ZIEGLER SPECTRA FOR SELF-INJECTIVE ALGEBRAS OF POLYNOMIAL GROWTH

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Abstract

We describe the Ziegler spectra for a large class of self-injective algebras of polynomial growth representation type. To derive our results, we extend the methods and techniques used in studying the finite-dimensional representation theory of such algebras. In particular, the use of tubular extensions and covering functors, and the reduction into various (e.g. tilting, socle, or stable) equivalence classes.

Our broadest result reduces the description of the Ziegler topology, for an arbitrary finite-dimensional self-injective algebra of polynomial growth, to such a description over a tame hereditary or tubular algebra. In the case of trivial extensions, we are able to give a complete description and a construction of the infinite-dimensional points lying in the closure of each Auslander-Reiten component.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction and Background

1.1 Introduction

For a ring R, representation theory is the study of the category $\mathrm{Mod}\text{-}R$ consisting of R-modules and R-linear maps. As this category is usually too complex to describe in detail, it is common to restrict attention to a full subcategory determined by a specific class of modules (e.g. injective, projective, flat, noetherian, artinian). In this thesis, we choose to look at the class of pure-injective modules.

Notions of purity arose from attempts to characterise injectivity in abelian groups (i.e. the question of when a submodule is actually a direct summand). In this sense, a pure submodule is almost a direct summand and a pure-injective module is almost an injective module. It is possible to characterise (pure-)injectivity in terms of solutions to R-linear equations (with 'pure-injective' being equivalent to 'algebraically compact'); this is a viewpoint natural to model theory.

It was in the context of model theory where a topological space Zg_R was introduced and associated to $\mathrm{Mod}\text{-}R$. Now called the Ziegler spectrum, this space is an invariant of the module category, containing much of its model theoretic information. It is defined to be the set of isomorphism classes of indecomposable pure-injective R-modules, with a topology relating to certain axiomatizable subclasses of $\mathrm{Mod}\text{-}R$. Despite this origin, we don't take the model theoretic approach here, using instead an algebraic and category theoretic language.

For a finite-dimensional algebra R, the class of finite-dimensional R-modules is usually the object of study. This class of modules forms an abelian subcategory $\text{mod-}R$ which is often approachable using the techniques of Auslander-Reiten theory. A quiver Γ_R is constructed, having vertices corresponding to the isomorphism classes of indecomposable objects in $\text{mod-}R$, and arrows corresponding to irreducible morphisms. Since every finite-dimensional module has a well-behaved decomposition into indecoposable summands, this Auslander-Reiten quiver Γ_R provides a good visualisation and reflects some of the complexity of the category.

The exclusion of all infinite-dimensional modules is one of convenience (indeed, they behave very differently even in regards to decomposition, see [59, p. 7] for many unintuitive examples). However, it is a nice coincidence that all finite-dimensional modules are pure-injective and, in fact, the finite-dimensional points of Zg_R form a dense and discrete subspace. Thus we have a bridge into the infinite-dimensional representation theory of R. Indeed, the existence of infinite-dimensional indecomposable (pure-injective) modules has long been known to be equivalent to Γ_A (hence Zg_R) being infinite [7].

For the algebras we study here, the structure of Γ_R and the corresponding modules will be well known. Our task then is to determine the infinite-dimensional points of Zg_R and to describe the topology. We will see there are close connections between the finite- and infinite-dimensional points. A common theme of this work will be to adapt the methods and techniques, used in studying finite-dimensional modules, to modules of infinite dimension.

Algebras for which the Ziegler spectrum is well understood include the tame hereditary algebras, the tubular algebras, and the domestic string algebras. From the results of this thesis, we can add to this list the finite-dimensional self-injective algebras of polynomial growth representation type.

1.1.1 Summary of content

In the remainder of Chapter 1, we give an overview of the background material the reader is assumed to be familiar with. We also introduce notation that will be used throughout; most is standard within the literature, particularly in the referenced material. The content of this paper relies heavily upon the finite-dimensional representation theory of certain algebras. Some generalities are given in this chapter and results pertaining to a particular algebra will be quoted and/or summarised as and when needed. We hope the details given here are sufficient to keep the document self-contained in this regard.

In Chapter 2, we look at trivial extensions of algebras. After recalling the basic concepts, we extend the results of Tachikawa [70] (concerning finite-dimensional representation theory) and give a description of the Ziegler spectrum for trivial extensions of hereditary algebras. As a consequence, we deduce the Ziegler spectrum for the trivial extension of any algebra tilted from a tame hereditary algebra.

In Chapter 3, we consider tubes and tubular enlargements. In particular, we investigate how the Ziegler closure of a tube is affected by a one-point tubular extension. These results are seen to be crucial in Chapter 5, where we study algebras with the property that every infinite-dimensional point of the Ziegler spectrum lies in the closure of a tube.

In Chapter 4, we look at covering theory for self-injective algebras. After a careful examination of covering and push-down functors, we prove a number of results useful for studying pure-injective modules and Ziegler spectra.

In Chapter 5, we study the canonical algebras and certain self-injective algebras of canonical type. Using the techniques developed in Chapters 3 and 4, we are able to describe their Ziegler spectra.

In Chapter 6, using the results of Chapter 5, we are able to describe the Ziegler spectrum for any 'non-exceptional' finite-dimensional self-injective algebra of polynomial growth representation type. We conclude with some remarks concerning the exceptional case and conjecture on further results.

Regarding the original work contained in this thesis. All main theorems presented here, i.e. Theorems 2.2.10, 3.4.4, 3.4.5, 6.1.5, 6.1.10, 6.1.18, and their corollaries, are completely original. With the exception of 6.1.10(4) which is proven in [47] independently and by different methods.

Our approach to tubular enlargements in Chapter 3 and covering functors in Chapter 4, is original to the extent that we adapt, to infinite-dimensional modules, methods only previously applied to finite-dimensional modules. In particular, the use of Galois coverings in Sections 4.1.2 and 4.1.3 is original. There is overlap in Proposition 3.5.4 and [27, 8.1] in regards to the Ziegler closure of ray tubes of the form that we denote by $\Gamma(p, n, 0)$. However, from our narrower perspective, we are able to give a stronger result.

In a few instances, for the sake of clarity, we have included a proof from elsewhere; in all cases we clearly reference its source. For example, we provide an English translation of Lemma 1.11.5 from its original French.

1.2 Rings and modules

Throughout $\mathbf{k}$ denotes a fixed algebraically closed field. For a $\mathbf{k}$-vector space V we denote by $\dim_{\mathbf{k}} V$ its dimension over $\mathbf{k}$ and by $V^\star$ we always mean the dual space $\mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$ consisting of $\mathbf{k}$-linear maps $V \to \mathbf{k}$. We define $\mathbb{P}^1(\mathbf{k}) := \mathbf{k} \cup \{\infty\}$.

Let $\mathbb{Q}^+$ and $\mathbb{R}^+$ denote the set of positive rational and positive real numbers, respectively. Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and set of positive integers, respectively.

We assume some familiarity with rings, modules, and basic concepts in (additive) category theory, e.g. functors, natural transformations, limits and colimits (in particular, inverse and direct limits).

For a ring R the category of all right (resp. left) R-modules is denoted by $\mathrm{Mod}\text{-}R$ (resp. $R\text{-}\mathrm{Mod}$) and its full subcategory of finitely-presented R-modules is denoted by $\mathrm{mod}\text{-}R$ (resp. $R\text{-}\mathrm{mod}$). We define $\mathrm{Ab} := \mathrm{Mod}\text{-}\mathbb{Z}$ — the category of abelian groups. All functors between module categories are assumed to be additive. For two R-modules M and N, the set of R-linear maps from M to N is denoted by $\mathrm{Hom}_R(M, N)$ or simply by (M, N) when context is clear. Given a class $\mathcal{C}$ of R-modules, denote by $\mathrm{add}(\mathcal{C})$ the class of all R-modules obtained as finite direct sums of modules in $\mathcal{C}$, and denote by $\mathrm{ind}(\mathcal{C})$ the class of all indecomposable R-modules belonging to $\mathcal{C}$. We may consider any class of R-modules as a full subcategory of $\mathrm{Mod}\text{-}R$. We occasionally use the abbreviation $\mathrm{ind}\text{-}R := \mathrm{ind}(\mathrm{mod}\text{-}R)$.

A ring is **connected** if it cannot be written as a cartesian product $R \times S$ of two rings R and S. Note an equivalence $\mathrm{Mod}\text{-}(R \times S) \simeq \mathrm{Mod}\text{-}R \times \mathrm{Mod}\text{-}S$ is given by $M \mapsto (Me_R, Me_S)$ where $e_R = (1_R, 0)$ and $e_S = (0, 1_S)$.

Two rings R and S are said to be **Morita equivalent** if there exists an equivalence $\mathrm{Mod}\text{-}R \simeq \mathrm{Mod}\text{-}S$.

Given two rings R and S by an **R-S-bimodule** M we always mean M has both a *left R-module* and *right S-module* structure such that $r(ms) = (rm)s$ for all $r \in R$, $s \in S$ and $m \in M$.

1.2.1 Algebras and quivers

The general term **algebra** will mean an associative algebra over the field $\mathbf{k}$. All finite-dimensional algebras will be unital, but we do consider non-unital infinite-dimensional algebras.

If A is a finite-dimensional algebra, the $\mathbf{k}$-dual space yields the **standard duality** between right and left finite-dimensional A-modules, i.e. the map $M \mapsto M^\star$ defines an equivalence $(\mathrm{mod}\text{-}A)^{\mathrm{op}} \to A\text{-}\mathrm{mod}$ [63, I.2.9]. In this way, any result for finite-dimensional A-modules dualises to a result for A^{op}-modules, and given say "Proposition x" we refer to the dual result by "Proposition x*" if it is not otherwise explicitly stated.

A **complete set of local idempotents** for an algebra A is a set of elements $\{e_i\}_{i\in I} \subseteq A$ with the following properties: $e_i^2 = e_i$ and e_iAe_i is a local ring for all $i \in I$, $e_ie_j = 0$ if $i \neq j$, and $A = \bigoplus_{i,j} e_iAe_j$. An algebra is called **locally bounded** if there exists a complete set of local idempotents $\{e_i\}_{i\in I}$ such that $\dim_{\mathbf{k}} e_iA + \dim_{\mathbf{k}} Ae_i < \infty$ for all $i \in I$. An algebra is called **basic** if there exists a complete set of local idempotents $\{e_i\}_{i\in I}$ such that $e_iA \not\simeq e_jA$ for all $i \neq j$. A finite-dimensional algebra A is basic if and only if the quotient algebra $A/\mathrm{rad}(A)$ is isomorphic to a product $\mathbf{k} \times \cdots \times \mathbf{k}$ of copies of $\mathbf{k}$ [63, I.6.2]. We write $\mathrm{rad}(A)$ for the **(Jacobson) radical** of A (i.e. the intersection of all maximal ideals of A). Every finite-dimensional algebra is Morita equivalent to a basic algebra [63, I.6.8].

The modules we always denote by $P_A(i) := e_iA$, resp. $I_A(i) := (Ae_i)^\star$, are (up to isomorphism) a complete set of indecomposable projective, resp. injective, A-modules. We denote by $\mathrm{proj}\text{-}A$, resp. $\mathrm{inj}\text{-}A$, the subcategories of $\mathrm{mod}\text{-}A$ consisting of all finite-dimensional projective, resp. injective, A-modules. The modules $S_A(i) := e_i\mathbf{k}$ are (up to isomorphism) a complete set of simple A-modules.

In this work, an algebra A is called **self-injective** if the projective A-modules coincide with the injective A-modules. For A locally bounded this is equivalent to, for any complete set of local idempotents $\{e_i\}_{i\in I}$, the existence of a permutation $\nu : I \to I$ and isomorphisms $P_A(i) \simeq I_A(\nu(i))$ for all $i \in I$. When A is finite-dimensional, this is equivalent to either of (hence both) A_A or ${}_AA$ being injective [1, Ex. 24.15].

By a **graph** we mean a set of vertices and a set of edges specified by unordered pairs of vertices (we allow for loops and multiple edges between vertices). A graph is **locally finite** if any vertex is incident to only finitely many edges. A **quiver** is a directed graph (i.e. a graph whose edges are specified by ordered pairs). A **simple graph** is a graph without loops or multiple edges; a **tree** is an acyclic and connected simple graph. The **Dynkin graphs** $\mathbb{A}_p$ for $p \geq 1$, $\mathbb{D}_m$ for $m \geq 4$, $\mathbb{E}_n$ for $n = 6, 7, 8$, and **Euclidean (= extended Dynkin) graphs** $\widetilde{\mathbb{A}}_p$ for $p \geq 1$, $\widetilde{\mathbb{D}}_m$ for $m \geq 4$, $\widetilde{\mathbb{E}}_n$ for $n = 6, 7, 8$, are defined at the beginning of Chapter 5 (see also [63, VII.2]).

We will use the language of quivers and relations from [63]. Given a locally finite quiver $\mathscr{Q}$ denote by $\mathbf{k}\mathscr{Q}$ its **path algebra**, see [63, II.1.2] for a precise definition. An ideal $I \leq \mathbf{k}\mathcal{Q}$ is **admissible** if $(\mathbf{k}\mathscr{Q})^m \subseteq I \subseteq (\mathbf{k}\mathscr{Q})^2$ for some $m \geq 2$. If $\mathscr{Q}$ is locally finite and I is an admissible ideal of $\mathbf{k}\mathscr{Q}$, then $\mathbf{k}\mathscr{Q}/I$ is locally bounded. Any algebra of this form is called a **bound quiver algebra**. In fact, the bound quiver algebras are precisely the basic locally bounded algebras [9, 2.1], cf. [63, II.3.7].

Specifically, every basic and connected locally bounded algebra A has a presentation $A \simeq \mathbf{k}\mathscr{Q}_A/I$ as a bound quiver algebra (with $\mathscr{Q}_A$ a connected quiver). The quiver $\mathscr{Q}_A$ in such a presentation is unique (up to isomorphism). If $\{e_i\}_{i\in J}$ is a complete set of local idempotents for A, then $\mathscr{Q}_A$ is defined to have vertices $\{i\}_{i\in J}$ and arrows $i \to j$ equal in number to $\dim_{\mathbf{k}} e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad}^2 A)e_j$. If $A = \mathbf{k}\mathscr{Q}/I$ is such an algebra, then the category $\operatorname{Mod}\text{-}A$ is equivalent to the category of representations of $\mathscr{Q}$ bound by I [63, III.1.6].

We will also use the following categorical language. If A is a small $\mathbf{k}$-linear category, let $A\text{-}\operatorname{Mod} := (A, \operatorname{Ab})$ denote the category of all additive functors from A to Ab (it is equivalent to the category of all $\mathbf{k}$-linear functors from A to $\operatorname{Mod}\text{-}\mathbf{k}$). Similarly $\operatorname{Mod}\text{-}A := (A^{\mathrm{op}}, \operatorname{Ab})$. This notation is consistent with the idea of considering a finite-dimensional algebra A as a category with a single object $*$ and hom-space $(*, *) = A$.

However, we associate a different category to a locally bounded algebra as follows.

Given a locally bounded algebra A with a complete set of local idempotents $\{e_i\}_{i\in I}$, define the category — also denoted A — with object set I and morphisms $A(i,j) := e_jAe_i$ for all $i,j \in I$. Composition is given by multiplication in A, i.e. $g \circ f := gf$ for $f \in A(i,j)$ and $g \in A(j,k)$. In this way, we consider A as a $\mathbf{k}$-linear category. The functor category $(A^{\mathrm{op}}, \mathrm{Ab})$ is then equivalent to the category of right A-modules, and $\mathrm{Mod}\text{-}A$ may refer to both. The representable functor $A(-,i)$ corresponds to the indecomposable projective A-module $P_A(i) = e_iA$. We will move freely between modules, functors, and quiver representations, as convenient. Note if $A = \mathbf{k}\mathscr{Q}_A/I$, then the category constructed in this way (or its opposite, depending on convention) is the $\mathbf{k}$-linear **path category** of $\mathscr{Q}$ bound by the relations in I.

Given locally bounded algebras A and B, an additive functor $A \otimes_{\mathbf{k}} B^{\mathrm{op}} \to \mathrm{Ab}$ (or equivalently $B^{\mathrm{op}} \otimes_{\mathbf{k}} A \to \mathrm{Ab}$) defines an A-B-bimodule for which $\mathbf{k}$ acts centrally (and vice versa).

For an introduction to **Auslander-Reiten theory** see [54, §2] and for further details see [55] [63] [8]. If A is a locally bounded algebra, then the category $\mathrm{mod}\text{-}A$ has almost-split (= Auslander-Reiten) sequences. A translation quiver Γ_A is defined called the **AR (Auslander-Reiten) quiver** of A [63, §IV]. We denote by τ_A (or just τ) the translate of Γ_A. We use the following terminology from [55, p.55]: each connected component Γ' of Γ_A defines a **component** of $\mathrm{mod}\text{-}A$, namely, the full additive subcategory generated by the indecomposable modules belonging to Γ'.

If $\mathcal{C} \subseteq \mathrm{mod}\text{-}A$ is such a component, we write $\Gamma(\mathcal{C})$ for the corresponding connected component of Γ_A. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two components (or any two module classes) of $\mathrm{mod}\text{-}A$, we define $\mathcal{C}_1 \vee \mathcal{C}_2 := \mathrm{add}(\mathcal{C}_1 \cup \mathcal{C}_2)$ as their additive closure. In this way, we can write $\mathrm{mod}\text{-}A = \bigvee_{i\in I} \mathcal{C}_i$ when $(\mathcal{C}_i)_{i\in I}$ are the components of $\mathrm{mod}\text{-}A$. We typically describe $\mathrm{mod}\text{-}A$ insofar as detailing its components and their interrelations, see for example the description given for tame hereditary algebras in Section 1.8 and tubular algebras in Section 1.10.

A component $\mathcal{C}$ is called **stable** if the translates τ and τ^{-1} are defined for every vertex of $\Gamma(\mathcal{C})$ (i.e. $\Gamma(\mathcal{C})$ is a stable translation quiver). A component $\mathcal{C}$ is called **standard** if $\mathcal{C}$ is equivalent to the (additive closure of the) path category of $\Gamma(\mathcal{C})$ bound by the "mesh relations" (see [55, 2.3] for the definition). Additionally, **preprojective** and **preinjective** components are defined at [55, 2.1]; **separating** components and **tubular families** are defined at [55, 3].

For the definition of a **tube** see [55, 3.1] or [17]. Concisely, a tube is a component $\mathcal{C}$ such that $\Gamma(\mathcal{C})$ contains a cyclic path and has as a geometric realisation the half-infinite cylinder $S^1 \times (\mathbb{R}^+ \cup \{0\})$. The **mouth** of a tube is then defined as the vertices (and arrows) belonging to the subset $S^1 \times \{0\}$. If $\mathcal{C}$ is a stable tube, the number of vertices belonging to the mouth of $\Gamma(\mathcal{C})$ is called the **rank** of $\mathcal{C}$. A **homogenous tube** is a stable tube of rank 1. A stable tube of rank $r \geq 1$ has the translation quiver $\Gamma(r-1, 0, 0)$ as defined in Section 3.5. The **tubular type** of a stable tubular family is the non-decreasing sequence of positive integers $(n_1, \ldots, n_t)$ specifying the ranks of the inhomogenous tubes (or is (1) in case all tubes are homogenous). Example 6.1.13 contains pictures of tubes of various shapes.

For the definition of **tame**, **wild**, **(n-parametric) domestic**, and **polynomial growth**, representation-type see [13] [65, XIX.3.6] for finite-dimensional algebras, and [19] for locally bounded algebras in general. Roughly speaking, a finite-dimensional algebra A is tame if, for all $d \in \mathbb{N}$, the category $\operatorname{ind}_d A$ — of all indecomposable A-modules of dimension d — consists of a finite number of discrete families and a finite number of one-parameter families. If $\mu_A(d)$ denotes the (minimal) number of one-parameter families in $\operatorname{ind}_d A$, then A is n-parametric domestic (resp. of polynomial growth) if $\mu_A(d) \leq n$ (resp. $\mu_A(d) \leq d^k$ for $k \in \mathbb{N}$ fixed) for all $d \in \mathbb{N}$. We don't require any technical knowledge of these terms. In all examples below, representation-infinite domestic (resp. of polynomial growth, but non-domestic) implies $\operatorname{mod}\text{-}A$ consists of finitely (resp. countably) many distinct tubular $\mathbb{P}^1(\mathbf{k})$-families and a finite number of additional components.

A finite-dimensional algebra A is said to be **representation finite** or **representation infinite**, according to whether there are finitely- or infinitely- many isomorphism classes of indecomposable finite-dimensional A-modules.

1.3 Ziegler spectra of rings

Let R be a ring. A short exact sequence of R-modules

$$0 \to L \xrightarrow{f} M \to N \to 0 \tag{1.3.1}$$

is said to be **pure-exact** if, for all $X \in R\text{-}\operatorname{mod}$, the induced sequence

$$0 \to L \otimes_R X \to M \otimes_R X \to N \otimes_R X \to 0$$

is exact in Ab. Given a pure-exact sequence of the form (1.3.1), we call $f : L \to M$ a **pure monomorphism**. This is one of many equivalent definitions for pure-exactness, see for instance [36, 6.4] and [51, 5.2]. An R-module L is called **pure-injective** if, for all $M \in \operatorname{Mod}\text{-}R$, every pure monomorphism $f : L \to M$ is a split monomorphism (equiv. every pure-exact sequence beginning at L is split exact). The image of a pure monomorphism $f : L \to M$ is called a **pure submodule** of M.

The isomorphism classes of indecomposable pure-injective (right) R-modules form a set, denoted Zg_R (or sometimes $\operatorname{Zg}\text{-}R$), called the (right) **Ziegler spectrum** of R. This set was first topologised in [74] in model theoretic terms, but we use the following equivalent formulation. A **definable subcategory** $\mathcal{D} \subseteq \operatorname{Mod}\text{-}R$ is a full subcategory that is closed under direct limits, products, and pure submodules. The closed subsets of Zg_R are defined to be those of the form $\operatorname{Zg}(\mathcal{D}) := \operatorname{Zg}_R \cap \mathcal{D}$, as $\mathcal{D}$ ranges over the definable subcategories of $\operatorname{Mod}\text{-}R$ — see [50, §5] for more information.

An indecomposable module that is not finitely-presented, but does have finite length when considered as a module over its endomorphism ring, is called a **generic module**. Every generic module is pure-injective and gives a closed point of the Ziegler spectrum [50, 4.4.24, 4.4.30].

1.3.1 Interpretation functors

For rings R and S, and definable subcategories $\mathcal{D} \subseteq \operatorname{Mod}\text{-}R$, $\mathcal{E} \subseteq \operatorname{Mod}\text{-}S$, an **interpretation functor** (elsewhere a **definable functor**) is a functor $F : \mathcal{D} \to \mathcal{E}$ that commutes with direct limits and products.

Proposition 1.3.2. *[40, 7.1, 7.2] If $F : \mathcal{D} \to \mathcal{E}$ is an interpretation functor, then* $\ker(F) := \{M \in \mathcal{D} \mid F(M) = 0\}$ *is a definable subcategory of* Mod-R.

Example 1.3.3. Examples of such definable subcategories include the hom-orthogonal class for any finitely-presented $X \in \text{mod-}R$, i.e. $\{\, M \in \text{Mod-}R \mid (X, M) = 0 \,\}$ — recall if X is finitely-presented, then $(X, -) : \text{Mod-}R \to \text{Ab}$ commutes with direct limits [69, V.3.4] and is an interpretation functor.

For some rings the kernel of $(-, X)$ is also a definable subcategory [16, §2.2]. When R is a finite-dimensional algebra, this is easily verified for any finite-dimensional $X \in \text{mod-}R$, i.e. $\{\, M \in \text{Mod-}R \mid (M, X) = 0 \,\}$ is definable. Note $(M, X) = (M, X^{\star\star}) = (M \otimes X^\star)^\star$ since X is finite-dimensional, and $(M, X) = 0$ if and only if $M \otimes_R X^\star = 0$. As $X^\star$ is finite-dimensional, $- \otimes_R X^\star$ commutes with products [69, I.13.2] and is therefore an interpretation functor with definable kernel.

Say an interpretation functor $F : \mathcal{D} \to \mathcal{E}$ "preserves indecomposability for pure-injectives" if $F(M)$ is indecomposable for all $M \in \text{Zg}(\mathcal{D})$; and "reflects isomorphism for pure-injectives" if $F(M) \simeq F(M')$ whenever $M = M'$ in $\text{Zg}(\mathcal{D})$.

An **embedding** $\varphi : X \to Y$ of topological spaces is a continuous injective map that yields a homeomorphism $X \simeq \varphi(X)$ (where $\varphi(X)$ is given the subspace topology inherited from Y).

Proposition 1.3.4. *[49, 15.2, 15.3] Let $F : \mathcal{D} \to \mathcal{E}$ be an interpretation functor. If F preserves indecomposability for pure-injectives, then F induces a closed and continuous map* $\text{Zg}(\mathcal{D}) \to \text{Zg}(\mathcal{E})$. *The induced map is a closed embedding if F also reflects isomorphism for pure-injectives.*

Example 1.3.5. Suppose $f : S \to R$ is a ring morphism, then restriction along f (a functor we denote by $\text{res}_f : \text{Mod-}R \to \text{Mod-}S$) is an interpretation functor (it has both left and right adjoints, thus preserves all limits and colimits). It is fully faithful if and only if f is a ring epimorphism [69, XI.1.2] and, if this is the case, then it induces a closed embedding $\text{Zg}_R \to \text{Zg}_S$.

1.3.2 Ziegler spectra of algebras

We collect some known results concerning the Ziegler spectrum of a finite-dimensional algebra R in the following proposition. For any subset $X \subseteq \mathrm{Zg}_R$, let $\mathrm{cl}(X)$ denote the closure of X in Zg_R. The phrase "almost all" means "all but finitely many".

Proposition 1.3.6. *Let R be a finite-dimensional algebra.*

(i) Every indecomposable finite-dimensional R-module is pure-injective, giving a point in Zg_R that is both open (i.e. isolated) and closed.

(ii) Any set of finite-dimensional points in Zg_R carries the discrete topology.

(iii) The closure in Zg_R of any infinite set of finite-dimensional points contains at least one infinite-dimensional point and no additional finite-dimensional points.

(iv) The finite-dimensional points form a dense subset of Zg_R.

(v) The closure of any set containing almost all finite-dimensional points, contains all infinite-dimensional points.

Proof. If M is a finite-dimensional R-module, then M has finite endo-length and is therefore pure-injective by [50, 4.4.24]. If M is also indecomposable, then $\{M\}$ is a closed subset of Zg_R by [50, 4.4.30] and an open subset by [50, 5.3.33]. This proves (i) and (ii) is an immediate consequence. The spectrum Zg_R is compact by [50, 5.1.33]. If $X \subseteq \mathrm{Zg}_R$ is an infinite subset, the closure $\mathrm{cl}(X)$ of X contains a limit point. If X consists only of finite-dimensional points, it follows from (ii) that such a limit point must be infinite-dimensional. Additionally, the closure $\mathrm{cl}(X)$ cannot contain any finite-dimensional point that wasn't already in X (removing such a point would result in closed set still containing X). This proves (iii). Finally, (iv) is given by [50, 5.3.36] and for (v), if $X \subseteq \mathrm{Zg}_R$ contains all finite-dimensional points except $M_1, \ldots, M_n$ say, then $\mathrm{cl}(X \cup \{M_1, \ldots, M_n\}) = \mathrm{Zg}_R$ by (iv) and hence $\mathrm{cl}(X) = \mathrm{Zg}_R \setminus \{M_1, \ldots, M_n\}$ contains all infinite-dimensional points. □

This leads to the following approach to describing the Ziegler spectrum of a finite-dimensional algebra.

Corollary 1.3.7. *Let R be a finite-dimensional algebra with $X_1, \ldots, X_n$ closed subsets of Zg_R such that $X := \bigcup_{i=1}^{n} X_i$ contains almost all finite-dimensional points, then there* *exists a finite set Y and a homeomorphism*

$$\mathrm{Zg}_R \simeq X \sqcup Y$$

where the right-hand side is a disjoint union of topological spaces (with X having the subspace topology and Y the discrete topology).

Let R be a finite-dimensional algebra. Given a component $\mathcal{C} \subseteq \mathrm{mod}\text{-}R$ we denote by $\mathrm{cl}(\mathcal{C})$ the closure of $\mathcal{C} \cap \mathrm{Zg}_R$ in Zg_R, i.e. the closure of the set of (isomorphism classes of) modules belonging to $\mathrm{ind}(\mathcal{C})$.

1.3.3 Stable equivalence of algebras

Let R be a finite-dimensional algebra and denote by $\underline{\mathrm{mod}}\text{-}R$ the projectively stable category of $\mathrm{mod}\text{-}R$, i.e. the category with objects $\underline{M}$ for $M \in \mathrm{mod}\text{-}R$ and with hom-spaces $(\underline{M}, \underline{N}) := \mathrm{Hom}_R(M, N)/P(M, N)$, where $P(M, N)$ denotes the subspace of all homomorphisms that factor through a projective module.

Two algebras R and S are said to be **stably equivalent** if there exists an equivalence $\underline{\mathrm{mod}}\text{-}R \simeq \underline{\mathrm{mod}}\text{-}S$.

Recall there exists a duality $\mathrm{Tr} : (\underline{\mathrm{mod}}\text{-}R)^{\mathrm{op}} \to \underline{\mathrm{mod}}\text{-}R^{\mathrm{op}}$ [63, IV.2.2] and therefore two algebras R and S are stably equivalent if and only if there exists an equivalence $\underline{\mathrm{mod}}\text{-}R^{\mathrm{op}} \to \underline{\mathrm{mod}}\text{-}S^{\mathrm{op}}$.

Given a finite-dimensional algebra R define $\mathrm{Zg}^s_R := \mathrm{Zg}_R \setminus \mathrm{proj}\text{-}R$. Note Zg^s_R is a cofinite subset of Zg_R, containing all but the set $\mathfrak{P}$ of indecomposable projective modules, and satisfies $\mathrm{Zg}_R = \mathrm{Zg}^s_R \sqcup \mathfrak{P}$ by Corollary 1.3.7. The following result is (essentially) due to Krause [40].

Proposition 1.3.8. *If R and S are stably equivalent algebras, then there exists a homeomorphism $\mathrm{Zg}^s_R \simeq \mathrm{Zg}^s_S$. Moreover, if $F : \underline{\mathrm{mod}}\text{-}R \to \underline{\mathrm{mod}}\text{-}S$ is an equivalence, then there is a homeomorphism $\varphi : \mathrm{Zg}^s_R \to Zg^s_S$ satisfying $\varphi(M) = F(M)$ for all non-projective $M \in \mathrm{ind}\text{-}R$.*

Proof. The canonical projection $\mathrm{mod}\text{-}R \to \underline{\mathrm{mod}}\text{-}R$ extends (essentially uniquely) to a functor $G : \mathrm{Mod}\text{-}R \to \mathrm{Flat}((\underline{\mathrm{mod}}\text{-}R)^{\mathrm{op}}, \mathrm{Ab})$ that commutes with direct limits and satisfies $G(M) = (-, \underline{M})$ for all $M \in \mathrm{mod}\text{-}R$ [38, 2.1]. Now G induces a homeomorphism

$\mathrm{Zg}_R^s \to \mathrm{Zg}(\mathrm{Flat}((\underline{\mathrm{mod}}\text{-}R)^{\mathrm{op}}, \mathrm{Ab}))$ [38, 2.2] and, in this way, Zg_R^s is determined (up to homeomorphism) by $\underline{\mathrm{mod}}\text{-}R$.

Now given an equivalence $F : \underline{\mathrm{mod}}\text{-}R \to \underline{\mathrm{mod}}\text{-}S$, then restriction along F gives an equivalence $\mathrm{res}_F : ((\underline{\mathrm{mod}}\text{-}S)^{\mathrm{op}}, \mathrm{Ab}) \to ((\underline{\mathrm{mod}}\text{-}R)^{\mathrm{op}}, \mathrm{Ab})$. Note res_F has a left adjoint, namely $F_L : ((\underline{\mathrm{mod}}\text{-}R)^{\mathrm{op}}, \mathrm{Ab}) \to ((\underline{\mathrm{mod}}\text{-}S)^{\mathrm{op}}, \mathrm{Ab})$, that is also an equivalence [43, IV.4.1] and satisfies

$$F_L((-, \underline{M})) = (?, F-) \otimes_{\underline{\mathrm{mod}}\text{-}R} (-, \underline{M}) \simeq (?, F(\underline{M}))$$

by Proposition 1.11.1 below. As F_L preserves flat functors (they are direct limits of representable functors), a homeomorphism $\mathrm{Zg}_R^s \to \mathrm{Zg}_S^s$ is thus obtained as a composition of homeomorphisms

$$\mathrm{Zg}_R^s \to \mathrm{Zg}(\mathrm{Flat}((\underline{\mathrm{mod}}\text{-}R)^{\mathrm{op}}, \mathrm{Ab})) \to \mathrm{Zg}(\mathrm{Flat}((\underline{\mathrm{mod}}\text{-}S)^{\mathrm{op}}, \mathrm{Ab})) \to \mathrm{Zg}_S^s$$

with the desired property. $\square$

In the above proof $\mathrm{Flat}((\underline{\mathrm{mod}}\text{-}R)^{\mathrm{op}}, \mathrm{Ab})$ — the category of flat additive functors $(\underline{\mathrm{mod}}\text{-}R)^{\mathrm{op}} \to \mathrm{Ab}$ — is a "locally finitely-presented" additive category with products [14, 2.1] (elsewhere called "finitely accessible" with products). A notion of purity (and Ziegler spectra) exists for such categories, see e.g. [38, §1], [14, §3], [50, §16]. The functor F_L is seen to commute with direct limits and products; it is an interpretation functor, in a general sense.

Corollary 1.3.9. *If R and S are stably equivalent finite-dimensional algebras with the same number of (isomorphism classes of) indecomposable projective modules, then there exists a homeomorphism $\mathrm{Zg}_R \simeq \mathrm{Zg}_S$.*

1.3.4 PP functors and the isolation condition

The following definitions are algebraic variants of concepts more readily understandable in the model theory of modules. A comprehensive account of both algebraic and model theoretic approaches is given in [50]. However, we only need the two simple Lemmas 1.3.10 and 1.3.12 — to ultimately apply Proposition 1.3.13 — so the reader is not expected to be fluent in model theory.

Let R be a ring, given $C \in \text{mod-}R$ and $\bar{c} = (c_1, \ldots, c_n) \in C^n$ for some $n \geq 1$, define the functor $\phi_{C,\bar{c}} : \text{Mod-}R \to \text{Ab}$ by

$$\phi_{C,\bar{c}}(M) := \{ (f(c_1), \ldots, f(c_n)) \mid f \in (C, M) \}$$

for all $M \in \text{Mod-}R$. Note $\phi_{C,\bar{c}}(M)$ is a subgroup of M^n. Given $M \in \text{Mod-}R$, any subgroup $\phi(M) \leq M^n$ of this form — i.e. $\phi = \phi_{C,\bar{c}}$ for some C and $\bar{c}$ — is called a **pp-definable subgroup**.

A functor $F : \text{Mod-}R \to \text{Ab}$ is called a **pp functor** if there exist two functors $\phi := \phi_{C_\phi,\bar{c}_\phi}$ and $\psi := \psi_{C_\psi,\bar{c}_\psi}$ satisfying $\psi(M) \leq \phi(M)$ and a natural equivalence $F(M) \simeq \phi(M)/\psi(M)$ for all $M \in \text{Mod-}R$ (note the assignment $M \mapsto \phi(M)/\psi(M)$ is functorial). Such a functor will be denoted by ϕ/ψ. Note, these are just the interpretation functors $\text{Mod-}R \to \text{Ab}$ [15, 2.1], cf. [50, 1.2.3, 1.2.31].

Given a pp functor ϕ/ψ denote by (ϕ/ψ) the following open subset of Zg_R. Such subsets define a basis of open sets for Zg_R [50, 5.1.8].

$$(\phi/\psi) := \{ M \in \text{Zg}_R \mid \phi(M)/\psi(M) \neq 0 \}$$

For $M \in \text{Mod-}R$, a pp functor ϕ/ψ is called M**-minimal** if there exists no pp-definable subgroup strictly between $\psi(M)$ and $\phi(M)$. Given a closed subset $X \subseteq \text{Zg}_R$, a pp functor ϕ/ψ is X**-minimal** provided ϕ/ψ is M-minimal for every $M \in \text{Mod-}R$ such that $\langle M \rangle \cap \text{Zg}_R = X$ (here $\langle M \rangle$ denotes the smallest definable subcategory containing M). By [50, 3.4.11] it is enough to check the condition on one such module M. Hence, if X is the closure of a point $M \in \text{Zg}_R$, then a pp functor ϕ/ψ is X-minimal if and only if it is M-minimal.

