline{\Pi}_H(x)} \\
 &= \lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \int_0^K P(\tau_y^H < \infty) \frac{P(Z \in x - dy)}{P(Z > x)} \frac{P(Z > x)}{\overline{\Pi}_H(x)} \\
 &= \lim_{K \rightarrow \infty} \int_0^K P(\tau_y^H < \infty) \alpha e^{\alpha y} dy \int_0^1 e^{\alpha z} V(dz) \\
 &= \int_0^\infty P(\tau_y^H < \infty) \alpha e^{\alpha y} dy \int_0^1 e^{\alpha z} V(dz) < \infty
 \end{aligned}$$

by (1.21) and (4.16). Thus by (1.21), (4.18), (4.19) and (4.20),

$$\lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \int_K^{x-K} P(Z \in dy) \frac{P(\tau_{x-y}^H < \infty)}{\overline{\Pi}_H(x)} \quad \text{exists.}$$

By (1.21) and (4.15), it then follows that

$$\lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \int_K^{x-K} P(Z \in dy) \frac{P(Z > x - y)}{P(Z > x)} \quad \text{exists.}$$

We can now repeat the random walk argument following (2.4), with  $Z$  replacing  $H_1$ , to see that that  $Z \in \mathcal{S}^{(\alpha)}$ . Since  $\mathcal{S}^{(\alpha)}$  is closed under tail equivalence, this in turn implies  $\overline{\Pi}_H \in \mathcal{S}^{(\alpha)}$ .

The converse holds by Lemma 3.5 of [11], where the value of  $L'$  is also calculated.  $\square$

We conclude this section by discussing other possible values of the the limit in (1.20) under the assumption  $\overline{\Pi}_X \in \mathcal{L}^{(\alpha)}$ , which will remain in place for the remainder of this section. Fix  $x > 0, t > 0$  and write  $X = Y + Z$  where  $Y$  is the sum of the jumps of  $X$  which are larger than  $x$ . Then choose  $K > 0$  large enough that  $P(\inf_{0 \leq s \leq t} Z_s \geq -K) = c > 0$ . Then

$$P(X_t > x) \geq P\left(Y_t > x + K, \inf_{0 \leq s \leq t} Z_s \geq -K\right) \geq ct \overline{\Pi}_X(x + K).$$

Thus

$$(4.21) \quad \liminf_{x \rightarrow \infty} \frac{P(\tau_x < \infty)}{\overline{\Pi}_X(x)} \geq \liminf_{x \rightarrow \infty} \frac{P(X_t > x)}{\overline{\Pi}_X(x)} \geq cte^{-\alpha K},$$

and so the limit in (1.20) can not be 0.

If  $Ee^{\alpha X_1} < 1$ , the limit in (1.20) is finite and nonzero precisely when  $\overline{\Pi}_X \in \mathcal{S}^{(\alpha)}$  by Theorem 1.5. This does not preclude the possibility that the limit does not exist when  $\overline{\Pi}_X \in \mathcal{L}^{(\alpha)} \setminus \mathcal{S}^{(\alpha)}$  and  $Ee^{\alpha X_1} < 1$ . We should note that while it is possible to have a random variable  $Z \in \mathcal{L}^{(\alpha)} \setminus \mathcal{S}^{(\alpha)}$  with  $Ee^{\alpha Z} < \infty$ , by far the most frequently encountered case is when  $Ee^{\alpha Z} = \infty$ . For example, if

$$P(Z > x) \sim x^\beta e^{-\alpha x},$$

then  $Z \in \mathcal{S}^{(\alpha)}$  if  $\beta < -1$  and  $Z \in \mathcal{L}^{(\alpha)} \setminus \mathcal{S}^{(\alpha)}$  if  $\beta \geq -1$ . In the case that  $Ee^{\alpha X_1} = \infty$ , the limit in (1.20) is  $\infty$ . To see this, we first calculate the limit in (1.15). By Fatou,

$$\liminf_{x \rightarrow \infty} \frac{\overline{G * G}(x)}{2\overline{G}(x)} = \liminf_{x \rightarrow \infty} \int_0^x \frac{G(x-y)}{G(x)} G(dy) \geq \int_0^\infty e^{\alpha y} G(dy) = \infty,$$

since  $Ee^{\alpha X_1} = \infty$  implies  $\int_1^\infty e^{\alpha y} \Pi_X(dy) = \infty$ . Thus by Braverman [5]

$$\lim_{x \rightarrow \infty} \frac{P(X_t > x)}{P(X_r > x)} = 0 \quad \text{for all } 0 < t < r.$$

Hence by (4.21), if  $0 < t < r$ ,

$$\liminf_{x \rightarrow \infty} \frac{P(\tau_x < \infty)}{\overline{\Pi}_X(x)} \geq \liminf_{x \rightarrow \infty} \frac{P(X_r > x)}{\overline{\Pi}_X(x)} \geq cte^{-\alpha K} \liminf_{x \rightarrow \infty} \frac{P(X_r > x)}{P(X_t > x)} = \infty.$$

There remains the possibility that  $Ee^{\alpha X_1} \in [1, \infty)$ . If  $Ee^{\alpha X_1} = 1$  then from (1.10), if  $\Gamma^* > 0$ , the limit in (1.20) is  $\infty$  since  $\overline{\Pi}_X(x) = o(e^{-\alpha x})$ . If  $\Gamma^* = 0$ , we are unable to say anything. For the remaining case,  $Ee^{\alpha X_1} \in (1, \infty)$ , we may assume  $X_t \rightarrow -\infty$  else the limit in (1.20) is trivially  $\infty$ . Then there exists  $\gamma \in (0, \alpha)$  such  $Ee^{\gamma X_1} = 1$ . Thus from (1.10) the limit in (1.20) is again  $\infty$ , since  $\Gamma^* > 0$  in this case.

