Value Allocation Under Ambiguity

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Abstract

We consider a pure exchange economy with asymmetric information where individual behavior exhibits ambiguity aversion along the line of maximin expected utility decision making. For such economies we introduce different notions of maximin value allocations. We also introduce a strong notion of (maximin) incentive compatibility. We prove existence and incentive compatibility of the maximin value allocation, when the economy's state space is either finite or non-finite. In the latter case, we provide two different existence results: assuming first countable and then uncountable infinitely many states of nature of the world. We conclude that unlike the Bayesian value allocation approach, incentive compatibility is related to efficiency rather than to direct exchange of information.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Introduction

Based on the maximin expected utility modeling of Gilboa and Schmeilder (1989), de Castro and Yannelis (2009) assume maximin preferences for the agents of a partition (of the state space) type asymmetric information exchange economy. As a natural consequence, allocations for this economy need not be private information measurable. Non-Bayesian agents enjoy higher (first best) Pareto efficiency in equilibrium, while informationally unconstrained efficiency leads to incentive compatibility. The Bayesian conflict between efficiency and incentive compatibility is resolved: every maximin efficient allocation is maximin incentive compatible (see in de Castro and Yannelis, 2009, Theorem 4.1).

Major general equilibrium concepts - in particular, the Walrasian equilibrium, the core and the value allocation - were brought forward for examination in the previous non-Bayesian context. While the core and the competitive equilibrium were thoroughly studied [see in de Castro et al. (2011, 2012) and He and Yannelis (2013)], little attention was drawn to the value allocation. Indeed, the notion of the maximin value allocation was only defined in de Castro and Yannelis (2009).

This thesis is devoted to the Shapley (1969) value allocation under ambiguity. The three chapters of the thesis correspond to three respective research articles. In the first chapter (co-authored by Leonidas Koutsougeras), ex ante and interim maximin value allocations are introduced, where: (i) agents may have any kind of information within their groups and (ii) agents' priors and preferences become coalitional dependent. At the same time, a stronger notion of maximin incentive compatibility is introduced, where transfers are allowed inside the coalition of the cheating agents. Existence and incentive compatibility of the maximin value allocation are established, but only with a finite set of states. This gives rise to the next two chapters of the thesis, given that real life economies do not always contain a finite number of states. By focusing on the private (information) maximin value allocation only, chapter 2 provides an existence result when the set of states is countable. The same is done in chapter 3, but when the state space is uncountable.

Chapter 1

Maximin Value Allocation

1.1 Preliminaries

In this chapter we continue and extend the current line of research of de Castro and Yannelis (2009) and de Castro et al. (2010, 2011). Primarily, these three papers introduce maximin expected utility multi prior modeling into a differential information economy. It is being established, thereafter, that this analytical tactic leads to allocations with more desirable general equilibrium properties, than the ones of the standard Bayesian case.

In de Castro and Yannelis (2009), it is suggested that in a partition type differential information economy it is sensible to assume that individuals do not present Bayesian preferences. The authors argue that it seems unreasonable to take for granted that the agents are always in a position to attach a (strictly positive) probability to the states they cannot distinguish. Therefore, individuals may not be in a position to update their prior in the standard Bayesian manner. Instead, they propose that agents choose their prior from a class of probability measures according to a maximin criterion. In this way, individuals evaluate random state contingent plans according to the maximin expected utility, instead of the standard Bayesian (or subjective) one[1]. This, in turn, brings about very desirable properties to the general equilibrium of the economy.

One important property of equilibria in the context of differential information is group incentive compatibility. This term refers to the truthful disclosure of information and concerns groups of individuals, i.e., coalitions, rather than merely single individuals. The underlying principle is that in an uncertainty context, contractual

[1]See also in de Castro and Yannelis (2013). For further developments on the maximin expected utility the reader is referred to Even and Lehrer (2014).

agreements for future deliveries of commodities are possible only to the extend that, given that the parties are differentially informed, contracts do not create an incentive for a group of agents to profit by untruthfully disclosing information. In other words, a crucial effect of the presence of differential information is that it restricts the possible contracts that can be exchanged among individuals at equilibrium, to those that are incentive compatible. Indeed, an equilibrium which is based on contractual arrangements which are not incentive compatible is not very sensible (and unstable). Therefore, in the differential information context it becomes essential to test equilibrium notions against the criterion that equilibrium outcomes are implemented via incentive compatible contracts, i.e., equilibrium allocations are incentive compatible. Efficiency is of course another desirable property of an equilibrium notion, which is often difficult to obtain in the presence of differential information, at least in the Bayesian framework.

Earlier studies of equilibrium concepts in differential information economies adopted a formulation of information where consumption sets were restricted to contingent consumption assignments compatible with the private information of individuals. This approach to differential information fitted well the Bayesian framework of decision making by individuals. Following that approach, the study of equilibrium notions such as the value allocation [Krasa and Yannelis (1994, 1996)] or the core [Yannelis (1991), Koutsougeras and Yannelis (1993)], had succeeded in establishing incentive compatibility properties of these notions but efficiency was from the outset an elusive issue. Roughly speaking, the restriction of the space of allocations to those compatible with the information of individuals (i.e., private information measurable ones) excluded many opportunities for trade and this resulted in outcomes which are only (information) constraint efficient.

In a more recent alternative approach to the formulation of differential information, de Castro and Yannelis (2009) propose to abandon the private information measurability restriction on consumption allocations. Such a restriction was essential in the Bayesian expected utility framework and was economically justified by the incentive compatibility properties that it inherited to equilibrium outcomes. However, the private information measurability restriction of consumption allocations is not (in general) necessary for incentive compatibility and from the viewpoint of efficiency it proves to be too strong. On the other hand, abandoning private information measurability restrictions altogether is incompatible with Bayesian expected utility. It becomes evident then that dropping measurability restrictions of consumption allocations must be accompanied by a change of the Bayesian expected utility behavioral hypothesis. This idea has led de Castro and Yannelis (2009) to adopt an alternative behavioral model, namely the maximin expected utility one[2]. In that formulation, the asymmetric information is built into the definition of preferences, so there is no need for measurability restrictions on allocations. The non Bayesian-maximin agents become able to complete their (incomplete Bayesian) preferences, since they are no longer restricted to consume private information measurable allocations only[3]. Because of this, they become capable of reaching Pareto superior outcomes. Furthermore, even without private information measurable consumption, incentive compatibility is maintained, as it is shown in de Castro and Yannelis (2009) and de Castro et al. (2010, 2011). That is to say, incentive compatibility is no longer necessarily associated with private information measurability of allocations, as in the Bayesian case. This actually happens because the Bayesian conflict between efficiency and incentive compatibility is naturally resolved by the agents' maximin ambiguity aversion. Indeed, it has been shown in de Castro and Yannelis, 2009 (Theorem 4.1) that every maximin efficient allocation is maximin incentive compatible. This has led de Castro et al. (2010) as far as concluding that the agents' maximin ambiguity averse preferences do not (at least in general equilibrium terms) reflect pessimism, but incentive compatibility.

Motivated by this growing literature, in this paper we extend the notion of the non-Bayesian value allocation introduced in de Castro and Yannelis (2009), by allowing individuals to exchange information within their groups. Specifically, we introduce information exchange protocols within coalitions, which determine the new information available to the members of coalitions. In this way, the maximin (expected or not) utility of an agent becomes a function of his/her coalition membership. Hence,

[2]Da-Silva and Beloso (2009, 2012) were the first to adopt the maximin expected utility in a context of general equilibrium with asymmetric information.

[3]For recent papers adopting a similar framework in equilibrium theory with asymmetric information see Condie and Ganguli (2011), de Castro and Chateauneuf (2011) and Jungbauer and Ritzberger (2011), among others.

we are able to generalize the concept of a value allocation in differential information economies for arbitrary communication protocols, which may include full or partial information sharing within coalitions. We also refine the notion of group incentive compatibility by allowing transfers between members of a coalition misreporting a state of nature. This possibility provides coalitions with more flexibility in misreporting a state of nature, so it enlarges the set of allocations which are not incentive compatible, thereby resulting in a stronger notion of incentive compatibility.

The aforementioned extension regarding individuals' information is motivated by a subtle observation stemming from the work of de Castro and Yannelis (2009): neither the technical properties of the characteristic function of the value allocation introduced in this paper nor the incentive compatibility result of this paper depend on whether the information partition that individuals use is the initial one they are endowed with, or some (any) other.[4] Especially regarding incentive compatibility, what is essential is the Pareto efficiency properties of an allocation with respect to the information available to individuals, regardless of whether the available information is private or not. On the other hand, Pareto efficiency is built in the definition of the value allocation. This is the key observation that motivates the main extension proposed in this paper.

Our analysis proceeds as follows: after setting up the appropriate model framework in section 2, we define and explain in section 3 the maximin value allocation. We provide definitions for two maximin value notions - the ex ante and the interim. Both notions attract (for different purposes) equal analytical attention and are considered equally significant for our purposes. Section 4 contains an existence result for the maximin value allocation, when the economy is extended to an infinite dimensional commodity vector space one. In particular, we model the commodity space via an ordered Banach space and prove existence of the maximin value allocation under the standard assumptions of a well behaved economy. In section 5, we provide a new definition of transfer maximin coalitional incentive compatibility and prove that every maximin value allocation satisfies it. In section 6, we summarize our conclusions.

[4]This is a notable difference with the Bayesian approach, where the information partition that individuals use may affect the technical properties of characteristic functions. For some information sharing rules, for instance, the endowment of a coalition may lie outside the consumption sets of its members, so the value of a coalition may not be well defined.

1.2 The Ambiguous Economy

The *ambiguous economy* is an economy with differential information and maximin preferences, called here the *Maximin Differential Information Economy* (MDIE). The model proposed in this section generalizes the one of de Castro and Yannelis (2009) in various directions, which are revealed as the description of the MDIE unravels.

In what follows we consider an economy characterized by uncertainty, over which the participants are asymmetrically informed. The uncertainty and its resolution allows us to distinguish three segments of the time line: the period elapsing before the occurrence of any event (*ex ante*), the period during which an event occurs and individuals receive signals about it (*interim*) and the period elapsing after the occurrence of an event and the resolution of the uncertainty (*ex post*). In this chapter we will be interested in consumption decision making in the ex ante and interim stages. It is implicitly assumed that the economy features institutions such as contracts for contingent deliveries or other assets, so that agents can arrange trades that will be conducted in the future

Uncertainty is summarized by a set Ω, which is the common to all the agents finite set of states, along with the algebra $\mathcal{F} = 2^{\Omega}$ of all the subsets of Ω. For the sake of the exposition we start here with a finite number $m \in \mathbb{N}$ of commodities, which give rise to a commodity space identified with $\Re^m$ [5], and a finite set of agents $I = \{1, 2, ..., n\}$. The agents in I are associated with certain characteristics:

(i) Information. The private information about the occurrence of an event available to an individual $i \in I$ is modeled as a partition $\mathcal{F}_i$ of Ω. This partition summarizes the events discernable by an individual. The interpretation as usual is that upon realization of a state of nature $\omega \in \Omega$ the individual i becomes aware of the event $\mathcal{F}_i(\omega)$, the element of his/her partition $\mathcal{F}_i$ which contains ω, i.e., there may be insufficient information to allow the agent to distinguish the 'true state' ω from others

[5] Throughout the whole dissertation, $\mathbb{R}^m$ and $\Re^m$ interchangeably represent the m-dimensional Euclidean space.

$\omega' \in \mathcal{F}_i(\omega)$. By abuse of notation, $\mathcal{F}_i \subseteq \mathcal{F}$ will be also denoting the algebra generated by the partition $\mathcal{F}_i$. Based on the ideas of Allen (2006), we allow individuals to share information within groups through some information exchange protocol. To this end, for a given coalition $S \subseteq I$ we assign the partition $\mathcal{K}_i^S$ of Ω to each $i \in S$. That is, $\mathcal{K}_i^S$ represents the new information available[6] to each individual as a member of the coalition $i \in S$, according to the underlying information exchange protocol within S. In principle, $\mathcal{K}_i^S$ can be arbitrary, but it serves our intuition to impose: (i) $\mathcal{K}_i^{\{i\}} = \mathcal{F}_i$, for every $i \in I$ and (ii) $\bigwedge_{i \in S} \mathcal{F}_i \subseteq \mathcal{K}_i^S \subseteq \bigvee_{i \in S} \mathcal{F}_i$, for every $i \in S \subseteq I$, i.e., the information available to each member of a coalition is not superior to the pooled information of all the coalition's members, but at the same time, cannot be worse than the common knowledge information between all the coalition's members.
The example that follows demonstrates how information exchange protocols can work inside different coalitions.

Example 1. Let $\Omega = \{a, b, c, d\}$, let $I = \{1, 2, 3\}$ and assume that the three agents begin with the following information (partitions of Ω):
$\mathcal{F}_1 = \{\{a, b, c\}, \{d\}\}$, $\mathcal{F}_2 = \{\{a\}, \{b, c, d\}\}$ and $\mathcal{F}_3 = \{\{a, b, c, d\}, \{\emptyset\}\}$.
Then, depending on the coalition and the corresponding information exchange protocol adopted, the agents' information can be reformed in various ways.
For example:
• $\mathcal{K}_1^{\{1,3\}} = \mathcal{F}_1$ and $\mathcal{K}_3^{\{1,3\}} = \{\{a, b, c\}, \{d\}\}$ (in the coalition $\{1, 3\}$, the protocol dictates agent 1 to inform agent 3).
• $\mathcal{K}_1^{\{1,2,3\}} = \mathcal{F}_1$, $\mathcal{K}_2^{\{1,2,3\}} = \mathcal{F}_2$, but $\mathcal{K}_3^{\{1,2,3\}} = \{\{a, b, c\}, \{d\}\}$ (in the coalition $\{1, 2, 3\}$, the protocol demands that agent 1 should inform agent 3 but agent 2 should not; no information exchange should occur between agent 1 and 2).

Some particular cases of information exchange protocols within coalitions stand out as they correspond to the prior literature:
(a) $\mathcal{K}_i^S = \mathcal{F}_i$, for every $i \in S \subseteq I$, i.e., no information exchange within coalitions.

[6]As before, by abuse of notation, we use the same notation for the algebra generated by this partition.

(b) $\mathcal{K}_i^S = \bigvee_{i \in S} \mathcal{F}_i$, for every $i \in S \subseteq I$, i.e., full information exchange within coalitions.
(c) $\mathcal{K}_i^S = \bigwedge_{i \in S} \mathcal{F}_i$, for every $i \in S \subseteq I$, i.e., only publicly known information within coalitions.

(ii) Prior Beliefs. Once it is acknowledged that agents' information varies according to the coalition they participate, inevitably this spills over to their decision under uncertainty process. Indeed, given a coalition $S \subseteq I$, individuals $i \in S$ are endowed with a probability assessment $q_i^S : \mathcal{K}_i^S \to [0,1]$, which is an additive probability measure. Note that whereas $q_i^S(A)$ is defined for $A \in \mathcal{K}_i^S$, it need not be defined for every proper subset of A. The interpretation is that agents' priors do not assign a probability to events (subsets) of Ω that their information does not identify, in other words, agents face ambiguity.

Remark 1. It is worth pointing out that whenever for each $i \in S$ and $S \subseteq I$ we have that $\mathcal{K}_i^S = \mathcal{F}_i$, our modeling of q_i^S (the ambiguity related prior) reduces to the one of de Castro and Yannelis (2009). On the other end, $\mathcal{K}_i^S = \bigvee_{i \in S} \mathcal{F}_i$ for each $i \in S$ and $S \subseteq I$ represents a situation that could be appropriately termed 'minimum ambiguity'. Of course alternative hypotheses could be adopted, leading to varying degrees of ambiguity.

(iii) Consumption set and initial endowment. The set valued map $X_i : \Omega \to 2^{\Re_+^m}$ gives the (random-state dependent) consumption set, $X_i(\omega) \subset \Re_+^m$, of the *ith* agent. A map $x_i : \Omega \to \Re_+^m$ gives a (random) individual consumption allocation (or bundle) of the *ith* agent across all the states $\omega \in \Omega$. Then, the set

$$L_{X_i} = \{x_i : \Omega \to \Re_+^m | x_i(\omega) \in X_i(\omega), \forall \omega \in \Omega\}$$

is the set of all the (random) individual consumption allocations available to the *ith* agent. We refer to $x_i \in L_{X_i}$ as a (random) assignment of agent i and denote $\prod_{i \in I} L_{X_i} = L_X$. The (random) initial endowment of the *ith* agent is given by $e_i \in L_{X_i}$.