Lemma 1.3.10. *Let A be a finite-dimensional algebra. For all $M \in \text{mod-}A$, the hom-functor $(M, -) : \text{Mod-}A \to \text{Ab}$ is (equivalent to) a pp functor. Furthermore, this pp functor is N-minimal for any $N \in \text{Mod-}R$ satisfying $\dim_{\mathbf{k}}(M, N) = 1$.*

Proof. As noted above, $(M, -)$ is an interpretation functor and thus a pp functor, i.e. $(M, -) \simeq \phi/\psi$ for some ϕ and ψ. If $\dim_{\mathbf{k}}(M, N) = 1$, then $\psi(N) \leq Y \leq \phi(N)$ implies $Y = \psi(N)$ or $Y = \phi(N)$ — as any pp-definable subgroup Y is necessarily a $\mathbf{k}$-vector space — and thus ϕ/ψ is N-minimal. □

Given a closed set $X \subseteq \mathrm{Zg}_R$ and a point $M \in X$, we say M **is isolated in** X **by an** X**-minimal pp functor** ϕ/ψ if $(\phi/\psi) \cap X = \{M\}$. By [50, 5.3.2] every X-minimal pp functor isolates some point in X. In general, it is unknown whether every isolated point is isolated by a minimal pp functor in this way. The **isolation condition** holds for a closed subset $X \subseteq \mathrm{Zg}_R$ if for every closed subset $Y \subseteq X$ (with respect to the subspace topology) any isolated point of Y is isolated by a Y-minimal pp functor.

Proposition 1.3.11. *[50, 5.3.16] Given a closed subset $X \subseteq \mathrm{Zg}_R$, then the isolation condition holds for X if and only if every isolated point M in some closed subset of X is isolated in its own closure by an M-minimal pp functor.*

Lemma 1.3.12. *If $M \in \mathrm{Zg}_R$ is indecomposable and has finite length over its endomorphism ring, then M is a closed point of Zg_R and is isolated in its closure $\{M\}$ by an M-minimal pp functor.*

Proof. Let $S = \mathrm{End}_R(M)$ and suppose ${}_SM$ has finite length, then M is pure-injective by [50, 4.4.24] and a closed point of Zg_R by [50, 4.4.30]. Additionally, the pp-definable groups of M coincide with the S-submodules of ${}_SM$ by [50, 4.4.25]. Choose a pp formula ϕ such that $\phi(M)$ has minimal but non-zero length over S, then the pp functor $\phi/(x = 0)$ isolates M in its closure. (See [50, §1] for "pp formula"). □

1.3.5 Cantor-Bendixson rank

Let Z be a topological space, denote by $Z' \subseteq Z$ the closed subset of non-isolated points, this set is the **CB (Cantor-Bendixson) derivative** of Z. Set $Z^{(0)} := Z$ and $Z^{(n)} := (Z^{(n-1)})'$ for all $n \geq 1$. These derivatives can be continued transfinitely [50, §5.3.6] and must eventually stabilize. There exists an ordinal α such that $Z^{(\beta)} = Z^{(\alpha)}$ for all $\beta > \alpha$ — define $Z^{(\infty)} := Z^{(\alpha)}$.

If there exists an ordinal α such that $Z^{(\alpha-1)} \neq \emptyset$ and $Z^{(\alpha)} = \emptyset$, then say Z has **CB rank** $\alpha - 1$, denoted $\mathrm{CB}(Z) = \alpha - 1$. Otherwise, say Z has undefined rank, denoted $\mathrm{CB}(Z) = \infty$. For a point $p \in Z$, define $\mathrm{CB}(p) := \alpha$ if $p \in Z^{(\alpha)} \backslash Z^{(\alpha+1)}$ (i.e. if p is isolated in $Z^{(\alpha)}$) and $\mathrm{CB}(p) := \infty$ if no such α exists.

1.3.6 Krull-Gabriel dimension

The **Krull-Gabriel (KG) dimension** of a finite-dimensional algebra A, denoted $\mathrm{KG}(A)$, is a measure of complexity of the category $((\mathrm{mod}\text{-}A)^{\mathrm{op}}, \mathrm{mod}\text{-}\mathbf{k})$ of contravariant $\mathbf{k}$-linear functors from $\mathrm{mod}\text{-}A$ to $\mathrm{mod}\text{-}\mathbf{k}$.

The definition (originating in [26]) and motivation for this dimension is given in [47]. The main result of [47] determines the KG dimension of all standard self-injective algebras of polynomial growth. Such algebras are defined in Chapter 6 and we prove independently (by different methods) the same KG dimension for all 'non-exceptional' algebras of this kind; we do so using the following result.

Proposition 1.3.13. *[50, 5.3.60] Given a ring R, if the isolation condtion holds for Zg_R, then* $\mathrm{CB}(\mathrm{Zg}_R) = \mathrm{KG}(R)$.

1.4 Self-injective algebras

Given an algebra A and a module $M \in \mathrm{Mod}\text{-}A$, define the **radical** of M to be $\mathrm{rad}(M) := M\mathrm{rad}(A)$. Define also the **socle** $\mathrm{soc}(M)$ of M to be the (direct) sum of all simple submodules of M.

Lemma 1.4.1. *[63, IV.3.1] If P is a (non-simple) indecomposable projective-injective module over a locally bounded algebra, then the following exact sequence is an Auslander-Reiten sequence.*

$$0 \to \mathrm{rad}(P) \xrightarrow{\begin{pmatrix} i \\ p \end{pmatrix}} P \oplus \mathrm{rad}(P)/\mathrm{soc}(P) \xrightarrow{(q\ -j)} P/\mathrm{soc}(P) \to 0$$

Here i, j are inclusions and p, q are projections.

Lemma 1.4.2. *If R is a representation-infinite self-injective algebra and P is an indecomposable projective-injective R-module, then* $\mathrm{rad}(P)/\mathrm{soc}(P) \neq 0$. *Equivalently, there exists no AR sequence with projective (= injective) middle term.*

Proof. Lemma 1.4.1 gives the equivalence of the assertions. By [8, X.1.8] the existence of an AR sequence with projective middle term implies R is a Nakayama algebra (with Loewey length 2) and such algebras are representation-finite [8, VI.2.1]. □

Let A be a representation-infinite self-injective algebra and let Γ_A be its Auslander-Reiten quiver. Define the **stable AR quiver** Γ_A^s to be the full translation subquiver of Γ_A given by the stable points (equiv. obtained from Γ_A by removing the projective-injective vertices). As a consequence of Lemma 1.4.2, the connected components of Γ_A^s remain in bijection with the connected components of Γ_A.

Corollary 1.4.3. *If $F : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}B$ is a stable equivalence for representation-infinite self-injective algebras, then the induced homeomorphism $\varphi : \mathrm{Zg}_A^s \to \mathrm{Zg}_B^s$ that extends F — given by Proposition 1.3.8 — satisfies $\varphi(\tau_A M) = \tau_B \varphi(M)$ for all non-projective $M \in \mathrm{ind}\text{-}A$.*

Proof. By [8, X.1.9], F commutes with the AR translate wherever it is defined. □

As A is a self-injective algebra, the (right) socle $\mathrm{soc}(A) := \mathrm{soc}(A_A)$ is a two-sided ideal of A that coincides with the left socle $\mathrm{soc}({}_AA)$ and is both the left and right annihilator of $\mathrm{rad}(A)$ [34, 4.3.3, 4.5.1].

Lemma 1.4.4. *[36, 8.69] If A is a self-injective algebra and M an A-module, then $M = N \oplus E$ where E is injective and N satisfies $N\mathrm{soc}(A) = 0$.*

Thus every non-projective (= non-injective) A-module can be considered as a module over the quotient algebra $A/\mathrm{soc}(A)$.

Corollary 1.4.5. *Given a self-injective finite-dimensional algebra A, there exists a homeomorphism $\mathrm{Zg}_A^s \simeq \mathrm{Zg}_{A/\mathrm{soc}(A)}$.*

Proof. Restriction along the projection $\pi : A \to A/\mathrm{soc}(A)$ is a fully faithful interpretation functor $\mathrm{res}_\pi : \mathrm{Mod}\text{-}A/\mathrm{soc(A)} \to \mathrm{Mod}\text{-}A$, inducing a closed embedding $\mathrm{Zg}_{A/\mathrm{soc}(A)} \to \mathrm{Zg}_A$ by Proposition 1.3.4. The image of this embedding contains all but the projective-injective modules and so is equal to Zg_A^s by Lemma 1.4.4. □

Two algebras A and B are **socle equivalent** if $A/\mathrm{soc}(A) \simeq B/\mathrm{soc}(B)$.

Corollary 1.4.6. *If A and B are socle equivalent self-injective algebras, then there exists a homeomorphism $\mathrm{Zg}_A^s \simeq \mathrm{Zg}_B^s$.*

Proof. Immediate from Corollary 1.4.5. □

We show in the following results that the homeomorphism given by Corollary 1.4.5 commutes with the Auslander-Reiten translation of the finite-dimensional points (wherever it is defined).

Lemma 1.4.7. *If A is a self-injective algebra, then every AR sequence of $A/\mathrm{soc}(A)$-modules is an AR sequence of A-modules.*

Proof. Let $0 \to L \to M \xrightarrow{g} N \to 0$ be an AR sequence in $\mathrm{mod}\text{-}A/\mathrm{soc}(A)$, we claim it is an AR sequence in $\mathrm{mod}\text{-}A$. Restriction along $\pi : A \to A/\mathrm{soc}(A)$ is a fully faithful and exact functor, so L remains indecomposable and g is not a split epimorphism in $\mathrm{mod}\text{-}A$. By [63, IV.1.13] it is enough to show g is right almost split in $\mathrm{mod}\text{-}A$, i.e. any non-split epimorphism $h : X \to N$ in $\mathrm{mod}\text{-}A$ factors through g. If X is non-projective, then h belongs to $\mathrm{mod}\text{-}A/\mathrm{soc}(A)$ by Lemma 1.4.4 and the factorisation is given by assumption. Otherwise X is projective and the factorisation is given by the well known characterisation of projective modules. □

Lemma 1.4.8. *[34, 4.15.3] If A is a self-injective algebra and P a (non-simple) indecomposable projective A-module, then* $\mathrm{rad}(P)$*, resp.* $P/\mathrm{soc}(P)$*, is a injective, resp. projective, module over* $A/\mathrm{soc}(A)$*.*

Corollary 1.4.9. *For a representation-infinite self-injective algebra A, the AR quiver $\Gamma_{A/\mathrm{soc}(A)}$ is obtained from Γ^s_A by forgetting the translate of all points corresponding to modules of form $P/\mathrm{soc}(P)$ where P is projective-injective A-module.*

Proof. The claim is that the underlying quivers of $\Gamma_{A/\mathrm{soc}(A)}$ and Γ^s_A are the same and only their translations differ as specified.

By Lemma 1.4.4 we can identify the vertices of $\Gamma_{A/\mathrm{soc}(A)}$ with those of Γ^s_A. By Lemma 1.4.7 we can also identfy all arrows with non-injective source and all arrows of non-projective target in $\Gamma_{A/\mathrm{soc}(A)}$ with corresponding arrows, having the same source, resp. target, in Γ^s_A.

By Lemma 1.4.8 the indecomposable injective, resp. projective, $A/\mathrm{soc}(A)$-modules are of the form $\mathrm{rad}(P)$, resp. $P/\mathrm{soc}(P)$, as P ranges over the indecomposable projective-injective A-modules. If P is such an A-module, then Lemma 1.4.1 gives the following AR sequence in $\mathrm{mod}\text{-}A$.

$$0 \to \mathrm{rad}(P) \to P \oplus \mathrm{rad}(P)/\mathrm{soc}(P) \to P/\mathrm{soc}(P) \to 0$$

As an epimorphism in $\text{mod-}A/\text{soc}(A)$ the map $\text{rad}(P) \to \text{rad}(P)/\text{soc}(P)$ is left-minimal almost-split, since $\text{rad}(P)$ is injective [63, IV.3.5]. Dually, as a monomorphism in $\text{mod-}A/\text{soc}(A)$ the map $\text{rad}(P)/\text{soc}(P) \to P/\text{soc}(P)$ is right-minimal almost-split as $P/\text{soc}(P)$ is projective. Thus we have a clear bijection between the arrows with source $\text{rad}(P)$ in $\Gamma_{A/\text{soc}(A)}$ and Γ_A^s respectively, and similarly a bijection between the arrows with target $P/\text{soc}(P)$ in $\Gamma_{A/\text{soc}(A)}$ and Γ_A^s respectively. □

1.5 Quasi-stable components

Let A be a locally bounded algebra, for each component $\mathcal{C} \subseteq \text{mod-}A$ denote by $\Gamma^s(\mathcal{C})$ the translation quiver obtained from $\Gamma(\mathcal{C})$ by removing the projective vertices. A component $\mathcal{C}$ is called **quasi-stable** if $\Gamma^s(\mathcal{C})$ is stable (equiv. all projective modules in $\mathcal{C}$ are injective, and vice versa).

If A is self-injective, then every component of $\text{mod-}A$ is quasi-stable. Indeed, for any algebra, the AR translate (resp. its inverse) is defined for all non-projective (resp. non-injective) modules in $\text{ind-}A$.

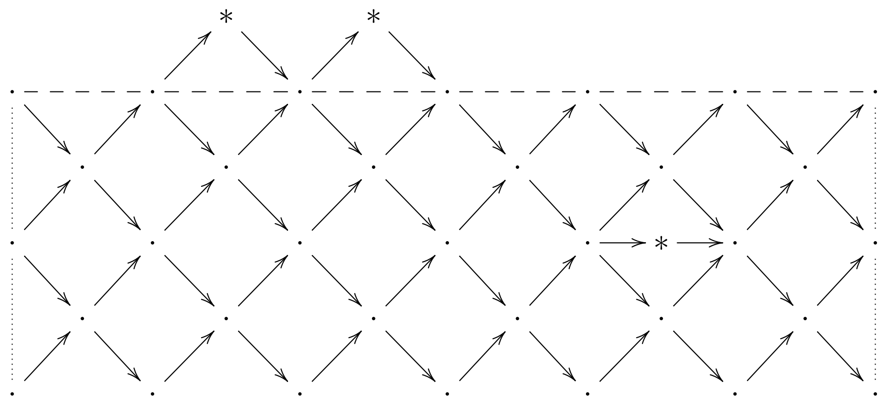
Given a connected quiver Δ with vertex set I define the translation quiver $\mathbb{Z}\Delta$ as follows. The vertices of $\mathbb{Z}\Delta$ are $\mathbb{Z} \times I = \{(n,i) \mid n \in \mathbb{Z}, i \in I\}$ and arrows are $(n,i) \to (n,j)$ and $(n+1,j) \to (n,i)$ for each arrow $i \to j$ in Δ and all $n \in \mathbb{Z}$. The translate is defined by $\tau_{\mathbb{Z}\Delta}(n,i) := (n+1,i)$ for all $(n,i) \in \mathbb{Z} \times I$.

Lemma 1.5.1. *[63, VIII.1.7–1.8] If Δ and Δ' are two quivers sharing a common underlying tree, or if both are of Euclidean type $\widetilde{\mathbb{A}}_p$ and have the same number of clockwise and anti-clockwise arrows, then $\mathbb{Z}\Delta \simeq \mathbb{Z}\Delta'$.*

Examples of quasi-stable components $\mathcal{C}$ of the form $\Gamma^s(\mathcal{C}) = \mathbb{Z}\Delta$, where Δ is Euclidean quiver, are given in Section 5.1.

A component $\mathcal{C} \subseteq \text{mod-}A$ is called a **quasi-tube** (resp. **quasi-stable tube**) if $\Gamma^s(\mathcal{C})$ is a tube (resp. stable tube) and the **rank** of a quasi-stable tube $\mathcal{C}$ is just the rank of $\Gamma^s(\mathcal{C})$.

Example 1.5.2. The following diagram depicts (a finite part of) a quasi-stable tube. The projective-injective vertices are denoted by $*$ and only the translate at the mouth (of the corresponding stable tube) is drawn, by a dashed line. The left and right edges are to be identified along the dotted lines.



(1.5.3)

Technically, a quasi-stable tube need not be a tube (in the actual module category mod-A, as opposed to the stable module category $\underline{\text{mod}}$-A) and therefore may not enjoy all the nice properties a tube has. The following result gives a criterion for being a genuine tube.

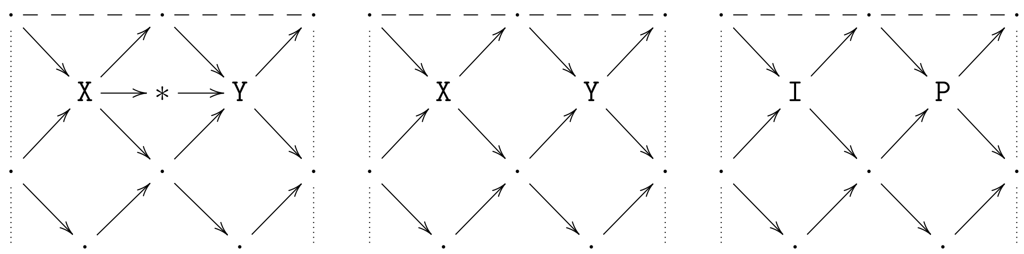
Lemma 1.5.4. *A quasi-stable tube $\mathcal{C} \subseteq$ mod-A is a tube if and only if, for each projective-injective $P \in \text{ind}(\mathcal{C})$, the module $\text{rad}(P)/\text{soc}(P)$ is indecomposable. If $\mathcal{C}$ is a tube, then each projective-injective vertex belongs to the mouth of $\Gamma(\mathcal{C})$.*

Proof. If $\mathcal{C}$ is a tube, then a middle term in any AR sequence belonging to $\mathcal{C}$ has at most 2 indecomposable direct summands (in any given decomposition). By Lemma 1.4.1 a projective-injective module $P \in \text{ind}(\mathcal{C})$ lies in the AR sequence

$$0 \to \text{rad}(P) \to P \oplus \text{rad}(P)/\text{soc}(P) \to P/\text{soc}(P) \to 0$$

hence $\text{rad}(P)/\text{soc}(P)$ must be indecomposable. Note also, as $\Gamma^s(\mathcal{C})$ is a stable tube, the modules $\text{rad}(P)$ and $P/\text{soc}(P)$ must belong to the mouth of $\Gamma^s(\mathcal{C})$. Since $\Gamma(\mathcal{C})$ is obtained from $\Gamma^s(\mathcal{C})$ by adding back the projective-injective vertices, it follows that P must belong to the mouth of $\Gamma(\mathcal{C})$. The converse is now clear, if each $\text{rad}(P)/\text{soc}(P)$ is indecomposable, then $\mathcal{C}$ is a tube containing a "crown" of projective-injective vertices at the mouth of $\Gamma(\mathcal{C})$. $\square$

Let A be a finite-dimensional self-injective algebra. By Corollary 1.4.9, the components of Γ^s_A and $\Gamma_{A/\mathrm{soc}(A)}$ are in bijection. By definition, a quasi-stable tube $\mathcal{C}$ in mod-A determines a stable tube $\Gamma^s(\mathcal{C}) \subseteq \Gamma^s_A$. However the corresponding component of $\Gamma_{A/\mathrm{soc}(A)}$ is neither a stable nor a quasi-stable tube (it may not even be a tube). This is illustrated in the following diagram. The point to stress here is that quasi-stable tubes over A cannot simply be reduced to (stable) tubes over $A/\mathrm{soc}(A)$.



The left-most translation quiver depicts a quasi-stable tube (with translate drawn at the mouth only by a dashed line, and the dotted edges to be identified); the vertex $*$ is projective-injective and $\tau\mathtt{Y} = \mathtt{X}$. The centre quiver is the stable tube obtained by removing $*$. The right-most quiver is the translation quiver obtained by forgetting the translation $\tau\mathtt{Y} = \mathtt{X}$, resulting in a projective vertex $\mathtt{P} = \mathtt{Y}$ and an injective vertex $\mathtt{I} = \mathtt{X}$. The resulting translation quiver is not a tube, in particular, its geometric realisation (i.e. the triangulation given by the arrows and translates) is no longer homeomorphic to the half-infinite cylinder $S^1 \times (\mathbb{R}^+ \cup \{0\})$ — there is a hole between $\mathtt{I}$ and $\mathtt{P}$.

1.6 Torsion pairs

Let $\mathcal{A}$ be an abelian category, then a pair $(\mathcal{F}, \mathcal{T})$ of full subcategories of $\mathcal{A}$ is said to be a **torsion pair** (or a **torsion theory**) if the following conditions are satisfied:

(a) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$.

(b) If $\mathrm{Hom}_{\mathcal{A}}(M, \mathcal{F}) = 0$, then $M \in \mathcal{T}$.

(c) If $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, M) = 0$, then $M \in \mathcal{F}$.

If $(\mathcal{F}, \mathcal{T})$ is a torsion pair, then $\mathcal{F}$ is called the class of **torsion-free** objects and $\mathcal{T}$ is called the class of **torsion** objects. We will always write a torsion pair ordered with the torsion-free class in the first position. Each of the subcategories $\mathcal{F}$ and $\mathcal{T}$ determines the other as a hom-orthogonal subcategory.

Lemma 1.6.1. *[69, VI.2.1–2.2] [63, VI.1.4] If $\mathcal{A}$ is an abelian category, then*

(i) A full subcategory $\mathcal{F} \subseteq \mathcal{A}$ is a torsion-free class of a torsion pair in $\mathcal{A}$ if and only if $\mathcal{F}$ is closed under subobjects, products, and extensions.

(ii) A full subcategory $\mathcal{T} \subseteq \mathcal{A}$ is a torsion class of a torsion pair in $\mathcal{A}$ if and only if $\mathcal{T}$ is closed under images, coproducts, and extensions.

A torsion pair $(\mathcal{F}, \mathcal{T})$ in an abelian category $\mathcal{A}$ is said to be **split** (or **splitting**) in case every indecomposable object of $\mathcal{A}$ lies in $\mathcal{F}$ or $\mathcal{T}$. In general, given a torsion pair $(\mathcal{F}, \mathcal{T})$ and object $M \in \mathcal{A}$, there exists a canonical exact sequence

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0$$

where $t(M) \in \mathcal{T}$ is the largest torsion subobject of M and $M/t(M) \in \mathcal{F}$ a torsion-free quotient. A torsion pair is then split whenever each of these canonical exact sequences is split exact in $\mathcal{A}$.

1.7 Tilting theory

Let A be a finite dimensional algebra, then a finite-dimensional A-module M is a **tilting module** if it satisfies the following conditions:

(a) $\mathrm{pd}(M) \leq 1$.

(b) $\mathrm{Ext}^1_A(M, M) = 0$.

(c) The number of isomorphism classes of indecomposable direct summands of M is equal to the number of isomorphism classes of indecomposable direct summands of A_A.

Let M be a tilting A-module and let $B := \mathrm{End}_A(M)$, we say B **is tilted from** A. Consider M naturally as a B-A-bimodule and define the following module classes:

- $\mathcal{F} = \ker(\mathrm{Hom}_A(M, -))$,

- $\mathcal{G} = \ker(\operatorname{Ext}^1_A(M, -))$,
- $\mathcal{X} = \ker((-) \otimes_B M)$.
- $\mathcal{Y} = \ker(\operatorname{Tor}^1_B(M, -))$,

Rather than introduce extra notation, we will consider $\mathcal{F}$ and $\mathcal{G}$ as full subcategories of either mod-A or Mod-A depending on context. Similarly, we may consider $\mathcal{X}$ and $\mathcal{Y}$ as full subcategories of mod-B or Mod-B.

Proposition 1.7.1. *If M is a tilting A-module and $B = \operatorname{End}_A(M)$, then*

(i) $(\mathcal{F}, \mathcal{G})$ *is a torsion pair of* mod-A *(or* Mod-A*),*

(ii) $(\mathcal{Y}, \mathcal{X})$ *is a torsion pair of* mod-B *(or* Mod-B*),*

(iii) The functors $\operatorname{Hom}_A(M, -)$ *and* $(-) \otimes_B M$ *restrict to mutually inverse equivalences* $\mathcal{G} \simeq \mathcal{Y}$.

(iv) The functors $\operatorname{Ext}^1_A(M, -)$ *and* $\operatorname{Tor}^1_B(M, -)$ *restrict to mutually inverse equivalences* $\mathcal{F} \simeq \mathcal{X}$.

Proof. For finitely-presented modules, these statements are given by the classic Brenner-Butler tilting theorems [10, §3] (see also [55, §4.1] and [63, §VI]). For arbitrary modules see [12, §1]. □

We now give some results concerning torsion pairs for modules over a hereditary algebra (see Section 1.8 for the definition of "hereditary algebra").

Lemma 1.7.2. *[12, 1.6] If M is a tilting A-module for a hereditary algebra A, then $(\mathcal{Y}, \mathcal{X})$ is a split torsion pair (in* mod-B *or* Mod-B*), in particular,* mod-$B = \mathcal{X} \vee \mathcal{Y}$.

Lemma 1.7.3. $\mathcal{F}$ *(resp. $\mathcal{G}$) is definable subcategory of* Mod-A *and $\mathcal{X}$ (resp. $\mathcal{Y}$) is definable subcategory of* Mod-B.

Proof. As M is a finitely-presented R-module, both $\operatorname{Hom}_R(M, -)$ and $\operatorname{Ext}^1_R(M, -)$ are interpretation functors and therefore have definable kernels $\mathcal{F}$ and $\mathcal{G}$ respectively. Similarly $\mathcal{Y}$ and $\mathcal{X}$ are definable subcategories of Mod-B [50, §18.2.3]. □

Corollary 1.7.4. *Let A be a hereditary algebra, M a tilting A-module, and $B = \operatorname{End}_A(M)$. If $(\mathcal{F}, \mathcal{G})$ is the torsion pair of* Mod-A *determined by M, then there is a homeomorphism* $\operatorname{Zg}_B \simeq \operatorname{Zg}(\mathcal{F}) \sqcup \operatorname{Zg}(\mathcal{G})$.

Proof. By Lemma 1.7.2, M determines a split torsion pair $(\mathcal{Y}, \mathcal{X})$ in Mod-B. In particular, every indecomposable pure-injective B-module lies in one of the (mutually disjoint) definable subcategories $\mathcal{X}$ or $\mathcal{Y}$. Therefore

$$\mathrm{Zg}_B = \mathrm{Zg}(\mathcal{X}) \sqcup \mathrm{Zg}(\mathcal{Y}) \simeq \mathrm{Zg}(\mathcal{F}) \sqcup \mathrm{Zg}(\mathcal{G})$$

by the equivalences of Proposition 1.7.1. □

Say two algebras A and B are **tilting-cotilting equivalent** if there is a sequence of algebras $A_1, \dots, A_n$ such that $A_1 = A$, $A_n = B$ and for each $i = 1, \dots, n-1$, either A_i is tilted from A_{i+1} or A_{i+1} is tilted from A_i.

We can usually assume a tilting module M is **multiplicity-free**, i.e. no two indecomposable direct summands of M are isomorphic. For such tilting modules, $\mathrm{End}_A(M)$ is a basic algebra.

1.8 Tame hereditary algebras

An algebra is **hereditary** if any submodule of a projective module is again projective. Equivalently, an algebra is hereditary if and only if every quotient of an injective module is again injective [63, VII.1.4].

Every finite dimensional hereditary algebra is Morita equivalent to the path algebra of a finite acyclic quiver [63, VII.1.7]. If $A = \mathbf{k}\mathscr{Q}$ is the path algebra of a finite, connected, and acyclic quiver $\mathscr{Q}$ (equiv. A is basic and connected), then A is representation-finite if and only if $\mathscr{Q}$ has Dynkin type; A is representation-infinite tame (equiv. domestic) if and only if $\mathscr{Q}$ has Euclidean type; and A is representation wild otherwise [65, VII.5.10, XIX.3.15].

We give a brief overview of the representation theory of (representation-infinite) tame hereditary algebras. For futher details on the finite-dimensional representation theory we refer to [55] [63] [64] [65] and for the Ziegler spectra of these algebras we refer to [49] [58] [50, §8.1.2].

Henceforth, by a **(tame) hereditary algebra of Euclidean type** Δ we mean a path algebra $\mathbf{k}\mathscr{Q}$ of a quiver $\mathscr{Q}$ of Euclidean type Δ (i.e. one of $\widetilde{\mathbb{A}}_n$ for $n \geq 1$, $\widetilde{\mathbb{D}}_m$ for $m \geq 4$, or $\widetilde{\mathbb{E}}_n$ for $n = 6, 7, 8$) — in the case of $\widetilde{\mathbb{A}}_n$ we forbid the cyclic orientation.

We also divide the class of hereditary algebras of type $\widetilde{\mathbb{A}}_n$ and say such an algebra has type $\widetilde{\mathbb{A}}_{p,q}$ if its quiver has p anti-clockwise arrows and q clockwise arrows such that $1 \leq p \leq q$ (take the opposite quiver if necessary).

Proposition 1.8.1. *[55, 3.6.5] If H is a hereditary algebra of Euclidean type Δ, then*

$$\operatorname{mod}\text{-}H = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q}$$

where $\mathcal{P}$ is a preprojective component, containing all projective modules; $\mathcal{Q}$ is a preinjective component, containing all injective modules; and $\mathcal{T}$ is a stable tubular $\mathbb{P}^1(\mathrm{k})$-family separating $\mathcal{P}$ from $\mathcal{Q}$. Furthermore

(i) $\mathcal{T}$ has tubular type (p,q) iff $\Delta = \widetilde{\mathbb{A}}_{p,q}$ for $1 \leq p \leq q$.

(ii) $\mathcal{T}$ has tubular type $(2,2,n-2)$ iff $\Delta = \widetilde{\mathbb{D}}_n$ for $n \geq 4$.

(iii) $\mathcal{T}$ has tubular type $(2,3,n-3)$ iff $\Delta = \widetilde{\mathbb{E}}_n$ for $n = 6,7,8$.

Additionally, $\mathcal{T}$ forms an abelian subcategory of $\operatorname{mod}\text{-}H$*. Modules belonging to $\mathcal{T}$ are called **regular**. Modules that are simple as objects in $\mathcal{T}$ are called **simple regular**.*

Proposition 1.8.2. *[49] [58] Let H be a hereditary algebra of Euclidean type, then the infinite-dimensional points of Zg_H consist of a unique generic module G, and for each tube of rank $n \geq 1$, say, there are n modules $S_i[\infty]$ — called **Prüfer modules** — and n modules $\hat{S}_i$ — called **adic modules** — corresponding to the simple regular modules S_i for $i = 1, \ldots, n$ at the mouth of the tube. Furthermore, a subset $\mathfrak{C} \subseteq \mathrm{Zg}_H$ is closed if and only if the following conditions hold:*

(a) For each simple regular module S, if $\mathfrak{C}$ contains infinitely many points M satisfying $(S,M) \neq 0$, then $\mathfrak{C}$ contains $S[\infty]$.

(b) For each simple regular module S, if $\mathfrak{C}$ contains infinitely many points M satisfying $(M,S) \neq 0$, then $\mathfrak{C}$ contains $\hat{S}$.

(c) If $\mathfrak{C}$ is infinite or contains any infinite-dimensional point, then $\mathfrak{C}$ contains G.

As described in the above proposition, the infinite-dimensional points of Zg_H correspond to the tubes of finite-dimensional modules; this connection is made explicit later in Proposition 3.4.10.

Example 1.8.3. The path algebra $H := \mathbf{k}(\cdot \rightleftarrows \cdot)$ is called the **(tame) Kronecker algebra**, it is a domestic hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_1$ and all tubes in $\text{mod-}H$ have rank 1. For future reference, the simple regular H-modules $(S_\lambda)_{\lambda \in \mathbb{P}^1(\mathbf{k})}$ are as follows: for $\lambda \in \mathbf{k}$ we have $S_\lambda := \mathbf{k} \underset{\lambda}{\overset{1}{\leftleftarrows}} \mathbf{k}$ and finally $S_\infty := \mathbf{k} \underset{1}{\overset{0}{\leftleftarrows}} \mathbf{k}$. See for instance [55, 3.2].

1.8.1 APR reflections

Let $\mathscr{Q}$ be a quiver with $v \in \mathscr{Q}$ a vertex. Define the quiver $\sigma_v\mathscr{Q}$ — called the **reflection of $\mathscr{Q}$ at** v — to be the same as $\mathscr{Q}$ except any arrow $u \to v$ (resp. $v \to u$) is to be replaced by an arrow $v \to u$ (resp. $u \to v$) in the opposite direction. For instance, if v is a source in $\mathscr{Q}$, then v is a sink in $\sigma_v\mathscr{Q}$, and vice versa.

Proposition 1.8.4. *Let $A = \mathbf{k}\mathscr{Q}_A$ be an hereditary algebra. If $v \in \mathscr{Q}_A$ is a sink vertex and $B = \mathbf{k}(\sigma_v\mathscr{Q}_A)$, then B is tilted from A and there exists a homemorphism $\mathrm{Zg}_A \simeq \mathrm{Zg}_B$.*

Proof. There exists a tilting A-module M (commonly called an **APR tilting** module) such that $B \simeq \mathrm{End}_A(M)$ [63, VII.5.3]. For such a tilting module the torsion pair induced on $\text{mod-}A$ is split — in fact, the simple projective $S(v)$ is the only torsion-free indecomposable A-module [63, VI.2.8(c)]. The corresponding torsion pair $(\mathcal{F}, \mathcal{G})$ in $\text{Mod-}A$ then gives two definable subcategories which together contain all indecomposable finite-dimensional modules, thus $\mathrm{Zg}_A = \mathrm{Zg}(\mathcal{F}) \sqcup \mathrm{Zg}(\mathcal{G})$ by Proposition 1.3.6 and $\mathrm{Zg}_A \simeq \mathrm{Zg}_B$ by Corollary 1.7.4. □

Corollary 1.8.5. *Two hereditary algebras of the same Euclidean type (equiv. same tubular type) are tilting-cotilting equivalent and have homeomorphic Ziegler spectra.*

Proof. The result follows by repeated application of Proposition 1.8.4 and the fact that, in a finite sequence of such reflections at sinks, we can move between any two quivers of the same Euclidean type. This follows from [63, VII.5.2] for types $\widetilde{\mathbb{D}}_n$ (for $n \geq 4$) and $\widetilde{\mathbb{E}}_m$ (for $m = 6, 7, 8$), and from [63, VIII.1.8] for types $\widetilde{\mathbb{A}}_{p,q}$ (for $1 \leq p \leq q$). □

1.9 Euclidean algebras

In this work by a **Euclidean algebra (of type Δ)** we mean an algebra of the form $B = \mathrm{End}_H(M)$ where H is a hereditary algebra of Euclidean type Δ and M is a tilting H-module that does not contain both a preprojective and a preinjective direct summand.

This assumption on M is equivalent to assuming that the tilted algebra B is also representation-infinite [55, 4.2.8]. Thus we use "Euclidean algebra" as short-hand for a representation-infinite tilted (from hereditary) algebra of Euclidean type — note this differs from [66] for our "Euclidean algebra" class is closed under taking opposite rings. An alternate characterisation of these algebras is given below. Their representation theory is well understood [55, 4.9] [65, XVII]. They form a class of 1-parametric domestic algebras and we give a brief overview here. Their Ziegler spectra can be determined by Corollary 1.7.4.

1.9.1 Tame concealed algebras

A **tame concealed algebra** is a tilted algebra of the form $B = \mathrm{End}_H(M)$ where H is a representation-infinite tame hereditary algebra and M is a preprojective tilting H-module (equiv. a preinjective tilting H-module [55, 4.3.1]).

Thus tame concealed algebras are Euclidean algebras. If B is a tame concealed algebra, then the structure of mod-B is given as in Proposition 1.8.1 (with H replaced by B). This follows from Lemma 1.7.2 — see [55, §4.3] for further details. We only note that if $(\mathcal{F}, \mathcal{G})$ is the torsion-pair induced by M in mod-H, then the tubular family over H belongs to $\mathcal{G}$ (assuming M is preprojective) and remains intact upon tilting, i.e. the tilting functor $\mathrm{Hom}_H(M, -)$ maps the tubular family of mod-H to a tubular family of mod-B.

1.9.2 Domestic tubular extensions of tame concealed algebras

A Euclidean algebra that is not tame concealed is of the form $B = \mathrm{End}_H(M)$ where M is a tilting H-module containing no preinjective direct summand but containing a non-zero regular direct summand, or by duality, is opposite to such an algebra. Note, since the tilting module M contains a regular direct summand, unlike the concealed case, the tubular family of mod-H is not (entirely) preserved under tilting. For the following characterisation of Euclidean algebras, we use the definition of a "tubular branch extension" given in Section 3.6.2.