**5. The case  $\alpha = 0$ .** The definitions of  $\mathcal{L}^{(\alpha)}$  and  $\mathcal{S}^{(\alpha)}$  are valid for  $\alpha = 0$ , in which case they are called the long-tailed and subexponential classes respectively. Our methods give some partial results in this case, and here we describe them for the Lévy case.

REMARK 5.1. When  $\alpha = 0$ , (1.17) implies (1.19) and (1.18) implies (1.19), but (1.17) and (1.18) are not necessarily equivalent since it is possible that  $\widehat{k}(0) = 0$ . To see this, by (4.14) for any  $x > 0$  without any assumptions on  $\overline{\Pi}_X$  or  $\overline{\Pi}_H$ ,

$$\frac{\overline{\Pi}_H(x)}{\overline{\Pi}_X(x)} \leq \widehat{V}(\infty) = \frac{1}{\widehat{k}(0)}.$$

If  $\overline{\Pi}_X \in \mathcal{L}^{(0)}$ , then applying Fatou to (4.14) proves (1.19). If  $\overline{\Pi}_H \in \mathcal{L}^{(0)}$ , then for any  $K > 0$ ,

$$\frac{\overline{\Pi}_H(x)}{\overline{\Pi}_H(x+K)} \geq \int_0^K \widehat{V}(dy) \frac{\overline{\Pi}_X(x+y)}{\overline{\Pi}_H(x+K)} \geq \widehat{V}(K) \frac{\overline{\Pi}_X(x+K)}{\overline{\Pi}_H(x+K)}.$$

Letting  $x \rightarrow \infty$  and then  $K \rightarrow \infty$  proves (1.19).

REMARK 5.2. If  $X_t \rightarrow -\infty$  a.s., then  $P(\tau_x < \infty) = 1$  for all  $x \geq 0$ , so (1.12) trivially holds when  $\alpha = 0$ . Since this provides no useful information about the asymptotic behaviour of  $P(\tau_x < \infty)$ , we must also include the condition  $X_t \rightarrow -\infty$  when considering (1.12) in the  $\alpha = 0$  case. In that case, the proof for  $\alpha > 0$  is easily modified, and is in fact much simpler, to show that (1.14) holds with  $\alpha = 0$ , the limit being  $\widehat{\kappa}(0) = 0$  since  $X_t \rightarrow -\infty$ . However, this does not enable us to conclude anything about (1.13). Conversely, if (1.13) holds with  $\alpha = 0$ , then we can divide through (4.10) by  $\widehat{n}(h > x)$  and apply Fatou to obtain

$$\liminf_{x \rightarrow \infty} \frac{P(\tau_x < \infty)}{\widehat{n}(h > x)} \geq \int_0^\infty dt \int_{y \geq 0} P(|\underline{X}_{\widehat{L}_t^{-1}}| \in dy) = \widehat{V}(\infty) = \frac{1}{\widehat{\kappa}(0)}.$$

Again using (4.10), the corresponding upper bound holds trivially for every  $x > 0$  without taking the limit. Thus (1.14) holds with  $\alpha = 0$ , but we are unable to conclude anything about (1.12) unless  $\widehat{q} > 0$ . In this direction, there is no need to assume  $X_t \rightarrow -\infty$  a.s. If  $X_t \rightarrow -\infty$  a.s., then (1.14) simply reduces to  $\widehat{n}(h = \infty) = \widehat{q}$ .

REMARK 5.3. Theorem 1.6 continues to hold when  $\alpha = 0$  with the interpretation that  $Ee^{\alpha H_1} < 1$  means  $H$  is defective. The proof is an obvious modification of the proof in the  $\alpha > 0$  case. Theorem 1.5 as stated does not hold for  $\alpha = 0$ , where we interpret  $Ee^{\alpha X_1} < 1$  to mean  $X_t \rightarrow -\infty$ . This is because when  $\overline{\Pi}_X \in \mathcal{L}^{(0)}$  one can show

$$\liminf_{x \rightarrow \infty} \frac{\widehat{n}(h > x)}{\overline{\Pi}_X(x)} \geq \frac{1}{q}.$$

Thus if in addition (1.20) holds, then  $P(\tau_x < \infty) \in \mathcal{L}^{(0)}$  and  $X_t \rightarrow -\infty$ , hence by Remark 5.2,

$$\lim_{x \rightarrow \infty} \frac{\widehat{n}(h > x)}{P(\tau_x < \infty)} = 0.$$

This then implies

$$\lim_{x \rightarrow \infty} \frac{P(\tau_x < \infty)}{\overline{\Pi}_X(x)} = \infty,$$

which contradicts (1.20).

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