(iv) Preferences. Each individual $i \in I$ is endowed with a primitive preference relation over commodities, represented by the utility function $u_i : \Re^m_+ \to \Re_+$[7]. These preferences give rise to preferences over assignments in a way that we develop below. We begin by defining Δ, which is a finite class of (additive) probability measures $\mu : \mathcal{F} \to [0,1]$, $\mathcal{F} = \mathcal{P}(\Omega)$. We then fix a coalition $S \subseteq I$ and define for each agent $i \in S$ the class

$$\mathcal{P}_i^S = \{\mu_i \in \Delta : \mu_i(A) = q_i^S(A), \forall A \in \mathcal{K}_i^S\}.$$

For each $i \in S$, $\mathcal{P}_i^S$ is the class of all extensions of the probability measure q_i^S to the measurable space $(\Omega, \mathcal{F})$. Note that the probability measures $\mu_i \in \mathcal{P}_i^S$: (a) agree with q_i^S in the elements of the algebra $\mathcal{K}_i^S$ and (b) unlike the prior belief q_i^S, give the probability of all the events of Ω, so that ambiguity no longer exists. Note, however, that it is not guaranteed that $\mu_i(\omega) > 0$ for all $\omega \in \Omega$, which means that agents' Bayesian (or subjective) expected utilities cannot be defined.

We can, therefore, proceed to define agents' preferences over assignments based on the idea of maximin expected utility. For what follows in the sequel it will be useful to qualify preferences for decision making at the ex ante and interim stages.

(A) Ex ante preferences over assignments can be defined as follows: for the fixed coalition $S \subseteq I$ and given any two assignments $x_i, y_i \in L_{X_i}$, an ex ante preference relation $\succeq$ is defined by

$$x_i \succeq y_i \Leftrightarrow \min_{\mu_i \in \mathcal{P}_i^S} \sum_{\omega \in \Omega} u_i(x_i(\omega))\mu_i(\omega) \geq \min_{\mu_i \in \mathcal{P}_i^S} \sum_{\omega \in \Omega} u_i(y_i(\omega))\mu_i(\omega).$$

[7]We define non random (ex post) utility functions. The analysis and all the results of the paper would not be affected if a random (state dependent) utility function $u_i : \Omega \times \Re^m_+ \to \Re_+$ were adopted for each agent $i \in I$.

For $S \subseteq I$ fixed, it follows from de Castro and Yannelis (2009) that the above preference relation is equivalent to the following one (see also in He-Yannelis, 2013):

$$x_i \succeq y_i \Leftrightarrow \sum_{E \in \mathcal{K}_i^S} q_i^S(E) \min_{\omega \in E} u_i(x_i(\omega)) \geq \sum_{E \in \mathcal{K}_i^S} q_i^S(E) \min_{\omega \in E} u_i(y_i(\omega)).$$

Since both the information and the beliefs available to individuals depend on their coalition membership, we need to enrich the formulation of ex ante maximin expected utility by parameterizing it with respect to group membership. Thus, the map $\upsilon_i : L_{X_i} \times 2^I \to \Re_+$, defined for any $(x_i, S) \in L_{X_i} \times 2^I$ by

$$\upsilon_i(x_i, S) = \min_{\mu_i \in \mathcal{P}_i^S} \sum_{\omega \in \Omega} u_i(x_i(\omega))\mu_i(\omega), \tag{1}$$

or equivalently by

$$\upsilon_i(x_i, S) = \sum_{E \in \mathcal{K}_i^S} q_i^S(E) \min_{\omega \in E} u_i(x_i(\omega)), \tag{2}$$

gives the *ex ante maximin expected utility* of the *ith* agent.
Both (1) and (2) are well defined. In (1) specifically, since the class $\mathcal{P}_i^S$ is finite, the minimum sum is attained and there exists a minimizing prior $\mu_i^\star \in \mathcal{P}_i^S$, for each $i \in S \in 2^I$. Also, since Ω is finite, the minimum utility in (2) is attained.

(B) Following the footsteps of de Castro and Yannelis (2009), agents lose their priors in the interim and their maximin utility becomes non-expected. In the interim, when the state $\omega \in \Omega$ occurs and individuals receive signals about it, each agent $i \in S$ naturally accumulates the probability distribution of his/her prior q_i^S to the event $\mathcal{K}_i^S(\omega) \in \mathcal{K}_i^S$. However, as previously discussed, q_i^S will not provide the probability of events $B \subset \mathcal{K}_i^S(\omega)$. In the interim, therefore, individuals cannot assess the probability of any (non trivial) event in the first place. That is, in the interim, agents do not need priors to begin with. From expression (2) above, we derive the *interim maximin utility* map $\underline{u}_i : \Omega \times L_{X_i} \times 2^I \to \Re_+$ of the agent $i \in S \subseteq I$, defined for any $(\omega, x_i, S) \in \Omega \times L_{X_i} \times 2^I$ by the formula

$$\underline{u}_i(\omega, x_i, S) = \min_{\omega' \in \mathcal{K}_i^S(\omega)} u_i(x_i(\omega')). \tag{3}$$

Clearly, (3) is well defined; the minimum utility is attained because Ω is finite.

Remark 2. The maximin (expected or non-expected) utility of expressions (1), (2) and (3) above extends the one of the corresponding formulas in de Castro and Yannelis (2009) in two ways. First, it accommodates any kind of information for individuals (decision makers) as group members, not necessarily their initial private information. Second, it is coalition dependent, so that which coalition the maximin agent is into affects the utility outcome and, thus, does matter in his/her decision making. Nevertheless, this does not create any complications because in the sequel we will proceed by considering coalitions of individuals, i.e., individuals will always be viewed as members of some coalition.

We can now define a pure exchange economy with differential information and ambiguity (i.e., a MDIE) as

$$\mathcal{E} = \{(\mathcal{F}_i, q_i, X_i, e_i, u_i) : i \in I\}.$$

In order to emphasize the (ex ante or interim, respectively) maximin postulate of decision making by individuals we can define

$$\mathcal{E}^{ea} = \{(L_{X_i}, e_i, \upsilon_i) : i \in I\},$$

called the *ex ante Maximin Differential Information Economy* (eaMDIE), and

$$\mathcal{E}^i = \{(L_{X_i}, e_i, \underline{u}_i) : i \in I\},$$

which we call *interim Maximin Differential Information Economy* (iMDIE).

In each case, a (random) allocation of commodities is

$$x = (x_1, x_2, ..., x_n) \in L_X$$

and is said to be *feasible*, if

$$\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega), \forall \omega \in \Omega.$$

Remark 3. For clarity, a few final remarks on the advantage of the current formulation of differential information over others based on Bayesian expected utility previously suggested are useful. Bayesian agents exhibiting the standard subjective expected utility are allowed to consume only allocations adapted to their private information, i.e., $\mathcal{F}_i$ - measurable, because Bayesian agents' preferences are incomplete in the space consisting of all possible assignments across Ω. By contrast, non-Bayesian (maximin) agents are allowed to make non private information measurable consumption choices as well. A maximin agent has complete (or more accurately, completed) preferences, as opposed to the incomplete preferences of the same agent when being Bayesian.

1.3 The Maximin Value Allocation

We define in this section two maximin cardinal value allocation notions, an ex ante and an interim one.

Both of them are extensions of the Bayesian value allocation concepts of Krasa and Yannelis (1994, 1996). Our motivation in defining them, however, is more subtle. Indeed, the ex ante maximin value allocations result in superior Pareto optimal outcomes, since they are defined without the private information measurability requirement on agents' consumption and initial endowments. The interim maximin value allocations, on the other hand, refine the (existing within the Bayesian value allocation concepts) Bayesian relationship between efficiency and incentive compatibility.

At the same time, the definitions provided here extend a similar maximin value allocation notion introduced by de Castro and Yannelis (2009).

Given an economy $\mathcal{E}^{ea}$, we derive an *ex ante maximin TU game* $\Gamma = (I, V_{\lambda,\upsilon})$, whose characteristic function $V_{\lambda,\upsilon}$ is given for every coalition $S \subseteq I$ by

$$V_{\lambda,\upsilon}(S) = \left\{ \max_{x_i \in L_{X_i}} \sum_{i \in S} \lambda_i \upsilon_i(x_i, S) : \sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega), \forall \omega \in \Omega \right\}. \tag{4}$$

Given such a TU game, the Shapley (1953) value of each individual is defined as

$$Sh_i(V_{\lambda,\upsilon}) = \sum_{i \in S} \frac{(|S|-1)!(|I|-|S|)!}{|I|!} [V_{\lambda,\upsilon}(S) - V_{\lambda,\upsilon}(S \setminus \{i\})], \tag{5}$$

where $S \subseteq I$. Briefly, the interpretation of this formula is that the payoff $Sh_i(V_{\lambda,\upsilon})$ each player receives, corresponds to the contributions this individual makes to all the possible coalitions he/she can join. The combinatoric weight of these contributions refers to all the possible ways a coalition $S \subseteq I$ can form, when the players of Γ

enter in equiprobable random order into the various coalitions they participate, i.e., when the various coalitions in Γ are equiprobably sequentially formed. One important property which will be useful to us in the sequel is the group rationality:

$$\sum_{i \in I} Sh_i(V_{\lambda,\upsilon}) = V_{\lambda,\upsilon}(I).$$

Given that the TU game defined above is derived from an exchange economy, we are interested in state contingent commodity allocations. That is, consumption outcomes for which the corresponding utilities of individuals are related to their Shapley value. We call these outcomes value allocations and we define them (with ex ante maximin preferences) as follows:

Definition 1. An allocation $x \in L_X$ of $\mathcal{E}^{ea}$ is said to be an *ex ante maximin value allocation* if the following two conditions are satisfied:

(i) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$, for all $\omega \in \Omega$,

(ii) there exists $\lambda \in \Re^I_+ \setminus \{0\}$, such that $\lambda_i \; \upsilon_i(x_i, I) = Sh_i(V_{\lambda,\upsilon})$, for all $i \in I$.

Given, now, an economy $\mathcal{E}^i$, we derive for each state $\omega \in \Omega$ an *interim maximin TU game* $\Gamma = (I, V_{\lambda,\underline{u},\omega})$, whose characteristic function $V_{\lambda,\underline{u},\omega}$ is given for every coalition $S \subseteq I$ by

$$(6) \qquad V_{\lambda,\underline{u},\omega}(S) = \left\{ \max_{x_i \in L_{X_i}} \sum_{i \in S} \lambda_i(\omega) \underline{u}_i(\omega, x_i, S) : \sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega), \forall \omega \in \Omega \right\}.$$

The Shapley value of each individual in this game is defined exactly as in (5) with $V_{\lambda,\underline{u},\omega}$ replacing $V_{\lambda,\upsilon}$. The corresponding definition of a value allocation in this case (i.e., with interim maximin preferences) is as follows:

Definition 2. An allocation $x \in L_X$ of $\mathcal{E}^i$ is said to be an *interim maximin value allocation* if the following two conditions are satisfied for all $\omega \in \Omega$:

(i) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$,

(ii) there exists $\lambda(\omega) \in \Re^I_+ \setminus \{0\}$, such that for all $i \in I$

$$\lambda_i(\omega)\underline{u}_i(\omega, x_i, I) = Sh_i(V_{\lambda,\underline{u},\omega}).$$

Remark 4. It is easy to verify that in both definitions the characteristic function [of expressions (4) and (6), respectively] satisfies monotonicity, superadditivity and becomes zero for the empty set. Since for each $S \subseteq I$ the two characteristic functions have all the required properties, the dependence of maximin (expected or not) utility on coalitions does not affect in any way the technical properties of the model.

Some notable specifications of the above definitions are the following:

(a) If $\mathcal{K}_i^S = \mathcal{F}_i$ for every $i \in S$ and $S \subseteq I$, the above definitions are specified into the (ex ante and interim respectively) *private (information) maximin value allocation*[8]. Further, if agents' maximin expected utilities reduce (appropriately) to Bayesian ones, agents' differential Bayesian priors reduce to a common one and (specifically) measurability restrictions are imposed on agents' net trades, the ex ante private maximin value allocation reduces to the ex ante private (information) Bayesian value allocation of Krasa and Yannelis (1994, 1996)[9].
(b) If $\mathcal{K}_i^S = \bigvee_{i \in S} \mathcal{F}_i$ for every $i \in S$ and $S \subseteq I$, the above definitions are specified into the (ex ante and interim respectively) *fine maximin value allocation*. The ex ante fine maximin value allocation can then (as previously) be reduced to the ex ante fine Bayesian value allocation of Krasa and Yannelis (1994, 1996).

[8]A version of which was introduced in de Castro and Yannelis (2009).
[9]The private value allocation bears distinct significance because it awards the informational superiority of agents.

(c) If $\mathcal{K}_i^S = \bigwedge_{i \in S} \mathcal{F}_i$ for every $i \in S$ and $S \subseteq I$, the above definitions are specified into the (ex ante and interim respectively) *coarse maximin value allocation.* The ex ante coarse maximin value allocation can then (as previously) be reduced to the ex ante coarse Bayesian value allocation of Krasa and Yannelis (1994, 1996).

1.4 Existence of a Maximin Value Allocation

In section 2, recall that formula (3) was derived as a special case of formula (2) [which is equivalent to formula (1)]. For this reason, given Definitions 1 and 2 of section 3, it suffices to prove existence of an ex ante maximin value allocation only. Existence of an interim maximin value allocation is then implied.

Consider an eaMDIE $\mathcal{E}^{ea} = \{(L_{X_i}, e_i, \upsilon_i) : i \in I\}$. W.l.o.g., we can contrast and one-to-one correspond this economy with a deterministic economy

$$\mathcal{E}' = \{(X_i, e_i, u_i) : i = 1, 2, ..., n\}.$$

In Emmons and Scafuri (1985), it is shown that for a given economy $\mathcal{E}'$ with commodity space $\Re^m$, a cardinal value allocation exists if: (i) The consumption sets are closed, bounded below and convex subsets of $\Re^m_+$, while the initial endowments are elements of the consumption sets and (ii) utility functions are continuous and concave in $\Re^m_+$.

This means that when: (i) The commodity space of $\mathcal{E}^{ea}$ is identified with $\Re^m$ or any other finite dimensional vector space and (ii) the aforementioned (standard) assumptions for the agents' ex post utility function and random consumption set are satisfied, an ex ante maximin value allocation exists for $\mathcal{E}^{ea}$.

However, when the consumption set of every agent is replaced by the positive cone of an infinite dimensional vector space, an existence result is meaningful. We will provide such a result by restricting ourselves to separable ordered Banach spaces and by following the known argumentative path of the existence proofs of Yannelis, 1991 (for the core allocation) and Krasa and Yannelis, 1996 (for the value allocation).

First, nevertheless, we have to construct an infinite dimensional commodity space eaMDIE. For this, we have to modify appropriately the model of section 2. Let, therefore, B be the commodity space of $\mathcal{E}^{ea}$, where B is a separable ordered Banach space and let B_+ denote the positive cone of B. Let again Ω be finite, $\mathcal{F} = \mathcal{P}(\Omega)$ and consider for each agent $i \in S \subseteq I$ the minimizing prior $\mu_i^\star : \mathcal{F} \to [0, 1]$ [of expression (1) in section 2], which is the counting measure on $(\Omega, \mathcal{F})$. For our purposes, it is sufficient to fix a coalition $S \subseteq I$. Define now the correspondence $X_i : \Omega \to 2^{B+}$ giving the random consumption set $X_i(\omega) \subset B_+$ of the ith agent. If a map $x_i \in L_{X_i}$ is a random individual consumption allocation of the ith agent, we now assume that

$$L_{X_i} = \{x_i | x_i : \Omega \to B_+ \text{ and } x_i(\omega) \in X_i(\omega), \forall \omega \in \Omega\} \subset L^1(\Omega, \mathcal{F}, \mu_i^\star; B).$$

For each agent i, therefore, we maintain that his/her set of random assignments L_{X_i} is a strict subset of the standard L^1 Bochner space of (equivalence classes of) all Bochner measurable B valued maps, which are Bochner integrable bounded, normed by the usual Bochner one norm, i.e., by the usual functional $|| \cdot ||_1$. Of course, for all i, $L^1(\Omega, \mathcal{F}, \mu_i^\star \,; B)$ is a Banach space. Let the map $e_i : \Omega \to B_+$ giving the random initial endowment of the ith agent, where $e_i \in L_{X_i}$ for all i. The ex post utility of the ith agent is given by the function $u_i : B_+ \to \Re_+$. The map $\upsilon_i : L_{X_i} \times 2^I \to \Re_+$, defined by expression (1) in section 2, gives the ex ante maximin expected utility of the ith agent. Since 2^I is a finite class of finite sets, the technical properties of L_{X_i}, $i \in I$, are not affected by 2^I in the respective set product. W.l.o.g., therefore, we can ignore 2^I form the domain of υ_i, $i \in I$, when technical properties (like continuity or concavity) of the latter are examined on its domain. Finally, when u_i is bounded the same holds for υ_i, $i \in I$, so we can safely use supinf and infsup arguments for the latter.

We have thus constructed the following ordered Banach commodity space eaMDIE:

$$\mathcal{E}^{ea} = \{(L_{X_i}, \upsilon_i, e_i) : i = 1, 2, ..., n\}.$$

In its finite dimensional trace, $\mathcal{E}^{ea}$ can be directly related with a finite dimensional commodity space economy. We will take advantage of this fact, by approximating $\mathcal{E}^{ea}$ via finite dimensional commodity space economies and obtaining an ex ante maximin value allocation in $\mathcal{E}^{ea}$ as a limit of (ex ante maximin) value allocations drawn along the approximating (finite dimensional commodity space) economies.