Proposition 1.9.1. *[55, 4.9.1] A tubular branch extension or a tubular branch co-extension of a tame concealed algebra of extension type $\bar{n} = (p, q)$ for $1 \leq p \leq q$, $(2, 2, m-2)$ for $m \geq 4$, or $(2, 3, n-3)$ for $n = 6, 7, 8$, is a Euclidean algebra (and is tilted from a tame hereditary algebra of tubular type $\bar{n}$). Conversely, every Euclidean algebra is of this form.*

Proposition 1.9.2. *[55, 4.9.2] If B is a Euclidean algebra, then*

$$\text{mod-}B = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q}$$

where $\mathcal{P}$ is a preprojective component; $\mathcal{Q}$ is a preinjective component; and $\mathcal{T}$ is a tubular $\mathbb{P}^1(\mathbf{k})$-family separating $\mathcal{P}$ from $\mathcal{Q}$.

Remark We point out the following facts proven in the above results, cf. [65, XVII.3.5]. Suppose that $B = \text{End}_H(M)$ is a Euclidean algebra such that B is the tubular extension of a tame concealed algebra B_0 as described by Proposition 1.9.1. Writing mod-$B = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q}$ in the form given by Proposition 1.9.2, then the following facts hold.

- $\mathcal{P}$ consists entirely of B_0-modules and coincides with the preprojective component of mod-B_0.
- $\mathcal{Q}$ is the image of the preinjective component of mod-H under the tilting functor $\text{Hom}_H(M, -)$; it contains all preinjective B_0-modules.
- $\mathcal{T}$ contains all regular B_0-modules with almost all tubes of $\mathcal{T}$ being stable and belonging to mod-B_0. The remaining tubes contain projective modules, they are non-stable but "coherent" in the sense of [17].

Dual statements hold for domestic tubular cotextensions of a tame concealed algebra.

1.10 Tubular algebras

Following Ringel [55, §5] we define a **tubular algebra** to be a tubular branch extension of a tame concealed algebra of extension type $\bar{n} = (2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$. For the definition of "tubular branch extension" see Section 3.6.2.

Unlike the domestic tubular extensions of Section 1.9.2, we do not get anything new by considering also tubular coextensions, that is to say, the opposite of a tubular algebra is again a tubular algebra [55, 5.3.2]. Thus, if A is a tubular algebra of type $\bar{n}$, then there exists tame concealed algebras A_0 and A_∞ such that A is simultaneously a tubular extension of A_0 and tubular coextension of A_∞.

Proposition 1.10.1. *[55, 5.2.4] If A is a tubular algebra, defined as a tubular branch extension of tame concealed A_0 and tubular branch coextension of tame concealed A_∞, then*

$$\operatorname{mod}\text{-}A = \mathcal{P} \vee \mathcal{T}_0 \vee \bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q \vee \mathcal{T}_\infty \vee \mathcal{Q}$$

with the following properties:

- *$\mathcal{P}$ is a preprojective component and is the preprojective component of* $\operatorname{mod}\text{-}A_0$.
- *$\mathcal{Q}$ is a preinjective component and is the preinjective component of* $\operatorname{mod}\text{-}A_\infty$.
- *For all $q \in \mathbb{Q}^+ \cup \{0, \infty\}$ the class $\mathcal{T}_q$ is a tubular $\mathbb{P}^1(\mathrm{k})$-family that separates $\mathcal{P} \vee \bigvee_{q'<q} \mathcal{T}_{q'}$ from $\bigvee_{q'>q} \mathcal{T}_{q'} \vee \mathcal{Q}$.*
- *For all $q \in \mathcal{Q}^+$ the tubular family $\mathcal{T}_q$ is stable.*
- *The tubular family $\mathcal{T}_0$ contains projective modules and is obtained from the regular A_0-modules by tubular extension.*
- *The tubular family $\mathcal{T}_\infty$ contains injective modules and is obtained from the regular A_∞-modules by tubular coextension.*

Let A be a tubular algebra. For $r \in \mathbb{R}^+ \cup \{0, \infty\}$ define $\mathcal{P}_r := \mathcal{P} \vee \bigvee_{q<r} \mathcal{T}_q$ and $\mathcal{Q}_r := \bigvee_{q>r} \mathcal{T}_r \vee \mathcal{Q}$. We say a module $M \in \operatorname{Mod}\text{-}A$ has **slope** $r \in \mathbb{R}^+ \cup \{0, \infty\}$ if $\operatorname{Hom}_A(M, \mathcal{P}_r) = 0$ and $\operatorname{Hom}_A(\mathcal{Q}_r, M) = 0$. By the above proposition, $\mathcal{T}_q$ separates $\mathcal{P}_q$ from $\mathcal{Q}_q$, so the finite-dimensional modules of slope $q \in \mathbb{Q}^+ \cup \{0, \infty\}$ are precisely those belonging to $\mathcal{T}_q$.

Proposition 1.10.2. *[53, 13.6] If A is a tubular algebra, then any indecomposable A-module not belonging to $\mathcal{P}_0$ or $\mathcal{Q}_\infty$ has a unique slope.*

Thus the points of Zg_A—for A a tubular algebra—are partitioned by their slope.

Corollary 1.10.3. *Let A be a tubular algebra and let $\mathcal{D}_r$ be the full subcategory of* $\mathrm{Mod}\text{-}A$ *given by the modules of slope r, for all $r \in \mathbb{R}^+ \cup \{0, \infty\}$. Then each $\mathcal{D}_r$ is a definable subcategory of* $\mathrm{Mod}\text{-}A$ *and (as a disjoint union of sets) we have*

$$\mathrm{Zg}_A = \mathcal{P}_0 \sqcup \left(\bigsqcup_{r \in \mathbb{R}^+ \cup \{0,\infty\}} \mathrm{Zg}(\mathcal{D}_r) \right) \sqcup \mathcal{Q}_\infty \tag{1.10.4}$$

with each subset $\mathrm{Zg}(\mathcal{D}_r)$ *non-empty.*

Proof. Each $M \in \mathcal{Q}_r$ is finite-dimensional and so $\mathrm{Hom}_A(M, -)$ is an interpretation functor with definable kernel. Dually, for $M \in \mathcal{P}_r$ is finite-dimensional and $\mathrm{Hom}_A(-, M)$ has definable kernel. It follows that each $\mathcal{D}_r$ is a definable subcategory of $\mathrm{Mod}\text{-}A$. Now every non-zero definable subcategory contains an indecomposable pure-injective module by [50, 5.1.5] and $\mathcal{D}_r$ is non-zero by [53, §13], hence $\mathrm{Zg}(\mathcal{D}_r)$ is non-empty. Finally, every (non-preprojective, non-preinjective) point of Zg_A has a slope by Proposition 1.10.2 and so belongs to the right-hand side of (1.10.4). Equality follows from Proposition 1.3.6. □

For $q \in \mathbb{Q}^+ \cup \{0, \infty\}$ the points in $\mathrm{Zg}(\mathcal{D}_q)$ — the indecomposable pure-injective modules of slope q — can be described in terms of the tubes belonging to the tubular family $\mathcal{T}_q$. In particular, every infinite-dimensional point of $\mathrm{Zg}(\mathcal{D}_q)$ belongs to the closure of such a tube, and there is a unique generic module in $\mathrm{Zg}(\mathcal{D}_q)$ common to the closure of all tubes in $\mathcal{T}_q$. See Corollary 3.6.7 for this result. Detailed examples are given in Section 5.1.

The Ziegler spectrum of a tubular algebra is further studied in [32] [33] [28]. In particular, for $r \in \mathbb{R}^+$ irrational, the closed subset $\mathrm{Zg}(\mathcal{D}_r)$ consists entirely of infinite-dimensional points, but they are all topologically indistinguishable [33, 8.5].

1.11 Tensor products

The tensor product of two modules over a ring is a familiar construction. A direct generalisation gives the tensor product of two functors over a $\mathbb{k}$-linear category. We end this chapter with a few results concerning their use. In particular, the construction of adjoints to functors between module categories.

1.11.1 Tensor products and adjoint functors

If A is a (essentially) small $\mathbb{k}$-linear category, then there exists a pair of functors

$$\begin{aligned}[-,-]_A :&(A,V)^{\mathrm{op}} \otimes_{\mathbb{k}} (A,V) \to V \\ - \otimes_A - :&(A^{\mathrm{op}},V) \otimes_{\mathbb{k}} (A,V) \to V\end{aligned}$$

where $V := \mathrm{Mod}\text{-}\mathbb{k}$, with the following properties.

The functor $[-,-]_A$ is the **hom functor** defined by $[M,N]_A := \mathrm{Hom}_{(A,V)}(M,N)$ for all $M,N \in A\text{-}\mathrm{Mod}$. In the context of modules over a ring A, this is usually denoted $\mathrm{Hom}_A(M,N)$. This functor commutes with limits in both variables and satisfies the Yoneda isomorphisms $[A(a,-),N]_A \simeq M(a)$ for all $M \in A\text{-}\mathrm{Mod}$ and $a \in A$.

The functor $-\otimes_A-$ is the **tensor product** and generalises the tensor product over a $\mathbb{k}$-algebra. This functor is defined in [21, §1] and [45, §6] (and at a higher level of generality in [37, §3.1]). It preserves colimits in both variables and satisifes the Yoneda isomorphisms $M \otimes_A A(a,-) \simeq M(a)$ and $A(-,a) \otimes_A N \simeq N(a)$ for all $M \in (A^{\mathrm{op}},V)$, $N \in (A,V)$, and $a \in A$. Additionally, there exists a symmetry $M \otimes_A N \simeq N \otimes_{A^{\mathrm{op}}} M$ for all $M \in (A^{\mathrm{op}},V)$ and $N \in (A,V)$.

Given an B-A-bimodule $X : B \otimes_k A^{\mathrm{op}} \to V$ there are a pair of induced functors $[X,-]_B : (B,V) \to (A,V)$ and $X \otimes_A - : (A,V) \to (B,V)$, and for all $M \in (A,V)$, $N \in (B,V)$ there are natural isomorphisms

$$[X \otimes_A M, N]_B \simeq [M,[X,N]_B]_A$$

giving that $X \otimes_A -$ is left adjoint to $[X,-]_B$ — the **hom-tensor adjunction**.

Given a $\mathbb{k}$-linear functor $F : A \to B$ between small $\mathbb{k}$-linear categories, we have an A-B-bimodule $B(-,F-) : B^{\mathrm{op}} \otimes_{\mathbb{k}} A \to V$ — defined by $(b,a) \mapsto B(b,F(a))$ — and similarly a B-A-bimodule $B(F-,-) : A^{\mathrm{op}} \otimes_{\Bbbk} B \to V$. These bimodules induce the following functors:

$$B(F-,-)\otimes_A ? : (A,V) \to (B,V)$$
$$[B(-,F-),?]_A : (A,V) \to (B,V)$$

where ? is the placeholder for the variable in (A,V).

Proposition 1.11.1. *Let $F : A \to B$ be a $\Bbbk$-linear functor between small $\Bbbk$-linear categories, let $\mathrm{res}_F : (B,V) \to (A,V)$ be the functor defined by restriction along F and $\mathrm{res}_{F^{\mathrm{op}}} : (B^{\mathrm{op}},V) \to (A^{\mathrm{op}},V)$ by restriction along F^{op}.*

(i) The functor $F_L := B(F-,-)\otimes_A ?$ is left adjoint to res_F.

(ii) The functor $F_R := [B(-,F-),?]_A$ is right adjoint to res_F.

(iii) The functor res_F preserves all limits and colimits.

(iv) If F is full and surjective on objects, then res_F is fully faithful.

(v) If F is full and faithful, then both F_L and F_R are fully faithful.

(vi) The following are equivalent:

(a) F_L commutes with products,

(b) F_L is an interpretation functor,

(c) $B(F-,b)$ is finitely-presented for all $b \in B$,

(d) $\mathrm{res}_{F^{\mathrm{op}}}$ preserves finitely-presented objects,

(vii) The following are equivalent:

(e) F_R commutes with products,

(f) F_R is an interpretation functor,

(g) $B(b,F-)$ is finitely-presented for all $b \in B$,

(h) res_F preserves finitely-presented objects,

Proof. We have natural Yoneda isomorphisms

$$[B(F(a),-),M]_B \simeq M(F(a)) \simeq B(-,F(a)) \otimes_B M$$

for all $M \in (B,V)$ and $a \in A$. Thus $\mathrm{res}_F \simeq [B(F-,-),?]_B \simeq B(-,F-)\otimes_B ?$. Claims (i) and (ii) are instances of the hom-tensor adjunction. Claim (iii) follows immediately, since left adjoints commute with colimits and right adjoints commute with limits. Claim (iv) is a simple exercise. Claim (v) is by [6, 3.4(e), 3.5(d)].

For (vi) (a) $\Leftrightarrow$ (b) note that, as a left adjoint, F_L commutes with direct limits, and is therefore an interpretation functor if and only if it commutes with products.

For (vi) (a) $\Leftrightarrow$ (c) the functor F_L commutes with products if and only if, given any set $(M_i)_{i\in I}$ of A-modules, the canonical morphism

$$B(F-,b)\otimes_A\left(\prod_{i\in I} M_i\right) \to \prod_{i\in I}\left(B(F-,b)\otimes_A M_i\right)$$

is an isomorphism for all $b \in B$ [43, V.4 Ex.6]. Equivalently, that $B(F-,b)$ is finitely-presented for all $b \in B$ by [69, I.13.2].

For (vi) (c) $\Leftrightarrow$ (d) note that $B(F-,b) = \mathrm{res}_{F^{\mathrm{op}}}B(-,b)$ and $M \in (B^{\mathrm{op}},V)$ is finitely-presented if there exists an exact sequence

$$\bigoplus_{i=1}^{k} B(-,x_i) \to \bigoplus_{j=1}^{l} B(-,y_i) \to M \to 0$$

for some $x_i, y_j \in B$. Since $\mathrm{res}_{F^{\mathrm{op}}}$ is right exact, this induces an exact sequence

$$\bigoplus_{i=1}^{k} B(F-,x_i) \to \bigoplus_{j=1}^{l} B(F-,y_i) \to \mathrm{res}_{F^{\mathrm{op}}}M \to 0$$

and hence $\mathrm{res}_{F^{\mathrm{op}}}M$ is finitely-presented when each of $B(F-,x_i)$ and $B(F-,y_i)$ is finitely-presented, since $\mathrm{fp}(A^{\mathrm{op}},V)$ is closed under cokernels [50, E.1.16]. The converse is clear.

For (vii) (e) $\Leftrightarrow$ (f) note that, as a right adjoint, F_R commutes with products, and is therefore an interpretation functor if and only if it commutes with direct limits.

For (vii) (e) $\Leftrightarrow$ (g) the functor F_R commutes with direct limits if and only if, given any directed set $(M_i)_{i\in I}$ of A-modules, the canonical morphism

$$\varinjlim [B(b,F-),M_i] \to [B(b,F-),\varinjlim M_i]_A$$

is an isomorphism for all $b \in B$. Equivalently, that $B(b,F-)$ is finitely-presented for all $b \in B$ by [69, V.3.4].

For (vii) (g) $\Leftrightarrow$ (h) the proof is similar to (vi) (c) $\Leftrightarrow$ (d). $\square$

Corollary 1.11.2. *If $F : A \to B$ is a $\Bbbk$-linear functor between small $\Bbbk$-linear categories, then an adjunction $F \dashv G$ lifts to an adjunction $\mathrm{res}_G \dashv \mathrm{res}_F$.*

Proof. An adjunction $F \dashv G$ is defined by a bimodule isomorphism $B(F-,-) \simeq A(-,G-)$. Thus $F_L = B(F-,-)\otimes_A ? \simeq A(-,G-)\otimes_A ? \simeq \mathrm{res}_G$ by Proposition 1.11.1. $\square$

Corollary 1.11.3. *If $F : A \to B$ and $G : B \to C$ are $\Bbbk$-linear functors between small $\Bbbk$-linear categories, then $(GF)_L = G_F \circ F_L$ and $(GF)_R = G_R \circ F_R$ are the left and right adjoints of $\mathrm{res}_{G\circ F}$ respectively.*

Proof. This follows from Proposition 1.11.1 and the bimodule isomorphisms

$$C(G-,-)\otimes_B B(F-,-) \simeq C(GF-,-)$$
$$B(-,F-)\otimes_B C(-,G-) \simeq C(-,GF-)$$

cf. [37, Th. 4.47]. $\square$

1.11.2 Tensor products and inverse limits

Let $\mathcal{X} := ((X_i)_{i\in\mathbb{N}}, (\gamma_{i,j} : X_j \to X_i)_{j\geq i})$ be an inverse system of sets. Such a system is defined by the chain of functions $\cdots \to X_3 \xrightarrow{\gamma_{2,3}} X_2 \xrightarrow{\gamma_{1,2}} X_1$ and the equalities $\gamma_{i,i} = 1_{X_i}$ and $\gamma_{i+n,i} = \gamma_{i+n,i+n-1}\cdots\gamma_{i+2,i+1}\gamma_{i+1,i}$ for all $i, n \in \mathbb{N}$. The limit of this system — denoted $\varprojlim X_i$ — can be defined as the set of all sequences $(x_i)_{i\in\mathbb{N}}$ satisfying $x_i \in X_i$ and $x_i = \gamma_{i,i+1}(x_{i+1})$ for all $i \in \mathbb{N}$ [43, §VI.1].

Following Grothendieck [29, 13.1.2] we call $\mathcal{X}$ a **Mittag-Leffler system** if for all $i \geq 1$ there exists $j \geq 1$ such that for all $k \geq j$ we have $\gamma_{ik}(X_k) = \gamma_{ij}(X_j)$. Equivalently, for all $i \geq 1$, the descending sequence $\cdots \subseteq \gamma_{i,i+2}(X_{i+2}) \subseteq \gamma_{i,i+1}(X_{i+1}) \subseteq X_i$ eventually stabilizes.

Lemma 1.11.4. *If $\mathcal{X}$ is a Mittag-Leffler system of non-empty sets, then the inverse limit $\varprojlim X_i$ is non-empty.*

Proof. For $i \in \mathbb{N}$ define $Y_i := \bigcap_{j\geq i} \gamma_{i,j}(X_j)$, then for $j \geq i$ the function $\gamma_{i,j} : X_j \to X_i$ restricts to a surjection $\delta_{i,j} : Y_j \to Y_i$. The limits $\varprojlim X_i$ and $\varprojlim Y_i$ coincide, with the latter easily seen to be non-empty: choose $y_1 \in Y_1$ and inducitvely $y_{i+1} \in \gamma_{i,i+1}^{-1}(y_i)$ for $i \geq 1$, then $(y_i)_{i\in\mathbb{N}} \in \varprojlim Y_i$. $\square$

A morphism $f : \mathcal{X} \to \mathcal{Y}$ for inverse systems $\mathcal{X} := ((X_i)_{i\in\mathbb{N}}, (\gamma_{i,j} : X_j \to X_i)_{j\geq i})$ and $\mathcal{Y} := ((Y_i)_{i\in\mathbb{N}}, (\delta_{i,j} : Y_j \to Y_i)_{j\geq i})$ is a collection of functions $f_i : X_i \to Y_i$ satisfying $\delta_{i,j} f_j = f_i \gamma_{i,j}$ for all $i, j \in \mathbb{N}$. Given such a morphism, the induced morphism between limits $f : \varprojlim X_i \to \varprojlim Y_i$ is defined by $f((x_i)_{i\in\mathbb{N}}) = (f_i(x_i))_{i\in\mathbb{N}}$ for all $(x_i)_{i\in\mathbb{N}} \in \varprojlim X_i$. The following result is given (essentially) in [29, 13.2.2] and the proof is translated here for convenience.

Lemma 1.11.5. *Given an algebra R, let $(f_i : X_i \to Y_i)_{i\in\mathbb{N}}$ be a morphism between the inverse systems $((X_i)_{i\in\mathbb{N}}, (\gamma_{i,j} : X_j \to X_i)_{j\geq i})$ and $((Y_i)_{i\in\mathbb{N}}, (\delta_{i,j} : Y_j \to Y_i)_{j\geq i})$ of finite-dimensional R-modules. Let $f : \varprojlim X_i \to \varprojlim Y_i$ denote the induced morphism between limits. If $(y_i)_{i\in\mathbb{N}} \in \varprojlim Y_i$, then $(y_i)_{i\in\mathbb{N}} \in \operatorname{im}(f)$ if and only if $f_i^{-1}(y_i) \neq \emptyset$ for all $i \in \mathbb{N}$.*

Proof. Let $\bar{y} := (y_i)_{i\in\mathbb{N}} \in \varprojlim Y_i$ be given, set $U_i := f_i^{-1}(y_i)$ and suppose $U_i \neq \emptyset$ for all $i \in \mathbb{N}$. We have an inverse system $\mathcal{U} := ((U_i)_{i\in\mathbb{N}}, (\alpha_{i,j} : U_j \to U_i)_{j\geq i})$ of sets, where $\alpha_{i,j}$ is the restriction of $\gamma_{i,j}$ for all $j \geq i$. Note $\varprojlim U_i \subseteq \varprojlim X_i$ and if $\bar{x} \in \varprojlim U_i$, then $f(\bar{x}) = \bar{y}$. Thus, by Lemma 1.11.4, the proof is complete by showing $\mathcal{U}$ is a Mittag-Leffler system.

Let $L_i := \ker(f_i)$ for $i \in \mathbb{N}$ and $\mathcal{L} := ((L_i)_{i\in\mathbb{N}}, (\beta_{i,j} : L_j \to L_i)_{j\geq i})$ where $\beta_{i,j}$ is given by the restriction of $\gamma_{i,j}$ for all $j \geq i$. As each L_i is a finite-dimensional $\mathbf{k}$-vector space, $\mathcal{L}$ is a Mittag-Leffler system. Given $i \in \mathbb{N}$ we can choose $j \geq i$ such that $\gamma_{i,k}(L_k) = \gamma_{i,j}(L_j)$ for all $k \geq j$ — we claim that $\gamma_{i,k}(U_k) = \gamma_{i,j}(U_j)$ also. We know $\gamma_{i,k}(U_k) \subseteq \gamma_{i,j}(U_j)$, so we prove $\gamma_{i,j}(U_j) \subseteq \gamma_{i,k}(U_k)$.

Let $u_i \in \gamma_{i,j}(U_j)$ be given, then $u_i = \gamma_{i,j}(u_j)$ for some $u_j \in U_j$. Choose any $u'_k \in U_k$ and set $u'_j := \alpha_{j,k}(u'_k)$ and $u'_i := \alpha_{i,j}(u'_j)$. Now $f_j(u_j - u'_j) = y_j - y_j = 0$, so $u_j - u'_j \in L_j$. Hence $u_i - u'_i \in \gamma_{i,j}(L_j) = \gamma_{i,k}(L_k)$ and $u_i - u'_i = \gamma_{i,k}(x_k)$ for some $x_k \in L_k$. Now $f_k(u'_k + x_k) = y_k$ (so $u'_k + x_k \in U_k$) and $\gamma_{i,k}(u'_k + x_k) = u'_i + (u_i - u'_i) = u_i$, thus $u_i \in \gamma_{i,k}(U_k)$ as desired.

This proves one direction of the lemma, the other direction is trivial. $\square$

Proposition 1.11.6. *Let R be an algebra and $((X_i)_{i\in\mathbb{N}}, (\gamma_{i,j} : X_i \to X_j)_{j\geq i})$ an inverse system of finite-dimensional right R-modules. Given any finite-dimensional left R-module M, we have $(\varprojlim X_i) \otimes_R M \simeq \varprojlim(X_i \otimes_R M)$.*

Proof. Take a finite presentation of M of the form

$$R^k \xrightarrow{f} R^l \xrightarrow{g} M \longrightarrow 0 \tag{1.11.7}$$

The tensor product $- \otimes_R M$ is right exact and induces the following commutative diagram, where all columns (except possibly the left-most) are exact.

$$\begin{tikzcd}
\varprojlim(X_i \otimes R^k) \arrow[d, "\bar{f}=(X_i\otimes f)_{i\in\mathbb{N}}"] & \cdots \arrow[r] & X_2 \otimes R^k \arrow[r] \arrow[d, "X_2\otimes f"] & X_1 \otimes R^k \arrow[d, "X_1\otimes f"] \\
\varprojlim(X_i \otimes R^l) \arrow[d, "\bar{g}=(X_i\otimes g)_{i\in\mathbb{N}}"] & \cdots \arrow[r] & X_2 \otimes R^l \arrow[r] \arrow[d, "X_2\otimes g"] & X_1 \otimes R^l \arrow[d, "X_1\otimes g"] \\
\varprojlim(X_i \otimes_R M) \arrow[d] & \cdots \arrow[r] & X_2 \otimes_R M \arrow[r] \arrow[d] & X_1 \otimes_R M \arrow[d] \\
0 & & 0 & 0
\end{tikzcd}$$

We claim the left-most column is also exact. The map $\bar{g}$ is surjective by Lemma 1.11.5 applied to the bottom two rows. It is clear that $\bar{g}\bar{f} = 0$ since $g_i f_i = 0$ for all $i \in \mathbb{N}$. Finally, given $(x_i)_{i\in\mathbb{N}} \in \ker(\bar{g})$, then $x_i \in \ker(g_i)$ and $f_i^{-1}(x_i) \neq \emptyset$ for all $i \in \mathbb{N}$. Applying Lemma 1.11.5 to the top two rows gives $(x_i)_{i\in\mathbb{N}} \in \mathrm{im}(\bar{f})$ as required.

Let $X := \varprojlim X_i$ and apply $X \otimes_R -$ to the exact sequence (1.11.7) to give the top row of the following commutative diagram (the bottom row is the left-most column from above).

$$\begin{tikzcd}
(\varprojlim X_i) \otimes R^k \arrow[r, "X\otimes f"] \arrow[d] & (\varprojlim X_i) \otimes R^l \arrow[r, "X\otimes g"] \arrow[d] & (\varprojlim X_i) \otimes_R M \arrow[r] & 0 \\
\varprojlim(X_i \otimes R^k) \arrow[r, "\bar{f}"] & \varprojlim(X_i \otimes R^l) \arrow[r, "\bar{g}"] & \varprojlim(X_i \otimes_R M) \arrow[r] & 0
\end{tikzcd}$$

The vertical arrows are the canonical isomorphisms, defined by $(x_i)_{i\in\mathbb{N}} \otimes (r_1, \ldots, r_t) \mapsto (x_i \otimes (r_1, \ldots, r_t))_{i\in\mathbb{N}}$ for $t = k$ or l. It follows that there exists a canonical isomorphism of cokernels $(\varprojlim X_i) \otimes_R M \simeq \varprojlim(X_i \otimes_R M)$ as desired. $\square$

Chapter 2

Trivial Extensions

2.1 Trivial extensions of an algebra

Let A be a finite-dimensional algebra and M a finite-dimensional A-A-bimodule. The $\mathbb{k}$-vector space $A \oplus M$ becomes a $\mathbb{k}$-algebra under pointwise addition and multiplication defined by $(r, x)(s, y) := (rs, ry + xs)$ for all $r, s \in A$ and $x, y \in M$. This algebra is called the **trivial extension of** A **by** M and is denoted by $A \ltimes M$. When $M = A^\star$ (the $\mathbb{k}$-dual of A with its natural bimodule structure) we call $A \ltimes A^\star$ simply the **trivial extension of** A.

We consider M as a two-sided ideal of $A \ltimes M$ (identifying M with the image of the canonical inclusion $M \to A \ltimes M$, $m \mapsto (0, m)$). In this way, the canonical projection $A \ltimes M \to A$ has kernel M, satisfying $M^2 = 0$.

Similarly, we can identify A as a subalgebra of $A \ltimes M$ (via the canonical inclusion $A \to A \ltimes M$, $r \mapsto (r, 0)$). It follows from [63, I.1.4] that $I := \mathrm{rad}(A) \oplus M$ is the radical of $A \ltimes M$ and $I^2 = \mathrm{rad}^2(A) \oplus (\mathrm{rad}(A)M + M\mathrm{rad}(A))$. A complete set of local idempotents $\{e_1, \ldots, e_n\}$ for A gives such a set $\{f_i = (e_i, 0) \mid i = 1, \ldots n\}$ for $A \ltimes M$ satisfying

$$f_i(I/I^2)f_j \simeq e_i(\mathrm{rad}(A)/\mathrm{rad}^2(A))e_j \oplus e_i(M/(\mathrm{rad}(A)M + M\mathrm{rad}(A)))e_j \qquad (2.1.1)$$

for all $i, j \in \{1, \ldots, n\}$. It follows that the quiver $\mathscr{Q}_{A \ltimes M}$ of $A \ltimes M$ is obtained from the quiver $\mathscr{Q}_A$ of A by adding new arrows (determined by the bimodule structure of M) but no additional vertices. We refer to [20] for a general description of the quiver and relations of a trivial extension (in case $M = A^\star$). All instances appearing in this work can be calculated explicitly, as in Example 2.1.3 below.

Lemma 2.1.2. *Given a finite-dimensional algebra A, its trivial extension $A \ltimes A^\star$ is a (symmetric) self-injective algebra.*

Proof. The bilinear form $\langle -, - \rangle : V \times V \to \Bbbk$ on $V := A \ltimes A^\star$ defined by

$$\langle (r, f), (s, g) \rangle := g(r) + f(s)$$

for all $(r, f), (s, g) \in V$, is easily checked to be a (symmetric) Frobenius form (i.e. associative and non-degenerate). Any finite-dimensional algebra with such a Frobenius form is self-injective [1, Th. 31.9, §31 Ex. 5]. □

2.1.1 Trivial extensions of a module category

Let $T := R \ltimes M$ be a trivial extension. Let $F : \mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}R$ be the functor $F := - \otimes_R M$ and define the category $(\mathrm{Mod}\text{-}R) \ltimes F$ having as objects all R-linear maps $\phi : FX \to X$ satisfying $\phi \circ F(\phi) = 0$; and having as morphisms all commuting squares of the following form, where $\alpha : X \to Y$ is an R-linear map. See [22, §1] for further details.

$$\begin{array}{ccc} FX & \xrightarrow{F\alpha} & FY \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ X & \xrightarrow{\alpha} & Y \end{array}$$

The category $(\mathrm{Mod}\text{-}R) \ltimes F$ is seen to be equivalent to $\mathrm{Mod}\text{-}(R \ltimes M)$ [22, p. 18]. If $\phi : FX \to X$ is an object of $(\mathrm{Mod}\text{-}R) \ltimes F$, then X is given the structure of an $R \ltimes M$-module by defining

$$x \cdot (r, m) := xr + \phi(x \otimes m)$$

for all $x \in X$ and $(r, m) \in R \ltimes M$. Conversely, given $X \in \mathrm{Mod}\text{-}(R \ltimes M)$, the R-linear map $X \otimes_R M \to X$ is defined by $x \otimes_R m \mapsto x \cdot (0, m)$.

There is an equivalent category $G \ltimes (\mathrm{Mod}\text{-}R)$ for $G := \mathrm{Hom}_R(M, -)$, having as objects all R-linear maps $\phi : X \to GX$ satisfying $G(\phi) \circ \phi = 0$. Given such an object

$\phi : X \to GX$, then X becomes an $R \ltimes M$-module by defining

$$x \cdot (r, m) := xr + \phi_x(m)$$

for all $x \in X$ and $(r, m) \in R \ltimes M$.

In this way, we can consider any $R \ltimes M$-module in the form of an R-linear map $X \otimes_R M \to X$ or an R-linear map $X \to \mathrm{Hom}_R(M, X)$. The correspondence between these two forms is the usual hom-tensor adjunction.

2.1.2 One-point extensions

Let S and R be finite-dimensional algebras and M a finite-dimensional S-R-bimodule. Restricting along the canonical projections $S \times R \to S$ and $S \times R \to R$ gives M the structure of an $(S \times R)$-$(S \times R)$-bimodule, i.e. $(s, r)m := sm$ and $m(s, r) := mr$ for all $m \in M$ and $(s, r) \in S \times R$. The trivial extension $(S \times R) \ltimes M$ is then easily seen to be isomorphic to the following algebra of matrices.

$$\begin{pmatrix} R & 0 \\ M & S \end{pmatrix} := \left\{ \begin{pmatrix} r & 0 \\ m & s \end{pmatrix} \mid m \in M, r \in R, s \in S \right\}$$

A special case of this construction is when R is a finite-dimensional $\mathbf{k}$-algebra, $S = \mathbf{k}$, and M a finite-dimensional right R-module (then M is naturally a $\mathbf{k}$-R-bimodule by $\lambda m := m\lambda$ for all $\lambda \in \mathbf{k}$ and $m \in M$). The trivial extension $(\mathbf{k} \times R) \ltimes M$ is then called the **one-point extension of R by M** and is by denoted $R[M]$. In this case the quiver $\mathscr{Q}_S$ is just a single vertex ω and the quiver $\mathscr{Q}_{R[M]}$ is obtained from $\mathscr{Q}_R \sqcup \{\omega\}$ by additional arrows from ω to $\mathscr{Q}_R$ only, hence the name "one-point extension". Indeed, the right-hand side of equation (2.1.1) becomes

$$e_i(\mathrm{rad}(R)/\mathrm{rad}^2(R))e_j \oplus e_i(M/M\mathrm{rad}(R))e_j$$

with the second summand non-zero only when $e_i = 1_S$ (and in this case the first summand vanishes). Note $M/M\mathrm{rad}(R)$ is semi-simple [63, I.3.8] and its support determines the targets of arrows with source ω in $\mathscr{Q}_{R[M]}$. We give the following example and we will meet one-point extensions again in Section 3.1 (see [55, p. 90] and [65, XV.1] for further details).

Example 2.1.3. Let R be the path algebra of the quiver $\mathscr{Q}_R$ (depicted below left) of Euclidean type $\widetilde{\mathbb{E}}_6$ and let M be the right R-module (depicted below middle). Since $M/M\mathrm{rad}(M) = S(1)\oplus S(2)\oplus S(3)$, the quiver $\mathscr{Q}_{R[M]}$ (depicted below right) is obtained from $\mathscr{Q}_R$ be adding a vertex ω and arrows $\omega \to i$ for $i = 1, 2, 3$.

$\mathscr{Q}_R$
$$\begin{tikzcd}
 & 1' \arrow[dl] & 1 \arrow[l] \\
0 & 2' \arrow[l] & 2 \arrow[l] \\
 & 3' \arrow[ul] & 3 \arrow[l]
\end{tikzcd}$$

M
$$\begin{tikzcd}
 & \mathbf{k} \arrow[dl, "\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)"'] & \mathbf{k} \arrow[l, "1"'] \\
\mathbf{k}^2 & \mathbf{k} \arrow[l, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"] & \mathbf{k} \arrow[l, "1"'] \\
 & \mathbf{k} \arrow[ul, "\left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)"] & \mathbf{k} \arrow[l, "1"']
\end{tikzcd}$$

$\mathscr{Q}_{R[M]}$
$$\begin{tikzcd}
 & 1' \arrow[dl, "\alpha_1"'] & 1 \arrow[l, "\beta_1"'] & \\
0 & 2' \arrow[l, "\alpha_2"] & 2 \arrow[l, "\beta_2"] & \omega \arrow[l, "\gamma_2"] \arrow[ul, "\gamma_1"'] \arrow[dl, "\gamma_3"] \\
 & 3' \arrow[ul, "\alpha_3"] & 3 \arrow[l, "\beta_3"] &
\end{tikzcd}$$

Choose elements $m_i \in M$ such that $m_i + M\mathrm{rad}(M)$ is a basis vector for the summand $S(i)$ of $M/M\mathrm{rad}(M)$ for each $i = 1, 2, 3$. Then a projection $\mathbf{k}\mathscr{Q}_{R[M]} \to R[M]$ can be chosen such that $\gamma_i \mapsto m_i$. The elements m_i satisfy only one relation in M, namely $m_1\beta_1\alpha_1 + m_2\beta_2\alpha_2 + m_3\beta_3\alpha_3 = 0$, therefore $R[M]$ is given by the quiver $\mathscr{Q}_{R[M]}$ and relation $\gamma_1\beta_1\alpha_1 + \gamma_2\beta_2\alpha_2 + \gamma_3\beta_3\alpha_3 = 0$.

2.1.3 Trivial extensions under tilting-cotilting equivalence

The following result is due to Tachikawa and Wakamatsu.

Proposition 2.1.4. *[71, 1.4] Given finite-dimensional algebras A and B, if A and B are tilting-cotilting equivalent, then $A \ltimes A^\star$ and $B \ltimes B^\star$ are stably equivalent.*

We derive the following result.

Corollary 2.1.5. *Given tilting-cotilting equivalent finite-dimensional algebras A and B, there exists a homeomorphism $\mathrm{Zg}^s_{A\ltimes A^\star} \simeq \mathrm{Zg}^s_{B\ltimes B^\star}$ which, if A and B are basic, extends to a homeomorphism $\mathrm{Zg}_{A\ltimes A^\star} \simeq \mathrm{Zg}_{B\ltimes B^\star}$.*

Proof. The first statement is immediate from Proposition 1.3.8 and 2.1.4. The second statement follows from Corollary 1.3.9 — in this case $\mathscr{Q}_A$, $\mathscr{Q}_{A\ltimes A^\star}$, $\mathscr{Q}_B$, and $\mathscr{Q}_{B\ltimes B^\star}$, all have the same number of vertices, hence the corresponding algebras have the same number of indecomposable projective modules. $\square$

By this result, up to homeomorphism of their Ziegler spectra, we only need to consider trivial extensions of one algebra per tilting-cotilting equivalence class. We take this approach in the following section.