We are now ready to state and prove the existence result of this paper.

Theorem 1. Let $\mathcal{E}^{ea}$ be the previously defined ordered Banach commodity space eaMDIE, which satisfies the following assumptions:
(A_1) $X_i : \Omega \to 2^{B_+}$ is a closed, convex, non empty valued correspondence.
(A_2) $u_i : B_+ \to \Re_+$ is a continuous, concave and bounded function.

Then an ex ante maximin value allocation exists in $\mathcal{E}^{ea}$.

Proof. The proof follows the one of Krasa and Yannelis (1996). For the sake of completeness we repeat all the steps.

Step I

By hypothesis (A_1) of the theorem, L_{X_i} is convex, norm closed and bounded from below and furthermore $e_i \in L_{X_i}$, so that $L_{X_i} \neq \emptyset$ for every $i \in I$. Hence $L_X \neq \emptyset$ is non empty as well. By hypothesis (A_2) of the theorem, the concavity of u_i implies the concavity of υ_i for each $i \in I$. Also, by Theorem 2.8 of Balder and Yannelis (1993), since u_i is continuous, concave and bounded, it follows that υ_i is weakly upper semi continuous on L_{X_i} for each $i \in I$. Let $\mathcal{A}$ be the (same for all the agents) set of all the finite dimensional subspaces of their $L^1(\Omega, \mathcal{F}, \mu_i^\star; B)$, each one of these subspaces containing all the agents' projections of initial endowments e_i. Then, for each $\alpha \in \mathcal{A}$ we define the $L^\alpha_{X_i} = L_{X_i} \cap \alpha$ be the restriction of the consumption set, $\upsilon_i^\alpha = \upsilon_i \cap \alpha$ is the restriction of the utility function and $e_i^\alpha = e_i \cap \alpha$ the restriction of the endowment of the *ith* agent on the finite dimensional subspace $\alpha \in \mathcal{A}$. We can construct, therefore, the α - finite dimensional commodity space economy

$$\mathcal{E}^\alpha = \{(L^\alpha_{X_i}, \upsilon_i^\alpha, e_i^\alpha) : i = 1, 2, ..., n\}.$$

Step II

Certainly, the standard results of existence of a cardinal value allocation apply to $\mathcal{E}^\alpha$, as long as $\mathcal{E}^\alpha$ satisfies all the assumptions of Emmons and Scafuri (1985). Continuity of υ_i^α is the only one which is not immediate. However, it follows by the hypothesis (A_2) of the theorem via the Dominated Convergence Theorem, and because $L_{X_i}^\alpha$ is a finite dimensional space, that $\upsilon_i^\alpha : L_{X_i}^\alpha \times 2^I \to \Re_+$ is continuous on $L_{X_i}^\alpha$ for each $S \in 2^I$. Hence, there exists a cardinal (in our case, ex ante maximin) value allocation for $\mathcal{E}^\alpha$, i.e., there exists $x^\alpha \in L_X^\alpha = \prod_{i=1}^n L_{X_i}^\alpha$ such that:

(i) $\sum_{i\in I} x_i^\alpha = \sum_{i\in I} e_i^\alpha$,
(ii) there exists a $\lambda_i^\alpha \geq 0$ for every agent i, with $\sum_{i\in I} \lambda_i^\alpha = 1$, such that λ_i^α $\upsilon_i^\alpha(x_i^\alpha, I) = Sh_i(V_{\lambda^\alpha,\upsilon^\alpha})$, for all $i \in I$, where $Sh_i(V_{\lambda^\alpha,\upsilon^\alpha})$ is the Shapley value of the ith agent, derived from the TU game $(I, V_{\lambda^\alpha,\upsilon^\alpha})$, whose characteristic function $V_{\lambda^\alpha,\upsilon^\alpha}$ is defined for every coalition $S \subseteq I$ as follows:

$$V_{\lambda^\alpha,\upsilon^\alpha}(S) = \left\{ \max_{x^\alpha \in \prod_{i\in S} L_{X_i}^\alpha} \sum_{i\in S} \lambda_i^\alpha \upsilon_i^\alpha(x_i^\alpha, S) : \sum_{i\in S} x_i^\alpha = \sum_{i\in S} e_i^\alpha \right\}.$$

By denoting $\sum_{i\in I} e_i = e \geq 0$, we have that $0 \leq \sum_{i\in I} x_i^\alpha = \sum_{i\in I} e_i^\alpha$, which implies that for each $\mathcal{E}^\alpha$ economy and for each cardinal value allocation $x^\alpha \in L_X^\alpha$ of this economy, every x_i^α lies in the order interval $[0, e]$ of $\sum_{i\in I} L_{X_i} \subset L^1(\Omega, \mathcal{F}, \mu_i^\star\ ;\ B)$, $i \in I$. We direct the set $\mathcal{A}$ by inclusion, so that $\{(x_1^\alpha, ..., x_n^\alpha, \lambda_1^\alpha, ..., \lambda_n^\alpha) : \alpha \in \mathcal{A}\}$ forms a net in $C = \prod_{i=1}^n [0, e] \times \Delta$, where Δ is the $(n-1)$ - dimensional simplex. Since B is an ordered Banach space so is $L^1(\Omega, \mathcal{F}, \mu_i^\star\ ;\ B)$, $i \in I$, which implies (see, for example, in Aliprantis and Burkinshaw, 1985) that this space has weakly compact order intervals. Hence, the order interval $[0, e]$ is weakly compact and therefore C is compact. It follows that there is a subnet (still indexed by $\mathcal{A}$ for notational simplicity) $\{x^\alpha, \lambda^\alpha\}_{\alpha\in\mathcal{A}}$, which converges to a point $(\bar{x}, \bar{\lambda}) = (\bar{x}_1, ..., \bar{x}_n, \bar{\lambda}_1, ..., \bar{\lambda}_n)$ in C. We shall show that this limit constitutes an ex ante maximin value allocation for the infinite dimensional commodity space economy $\mathcal{E}^{ea} = \{(L_{X_i}, \upsilon_i, e_i) : i = 1, 2, ..., n\}$.

Step III

First notice that since L_{X_i} is convex and norm closed, by Mazur's Theorem it is weakly closed as well, hence the weak limit $\bar{x}_i$ lies actually in L_{X_i}, for each $i \in I$, so we conclude that $\bar{x} \in L_X$. Moreover, since for each $\alpha \in \mathcal{A}$ we have $\sum_{i\in I} x_i^\alpha = \sum_{i\in I} e_i^\alpha$, by taking weak limits we conclude that

$$\sum_{i\in I} \bar{x}_i = \sum_{i\in I} e_i. \tag{7}$$

Step IV

In order to define the ex ante maximin value allocation of the original economy we verify first that the TU game $(I, V_{\bar{\lambda},\upsilon})$, whose characteristic function $V_{\bar{\lambda},\upsilon}$ is given for every coalition $S \subseteq I$ as in (4), is well defined. Indeed, for every $S \subseteq I$, since $[0, \sum_{i\in S} e_i]$ is weakly compact and υ_i is weakly upper semi continuous on this set it follows that $V_{\bar{\lambda},\upsilon}(S)$ is well defined and so is the Shapley value $Sh_i(V_{\bar{\lambda},\upsilon})$ of each individual $i \in I$. Next, let $x^\star \in \prod_{i\in S} L_{X_i}$ be such that $\sum_{i\in S} x_i^\star = \sum_{i\in S} e_i$ and $V_{\bar{\lambda},\upsilon}(S) = \sum_{i\in S} \bar{\lambda}_i \upsilon_i(x_i^\star, S)$. We have that the sequence $\{V_{\lambda^\alpha,\upsilon^\alpha}(S)\}_{\alpha\in\mathcal{A}}$ is bounded, so we may assume, by passing to a subnet if necessary, that it converges. Since $\sum_{i\in S} x_i^\star = \sum_{i\in S} e_i$, by denoting $x_i^{\star\alpha} = x_i^\star \cap \alpha \in \prod_{i\in S} L^\alpha_{X_i}$, we have that: $x_i^{\star\alpha} \to x^\star$ weakly and $\sum_{i\in S} x_i^{\star\alpha} = \sum_{i\in S} e_i^\alpha$. Consequently, for each $\alpha \in \mathcal{A}$, $V_{\lambda^\alpha,\upsilon^\alpha}(S) \geq \sum_{i\in S} \lambda_i^\alpha \upsilon_i^\alpha(x_i^{\star\alpha}, S)$. Further, by the definition of the weak upper semicontinuity of υ_i at $x_i^\star$, for every $\epsilon > 0$, there exists an α_ϵ, such that for every $\alpha > \alpha_\epsilon$ we have

$$\sum_{i\in S} \lambda_i \upsilon_i(x_i^\star, S) \leq \sum_{i\in S} \lambda_i^\alpha \upsilon_i^\alpha(x_i^{\star\alpha}, S) + \epsilon.$$

By considering a net $\epsilon \to 0$ we can extract a subnet (still indexed by α) such that $\liminf_\alpha \sum_{i\in S} \lambda_i^\alpha \upsilon_i^\alpha(x_i^{\star\alpha}, S) \geq \sum_{i\in S} \lambda_i \upsilon_i(x_i^\star, S)$. Hence, we conclude that

$$
\begin{aligned}
\lim_\alpha V_{\lambda^\alpha,\upsilon^\alpha}(S) &= \liminf_\alpha V_{\lambda^\alpha,\upsilon^\alpha}(S) \\
&\geq \liminf_\alpha \sum_{i\in S} \lambda_i^\alpha \upsilon_i^\alpha(x_i^{\star\alpha}, S) \\
&\geq \sum_{i\in S} \bar{\lambda}_i \upsilon_i(x_i^\star, S) \\
&= V_{\bar{\lambda},\upsilon}(S). \qquad (8)
\end{aligned}
$$

On the other hand, let $y^\alpha \in \prod_{i\in S} L^\alpha_{X_i}$, for each $\alpha \in \mathcal{A}$, be such that it satisfies $\sum_{i\in S} y_i^\alpha = \sum_{i\in S} e_i^\alpha$ and $\sum_{i\in S} \lambda_i^\alpha \upsilon_i^\alpha(y_i^\alpha, S) = V_{\lambda^\alpha,\upsilon^\alpha}(S)$. Certainly y^α converges weakly to $y \in \prod_{i\in S} L_{X_i}$ and $\sum_{i\in S} y_i = \sum_{i\in S} e_i$. It follows that

$$
\begin{aligned}
\lim_\alpha V_{\lambda^\alpha,\upsilon^\alpha}(S) &= \limsup_\alpha V_{\lambda^\alpha,\upsilon^\alpha}(S) \\
&= \limsup_\alpha \sum_{i\in S} \lambda_i^\alpha \upsilon_i^\alpha(y_i^\alpha, S) \\
&= \sum_{i\in S} \bar{\lambda}_i \limsup_\alpha \upsilon_i^\alpha(y_i^\alpha, S) \\
&\leq \sum_{i\in S} \bar{\lambda}_i \upsilon_i(y_i, S) \\
&\leq V_{\bar{\lambda},\upsilon}(S), \qquad (9)
\end{aligned}
$$

where the fourth line follows from the weak upper semi continuity of υ_i and the last from the fact that $y \in L_{X_i}$ is feasible and the definition of $V_{\bar{\lambda},\upsilon}(S)$. Therefore, from (8) and (9) we conclude that $\lim_\alpha V_{\lambda^\alpha,\upsilon^\alpha}(S) = V_{\bar{\lambda},\upsilon}(S)$ and therefore

$$\lim_\alpha Sh_i(V_{\lambda^\alpha,\upsilon^\alpha}) = Sh_i(V_{\bar\lambda,\upsilon}), \forall i \in I.$$

Step V

We finally prove that $\bar\lambda_i \upsilon_i(\bar x_i, I) = Sh_i(V_{\bar\lambda,\upsilon})$, for all $i \in I$.
By definition of the (ex ante maximin) value allocation for each $\alpha \in \mathcal{A}$, we have:

$$\begin{aligned} \lambda_i^\alpha \upsilon_i^\alpha(x_i^\alpha, I) &= Sh_i(V_{\lambda^\alpha,\upsilon^\alpha}), \forall i \in I, \\ \sum_{i\in S} x_i^\alpha &= \sum_{i\in S} e_i^\alpha. \end{aligned}$$

Therefore, $\lim_\alpha \sup \lambda_i^\alpha \upsilon_i^\alpha(x_i^\alpha, I) = \lim_\alpha \sup Sh_i(V_{\lambda^\alpha,\upsilon^\alpha})$.
It follows that $\bar\lambda_i \lim_\alpha \sup \upsilon_i^\alpha(x_i^\alpha, I) = \lim_\alpha \sup Sh_i(V_{\lambda^\alpha,\upsilon^\alpha})$. By the weak upper semi continuity of υ_i, since $x_i^\alpha \to \bar x_i$ weakly, it follows that $\bar\lambda_i \upsilon_i(\bar x_i, I) \geq \bar\lambda_i \lim_\alpha \sup \upsilon_i^\alpha(x_i^\alpha, I)$, $\forall i \in I$. Hence, we conclude that $\bar\lambda_i \upsilon_i(\bar x_i, I) \geq \lim_\alpha \sup Sh_i(V_{\lambda^\alpha,\upsilon^\alpha})$ and further, by the previous step, we infer that $\bar\lambda_i \upsilon_i(\bar x_i, I) \geq Sh_i(V_{\bar\lambda,\upsilon}), \forall i \in I$. Suppose that the above inequality is strict for some $i \in I$. Summing up over $i \in I$, we have $\sum_{i\in I} \bar\lambda_i \upsilon_i(\bar x_i, I) > \sum_{i\in I} Sh_i(V_{\bar\lambda,\upsilon})$ and further, since $\sum_{i\in I} Sh_i(V_{\bar\lambda,\upsilon}) = V_{\bar\lambda,\upsilon}(I)$, it follows that

$$\sum_{i\in I} \bar\lambda_i \upsilon_i(\bar x_i, I) > V_{\bar\lambda,\upsilon}(I). \tag{10}$$

However, by (7) and the definition of $V_{\bar\lambda,\upsilon}(I)$, it must be $\sum_{i\in I} \bar\lambda_i \upsilon_i(\bar x_i, I) \leq V_{\bar\lambda,\upsilon}(I)$, which together with (10) imply $V_{\bar\lambda,\upsilon}(I) \geq \sum_{i\in I} \bar\lambda_i \upsilon_i(\bar x_i, I) > V_{\bar\lambda,\upsilon}(I)$, which is a clear contradiction. Therefore, it must be

$$\bar{\lambda}_i \upsilon_i(\bar{x}_i, I) = Sh_i(V_{\bar{\lambda},\upsilon}), \forall i \in I, \tag{11}$$

as desired[10]. Now, (7) along with (11) prove the claim of the theorem. □

1.5 Incentive Compatibility of the Maximin Value Allocation

In this section, we consider an iMDIE $\mathcal{E}^i = \{(L_{X_i}, e_i, \underline{u}_i) : i \in I\}$ only, for reasons explained further down.

We shall be interested in the notions of maximin Pareto efficiency, maximin individual rationality and transfer maximin incentive compatibility of an allocation of $\mathcal{E}^i$. Specifically in the case of incentive compatibility, our attention is focused on coalitional incentive compatibility. As Koutsougeras and Yannelis (1993) and Krasa and Yannelis (1994) point out for the Bayesian case, it is the coalitional aspect of the incentive compatibility notion that ensures the viability of an efficient contract. The same can be argued for the maximin case as well. Besides, as in the Bayesian case, every maximin coalitional incentive compatible contract is a fortiori maximin individual incentive compatible (but the reverse is not valid).

The definitions provided here, however, are generalizations of the respective definitions in de Castro et al. (2010). Indeed, it is inherent in our definitions the possibility that the information available to individuals as members of coalitions is not necessarily the one they are initially endowed with. The definitions of de Castro et al. (2010), consequently, which are based on the private information of individuals only, become a special case of ours.

[10]Incidentally, (11) proves that indeed $\sum_{i \in I} \bar{\lambda}_i \upsilon_i(\bar{x}_i, I) = \sum_{i \in I} Sh_i(V_{\bar{\lambda},\upsilon}) = V_{\bar{\lambda},\upsilon}(I)$, i.e., the Pareto optimality of the value allocation $\bar{x} \in L_X$.

At the same time, our notion of incentive compatibility refines the one of de Castro et al. (2010), since our definition restricts the set of incentive compatible allocations. Unlike de Castro et al. (2010), where transfers are not allowed between the cheating coalitions, we provide coalitions with the possibility of redistributing among their members the benefits from misreporting a state of nature. In that way, there is more potential in gaining from lying, the set of the no incentive compatible allocations enlarges, so the set of the incentive compatible ones shrinks. Any allocation that satisfies incentive compatibility in our (stronger) sense is also incentive compatible in the (weaker) sense of de Castro et al., 2010 (but the opposite is not true).

It is sufficient to provide all the definitions for the interim period only. This is because the minimum in formula (3) of section 2 is the key element to the proof of Theorem 3, given at the end of this section. Of course, one could choose to cast the same definitions in an ex ante sense, using formula (2) of section 2. Again, with minor alternations to the proof of the theorem below, maximin efficiency of the maximin value allocation would lead to its maximin incentive compatibility.

Definition 3. A feasible allocation $x \in L_X$ is

(i) (interim) maximin Pareto optimal [i.e., (interim) maximin efficient], if there does not exist a state $\bar{\omega} \in \Omega$ and another feasible allocation $y \in L_X$, such that $\underline{u}_i(\bar{\omega}, y_i, I) \geq \underline{u}_i(\bar{\omega}, x_i, I)$ for all $i \in I$ and $\underline{u}_i(\bar{\omega}, y_i, I) > \underline{u}_i(\bar{\omega}, x_i, I)$ for some $i \in I$,

(ii) (interim) maximin individually rational, if $\underline{u}_i(\omega, x_i, \{i\}) \geq \underline{u}_i(\omega, e_i, \{i\})$ for all $i \in I$ and for all $\omega \in \Omega$.

Definition 4. A feasible allocation $x \in L_X$ is said to be transfer (interim) maximin coalitionally incentive compatible if the following does not hold:
There exist a coalition $S \subset I$, two different states a and b of Ω and $z \in \Re_+^{mS}$, where $\sum_{i \in S} z_i = \sum_{i \in S}[e_i(a) + x_i(b) - e_i(b)]$, such that

(i) $\mathcal{K}_i^I(a) = \mathcal{K}_i^I(b)$ if and only if $i \in I \setminus S$,

(ii) $u_i(x_i(a)) = u_i(x_i(b))$, for all $i \in I \setminus S$,

(iii) $\underline{u}_i(a, y_i, I) > \underline{u}_i(a, x_i, I)$ for all $i \in S$, where for all $i \in S$

$$y_i(\omega) = \begin{cases} z_i, & \text{if } \omega = a \\ x_i(\omega), & \text{otherwise.} \end{cases}$$

Remark 5. Clearly, Definitions 3 and 4 are appropriate adjustments to the respective definitions of the Bayesian case, so that the agents' maximin preferences can be now accommodated. Both of them, therefore, have the familiar interpretation. Focusing on Definition 4 solely, the new requirement relative to the definition in de Castro et al. (2010) is that individuals in the coalition S, who may not retain their private information, may also redistribute the allocation they receive by misreporting the state. That is, if a is the true state known only by S, who reports to $I \setminus S$ the false state b, then all the members of S can become strictly better off in terms of their interim maximin utility, as indicated by condition (iii), by redistributing (if necessary) the allocation they receive. If in Definition 4 it specifically happens that: (a) $z_i = e_i(a) + x_i(b) - e_i(b)$ for all $i \in S \subset I$ and (b) for every $i \in I$ we have that $\mathcal{K}_i^S = \mathcal{F}_i$ whenever $i \in S \subseteq I$, then Definition 4 coincides with the one of de Castro et al. (2010). Observe also that if $S = \{i\}$ in Definition 4, then this definition reduces to the one of the (interim) maximin individual incentive compatibility. Note, finally, that both Definition 3 and Definition 4 work for an infinite dimensional commodity vector space economy.

Remark 6. Any (interim) maximin value allocation of $\mathcal{E}^i$ is (interim) maximin individually rational and (interim) maximin Pareto optimal.

It can be easily shown that the (interim) maximin value allocation meets the requirements of Definition 3, so we only state the result of the previous remark for reference. Finally, we prove the transfer (interim) maximin coalitional incentive compatibility of the (interim) maximin value allocation. The proof reflects the result that every maximin Pareto optimal allocation is transfer maximin coalitionally incentive compatible. It should be noted that our result requires no additional assumptions than those in de Castro and Yannelis (2009). Hence, the following theorem shows that the de Castro and Yannelis (2009) result is actually much stronger than previously thought.

Theorem 2. Let $\mathcal{E}^i$ be an economy where $\sigma(e_i) \subseteq \mathcal{K}_i^S$, for each $i \in S$ and $S \subseteq I$[11]. Any (interim) maximin value allocation of $\mathcal{E}^i$ is transfer (interim) maximin coalitionally incentive compatible.

Proof. Let $x \in L_X$ be an interim maximin value allocation for $\mathcal{E}^i$. According to Remark 6, x is a Pareto optimal allocation. Suppose that x is not transfer (interim) maximin coalitional incentive compatible. Then, there exist a coalition $S \subset I$, two different states a and b of Ω and $z \in \Re_+^{mS}$, where $\sum_{i \in S} z_i = \sum_{i \in S}[e_i(a) + x_i(b) - e_i(b)]$, such that

(i) $\mathcal{K}_i^I(a) = \mathcal{K}_i^I(b)$ if and only if $i \in I \setminus S$,

(ii) $u_i(x_i(a)) = u_i(x_i(b))$, for all $i \in I \setminus S$,

(iii) $\underline{u}_i(a, y_i, I) > \underline{u}_i(a, x_i, I)$ for all $i \in S$, where for all $i \in S$

$$y_i(\omega) = \begin{cases} z_i, \text{ if } \omega = a \\ x_i(\omega), \text{ otherwise.} \end{cases}$$

We will construct a feasible allocation $w \in L_X$ for $\mathcal{E}^i$ that Pareto improves upon x at the state a, which is a contradiction. To this end, for each $i \in I \setminus S$ define the assignment w_i, such that

[11]This is the $\mathcal{K}_i^S$ - measurability assumption imposed on e_i, $i \in S \subseteq I$, which means that the endowment itself serves as one informative signal for individuals in any coalition. In the particular case where for each $i \in S \subseteq I$ we have that $\mathcal{F}_i \subseteq \mathcal{K}_i^S$, i.e., whenever individuals use at least their private information in each coalition, the private information ($\mathcal{F}_i$ -) measurability of endowments secures their $\mathcal{K}_i^S$ - measurability as well. Indeed, $\sigma(e_i) \subseteq \mathcal{F}_i \subseteq \mathcal{K}_i^S$.

$$w_i(\omega) = \begin{cases} e_i(a) + x_i(b) - e_i(b), & \text{if } \omega = a \\ x_i(\omega), & \text{otherwise,} \end{cases}$$

and for each $i \in S$ define the assignment w_i, such that

$$w_i(\omega) = y_i(\omega) = \begin{cases} z_i, & \text{if } \omega = a \\ x_i(\omega), & \text{otherwise.} \end{cases}$$

By construction, we have that $w_i \in L_{X_i}$ for all $i \in S$. Moreover, by the measurability of the endowments and (i) above, we have for all $i \in I \setminus S$ that $e_i(a) = e_i(b)$, so

$$w_i(a) = e_i(a) + x_i(b) - e_i(b) = x_i(b). \tag{12}$$

Therefore, $w_i \in L_{X_i}$ for all $i \in I \setminus S$ as well, so that finally $w \in L_X$. For this allocation we have at the state a: $w_i(a) = e_i(a) + x_i(b) - e_i(b)$, for all $i \in I \setminus S$ and $w_i(a) = z_i$, for all $i \in S$. Therefore,

$$\begin{aligned} \sum_{i \in I} w_i(a) &= \sum_{i \in I \setminus S} [e_i(a) + x_i(b) - e_i(b)] + \sum_{i \in S} z_i \\ &= \sum_{i \in I \setminus S} [e_i(a) + x_i(b) - e_i(b)] + \sum_{i \in S} e_i(a) + \sum_{i \in S} x_i(b) - \sum_{i \in S} e_i(b) \\ &= \sum_{i \in I} e_i(a) + \sum_{i \in I} x_i(b) - \sum_{i \in I} e_i(b) \\ &= \sum_{i \in I} e_i(a), \end{aligned}$$

where the last equality follows from the feasibility of the interim maximin value allocation x at the state b. That is, w is feasible in the state $a \in \Omega$, which is an arbitrary state, so w is feasible for all other states as well. We conclude, therefore, that w is a feasible allocation for $\mathcal{E}^i$. Next, by (12) and condition (ii) above, it follows that $u_i(w_i(a)) = u_i(x_i(b)) = u_i(x_i(a))$, for all $i \in I \setminus S$. This in turn implies the following for all $i \in I \setminus S$:

$$\begin{aligned}\underline{u}_i(a, w_i, I) &= \min_{\omega \in \mathcal{K}_i^I(a)} u_i(w_i(\omega)) \\ &= \min_{\omega \in \mathcal{K}_i^I(a) \setminus \{a\}} u_i(x_i(\omega)) \\ &\geq \min_{\omega \in \mathcal{K}_i^I(a)} u_i(x_i(\omega)) \\ &= \underline{u}_i(a, x_i, I).\end{aligned}$$

Therefore, $\underline{u}_i(a, w_i, I) \geq \underline{u}_i(a, x_i, I)$, for all $i \in I \setminus S$, while by (iii) above $\underline{u}_i(a, w_i, I) > \underline{u}_i(a, x_i, I)$, for all $i \in S$, which (according to Definition 3) contradicts the Pareto optimality of the interim maximin value allocation x. □

1.6 Conclusion

In this chapter we generalized the model and the non-Bayesian value allocation concept of de Castro and Yannelis (2009). We then studied the normative properties of the maximin value allocation (existence and maximin individual rationality, Pareto efficiency, incentive compatibility), by enriching all the respective definitions.

We first constructed (in section 2) the MDIE, at which the agents had ex ante or interim maximin preferences, parametrized with respect to group membership. We fixed a finite set of agents and states, but we allowed for a commodity space of either finite or infinite dimensions. Agents were allowed to cooperate and form coalitions. Within coalitions, information exchange protocols determined the availability of their members' information..

Upon a MDIE associated with a Shapley-value-solved maximin TU game, we defined (in section 3) two versions of the maximin value allocation, the ex ante and the interim one. Both of them were proposed as both an analytical extension and a conceptual refinement of the Bayesian value allocation notions of Krasa and Yannelis (1994, 1996). When (in section 4) the MDIE had an infinite dimensional commodity space, it was sufficient to prove existence of the ex ante maximin value allocation only. On the other hand, in section 5, it was enough to show how maximin efficiency leads to maximin incentive compatibility for the interim maximin value allocation only.

In general, due to its fairness characteristic, the Shapley (1969) value allocation is an attractive general equilibrium concept. Given an economy with cooperating agents, it is a commodities' outcome that rewards and reflects the contribution of agents to all the coalitions they participate. In a differential information economy, particular attention is usually drawn to the Krasa and Yannelis' (1994, 1996) private (information) Bayesian value allocation. That is because it additionally manages to capture and measure the informational superiority/advantage of an agent, in terms of the higher consumption quantity this agent gets allocated with.

The private maximin value allocation is not a trivial variation (extension) of its Bayesian counterpart. The short discussion that follows contains conclusive remarks that substantiate this statement.

The private (Bayesian) value allocation, despite its enhanced interpretative ability and despite the fact that it is equipped with desirable general equilibrium properties, acquires one distinct shortcoming with considerable social impact: it is an informationally restricted (i.e., second best) efficient incentive compatible allocation. Due to the Bayesian nature of the private value allocation, this feature that accompanies that notion is unavoidable. The private value allocation, a notion with built in efficiency, is incentive compatible viable only with the private information measurability requirement. However, while the private information measurability deals successfully with incentive compatibility issues, it reduces the efficiency of this concept. Therefore, the private value allocation exists under the well known (Bayesian) conflict between first best efficiency and incentive compatibility.

The private maximin value allocation, on the other hand, has the inherent ability to overcome and amend this problematic issue that the private value allocation is endowed with. To begin with, the private maximin value allocation is no longer tied with the efficiency reducing private information measurability condition, so that efficiency is increased beyond the second (and up to the first) best sense. Further, the private maximin value allocation is a 'less' informationally restricted efficient incentive compatible allocation. Maximin efficiency implies maximin incentive compatibility, without needing the private information measurability assumption on agents' consumption. The notions of efficiency and incentive compatibility can coexist in non-private information measurable allocations.

The results in this paper strengthen these arguments in two directions:

First, by pointing out that the possibility of information exchange within coalitions does not affect the existence of value allocations. Indeed, in the Bayesian approach, some information sharing rules could result to non-existence of value allocations, notably when the endowment of coalitions failed to be measurable with respect to the information available to individuals, in which case characteristic functions of coalitions failed to be well defined. By contrast, such problems do not arise in the non - Bayesian approach, for any arbitrary rule of information exchange.

Second, by pointing out that in the non-Bayesian context, and even when allowing for inter coalition information exchange, value allocations are incentive compatible in a stronger sense than the one detected elsewhere. In the Bayesian approach, whenever information exchange was allowed within coalitions, one could easily construct examples where value allocations failed to be incentive compatible even in very weak terms. In the non-Bayesian approach, by contrast, although the set of incentive compatible allocations is restricted because transfers between the members of the cheating coalitions are allowed, value allocations are incentive compatible (in a strong sense) for any arbitrary information sharing rule.

One interesting extension of this line of work is towards a non-finite number of states. This is a nontrivial problem because it involves delicate technical issues.

In particular, the space of allocations must be carefully chosen, with a view to a suitable duality with a space of measures, that will allow a meaningful minimization of expected utility over probability measures, as required in the definition of (ex ante) maximin preferences.

Chapter 2

Countable Set of States

2.1 Preliminaries

De Castro and Yannelis (2009) introduce a private information maximin value allocation notion in an exchange economy with partition type differential information and ambiguity. The economy was assumed to be finite. That is, containing a finite number of states and commodities, while being comprised of a finite number of non-Bayesian agents.

Angelopoulos and Koutsougeras (2014) enrich and generalize this maximin general equilibrium concept in three directions. First, they assume arbitrary information exchange protocols within agents' coalitions, so that agents' prior beliefs and maximin utilities become coalitional dependent, accommodating any kind of information for individuals as group members, not necessarily and only their private information. In this way, the private (information) maximin value allocation becomes just a special (yet, notable) case. Second, with these conceptual tools in hand, the authors of this paper introduce both ex ante and interim maximin value allocation notions. Finally, they allow for an infinite dimensional commodity space in the underlying economy.

In this chapter we restrict our attention to the (ex ante) private maximin value allocation only, which is worth pursuing for the following sequence of reasons of increasing significance:

To begin with, it is a Shapley (1969) value allocation notion, hence it is a fair cooperative equilibrium concept. Indeed, each agent is assigned with a utility level, which is the expected marginal contribution of this agent to all the coalitions he participates.

Thereafter, it is a direct extension of the private (Bayesian) value allocation of Krasa and Yannelis (1994, 1996). Thus, within it, fairness is strengthened; better informed agents are assigned with higher utility.

Unlike the private value allocation, however, the maximin value allocation exists without being necessarily tied with the private information measurability assumption on agents' net trades (see in Angelopoulos and Koutsougeras, 2014). Thereby, all the maximin (Pareto efficient) value allocations are taken into account and there is no efficiency loss in equilibrium.

More importantly, for the accomplishment of incentive compatibility of this notion[12], less informational measurability restrictions are imposed. That is, assuming only informational measurable initial endowments, every maximin value allocation is maximin efficient incentive compatible (see again in Angelopoulos and Koutsougeras, 2014). In that sense, no conflict between efficiency and incentive compatibility occurs.

Many real life economies are better modeled and explained via a non-finite number of states, countable or uncountable depending on the case. In the maximin preferences framework, nevertheless, this poses elemental analytical obstacles: agents' maximin utilities (minimized over the states) are not well defined to begin with. Existence issues of the maximin value allocation with a non-finite set of states were raised in Angelopoulos and Koutsougeras (2014). This chapter attempts to deal with this matter when, specifically, the set of states is countable.

In sections 2 and 3 the appropriate analytical framework is established; the ambiguous economy and the maximin value allocation are, respectively, defined. The existence result of this chapter is provided in section 4. Existence of the maximin value allocation is proved by truncating the countability of the set of states; a technique also adopted in He and Yannelis (2013), but for the maximin Walrasian expectations equilibrium. In section 5 we conclude.

[12]Incentive compatibility is a contract theoretic property of an allocation. It is originated by the contracts' ex post fulfillment issue. See, for example, in the introduction of de Castro et al (2011) or of Angelopoulos and Koutsougeras (2014).

2.2 The Ambiguous Economy

The ambiguous economy is a two (ex ante - ex post) period exchange economy, within which the non-Bayesian asymmetrically informed agents are, in particular, maximin ambiguity averse. There is a finite number of individuals (maximin agents) participating into the ambiguous economy, who are allowed to cooperate and form alliances. Their trade (contract writing) occurs in Euclidean spaces. They write their consumption contacts facing, specifically, countable infinitely many states of nature of the world.

We construct such an economy following the footsteps of de Castro and Yannelis (2009), de Castro et al. (2011, 2012), Angelopoulos and Koutsougeras (2014) and He and Yannelis (2013).