2.2 Trivial extensions of Euclidean algebras

In [70] Tachikawa considered the representation theory of the trivial extension $H \ltimes H^\star$ of a hereditary algebra H. It was shown that, regardless of the representation type of H, the trivial extension possessed only "twice as many" indecomposable finite-dimensional modules — this is Proposition 2.2.1 below. We prove a similar statement holds for pure-injective modules. In Section 2.2.1, we present some details from Tachikawa's work. In Section 2.2.2, we then determine the Ziegler spectrum for the trivial extension of a hereditary algebra with radical square zero, i.e. $\mathrm{rad}^2(H) = 0$. In Section 2.2, as a corollary, we derive a description of the Ziegler spectrum for the trivial extension of any Euclidean algebra.

2.2.1 Trivial extensions of hereditary algebras

Let H be a finite-dimensional hereditary algebra and let $T := H \ltimes H^\star$ be its trivial extension. Given $M \in \mathrm{Mod}\text{-}T$ we have, using the canonical inclusion $H \to T$, the restriction of M to its underlying H-module M_H. As observed by Tachikawa [70] the exact sequence

$$0 \to MH^\star \to M \to M/MH^\star \to 0$$

splits when restricted to $\mathrm{Mod}\text{-}H$, since $MH^\star$ is an injective H-module (indeed, the map $H^{\star(M)} \to MH^\star$ defined by $(q_m)_{m\in M} \mapsto \sum_{m\in M} mq_m$ is a H-linear epimorphism; and as $H^{\star(M)}$ is an injective H-module[1], so too is $MH^\star$, since H is hereditary). Thus $M_H = U \oplus V$ where $U := M/MH^\star$ and $V := MH^\star$. Expressing M as a H-linear map $\phi : M \to (H^\star, M)$ — as in Section 2.1.1 — it is easily seen that $\phi(V) = 0$ and $\mathrm{im}(\phi) \subseteq V$. Thus ϕ is completely determined by the component $\phi_M : U \to (H^\star, V)$ which we will call the **reduced form** of M.

[1] Direct sums of injective H-modules are again injective [1, Prop. 18.13].

Proposition 2.2.1. *[70, 1.2] Let $\phi_M : U \to (H^\star, V)$ be a finite-dimensional T-module M in reduced form. Assuming $\phi_M \neq 0$, then M is indecomposable if and only if one of the following conditions hold:*

(i) *ϕ_M is an isomorphism with U an indecomposable projective H-module.*

(ii) *ϕ_M is a proper monomorphism such that $W = \operatorname{cok}(\phi)$ is indecomposable and the exact sequence*

$$0 \longrightarrow U \xrightarrow{\phi_M} (H^\star, V) \longrightarrow W \longrightarrow 0 \tag{2.2.2}$$

is a minimal projective presentation in mod-H.

Recall the functor $-\otimes_H H^\star : \text{mod-}H \to \text{mod-}H$ restricts to an equivalence $\text{proj-}H \to \text{inj-}H$ satisfying $P(i) \otimes_H H^\star \simeq I(i)$ for each vertex $i \in \mathscr{Q}_H$ and with quasi-inverse $(H^\star, -)$ [63, III.2.10].

The modules $M \in \text{ind-}T$ with reduced form $\phi_M = 0$ are precisely those of ind-H under the zero embedding (i.e. restriction along the canonical projection $T \to H$). Proposition 2.2.1 then provides a bijection with the remaining points of ind-T [70, 1.4] as follows: map each projective $W \in \text{ind-}H$ to the T-module whose reduced form is the isomorphism $W \to (H^\star, W \otimes_H H^\star)$; and map each non-projective $W \in \text{ind-}H$ to the T-module whose reduced form induces a minimal projective presentation of W as in (2.2.2).

2.2.2 Trivial extensions of source-sink oriented quivers

Unless otherwise stated, the results in this section are original work.

Say a quiver $\mathscr{Q}$ has **source-sink orientation** if every vertex of $\mathscr{Q}$ is either a source or a sink. For example, any finite tree can be given source-sink orientation. There are Euclidean quivers of types $\widetilde{\mathbb{D}}_n$ (for $n \geq 4$) and $\widetilde{\mathbb{E}}_m$ (for $m = 6, 7, 8$) having source-sink orientation, as well as quivers of type $\widetilde{\mathbb{A}}_{p,q}$ provided $p + q$ is even.

Lemma 2.2.3. *(i) Any finite tree can be given source-sink orientation.*

(ii) Given two finite quivers with the same underlying tree, then one can be obtained from the other by a finite sequence of reflections at sinks.

Proof. The first statement is clear and the second statement is [63, VII.5.2]. $\square$

Say an algebra A has **radical square zero** if $\mathrm{rad}^2(A) = 0$.

Lemma 2.2.4. *A finite-dimensional path algebra $A = \Bbbk\mathscr{Q}_A$ has radical square zero if and only if $\mathscr{Q}_A$ has source-sink orientation.*

Proof. Recall that $\mathrm{rad}(A)$ is generated by the arrows of $\mathscr{Q}_A$ [63, II.1.10]. Thus A has radical square zero if and only if $\alpha\beta = 0$ for any two arrows α and β in $\mathscr{Q}_A$. The latter condition clearly holds if $\mathscr{Q}_A$ has source-sink orientation. Conversely, suppose j is neither a source nor a sink in $\mathscr{Q}_A$, then there exists (without loss of generality) arrows $\alpha : i \to j$ and $\beta : j \to k$ with $\alpha\beta \neq 0$ in A — a contradiction, since A is defined freely over $\mathscr{Q}_A$ with no relations. $\square$

Until the end of this section, unless otherwise stated, let $H = \Bbbk\mathscr{Q}_H$ be the path algebra of a finite quiver $\mathscr{Q}_H$ having source-sink orientation. Let $T = H \ltimes H^\star$ be the trivial extension of H and let $S = H^{\mathrm{op}} \ltimes (H^{\mathrm{op}})^\star$ be the trivial extension of the opposite algebra H^{op}. Recall $\mathscr{Q}_H$ (resp. $\mathscr{Q}_{H^{\mathrm{op}}}$) is a subquiver of $\mathscr{Q}_T$ (resp. $\mathscr{Q}_S$) via the inclusion $H \to T$ (resp. $H^{\mathrm{op}} \to S$).

Lemma 2.2.5. *The trivial extensions T and S are isomorphic and, after identifying $\mathscr{Q}_S$ with $\mathscr{Q}_T$, the arrows of $\mathscr{Q}_T$ partition into a disjoint union of the arrows of $\mathscr{Q}_H$ and the arrows of $\mathscr{Q}_{H^{\mathrm{op}}}$.*

Proof. The assumption that $\mathscr{Q}_H$ has source-sink orientation implies that $\mathrm{rad}(H)$ has as a basis the arrows of $\mathscr{Q}_H$. Then $\mathrm{rad}(H)H^\star = \langle e_i^\star \mid i \text{ is a source in } \mathscr{Q}_H\rangle$ and $H^\star\mathrm{rad}(H) = \langle e_i^\star \mid i \text{ is a sink in } \mathscr{Q}_H\rangle$. Thus $H^\star/(\mathrm{rad}(H)H^\star + H^\star\mathrm{rad}(H))$ has basis $\{\bar{\alpha}^\star \mid \alpha \text{ an arrow of } \mathscr{Q}_H\}$. It follows that each arrow $\alpha : i \to j$ in $\mathscr{Q}_H$ gives rise to two arrows $\alpha : i \to j$ and $\alpha^{\mathrm{op}} : j \to i$ in $\mathscr{Q}_T$ — recall Equation (2.1.1). This accounts for all arrows of $\mathscr{Q}_T$ and clearly the arrows of the form α^{op} give a subquiver of the shape $\mathscr{Q}_H^{\mathrm{op}} = \mathscr{Q}_{H^{\mathrm{op}}}$.

By duality the above paragraph holds true with H and H^{op} interchanged, and there is an obvious isomorphism $\mathscr{Q}_T \simeq \mathscr{Q}_S$. It follows that $T \simeq S$ since the relations for T and S depend only on their quiver [20, 3.12]. $\square$

Identifying S with T as in the above lemma, we have the canonical inclusion $H \to T$ satisfying $\alpha \mapsto (\alpha, 0)$ and an inclusion $H^{\mathrm{op}} \to T$ satisfying $\alpha^{\mathrm{op}} \mapsto (0, \alpha^\star)$ for each arrow

$\alpha \in \mathscr{Q}_H$. Given $M \in \text{Mod-}T$ let M_H, resp. $M_{H^{\text{op}}}$, denote the restriction of M along the respective inclusion. Similarly, we have the canonical projection $T \to H$ and a projection $T \to H^{\text{op}}$. Given $M \in \text{Mod-}H$ or $\text{Mod-}H^{\text{op}}$ let $Z(M)$ denote the T-module given by restriction along the respective projection (this is the zero embedding, given by "extending by zero").

Lemma 2.2.6. *Let M be a T-module, then*

(i) M is semi-simple if and only if M_H and $M_{H^{\text{op}}}$ are semi-simple.

(ii) $M = Z(M_H)$ if and only if $M_{H^{\text{op}}}$ is semi-simple.

(iii) $M = Z(M_{H^{\text{op}}})$ if and only if M_H is semi-simple.

Proof. Since $\mathscr{Q}_H$ has source-sink orientation, the ideal $H^\star \leq T$ is generated by elements of the form $(0, \alpha^\star)$ for α an arrow of $\mathscr{Q}_H$. After identfying H^{op} with its image under the inclusion $H^{\text{op}} \to T$, we see $H^\star$ is generated by $\text{rad}(H^{\text{op}})$. Expressing $M \in \text{Mod-}T$ as a H-linear map $\phi_M : M \otimes_H H^\star \to M$ it is clear the following conditions are equivalent: (a) $M = Z(M_H)$; (b) $\phi_M = 0$; (c) $MH^\star = 0$; (d) $M\text{rad}(H^{\text{op}}) = 0$; (e) $M_{H^{\text{op}}}$ is semi-simple. This proves (ii) and dually (iii). Now (i) follows since $\text{rad}(T) = \text{rad}(H) \oplus H^\star$. $\square$

Recall for all $M \in \text{mod-}H$ the quotient $\text{top}(M) := M/\text{rad}(M)$ is semi-simple and determines the projective cover $P(M)$ of M [63, §I.5].

Lemma 2.2.7. *Let M be a finite-dimensional H-module that is indecomposable and non-projective. If M has minimal projection presentation*

$$0 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{f} M \longrightarrow 0$$

then both P_1 and $P_0 \otimes_H H^\star$ are semi-simple H-modules.

Proof. Let I, resp. I', denote the source, resp. sink, vertices in the support of M. As $\mathscr{Q}_H$ is assumed to have source-sink orientation, $\text{supp}(M) = I \sqcup I'$. Note $I \neq \emptyset$, otherwise M would be a semi-simple projective module, contrary to our assumption. For a sink $i \in I'$, any non-zero element $\bar{m} \in \text{top}(M)e_i$ gives rise to a (simple) projective direct summand $mH \simeq P(i)$ of M. Since M is assumed indecomposable and non-projective, it follows that $\text{top}(M)e_i = 0$ for all $i \in I'$. For a source $i \in I$, we have

$\operatorname{rad}(M)e_i = 0$ and so $\operatorname{top}(M)e_i = Me_i$. Hence, by [63, I.5.8], we have

$$P_0 \simeq P(\operatorname{top}(M)) = P\left(\bigoplus_{j\in I} S(j)^{d_j}\right) = \bigoplus_{j\in I} P(j)^{d_j}$$

where $d_j := \dim_{\mathrm{k}} Me_j$ for $j \in I$, and thus

$$P_0 \otimes_H H^\star \simeq \bigoplus_{j\in I} P(j)^{d_j} \otimes H^\star = \bigoplus_{j\in I} I(j)^{d_j}$$

is semi-simple. Now since $\mathcal{Q}_H$ is assumed to have source-sink orientation, there are no paths between two distinct source vertices, so $\dim_{\mathrm{k}}(e_j He_i) = \delta_{i,j}$ for all $i, j \in I$, and

$$\dim_{\mathrm{k}}(P_0 e_i) = \dim_{\mathrm{k}} \bigoplus_{j\in I} P(j)^{d_j} e_i = \sum_{j\in I} d_j \dim_{\mathrm{k}}(e_j He_i) = d_i = \dim_{\mathrm{k}} Me_i \tag{2.2.8}$$

for all $i \in I$. Now the epimorphism $f : P_0 \to M$ is given by surjective components $f_v : P_0 e_v \to Me_v$ for vertices $v \in \mathcal{Q}_H$. By (2.2.8) we see f_i is a bijection for each source $i \in I$. Observe that $P_0 e_i = 0$ for any source vertex i of $\mathcal{Q}_H$ not belonging to I. It follows that $P_1 = \ker(f)$ is supported entirely on sink vertices and must therefore be a direct sum of simple projective modules. □

Proposition 2.2.9. *Let M be a finite-dimensional T-module that is indecomposable and non-projective, then $M = Z(M_H)$ or $M = Z(M_{H^{\mathrm{op}}})$.*

Proof. Let $\phi : U \to (H^\star, V)$ be the reduced form of M. If $\phi = 0$, then $M = Z(M_H)$. Otherwise, by Proposition 2.2.1 and the assumption that M is non-projective, the induced exact sequence $U \to (H^\star, V) \to \operatorname{coker}(\phi) \to 0$ is a minimal projective presentation in mod-H. Then both U and $V \simeq (H^\star, V) \otimes_H H^\star$ are semi-simple H-modules by Lemma 2.2.7. Therefore $M_H = U \oplus V$ is semi-simple and $M = Z(M_{H^{\mathrm{op}}})$ by Lemma 2.2.6. □

Theorem 2.2.10. *Let $H = \mathrm{k}\mathcal{Q}_H$ be the path algebra of a finite quiver $\mathcal{Q}_H$ having source-sink orientation, and let $T = H \ltimes H^\star$ be its trivial extension. There exists a homeomorphism of Ziegler spectra $\mathrm{Zg}_T \simeq \mathrm{Zg}_H \sqcup \mathrm{Zg}_{H^{\mathrm{op}}}$.*

Proof. By Proposition 1.3.4, restriction along the canonical projections $T \to H$ and $T \to H^{\mathrm{op}}$ induces closed embeddings $\mathrm{Zg}_H \to \mathrm{Zg}_T$ and $\mathrm{Zg}_{H^{\mathrm{op}}} \to \mathrm{Zg}_T$ respectively. Let

$\mathfrak{C}_1$ and $\mathfrak{C}_2$ denote the image of these embeddings in Zg_T. By Proposition 2.2.9 the closed subset $\mathfrak{C}_1 \cup \mathfrak{C}_2 \subseteq \mathrm{Zg}_T$ contains all finite-dimensional points except for the finite set $\mathfrak{P}$ of projective-injective modules. Thus $\mathrm{Zg}_T = \mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \mathfrak{P}$ by Corollary 1.3.7. By Lemma 2.2.6 the intersection $\mathfrak{S} = \mathfrak{C}_1 \cap \mathfrak{C}_2$ consists of the simple T-modules, hence

$$\mathrm{Zg}_T = \mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \mathfrak{P} = \mathfrak{C}_1 \sqcup (\mathfrak{C}_2 \backslash \mathfrak{S}) \sqcup \mathfrak{P} \simeq \mathfrak{C}_1 \sqcup \mathfrak{C}_2 \simeq \mathrm{Zg}_H \sqcup \mathrm{Zg}_{H^{\mathrm{op}}}$$

because $\mathfrak{S}$ and $\mathfrak{P}$ are two discrete subsets of the same size. $\square$

Corollary 2.2.11. *Let $A = \Bbbk\mathscr{Q}_A$ be a path algebra whose quiver $\mathscr{Q}_A$ has underlying graph a finite tree, and let $T = A \ltimes A^\star$ be its trivial extension. The Ziegler spectrum Zg_T of T is homeomorphic to $\mathrm{Zg}_A \sqcup \mathrm{Zg}_A$ — a disjoint union of two copies of the Ziegler spectrum Zg_A of A.*

Proof. If $\mathscr{Q}_A$ does not have source-sink orientation, then by Lemma 2.2.3 we can, by a finite sequence of reflections, obtain a quiver $\mathscr{Q}_B$ having source-sink orientation with same underlying graph as $\mathscr{Q}_A$. Let $B := \Bbbk\mathscr{Q}_B$ and $S := B \ltimes B^\star$, then A and B are tilting-cotilting equivalent, implying $\mathrm{Zg}_A \simeq \mathrm{Zg}_B$ and $\mathrm{Zg}_T \simeq \mathrm{Zg}_S$, by Proposition 1.8.4 and Corollary 2.1.5. By Theorem 2.2.10 we have

$$\mathrm{Zg}_T \simeq \mathrm{Zg}_S \simeq \mathrm{Zg}_B \sqcup \mathrm{Zg}_{B^{\mathrm{op}}} \simeq \mathrm{Zg}_A \sqcup \mathrm{Zg}_{A^{\mathrm{op}}} \simeq \mathrm{Zg}_A \sqcup \mathrm{Zg}_A$$

as claimed. $\square$

Remarks

- Note Theorem 2.2.10 does not depend on the representation type of H. In particular, it applies even when H is wild (of course, in this case, Zg_H isn't particularly well-understood)
- Whether there exists a homeomorphism $\mathrm{Zg}_H \simeq \mathrm{Zg}_{H^{\mathrm{op}}}$ in general is an open question, see [50, §5.4]. For rings with finite KG-dimension, such a homeomorphism is known to exist [50, 5.4.20].

2.2.3 Types $\widetilde{\mathbb{D}}_m$ for $m \geq 4$ and $\widetilde{\mathbb{E}}_n$ for $n = 6, 7, 8$

As Euclidean graphs of types $\widetilde{\mathbb{D}}_m$ and $\widetilde{\mathbb{E}}_n$ are trees we have proven the following result.

Corollary 2.2.12. *If B is a Euclidean algebra of the form $B = \mathrm{End}_A(M)$ for A a hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_m$ (for $m \geq 4$) or $\widetilde{\mathbb{E}}_n$ (for $n = 6, 7, 8$), then there exists a homeomorphism $\mathrm{Zg}_{B \ltimes B^\star} \simeq \mathrm{Zg}_A \sqcup \mathrm{Zg}_A$.*

Proof. By definition B is tilting-cotilting equivalent to A, then

$$\mathrm{Zg}_{B \ltimes B^\star} \simeq \mathrm{Zg}_{A \ltimes A^\star} \simeq \mathrm{Zg}_A \sqcup \mathrm{Zg}_A$$

by Corollary 2.1.5 and 2.2.11 respectively. $\square$

2.2.4 Types $\widetilde{\mathbb{A}}_{p,q}$ for $1 \leq p \leq q$

Let $C = C(p, q)$ be the path algebra of the following quiver (2.2.13), then C is a tame hereditary algebra of Euclidean type $\Delta := \widetilde{\mathbb{A}}_{p,q}$. Let $T := C \ltimes C^\star$ be its trivial extension. In case $p + q$ is even, then we have an analogous result to Corollary 2.2.12 (for there exists a source-sink orientation of $\mathscr{Q}_C$), but for $p + q$ odd one must work a little harder. The results of Section 5.2 will describe Zg_T in both cases, so in this section we will merely point to a reduction technique that could also be used.

$$\mathscr{Q}_C \qquad \begin{tikzcd} & a_1 \arrow[dl, "\alpha_1"'] & a_2 \arrow[l, "\alpha_2"'] \arrow[r, dotted, no head] & \cdot & a_{p-1} \arrow[l, "\alpha_{p-1}"'] & \\ v_0 & & & & & v_\omega \arrow[ul, "\alpha_p"'] \arrow[dl, "\beta_q"] \\ & b_1 \arrow[ul, "\beta_1"] & b_2 \arrow[l, "\beta_2"] \arrow[r, dotted, no head] & \cdot & b_{q-1} \arrow[l, "\beta_{q-1}"] & \end{tikzcd} \tag{2.2.13}$$

The quiver $\mathscr{Q}_T$ of T is obtained from $\mathscr{Q}_C$ by adding two arrows $\alpha_0, \beta_0 : v_0 \to v_\omega$ (corresponding to the two maximal paths $\alpha_p \alpha_{p-1} \cdots \alpha_1$ and $\beta_q \beta_{q-1} \cdots \beta_1$ in $\mathscr{Q}_C$) as follows.

$$\mathscr{Q}_T \qquad \begin{tikzcd} & a_1 \arrow[dl, "\alpha_1"'] & a_2 \arrow[l, "\alpha_2"'] \arrow[r, dotted, no head] & \cdot & a_{p-1} \arrow[l, "\alpha_{p-1}"'] & \\ v_0 \arrow[rrrrr, bend left=8, "\alpha_0"] \arrow[rrrrr, bend right=8, "\beta_0"'] & & & & & v_\omega \arrow[ul, "\alpha_p"'] \arrow[dl, "\beta_q"] \\ & b_1 \arrow[ul, "\beta_1"] & b_2 \arrow[l, "\beta_2"] \arrow[r, dotted, no head] & \cdot & b_{q-1} \arrow[l, "\beta_{q-1}"] & \end{tikzcd}$$

Define an α-**path** (resp. β-**path**) to be a path $\gamma_{i_1} \cdots \gamma_{i_n}$ in $\mathscr{Q}_T$ consisting of arrows $\gamma_{i_j} = \alpha_{i_j}$ (resp. $\gamma_{i_j} = \beta_{i_j}$) for $j = 1, \ldots, n$. An α-**cycle** (resp. β-**cycle**) is then an

α-path (resp. β-path) of length $p+1$ (resp. $q+1$); necessarily cyclic. Now a set of generators for the relations of T are as follows:

- $\alpha_i\beta_j$ for all i, j,
- $\rho - \sigma$ for any α-cycle ρ and β-cycle σ with common source,
- Any α-path of length $\geq p+2$,
- Any β-path of length $\geq q+2$.

Recall, from Section 1.4, that as T is self-injective its right (= left) socle $\mathrm{soc}(T_T)$ is a two-sided ideal. For an introduction to (and definition of) **string algebras** (and **string/band** modules) see [11] [62] [57] and for their Ziegler spectra see [52] [41] [42].

Lemma 2.2.14. *The quotient algebra $S := T/\mathrm{soc}(T)$ is a 2-parametric domestic string algebra.*

Proof. Let $\mathcal{S}$ be the set of all α-cycles and β-cycles in $\mathscr{Q}_T$, then $\mathcal{S}$ generates the socle $\mathrm{soc}(T)$ of T. The quotient algebra $S = T/\mathrm{soc}(T)$ is then given by the quiver $\mathscr{Q}_T$ with a generating set of relations given all $\alpha_i\beta_i$ and the elements of $\mathcal{S}$. Since these are all monomial relations, S is by definition a string algebra. S is (2-parametric) domestic since there are just two bands, namely, $\alpha_0\beta_0^{-1}$ and $\alpha_p\alpha_{p-1}\cdots\alpha_1\beta_1^{-1}\beta_2^{-1}\cdots\beta_{q-1}^{-1}$. $\square$

By Corollary 1.4.5 we know $\mathrm{Zg}_T^s \simeq \mathrm{Zg}_S$ and so every non-projective point of Zg_T is a (possibly infinite) string or band module over S.

Remark We claim there is a homeomorphism $\mathrm{Zg}_T \simeq \mathrm{Zg}_C \sqcup \mathrm{Zg}_C$. A bijection of points is fairly clear given some familiarity with string algebras, but proving the topologies coincide under such a bijection is beyond the scope of this work. This claim can be established from results of Chapter 5. For now we settle for the following conclusion.

Corollary 2.2.15. *If B is a Euclidean algebra of the form $B = \mathrm{End}_A(M)$ where A is a hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_{p,q}$ (for $1 \leq p \leq q$), then there exists a 2-parametric domestic string algebra S and a homeomorphism $\mathrm{Zg}^s_{B\ltimes B^\star} \simeq \mathrm{Zg}_S$.*

Proof. By definition B is tilting-cotilting equivalent to A, and A is tilting-cotilting equivalent to $C := C(p,q)$ by Corollary 1.8.5. Let $T = C \ltimes C^\star$ and $S = T/\mathrm{soc}(T)$, then $\mathrm{Zg}_T^s \simeq \mathrm{Zg}_S$ by Corollary 1.4.5 and S is a domestic string algebra by Lemma 2.2.14. Now $\mathrm{Zg}_{B\ltimes B^\star} \simeq \mathrm{Zg}_T$ by Corollary 2.1.5 and the result follows. $\square$

Chapter 3

Closure of Tubes

For a finite-dimensional algebra R of infinite representation-type, it is often fruitful to consider the closure $\mathrm{cl}(\mathcal{C})$ in Zg_R of a single component $\mathcal{C}$ of $\mathrm{mod}\text{-}R$. By Proposition 1.3.6 this closure contains no additional finite-dimensional points and at least one infinite-dimensional point. However, since $\mathrm{mod}\text{-}R$ has infinitely many components, not all infinite-dimensional points of Zg_R will necessarily be found this way.

Closure of tubes in the Ziegler spectrum has been considered for stable tubes in [39] [60] and for certain (standard) ray and coray tubes in [27]. In this section we consider the closure of tubes that are constructed from standard stable tubes by a process of iterated ray and coray insertion (i.e. by the one-point tubular (co)extensions defined below). In this way, we are able to describe the closure of certain quasi-stable tubes (later seen to appear over self-injective algebras).

3.1 One-point extensions

Let R be a finite-dimensional algebra and X a finite-dimensional right R-module. The **one-point extension of R by X** is the following algebra of matrices:

$$R[X] := \begin{pmatrix} R & 0 \\ X & \mathbf{k} \end{pmatrix} = \left\{ \begin{pmatrix} r & 0 \\ x & \lambda \end{pmatrix} \mid r \in R, x \in X, \lambda \in \mathbf{k} \right\}$$

We have already seen these algebras as a special case of trivial extensions in Section 2.1.2. The dual concept is the **one-point coextension of R by X**, it is the algebra $[X]R := (R^{\mathrm{op}}[X^\star])^{\mathrm{op}}$ and is isomorphic to the following algebra of matrices:

$$\begin{pmatrix} \mathbf{k} & 0 \\ X^\star & R \end{pmatrix} = \left\{ \begin{pmatrix} \lambda & 0 \\ x & r \end{pmatrix} \mid r \in R, x \in X^\star, \lambda \in \mathbf{k} \right\}$$

In these definitions we are using the $\mathbf{k}$-R-bimodule (resp. R-$\mathbf{k}$-bimodule) structure of X (resp. $X^\star$) induced by the commutativity of $\mathbf{k}$. Using the standard duality $\mathrm{mod}\text{-}[X]R \simeq (\mathrm{mod}\text{-}R^{\mathrm{op}}[X^\star])^{\mathrm{op}}$ all results concerning finite-dimensional modules over one-point extensions dualise to one-point coextensions. However, we must be careful when dealing with infinite-dimensional modules.

If we choose for R a complete set of local idempotents $\{e_1, \dots, e_n\}$, then for $R[X]$ we get the following complete set of local idempotents:

$$\left\{ f_i = \begin{pmatrix} e_i & 0 \\ 0 & 0 \end{pmatrix} \text{ for } i = 1, \dots, n, \text{ and } f_\omega = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

The vertex of the quiver $\mathscr{Q}_{R[X]}$ of $R[X]$ that corresponds to the idempotent f_ω is called the **extension vertex**. The ideal $I = \langle f_\omega \rangle$ is such that $R[X]/I \simeq R$ and the projection $\pi : R[X] \to R$ induces a full embedding $\mathrm{res}_\pi : \mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}R[X]$ which we call the **zero embedding** (it is given by "extending by zero").

Consider R and $R[X]$ as $\mathbf{k}$-linear categories with objects $\{1, \dots, n\}$ and $\{1, \dots, n, \omega\}$ respectively. We may regard R as a full subcategory of $R[X]$ by the following identification of hom-spaces:

$$R[X](i, j) = f_j R[X] f_i \simeq e_j R e_i = R(i, j) \tag{3.1.1}$$

for all $i, j \in \{1, \dots, n\}$. Let $F : R \to R[X]$ be the inclusion functor, then restriction along F defines a functor $\mathrm{res}_F : \mathrm{Mod}\text{-}R[X] \to \mathrm{Mod}\text{-}R$ (strictly speaking, this is restriction along F^{op}). Similarly, for left modules $\mathrm{res}_F : R[X]\text{-}\mathrm{Mod} \to R\text{-}\mathrm{Mod}$. We will be using both the left and right adjoints to these functors, they are constructed as follows. Recall, the representable functors $R(-, i)$ (resp. $R(i, -)$) for $i = 1, \dots, n$ correspond to the indecomposable projective right (resp. left) R-modules.

Lemma 3.1.2. *For all $i \in \{1, \ldots, n, \omega\}$ we have*

$$\mathrm{res}_F R[X](-,i) \simeq \begin{cases} R(-,i) & \text{if } i \neq \omega, \\ X & \text{if } i = \omega. \end{cases}$$

$$\mathrm{res}_F R[X](i,-) \simeq \begin{cases} R(i,-) & \text{if } i \neq \omega, \\ 0 & \text{if } i = \omega. \end{cases}$$

Proof. Pointwise we have equation (3.1.1) and the following isomorphisms:

$$\begin{aligned} R[X](j,\omega) &= f_\omega R[X] f_j \simeq X e_j = X(j), \\ R[X](\omega,j) &= f_j R[X] f_\omega \simeq 0. \end{aligned}$$

for all $j \in \{1, \ldots, n\}$, easily checked to be natural in both variables. □

Proposition 3.1.3. *Let $F : R \to R[X]$ be the inclusion functor for a one-point extension, then $\mathrm{res}_F : \mathrm{Mod}\text{-}R[X] \to \mathrm{Mod}\text{-}R$ has fully faithful left and right adjoints.*

The left adjoint $F_L : \mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}R[X]$ is defined by

$$F_L M(i) := \begin{cases} M(i) & \text{if } i \neq \omega, \\ 0 & \text{if } i = \omega, \end{cases}$$

for all $M \in \mathrm{Mod}\text{-}R$ and coincides with the zero embedding.

The right adjoint $F_R : \mathrm{Mod}\text{-}R[X] \to \mathrm{Mod}\text{-}R$ is defined by

$$F_R M(i) := \begin{cases} M(i) & \text{if } i \neq \omega, \\ (X,M) & \text{if } i = \omega. \end{cases}$$

Both adjoints F_L and F_R are interpretation functors, they coincide on modules M satisfying $(X,M) = 0$, and preserve any Auslander-Reiten sequence in $\mathrm{mod}\text{-}R$ that begins on such a module.

Proof. The adjoints F_L and F_R can be constructed by Proposition 1.11.1 and simplified using Lemma 3.1.2 and the Yoneda isomorphisms. Moreover, since each representable $R[X]$-module restricts to a finitely-presented R-module (this is why we require

$X = \mathrm{res}R[X](-,\omega)$ to be finite-dimensional), we know F_L commutes with products, F_R commutes with direct limits, and both functors are interpretation functors. Furthermore, since F is an inclusion, both F_L and F_R are fully faithful — this all follows from Proposition 1.11.1.

Note the projection $\pi : R[X] \to R$ is left adjoint to F, indeed, we have already established natural bijections $R(\pi(i), j) \simeq R[X](i, F(j))$. Therefore π^{op} is *right* adjoint to F^{op} (we must be careful to use "op" here) and the adjunction $F^{\mathrm{op}} \dashv \pi^{\mathrm{op}}$ lifts to an adjunction $\mathrm{res}_{\pi^{\mathrm{op}}} \dashv \mathrm{res}_{F^{\mathrm{op}}}$ by Corollary 1.11.2. Therefore (after forgoing the "op" again), we have $\mathrm{res}_\pi \simeq F_L$ by uniqueness of adjoints.

The functors F_L and F_R restrict to functors $\mathrm{mod}\text{-}R \to \mathrm{mod}\text{-}R[X]$. These are just the two embeddings defined in [64, XV.1] and the final statement of this proposition is given by [64, XV.1.7]. $\square$

The analogous result for one-point coextensions is as follows. Observe the role of the left and right adjoints is reversed (in regards to which is the zero embedding).

Proposition 3.1.4. *Let $F : R \to [X]R$ be the inclusion functor for a one-point coextension, then* $\mathrm{res}_F : \mathrm{Mod}\text{-}[X]R \to \mathrm{Mod}\text{-}R$ *has fully faithful left and right adjoints.*

The left adjoint $F_L : \mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}[X]R$ *is defined pointwise by*

$$F_L M(i) := \begin{cases} M(i) & \text{if } i \neq \omega, \\ M \otimes X^\star & \text{if } i = \omega, \end{cases}$$

for all $M \in \mathrm{Mod}\text{-}R$.

The right adjoint $F_R : \mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}[X]R$ *is defined pointwise by*

$$F_R M(i) := \begin{cases} M(i) & \text{if } i \neq \omega, \\ 0 & \text{if } i = \omega, \end{cases}$$

and coincides with the zero embedding. Both adjoints F_L and F_R are interpretation functors, they coincide on modules M satisfying $(M, X) = 0$, and preserve any Auslander-Reiten sequence in $\mathrm{mod}\text{-}R$ *that ends on such a module.*

We will need the following result.

Lemma 3.1.5. *Let $F : R \to [X]R$ be a one-point coextension. For any inverse system $((M_i)_{i\in\mathbb{N}}, (\alpha_{j,i} : M_j \to M_i)_{j\geq i})$ in* mod *-R, we have $F_L(\varprojlim M_i) = \varprojlim F_L(M_i)$.*

Proof. Let $A = [X]R$. The left adjoint F_L commutes with the given inverse limit if and only if, for all $i \in \{1, \ldots, n, \omega\}$, the canonical map

$$\left(\varprojlim M_i\right) \otimes_R A(i, F(-)) \to \varprojlim \left(M_i \otimes_R A(i, F(-))\right) \tag{3.1.6}$$

is a $\mathbf{k}$-linear isomorphism [43, §V.4 Ex. 5]. For $i \neq \omega$, we have $A(i, F(-)) = R(i, -)$ and, under the Yoneda isomorphisms, the map (3.1.6) becomes the obvious identity $\left(\varprojlim M_i\right)(i) \to \varprojlim M_i(i)$. For $i = \omega$, the canonical map (3.1.6) is the canonical morphism $\left(\varprojlim M_i\right) \otimes_R X^\star \to \varprojlim \left(M_i \otimes_R X^\star\right)$ which is an isomorphism by Proposition 1.11.6. □

3.2 Rays, corays, and tubular enlargements

Let R be an algebra and $\mathbb{T}$ a tube of mod-R. Every arrow in $\Gamma(\mathbb{T})$ either "points to infinity" or "points to the mouth" (this is made precise in [55, §4.6]). A **ray** in $\Gamma(\mathbb{T})$ is an infinite path $X[1] \to X[2] \to \cdots$ with pairwise distinct vertices and all arrows pointing to infinity. Dually, a **coray** in $\Gamma(\mathbb{T})$ is an infinite path $\cdots \to [2]X \to [1]X$ with pairwise distinct vertices and all arrows pointing to the mouth. A ray (resp. coray) is **maximal** if it is not properly contained in another ray (resp. coray). Every ray (resp. coray) is contained in a unique maximal ray (resp. coray) [55, 4.6.2]. We define a ray, resp. coray, in $\mathbb{T}$ itself to be any sequence of indecomposable modules and irreducible morphisms, that represent a ray, resp. coray, in $\Gamma(\mathbb{T})$.

A vertex X in $\Gamma(\mathbb{T})$ is a **ray vertex** if there exists an infinite **sectional path** $X[1] \to X[2] \to \cdots$ (i.e. $X[i] \neq \tau X[i+2]$ for all $i \geq 1$) beginning at $X[1] = X$ and containing every sectional path that begins at X. If X is a ray vertex and $X[1] \to X[2] \to \cdots$ is the infinite sectional path beginning at X, then this path is indeed a ray in $\Gamma(\mathbb{T})$ [55, 4.6.3]. However, in this terminology the converse need not be true, i.e. not every ray in $\Gamma(\mathbb{T})$ need begin at a ray vertex. A **coray vertex** is defined dually.

Let A and B be locally bounded algebras and $F : \text{mod-}A \to \text{mod-}B$ a fully faithful functor. Let $\mathbb{S}$ and $\mathbb{T}$ be a (quasi-) tubes in mod-A and mod-B respectively, then we say $\mathbb{T}$ is a **tubular enlargement** of $\mathbb{S}$ (with respect to F) if $F(\mathbb{S}) \subseteq \mathbb{T}$, i.e. $F(M) \in \mathbb{T}$ for all $M \in \mathbb{S}$. A tubular family $\mathcal{T} = (\mathbb{T}_\lambda)_{\lambda\in\Lambda}$ is a **tubular enlargement** (with respect to F) of a tubular family $\mathcal{S} = (\mathbb{S}_\lambda)_{\lambda\in\Lambda}$ provided $\mathbb{T}_\lambda$ is a tubular enlargement (with respect to F) of $\mathbb{S}_\lambda$ for all $\lambda \in \Lambda$.

Suppose $\mathbb{T}$ is a tubular enlargement of $\mathbb{S}$ with respect to F. We say the tubular enlargement is **ray preserving** if for every ray $X[1] \to X[2] \to \cdots$ in $\mathbb{S}$, the image $FX[1] \to FX[2] \to \cdots$ is a ray in $\mathbb{T}$. We say the tubular enlargement is **ray conservative** if for every ray $X[1] \to X[2] \to \cdots$ in $\mathbb{S}$, there exists a ray $Y[1] \to Y[2] \to \cdots$ in $\mathbb{T}$ and a (strictly increasing) sequence of integers $(k_i)_{i\in\mathbb{N}}$ such that, for all $i \in \mathbb{N}$, we have $FX[i] = Y[k_i]$ and for any irreducible morphism $X[i] \to X[i+1]$, the image $FX[i] \to FX[i+1]$ factors into a composition of irreducible morphisms $Y[k_i] \to Y[k_i+1] \to \cdots \to Y[k_{i+1}]$. The definitions of **coray preserving** and **coray conservative** are dual.

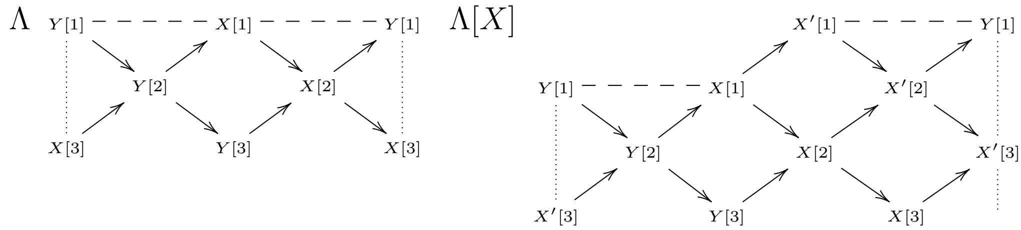
3.3 One-point tubular extensions

Let A be a locally bounded algebra and let $\mathbb{T}$ be a tube in mod-A. A module $M \in \mathbb{T}$ is called a **ray module** if M represents a ray vertex in $\Gamma(\mathbb{T})$. If M is a ray module, then we say the one-point extension $R[M]$ is a **one-point tubular extension** and that $R[M]$ is obtained from R by **ray insertion** at M (a more general *n-fold ray insertion* is defined in [17, §2]). Dually, a module $M \in \mathbb{T}$ is a called a **coray module** if M represents a coray vertex in $\Gamma(\mathbb{T})$, and if M is a coray module, then the one-point coextension $[M]R$ is a **one-point tubular coextension** and $[M]R$ is said to be obtained from R by **coray insertion** at M.

Let Λ be a tube and suppose $X[1] \to X[2] \to \cdots$ is a ray in Λ beginning at a ray vertex $X = X[1]$. Define the translation quiver $\mathbf{\Lambda[X]}$ — said to be obtained from Λ by **ray insertion** at X — as follows. The vertices of $\Lambda[X]$ are those of Λ and additional vertices $X'[i]$ for $i \geq 1$. The arrows of $\Lambda[X]$ are obtained from the arrows of Λ by replacing any arrow of the form $X[i] \to Y$ where $Y \neq X[i+1]$ with an arrow

$X'[i] \to Y$, and adding arrows $X[i] \to X'[i]$ and $X'[i] \to X'[i+1]$ for all $i \geq 1$. Finally, the translation of $\Lambda[X]$ is obtained from the translation of Λ by replacing any $\tau_\Lambda Y = X[i]$ with $\tau_{\Lambda[X]} Y = X'[i]$ and defining $\tau_{\Lambda[X]} X'[i+1] = X[i]$ for $i \geq 1$, leaving $X'[1]$ as a projective vertex.

Example 3.3.1. A ray insertion at $X := X[1]$ in the stable tube Λ (below left), results in the tube $\Lambda[X]$ (below right) as depicted in the following diagram (only a finite portion of each tube is drawn).



As usual, for each tube, only the translate at the mouth is drawn (by a dashed line) and left/right edges are to be identified (along the dotted lines).

Proposition 3.3.2. *Let R be a finite-dimensional algebra and let M be ray module belonging to a standard tube $\mathbb{T}$ of* mod -R. *Define $\Lambda := \Gamma(\mathbb{T})$ and let $M[1] \to M[2] \to \cdots$ denote the ray beginning at $M[1] := M$.*

(i) If $N \in \operatorname{ind}(\mathbb{T})$, then $(M, N) = 0$ unless $N \simeq M[j]$ for some $j \geq 1$, in which case $\dim_{\mathrm{k}}(M, M[j]) = 1$.

(ii) The module class $\mathbb{U} := F_L(\mathbb{T}) \vee F_R(\mathbb{T})$ is a standard tube in mod -$R[M]$ *of the form $\Gamma(\mathbb{U}) = \Lambda[M]$. Moreover, $\mathbb{U}$ is tubular enlargement of $\mathbb{T}$ — with respect to both F_L and F_R — that is ray preserving and coray conservative.*

Here $F_L, F_R :$ Mod -$R \to$ Mod -$R[M]$ *are the two embeddings for the one-point extension given by Proposition 3.1.3.*

Proof. The result follows (for the most part) as a special case of [55, 4.5.1]. However, our notation and terminlogy differs.

For (i), since $\mathbb{T}$ is standard, we can work in the mesh category of Λ. In the proof of [55, 4.5.1, p. 217] it is shown that any map given by a non-sectional path starting at M is zero. This implies $(M, N) = 0$ when $N \neq M[j]$ for any $j \geq 1$. Additionally, since M is a ray module, any sectional path from $M[1]$ to $M[j]$ belongs to the ray at M, and consequently $\dim_{\Bbbk}(M, M[j]) = 1$.

We leave the proof of statement (ii) to [55, 4.5.1] but we note the following facts. As F_L is just the zero embedding, the module class $F_L(\mathbb{T})$ consists of R-modules and may be identified with $\mathbb{T}$. Furthermore, F_L maps any irreducible morphism belonging to a ray in $\mathbb{T}$ to an irreducible morphism belonging to a ray in $\mathbb{U}$. By (i) and Proposition 3.1.3, we know F_L and F_R coincide on all modules in $\mathbb{T}$ except for those on the ray beginning at M, i.e. $M[i]$ for $i \geq 1$. Applying F_R to the ray $M[1] \to M[2] \to \cdots$ in $\mathbb{T}$ gives a ray $F_R M[1] \to F_R M[2] \to \cdots$ in $\mathbb{U}$ that represents the ray denoted $M'[1] \to M'[2] \to \cdots$ in $\Lambda[M]$. That $\mathbb{U}$ is a ray preserving, coray conservative, tubular enlargemenet of $\mathbb{T}$ (with respect to either F_L or F_R) follows easily from the construction of $\Lambda[M]$ from Λ by ray insertion. □

Example 3.3.3. Continuing Example 3.3.1, for definiteness, take R to be any tame hereditary algebra with a stable tube $\mathbb{T}$ of rank 2 (by Proposition 1.8.1, there are many choices). Let X be a simple regular module belonging to $\mathbb{T}$ and identify $\Gamma(\mathbb{T})$ with Λ so that X represents $X[1]$. Consider the tube $\mathbb{U}$ over the algebra $R[X]$ given by Proposition 3.3.2, it is a tubular enlargement of $\mathbb{T}$ of the form $\Lambda[X]$. Let F_L and F_R be the two embeddings $\operatorname{Mod}\text{-}R \to \operatorname{Mod}\text{-}R[X]$ of the one-point extension. Then for all $i \in \mathbb{N}$, we have $F_L Y[i] = F_R Y[i] = Y[i]$, $F_L X[i] = X[i]$ and $F_R X[i] = X'[i]$, from which it is easily seen that $\mathbb{U}$ is a ray preserving and coray conservative tubular enlargement of $\mathbb{T}$ with respect to both F_L and F_R.

Proposition 3.3.4. *[55, 4.7.1] Let R be a finite-dimensional algebra with a separating tubular family $\mathcal{T}$. If M is a ray module in a tube $\mathbb{T}$ of $\mathcal{T}$, let $\mathbb{U}$ be the tube of* $\operatorname{mod}\text{-}R[M]$ *containing M, then* $\operatorname{mod}\text{-}R[M]$ *contains a separating tubular family $\mathcal{U}$ consisting of $\mathbb{U}$ and the tubes of $\mathcal{T}$, excluding $\mathbb{T}$.*

In this section, we have considered one-point extensions corresponding to single ray insertions. Propositions 3.3.2 and 3.3.4 are special cases of Ringel's results [55, 4.5.1] and [55, 4.7.1] respectively. The presentation here differs, as we focus on the two embedding functors and there ray/coray preserving properties — these play a crucial role in the theory of tubular enlargements that we develop in subsequent sections.

3.4 Closure of tubes

3.4.1 Prüfer and adic modules

Let R be a locally bounded algebra and let $\mathbb{T}$ be a tube in the module category mod-R. Given a ray $M[1] \to M[2] \to \cdots$ in $\Gamma(\mathbb{T})$ (and a choice of representative irreducible morphisms $M[i] \to M[i+1]$ in mod-R), let $M[\infty] := \varinjlim M[j]$ denote the direct limit of this system in Mod-R. Define the following properties:

(R1) The module $M[\infty]$ is indecomposable and pure-injective.

(R2) Property (R1) holds and, if $X \in \mathrm{cl}(\mathbb{T})$, then $(M[1], X) = 0$ unless $X \simeq M[i]$ for some $i \in \mathbb{N} \cup \{\infty\}$.

Note if (R1) holds, then $M[\infty]$ belongs to $\mathrm{cl}(\mathbb{T})$ (indeed, any definable subcategory containing $M[i]$ for $i \in \mathbb{N}$ is closed under direct limits and contains $M[\infty]$), and if (R2) holds, then $\{M[\infty], M[1], M[2], \ldots,\}$ is an open subset in $\mathrm{cl}(\mathbb{T})$.

Strictly speaking, these properties are defined relative to a choice of irreducible maps. However, we say a ray satisfies one of these properties if that property holds for some choice.

Lemma 3.4.1. *If $\mathbb{T}$ is a standard component and $M[1] \to M[2] \to \cdots$ is a ray in $\mathbb{T}$ with $M[1]$ a ray module, then $\dim_{\mathrm{k}}(M[i], M[i+1]) = 1$ for all $i \geq 1$. Consequently, the module $M[\infty]$ is defined uniquely up to isomorphism.*

Proof. The first claim is Proposition 3.3.2(i) and the second follows. □

Dually, given a coray $\cdots \to [2]M \to [1]M$ in $\Gamma(\mathbb{T})$ (and a choice of representative irreducible morphism $M[i+1] \to M[i]$ in mod-R), let $[\infty]M := \varprojlim M[j]$ denote the the inverse limit of this system in Mod-R. Define the following properties:

(R1*) The module $[\infty]M$ is indecomposable and pure-injective.

(R2*) Property (R1*) holds and, if $X \in \mathrm{cl}(\mathbb{T})$, then $(X, [1]M) = 0$ unless $X \simeq [i]M$ for some $i \in \mathbb{N} \cup \{\infty\}$.

Extending the terminology used for hereditary algebras, we will call a module of the form $M[\infty]$ a **Prüfer module** of $\mathbb{T}$ and specifically, if not ambiguous (e.g. if $\mathbb{T}$ is standard stable), the $M[1]$**-Prüfer**. Similarly, a module of the form $[\infty]M$ is called an **adic module** or the $M[1]$**-adic** of $\mathbb{T}$.

Observe the standard $\mathbb{k}$-duality maps rays (resp. corays) in mod-R^{op} to corays (resp. rays) in mod-R, and vice versa. Furthermore, given a ray $M[1] \to M[2] \to \cdots$ of R^{op}-modules and its dual coray $\cdots \to [2]N \to [1]N$ of R-modules (i.e. $[i]N := M[i]^{\star}$), we have $(M[\infty])^{\star} = (\varinjlim M[i])^{\star} = \varprojlim(M[i]^{\star}) = [\infty]N$. Thus every adic R-module is a $\mathbb{k}$-dual of a Prüfer R^{op}-module. The following result (and its dual) is useful for calculating Prüfer and adic modules.

Lemma 3.4.2. *[43, IX.3.1] For any strictly increasing sequence of integers $(j_i)_{i\geq 1}$ (i.e. $j_i < j_{i+1}$ for all $i \geq 1$) we have $M[\infty] = \varinjlim_{i\geq 1} M[j_i]$. In particular, we have $M[\infty] = \varinjlim_{i\geq n} M[i]$ for any $n \in \mathbb{N}$.*

Corollary 3.4.3. *Suppose $\mathbb{T}$ is a tubular enlargement of a tube $\mathbb{S}$ with respect to a functor $F : \mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}B$.*

(i) If F preserves direct limits and the tubular enlargement is ray conservative, then F maps Prüfer modules of $\mathbb{S}$ to Prüfer modules of $\mathbb{T}$.

(ii) If F preserves inverse limits and the tubular enlargement is coray conservative, then F maps adic modules of $\mathbb{S}$ to adic modules of $\mathbb{T}$.

Proof. We are assuming $\mathbb{S}$ is a tube in mod-A and $\mathbb{T}$ is a tube in mod-B. The result follows immediately from Lemma 3.4.2, its dual Lemma 3.4.2*, and the definition of ray/coray conservative tubular enlargements. □

Prüfer and adic modules for quasi-stable tubes

Let A be a locally bounded algebra and let $\mathbb{T} \subseteq$ mod-A be a quasi-stable tube (recall, from Section 1.5, the projective and injective modules in $\mathbb{T}$ coincide). The definition of rays and corays, as given above, is only applicable when $\mathbb{T}$ is a genuine tube (equiv. the criterion of Lemma 1.5.4 is met). However, there is a clear generalisation for quasi-stable tubes, using the fact that $\Gamma^s(\mathbb{T})$ is stable.

We will define a **ray** in $\Gamma(\mathbb{T})$ to be an infinite path $X[1] \to X[2] \to \cdots$ satisfying one of the following conditions:

(i) $X[1] \to X[2] \to \cdots$ defines a ray in $\Gamma^s(\mathbb{T})$, in particular, this means $X[i]$ is not projective (= injective) for all $i \geq 1$.

(ii) $X[1]$ is projective with $\text{rad}(X[1])/\text{soc}(X[1])$ an indecomposable A-module, and $X[2] \to X[3] \to \cdots$ defines a ray in $\Gamma^s(\mathbb{T})$.

Condition (ii) is a technicality ensuring that, when $\mathbb{T}$ is indeed a tube, this definition of a ray coincides with the original definition given in Section 3.3. Note, if given a ray $X[1] \to X[2] \to \cdots$ satisfying condition (ii), then $X[2] \to X[3] \to \cdots$ is a ray satisfying condition (i).

Every ray of $\Gamma(\mathbb{T})$ is contained in a maximal ray, and these are in bijective correspondence with the maximal rays of $\Gamma^s(\mathbb{T})$. A quasi-stable tube is a tube if and only if $\Gamma(\mathbb{T})$ is partitioned by its maximal rays (i.e. $\mathbb{T}$ satisfies property (T1) defined below), this follows easily from Lemma 1.5.4.

We define a **coray** for $\Gamma(\mathbb{T})$ dually. Now Prüfer and adic modules for $\mathbb{T}$ can be defined as above.

3.4.2 Properties (T1)–(T4)

Let R be a locally bounded algebra and let $\mathbb{T}$ be a tube (or quasi-stable tube) of $\text{mod-}R$. Define the following properties:

(T1) The vertices of $\Gamma(\mathbb{T})$ are partitioned by a set of maximal rays.

(T1*) The vertices of $\Gamma(\mathbb{T})$ are partitioned by a set of maximal corays.

(T2) Each (maximal) ray of $\mathbb{T}$ satisfies (R1).

(T2*) Each (maximal) coray of $\mathbb{T}$ satisfies (R1*).

(T3) $\mathbb{T}$ satisfies (T2) and (T2*). Additionally, the closure $\text{cl}(\mathbb{T})$ of $\mathbb{T}$ in the Ziegler spectrum Zg_R contains the following complete and irredundant list of modules.

- The finite-dimensional indecomposable modules of $\mathbb{T}$.
- A Prüfer module $M[\infty]$ for each maximal ray.
- An adic module $[\infty]M$ for each maximal coray.
- A unique generic module G.

(T4) $\mathbb{T}$ satisfies (T3) and a subset $\mathfrak{C} \subseteq \text{cl}(\mathbb{T})$ is closed if and only if the following conditions hold:

(i) $M[\infty]$ belongs to $\mathfrak{C}$ if infinitely many modules of the form $M[j]$ for $j \in \mathbb{Z}$ belong to $\mathfrak{C}$, where $M[1] \to M[2] \to \cdots$ is the ray for $M[\infty]$.

(ii) $[\infty]M$ belongs to $\mathfrak{C}$ if infinitely many modules of the form $[j]M$ for $j \in \mathbb{Z}$ belong to $\mathfrak{C}$, where $\cdots \to [2]M \to [1]M$ is the coray for $[\infty]M$.

(iii) G belongs to $\mathfrak{C}$ if $\mathfrak{C}$ contains infinitely many modules or at least one infinite-dimensional module.

Examples of tubes satisfying (T1), (T1*), and (T4), include all stable tubes over a Euclidean or a tubular algebra—see Corollary 3.6.7 below. Examples of non-stable tubes satisfying (T3) and one of (T1) or (T1*) are given in Section 3.6.3. Examples of quasi-stable tubes satisfying (T1), (T1*), and (T4), are given in Sections 3.5 and 5.2–5.3.

We now investigate how these properties are preserved under one-point tubular extensions and coextensions. The following is the main result of this chapter and it will be used extensively, in the sequel, to compute the Ziegler-closure of many tubes (for example, see Section 3.5 below).

Theorem 3.4.4. *Let R be a finite-dimensional algebra with M a ray module in a standard tube*[1] *$\mathbb{T}$ of* mod-R *and let $\mathbb{U}$ be the tube of* mod-$R[M]$ *containing M. If $\mathbb{T}$ satisfies one of (T1), (T1*), (T2), or (T2*), then $\mathbb{U}$ satisfies the same property. Furthermore, if M satisfies (R2) and $\mathbb{T}$ satisfies either (T3) or (T4), then $\mathbb{U}$ satisfies the same property.*

Proof. We make use of the Proposition 3.3.2 and the notation therein. Let $\Lambda := \Gamma(\mathbb{T})$ so that $\Lambda[M] = \Gamma(\mathbb{U})$. Let $F_L, F_R : \text{Mod-}R \to \text{Mod-}R[M]$ be the two embeddings for the one-point extension. Let $M[1] \to M[2] \to \cdots$ denote the (not necessarily maximal) ray of Λ beginning at $M = M[1]$ and set $M'[i] := F_R M[i]$ for $i \geq 1$.

If $\mathbb{T}$ satisfies (T1), then Λ is partitioned into say $p \geq 1$ maximal rays. By definition of ray insertion, $\Lambda[M]$ is partitioned into $p+1$ maximal rays, i.e. the p maximal rays coming from Λ (one of which contains the ray beginning at M) and the ray $M'[1] \to M'[2] \to \cdots$ which is seen to be maximal. Thus $\mathbb{U}$ satisfies (T1).

If $\mathbb{T}$ satifies (T1*), then Λ is partitioned into say $q \geq 1$ maximal corays. From the definition of $\Lambda[M]$ one can check that it is also partitioned into q maximal corays, and

[1] A "standard tube" is just a tube that is standard as a component of the module category.

thus $\mathbb{U}$ satisfies (T1*).

Recall, by Proposition 3.1.3, both F_L and F_R are fully faithful interpretation functors and, in particular, they preserve indecomposability, pure-injectivity, and commute with direct limits. Note also that both functors commute with inverse limits (for F_L is just the zero embedding and F_R is a right adjoint). Additionally, by Proposition 3.3.2, the functors F_L and F_R are ray preserving (so are, in particular, ray conservative) and coray conservative. Thus, by Corollary 3.4.3, both functors map the Prüfer, resp. adic, modules of $\mathbb{T}$ to corresponding Prüfer, resp. adic, modules of $\mathbb{U}$.

If $\mathbb{T}$ satisfies (T2), then the p Prüfer modules of $\mathbb{T}$ are mapped by F_L to p corresponding (indecomposable and pure-injective) Prüfer modules of $\mathbb{U}$. By Proposition 3.3.2(i), it follows that F_R coincides with F_L on these Prüfer modules except for $M[\infty]$, where $F_R(M[\infty]) = \varinjlim F_R(M[i]) = \varinjlim M'[i] = M'[\infty]$—the remaining Prüfer module of $\mathbb{U}$. Thus $\mathbb{U}$ satisfies (T2).

If $\mathbb{T}$ satisfies (T2), then F_L maps the q adic modules of $\mathbb{T}$ to q corresponding (indecomposable and pure-injective) adic modules of $\mathbb{U}$. Thus $\mathbb{U}$ satisfies (T2*). Note that F_R defines the same map of adic modules. Indeed, given a coray in $\mathbb{T}$, we see F_L and F_R coincide on infinitely many points along this coray, and therefore (as they are coray conservative) must map this coray into the same coray of $\mathbb{U}$.

By Proposition 1.3.4, both F_L and F_R induce closed embeddings $\mathrm{Zg}_R \to \mathrm{Zg}_{R[M]}$ which restrict to maps $\mathrm{cl}(\mathbb{T}) \to \mathrm{cl}(\mathbb{U})$ by continuity, since $F_L(\mathbb{T}), F_R(\mathbb{T}) \subseteq \mathbb{U}$. In fact, by Proposition 3.3.2, we know $\mathbb{U} = F_L(\mathbb{T}) \vee F_R(\mathbb{T})$ and it follows that $\mathrm{cl}(\mathbb{U}) = F_L(\mathrm{cl}(\mathbb{T})) \cup F_R(\mathrm{cl}(\mathbb{T}))$.

If $\mathbb{T}$ satisfies (T3) and M satisfies (R2), then, by Proposition 3.1.3, we know F_L and F_R must coincide on all infinite-dimensional points of $\mathrm{cl}(\mathbb{T})$ except for the Prüfer module $M[\infty]$. In particular, they coincide on the generic module of $\mathrm{cl}(\mathbb{T})$. From the equality $\mathrm{cl}(\mathbb{U}) = F_L(\mathrm{cl}(\mathbb{T})) \cup F_R(\mathrm{cl}(\mathbb{T}))$ and the above determination of the Prüfer and adic modules for $\mathbb{U}$, it follows immediately that $\mathbb{U}$ satisfies (T3).

If $\mathbb{T}$ satisfies (T4) and M satisfies (R2), then, by the above considerations, we know a subset $\mathfrak{C} \subseteq \mathrm{cl}(\mathbb{U})$ is closed in $\mathrm{cl}(\mathbb{U})$ if and only if both subsets $\mathfrak{C}_L := F_L^{-1}(\mathfrak{C} \cap F_L(\mathrm{cl}(\mathbb{T})))$ and $\mathfrak{C}_R := F_R^{-1}(\mathfrak{C} \cap F_R(\mathrm{cl}(\mathbb{T})))$ are closed in $\mathrm{cl}(\mathbb{T})$. From this, it follows that $\mathbb{U}$ satisfies (T4) also. This completes the proof. $\square$

We note Propositions 3.3.2–3.3.4 immediately dualise to results concerning one-point tubular coextensions and Theorem 3.4.4 has the following dual.

Theorem 3.4.5. *Let R be a finite-dimensional algebra with M a coray module in a standard tube $\mathbb{T}$ of* $\operatorname{mod}\text{-}R$ *and let $\mathbb{U}$ be the tube of* $\operatorname{mod}\text{-}[M]R$ *containing M. If $\mathbb{T}$ satisfies one of (T1), (T1*), (T2), or (T2*), then $\mathbb{U}$ satisfies the same property. Furthermore, if M satisfies (R2*) and $\mathbb{T}$ satisfies either (T3) or (T4), then $\mathbb{U}$ satisfies the same property.*

Proof. The proof is essentially similar to the proof of Theorem 3.4.4, but some crucial differences are to be noted. If $F_L, F_R : \operatorname{Mod}\text{-}R \to \operatorname{Mod}\text{-}[M]R$ are the two full embeddings of the one-point coextension, given by Proposition 3.1.4, then F_R is the zero embedding (in contrast to Theorem 3.4.4 where the left adjoint F_L was the zero embedding). Both F_L and F_R are ray conservative and coray preserving for the tubular enlargment $\mathbb{U}$ of $\mathbb{T}$. Both functors are interpretation functors and commute with direct limits, but here Lemma 3.1.5 is necessary to ensure that F_L commutes with the relevant inverse limits (i.e. inverse limits along corays in $\mathbb{T}$). Otherwise, there are no further complications in deriving this result. □

Lemma 3.4.6. *If $\mathbb{T}$ is a tube satisfying (T4), then its Ziegler-closure* $\operatorname{cl}(\mathbb{T})$ *has CB rank 2, and the CB ranks of points in* $\operatorname{cl}(\mathbb{T})$ *are as follows:*

- *Each finite-dimensional module has CB rank 0.*
- *Each Prüfer and adic module has CB rank 1.*
- *The generic module has CB rank 2.*

Proof. Let $\mathfrak{C} := \operatorname{cl}(\mathbb{T})$. If $X \in \operatorname{cl}(\mathbb{T})$ is finite-dimensional, then $\{X\}$ is open in $\mathfrak{C}$ by Proposition 1.3.6, thus X has CB rank 0. If $X = M[\infty]$ is a Prüfer module, then $\{X\}$ cannot be open in $\mathfrak{C}$, for else $\mathfrak{C}\backslash\{X\}$ would be a closed set containing all points $M[j]$ for $j \geq 1$ but not $M[\infty]$, contradicting (T4). Thus X has CB rank ≥ 1. Similarly, each adic module and the generic module has CB rank ≥ 1. The first CB derivative $\mathfrak{C}^{(1)}$ therefore consists precisely of the infinite-dimensional modules in $\mathfrak{C}$. Now it is easily seen that $\{X\}$ is open in $\mathfrak{C}^{(1)}$ whenever X is a Prüfer or an adic module, but not when X is the generic module. Therefore each Prüfer and adic module has CB rank 1 and the generic module has CB rank 2 in $\mathfrak{C}$. □

Lemma 3.4.7. *Let M be a ray module in a standard tube $\mathbb{T}$ of* mod-R*. If $\mathbb{T}$ satisfies (T4) and M satisfies (R2), then the Prüfer module $M[\infty]$ is isolated in its own closure by an $M[\infty]$-minimal pp functor.*

Proof. By Proposition 3.3.2, we know $\dim_{\mathrm{k}}(M[1], M[\infty]) = 1$. Since $M[1]$ is finite-dimensional, $(M[1], -)$ is a pp functor that is $M[\infty]$-minimal, by Lemma 1.3.10. By (T4) the closure of $M[\infty]$ in Zg_R is the set $\{M[\infty], G\}$ where G is the unique generic module in $\mathrm{cl}(\mathbb{T})$. Note $(M[1], G) = 0$ by (R2), hence $(M[1], -)$ isolates $M[\infty]$ in its own closure. $\square$

Lemma 3.4.8. *Let M be a coray module in a standard tube $\mathbb{T}$ of* mod-R*. If $\mathbb{T}$ satisfies (T4) and M satisfies (R2*), then the adic module $[\infty]M$ is isolated in its own closure by an $[\infty]M$-minimal pp functor.*

Proof. By Proposition 3.3.2*, we know $\dim_{\mathrm{k}}([\infty]M, [1]M) = 1$. Since $[1]M$ is finite-dimensional, $- \otimes_R [1]M^\star$ is a pp functor that is $[\infty]M$-minimal by Lemma 1.3.10* (see Example 1.3.3). By (T4) the closure of $[\infty]M$ in Zg_R is the set $\{[\infty]M, G\}$ where G is the unique generic module in $\mathrm{cl}(\mathbb{T})$. Note $G \otimes_R [1]M^\star = (M[1], G)^\star = 0$ by (R2*), hence $- \otimes_R [1]M$ isolates $[\infty]M$ in its own closure. $\square$

To make repeated application of Theorems 3.4.4–3.4.5 it helps to determine which ray and coray modules exist and when they satisfy properties (R2) and (R2*) respectively. This is done in the following lemma (and its dual).

We say two ray modules M and N belong to the same **ray class** if M belongs to the ray beginning at N or vice versa.

Lemma 3.4.9. *Let R be a finite-dimensional algebra with M a ray module in a standard tube $\mathbb{T}$ of* mod-R *and suppose $\mathbb{T}$ satisfies (T4). Let $\mathbb{U}$ be the tube of* mod-$R[M]$ *containing M and let ω be the extension vertex of $R[M]$.*

(i) The projective module $P(\omega)$ is a ray module of $\mathbb{U}$. If M satisfies (R2) in $\mathbb{T}$, then $P(\omega)$ satisfies (R2) in $\mathbb{U}$.

(ii) If N is a ray module in $\mathbb{T}$ with different ray class than M, then N is a ray module of $\mathbb{U}$. If N satisfies (R2) in $\mathbb{T}$, then N satisfies (R2) in $\mathbb{U}$.

(iii) If M is a coray module in $\mathbb{T}$, then $P(\omega)$ is a coray module of $\mathbb{U}$. If M satisfies (R2*) in $\mathbb{T}$, then $P(\omega)$ satisfies (R2*) in $\mathbb{U}$.

(iv) *If N is a coray module in $\mathbb{T}$, then N is a coray module in $\mathbb{U}$. If N satisfies (R2*) in $\mathbb{T}$, then N satisfies (R2*) in $\mathbb{U}$.*

Proof. We use the notation of Theorem 3.4.4 and by that theorem $\mathbb{U}$ satisfies (T4). The claims about being a ray/coray module all follow easily from definition (or see [17, 2.9, 2.10]), so we concentrate on properties (R2) and (R2*). These statements all have similar proofs and we become decidedly briefer in proving each successive one.

Suppose M satisfies property (R2) in $\mathbb{T}$, we prove $P(\omega)$ satisfies (R2) in $\mathbb{U}$. Recall $P(\omega) = M' = M'[1] = F_R(M[1])$. Let $X \in \mathrm{cl}(\mathbb{U})$ be given and suppose $(M', X) \neq 0$. If X is finite-dimensional, then Proposition 3.3.2 gives $X \simeq M'[i]$ for some $i \geq 1$ as desired. So suppose X is infinite-dimensional. If $X = F_R(Y)$ for some $Y \in \mathrm{cl}(\mathbb{T})$, then $(M', X) = (F_R M, F_R Y) \simeq (M, Y)$ and property (R2) for M implies $Y = M[\infty]$, hence $X = M'[\infty]$ as desired. The remaining possibility is $X = M[\infty]$, but then $(M', X) = \varinjlim(M', M[i]) = 0$ — as M' has a different ray class than M — in contradiction to the assumption $(M', X) \neq 0$. Therefore $P(\omega) = M'$ satisfies (R2) in $\mathbb{U}$.

Let N be a ray module in $\mathbb{T}$ with different ray class than M and suppose N satisfies property (R2) in $\mathbb{T}$. Let $X \in \mathrm{cl}(\mathbb{T})$ be given and suppose $(N, X) \neq 0$. Again, by Proposition 3.3.2(i), we may assume X is infinite-dimensional. If $X = F_L(Y)$ for $Y \in \mathrm{cl}(\mathbb{T})$, then $\mathrm{Hom}_{R[M]}(N, X) = (F_L N, F_L Y) = \mathrm{Hom}_R(N, Y)$ (recall F_L is the zero embedding) and property (R2) for N in $\mathbb{T}$ gives $Y = N[\infty]$, hence $X = N[\infty]$ as desired. Otherwise $X = M'[\infty]$, but then $(N, X) = \varinjlim(N, M'[i]) = 0$ (as N has a different ray class to M in $\mathbb{T}$, and thus to M' in $\mathbb{U}$) — a contradiction. Therefore N satisfies (R2) in $\mathbb{U}$.

Supppose M is a coray module and satisfies (R2*) in $\mathbb{T}$, we show $P(\omega) = M'$ satisfies (R2*) in $\mathbb{U}$. Let $X \in \mathrm{cl}(\mathbb{U})$ – infinite-dimensional – be given and suppose $(X, M') \neq 0$. If $X = F_R Y$, then $(X, M') \simeq (Y, M)$ and $Y = [\infty]M$, hence $X = [\infty]M'$. Otherwise $X = M[\infty]$ and $(X, M') = \varprojlim(M[i], M') = 0$ — a contradiction. Therefore $P(\omega)$ satisfies (R2*) in $\mathbb{U}$.

Suppose N is a coray module and satisfies (R2*) in $\mathbb{T}$. Let $X \in \mathrm{cl}(\mathbb{U})$ – infinite-dimensional – be given and suppose $(X, N) \neq 0$. If $X = F_L Y$, then $\mathrm{Hom}_{R[M]}(X, N) \simeq \mathrm{Hom}_R(X, Y)$ and $X = Y = [\infty]N$ by property (R2*) for N in $\mathbb{T}$. Otherwise $X = M'[\infty]$ and $\mathrm{Hom}_{R[M]}(X, N) \simeq \mathrm{Hom}_R(M[\infty], N) = 0$ by property (R2*) for N in $\mathbb{T}$ — a contradiction. Therefore N satisfies (R2*) in $\mathbb{U}$. □

Proposition 3.4.10. *Let A be a tame concealed algebra and $\mathbb{T}$ a tube of* mod-A*, then* $\mathbb{T}$ *satisfies (T4). Every simple regular A-module is both a ray and a coray module, satisfying properties (R2) and (R2*).*

Proof. $\mathbb{T}$ satisfies (T4) by Proposition 1.8.2. Since $\mathbb{T}$ is stable, every simple regular module S in $\mathbb{T}$ is both a ray and a coray module. Let S be such a module and let $S[1] \to S[2] \to \cdots$ denote the (maximal) ray beginning at $S = S[1]$. To prove (R2) holds for S, let $X \in \operatorname{cl}(\mathbb{T})$ be given and suppose $(S, X) \neq 0$. If X is finite-dimensional, then Proposition 3.3.2 implies $X \simeq S[i]$ for some $i \geq 1$ as required. So assume X is infinite-dimensional, i.e. X is a Prüfer, an adic, or the generic module G.

By [64, X.2.8] any non-zero map $S \to Y$, with $Y \notin \mathbb{T}$ indecomposable and finite-dimensional, factors through the initial part $S[1] \to \cdots \to S[n]$ of the ray at S, for arbitrarily large $n \geq 1$. By [50, 5.3.31] the same result holds for infinite-dimensional Y, in particular, for $Y = X$. Thus, a non-zero map $S \to X$ induces a non-zero map $S[\infty] \to X$. However, it is shown in [53] that $(S[\infty], Z) = 0$ for all $Z \in \mathbb{T}$ and $Z = G$. Then also

$$(S[\infty], [\infty]V) = (S[\infty], \varprojlim [j]V) = \varprojlim (S[\infty], [j]V) = 0$$

for any adic module $[\infty]V$. Thus, as $(S, X) \neq 0$, we know X is a Prüfer module $U[\infty]$ for some simple regular U in $\mathbb{T}$. However

$$(S, U[\infty]) = (S, \varinjlim U[j]) = \varinjlim (S, U[j]) = 0$$

unless $U = S$ (i.e. $X = S[\infty]$) by (R1). This proves (R2) holds for S.

To prove (R2*) holds for S, take $X \in \operatorname{cl}(\mathbb{T})$, we may again assume X is infinite-dimensional. We know $(X, S) = 0$ if X is a Prüfer or generic module by [53]. Thus, if $(X, S) \neq 0$, then M is an adic module. Suppose $X = [\infty]V$ for $V \neq S$, then $(X, S) \simeq (X \otimes S^\star)^\star$ and

$$X \otimes S^\star = (\varprojlim [i]V) \otimes S^\star \simeq \varprojlim ([i]V \otimes S^\star) \simeq \varprojlim ([i]V, S)^\star$$

by Proposition 1.11.6. By duality $([i]V, S)^\star = (S^\star, [i]V^\star) = 0$ (since $S^\star$ is a ray module over A^{op} and $[i]V^\star$ does not belong to the ray beginning at $S^\star$) and thus $([\infty]V, S) = 0$. Proving property (R2*) for S. □

3.5 Closure of some quasi-stable tubes

In this section we define a class of algebras possessing a separating tubular family that contains a quasi-stable tube of a specified form. The results on one-point tubular extensions are then applied to compute the Ziegler closure of these tubes.