$I = \{1, 2, ..., s\}$ is the finite set of agents of the economy and an $S \in \mathcal{P}(I)$ is a coalition of agents. There is a finite number, l, of commodities traded in the market and $\mathbb{R}^l$ is the economy's commodity space. The underlying state contingent uncertainty, state dependent randomness and informational structure in the economy are summarized by a countable set of states $\Omega = \mathbb{N} = \{\omega_n\}_{n\in\mathbb{N}}$. $\mathcal{F} = \mathcal{P}(\Omega)$ is the natural σ - algebra of Ω, containing all the events of the economy.

$\mathbb{R}^l$, Ω and $\mathcal{F}$ are common (factors) to all the economy's agents. Agents, particularly, are assigned with the following differential characteristics:

1. *Informational sets.* Π_i is the agent's i partition of Ω and $\mathcal{F}_i \subseteq \mathcal{F}$ is the same agent's σ - algebra, generated by Π_i. Both of them interchangeably represent the private (or asymmetric) information of the i agent. It is specifically assumed that the agent's i partition of $\mathbb{N}$ contains countable infinitely many finite (only) sets. That is, the states between which the agent i cannot distinguish are always of finite number. In other words, the privately informed agent i cannot be "too uninformed". Finally, it is maintained that the agent i retains his private information within his coalitions. There are no underlying information exchange protocols inside agents' groups. Consequently, the agents' priors and preferences are not coalitional dependent (of course, one can proceed as in Angelopoulos and Koutsougeras

(2014): allow individuals to exchange information within their groups by obeying to arbitrary information sharing rules and, thereby, examine generalized informational aspects of the (ex ante) maximin value allocation).

2. *Prior beliefs*. The (σ - additive) probability measure $q_i : \mathcal{F}_i \to [0,1]$ is the informationally restricted private prior of the i agent. By definition, q_i satisfies the following incompleteness property: $q_i(B_i)$ may be unknown for a $\emptyset \neq B_i \subset A_i \in \mathcal{F}_i$, even though $q_i(A_i)$ is provided (known) by q_i. That is, the economy's agents may be unable to completely form a prior belief. In other words, agents face ambiguity.

3. *Preferences*, *consumption sets* and *endowments*. With $u_i(\omega) := u_i^\omega : \mathbb{R}_+^l \to \mathbb{R}_+$, $\omega \in \Omega$, we denote the agent's i random state dependent (r.s.d.) utility function(s), representing the same agent's preferences (over r.s.d. consumption). For each agent i, u_i^ω, $\omega \in \Omega$, is taken to be continuous and concave. Additionally, for each agent i, the family of utility functions $\{u_i^\omega : \mathbb{R}_+^l \to \mathbb{R}_+, \omega \in \Omega\}$ is assumed to be uniformly bounded. An $x_i(\omega) \in \mathbb{R}_+^l$, $\omega \in \Omega$, is a r.s.d. consumption bundle of the i agent. Then, an $\{x_i(\omega)\}_{\omega\in\Omega} := x_i \in (\mathbb{R}_+^l)^\infty$ is a r.s.d. consumption plan of the i agent. Followingly, $X_i = \{x_i : x_i \in (\mathbb{R}_+^l)^\infty\} \subset (\mathbb{R}_+^l)^\infty$ is the (feasible) r.s.d. consumption set of this agent, which is assumed to be closed and convex. An $e_i(\omega) \in \mathbb{R}_+^l$, $\omega \in \Omega$, is a r.s.d. initial endowment of the agent i and $\{e_i(\omega)\}_{\omega\in\Omega} := e_i \in \ell_+^1 \cap X_i \neq \emptyset$ is the same agent's r.s.d. initial endowment plan[13]. Agents' preferences over r.s.d. consumption bundles give rise to their maximin preferences as well, over r.s.d. consumption plans. Agent's i (ex ante) maximin preferences are represented by his (*ex ante*) *maximin* (*expected*) *utility* function $\upsilon_i : X_i \to \mathbb{R}_+$, which is defined by

$$\upsilon_i(x_i) = \sum_{A_i \in \Pi_i} [\min_{\omega \in A_i} u_i^\omega(x_i(\omega))] q_i(A_i).$$

[13]Note that $(\mathbb{R}^l)^\infty = \mathbb{R}^{l\times\infty}$ is an infinite dimensional Euclidean space (i.e., of dimension $l.\infty = \infty$). It is the space of all $l \times \infty$ matrices with real entries. Equivalently viewed, it is the space of all sequences of real $l \times 1$ vectors. Henceforth, ℓ^1 is thought of as the subspace of $(\mathbb{R}^l)^\infty$ containing all the summable sequences of real $l \times 1$ vectors.

Agents are thought to be rational, that is, utility maximizers. Hence, agents are maximin utility maximizers as well. Lastly, agents are assumed to have monotone (increasing) maximin preferences[14].

Remark 1. The formulation above was established in de Castro and Yannelis (2010) with a finite Ω. With a countable Ω, He and Yannelis (2013) (well) define and use it as well, in a close (but different) to ours manner[15]. In our setting, this format is well defined because, for each agent i: (i) each $A_i \in \Pi_i$ is finite (so the minimum in the expression above is attained) and (ii) $\{u_i^\omega : \mathbb{R}_+^l \to \mathbb{R}_+,\ \omega \in \Omega\}$ is uniformly bounded (so the the sum above is finite).

Remark 2. It is worth observing that if (i) agents are assumed to have private information measurable r.s.d. utility functions, i.e., *if for any* $i \in I$ *we have that:* $\omega, \bar{\omega} \in A_i \in \Pi_i$, *then* $u_i^\omega(x_i(\omega)) = u_i^{\bar{\omega}}(x_i(\bar{\omega}))$ and (ii) agents' priors are assumed to be non-informationally restricted and of full support, the previous formula reduces to the standard Bayesian (or subjective) expected utility. As it was to be expected: (i) the private information measurability condition is necessary (and unavoidable) in the Bayesian context, while (ii) Bayesian agents (i.e., agents with Bayesian preferences) are accommodated in our model as a special case.

We have finally derived the following (*ex ante*) *ambiguous economy*:

$$\mathcal{E} = \{\ (\mathbb{R}^l)^\infty\ ;\ (\Omega, \mathcal{F})\ ;\ (\ [\ \mathcal{F}_i(\Pi_i), X_i, e_i, v_i(u_i^\omega, q_i)\ ] : i \in I\ )\ \}.$$

A r.s.d. allocation (contract) of $\mathcal{E}$ is a list of all the economy's agents' r.s.d. consumption plans. It is notated as

[14] First the column wise and then the coordinate wise ordering is assumed on $(\mathbb{R}^l)^\infty$.

[15] He and Yannelis (2013) define (on $\Omega \times \mathbb{R}_+^l$) one r.s.d. utility function for each agent, instead of defining a class of r.s.d. utility functions (one for each state) for each agent. They also use a different assumption to derive the fact that all the elements of an agent's partition are finite. The essential difference, however, is that He and Yannelis (2013) allow for the maximin utility of an agent to be infinity.

$$x = (x_1, x_2, ..., x_i, ..., x_s) \in X = \prod_{i \in I} X_i \subset ((\mathbb{R}^l_+)^\infty)^s$$

and is said to be *feasible* if

$$\sum_{i \in I} x_i = \sum_{i \in I} e_i \iff \sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega), \text{ for all } \omega \in \Omega.$$

According to the feasibility condition, the market (i.e., the economy) clears without free disposal.

2.3 The Maximin Value Allocation

Upon the ambiguous economy constructed in the previous section, we can now define the (ex ante private) maximin value allocation of Angelopoulos and Koutsougeras, 2014.

First, however, we have to define a Shapley (1953)-value-solvable (*ex ante*) *Maximin Transferable Utility* (MTU) *game*.

Let the MTU game $\Gamma = (I, V_{\lambda,\upsilon}, Sh)$. Γ is a cooperative (coalitional) game, allowing for side payments among its finitely many players $1, 2, ..., s \in I$. Within Γ, the players' payoffs are identified with maximin utilities and:

1. υ is the set of all the players' maximin utility functions. The players' maximin utilities $\upsilon_i(\cdot)$, $i \in I$, become common scaled (hence, interpersonally comparable) and transferable by a personal factor $\lambda_i \geq 0$ assigned to each player i, such that not all λ_i are equal to zero. $\lambda \in \mathbb{R}^s_+ \setminus \{0\}$ is the vector of all the players' factors. At the same time, λ_i is the player's i weight to Γ, so that $\sum_{i \in I} \lambda_i = 1$. Finally, $V(\lambda, \upsilon) := V_{\lambda,\upsilon} : 2^I \to \mathbb{R}_+$ is a set function, called the maximin characteristic function of Γ. It measures the gain (i.e., the maximin utility level) of every coalition

$S \subseteq I$[16]. The $V_{\lambda,\upsilon}$ of Γ must have a specific (any) functional form, satisfying monotonicity, superadditivity and normalized to become zero for the empty set (coalition).

2. $\mathcal{V}$ is the class of all the $V_{\lambda,\upsilon}$ of Γ. Then, $Sh : \mathcal{V} \to \mathbb{R}^s_+$ is the maximin Shapley value function of Γ, assigning: (i) to Γ the maximin Shapley (1953) value $Sh(V_{\lambda,\upsilon})$, which is a vector of $\mathbb{R}^s_+$ and (ii) to each player i of Γ the respective coordinate $Sh_i(V_{\lambda,\upsilon})$ of the previous vector. The latter is the maximin Shapley value of the i player, a proposed (positive) maximin utility level to be received by this player. For each player i of Γ, his $Sh_i(V_{\lambda,\upsilon})$ is given by the formula

$$Sh_i(V_{\lambda,\upsilon}) = \sum_{S \subseteq I, i \in S} \tfrac{(|S|-1)!(|I|-|S|)!}{|I|!}[V_{\lambda,\upsilon}(S) - V_{\lambda,\upsilon}(S \setminus \{i\})], \text{ where } |I| = s,$$

which conveys the following interpretation: each player i is assigned with a maximin utility $Sh_i(V_{\lambda,\upsilon}) \in \mathbb{R}_+$, which is the expected marginal contribution of this player to all the different(ly sized) coalitions S he becomes a member of. Therefore, for any $V_{\lambda,\upsilon} \in \mathcal{V}$, $Sh(V_{\lambda,\upsilon})$ is a fair solution to Γ. $Sh(V_{\lambda,\upsilon})$ is a normative solution as well, satisfying: (i) (group rationality) $\sum_{i \in I} Sh_i(V_{\lambda,\upsilon}) = V_{\lambda,\upsilon}(I)$ and (ii) (individual rationality) $Sh_i(V_{\lambda,\upsilon}) \geq V_{\lambda,\upsilon}(\{i\})$, for all $i \in I$.

Now, by associating $\mathcal{E}$ with Γ and by appropriately defining the $V_{\lambda,\upsilon}$ of Γ, i.e., by attaching a specific functional form to $V_{\lambda,\upsilon}$, we define the maximin value allocation of $\mathcal{E}$ as follows:

Definition. Let the $V_{\lambda,\upsilon}$ of Γ be given for every coalition $S \subseteq I$ by

$$V_{\lambda,\upsilon}(S) = \max_{x_i \in X_i} \sum_{i \in S} \lambda_i \sum_{A_i \in \Pi_i} [\min_{\omega \in A_i} u_i^{\omega}(x_i(\omega))] q_i(A_i),$$

$$\text{subject to } \sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega), \text{ for all } \omega \in \Omega.$$

[16]Hence, $V_{\lambda,\upsilon}$ measures the worth or power of every coalition.

Then, an allocation $x \in X$ of $\mathcal{E}$ is said to be an (*ex ante private*) *maximin value allocation* if the following two conditions are satisfied:

1. $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$, for all $\omega \in \Omega$.

2. For all $i \in I$, we have that $\lambda_i \sum_{A_i \in \Pi_i} [\min_{\omega \in A_i} u_i^{\omega}(x_i(\omega))] q_i(A_i) = Sh_i(V_{\lambda,v})$,

   for a $\lambda_i \geq 0$ for all i, such that $\sum_{i \in I} \lambda_i = 1$.

The following series of remarks is in order:

Remark 3. The maximin value allocation is a cardinal value allocation. It is easy to show that the group rationality (individual rationality, respectively) of $Sh(V_{\lambda,v})$ guarantees the maximin Pareto efficiency (maximin individual rationality, respectively) of the maximin value allocation. Let $\Pi_i(\omega)$ be the element of the agent's i partition Π_i containing ω, the state realized in the economy's second period. Then, by appropriately adjusting Theorem 2 of Angelopoulos and Koutsougeras (2014), it is straightforward to prove that (i) the maximin efficiency and (ii) the condition *if for any $i \in I$ we have that: $\omega, \bar{\omega} \in A_i \in \Pi_i$, then $e_i(\omega) = e_i(\bar{\omega})$*[17] secure the transfer maximin coalitional incentive compatibility of the maximin value allocation[18].

Remark 4. It can be easily verified that the $V_{\lambda,v}$ of Γ is (indeed) monotone, superadditive and becomes zero for the empty coalition (of none agent). A coalition S obtains for its members a total gain of $V_{\lambda,v}(S)$. By the way $V_{\lambda,v}$ is defined, it is secured that every coalition S of $\mathcal{E}$ acts rationally, i.e., maximizes its maximin utility, subject to the feasibility of consumption within S. The coalition's S maximin

[17]This is the private information measurability assumption, imposed on the agents' initial endowments. Private information measurable agents' consumption is not, on the other hand, demanded.
[18]The reader is referred to Angelopoulos and Koutsougeras (2014), for the definition of the notions of maximin: Parato optimality, individual rationality and (transfer coalitional) incentive compatibility of an allocation. De Castro and Yannelis (2010) first introduced the maximin version of these properties an allocation should (desirably) satisfy. Angelopoulos and Koutsougeras (2014) enrich them and provide a stronger notion of incentive compatibility.

utility is its agents' aggregate common scaled (and weighted) maximin utility.

Remark 5. When agents have monotone maximin preferences, it can be easily understood that the following property is valid for every (feasible and maximin individually rational Pareto optimal) maximin value allocation of $\mathcal{E}$: *Every coalition maximizes its maximin utility subject to its consumption constraints* if and only if *every agent in a coalition independently maximizes his maximin utility subject to the feasibility of consumption within this coalition.* This allows us to deduce that if $x \in X$ is a maximin value allocation of $\mathcal{E}$, then the $V_{\lambda,v}$ of Γ ends up being defined by

$$V_{\lambda,v}(S) = \max_{x_i \in X_i} \sum_{i \in S} \lambda_i \sum_{A_i \in \Pi_i} [\min_{\omega \in A_i} u_i^{\omega}(x_i(\omega))] q_i(A_i) =$$

$$= \sum_{i \in S} \lambda_i \max_{x_i \in X_i} \sum_{A_i \in \Pi_i} [\min_{\omega \in A_i} u_i^{\omega}(x_i(\omega))] q_i(A_i) =$$

$$= \sum_{i \in S} \lambda_i \sum_{A_i \in \Pi_i} [\min_{\omega \in A_i} \left(\max_{x_i(\omega)} u_i^{\omega}(x_i(\omega)) \right)] q_i(A_i),$$

$$\text{subject to } \sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega), \text{ for all } \omega \in \Omega,$$

$$\text{that is, finally, by } \; V_{\lambda,v}(S) = \sum_{i \in S} \lambda_i \sum_{A_i \in \Pi_i} [\min_{\omega \in A_i} u_i^{\omega}(x_i^{\star}(\omega))] q_i(A_i).$$

Remark 6. Within any maximin value allocation, a coalition S cannot redistribute among its members its allocated maximin utility. That is to say, side payments are not allowed within a maximin value allocation[19]. Finally, although an agent's i λ_i may be zero, the maximin value allocation may still generate a strictly positive maximin utility for this agent. However, an agent's zero weight leads, irrespective of the agent's maximin utility level, to the same agent's zero Shapley value.

Remark 7. Assuming that: (i) the agents' maximin expected utilities are specified to Bayesian ones as in Remark 2 and (ii) the agents' consumption and initial endowments (i.e., net trades) are, specifically, private information measurable, the

[19]Despite it being associated with a λ - transferable utility game.

definition above reduces to the one of the private (Bayesian) value allocation of Krasa and Yannelis (1994).

2.4 Existence

The analysis has revealed that although we begun with a finite dimensional Euclidean space, $\mathbb{R}^l$, as the economy's commodity space, $\mathcal{E}$ ended up being defined with the infinite dimensional Euclidean commodity space $(\mathbb{R}^l)^\infty$. $(\mathbb{R}^l)^\infty$ is a separable, partially ordered vector space. The product topology, that is, the topology of point wise (or coordinate wise) convergence, can be supplied to $(\mathbb{R}^l)^\infty$. $(\mathbb{R}^l)^\infty$, however, does not carry any norm. Thus, the standard separable Banach space methods are not applicable for existence purposes, now that the economy is underpinned by countably many states of nature.

Other techniques, therefore, have to be adopted. Towards this objective, instead of truncating the infinite dimension of the commodity space à la Bewley (1972), we truncate (as in He and Yannelis, 2013) the countability of the set of states. By doing so, the infinite - dimensionality of the commodity space is automatically reduced (to sequential finite dimensions) as well.

To be more precise: (i) We sequentially reduce the economy into its Ω - finite traces, (ii) we prove existence of the maximin value allocation in each one of these truncated economies and (iii) we use limiting arguments to prove existence of the maximin value allocation under the desirable infinity of Ω.

Therefore, given the economy with countable states, we first have to appropriately define a sequence of (truncated) economies with finitely many states. We begin by considering the (countable) set of all the finite subsets of $\Omega = \mathbb{N}$. We then assume any sequence E_n, $n \in \mathbb{N}$, of (not all the) finite subsets of Ω, satisfying the following condition:

$(\mathcal{C})$ For each $n \in \mathbb{N}$, for each $i \in I$ and for each $A_i \in \Pi_i$, we have that either $A_i \cap E_n = \emptyset$ or $A_i \subseteq E_n$.

Finally, for each $n \in \mathbb{N}$, we construct a truncated ambiguous economy $\mathcal{E}^n$ with a finite number of states, such that the finite set of states of $\mathcal{E}^n$ coincides with the term E_n of the aforementioned sequence. Given $\mathcal{E}$, therefore, we construct a sequence $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ of ambiguous economies containing a finite set of states as follows:

For each $n \in \mathbb{N}$, we define the economy $\mathcal{E}^n$ as

$$\mathcal{E}^n = \{ \ (\mathbb{R}^l)^{|\Omega^n|} \ ; \ (\Omega^n, \mathcal{F}^n) \ ; \ (\ [\ \mathcal{F}_i^n(\Pi_i^n), X_i^n, e_i^n, \upsilon_i^n((u_i^\omega)^n, q_i^n) \ ] : i \in I \ ) \ \},$$

where $\Omega^n = E_n$, $\mathcal{F}^n = \{A \subseteq \Omega^n : A \in \mathcal{F}\} \subset \mathcal{F}$ [20] and for each agent i:

1. The private informational sets Π_i^n and $\mathcal{F}_i^n$ are now finite and defined as

$$\Pi_i^n = \{A_i \subseteq \Omega^n : A_i \in \Pi_i\} \subset \Pi_i \text{ and } \mathcal{F}_i^n = \mathcal{F}_i^n(\Pi_i^n) = \{A_i \subseteq \Omega^n : A_i \in \mathcal{F}_i\} \subset \mathcal{F}_i,$$

i.e., they are the restrictions of Π_i and $\mathcal{F}_i$ to Ω^n respectively[21] [22].

2. The private prior q_i^n is now a finitely additive probability measure, defined as $q_i^n = q_i|_{\Omega^n}$, satisfying $q_i^n(A_i \in \Pi_i^n) = q_i(A_i \in \Pi_i)$.

3. $X_i^n = X_i|_{\Omega^n} = \{\{x_i(\omega)\}_{\omega\in\Omega^n} := x_i^n \mid x_i^n \in (\mathbb{R}_+^l)^{|\Omega^n|}\} \subset (\mathbb{R}_+^l)^{|\Omega^n|}$, so that $X_i^n \subset X_i$; observe that X_i^n is closed and convex, while it is further assumed that $\{e_i(\omega)\}_{\omega\in\Omega^n} := e_i^n \in X_i^n$.

4. $\{(u_i^\omega)^n : \mathbb{R}_+^l \to \mathbb{R}_+, \omega \in \Omega^n\} \subset \{u_i^\omega : \mathbb{R}_+^l \to \mathbb{R}_+, \omega \in \Omega\}$, so that, for each $\omega \in \Omega^n$, $(u_i^\omega)^n = u_i^\omega : \mathbb{R}_+^l \to \mathbb{R}_+$(which is continuous, concave and has a uniform bound).

[20]That is, $\mathcal{F}^n$ is the power set (algebra) of Ω^n, which is the restriction of the power set (σ - algebra) $\mathcal{F}$ of Ω to Ω^n.

[21]Notice that condition $\mathcal{C}$ (i.e., the criterion of choosing the sequence $E_n = \Omega^n$, $n \in \mathbb{N}$) guarantees that Π_i^n is well defined, that is to say, Ω^n is well partitioned by every agent i. Consequently, condition $\mathcal{C}$ ensures that every truncated ambiguous economy $\mathcal{E}^n$ is well defined.

[22]Note also that $\mathcal{F}_i^n \subseteq \mathcal{F}^n$.

5. $\upsilon_i^n : X_i^n \to \mathbb{R}_+$, defined by $\upsilon_i^n(x_i^n) = \sum_{A_i \in \Pi_i^n} [\min_{\omega \in A_i} u_i^\omega(x_i(\omega))] q_i^n(A_i)$.

A r.s.d. allocation of $\mathcal{E}^n$ is denoted as

$$x^n = (x_1^n, x_2^n, ..., x_i^n, ..., x_s^n) \in X^n = \prod_{i \in I} X_i^n \subset ((\mathbb{R}_+^l)^{|\Omega^n|})^s$$

and is said to be feasible, if

$$\sum_{i \in I} x_i^n = \sum_{i \in I} e_i^n \iff \sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega), \text{ for all } \omega \in \Omega^n$$

We now state and prove the following existence result:

Theorem. A maximin value allocation exists in $\mathcal{E}$.

Proof. step 1

We prove that a maximin value allocation exists in $\mathcal{E}^n$, $n \in \mathbb{N}$. Wlog, we write $\mathcal{E}^n$ as $\mathcal{E}^n = \{ \ (\mathbb{R}^l)^{|\Omega^n|} \ ; \ (X_i^n, e_i^n, \upsilon_i^n) : i \in I \ \}$, $n \in \mathbb{N}$, which can be directly related with the deterministic economy $\mathcal{E}' = \{\mathbb{R}^{k<\infty} \ ; \ (\mathbb{X}_i, \epsilon_i, w_i) : i \in I\}$. According to Emmons and Scafuri (1985), a cardinal value allocation exists in $\mathcal{E}'$ if for each agent i: (i) $\epsilon_i \in \mathbb{X}_i \subset \mathbb{R}_+^k$, (ii) $\mathbb{X}_i$ is closed, below bounded and convex in $\mathbb{R}^k$ and (iii) $w_i : \mathbb{X}_i \to \mathbb{R}_+$ is continuous and concave. Hence, a (cardinal) maximin value allocation exists in $\mathcal{E}^n$, $n \in \mathbb{N}$, if the same conditions are accordingly satisfied. Consider the Euclidean norm $|| \cdot ||$, the standard topology and the point wise ordering on any finite dimensional Euclidean space. Fix a $n \in \mathbb{N}$ and an agent $i \in I$ of the (fixed) economy $\mathcal{E}^n$. The continuity of u_i^ω, $\omega \in \Omega^n$, secures continuity for υ_i^n as well. Indeed: u_i^ω, $\omega \in \Omega^n$, is continuous on $\mathbb{R}_+^l$ iff for all $x_i(\omega) \in \mathbb{R}_+^l$, $\omega \in \Omega^n$ and for all $\epsilon > 0$, there exists $\delta(x_i(\omega), \epsilon) > 0$, such that for all $y_i(\omega) \in \mathbb{R}_+^l$, $\omega \in \Omega^n$, with $0 < ||y_i(\omega) - x_i(\omega)|| < \delta$, we have that $0 < |u_i^\omega(y_i(\omega)) - u_i^\omega(x_i(\omega))| < \epsilon$. Now, assume that υ_i^n is continuous on X_i^n. This would mean that for all $x_i^n \in X_i^n$ and for all $\epsilon > 0$, there exists $\delta(x_i^n, \epsilon) > 0$, such that for all $y_i^n \in X_i^n$, with $0 < ||y_i^n - x_i^n|| < \delta$, we have that

$$0 < \mid \sum_{A_i \in \Pi_i^n} [\min_{\omega \in A_i} u_i^\omega(y_i(\omega))] q_i^n(A_i) - \sum_{A_i \in \Pi_i^n} [\min_{\omega \in A_i} u_i^\omega(x_i(\omega))] q_i^n(A_i) \mid < \epsilon, \text{ or}$$

$$0 < \mid \sum_{A_i \in \Pi_i^n} [\min_{\omega \in A_i} u_i^\omega(y_i(\omega)) - \min_{\omega \in A_i} u_i^\omega(x_i(\omega))] \; q_i^n(A_i) \mid < \epsilon, \text{ or}$$

$$0 < \sum_{A_i \in \Pi_i^n} |\min_{\omega \in A_i} u_i^\omega(y_i(\omega)) - \min_{\omega \in A_i} u_i^\omega(x_i(\omega))| \; q_i^n(A_i) < \epsilon, \text{ or}$$

$$0 < \sum_{A_i \in \Pi_i^n} \min_{\omega \in A_i} \mid u_i^\omega(y_i(\omega)) - u_i^\omega(x_i(\omega)) \mid q_i^n(A_i) < \epsilon.$$

From the last expression it is implied that for each $A_i \in \Pi_i^n$, we have that

$$0 < \min_{\omega \in A_i} \mid u_i^\omega(y_i(\omega)) - u_i^\omega(x_i(\omega)) \mid < \epsilon.$$

But this is true, because $0 < |u_i^\omega(y_i(\omega)) - u_i^\omega(x_i(\omega))| < \epsilon$, $\omega \in \Omega^n$. The concavity of u_i^ω, $\omega \in \Omega^n$, implies concavity for v_i^n on X_i^n as well (besides, by assumption, X_i^n is convex in $(\mathbb{R}^l)^{|\Omega^n|}$). Indeed: Since u_i^ω, $\omega \in \Omega^n$, is concave on $\mathbb{R}_+^l$, it holds that for every $x_i(\omega), y_i(\omega) \in \mathbb{R}_+^l$, $\omega \in \Omega^n$ and for every $t \in [0,1]$ we have that

$$u_i^\omega(tx_i(\omega) + (1-t)y_i(\omega)) \geq tu_i^\omega(x_i(\omega)) + (1-t)u_i^\omega(y_i(\omega)).$$

It follows from the last expression and for an $A_i \in \Pi_i^n$ that

$$\min_{\omega \in A_i} u_i^\omega(tx_i(\omega) + (1-t)y_i(\omega)) \geq \min_{\omega \in A_i} [\; tu_i^\omega(x_i(\omega)) + (1-t)u_i^\omega(y_i(\omega)) \;] =$$

$$t \min_{\omega \in A_i} u_i^\omega(x_i(\omega)) + (1-t) \min_{\omega \in A_i} u_i^\omega(y_i(\omega))$$

and hence that

$$\min_{\omega \in A_i} u_i^\omega(tx_i(\omega) + (1-t)y_i(\omega))\ q_i^n(A_i) \geq$$

$$t \min_{\omega \in A_i} u_i^\omega(x_i(\omega))\ q_i^n(A_i) + (1-t) \min_{\omega \in A_i} u_i^\omega(y_i(\omega))\ q_i^n(A_i).$$

Finally, we conclude that

$$\sum_{A_i \in \Pi_i^n} [\ \min_{\omega \in A_i} u_i^\omega(tx_i(\omega) + (1-t)y_i(\omega))\ ]\ q_i^n(A_i) \geq$$

$$t \sum_{A_i \in \Pi_i^n} [\min_{\omega \in A_i} u_i^\omega(x_i(\omega))] q_i^n(A_i) + (1-t) \sum_{A_i \in \Pi_i^n} [\min_{\omega \in A_i} u_i^\omega(y_i(\omega))] q_i^n(A_i), \text{ i.e., that}$$

$$\upsilon_i^n(tx_i^n + (1-t)y_i^n) \geq t\upsilon_i^n(x_i^n) + (1-t)\upsilon_i^n(y_i^n), \text{ for any } x_i^n, y_i^n \in X_i^n \text{ and } t \in [0,1].$$

By assumption, $e_i^n \in X_i^n$ (so that $X_i^n \neq \emptyset$). By construction, X_i^n is below (order) bounded by the zero element. By assumption, X_i^n is closed in $(\mathbb{R}^l)^{|\Omega^n|}$. Therefore, there exists an allocation $x^n \in X^n$ of the economy $\mathcal{E}^n$ such that:

1. $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$, for all $\omega \in \Omega^n$,

2. for all $i \in I$, we have that $\lambda_i^n \sum_{A_i \in \Pi_i^n} [\min_{\omega \in A_i} u_i^\omega(x_i(\omega))] q_i^n(A_i) = Sh_i(V_{\lambda^n, \upsilon^n}^n)$, where:
   (i) $\lambda_i^n \geq 0$ for all i, with $\sum_{i \in I} \lambda_i^n = 1$ and

   (ii) $Sh_i(V_{\lambda^n, \upsilon^n}^n) = \sum_{S \subseteq I, i \in S} \frac{(|S|-1)!(|I|-|S|)!}{|I|!} [V_{\lambda^n, \upsilon^n}^n(S) - V_{\lambda^n, \upsilon^n}^n(S \setminus \{i\})]$

is the Shapley value of the i agent, derived from the truncated maximin TU game $\Gamma^n = (I, V_{\lambda^n, \upsilon^n}^n, Sh)$, whose characteristic function $V_{\lambda^n, \upsilon^n}^n$ is defined by

$$V_{\lambda^n, \upsilon^n}^n(S) = \max_{x_i^n \in X_i^n} \sum_{i \in S} \lambda_i^n \sum_{A_i \in \Pi_i^n} [\min_{\omega \in A_i} u_i^\omega(x_i(\omega))] q_i^n(A_i),$$

$$\text{subject to } \sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega), \text{ for all } \omega \in \Omega^n.$$

step 2

We approximate the existence of a feasible allocation in $\mathcal{E}$ by all the existing maximin value allocations in the sequence of economies $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$. Consider a feasible allocation $x \in X$ of $\mathcal{E}$. From this allocation's feasibility condition

$$\sum_{i\in I} x_i(\omega) = \sum_{i\in I} e_i(\omega), \text{ for all } \omega \in \Omega,$$

the following condition is implied

$$\sum_{\omega\in\Omega}\sum_{i\in I} x_i(\omega) = \sum_{\omega\in\Omega}\sum_{i\in I} e_i(\omega) = e \ (< \infty, \text{ because } e_i \in \ell^1_+ \cap X_i, \text{ for all } i).$$

This means that for each agent i, each $x_i(\omega)$, $\omega \in \Omega$, belongs in the compact rectangle $[0, e]$ of $\mathbb{R}^l$. Define now, for each agent i, the set

$$C_i = C = \{x_i : 0 \leq x_i(\omega) \leq e, \, \omega \in \Omega \ \} = [0, e]^\infty \subset (\mathbb{R}^l_+)^\infty.$$

Clearly, C is compact in $(\mathbb{R}^l)^\infty$ with respect to the product topology of $(\mathbb{R}^l)^\infty$. Certainly, the set $\prod_{i\in I} C_i = C^{|I|=s}$ (which contains all the feasible allocations of $\mathcal{E}$) is also compact in $((\mathbb{R}^l)^\infty)^s$. Fix now again a $n \in \mathbb{N}$ and consider the corresponding (fixed) economy $\mathcal{E}^n$ with its (feasible) maximin value allocation $x^n \in X^n$. As previously,

$$\sum_{i\in I} x_i(\omega) = \sum_{i\in I} e_i(\omega), \text{ for all } \omega \in \Omega^n \Rightarrow \sum_{\omega\in\Omega^n}\sum_{i\in I} x_i(\omega) = \sum_{\omega\in\Omega^n}\sum_{i\in I} e_i(\omega) = e^n,$$

from which it is implied that for each agent i, each $x_i(\omega)$, $\omega \in \Omega^n$, belongs in the rectangle $[0, e^n]$ of $\mathbb{R}^l_+$. We define now the set

$$C_i^n = C^n = \{x_i^n : 0 \leq x_i(\omega) \leq e^n, \, \omega \in \Omega^n \ \} = [0, e^n]^{|\Omega^n|} \subset (\mathbb{R}^l_+)^{|\Omega^n|}.$$

Since by construction $[0, e^n] \subseteq [0, e]$ and, thus, $[0, e^n]^{|\Omega^n|} \subset [0, e]^\infty$, we conclude that for each agent i it holds that $x_i(\omega) \in [0, e]$, $\omega \in \Omega^n$ and $x_i^n \in C$. So that finally $x^n \in C^s$. Observe also that the agents' vector λ^n belongs in the unit $(s-1)$ - simplex of $\mathbb{R}^s$, which we denote as Δ. Then, by notating the maximin value allocation $x^n \in X^n$ of $\mathcal{E}^n$ with the augmented form $(x_1^n, ..., x_s^n\ , \lambda_1^n, ...\lambda_s^n) = (x^n, \lambda^n)$, we have that $(x^n, \lambda^n) \in C^s \times \Delta = K$. Evidently, K is compact in $((\mathbb{R}^l)^\infty)^s \times \mathbb{R}^s$, so that every sequence of K has a convergent subsequence in K. Consider the sequence $\{(x^m, \lambda^m) : m \in \mathbb{N}\}$ of K and its convergent subsequence $\{(x^n, \lambda^n) : n \in \mathbb{N}\}$ to the point (x, λ) of K. Since for each $n \in \mathbb{N}$ the maximin value allocation of the economy $\mathcal{E}^n$ belongs in K, we can wlog identify the previous (sub)sequence with the sequence of the existing maximin value allocations in the sequence of economies $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$. Concluding, the sequence of the maximin value allocations (of the truncated economies of the original economy) converges to the point $(x, \lambda) \in K$, which is a feasible allocation of the default economy $\mathcal{E}$.