We first define the form of the tubes we will consider. Given integers $p \geq 0$ and $0 \leq m \leq n$ define the translation quiver $\mathbf{\Gamma(p,n,m)}$. The vertices of $\Gamma(p,n,m)$ come in three kinds. For all $j \in \mathbb{N}$ we have $X_i[j]$ for $0 \leq i \leq n-m$, $Y_i[j]$ for $1 \leq i \leq m$, and $Z_i[j]$ for $1 \leq i \leq p$ (interpret an empty interval as there being no vertices of that kind).

The arrows of $\Gamma(p,n,m)$ are as follows: $X_i[j] \to X_i[j+1]$, $Y_i[j] \to Y_i[j+1]$, and $Z_i[j] \to Z_i[j+1]$ for all $j \in \mathbb{N}$ (and for all i in the relevant interval) — these arrows will form rays in $\Gamma(p,n,m)$. There are arrows $X_i[j] \to X_{i+1}[j]$ for all $0 \leq i < n-m$, and if $m = p = 0$, then $X_n[j+1] \to X_0[j]$ (in this case, no further arrows exist). If $m \geq 1$, there are arrows $Y_i[j+1] \to Y_{i+1}[j]$ for all $1 \leq i < m$, and if $p \geq 1$, then $Y_m[j+2] \to Z_1[j]$, or else $Y_m[j+2] \to X_0[j]$. Finally, if $p \geq 1$, there are arrows $Z_i[j+1] \to Z_{i+1}[j]$ for all $1 \leq i < p$, $Z_p[j+1] \to X_0[j]$, and if $m = 0$, then $X_n[j+1] \to Z_1[j]$.

The translation τ of $\Gamma(p,n,m)$ is defined as follows: $\tau X_i[j+1] := X_{i-1}[j]$ for all $1 < i \leq m-n$, and

$$\tau X_0[j] := \begin{cases} X_n[j] & \text{if } m = 0, p = 0, \\ Y_m[j+1] & \text{if } m \geq 1, \\ Z_m[j] & \text{if } p \geq 1. \end{cases}$$

Also $\tau Y_1[j+1] := X_{n-m}[j]$ and $\tau Y_i[j+1] := Y_{i-1}[j+1]$ for $1 < i \leq m$. Finally $\tau Z_i[j] := Z_{i-1}[j]$ for $1 < i \leq p$ and $\tau Z_1[j] = Y_m[j+1]$ if $m \geq 1$, otherwise $\tau Z_1[j] := Z_n[j]$.

The quivers $\Gamma(2,3,2)$ and $\Gamma(1,3,3)$ are depicted in Example 3.5.1 and Example 3.5.3 respectively, and Lemma 3.5.2 should help to clarify the general definition.

Example 3.5.1. $\Gamma(2,3,2)$ is depicted (partially) in the following diagram. It is a non-stable tube and only the translates at the mouth are drawn (by a dashed line).

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & & & & Y_1[1] \arrow[dr] & & Y_2[1] \arrow[dr] & & & \\
 & & & & & X_1[1] \arrow[ur] \arrow[dr] & & Y_1[2] \arrow[ur] \arrow[dr] & & Y_2[2] \arrow[dr] \arrow[rr, dashed, no head] & & Z_1[1] \arrow[dd, dotted, no head] \\
Z_1[1] \arrow[dr] \arrow[rr, dashed, no head] \arrow[dd, dotted, no head] & & Z_2[1] \arrow[dr] \arrow[rr, dashed, no head] & & X_0[1] \arrow[ur] \arrow[dr] & & X_1[2] \arrow[ur] \arrow[dr] & & Y_1[3] \arrow[ur] \arrow[dr] & & Y_2[3] \arrow[ur] \arrow[dr] & \\
 & Z_1[2] \arrow[ur] \arrow[dr] & & Z_2[2] \arrow[ur] \arrow[dr] & & X_0[2] \arrow[ur] \arrow[dr] & & X_1[3] \arrow[ur] \arrow[dr] & & Y_1[4] \arrow[ur] \arrow[dr] & & Y_2[4] \\
Y_2[4] \arrow[ur] & & Z_1[3] \arrow[ur] & & Z_2[3] \arrow[ur] & & X_0[3] \arrow[ur] & & X_1[4] \arrow[ur] & & Y_1[5] \arrow[ur] &
\end{tikzcd}$$

Note that $Y_1[1]$ and $Y_2[1]$ are projective-injective vertices. The left and right edges (dotted lines) are to be identified.

Lemma 3.5.2. *For all integers $p \geq 0$ and $0 \leq m \leq n$, the translation quiver $\Gamma(p,n,m)$ is a tube and the following properties hold.*

(i) $\Gamma(p,0,0)$ *is a stable tube of rank $p+1$.*

(ii) $X_n[1]$ *is a ray vertex in* $\Gamma(p,n,0)$ *and for $n \geq 1$ the quiver* $\Gamma(p,n,0)$ *is obtained from* $\Gamma(p,n-1,0)$ *by ray insertion at* $X_{n-1}[1]$.

(iii) $X_{n-m}[1]$ *is a coray vertex in* $\Gamma(p,n,m)$ *and for $m \geq 1$ the quiver* $\Gamma(p,n,m)$ *is obtained from* $\Gamma(p,n,m-1)$ *by coray insertion at* $X_{n-m+1}[1]$.

Proof. That $\Gamma(p,n,m)$ is a tube follows from (i)–(iii), since $\Gamma(p,n,m)$ can then be built from the stable tube $\Gamma(p,0,0)$ by finitely many ray insertions, followed by finitely many coray insertions (ray insertions were defined at the beginning of Section 3.3; coray insertions are dual). Statements (i)–(iii) can all be checked from definition. □

Remarks

(i) One should think of $\Gamma(p,0,0)$ as a stable tube (of rank $p+1$) with a particular ray (namely $X_0[1] \to X_0[2] \to \cdots$) singled out. $\Gamma(0,0,0)$ is a homogenous tube.

(ii) The vertices of $\Gamma(p,n,m)$ are partitioned into $p+n+1$ (maximal) rays, beginning at $X_0[1], \ldots, X_{n-m}[1]$, $Y_1[1], \ldots, Y_m[1]$, and $Z_1[1], \ldots, Z_p[1]$.

(iii) The vertices of $\Gamma(p,n,m)$ are partitioned into $p+m+1$ (maximal) corays, ending at $Y_1[1], \ldots, Y_m[1]$, $Y_m[2]$, and $Z_1[1], \ldots, Z_p[1]$ if $m \geq 1$, otherwise at $X_n[1]$, and $Z_1[1], \ldots, Z_p[1]$.

(iv) If $n \geq 1$ an $m = n$, then $\Gamma(p, n, n)$ is a quasi-stable tube with a "crown" of n projective-injective vertices $Y_1[1], \ldots, Y_n[1]$. It becomes a stable tube of rank $p + n + 1$ upon removing these points.

Example 3.5.3. The following diagram depicts the quasi-stable tube $\Gamma(1, 3, 3)$.

$$
\begin{tikzcd}[column sep=tiny]
 & & & Y_1[1] \arrow[dr] & & Y_2[1] \arrow[dr] & & Y_3[1] \arrow[dr] & & & \\
Z_1[1] \arrow[rr, dash, dashed] \arrow[dr] \arrow[dd, dash, dotted] & & X_0[1] \arrow[ur] \arrow[dr] & & Y_1[2] \arrow[ur] \arrow[dr] & & Y_2[2] \arrow[ur] \arrow[dr] & & Y_3[2] \arrow[rr, dash, dashed] \arrow[dr] & & Z_1[1] \arrow[dd, dash, dotted] \\
 & Z_1[2] \arrow[ur] \arrow[dr] & & X_0[2] \arrow[ur] \arrow[dr] & & Y_1[3] \arrow[ur] \arrow[dr] & & Y_2[3] \arrow[ur] \arrow[dr] & & Y_3[3] \arrow[ur] \arrow[dr] & \\
Y_3[4] \arrow[ur] & & Z_1[3] \arrow[ur] & & X_0[3] \arrow[ur] & & Y_1[4] \arrow[ur] & & Y_2[4] \arrow[ur] & & Y_3[4]
\end{tikzcd}
$$

The mouth of this tube contains what we informally refer to as a "crown" of 3 projective-injective vertices (namely $Y_1[1]$, $Y_2[1]$, and $Y_3[1]$). As claimed, it is a stable tube of rank $5 = 1 + 3 + 1$ upon the removal of these points.

Let A be a finite-dimensional algebra with S an A-module at the mouth of a stable tube belonging to a separating tubular family in mod-A (e.g. take A to be tame concealed and S a simple regular A-module).

Define the algebra $\mathbf{A}[\mathbf{S}, \mathbf{n}]$ for $n \geq 0$ inductively, by iterated one-point extensions, as follows:

$$
A[S, n] := \begin{cases} A & \text{if } n = 0, \\ (A[S, n-1])[P_{n-1}] & \text{if } n \geq 1, \end{cases}
$$

where $P_0 := S$ and for $i \geq 1$ $P_i := P(\omega_i)$ is the indecomposable projective module corresponding to the extension vertex ω_i of $A[S, i]$. Now define the algebra $\mathbf{A}[\mathbf{S}, \mathbf{n}, \mathbf{m}]$ for $0 \leq m \leq n$ inductively, by iterated one-point coextensions, as follows:

$$
A[S, n, m] := \begin{cases} A[S, n] & \text{if } m = 0, \\ [Q_{m-1}](A[S, n, m-1]) & \text{if } m \geq 1, \end{cases}
$$

where $Q_0 := P_n$ and for $i \geq 1$ $Q_i := \tau_{A[S,n,i]}\, Q_{i-1}$ is the translate of Q_{i-1} considered as an $A[S, n, i]$-module.

Proposition 3.5.4. *Let A be a finite-dimensional algebra with S an A-module at the mouth of a stable tube $\mathbb{S}$ belonging to a separating tubular family $\mathcal{S}$ in* mod-A*. Suppose*

S satisfies (R2) and (R2*) and $\mathbb{S}$ satisfies (T4) with rank $t \geq 1$.

For integers $0 \leq m \leq n$, then $\operatorname{mod}\text{-}A[S,n,m]$ *contains a separating tubular family $\mathcal{T}$ consisting of the tubes of $\mathcal{S}$, except $\mathbb{S}$, and an additional tube $\mathbb{T}$ containing S. Furthermore, $\mathbb{T}$ satisfies (T4) and has the form $\Gamma(\mathbb{T}) = \Gamma(t-1,n,m)$.*

Proof. We show $A[S,n,m]$ is obtained from A by a sequence of one-point tubular extensions, followed by a sequence of one-point tubular coextensions, of the type considered in Section 3.3.

For $n = m = 0$, $A[S,0,0] = A$, we take $\mathcal{T} := \mathcal{S}$, $\mathbb{T} := \mathbb{S}$ and the result holds by assumption. As $\mathbb{T}$ is a stable tube of rank t, we are free to identify $\Gamma(\mathbb{T}) = \Gamma(t-1,0,0)$ in such a way that $P_0 = S$ represents the vertex $X_0[1]$. By assumption P_0 satisfies (R2) and (R2*); this is necessary for the following argument.

We proceed inductively. For $n \geq 1$, $A[S,n] = (A[S,n-1])[P_{n-1}]$ and we assume P_{n-1} lies in a tube (with S) of the form $\Gamma(t-1,n-1,0)$, representing the vertex $X_{n-1}[1]$, and satisfying (R2) and (R2*). By Proposition 3.3.2 and Lemma 3.5.2, then P_n lies in a tube (with S) of the form $\Gamma(t-1,n,0)$ and represents the vertex $X_n[1]$. Proposition 3.3.4 and Theorem 3.4.4 give the desired result. Finally, Lemma 3.4.9 ensures P_n satisfies (R2) and (R2*), so the induction can continue.

For $n \geq m \geq 1$, $A[S,n,m-1] = [Q_{m-1}](A[S,n,m-1])$ and we assume Q_{m-1} lies in a tube (with S) of the from $\Gamma(t-1,n,m-1)$, representing the vertex $X_{n-m+1}[1]$, and satisfying (R2*). Note this holds when $m = 1$ since $Q_0 = P_n$. By Proposition 3.3.2* and Lemma 3.5.2, then Q_m lies in a tube (with S) of the form $\Gamma(t-1,n,m)$ and representd the vertex $X_{n-m}[1]$. Propostion 3.3.4* and Theorem 3.4.5 give the desired result. Finally, Lemma 3.4.9 ensures Q_m satisfies (R2*). $\square$

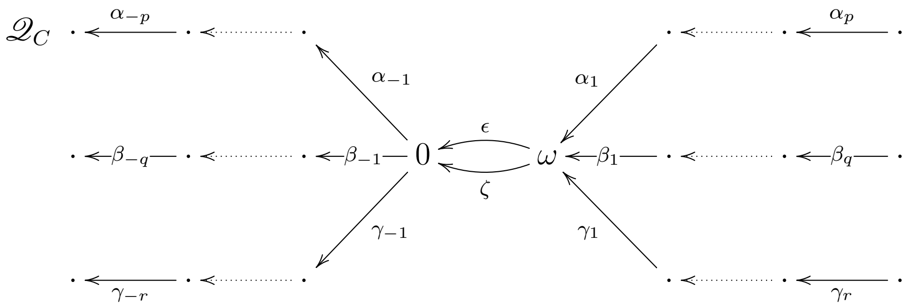
By Proposition 3.4.10 below, a tame concealed algebra A and a simple regular A-module S satisfy the hypothesis of the above proposition; we give the following examples of this kind. See also Example 6.1.13.

Example 3.5.5. Let A be the Kronecker algebra $\mathbf{k}(\cdot \leftleftarrows \cdot)$ and let S_∞ be the simple regular A-module $\mathbf{k} \underset{0}{\overset{1}{\leftleftarrows}} \mathbf{k}$. For $p \geq 1$ we find $B := A[S_\infty, p, p]$ has the following quiver, with $2p+1$ arrows labelled by α.

$$\cdot \xleftarrow{\alpha_{-p}} \cdot \cdots\cdots \cdot \xleftarrow{\alpha_{-1}} \cdot \underset{\beta_0}{\overset{\alpha_0}{\leftleftarrows}} \cdot \xleftarrow{\alpha_1} \cdot \cdots\cdots \cdot \xleftarrow{\alpha_p} \cdot$$

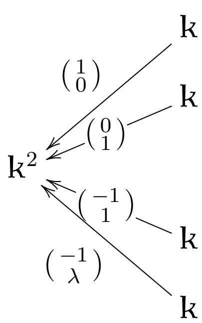
Relations for B are $\alpha_1\beta_0 = 0$, $\beta_0\alpha_{-1} = 0$, and $\alpha^{p+3} = 0$ (i.e. any path of $p+3$ α-arrows). By Proposition 3.5.4 we know S_∞ lies in a quasi-stable tube $\mathbb{T}$ of the form $\Gamma(0,p,p)$ in mod-B (for $p = 3$ it is depicted at [31, p. 162]). Hence $\Gamma(\mathbb{T})$ contains p projective-injective points and becomes a stable tube of rank $p+1$ upon their removal. We can continue this construction and follows.

For $\lambda \in \mathbf{k}$ let S_λ be the simple regular A-module $\mathbf{k} \underset{1}{\overset{\lambda}{\leftleftarrows}} \mathbf{k}$. Let $\lambda \in \mathbf{k}\backslash\{0\}$ and integers $q, r \geq 1$ be given. If $C := A[S_\infty, p, p][S_\lambda, q, q][S_0, r, r]$, then C is given by the following quiver with relations generated by $\alpha_1\zeta$, $\zeta\alpha_{-1}$, $\beta_1\zeta - \lambda\beta_1\epsilon$, $\zeta\beta_{-1} - \lambda\epsilon\beta_{-1}$, $\gamma_1\epsilon$, $\epsilon\gamma_{-1}$, any path of length $p+3$ consisting of α-arrows and ϵ, any path of length $q+3$ consisting of β-arrows and ϵ, any path of length $r+3$ consisting of γ-arrows and ζ.



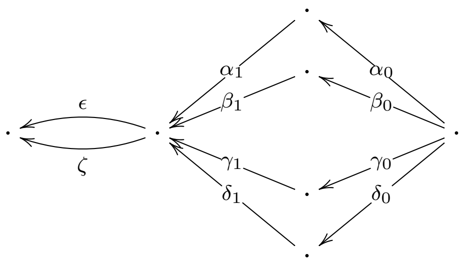
Then mod-C contains a separating tubular family with precisely three non-homogenous quasi-stable tubes of the form $\Gamma(0,p,p)$, $\Gamma(0,q,q)$, and $\Gamma(0,r,r)$. The homogenous tubes are the tubes of regular A-modules corresponding to S_κ for $\kappa \in \mathbf{k}\backslash\{0,\lambda\}$. By Proposition 3.5.4 all tubes in this family satisfy (T4).

Example 3.5.6. Let A be the path algebra of the subspace oriented quiver of Euclidean type $\widetilde{\mathbb{D}}_4$ and let M be the following A-module (for some fixed $\lambda \in \mathbf{k}\backslash\{0,1\}$).



As M is a simple regular A-module, lying at the mouth of a homogenous tube (see [64, XIII.2.4]), the algebra $A[M] = A[M,1]$ is a one-point tubular extension of A, and is a tubular algebra of type $(2,2,2,2)$. The module M lies in a tube of mod-$A[M]$ of the form $\Gamma(0,1,0)$ and has slope 0 over $A[M]$. Continuing, the algebra $A[M,1,1]$ has the following quiver with relations $\alpha_1\zeta = 0$, $\beta_1\epsilon = 0$, $\gamma_1\epsilon = \gamma_1\zeta$, and $\delta_1\epsilon = \lambda\delta_1\zeta$, $\alpha_0\alpha_1 + \beta_0\beta_1 + \gamma_0\gamma_1 = 0$, and $\alpha_0\alpha_1 + \lambda\beta_0\beta_1 + \delta_0\delta_1 = 0$.

$\mathscr{Q}_{A[M,1,1]}$



The tube $\mathbb{U}$ of mod-$A[M,1,1]$ containing M is a quasi-stable tube of the form $\Gamma(0,1,1)$, containing 1 projective-injective vertex, and is a stable tube of rank 2 upon removing this point (it is depicted at [31, p. 160]). By Proposition 3.5.4 the tube $\mathbb{U}$ satisfies (T4) and the Ziegler closure of $\mathbb{U}$ contains two Prüfer modules, two adics modules, and a unique generic module.

Corollary 3.5.7. *Let A and S be as in Proposition 3.5.4. Let $B := A[S,n,m]$ and denote by $\omega_1,\dots,\omega_n$ the extension vertices of B and by $\sigma_1,\dots,\sigma_m$ the coextension vertices of B. Let $\mathbb{T}$ be the tube of* mod-B *containing S, then*

(i) Every Prüfer module in $\mathrm{cl}(\mathbb{T})$ *is a module over $B/\langle\sigma_1,\dots,\sigma_m\rangle$.*

(ii) Every adic module in $\mathrm{cl}(\mathbb{T})$ *is a module over $B/\langle\omega_1,\dots,\omega_n\rangle$.*

The generic module in $\mathrm{cl}(\mathbb{T})$ *is an A-module.*

Proof. Suppose S has τ_A-periodicity $t \geq 1$. Then $\mathbb{T}$ has $t+n$ Prüfer modules, t of which are modules over A. The construction of $A[S,n]$ from A involves n consecutive one-point extensions, each introducing a new Prüfer module, thus all $t+n$ Prüfer modules are modules over $A[S,n]$. Now $\mathbb{T}$ has $t+m$ adic modules, t of which are modules over A, and B is obtained from $A[S,n]$ by m consecutive one-point coextensions, each introducing a new adic module supported on $\mathscr{Q}_A$ and the coextension vertices. It follows that all Prüfer modules are modules over $B/\langle\sigma_1,\dots,\sigma_m\rangle = A[S,n]$ and all adic modules are modules over $B/\langle\omega_1,\dots,\omega_n\rangle$. $\square$

Corollary 3.5.8. *Let A and S be as in Proposition 3.5.4. Let $B := A[S, n, m]$ and let $\mathbb{T}$ be the tube of* $\operatorname{mod}\text{-}B$ *containing S, then the isolation condition holds for* $\operatorname{cl}(\mathbb{T})$ *as a closed subset of* Zg_B.

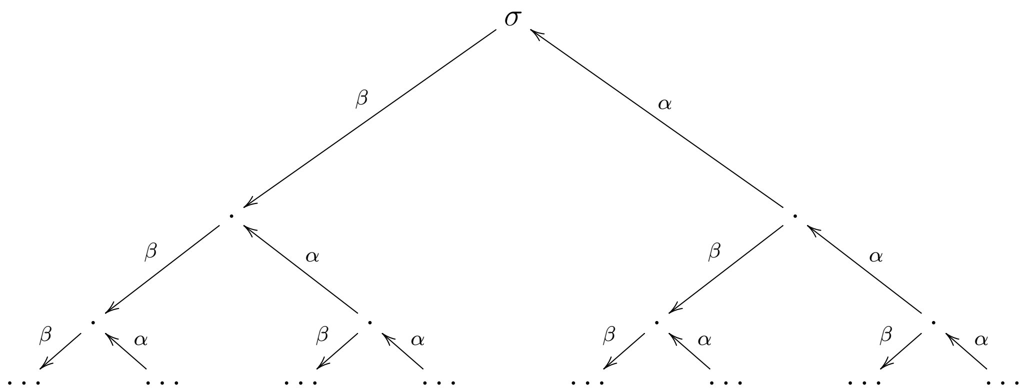
Proof. Let $\mathfrak{C} = \operatorname{cl}(\mathbb{T})$ be the Ziegler closure of $\mathbb{T}$. We show each point $M \in \mathfrak{C}$ is isolated in its own closure by an M-minimal pp functor, then $\mathfrak{C}$ satisfies the isolation condition by Proposition 1.3.11. If M is finite-dimensional, then $\{M\}$ is both open and closed in Zg_R (hence in $\mathfrak{C}$) by Proposition 1.3.6 and M is isolated in its closure by an M-minimal pp functor by Lemma 1.3.12. If $M = U[\infty]$ is a Prüfer module, then by Lemma 3.4.7 we know at some point in the construction of Proposition 3.5.4 (i.e. considered as an $A[S, n']$-module for some $0 \leq n' \leq n$) that M is isolated in its closure $\{M, G\}$ by the M-minimal pp functor determined by $(U[1], -)$. This same pp functor remains M-minimal over B and isolates M in its closure. Similarly, if $M = [\infty]V$ is an adic module, then by Lemma 3.4.8 we know M is isolated in its closure $\{M, G\}$ by an M-minimal pp functor. Finally, if $M = G$ is the generic module, then Lemma 1.3.12 applies. □

3.6 Tubular enlargements

The one-point tubular (co)extensions of Section 3.3 (and iterations of such tubular enlargements as in Section 3.5) are not enough to construct all tubes appearing over Euclidean or tubular algebras. Indeed, in defining such algebras, we have already made reference to a more general notion of tubular extension — namely, the branch extensions of [55, §4.7] (see also [65, §XV.3]). In this section we will describe such extensions and illustrate some of their properties.

3.6.1 Branch algebras

Let $\mathscr{Q}_K$ be a finite and connected full subquiver, containing the vertex σ, of the following infinitely repeating quiver (3.6.1). The vertex σ is called the **root** of $\mathscr{Q}_K$ and the algebra K defined by the quiver $\mathscr{Q}_K$ and all relations $\alpha\beta = 0$ is called a **branch algebra** (elsewhere a **truncated branch algebra**) with **branch** $\mathscr{Q}_K$. The **size** of a branch is its number of vertices; we allow for an empty branch of size 0.



(3.6.1)

The unique branch of size n and with all arrows pointing towards the root (i.e. α-arrows) is called the **subspace branch** of size n (equiv. length $n-1$).

3.6.2 Tubular branch extensions

Let A be a finite-dimensional algebra and M a ray module belonging to a separating tubular family of mod-A. Let ω denote the extension vertex in the quiver $\mathscr{Q}_{A[M]}$ (of the one-point extension $A[M]$) and let K be a branch with root σ. Define $\mathscr{Q}_{A[M,K]}$ as the quiver obtained from the disjoint union $\mathscr{Q}_{A[M]} \sqcup \mathscr{Q}_K$ by identifying the vertices ω and σ. The algebra $A[M,K]$ is defined to be the algebra with quiver $\mathscr{Q}_{A[M,K]}$ and all induced relations from $A[M]$ and K. Iterating this construction, one can define a general **tubular branch extension** $A[M_i, K_i]_{i=1}^{t} := (A[M_i, K_i]_{i=1}^{t-1})[M_t, K_t]$ for pairwise hom-orthogonal ray modules $M_1, \ldots, M_t$ and branches $K_1, \ldots, K_t$ (see [55, §4.7] and [65, §XVI.1] for further details).

If $B := A[M_i, K_i]_{i=1}^{t}$ be a tubular branch extension, let $\mathcal{T} := (\mathbb{T}_\lambda)_{\lambda\in\Lambda}$ denote the tubular family of mod-A containing the ray modules $M_1, \ldots, M_t$. Additionally, assume $\mathcal{T}$ is stable (i.e. each $\mathbb{T}_\lambda$ is a stable tube). For each $i = 1, \ldots, t$, take $\lambda_i \in \Lambda$ such that $M_i \in \mathbb{T}_{\lambda_i}$ and define $n_i := \operatorname{size}(K_i)$. Now for all $\lambda \in \Lambda$ define $r_\lambda := \operatorname{rank}(\mathbb{T}_\lambda)$ and $s_\lambda := \sum_{i\in I_\lambda} n_i$ where $I_\lambda = \{i \mid M_i \in \mathbb{T}_\lambda\}$. Let $\mu_1, \ldots, \mu_p \in \Lambda$ denote all indices satisfying $r_{\mu_i} + s_{\mu_j} > 1$ (there are only finitely many) and define $m_i := r_{\mu_i} + s_{\mu_j}$. By reordering if necessary, we assume $m_1 \leq m_2 \leq \cdots \leq m_p$, then the (non-decreasing) sequence $(m_1, \ldots, m_p)$ is called the **extension type** of the tubular extension.

Let $F : \text{Mod-}A \to \text{Mod-}B$ denote the zero embedding (i.e. F is the functor res_π

given by restriction along the projection $\pi : B \to A$ whose kernel is generated by the branches $K_1, \dots, K_t$). By [55, 4.7.1] the tubular family $\mathcal{T} \subseteq \text{mod-}A$ gives rise to a (separating) tubular family $\mathcal{U} := (\mathbb{U}_\lambda)_{\lambda\in\Lambda}$ in mod-B such that each $\mathbb{U}_\lambda$ is a tubular enlargement of $\mathbb{T}_\lambda$ with respect to F. Furthermore, each of these tubular enlargements is both ray preserving and coray conservative.

For $\lambda \in \Lambda$ the tube $\mathbb{U}_\lambda$ is obtained from $\mathbb{T}_\lambda$ by a sequence of "rectangular insertions" (see [65, XV.2.5] for a precise definition) and is called a **ray tube** of **rank** $r_\lambda + s_\lambda$. In case $s_\lambda = 0$ (which is true for almost all λ), then $\mathbb{U}_\lambda = \mathbb{T}_\lambda$ is a stable tube consisting of A-modules. The general form of a ray tube is given in [55, 4.6.4]. The ray tube $\mathbb{U}_\lambda$ is partitioned into $r_\lambda + s_\lambda$ maximal rays (where r_λ of these rays consist entirely of A-modules and coincide with the rays of $\mathbb{T}_\lambda$). Thus each ray tube satisfies (T1).

The notions of **tubular branch coextension** and **coray tube** are defined dually (for convenience, we will refer to the "extension type" of a tubular branch coextension, rather than the "coextension type").

Example 3.6.2. A one-point tubular extension $A[M]$ — as defined in Section 3.3 — coincides with the tubular branch extension $A[M, K]$ where K is the unique branch of size 1 (consisting of the root vertex σ and no arrows).

Example 3.6.3. Let A be a tame concealed algebra and $B := A[M_i, K_i]_{i=1}^t$ a tubular branch extension of A. Recall from Section 1.9.2, if B has extension type (p, q) for $1 \le p \le q$, $(2, 2, m-2)$ for $m \ge 4$, or $(2, 3, n-3)$ for $n = 6, 7, 8$, then B is a Euclidean algebra; and from Section 1.10, if B has extension type $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$, or $(2, 3, 6)$, then B is a tubular algebra (in this case, the tubular family of mod-B containing $M_1, \dots M_t$ is the family of slope 0).

Lemma 3.6.4. *Let M be a ray module at the mouth of a stable tube belonging to a separating tubular family in* mod-A. *For $n \ge 1$, if K_n is the subspace branch of size n, then $A[M, K_n] = A[M, n]$.*

Proof. If $n = 1$, then $A[M, K_1] = A[M] = A[M, 1]$. For $n \ge 2$, let ω_{n-1} be the unique source of $\mathscr{Q}_{K_{n-1}}$, so $K_n = K_{n-1}[P_{K_{n-1}}(\omega_{n-1})]$. Considering ω_{n-1} as a vertex of $\mathscr{Q}_{A[M,K_{n-1}]}$ — where, when $n = 2$, we identify ω_1 with the extension vertex of $A[M]$ — it is easily seen that $A[M, K_n] = (A[M, K_{n-1}])[P_{A[M,K_{n-1}]}(\omega_{n-1})]$. Hence, by definition, we see $A[M, K_n] = A[M, n]$. □

3.6.3 Closure of ray and coray tubes

Lemma 3.6.5. *Let $B := A[M_i, K_i]_{i=1}^t$ be a tubular branch extension where each K_i is a subspace branch. If each M_i satisfies (R2) and lies in a tube of* $\mathrm{mod}\text{-}A$ *satisfying (T4), then M_i belongs to a tube of* $\mathrm{mod}\text{-}B$ *satisfying (T4).*

Proof. As B can be obtained from A be repeated one-point tubular extensions, the result follows inductively from Theorem 3.4.4 and Lemma 3.4.9. The case $t = 1$ is illustrative and has already been proven: by Lemma 3.6.4 we have $A[M_1, K_1] = A[M_1, s_{\lambda_1}]$ and Proposition 3.5.4 proves the result. □

Proposition 3.6.6. *Let $B = A[M_i, K_i]_{i=1}^t$ be a tubular branch extension of a tame concealed algebra A. Let $\mathcal{T} = (\mathbb{T}_\lambda)_{\lambda\in\mathbb{P}^1(\mathbf{k})}$, resp. $\mathcal{U} = (\mathbb{U}_\lambda)_{\lambda\in\mathbb{P}^1(\mathbf{k})}$, denote the tubular families of* $\mathrm{mod}\text{-}A$, *resp.* $\mathrm{mod}\text{-}B$, *containing the modules $M_1, \ldots, M_t$, and indexed such that $\mathbb{U}_\lambda$ is a tubular enlargement of $\mathbb{T}_\lambda$ (with respect to the zero embedding F :* $\mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}B$*). For all $\lambda \in \mathbb{P}^1(\mathbf{k})$, the tube $\mathbb{U}_\lambda$ satisfies both (T2) and (T2*).*

Proof. By Proposition 3.4.10, each $\mathbb{T}_\lambda$ satisfies both (T2) and (T2*) over A. We show $\mathbb{U}_\lambda$ inherits the same property. Note F commutes with direct and inverse limits, and is a fully faithful interpretation functor, preserving indecomposability and pure-injectivity of all A-modules. If $\mathbb{U}_\lambda$ is a stable tube (equiv. does not contain any of the modules M_i), then $\mathbb{U}_\lambda = \mathbb{T}_\lambda$ and so clearly satisfies (T2) and (T2*) over B also. Hence, we can assume $\mathbb{U}_\lambda$ contains (at least) one of the modules M_i.

The maximal corays of $\mathbb{U}_\lambda$ are equal in number to the maximal corays of $\mathbb{T}_\lambda$. Since F is coray conservative, it follows from Corollary 3.4.3, that F maps the adic modules of $\mathbb{T}_\lambda$ to the adic modules of $\mathbb{U}_\lambda$; it follows that $\mathbb{U}_\lambda$ satisfies (T2*).

Note B is defined inductively $B = (A[M_i, K_i]_{i=1}^{t-1})[M_t, K_t]$ but the branch extensions can be done in any order (i.e. the ray modules M_i, with their branches K_i, can be arbitrarily ordered). For each $j \in \{1, \ldots, t\}$ we have the ray preserving zero embedding $F_j : \mathrm{Mod}\text{-}A[M_j, K_j] \to \mathrm{Mod}\text{-}B$. Any maximal ray in $\mathbb{U}_\lambda$ is then the image under some F_j of a maximal ray belonging to a tube of $\mathrm{mod}\text{-}A[M_j, K_j]$. In this way, each Prüfer module for $\mathbb{U}_\lambda$ is a Prüfer $A[M_j, K_j]$-module for some $j \in \{1, \ldots, t\}$. Hence, we can conclude property (T2) holds for $\mathbb{U}_\lambda$ provided we prove the result for tubular branch extensions that use a single branch, i.e. when $t = 1$.

Suppose $B = A[M, K]$ for a simple regular module M and branch K of size n. Let K' denote the subspace branch of size n, then by [55, 4.4.5] there exists a tilting module T_B such that $\mathrm{End}_B(T) = A[M, K']$. Set $B' := A[M, K']$ and let $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{Y}, \mathcal{X})$ denote the torsion pairs of Mod-B and Mod-B', respectively, induced by T. Recall by Proposition 1.7.1 the functor $F := \mathrm{Hom}_B(T, -)$ restricts to an equivalence of definable subcategories $\mathcal{G} \to \mathcal{Y}$.

Let $\mathbb{T}$ and $\mathbb{T}'$ be the tubes of mod-B and mod-B', respectively, containing M. It follows from [55, 4.4.5] that $\mathbb{T}' \subseteq \mathcal{Y}$ and $F(\mathbb{T} \cap \mathcal{G}) = \mathbb{T}'$. Furthermore, for each maximal ray $X[1] \to X[2] \to \cdots$ of $\mathbb{T}$ there exists $m \geq 1$ such that the ray $X[m] \to X[m+1] \to \cdots$ lies entirely in $\mathcal{G}$ and its image $FX[m] \to FX[m+1] \to \cdots$ is a maximal ray of $\mathbb{T}'$. Now $\mathbb{T}'$ satisfies (T2) by Proposition 3.4.10 and Lemma 3.6.5, in particular $\varinjlim_{i\geq 1} FX[i]$ is a Prüfer B'-module belonging to $\mathrm{cl}(\mathbb{T}') \subseteq \mathrm{Zg}(\mathcal{Y}) \subseteq \mathrm{Zg}_{B'}$. It follows that

$$F(X[\infty]) = F(\varinjlim_{i\geq 1} X[i]) = F(\varinjlim_{i\geq m} X[i]) = \varinjlim_{i\geq m} FX[i]$$

belongs to $\mathrm{Zg}(\mathcal{G}) \subseteq \mathrm{Zg}_B$. Therefore $\mathbb{T}$ satisfies (T2). The remaining tubes of $\mathcal{T}$ consist entirely of A-modules and satisfy (T2) by Proposition 3.4.10. This completes the proof. $\square$

Example 6.1.13 contains an example of a non-stable ray tube and indicates how, as in the proof of the above proposition, it may be tilted to a ray tube of the form $\Gamma(p, n, 0)$ for some $p, n \geq 1$.

We derive the following partial result concering the Ziegler closure of ray and coray tubes appearing over Euclidean and tubular algebras. A full description could be given but we don't require it here.

Corollary 3.6.7. *Let A be a Euclidean or tubular algebra, then*

(i) Every stable tube in mod-A *satisfies (T4).*

(ii) Every tube $\mathbb{T}$ *in* mod-A *satisfies (T2) and (T2***).*

Proof. Every tube of mod-A is stable or else is a ray or coray tube.

For A a Euclidean algebra, claim (i) follows from Proposition 3.4.10 or its dual, since every stable tube in mod-A is a tube over some tame concealed algebra. The same can be said for stable tubes consisting of modules of slopes 0 or ∞ when A is a tubular algebra, and for the (stable) tubes consisting of modules of positive rational slope (for these, the desired properties for establishing (T4) are given in [32, §3.5] and the proof is similar to Proposition 3.4.10).

If mod-A contains a ray tube $\mathbb{T}$, then A must be a tubular branch extension of a tame concealed algebra and properties (T2) and (T2*) for $\mathbb{T}$ are then given by Proposition 3.6.6.