<u>step 3</u>

We verify that (x, λ) is a maximin value allocation for $\mathcal{E}$, i.e., that conditions 1 and 2 of the Definition in section 2.3 are satisfied in the limit of the sequence $\{(x^n, \lambda^n) : n \in \mathbb{N}\}$. Condition 1, i.e., the feasibility of the allocation (x, λ), was derived in step 2. For condition 2, we need to show that for the existing (by step 2) $\lambda_i \geq 0$ for all $i \in I$, with $\sum_{i\in I} \lambda_i = 1$, the following condition is satisfied

$$\lambda_i \sum_{A_i \in \Pi_i} [\min_{\omega \in A_i} u_i^\omega(x_i(\omega))] q_i(A_i) = Sh_i(V_{\lambda,\upsilon}), \text{ for all } i.$$

For all $n \in \mathbb{N}$ and for any i, we have proven that

$$\lambda_i^n \sum_{A_i \in \Pi_i^n} [\min_{\omega \in A_i} u_i^\omega(x_i(\omega))] q_i^n(A_i) = Sh_i(V^n_{\lambda^n,\upsilon^n}).$$

Hence, for any i (and provided that the following limits exist), it holds that

$$\lim_{n\to\infty} \lambda_i^n \sum_{A_i \in \Pi_i^n} [\min_{\omega\in A_i} u_i^\omega(x_i(\omega))] q_i^n(A_i) = \lim_{n\to\infty} Sh_i(V^n_{\lambda^n,\upsilon^n}), \text{ or that}$$

$$\lambda_i \lim_{n\to\infty} \sum_{A_i \in \Pi_i^n} [\min_{\omega\in A_i} u_i^\omega(x_i(\omega))] q_i^n(A_i) =$$

$$\lim_{n\to\infty} \sum_{S\subseteq I, i\in S} \tfrac{(|S|-1)!(|I|-|S|)!}{|I|!} [V^n_{\lambda^n,\upsilon^n}(S) - V^n_{\lambda^n,\upsilon^n}(S\setminus\{i\})] =$$

$$\sum_{S\subseteq I, i\in S} \tfrac{(|S|-1)!(|I|-|S|)!}{|I|!} [\lim_{n\to\infty} V^n_{\lambda^n,\upsilon^n}(S) - \lim_{n\to\infty} V^n_{\lambda^n,\upsilon^n}(S\setminus\{i\})] = Sh_i(\lim_{n\to\infty} V^n_{\lambda^n,\upsilon^n}).$$

Since $\Omega = \mathbb{N}$, there exists an increasing (by containment) sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of Ω, such that A_n is finite for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} A_n = \Omega$. Since $\{A_n\}_{n\in\mathbb{N}}$ satisfies condition $\mathcal{C}$, we do not loose in generality if we identify the sequence $\{A_n\}_{n\in\mathbb{N}}$ with the sequence $\{\Omega^n\}_{n\in\mathbb{N}}$. Then, the increasing set sequence $\{\Omega^n\}_{n\in\mathbb{N}}$ is above bounded by and convergent to Ω, i.e.,

$$\limsup_n \Omega^n = \liminf_n \Omega^n = \lim_n \Omega^n = \Omega.$$

It is then implied that $\lim_n \mathcal{E}^n = \mathcal{E}$ and in particular (for any i) that:

As $n \to \infty$, $u_i^\omega(x_i(\omega))$, $\omega \in \Omega^n \to u_i^\omega(x_i(\omega))$, $\omega \in \Omega$, (ii) $\lim_{n\to\infty} q_i^n(A_i) = q_i(A_i)$,

(iii) $\lim_{n\to\infty} \Pi_i^n = \Pi_i$ and (iv) $\lim_{n\to\infty} X_i^n = X_i$ (hence, $\lim_{n\to\infty} x_i^n = x_i$, for any $x_i^n \in X_i^n$).

The previous establish the fact that the expression

$$\lambda_i \lim_{n\to\infty} \sum_{A_i \in \Pi_i^n} [\min_{\omega\in A_i} u_i^\omega(x_i(\omega))] q_i^n(A_i) = Sh_i(\lim_{n\to\infty} V^n_{\lambda^n,\upsilon^n}),\ i \in I,$$

leads to the desirable expression $\lambda_i \sum_{A_i\in\Pi_i} [\min_{\omega\in A_i} u_i^\omega(x_i(\omega))] q_i(A_i) = Sh_i(V_{\lambda,\upsilon})$, $i \in I$.

The left hand side limit is the well defined maximin utility of each agent. The proof concludes by showing that the right hand side limit is well defined as well. For this, we have to prove that the $V_{\lambda,\upsilon}$ of Γ[23] exists (is well defined). For every $n \in \mathbb{N}$, consider the existing maximin value allocation (x^n, λ^n) of $\mathcal{E}^n$, in which (by definition) feasibility of consumption is satisfied within any coalition. Then, for any $S \subseteq I$,

$$\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega), \text{ for all } \omega \in \Omega^n \Rightarrow \sum_{\omega \in \Omega^n} \sum_{i \in S} x_i(\omega) = \sum_{\omega \in \Omega^n} \sum_{i \in S} e_i(\omega) = \varepsilon^n,$$

which means that for every $i \in S$, each $x_i(\omega)$, $\omega \in \Omega^n$, belongs in the compact rectangle $[0, \varepsilon^n]$ of $\mathbb{R}^l$. For every $i \in S$, we know that u_i^ω, $\omega \in \Omega^n$ is continuous on $\mathbb{R}^l_+$, thus on $[0, \varepsilon^n] \subset \mathbb{R}^l_+$ as well. Then, for every $i \in S$ and for any $\omega \in \Omega^n$, it follows from the Weierstrass' Extreme Value Theorem that

$$\max_{x_i(\omega) \in [0, \varepsilon^n]} u_i^\omega(x_i(\omega)) = u_i^\omega(x_i^\star(\omega)) \text{ exists.}$$

For every $i \in S$, however, as $n \to \infty$, $u_i^\omega(x_i^\star(\omega))$, $\omega \in \Omega^n \to u_i^\omega(x_i^\star(\omega))$, $\omega \in \Omega$, thus $u_i^\omega(x_i^\star(\omega))$, $\omega \in \Omega$, exists as well[24]. Since for every $i \in S$ each $A_i \in \Pi_i$ is finite and the family $\{u_i^\omega : \mathbb{R}^l_+ \to \mathbb{R}_+, \omega \in \Omega\}$ is uniformly bounded, we conclude that

both the $\min_{\omega \in A_i} u_i^\omega(x_i^\star(\omega))$ exists and the $\sum_{A_i \in \Pi_i} [\min_{\omega \in A_i} u_i^\omega(x_i^\star(\omega))] q_i(A_i)$ (finitely) exists,

for each $i \in S$. So that then the $\sum_{i \in S} \lambda_i \sum_{A_i \in \Pi_i} [\min_{\omega \in A_i} u_i^\omega(x_i^\star(\omega))] q_i(A_i)$ also exists.

According to Remark 5, this finally means that

$$V_{\lambda,\upsilon}(S) = \max_{x_i \in X_i} \sum_{i \in S} \lambda_i \sum_{A_i \in \Pi_i} [\min_{\omega \in A_i} u_i^\omega(x_i(\omega))] q_i(A_i),$$

$$\text{subject to } \sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega), \text{ for all } \omega \in \Omega$$

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[23]In the way it was specified in the Definition of section 2.3.

[24]Note that $\lim_{n \to \infty} \varepsilon^n = \varepsilon = \sum_{\omega \in \Omega} \sum_{i \in S} e_i(\omega) = \sum_{\omega \in \Omega} \sum_{i \in S} x_i(\omega)$ ($< \infty$, since $e_i \in \ell^1_+ \cap X_i$, for all $i \in S$).

$$(\Rightarrow \text{subject to} \sum_{\omega\in\Omega}\sum_{i\in S} x_i(\omega) = \sum_{\omega\in\Omega}\sum_{i\in S} e_i(\omega) = \varepsilon),$$

exists. □

2.5 Conclusion

In this chapter we allowed for countably infinite states in an ambiguous economy and established the viability of a maximin value allocation in it.

One may reasonably argue that the maximin value allocation is a pessimistic equilibrium notion. Nevertheless, as in de Castro and Yannelis (2009), de Castro et al. (2011, 2012), He and Yannelis (2013) and Angelopoulos and Koutsougeras (2014), pessimism turns out to be a normative attitude in general equilibrium terms. Indeed, maximin-pessimistic agents enjoy higher (first best) efficiency in equilibrium; the maximin value allocation exists without (necessarily) private information measurable consumption and initial endowments. On top of that, less maximin (efficient) value allocations are lost for the achievement of incentive compatibility of this concept.

Given that uncountable set of states arise naturally in many real life economies, the issue of examining the possibility of existence of a maximin (efficient incentive compatible) value allocation with a continuum of states bears considerable importance.

Chapter 3

Uncountable Set of States

3.1 Preliminaries

Consider an economy with asymmetrically (i.e., privately) informed agents. Assume, specifically, that each agent's private information is a partition of the economy's state space or, interchangeably, the σ - algebra generated by this partition. Ambiguity arises naturally in such an economy and agents lose sensibly their Bayesian identity. This can be easily understood, by means of the following simple example of an economy with a finite number of states:

Say that $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and fix an agent α in this economy. For α, assume that $\Pi_\alpha = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$, so that $\mathcal{F}_\alpha = \sigma(\Pi_\alpha) = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \emptyset, \Omega\}$. It is reasonable to assume that since α is privately informed, the prior belief (additive probability measure) q_α of α is informationally restricted, i.e., that $q_\alpha : \mathcal{F}_\alpha \to [0, 1]$. Then although, for example, α assigns a probability to $\{\omega_2, \omega_3\}$, he is unable to attach a probability to ω_2 and ω_3. That is, $q_\alpha(\omega_2)$ and $q_\alpha(\omega_3)$ are unknown to α; α has ambiguity concerning the probability of occurrence of the states ω_2 and ω_3. Agent α is non-Bayesian, since his Bayesian (or subjective) expected utility cannot be defined.

The (non-Bayesian) agent α is specifically said to face extreme ambiguity, if he ignores the probability of occurrence of all the (non trivial) events in his informational algebra. In our example, if for all $A \in \mathcal{F}_\alpha$, with $\emptyset \neq A \subset \Omega$, it holds that $q_\alpha(A)$ is not unambiguously known to α. Clearly, an agent with extreme ambiguity concerning his probabilities does not have a prior belief at all. No probability measure makes sense to be assigned to him in the first place.

Whichever the case is, it is an experimental fact[25] that individuals are ambiguity averse. By assuming that agents are, in particular, maximin ambiguity averse, the

[25]First recognized by Ellsberg, 1961.

gain in the properties of general equilibrium outcomes is tremendous (see in de Castro and Yannelis, 2009, de Castro et al., 2011, 2012, He and Yannelis, 2013, Angelopoulos and Koutsougeras, 2014 and Angelopoulos, 2014).

The value allocation under ambiguity (namely, the maximin value allocation) is motivated along this analytical line. The private (information) maximin value allocation, specifically, was introduced in de Castro and Yannelis (2009). It was revisited and discussed in Angelopoulos and Koutsougeras (2014) as well, but only as a special case of an "informationally generalized" maximin value allocation notion. Indeed, Angelopoulos and Koutsougeras (2014) introduce maximin value allocations where the informational partition (or algebra) that individuals use inside coalitions is either the initial one they are endowed with, or some (any) other, depending on the coalition they are into and the underlying information exchange protocol within it. In this chapter, however, attention is specifically drawn back to the private maximin value allocation for the following reasons:

Any maximin value allocation is a cardinal (Shapley, 1969) value allocation, hence, a fair cooperative general equilibrium concept. Indeed, the level of the contribution of an agent in his coalitions reflects on the level of the utility this agent is assigned with. The private maximin value allocation, in particular, extends the private Bayesian value allocation of Krasa and Yannelis, 1994, 1996 (see in Angelopoulos and Koutsougeras, 2014 and Angelopoulos, 2014). Thereby, it inherits all the desirable properties of the latter notion and principally the fact that the informational superiority of an agent is rewarded (in consumption, hence, utility terms). At the same time, its chief advantages over the private Bayesian value allocation are two (see, again, in Angelopoulos and Koutsougeras, 2014 and Angelopoulos, 2014): (i) it exists without private information measurable net trades; thus, allows for informationally unconstrained (first best) Pareto efficiency and (ii) it is "less" informationally constrained efficient incentive compatible (in the sense that private information measurable initial endowments only (and not consumption) need to assumed).

Maximin value allocations are not necessarily viable in economies with a non-finite number of states (indeed, agents' maximin utilities (minimized over the states) are not well defined to begin with). Such economies, on the other hand, arise naturally in real life. In Angelopoulos (2014), existence of a private maximin value allocation was proved with countably infinite many states. In this chapter the same is done with an uncountably infinite state space.

3.2 The Ambiguous Economy

Let $\mathcal{E}$ be a two (interim - ex post) period exchange economy. The state space $\Omega = \mathbb{R}^k$, $k < \infty$, models the underlying state contingent uncertainty, state dependent randomness and informational structure of $\mathcal{E}$. $I = \{1, 2, ..., n\}$ is the finite set of agents of $\mathcal{E}$ and an $S \in \mathcal{P}(I)$ is a coalition of agents. $\mathbb{R}^l$, $l < \infty$, is the economy's commodity space. Agents of $\mathcal{E}$ trade by writing (coalitionally) consumption contracts. $\mathcal{F} = \mathcal{B}(\mathbb{R}^k)$ is the Borel σ - algebra of Ω and $(\Omega, \mathcal{F})$ is a Borel space.

Π_i is the informational partition of the agent i. Π_i is assumed to be a measurable partition of $\mathbb{R}^k$. $\mathcal{F}_i = \sigma(\Pi_i) \subset \mathcal{B}(\mathbb{R}^k)$ is the informational σ - algebra of the same agent. Agents of $\mathcal{E}$ trade in the economy's interim period, in which they receive signals regarding the ex post realized state. They are, therefore, endowed with advanced information: $\Pi_i(\omega) \in \Pi_i$ contains the actual (realised in the second period) state ω. $\Pi_i(\omega) \subseteq \mathbb{R}^k$ [and $\Pi_i(\omega) \in \mathcal{B}(\mathbb{R}^k)$], for any $\omega \in \Omega$, becomes now the new, refined informational set of the i agent, onto which he focuses.

The (Borel) probability measure $q_i : \mathcal{F}_i \to [0, 1]$ is the informationally restricted private prior of the i agent, satisfying (by definition) the following incompleteness property: $q_i(B_i)$ may be unknown for a $\emptyset \neq B_i \subset A_i \in \mathcal{F}_i$, even though $q_i(A_i)$ is provided (known) by q_i. That is, the economy's agents may be unable to completely form a prior belief. To put it differently, agents of $\mathcal{E}$ face ambiguity. Since for any $\omega \in \Omega$ the event $\Pi_i(\omega)$ and all its subevents are, actually, the only events that "matter" for the i agent, agent i accumulates the probability distribution of his prior q_i to $\Pi_i(\omega) \in \mathcal{F}_i$, or (w.l.o.g.) to $\Pi_i(\omega) \in \Pi_i$. Then, for any $i \in I$ and for any $\omega \in \Omega$, $q_i(B_i)$ is unknown for any B_i that satisfies (i) $\emptyset \neq B_i \subset \Pi_i(\omega)$ and (ii) B_i has the cardinality of the continuum[26], even though $q_i(\Pi_i(\omega))$ is known $[q_i(\Pi_i(\omega)) = 1]$. Thus, the non-Bayesian agents of $\mathcal{E}$ face extreme ambiguity and lose their priors.

[26]Clearly, condition (ii) is not needed when $\Pi_i(\omega)$ is finite or countable.