If mod-A contains a coray tube $\mathbb{T}$, then A must be a tubular branch coextension of a tame concealed algebra and properties (T2) and (T2*) for $\mathbb{T}$ are then given by Proposition 3.6.6*. $\square$

Chapter 4

Covering Techniques

Covering techniques have been successfully applied to the finite-dimensional representation theory of self-injective algebras (see [68] for a survey of many such applications). In this chapter we recall the key definitions and derive results useful for studying pure-injective modules and Ziegler spectra. The results of Sections 4.1.2–4.1.3 and their subsequent applications (e.g. Corollary 4.2.4) are original to this work. The preceding results on (Galois) coverings and push-down functors, although more or less known (see for instance [23] and [9]) aren't typically presented in explicit detail. We give this detail here since a number of key results (particularly Propositions 4.1.13 and 4.1.15) rely on their veracity.

4.1 Covering and push-down functors

Let A and B be small $\mathbf{k}$-linear categories. Given a functor $F : A \to B$ and an object $b \in B$, introduce the notation a/b as shorthand for $a \in F^{-1}(b)$ (i.e. $F(a) = b$). Given $a_1, a_2 \in A$ let $F_{a_1,a_2} : A(a_1, a_2) \to B(F(a_1), F(a_2))$ denote the $\mathbf{k}$-linear map of hom-spaces defined by F. In this chapter, for notational convenience, we will typically work with *left* modules (equiv. *covariant* functors).

A $\mathbf{k}$-linear functor $F : A \to B$ is called a **covering functor** if, for all $b_1, b_2 \in B$, the following two conditions are satisfied:

(a) For all a_1/b_1 the maps $\{F_{a_1,a_2} \mid a_2/b_2\}$ are the components of a bijection

$$\bigoplus_{a_2/b_2} A(a_1, a_2) \to B(b_1, b_2) \tag{4.1.1}$$

(b) For all a_2/b_2 the maps $\{F_{a_1,a_2} \mid a_1/b_1\}$ are the components of a bijection

$$\bigoplus_{a_1/b_1} A(a_1, a_2) \to B(b_1, b_2) \tag{4.1.2}$$

Given a covering functor $F : A \to B$ it is usual to call restriction along F (i.e. the functor $\mathrm{res}_F : B\text{-}\mathrm{Mod} \to A\text{-}\mathrm{Mod}$) the **pull-up functor**. The pull-up functor has a left adjoint, commonly denoted $F_\lambda : A\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}$, called the **push-down functor**[1]. The push-down functor is defined explicitly in [9, 3.2] and derived as follows.

By Proposition 1.11.1 we know F_λ can be defined, for left A-modules, by the B-A-bimodule $B(F-,-)$. The bijection (4.1.1) above is easily seen to be natural in a_1 and gives, for each $b_2 \in B$, an A-linear isomorphism

$$\bigoplus_{a_2/b_2} A(-, a_2) \to B(F(-), b_2) \tag{4.1.3}$$

Note the right-hand side of this equation is just $\mathrm{res}_F B(-, b_2)$. In this way, each representable right B-module lifts (or "pulls-up") to the coproduct of all representable A-modules lying above. For a_2/b_2 the right A-module $A(-, a_2)$ lies above $B(-, b_2)$ in the sense that

$$F_\lambda(A(-, a_2)) = A(-, a_2) \otimes_A B(-, F-) \simeq B(-, F(a_2)) = B(-, b_2)$$

The bijection (4.1.2) gives the analogous fact for left A-modules. Now the right-hand side of (4.1.3) is functorial in b_2, so there is a unique way to make the left-hand side functorial in b_2, such that equations (4.1.1) define a B-A-bimodule isomorphism, this is Lemma 4.1.5 below. To facilitate working with covering functors, let us introduce some additional notation.

[1]In Section 1.11.1, the left adjoint to F was denoted F_L. However, whenever F is a covering functor, we will use the notation F_λ for its left adjoint (i.e. the push-down functor); this is common within the literature.

Henceforth, fix a covering functor $F : A \to B$. Given $\beta : b_1 \to b_2$ in B, for each a_1/b_1 there exists, according to (4.1.1), a unique set $\{\beta^{a_1}_{(a_2)} : a_1 \to a_2 \mid a_2/b_2\}$ of morphisms in A (almost all zero) such that $\beta = \sum_{a_2/b_2} F(\beta^{a_1}_{(a_2)})$. We call this the "unique lifting property" of F. The notation $\beta^{a_1}_{(a_2)}$ is to suggest that the domain a_1 remains fixed and the codomain a_2 is allowed to vary, as we lift β to A via F. Similarly, for each a_2/b_2 the bijection (4.1.2) gives a unique set $\{\beta^{(a_1)}_{a_2} : a_1 \to a_2 \mid a_1/b_1\}$ of morphisms in A (again, almost all zero) such that $\beta = \sum_{a_1/b_1} F(\beta^{(a_1)}_{a_2})$. We are careful here to distinguish between $\beta^{(a_1)}_{a_2}$ and $\beta^{a_1}_{(a_2)}$.

We will denote a typical element of an infinite direct sum $\bigoplus_{i\in I} X_i$ as either an indexed sequence $(x_i)_{i\in I}$ – where $x_i \in X_i$ for each $i \in I$ and almost all entries are zero – or as a (ultimately finite) sum $\sum_{i\in I} x_i$.

Lemma 4.1.4. *Given $\beta : b_1 \to b_2$ and $\delta : b_2 \to b_3$ we have*

$$(\delta\beta)^{a_1}_{(a_3)} = \sum_{a_2/b_2} \delta^{a_2}_{(a_3)} \beta^{a_1}_{(a_2)}$$

for all a_1/b_1 and a_3/b_3.

Proof. We have

$$\begin{aligned}
\sum_{a_3/b_3} F\left(\sum_{a_2/b_2} \delta^{a_2}_{(a_3)} \beta^{a_1}_{(a_2)}\right) &= \sum_{a_3/b_3}\sum_{a_2/b_2} F(\delta^{a_2}_{(a_3)})F(\beta^{a_1}_{(a_2)}) \\
&= \sum_{a_2/b_2}\left(\sum_{a_3/b_3} F(\delta^{a_2}_{(a_3)})\right) F(\beta^{a_1}_{(a_2)}) \\
&= \sum_{a_2/b_2} \delta F(\beta^{a_1}_{(a_2)}) \\
&= \delta\beta
\end{aligned}$$

and the result follows by the unique lifting property. □

We are going to give the left-hand side of (4.1.3) the structure of a B-A-bimodule $\Theta : A^{\mathrm{op}} \otimes B \to \mathrm{Ab}$. For $a \in A$ and $b \in B$ define $\Theta(a, b) := \bigoplus_{x/b} A(a, x)$. As noted above $\Theta(a, b)$ is already functorial in a, i.e. $\Theta(-, b) : A \to \mathrm{Ab}$ is just a coproduct of representable A-modules. Now given $\beta : b_1 \to b_2$ in B and $x \in A$ define the map

$\theta(x,\beta) : \bigoplus_{a_1/b_1} A(x,a_1) \to \bigoplus_{a_2/b_2} A(x,a_2)$ by

$$(\alpha_{a_1})_{a_1/b_1} \mapsto \left(\sum_{a_1/b_1} \beta^{a_1}_{(a_2)} \alpha_{a_1} \right)_{a_2/b_2}$$

for all $(\alpha_{a_1})_{a_1/b_1} \in \Theta(x,b_1)$.

Lemma 4.1.5. Θ *is a well-defined* B*-*A*-bimodule and equations* (4.1.1) *define a* B*-*A*-bimodule isomorphism* $\Theta \simeq B(F(-),-)$*.*

Proof. For $x \in A$ we first prove $\Theta(x,-) : B \to \mathrm{Ab}$ is a well-defined functor. Given $b \in B$ and a/b the equality $F(1_a) = 1_b = \sum_{y/b} F((1_b)^a_{(y)})$ implies $(1_b)^a_{(a)} = 1_a$ and $(1_b)^a_{(y)} = 0$ for $y \neq a$. It follows that $\Theta(x,1_b) = 1_{\Theta(x,b)}$. Given $\beta : b_1 \to b_2$, $\delta : b_2 \to b_3$ in B, the equality $\Theta(x,\delta\beta) = \Theta(x,\delta)\Theta(x,\beta)$ follows easily from Lemma 4.1.4. So indeed $\Theta(x,-)$ is a functor.

The isomorphism $\phi_{a,b} : \Theta(a,b) \to B(F(a),b)$ given by equation (4.1.1) is defined by $(\alpha_x)_{x/b} \mapsto \sum_{x/b} F(\alpha_x)$ for all $(\alpha_x)_{x/b} \in \Theta(a,b)$. Naturality in a is easily confirmed, indeed, given $\alpha : a_1 \to a_2$ in A, $b \in B$, and $\bar{\gamma} = (\gamma_x)_{x/b} \in \Theta(a_2,b)$ we have

$$B(F(\alpha),b)\phi(a_2,b)(\bar{\gamma}) = \left(\sum_{x/b} F(\gamma_x) \right) F(\alpha) = \sum_{x/b} F(\gamma_x \alpha) = \phi(a_1,b)\Theta(\alpha,b)(\bar{\gamma})$$

as required. Naturality in b is also easily confirmed, given $a \in A$, $\beta : b_1 \to b_2$ in B, and $\bar{\gamma} = (\gamma_{a_1})_{a_1/b_1} \in \Theta(a,b_1)$ we have

$$\begin{aligned}
\phi(a,b_2)\Theta(a,\beta)(\bar{\gamma}) &= \sum_{a_2/b_2} F \left(\sum_{a_1/b_1} \beta^{a_1}_{(a_2)} \alpha_{a_1} \right) \\
&= \sum_{a_1/b_1} \left(\sum_{a_2/b_2} F(\beta^{a_1}_{(a_2)}) \right) F(\alpha_{a_1}) \\
&= \beta \sum_{a_1/b_1} F(\alpha_{a_1}) \\
&= B(F(a),\beta)\phi(a,b_1)(\bar{\gamma})
\end{aligned}$$

as required. □

Proposition 4.1.6. *If* $F : A \to B$ *is a covering functor, then the push-down functor*

$F_\lambda : A\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}$ *is defined for* $M \in A\text{-}\mathrm{Mod}$ *by*

$$F_\lambda M(b) := \bigoplus_{a/b} M(a)$$

for all $b \in B$*, and for* $\beta : b_1 \to b_2$ *in* B *by*

$$F_\lambda M(\beta)((m_{a_1})_{a_1/b_1}) := \left(\sum_{a_1/b_1} M(\beta^{a_1}_{(a_2)})(m_{a_1}) \right)_{a_2/b_2} \tag{4.1.7}$$

for all $(m_{a_1})_{a_1/b_1} \in F_\lambda M(b_1)$.

Proof. The push-down functor F_λ is left adjoint to res_F and is defined by the B-A-bimodule $B(F(?), -) \simeq \Theta(?, -) = \bigoplus_{a/(-)} A(?, a)$ by Proposition 1.11.1 and Lemma 4.1.5. The Yoneda isomorphisms give, for each $b \in B$, an isomorphism

$$F_\lambda M(b) := \bigoplus_{a/b} A(-, a) \otimes_A M \simeq \bigoplus_{a/b} M(a)$$

Hence, for $\beta : b_1 \to b_2$, we can chase around the following diagram

$$\begin{tikzcd}
\bigoplus_{a_1/b_1} M(a_1) \arrow[r] \arrow[d, dotted, "F_\lambda M(\beta)"] & \bigoplus_{a_1/b_1} A(-, a_1) \otimes_A M \arrow[d, "\Theta(-,\beta)\otimes M"] \\
\bigoplus_{a_2/b_2} M(a_2) & \bigoplus_{a_2/b_2} A(-, a_2) \otimes_A M \arrow[l]
\end{tikzcd}$$

and find (4.1.7) is the correct definition for the map $F_\lambda M(\beta)$. □

Corollary 4.1.8. *The push-down functor* $F_\lambda : A\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}$ *is an interpretation functor if and only if* $F^{-1}(b)$ *is finite for all* $b \in B$.

Proof. By Proposition 1.11.1 the push-down functor F_λ is an interpretation functor if and only if $\mathrm{res}_F B(-, b) = B(F-, b) = \bigoplus_{a \in F^{-1}(b)} A(-, a)$ is finitely-presented — equivalently $F^{-1}(b)$ is finite — for all $b \in B$. □

4.1.1 Galois coverings

Let A be a locally-bounded category, a group G of $\mathbf{k}$-linear automorphisms of A will be called **admissible** provided G acts freely (i.e. given $g \in G$, if $g \neq 1$, then $ga \neq a$ for all $a \in A$) and has finitely many orbits.

Given A and an admissible group G the **orbit category** A/G is defined with objects being the G-orbits of objects of A. Then a morphism $f : x \to y$ in A/G is a family of morphisms $({}_b f_a) \in \prod_{(a,b)\in x\times y} A(a,b)$ such that $g({}_b f_a) = {}_{gb} f_{ga}$ for all $g \in G$. See [23, §3] for further details. There is a projection functor $F : A \to A/G$ defined by $F(a) = G \cdot a$ with the following properties. The finite-dimensional algebra corresponding to A/G is called the **orbit algebra**.

Proposition 4.1.9. *[23, 3.1] If A is a locally-bounded category and G an admissible group of automorphisms of A, then the projection $F : A \to A/G$ is a covering functor. Furthermore, F is universal with respect to being G-invariant, i.e. $Fg = F$ for all $g \in G$ and if $E : A \to B$ is a $\Bbbk$-linear functor with this property, then $E = E'F$ for some unique $E' : A/G \to B$.*

Covering functors of the form $F : A \to A/G$ are called **Galois coverings**.

The following result is fundamental to the use of push-down functors in the representation theory of algebras. Given $g \in G$ and $M \in A\text{-}\mathrm{Mod}$ denote by ${}^gM$ the module Mg^{-1} — in this way G acts on $\mathrm{ind}\text{-}A$.

Proposition 4.1.10. *[23, 3.6] Let $F : A \to B$ be a Galois covering with $B = A/G$ and let $F_\lambda : A\text{-}\mathrm{mod} \to B\text{-}\mathrm{mod}$ be the corresponding push-down functor (restricted to finite-dimensional modules). If G acts freely on $\mathrm{ind}\text{-}A$, then*

(i) F_λ preserves indecomposability,

(ii) F_λ induces a Galois covering $A\text{-}\mathrm{ind} \to (B\text{-}\mathrm{ind})^0$ where $(B\text{-}\mathrm{ind})^0$ denotes the full subcategory of $B\text{-}\mathrm{ind}$ given by objects in the image of F_λ.

(iii) F_λ preserves Auslander-Reiten sequences and induces an isomorphism of Γ_A/G onto the union of some connected components of Γ_B.

A locally bounded category A is **locally support-finite** if, for each object $i \in A$, the set $\bigcup_M \mathrm{supp}(M)$ is finite, where the union ranges over $M \in \mathrm{ind}\text{-}A$ such that $M(i) \neq 0$.

Proposition 4.1.11. *[18] Let $F : A \to B$ be a Galois covering with $B = A/G$. If A locally support-finite and G acts freely on $\mathrm{ind}\text{-}A$, then $(B\text{-}\mathrm{ind})^0 = B\text{-}\mathrm{ind}$ and the push-down functor F_λ induces a Galois covering $A\text{-}\mathrm{ind} \to B\text{-}\mathrm{ind}$.*

Henceforth, fix a Galois covering $F : A \to B$ (with $B = A/G$). Let $F_\lambda : A\text{-}\operatorname{Mod} \to B\text{-}\operatorname{Mod}$ be the corresponding push-down functor. For $M \in A\text{-}\operatorname{Mod}$ let us use the abbreviation $M_\lambda := F_\lambda(M)$ where convenient. For $X \in B\text{-}\operatorname{Mod}$ note $\operatorname{res}_F(X) = XF$. We have an adjunction isomorphism $(M_\lambda, X) \simeq (M, XF)$ by definition of F_λ being left adjoint to res_F. Let us consider the unit $M \to M_\lambda F$ and counit $(XF)_\lambda \to X$ of this adjunction.

For $M \in A\text{-}\operatorname{Mod}$ and $a \in A$ observe that $(M_\lambda F)(a) = \bigoplus_{a'/F(a)} M(a')$. Define $\mu_{M,a} : M(a) \to (M_\lambda F)(a)$ to be the canonical inclusion map. For $X \in B\text{-}\operatorname{Mod}$ and $b \in B$ observe that $(XF)_\lambda(b) = \bigoplus_{a/b} XF(a) = \bigoplus_{a/b} X(b)$ is a direct sum of $|F^{-1}(b)| = |G|$ copies of $X(b)$. Define $\epsilon_{X,b} : (XF)_\lambda(b) \to X(b)$ to be the summation map $(x_a)_{a/b} \mapsto \sum_{a/b} x_a$.

Lemma 4.1.12. $\mu_M : M \to M_\lambda F$ *and* $\epsilon : (XF)_\lambda \to X$ *are the unit and counit, respectively, of the adjunction* $F_\lambda \dashv \operatorname{res}_F$.

Proof. One can check μ and ϵ — as defined by their components above — are well-defined natural transformations. Then by [43, IV.1.Th. 2] the result follows from proving the "triangle identities" $\epsilon_{X,F(a)} \circ \mu_{XF,a} = 1_{XF(a)}$ and $\epsilon_{M_\lambda,b} \circ F_\lambda(\mu_M)_b = 1_{M_\lambda(b)}$. However, these are easily verified. Each instance amounts to an inclusion followed by a summation, which, by definition, is the relevant identity morphism.

It must be checked that $\mu_M : M \to M_\lambda F$ is an A-linear map for each $M \in A\text{-}\operatorname{Mod}$ and that $\mu : 1 \to \operatorname{res}_F \circ F_\lambda$ is a natural transformation. Similarly, it must be checked that $\epsilon_X : (XF)_\lambda \to X$ is a B-linear map for each $X \in B\text{-}\operatorname{Mod}$ and that $\epsilon : F_\lambda \circ \operatorname{res}_F \to 1$ is a natural transformation. $\square$

Recall, for $g \in G$ and $M \in A\text{-}\operatorname{Mod}$ we define ${}^gM := Mg^{-1}$. It can be shown that $M_\lambda \simeq ({}^gM)_\lambda$ for all $g \in G$ and that $M_\lambda F \simeq \bigoplus_{g\in G} {}^gM$ [23, 3.2]. Under the latter isomorphism, the unit $\mu_M :\to M \to \bigoplus_{g\in G} {}^gM$ is the canonical inclusion map (identifying M with ${}^1M$ where $1 \in G$ is the group identity).

Although F_λ is a faithful functor, it is far from being full in general. The following result seems to be a new observation.

Proposition 4.1.13. *If* $M, N \in A\text{-}\operatorname{Mod}$ *are such that* $(M, {}^gN) = 0$ *for all* $g \neq 1$, *then* $F_\lambda : (M, N) \to (M_\lambda, N_\lambda)$ *is a bijection.*

Proof. As shown in [48, Prop. 5.1] a canonical map $t : \bigoplus_{g\in G}(M, {}^gN) \to (M, \bigoplus_{g\in G} {}^gN)$ can be defined by $t((f_g)_{g\in G}) := \sum_{g\in G} u_g f_g$, where $u_h : {}^hN \to \bigoplus_{g\in G} {}^gN$ denotes the canonical inclusion map. Furthermore, t is an injection whose image consists of all morphisms $f : M \to \bigoplus_{g\in G} {}^gN$ satisfying $p_h f = 0$ for almost all $h \in G$, where $p_h : \bigoplus_{g\in G} {}^gN \to {}^hN$ denotes the canonical projection map. In particular, if $(M, {}^gN) = 0$ for almost all $g \in G$, then t is a bijection; this is certainly the case by our assumption. Note also that the inclusion $u_1 : N \to \bigoplus_{g\in G} {}^gN$ coincides with the unit μ_N.

This means that $f \mapsto \mu_N f$ defines a bijection $(M, N) \to (M, N_\lambda F)$. Now composing with the adjunction isomorphism $(M, N_\lambda F) \to (M_\lambda, N_\lambda)$ gives a bijection $(M, N) \to (M_\lambda, N_\lambda)$ defined by

$$f \mapsto \epsilon_{N_\lambda} F_\lambda(\mu_N f) = \epsilon_{N_\lambda} F_\lambda(\mu_N) F_\lambda(f) = F_\lambda(f)$$

using the triangle identity $\epsilon_{N_\lambda} F_\lambda(\mu_N) = 1_{N_\lambda}$. □

The importance of Galois coverings lies in the following fact (elsewhere stated as a Galois covering being "balanced").

Lemma 4.1.14. *Let $F : A \to B$ be a Galois covering, then $\beta^{a_1}_{(a_2)} = \beta^{(a_1)}_{a_2}$ for all $\beta : b_1 \to b_2$ in B and a_1/b_1, a_2/b_2 in A.*

Proof. Let $\beta : b_1 \to b_2$ in B and $g \in G$ be given and fix a_1/b_1, then

$$\sum_{a_2/b_2} F(\beta^{a_1}_{(a_2)}) = \beta = \sum_{a_2/b_2} F(g\beta^{a_1}_{(a_2)})$$

by the G-invariance of F. Hence, for all a_2/b_2, since $g\beta^{a_1}_{(a_2)}$ is a morphism $ga_1 \to ga_2$, we have $g\beta^{a_1}_{(a_2)} = \beta^{ga_1}_{(ga_2)}$ by the unique lifting property. Similarly, $g\beta^{(a_1)}_{a_2} = \beta^{(ga_1)}_{ga_2}$. Now fixing a_1/b_1 and a_2/b_2 we have (using the principle that ha_1 runs through $F^{-1}(b_1)$ as h runs through G) the equality

$$\sum_{g\in G} F(\beta^{(ga_1)}_{a_2}) = \beta = \sum_{g\in G} F(\beta^{a_1}_{(g^{-1}a_2)}) = \sum_{g\in G} F(g\beta^{a_1}_{(g^{-1}a_2)}) = \sum_{g\in G} F(\beta^{ga_1}_{(a_2)})$$

from which we conclude $\beta^{a_1}_{(a_2)} = \beta^{(a_1)}_{a_2}$ by the unique lifting property. □

Given a covering functor $F : A \to B$, the right adjoint $F_\rho : A\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}$ to the restriction res_F exists and is derived similarly to F_λ.

Proposition 4.1.15. *Let $F : A \to B$ be a covering functor and $F_\rho : A\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}$ denote the right adjoint to res_F, then F_ρ is defined for $M \in A\text{-}\mathrm{Mod}$ by*

$$F_\rho M(b) := \prod_{a/b} M(a)$$

for all $b \in B$, and for $\beta : b_1 \to b_2$ by

$$F_\lambda M(\beta)((m_{a_1})_{a_1/b_1}) := \left(\sum_{a_1/b_1} M(\beta_{a_2}^{(a_1)})(m_{a_1}) \right) \tag{4.1.16}$$

for all $(m_{a_1})_{a_1/b_1} \in F_\rho M(b_1)$.

Proof. Similar to the B-A-bimodule isomorphism $B(F-,?) \simeq \bigoplus_{a/-} A(?,a)$ established in Lemma 4.1.5 (and defined using the bijections (4.1.1)), one can define an A-B-bimodule $\bigoplus_{a/-} A(a,?)$ (using bijections (4.1.2)) and an isomorphism $B(-,F?) \simeq \bigoplus_{a/-} A(a,?)$. Then, by Proposition 1.11.1, we know

$$F_\rho M(b) := \mathrm{Hom}_A(B(b,F-),M) \simeq \mathrm{Hom}_A\left(\bigoplus_{a/b} A(a,-),M\right) \simeq \prod_{a/b} M(a)$$

and working through the details, as in Proposition 4.1.6, one finds that (4.1.16) is the correct definition. □

The following fact is used in [23, p. 91] although no explicit proof is given.

Corollary 4.1.17. *If $F : A \to B$ is a Galois covering, then F_λ is a subfunctor of F_ρ.*

Proof. For all $M \in R\text{-}\mathrm{Mod}$ and $b \in B$ the canonical monomorphism

$$F_\lambda M(b) = \bigoplus_{a/b} M(a) \to \prod_{a/b} M(a) = F_\rho M(b)$$

is natural in b by Lemma 4.1.14 (for definitions (4.1.7) and (4.1.16) coincide). □

4.1.2 Finitely-supported modules

Given a collection of A-modules $\mathcal{M} = \{M_i\}_{i\in I}$ define

$$\mathrm{Supp}_A(\mathcal{M}) := \bigcup_{i\in I} \{a \in A \mid M_i(a) \neq 0\}$$

and say $\mathcal{M}$ is **finitely-supported** if $\mathrm{Supp}_A(\mathcal{M})$ is finite.

Lemma 4.1.18. *If $F : A \to B$ is a Galois covering and $F_\lambda : \mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}B$ its corresponding push-down functor, then $F_\lambda M = F_\rho M$ for any finitely-supported $M \in A\text{-}\mathrm{Mod}$.*

Proof. This is immediate from Corollary 4.1.17 (and the proof thereof) with the fact that finite coproducts and finite products coincide in Ab. □

Corollary 4.1.19. *Let $F : A \to B$ be a Galois covering and $F_\lambda : A\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}$ the corresponding push-down functor. If $\mathcal{M} = (\{M_i\}_{i\in I}, (\gamma_{i,j} : M_i \to M_j)_{i\leq j})$ is any (inverse) system in $A\text{-}\mathrm{Mod}$ such that $\{M_i\}_{i\in I}$ is a finitely-supported set of A-modules, then $F_\lambda \varprojlim M_i = \varprojlim F_\lambda M_i$.*

Proof. The set $\{M_i\}_{i\in I}$ being finitely-supported implies each M_i is finitely-supported and the limit $\varprojlim M_i$ is finitely-supported, hence $F_\lambda \varprojlim M_i = F_\rho \varprojlim M_i = \varprojlim F_\rho M_i = \varprojlim F_\lambda M_i$ by Lemma 4.1.18 and the fact that F_ρ — being a right adjoint — commutes with limits. □

Corollary 4.1.20. *Let $F : A \to B$ be a Galois covering. If $\mathcal{D} \subseteq A\text{-}\mathrm{Mod}$ is a definable subcategory whose class of modules is finitely-supported, then the restriction $F_\lambda|_{\mathcal{D}} : \mathcal{D} \to B\text{-}\mathrm{Mod}$, of the push-down functor, is an interpretation functor.*

Proof. By definition $\mathcal{D}$ is closed under direct limits and products of $A\text{-}\mathrm{Mod}$. Since F_λ is a left adjoint, it commutes with direct limits, and since $\mathcal{D}$ is assumed to be finitely-supported, the restriction $F_\lambda|_{\mathcal{D}}$ commutes with products by Corollary 4.1.19. Therefore $F_\lambda|_{\mathcal{D}} : \mathcal{D} \to B\text{-}\mathrm{Mod}$ is an interpretation functor. □

4.1.3 Convex subcategories

A subcategory $A' \subseteq A$ is **convex** if A' is closed under factorisation of morphisms in A, i.e. for any morphism $\alpha : x \to z$ in A', if $\alpha = \alpha_2\alpha_1$ for some $\alpha_1 : x \to y$ and $\alpha_2 : y \to z$ in A, then y, α_1, and α_2 belong to A'.

Given a *full* convex subcategory $A' \subseteq A$ define

$$I(x, y) := \begin{cases} A(x, y) & \text{if } x \notin A' \text{ or } y \notin A', \\ 0 & \text{if } x, y \in A', \end{cases}$$

for all $x, y \in A$. Then I is an ideal of A (it is the ideal generated by the set of objects $A \backslash A'$) and the quotient category A/I – defined with the same objects as A and with hom-spaces $(A/I)(x, y) := A(x, y)/I(x, y)$ – is isomorphic to A' (note here we assume A' has a null object 0). The projection $\pi : A \to A'$ is defined by $\pi(a) := a$ if $a \in A'$ and $\pi(a) = 0$ if $a \notin A'$, and has kernel I.

Restriction along $\pi : A \to A'$ gives a full embedding $\mathrm{res}_\pi : A'\text{-}\mathrm{Mod} \to A\text{-}\mathrm{Mod}$ which we call a **zero embedding** (it is given by "extending by zero"). The image of res_π is a full subcategory of $A\text{-}\mathrm{Mod}$ — closed under limits and colimits — which we identiy with $A'\text{-}\mathrm{Mod}$. In particular, we consider $A'\text{-}\mathrm{Mod}$ as a definable subcategory of $A\text{-}\mathrm{Mod}$.

Corollary 4.1.21. *Let $F : A \to B$ be a Galois covering. If $A' \subseteq A$ is a finite full convex subcategory, then the restriction $F_\lambda|_{A'\text{-}\mathrm{Mod}} : A'\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}$, of the push-down functor, is an interpretation functor. Furthermore, if $B = A/G$ and A' intersects each G-orbit of A at most once, then the restriction $F_\lambda|_{A'\text{-}\mathrm{Mod}} : A'\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}$ is full and faithful.*

Proof. Assuming A' is finite (i.e. has finitely many objects) then $A'\text{-}\mathrm{Mod}$ is a finitely-supported definable subcategory of $A\text{-}\mathrm{Mod}$ and Corollary 4.1.20 applies, giving that $F_\lambda|_{A'\text{-}\mathrm{Mod}} : A'\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}$ is an interpretation functor.

Assume B is of the form A/G and A' intersects each G-orbit of A at most once. Then for all $M, N \in A'\text{-}\mathrm{Mod}$ and $g \in G$ with $g \neq 1$ (the identity of G), the assumption implies $\mathrm{Supp}({}^gN)$ lies outside A'. Hence $\mathrm{Supp}(M) \cap \mathrm{Supp}({}^gN) = \emptyset$ and clearly $(M, {}^gN) = 0$. By Proposition 4.1.13 it follows that $F_\lambda|_{A'\text{-}\mathrm{Mod}}$ is fully faithful. $\square$

4.2 Repetitive algebras and k-fold trivial extensions

Given a finite-dimensional algebra R let $\{e_1, \ldots, e_t\}$ be a complete set of local idempotents for R and as usual consider R as the (locally bounded) category with objects $\{1, \ldots, t\}$ and hom spaces $R(i, j) := e_j R e_i$ for all $i, j \in \{1, \ldots, t\}$. The **repetitive category** $\widehat{R}$ of R is the category with objects $\{(i, m) \mid i \in \{1, \ldots, t\}, m \in \mathbb{Z}\}$ and hom spaces

$$\widehat{R}((i,m),(j,n)) := \begin{cases} R(i,j) & \text{if } n = m, \\ R(j,i)^{\star} & \text{if } n = m+1, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i, j \in \{1, \dots, t\}$ and $m, n \in \mathbb{Z}$.

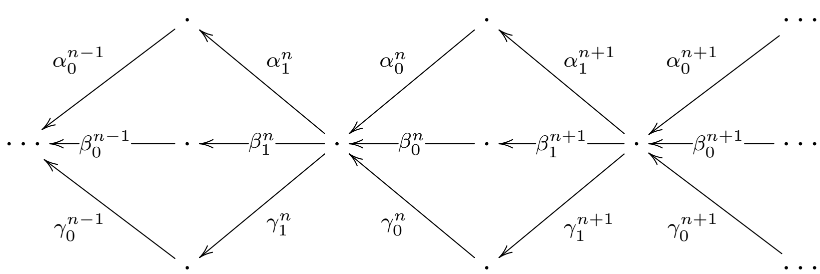
The repetitive category $\widehat{R}$ is a locally bounded category. It corresponds to an infinite-dimensional non-unital algebra of matrices, called the **repetitive algebra**, consisting of matrices of the following form having almost all entries zero (with copies of $R_n := R$ along the main diagonal) [35, 2.2–2.3].

$$\begin{pmatrix} & \ddots & \ddots & & & & & \\ \cdots & 0 & R_{n-1}^{\star} & R_{n-1} & 0 & \cdots & & \\ & \cdots & 0 & R_n^{\star} & R_n & 0 & \cdots & \\ & & \cdots & 0 & R_{n+1}^{\star} & R_{n+1} & 0 & \cdots \\ & & & & & \ddots & \ddots & \end{pmatrix}$$

The repetitive algebra is self-injective meaning, in this context, that the projective and injective $\widehat{R}$-modules coincide [30, 2.2].

The quiver $\mathscr{Q}_{\widehat{R}}$ of $\widehat{R}$ is easily seen to be obtained from $\bigsqcup_{n\in\mathbb{Z}} \mathscr{Q}_R^n$ — a disjoint union of copies of $\mathscr{Q}_R$ — by adding only arrows from $\mathscr{Q}_R^{n+1}$ to $\mathscr{Q}_R^n$ for $n \in \mathbb{Z}$. A general method for constructing the full quiver and relations for a repetitive algebra can be found in [61] and [20, 3.16].

Example 4.2.1. If $R = \mathrm{k}\mathscr{Q}_R$ is the path algebra of the subspace orientated quiver $\mathscr{Q}_R$ of Dynkin type $\mathbb{D}_4$, then $\mathscr{Q}_{\widehat{R}}$ is the following infinitely repeating quiver.



Relations for $\widehat{R}$ are $\alpha_1^n\alpha_0^{n-1} = \beta_1^n\beta_0^{n-1} = \gamma_1^n\gamma_0^{n-1}$, $\alpha_0^n\beta_1^n = \alpha_0^n\gamma_1^n = 0$, $\beta_0^n\alpha_1^n = \beta_0^n\gamma_1^n = 0$,

$\gamma_0^n \alpha_1^n = \gamma_0^n \beta_1^n = 0$ for all $n \in \mathbb{Z}$, and any path of length ≥ 3.

If $\widehat{R}$ is the repetitive category of an algebra R, then there exists an obvious automorphism $\nu : \widehat{R} \to \widehat{R}$ — defined on objects by $(i,n) \mapsto (i, n+1)$ — called the **Nakayama automorphism** of $\widehat{R}$. The infinite cyclic group $\langle \nu \rangle$, generated by ν, acts freely on $\widehat{R}$ with finitely many orbits (in bijection with the vertices of $\mathscr{Q}_R$). The orbit category $\widehat{R}/\langle \nu \rangle$ (stricly speaking, its corresponding algebra) is seen to be isomorphic to the trivial extension $R \ltimes R^\star$ and we have a universal Galois covering $\widehat{R} \to R \ltimes R^\star$ [35, 2.2]. For integers $k \geq 1$ we have an orbit category $\widehat{R}/\langle \nu^k \rangle$—called the ***k*-fold trivial extension of** R—and a factorisation of the universal covering $\widehat{R} \to \widehat{R}/\langle \nu \rangle$ into a chain of intermediate covering functors $\widehat{R} \to \cdots \to \widehat{R}/\langle \nu^{k+1} \rangle \to \widehat{R}/\langle \nu^k \rangle \to \cdots \to \widehat{R}/\langle \nu \rangle$. The following result, due to Wakamatsu, generalises Proposition 2.1.4.

Proposition 4.2.2. *[73] If A and B are tilting-cotilting equivalent algebras, then there exists stable equivalences $\underline{\mathrm{mod}}\text{-}\widehat{A} \simeq \underline{\mathrm{mod}}\text{-}\widehat{B}$ and $\underline{\mathrm{mod}}\text{-}\widehat{A}/\langle \nu^k \rangle \simeq \underline{\mathrm{mod}}\text{-}\widehat{B}/\langle \nu^k \rangle$ for all integers $k \geq 1$.*

4.2.1 ν-reflections

Let R be an algebra with $\widehat{R}$ its repetitive category. Let A be a finite and full convex subcategory of $\widehat{R}$, then the quiver $\mathscr{Q}_A$ of A is a full path-complete subquiver of $\mathscr{Q}_{\widehat{R}}$. If the vertices of $\mathscr{Q}_A$ intersect each ν-orbit of $\mathscr{Q}_R$ exactly once, then we say A is a **complete ν-slice** of $\widehat{R}$.

Given a sink vertex $(i,n) \in \mathscr{Q}_A$ define $\sigma^-_{(i,n)}\mathscr{Q}_A$ to be the full path-complete subquiver of $\mathscr{Q}_{\widehat{R}}$ determined by the vertices of $\mathscr{Q}_A$ except with (i,n) replaced by $(i, n+1)$. We call $\sigma^-_{(i,n)}\mathscr{Q}_A$ the **ν-reflection of $\mathscr{Q}_A$ at** (i,n) and denote by $S^-_{(i,n)}A$ the corresponding full convex subcategory of $\widehat{R}$. It is shown in [35, 2.8] that $S^-_{(i,n)}A$ is isomorphic to the quotient algebra $A[I(i,n)]/\langle e_{(i,n)} \rangle$ where $e_{(i,n)}$ denotes the idempotent of A corresponding to the vertex (i,n). Dually, for each source vertex $(i,n) \in \mathscr{Q}_A$ we have $\sigma^+_{(i,n)}\mathscr{Q}_A$ obtained from $\mathscr{Q}_A$ by replacing (i,n) with $(i, n-1)$ and the corresponding algebra $S^+_{(i,n)}A$. Any algebra B obtained from A by a finite sequence of such reflections is said be a **ν-reflection of** A and we say A and B are **ν-reflection equivalent**.