The (surjective) function $x_i : \Omega \rightarrow X_i \subset \mathbb{R}^l_+$ gives a random state dependent (r.s.d.) consumption plan $\{x_i(\omega) : \omega \in \Omega\}$ of the i agent. The agent's i r.s.d. consumption set is identified with the set of functions

$$L_{X_i} = \{ \ x_i \mid x_i : \Omega \rightarrow X_i \ \},$$

or (equivalently) with the class of sets

$$\mathbb{X}_i = \{ \ x_i(\Omega) = X_i : X_i \subset \mathbb{R}^l_+ \ \},$$

which is equivalent to the set of vectors

$$\mathbb{X}_i = \{ \ \{x_i(\omega) : \omega \in \Omega\} \ \}=\{x_i(\omega) : x_i(\omega) \in \mathbb{R}^l_+\} \subset \mathbb{R}^l_+.$$

The set $\mathbb{X}_i$ (or the set L_{X_i}) contains the feasible consumption of the i agent. The function $e_i \in L_{X_i}$ gives the r.s.d. initial endowment plan of the i agent.

The agent's i preferences are represented by the r.s.d. utility function

$$u_i : \Pi_i(\omega) \times \mathbb{X}_i \rightarrow \mathbb{R}_+, \text{ for any } \omega \in \Omega.$$

Agents are utility maximizers. They also are maximin ambiguity averse. Thus, the agent's i aforementioned preferences give rise to the same agent's (interim) maximin preferences, represented by the (interim) maximin utility map $\underline{u}_i : \Omega \times L_{X_i} \rightarrow \mathbb{R}_+$. According to de Castro and Yannelis (2009), $\underline{u}_i$ is given for any $(\omega, x_i) \in \Omega \times L_{X_i}$ by the formula

$$\underline{u}_i(\omega, x_i) = \min_{\omega' \in \Pi_i(\omega)} u_i(\omega', x_i(\omega')).$$

The interpretation of the above formulation is the following: *The i agent, considering the worst possible state, chooses the best possible utility*. In de Castro and Yannelis (2009), the agent's i (interim) maximin (non expected) utility is established as above, when (and because) Ω is finite. With an infinite Ω, however, the previous expression is not well defined, since the minimum may fail to exist. Towards overcoming this analytical obstacle, it seems natural to redefine the agent's i $\underline{u}_i$ as

$$\underline{u}_i(\omega, x_i) = \min_{\omega' \in \Pi_i(\omega)} \max_{x_i(\omega') \in \mathbb{X}_i} u_i(\omega', x_i(\omega')),$$

where $\Pi_i(\omega), \omega \in \Omega$ and $\mathbb{X}_i$ are compact, while u_i is continuous. This is an equivalent reformulation of the agent's i (interim) maximin utility, since the new formula carries the following interpretation: *The i agent minimizes (with respect to the states) his maximum (with respect to his consumption) utility.* More importantly, the previous reformulation will allow us to well define the agent's i (interim) maximin utility format. When the agent i maximizes his utility u_i, then the same agent maximizes his (interim) maximin utility $\underline{u}_i$ as well. The agents of $\mathcal{E}$, therefore, are (interim) maximin utility maximizers. Additionally, agents are assumed to have monotone/increasing (interim) maximin preferences[27].

Concluding, we define the economy

$$\mathcal{E}{=}\{ \ \mathbb{R}^l \ ; \ (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)) \ ; \ (\ [\ \Pi_i, \mathbb{X}_i, e_i, \underline{u}_i(u_i) \ ] : i \in I \ ) \ \},$$

for which the list of assigned (to all the economy's agents) functions

$$x = (x_1, x_2, ..., x_i, ..., x_n) \in L_X = \prod_{i \in I} L_{X_i},$$

$$\text{satisfying} \quad \sum_{i \in I} x_i = \sum_{i \in I} e_i \iff \sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega), \text{ for all } \omega \in \Omega,$$

is a feasible r.s.d. allocation (contract), i.e., a non free disposal general equilibrium.

[27]The pointwise partial ordering is assumed on both L_{X_i} and $\mathbb{X}_i$.

3.3 The Maximin Value Allocation

Let a Shapley (1953)-value-solvable (*interim*) *maximin transferable utility game* $\Gamma = (I, V_{\lambda,\underline{u},\omega}, Sh)$, within which the players' payoffs are identified with (interim) maximin utilities. Γ is a coalitional game with side payments, played by the finitely many players $1, 2, ..., n \in I$. An $S \subseteq I$ is a coalition of players. If Ω is the state space of Γ, $\omega \in \Omega$ is the actual state. $\underline{u}$ is the set of all the players' (interim) maximin utility functions. The players' (interim) maximin utilities $\underline{u}_i(\cdot)$, $i \in I$, become common scaled (hence, interpersonally comparable) and transferable by a personal r.s.d. factor $\lambda_i : \Omega \to \mathbb{R}_{++}$, assigned to each player i[28]. λ is the set of all the players' factors. $V(\lambda, \underline{u}, \omega) := V_{\lambda,\underline{u},\omega} : 2^I \to \mathbb{R}_+$ is a (monotone, superadditive and becoming zero for the null coalition - interim) maximin characteristic function of Γ. If $\mathcal{V}$ is the class of all the $V_{\lambda,\underline{u},\omega}$ of Γ, $Sh : \mathcal{V} \to \mathbb{R}^n_+$ is the (interim) maximin Shapley value function of Γ, which solves Γ by assigning:
(i) to Γ the (interim) maximin Shapley (1953) value $Sh(V_{\lambda,\underline{u},\omega}) \in \mathbb{R}^n_+$ and
(ii) to each player i of Γ the respective coordinate $Sh_i(V_{\lambda,\underline{u},\omega}) \in \mathbb{R}_+$ of the previous vector, where in particular

$$Sh_i(V_{\lambda,\underline{u},\omega}) = \sum_{S \subseteq I, i \in S} \tfrac{(|S|-1)!(|I|-|S|)!}{|I|!} [V_{\lambda,\underline{u},\omega}(S) - V_{\lambda,\underline{u},\omega}(S \setminus \{i\})], \; |I| = n.$$

Sh satisfies both group rationality $[\sum_{i \in I} Sh_i(V_{\lambda,\underline{u},\omega}) = V_{\lambda,\underline{u},\omega}(I)]$ and individual rationality $[Sh_i(V_{\lambda,\underline{u},\omega}) \geq V_{\lambda,\underline{u},\omega}(\{i\})$, for all $i \in I]$.

We now define the (interim private) maximin value allocation for $\mathcal{E}$, of Angelopoulos and Koutsougeras (2014), by associating $\mathcal{E}$ with Γ.

[28]In the standard interpretation of the value allocation, the players' factors $\lambda_i(\omega)$, $i \in I$ and $\omega \in \Omega$, are also their "weights" to Γ. One could, therefore, replace $\mathbb{R}_{++}$ with the interval $(0, 1)$ (and adittionally require that $\sum_{i \in I} \lambda_i(\omega) = 1$, for any $\omega \in \Omega$) in the range of each agent's i "weight function" λ_i. This, of course, would not affect the analysis in the sequel.

Definition. An allocation $x \in L_X$ of $\mathcal{E}$ is said to be an (*interim private*) *maximin value allocation* if the following two conditions are satisfied for any $\omega \in \Omega$:

1. $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$,

2. for all $i \in I$, we have that $\lambda_i(\omega) \min_{\omega' \in \Pi_i(\omega)} \max_{x_i(\omega') \in \mathbb{X}_i} u_i(\omega', x_i(\omega')) = Sh_i(V_{\lambda,\underline{u},\omega})$, where

   a. $\lambda_i(\omega) > 0$, for all i and

   b. $V_{\lambda,\underline{u},\omega}$ is defined by $V_{\lambda,\underline{u},\omega}(S) = \max_{x_i(\omega) \in \mathbb{X}_i} \sum_{i \in S} \lambda_i(\omega) \min_{\omega' \in \Pi_i(\omega)} \max_{x_i(\omega') \in \mathbb{X}_i} u_i(\omega', x_i(\omega'))$,

$$\text{subject to } \sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega).$$

Remark 1. Side payments are not assumed within any (interim private) maximin value allocation. The $V_{\lambda,\underline{u},\omega}$ of Γ as specified in the definition above is (indeed) monotone, superadditive and becomes zero for the empty coalition. Group and individual rationality of Sh of Γ imply the (interim) maximin Pareto efficiency and individual rationality of the (interim private) maximin value allocation. (Interim) maximin efficiency and private information measurable initial endowments secure transfer (interim) maximin coalitional incentive compatibility[29] for the (interim private) maximin value allocation. The proof of this statement is essentially the same with the one of Theorem 2 of Angelopoulos and Koutsougeras (2014). Nothing changes with uncountable infinitely many states.

Remark 2. When agents have monotone (interim) maximin preferences, the following property holds for every (feasible and interim maximin individually rational Pareto optimal) interim private maximin value allocation of $\mathcal{E}$: *Every coalition maximizes its (interim) maximin utility subject to its consumption constraints* if and only if *every agent in a coalition independently maximizes his (interim) maximin utility subject to the feasibility of consumption within this coalition.*

[29]See in Angelopoulos and Koutsougeras (2014) for the definition of this notion.

3.4 Existence

The theorem that follows provides the conditions both for the well definition of the agent's i (interim) maximin utility and for the existence of the corresponding (interim private) maximin value allocation in $\mathcal{E}$.

Theorem. If for each agent i and for any state ω the following assumptions hold:
(A_1) $\Pi_i(\omega)$ and $\mathbb{X}_i$ are compact in $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively,
(A_2) u_i is continuous on $\mathbb{R}^k \times \mathbb{R}^l$,
then $\min\limits_{\omega' \in \Pi_i(\omega)} \max\limits_{x_i(\omega') \in \mathbb{X}_i} u_i(\omega', x_i(\omega'))$ and an (interim private) maximin value allocation exist in $\mathcal{E}$.

Proof. For the whole proof: (i) Assume the standard topology and the pointwise ordering on any finite dimensional Euclidean space and (ii) fix an agent i and a state ω. Wlog, define u_i as $u_i : \mathbb{X}_i \times \Pi_i(\omega) \to \mathbb{R}_+$. Consider the correspondence $\phi_i : \Pi_i(\omega) \to \mathbb{X}_i$, defined by $\phi_i(\omega') = \mathbb{X}_i$. This is a constant correspondence, hence a continuous correspondence (both upper and lower hemicontinuous). Also, ϕ_i is nonempty valued (since $e_i \in L_{X_i}$) and compact valued [from (A_1)]. From (A_2), u_i is continuous on $\mathbb{X}_i \times \Pi_i(\omega)$, which is a subset of the Euclidean space $\mathbb{R}^l \times \mathbb{R}^k$. Then, from Berge's (1963, p. 116) Maximum Theorem, it follows that the maximum function $f_i : \Pi_i(\omega) \to \mathbb{R}_+$ exists, is defined by

$$f_i(\omega') = \max\{u_i(x_i(\omega'), \omega') : x_i(\omega') \in \phi_i(\omega')\} =$$

$$\max_{x_i(\omega') \in \phi_i(\omega')} u_i(x_i(\omega'), \omega') = \max_{x_i(\omega') \in \mathbb{X}_i} u_i(x_i(\omega'), \omega')$$

and is continuous on $\Pi_i(\omega)$, which [according to (A_1)] is compact. This means that, from the Weierstrass' Extreme Value Theorem, f_i attains its minimum value over

$\Pi_i(\omega)$, i.e., that the

$$\min_{\omega' \in \Pi_i(\omega)} \max_{x_i(\omega') \in \mathbb{X}_i} u_i(x_i(\omega'), \omega')$$

exists in $\mathcal{E}$. We now verify the validity of condition 1 of the Definition in section 3.3, i.e., we show that a feasible allocation exists in $\mathcal{E}$. Since $e_i \in L_{X_i}$, it holds that

$$\sum_{i \in I} x_i(\omega) \ , \sum_{i \in I} e_i(\omega) \in \sum_{i \in I} \mathbb{X}_i \neq \emptyset.$$

Therefore, it can be the case that $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$. We finally prove condition 2 of the same Definition, to conclude that an (interim private) maximin value allocation exists in $\mathcal{E}$. For this, we first have to show that the $V_{\lambda,\underline{u},\omega}$ of $\Gamma = (I, V_{\lambda,\underline{u},\omega}, Sh)$[30] exists (is well defined). By Remark 2, we have for any $S \subseteq I$ and for $\lambda_i(\omega) > 0$, for all $i \in S$, that

$$V_{\lambda,\underline{u},\omega}(S) = \sum_{i \in S} \lambda_i(\omega) \min_{\omega' \in \Pi_i(\omega)} \max_{x_i(\omega') \in \mathbb{X}_i} u_i(\omega', x_i(\omega')), \text{ subject to } \sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega),$$

which leads us to the expression

$$V_{\lambda,\underline{u},\omega}(S) = \sum_{i \in S} \lambda_i(\omega) \min_{\omega' \in \Pi_i(\omega)} \max_{x_i(\omega') \in [0,e]} u_i(\omega', x_i(\omega')), \text{ where } e = \sum_{i \in S} e_i(\omega) \in \mathbb{R}^l_+.$$

The rectangle $[0, e]$ is compact on $\mathbb{R}^l$ and since u_i is continuous on $\mathbb{R}^k \times \mathbb{R}^l$, it follows that u_i is continuous on $\mathbb{R}^k \times [0, e]$ as well. Then, by following the same argumentation as in the first part of the proof, it is implied that for any $i \in S$ the

$$\min_{\omega' \in \Pi_i(\omega)} \max_{x_i(\omega') \in [0,e]} u_i(\omega', x_i(\omega'))$$

[30]In the way it was specified in condition 2.b of the Definition of section 3.3

exists, hence $V_{\lambda,\underline{u},\omega}(S)$ exists for any S. Next, returning again to the fixed agent i, for any $\lambda_i(\omega) > 0$ of this agent and relying on Remark 2, we have (for any S) that

$$V_{\lambda,\underline{u},\omega}(S) - V_{\lambda,\underline{u},\omega}(S \setminus \{i\}) =$$

$$\sum_{i\in S} \lambda_i(\omega) \min_{\omega'\in\Pi_i(\omega)} \max_{x_i(\omega')\in[0,e]} u_i(\omega', x_i(\omega')) - \sum_{i\in S\setminus\{i\}} \lambda_i(\omega) \min_{\omega'\in\Pi_i(\omega)} \max_{x_i(\omega')\in[0,e]} u_i(\omega', x_i(\omega'))=$$

$$\lambda_i(\omega) \min_{\omega'\in\Pi_i(\omega)} \max_{x_i(\omega')\in[0,e]} u_i(\omega', x_i(\omega')).$$

Then,
$$Sh_i(V_{\lambda,\underline{u},\omega}) =$$

$$\sum_{S\subseteq I, i\in S} \tfrac{(|S|-1)!(|I|-|S|)!}{|I|!} \lambda_i(\omega) \min_{\omega'\in\Pi_i(\omega)} \max_{x_i(\omega')\in[0,e]} u_i(\omega', x_i(\omega'))=$$

$$\lambda_i(\omega) \min_{\omega'\in\Pi_i(\omega)} \max_{x_i(\omega')\in[0,e]} u_i(\omega', x_i(\omega')) \frac{\sum_{S\subseteq I, i\in S} (|S|-1)!(|I|-|S|)!}{|I|!},$$

where $|I| = n$. We finally verify that $\sum_{S\subseteq I, i\in S} (|S|-1)!(|I|-|S|)! = n!$. By definition,

$$\sum_{S\subseteq I, i\in S} (|S|-1)!(|I|-|S|)!=$$

$$0!(n-1)![\tbinom{n}{1} - \tbinom{n-1}{1}] + 1!(n-2)![\tbinom{n}{2} - \tbinom{n-1}{2}]+$$
$$2!(n-3)![\tbinom{n}{3} - \tbinom{n-1}{3}] + ... + (k-1)!(n-k)![\tbinom{n}{k} - \tbinom{n-1}{k}] + ... + (n-1)!,$$

where $3 < k < n$ and the quantity $[\tbinom{n}{k} - \tbinom{n-1}{k}]$ expresses the number of coalitions of cardinality $k \in \mathbb{N}$ that agent i participates in. Now, each term of the previous expanded sum is equal to $(n-1)!$. We can verify that with the general term. Indeed:

$$
\begin{aligned}
(k-1)!(n-k)![\binom{n}{k} - \binom{n-1}{k}] &= (k-1)!(n-k)![\frac{n!}{(n-k)!k!} - \frac{(n-1)!}{(n-k-1)!k!}] \\
&= \frac{n!}{k} - \frac{(n-k)(n-1)!}{k} \\
&= \frac{n!}{k} - \frac{n(n-1)! - k(n-1)!}{k} \\
&= \frac{n!}{k} - \frac{n! - k(n-1)!}{k} \\
&= (n-1)!.
\end{aligned}
$$

This, finally, means that $\sum_{S \subseteq I, i \in S} (|S|-1)!(|I|-|S|)! = n(n-1)! = n!$. □

3.5 Conclusion

The proof of the Theorem provided in section 3.4 relies heavily on the (Berge's, 1963) Maximum Theorem. Therefore, the analysis (and the Theorem of this chapter) can be easily generalized if we use ordered topological spaces to model the state space and the commodity space of the economy (instead of Euclidean spaces).

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