Proposition 4.2.3 ([35, 2.7, 2.10])**.** *Let R be a finite-dimensional algebra and $\widehat{R}$ its repetitive algebra. For an algebra A the following are equivalent:*

(i) A is ν-reflection equivalent to R.

(ii) $\widehat{A} \simeq \widehat{R}$.

(iii) $A \ltimes A^{\star} \simeq R \ltimes R^{\star}$.

(iv) A is a complete ν-slice of $\widehat{R}$.

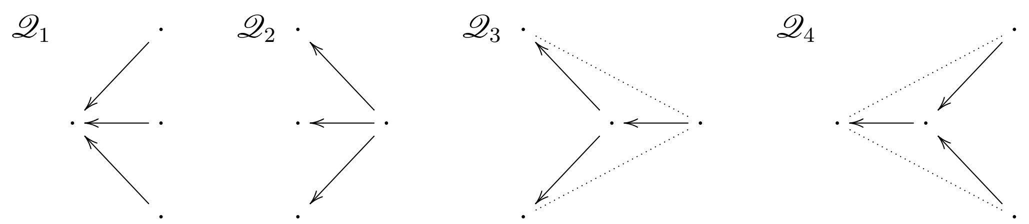
Corollary 4.2.4. *If R is a finite-dimensional algebra, for each ν-reflection A of R there exists a fully faithful interpretation functor* $\mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}R \ltimes R^{\star}$ *and an induced closed embedding* $\mathrm{Zg}_A \to \mathrm{Zg}_{R\ltimes R^{\star}}$.

Proof. Let $F_\lambda : \mathrm{Mod}\text{-}\widehat{R} \to \mathrm{Mod}\text{-}R \ltimes R^{\star}$ be the push-down functor for the Galois covering $\widehat{R} \to R \ltimes R^{\star}$. By Proposition 4.2.3 we know A is a complete ν-slice of $\widehat{R}$. Thus we can consider $\mathrm{Mod}\text{-}A$ a definable subcategory of $\mathrm{Mod}\text{-}\widehat{R}$ and the restriction of F_λ to $\mathrm{Mod}\text{-}A$ is a fully faithful interpretation functor $\mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}R \ltimes R^{\star}$ by Corollary 4.1.21. In fact, is easily seen to coincide with the composition of the zero embedding $\mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}A \ltimes A^{\star}$ and the Morita equivalence $\mathrm{Mod}\text{-}A \ltimes A^{\star} \to \mathrm{Mod}\text{-}R \ltimes R^{\ltimes}$. □

The notion of ν-reflection equivalence is a refinement of tilting-cotilting equivalence.

Proposition 4.2.5. *[72, 4.10] If two finite-dimensional algebras are ν-reflection equivalent, then they are titlting-cotilting equivalent.*

Example 4.2.6. Continuing the previous example, up to isomorphism the ν-reflections of the path algebra $\mathbf{k}\mathscr{Q}_1$ of the subspace orientated quiver of Dynkin type $\mathbb{D}_4$ are given by the following quivers (in each case relations for the algebra include all monomial relations for paths of length 2 indicated by dotted lines).



Note, although the quivers of these algebras are obtained by reflections at sinks, the corresponding tilts given by Proposition 4.2.5 are different from the APR-tilts of Proposition 1.8.4. For instance, the path algebras $\mathbf{k}\mathscr{Q}_1$ and $\mathbf{k}\mathscr{Q}_3$ are tilting-cotilting equivalent but they are not ν-reflection equivalent.

Chapter 5

Repetitive Algebras of Tame Canonical Algebras

In this chapter we investigate the repetitive algebra $\widehat{A}$ of a tame canonical algebra A (defined below). We describe all finitely-supported objects in $\mathrm{Zg}_{\widehat{A}}$ and the closed subsets consisting of such points. This will be sufficient to determine the Ziegler spectrum of self-injective algebras of the form $\widehat{A}/\langle\nu^k\rangle$ for $k \in \mathbb{N}$. In particular, this includes the trivial extension $A \ltimes A^\star \simeq \widehat{A}/\langle\nu\rangle$ of A.

5.1 Canonical algebras

Following Ringel [55, §3.7] we define, for a non-decreasing tuple $\bar{n} := (n_1, \ldots n_t)$ of $t \geq 2$ positive integers, a **canonical algebra** $C := C(\bar{n})$ of **tubular type** $\bar{n}$ to be an algebra given by the quiver $\mathscr{Q}_C$ (5.1.1) and the following relations. For $t = 2$ we have no relations. For $t = 3$ we impose the following relation:

$$(\alpha^1_{n_1}\alpha^1_{n_1-1}\cdots\alpha^1_1) + (\alpha^2_{n_2}\alpha^2_{n_2-1}\cdots\alpha^2_1) + (\alpha^3_{n_3}\alpha^3_{n_3-1}\cdots\alpha^3_1) = 0$$

For $t = 4$ we have a fixed parameter $\lambda \in \mathbf{k}\backslash\{0,1\}$ and the following relations:

$$(\alpha^1_{n_1}\alpha^1_{n_1-1}\cdots\alpha^1_1) + (\alpha^2_{n_2}\alpha^2_{n_2-1}\cdots\alpha^2_1) + (\alpha^3_{n_3}\alpha^3_{n_3-1}\cdots\alpha^3_1) = 0$$
$$(\alpha^1_{n_1}\alpha^1_{n_1-1}\cdots\alpha^1_1) + \lambda(\alpha^2_{n_2}\alpha^2_{n_2-1}\cdots\alpha^2_1) + (\alpha^4_{n_4}\alpha^4_{n_4-1}\cdots\alpha^4_1) = 0$$

Since we won't consider further cases we leave to [55, §3.7] a general description.

$$
\begin{tikzcd}
\mathscr{Q}_C & v_1^1 \arrow[ddl, "\alpha_1^1"'] & v_2^1 \arrow[l, "\alpha_2^1"'] \arrow[r, dotted, no head] & \cdot & v^1_{n_1-1} \arrow[l, "\alpha^1_{n_t-1}"'] & \\
 & v_1^2 \arrow[dl] & v_2^2 \arrow[l] \arrow[r, dotted, no head] & \cdot & v^2_{n_2-1} \arrow[l] & \\
v_0 & & \vdots & \vdots & & v_\omega \arrow[uul, "\alpha^1_{n_1}"'] \arrow[ul] \arrow[dl, "\alpha^t_{n_t}"] \\
 & v_1^t \arrow[ul, "\alpha_1^t"] & v_2^1 \arrow[l, "\alpha_2^t"] \arrow[r, dotted, no head] & \cdot & v^t_{n_t-1} \arrow[l, "\alpha^t_{n_t-1}"] &
\end{tikzcd}
\tag{5.1.1}
$$

A **star** of type $\bar{n} = (n_1, \dots, n_t)$ is the graph obtained from the disjoint union $\bigsqcup_{i=1}^{t} \Delta_i$ of graphs Δ_i of Dynkin type $\mathbb{A}_{n_i}$ respectively (i.e. the linear graph with n_i vertices), by choosing an endpoint of each Δ_i and identifying these vertices to a single vertex called the **centre**. A quiver with underlying graph a star is said to have **subspace orientation** if all arrows point towards the centre, and **factorspace orientation** if all arrows point away from the centre. Stars of type (n), $(2, 2, m)$ $m \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$ are precisely the Dynkin graphs $\mathbb{A}_n$, $\mathbb{D}_{m+2}$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, respectively. Similarly, stars of type $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ are precisely the Euclidean graphs $\widetilde{\mathbb{D}}_4$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, respectively.

Let $C := C(\bar{n})$ be a canonical algebra with quiver $\mathscr{Q}_C$ labelled as in (5.1.1), then observe that $\mathscr{Q}_C \backslash \{v_\omega\}$ is a star of type $\bar{n}$ in subspace orientation, and $\mathscr{Q}_C \backslash \{v_0\}$ is a star of type $\bar{n}$ in factorspace orientation. If $C_0 := \Bbbk(\mathscr{Q}_C \backslash \{v_\omega\})$, then C is a one-point extension of C_0, specifically $C = C_0[M]$ where module $M := \mathrm{rad} P_C(v_\omega)$. Dually, if $C_\infty := \Bbbk(\mathcal{Q}_C \backslash \{v_0\})$, then C is the one-point coextension $C = [N]C_\infty$ where $N := I_C(v_0)/\mathrm{soc}\, I_C(v_0)$.

If $t = 2$, then $C(\bar{n})$ is just a tame hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_{n_1,n_2}$. If $t = 3$ and $\bar{n}$ is one of $(2, 2, m)$ $m \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$, then $C(\bar{n})$ is a tame concealed algebra of tubular type $\bar{n}$ [55, 4.3.5], cf. [64, XII.1]. In these cases a canonical algebra is domestic. If $t = 3$ and $\bar{n}$ is one of $(3, 3, 3)$, $(2, 4, 4)$, or $(2, 3, 6)$, or $t = 4$ and $\bar{n} = (2, 2, 2, 2)$, then $C(\bar{n})$ is a tubular algebra (see Lemma 5.3.5 below). These algebras are tame (of polynomial growth) and the remaining canonical algebras have wild representation type.

5.2 Domestic canonical algebras

Henceforth, until the end of this section, let $A := C(\bar{n})$ be a canonical algebra of tubular type $\bar{n} := (p, q)$ for $1 \leq p \leq q$, $(2, 2, m)$ for $m \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$. Then A is a tame concealed algebra of Euclidean type $\Delta := \widetilde{\mathbb{A}}_{p,q}$, $\widetilde{\mathbb{D}}_m$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, or $\widetilde{\mathbb{E}}_8$, respectively. Let us relabel the quiver $\mathscr{Q}_A$ as follows.

$$\mathscr{Q}_A \qquad \begin{tikzcd}
& a_1 \arrow[dl, "\alpha_1"'] & a_2 \arrow[l, "\alpha_2"'] \arrow[r, dotted, no head] & \cdot & a_{p-1} \arrow[l, "\alpha_{p-1}"'] & \\
v_0 & b_1 \arrow[l, "\beta_1"] & b_2 \arrow[l, "\beta_2"] \arrow[r, dotted, no head] & \cdot & b_{q-1} \arrow[l, "\beta_{q-1}"] & v_\omega \arrow[l, "\beta_q"] \arrow[ul, "\alpha_p"'] \arrow[dl, "\gamma_r"] \\
& c_1 \arrow[ul, "\gamma_1"] & c_2 \arrow[l, "\gamma_2"] \arrow[r, dotted, no head] & \cdot & c_{r-1} \arrow[l, "\gamma_{r-1}"] &
\end{tikzcd} \tag{5.2.1}$$

For ease of notation, we will assume $\bar{n} = (p, q, r)$ but the results here hold without significant difference for $\bar{n} = (p, q)$.

5.2.1 The repetitive algebra

Let $\widehat{A}$ be the repetitive algebra of A, then $\mathscr{Q}_{\widehat{A}}$ is the following infinitely repeating quiver, obtained from $\bigsqcup_{i \in \mathbb{Z}} \mathscr{Q}_A^i$ — a disjoint union of copies of $\mathscr{Q}_A$ — by adding arrows $\epsilon^i, \zeta^i : v_0^{i+1} \to v_\omega^i$ for $i \in \mathbb{Z}$. We identify $\mathscr{Q}_A$ with the subquiver $\mathscr{Q}_A^0$.

$$\mathscr{Q}_{\widehat{A}} \qquad \begin{tikzcd}
& a_1^i \arrow[dl, "\alpha_1^i"'] & a_{p-1}^i \arrow[l, dotted] & & & a_1^{i+1} \arrow[dl, "\alpha_1^{i+1}"'] & a_{p-1}^{i+1} \arrow[l, dotted] & & \\
\cdots v_0^i & b_1^i \arrow[l, "\beta_1^i"] & b_{q-1}^i \arrow[l, dotted] & v_\omega^i \arrow[l, "\beta_q^i"] \arrow[ul, "\alpha_p^i"'] \arrow[dl, "\gamma_r^i"] & v_0^{i+1} \arrow[l, bend right, "\epsilon^{i+1}"'] \arrow[l, bend left, "\zeta^{i+1}"] & b_1^{i+1} \arrow[l, "\beta_1^{i+1}"] & b_{q-1}^{i+1} \arrow[l, dotted] & v_\omega^{i+1} \arrow[l, "\beta_q^{i+1}"] \arrow[ul, "\alpha_p^{i+1}"'] \arrow[dl, "\gamma_r^{i+1}"] & \cdots \arrow[l, bend right, "\epsilon^{i+2}"'] \arrow[l, bend left, "\zeta^{i+2}"] \\
& c_1^i \arrow[ul, "\gamma_1^i"] & c_{r-1}^i \arrow[l, dotted] & & & c_1^{i+1} \arrow[ul, "\gamma_1^{i+1}"] & c_{r-1}^{i+1} \arrow[l, dotted] & &
\end{tikzcd} \tag{5.2.2}$$

Define an α-**path** (or β-**path**, or γ-**path**, resp.) to be path in $\mathscr{Q}_{\widehat{A}}$ consisting of arrows of the form α_j^i (or β_j^i, or γ_j^i, resp.), ϵ^i, and ζ^i. Then a (not necessarily minimal) set of generators for the relations of $\widehat{A}$ is listed as follows:

- $\alpha_p^i \alpha_{p-1}^i \cdots \alpha_1^i + \beta_q^i \beta_{q-1}^i \cdots \beta_1^i + \gamma_r^i \gamma_{r-1}^i \cdots \gamma_1^i$,
- $\alpha_1^i \zeta^i$, $\zeta^i \alpha_p^{i-1}$, and any α-path of length $\geq p + 2$,

- $\beta_1^i\epsilon^i - \beta_1^i\zeta^i$, $\epsilon^{i+1}\beta_q^i - \zeta^{i+1}\beta_q^i$, and any β-path of length $\geq q+2$,
- $\gamma_1^i\epsilon^i$, $\epsilon^i\gamma_r^{i-1}$, and any γ-path of length $\geq r+2$,

for all $i \in \mathbb{Z}$.

Let $\nu_A : \widehat{A} \to \widehat{A}$ be the Nakayama automorphism of $\widehat{A}$ — defined by extending the obvious isomorphisms $\mathscr{Q}_A^i \mapsto \mathscr{Q}_A^{i+1}$. Let $\nu_A : \text{mod-}\widehat{A} \to \text{mod-}\widehat{A}$ also denote the translation (or *shift*) $M \mapsto M\nu_A^{-1}$. Observe, if M is a $\widehat{A}$-module, then the support of $\nu_A M$ (as a set of vertices in $\mathscr{Q}_{\widehat{A}}$) is a right shift, of the support of M, with respect to the quiver (5.2.2).

We begin by describing the category $\text{mod-}\widehat{A}$ of finite-dimensional $\widehat{A}$-modules. A brief overview is given at the beginning of Section 6.1.2 (although, in a more general setting). The structure of the stable module category $\underline{\text{mod}}\text{-}\widehat{A}$ is known from [30] and consists of tubular $\mathbb{P}^1(\mathbf{k})$-families and components of the form $\mathbb{Z}\Delta$ — recall A is a tame concealed algebra of Euclidean type Δ. The following description of $\text{mod-}\widehat{A}$ comes from [2] and elucidated in the subsequent paragraphs.

Proposition 5.2.3. *[2, 4.3] We have* $\text{mod-}\widehat{A} = \bigvee_{i\in\mathbb{Z}} (\mathcal{T}^i \vee \mathcal{W}^i)$ *for module classes* $\mathcal{W}^i$ *and* $\mathcal{T}^i$*, for* $i \in \mathbb{Z}$*, satisfying the following properties:*

(i) $\mathcal{W}^i$ *is a component of the form* $\Gamma^s(\mathcal{W}^i) = \mathbb{Z}\Delta$*,*

(ii) $\mathcal{T}^i$ *is a quasi-stable tubular* $\mathbb{P}^1(\mathbf{k})$*-family of tubular type* (p, q, r)*,*

(iii) $\mathcal{T}^i$ *separates* $\bigvee_{j<i} \mathcal{T}^j \vee \mathcal{W}^j$ *from* $\mathcal{W}^i \vee \bigvee_{j>i} (\mathcal{T}^j \vee \mathcal{W}^j)$*,*

(iv) $\nu_A(\mathcal{W}^i) = \mathcal{W}^{i+2}$ *and* $\nu_A(\mathcal{T}^i) = \mathcal{T}^{i+2}$*.*

Moreover, $\widehat{A}$ *is locally support-finite.*

We wish to describe the finitely-supported objects in $\text{Zg}_{\widehat{A}}$. To this end, we define subquivers of $\mathscr{Q}_{\widehat{A}}$ that will be shown to support these modules. As usual, we may consider $\widehat{A}$ as a locally bounded category with A a full convex subcategory (as determined by the identifcation of $\mathscr{Q}_A$ with $\mathscr{Q}_A^0 \subseteq \mathscr{Q}_{\widehat{A}}$ given above).

Define the following full subquivers of $\mathscr{Q}_{\widehat{A}}$ (i.e. those given by the following sets of vertices): $\mathscr{Q}B_1 := \mathscr{Q}_A^0$ and $\mathscr{Q}B_2 := (\mathscr{Q}_A^0 \cup \mathscr{Q}_A^1)\backslash\{v_0^0, v_\omega^1\}$. Similarly, let $\mathscr{Q}C_1 := \mathscr{Q}B_1^\pm := \mathscr{Q}_A^0$ (where $\pm$ stands for either $+$ or $-$), $\mathscr{Q}C_2 := \{v_\omega^0, v_0^1\}$, $\mathscr{Q}B_2^- :=$

$(\mathscr{Q}_A^0 \backslash \{v_0^0\}) \cup \{v_0^1\}$, and $\mathscr{Q}B_2^+ := (\mathscr{Q}_A^1 \backslash \{v_\omega^1\}) \cup \{v_\omega^0\}$. Now, given $i \in \mathbb{Z}$, define $\mathscr{Q}B_i := \nu_A^k(\mathscr{Q}B_j)$ if $i = j + 2k$ for $j \in \{1, 2\}$ and $k \in \mathbb{Z}$. Likewise, define $\mathscr{Q}C_i$ and $\mathscr{Q}B_i^\pm$ for all $i \in \mathbb{Z}$. Of course, the quivers $\mathscr{Q}B_i$, $\mathscr{Q}C_i$, and $\mathscr{Q}B_i^\pm$ for any $i \equiv 1 \pmod 2$ are all copies of $\mathscr{Q}_A$. The quiver $\mathscr{Q}B_i$ for any $i \equiv 2 \pmod 2$ is depicted in the following diagram.

$$\begin{tikzcd}
a_1^i & a_{p-1}^i \arrow[l, dotted] & & & a_1^{i+1} \arrow[dl] & a_{p-1}^{i+1} \arrow[l, dotted] \\
b_1^i & b_{q-1}^i \arrow[l, dotted] & v_\omega^i \arrow[l] \arrow[ul] \arrow[dl] & v_0^{i+1} \arrow[l, bend right] \arrow[l, bend left] & b_1^{i+1} \arrow[l] & b_{q-1}^{i+1} \arrow[l, dotted] \\
c_1^i & c_{r-1}^i \arrow[l, dotted] & & & c_1^{i+1} \arrow[ul] & c_{r-1}^{i+1} \arrow[l, dotted]
\end{tikzcd}$$

The quivers $\mathscr{Q}B_i^-$ and $\mathscr{Q}B_i^+$ for $i \equiv 2 \pmod 2$, which are subquivers of $\mathscr{Q}B_i$, are depicted, respectively, in the following diagram.

$$\begin{tikzcd}
a_1^i & a_{p-1}^i \arrow[l, dotted] & & & & & & a_1^{i+1} \arrow[dl] & a_{p-1}^{i+1} \arrow[l, dotted] \\
b_1^i & b_{q-1}^i \arrow[l, dotted] & v_\omega^i \arrow[l] \arrow[ul] \arrow[dl] & v_0^{i+1} \arrow[l, bend right] \arrow[l, bend left] & & v_\omega^i & v_0^{i+1} \arrow[l, bend right] \arrow[l, bend left] & b_1^{i+1} \arrow[l] & b_{q-1}^{i+1} \arrow[l, dotted] \\
c_1^i & c_{r-1}^i \arrow[l, dotted] & & & & & & c_1^{i+1} \arrow[ul] & c_{r-1}^{i+1} \arrow[l, dotted]
\end{tikzcd}$$

Let $\mathscr{Q}\xi$ denote any of the quivers defined above (i.e. ξ stands for one of B_i, C_i, $B_i^\pm$ and $i \in \mathbb{Z}$). Denote by ξ the (finite-dimensional) algebra defined by the quiver $\mathscr{Q}\xi$ and all induced relations from $\widehat{A}$ (we write $\mathscr{Q}\xi$ instead of $\mathscr{Q}_\xi$ for readability). As $\mathscr{Q}\xi$ is a full path-complete subquiver of $\mathscr{Q}_{\widehat{A}}$, we may also regard ξ as a full (finite) convex subcategory of $\widehat{A}$. There is then a corresponding zero embedding $\operatorname{Mod}\text{-}\xi \to \operatorname{Mod}\text{-}\widehat{A}$ by which we regard $\operatorname{Mod}\text{-}\xi$ as a definable subcategory of $\operatorname{Mod}\text{-}\widehat{A}$ and consequently $\operatorname{Zg}\text{-}\xi$ $(= \operatorname{Zg}_\xi)$ as a closed subset of $\operatorname{Zg}_{\widehat{A}}$ — see Section 4.1.3.

The seemingly redundant definitions $B_i = C_i = B_i^\pm$ for $i \equiv 1 \pmod 2$ make easier the statement of the following result (but also reveals a more general result which we give in Section 6.1.2).

Proposition 5.2.4. *For all $i \in \mathbb{Z}$, we have*

$$\text{mod-}B_i = \mathcal{P}_i \vee \mathcal{T}_i \vee \mathcal{Q}_i$$

where $\mathcal{P}_i$ is a preprojective component, $\mathcal{Q}_i$ is a preinjective component, and $\mathcal{T}_i$ is a separating quasi-stable tubular $\mathbb{P}^1(\mathrm{k})$-family of tubular type $\bar{n}$. Every tube of $\mathcal{T}_i$ satisfies (T4) and every infinite-dimensional point of $\mathrm{Zg}\text{-}B_i$ belongs to the closure of a tube. Moreover, in $\mathrm{Zg}\text{-}B_i$, every Prüfer module is a B_i^+-module, every adic module is a B_i^--module, and there is a unique generic (namely, the generic C_i-module) that is common to the closure of any tube.

Proof. Of course, we only need prove the result for $i \in \{1, 2\}$. For $i = 1$, the result follows immediately from Proposition 3.4.10 (and Proposition 1.8.2), since B_1 is just the (tame concealed) canonical algebra A.

The algebras $B_i, C_i, B_i^{\pm}$ are such that C_i is a tame concealed algebra and both $B_i^{\pm}$ are Euclidean algebras, with B_i^- a tubular branch coextension of C_i and B_i^+ a tubular branch extension of C_i, both of extension type (p, q, r). For $i = 1$, this is a trivial statement.

For $i = 2$, note that C_2 is just the tame Kronecker algebra. Let $(S_\lambda)_{\lambda \in \mathbb{P}^1(\mathrm{k})}$ denote the simple regular C_2-modules as defined in Example 1.8.3. Then B_2^+ is the tubular branch extension of C_2 at ray modules S_0, S_1, and S_∞ using the subspace branches of sizes $p-1$, $q-1$, and $r-1$, respectively. Thus B_2^+ is a Euclidean algebra by Proposition 1.9.1 and we have

$$\text{mod-}B_2^+ := \mathcal{P}_2^+ \vee \mathcal{T}_2^+ \vee \mathcal{Q}_2^+$$

for a preprojective component $\mathcal{P}_2^+$ (coinciding with the preprojective component of mod-C_2), a separating tubular $\mathbb{P}^1(\mathrm{k})$-family $\mathcal{T}_2^+$ (obtained as a tubular enlargement of the regular C_2-modules with extension type (p, q, r)), and a preinjective component $\mathcal{Q}_2^+$. In fact, by Lemma 3.6.4, we have $B_2^+ = C_2[S_0, p-1][S_1, q-1][S_\infty, r-1]$ and, by Proposition 3.5.4, the three non-stable tubes in $\mathcal{T}_2^+$ are ray tubes of the form $\Gamma(0, p-1, 0)$, $\Gamma(0, q-1, 0)$, and $\Gamma(0, r-1, 0)$.

The algebra B_2^- is constructed dually and

$$\text{mod-}B_2^- := \mathcal{P}_2^- \vee \mathcal{T}_2^- \vee \mathcal{Q}_2^-$$

for a preprojective component $\mathcal{P}_2^-$, a separating tubular $\mathbb{P}^1(\mathbf{k})$-family $\mathcal{T}_2^-$ (obtained as a tubular enlargement of the regular C_2-modules with extension type (p, q, r)), and a preinjective component $\mathcal{Q}_2^-$ (coinciding with the preinjective component of mod-C_2),.

Now (in the notation of Section 3.5) we claim that

$$B_2 = C_2[S_0, p-1, p-1][S_1, q-1, q-1][S_\infty, r-1, r-1]$$

and, indeed, we have already seen this algebra in Example 3.5.5. Thus, by Proposition 3.5.4, we know mod-B_2 contains a separating quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-family $\mathcal{T}_2$ of tubular type (p, q, r), obtained as a tubular enlargement of $\mathcal{T}_2^+$ (and dually, as a tubular enlargement of $\mathcal{T}_2^-$).

In fact, it is shown in [2, 4.3] that we have

$$\text{mod-}B_2 = \mathcal{P}_2 \vee \mathcal{T}_2 \vee \mathcal{Q}_2$$

where $\mathcal{P}_2 = \mathcal{P}_2^-$ and $\mathcal{Q}_2 = \mathcal{Q}_2^+$. Now, by Proposition 3.5.4, every tube in $\mathcal{T}_2$ satisfies (T4) and, by Corollary 3.5.7, every Prüfer B_2-module is a Prüfer B_2^+-module, and every adic B_2-module is an adic B_2^--module. If $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{T}_3$, are the three non-stable tubes of $\mathcal{T}_2$, then the closed subset

$$\text{Zg-}B_2^- \cup \text{Zg-}B_2^+ \cup \bigcup_{j=1,2,3} \text{cl}(\mathbb{T}_j) \subseteq \text{Zg-}B_2$$

contains all finite-dimensional B_2-modules (for the stable tubes of $\mathcal{T}_2$ are common to both $\mathcal{T}_2^-$ and $\mathcal{T}_2^+$), and so is the entirety of Zg-B_2 by Proposition 1.3.6. It follows that every infinite-dimensional point of Zg-B_2 lies in the closure of a tube (since the same is true for Zg-$B_2^\pm$). Finally, it is clear, by construction, that the generic module in the closure of any tube is just the generic C_2-module (which is also the generic $B_2^\pm$-module in Zg-$B_2^\pm$). $\square$

Proposition 5.2.5. *Let A be a domestic canonical algebra. Every tube of* mod-$\widehat{A}$

satisfies (T4) and every infinite-dimensional finitely-supported point of $\mathrm{Zg}_{\widehat{A}}$ *belongs to the closure of a tube. Moreover, the only finitely-supported points of* $\mathrm{Zg}_{\widehat{A}}$ *not belonging to a closed subset of the form* $\mathrm{Zg}\text{-}B_i$ *are the projective-injective modules* $P_{\widehat{A}}(v_0^i)$ *and* $P_{\widehat{A}}(v_\omega^i)$ *for* $i \in \mathbb{Z}$*.*

Proof. Write $\text{mod-}\widehat{A} := \bigvee_{i\in\mathbb{Z}} \mathcal{T}_i \vee \mathcal{W}_i$ as in Proposition 5.2.3 and, for all $i \in \mathbb{Z}$, write $\text{mod-}B_i := \mathcal{P}_i \vee \mathcal{T}_i \vee \mathcal{Q}_i$ as in Proposition 5.2.4. There is no conflict of notation here, as it is shown in [2, 4.3] that the tubular family $\mathcal{T}_i$ of $\text{mod-}B_i$ is preserved under the zero embedding $\text{Mod-}B_i \to \text{Mod-}\widehat{A}$, giving its namesake in $\text{mod-}\widehat{A}$. It is also shown in [2, 4.3] that $\mathcal{W}_i := \langle \mathcal{Q}_i \vee \mathcal{P}_{i+1}, \vee\{M_i\}\rangle$ for some projective-injective module $M_i \in \text{mod-}\widehat{A}$ with the direct summands of M_i being precisely the projective-injective indecomposable modules in the quasi-stable component $\mathcal{W}^i$ (in fact, for A canonical, such M_i are of the form $P_{\widehat{A}}(v_0^j)$ or $P_{\widehat{A}}(v_\omega^j)$ for some $j \in \mathbb{Z}$).

Since every tube of $\text{mod-}\widehat{A}$ is seen to be a tube belonging to a subcategory of the form $\text{mod-}B_i$, it follows from Proposition 5.2.4 that every tube of $\text{mod-}\widehat{A}$ satisfies (T4). Indeed, the zero embeddings $\text{Mod-}B_i \to \text{Mod-}\widehat{A}$ preserve and commute with all the necessary properties.

Now if $M \in \mathrm{Zg}_{\widehat{A}}$ is finitely-supported, then there exists a full path-complete subquiver $\mathscr{Q}\xi \subseteq \mathscr{Q}_{\widehat{C}}$ containing $\mathrm{Supp}(M)$ such that M is a module over the algebra ξ given by the quiver $\mathscr{Q}\xi$ and all induced relations. In this way, M belongs to $\mathrm{Zg}\text{-}\xi$. We may assume $\mathscr{Q}\xi = \bigcup_{i=1}^k \mathscr{Q}B^i$ for sufficiently large $k \geq 1$, then

$$\text{mod-}\xi = \mathcal{P}_1 \vee \left(\bigvee_{i=-k}^{k-1} \mathcal{T}_i \vee \mathcal{W}_i\right) \vee \mathcal{T}_k \vee \mathcal{Q}_k$$

and thus

$$\mathrm{Zg}\text{-}\xi = \bigcup_{i=-k}^{k} \mathrm{Zg}\text{-}B_k \cup \mathfrak{P}$$

where $\mathfrak{P}$ is the (finite) set of direct summands of the projective-injective modules $M_i \in \mathcal{W}_i$ for $-k \leq i \leq k-1$. Indeed, since the right hand-side of this equation is a closed subset containing all finite-dimensional points, we have equality by Proposition 1.3.6. If $M \notin \mathfrak{P}$, then M belongs to a closed subset of the form $\mathrm{Zg}\text{-}B_i$ and furthermore, if M is infinite-dimensional, then M belongs to the closure of a tube in $\mathrm{Zg}\text{-}B_i$ by Proposition 5.2.4. □

5.2.2 The k-fold trivial extensions

Let $A := C(p, q, r)$ be a domestic canonical algebra as above and let $T := A \ltimes A^\star$ be the trivial extension of A. The quiver $\mathscr{Q}_T$ of T is obtained from $\mathscr{Q}_A$ by adding two arrows $\epsilon, \zeta : v_0 \to v_\omega$, depicted below. A projection $\varphi : \mathbf{k}\mathscr{Q}_T \to T$ is then chosen satisfying $\varphi(\epsilon) = (\alpha_p\alpha_{p-1}\cdots\alpha_1)^\star$ and $\varphi(\zeta) = (\gamma_p\gamma_{p-1}\cdots\gamma_1)^\star$.

$$\mathscr{Q}_T \qquad \begin{tikzcd} & a_1 \arrow[dl, "\alpha_1"'] & a_2 \arrow[l, "\alpha_2"'] & \cdot \arrow[l, dotted, no head] & a_{p-1} \arrow[l, "\alpha_{p-1}"'] & \\ v_0 \arrow[rrrrr, bend left, "\epsilon"] \arrow[rrrrr, bend right, "\zeta"'] & b_1 \arrow[l, "\beta_1"'] & b_2 \arrow[l, "\beta_2"'] & \cdot \arrow[l, dotted, no head] & b_{q-1} \arrow[l, "\beta_{q-1}"'] & v_\omega \arrow[l, "\beta_q"'] \arrow[ul, "\alpha_p"'] \arrow[dl, "\gamma_r"] \\ & c_1 \arrow[ul, "\gamma_1"] & c_2 \arrow[l, "\gamma_2"] & \cdot \arrow[l, dotted, no head] & c_{r-1} \arrow[l, "\gamma_{r-1}"] & \end{tikzcd} \tag{5.2.6}$$

Define an α-**path** (or β-**path**, or γ-**path**, resp.) to be path in $\mathscr{Q}_T$ consisting of arrows of the from α_j (or β_j, or γ_j, resp.), ϵ, and ζ. Then a (not necessarily minimal) set of generators for the relations of T are as follows:

- $\alpha_p\alpha_{p-1}\cdots\alpha_1 + \beta_q\beta_{q-1}\cdots\beta_1 + \gamma_r\gamma_{r-1}\cdots\gamma_1$,
- $\zeta\alpha_p$, $\alpha_1\zeta$, and any α-path of length $\geq p+2$,
- $\beta_1\epsilon - \beta_1\zeta$, $\epsilon\beta_q - \zeta\beta_q$, and any β-path of length $\geq q+2$,
- $\epsilon\gamma_r$, $\gamma_1\epsilon$, and any γ-path of length $\geq r+2$.

We can now describe the Ziegler spectrum Zg_T. In fact, we give a more general result that will be needed in the next chapter, recall $T \simeq \widehat{A}/\langle\nu_A\rangle$ where $\nu_A : \widehat{A} \to \widehat{A}$ is the Nakayama automorphism of $\widehat{A}$.

For $k \in \mathbb{N}$, let $T^k := \widehat{A}/\langle\nu^k\rangle$ be the k-fold trivial extension of A. The structure of mod-T^k follows from Proposition 5.2.3 and it is known that mod-T^k consists of $2k$ quasi-stable components, each having stable form $\mathbb{Z}\Delta$ (where Δ is the Euclidean type of A), and $2k$ quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-families, each having tubular type $\bar{n} = (p, q, r)$ — this is outlined in the proof of the following result.

Theorem 5.2.7. *Let $A := C(\bar{n})$ be a domestic canonical algebra (of tubular type $\bar{n}$ and Euclidean type Δ) and let T^k be its k-fold trivial extension, for some $k \in \mathbb{N}$, then the following facts hold:*

(1) All (quasi-stable) tubes of $\operatorname{mod}\text{-}T^k$ *satisfy (T4). If two tubes belong to the same tubular family, then there is a unique generic module common to the closure of both tubes.*

(2) Every infinite-dimensional point of $\operatorname{Zg}\text{-}T^k$ *lies in the closure of some tube, i.e.* $\operatorname{Zg}\text{-}T^k$ *consists of the finite-dimensional points of* $\operatorname{ind}\text{-}T$*, the Prüfer/adic modules of each tube, and* $2k$ *generic modules.*

(3) The Ziegler spectrum $\operatorname{Zg}\text{-}T^k$ *satisfies the isolation condition and* $\operatorname{KG}(T^k) = 2$.

Proof. We use the notation and analysis of $\widehat{A}$ and $\operatorname{mod}\text{-}\widehat{A}$ given in Section 5.2.1 above. Recall, from Section 4.2, we have a Galois covering $F : \widehat{A} \to T^k$ and corresponding push-down functor $F_\lambda : \operatorname{Mod}\text{-}\widehat{A} \to \operatorname{Mod}\text{-}T^k$.

By Proposition 5.2.3, $\widehat{A}$ is locally support-finite and so, by Proposition 4.1.11, F_λ restricts to an essentially surjective functor $\operatorname{mod}\text{-}\widehat{A} \to \operatorname{mod}\text{-}T^k$ which, by Proposition 4.1.10, preserves AR sequences. The structure of $\operatorname{mod}\text{-}T^k$ now follows from the struture of $\operatorname{mod}\text{-}\widehat{A}$. Writing $\operatorname{mod}\text{-}\widehat{A} = \bigvee_{i\in\mathbb{Z}} \mathcal{T}^i \vee \mathcal{W}^i$ as in Proposition 5.2.3, then the equalities $\nu_A(\mathcal{W}^i) = \mathcal{W}^{i+2}$, $\nu_A(\mathcal{T}^i) = \mathcal{T}^{i+2}$ and the fact that F_λ is ν_A^k-invariant, implies $F_\lambda(\mathcal{W}^i) = F_\lambda(\mathcal{W}^{i+2k})$ and $F_\lambda(\mathcal{T}^i) = F_\lambda(\mathcal{T}^{i+2k})$ for all $i \in \mathbb{Z}$. Hence, we have

$$\operatorname{mod}\text{-}T^k = \bigvee_{i=1}^{2k} \mathcal{X}^i \vee \mathcal{Y}^i$$

where $\mathcal{X}^i = F_\lambda(\mathcal{W}^i)$ is a component of the form $\Gamma^s(\mathcal{X}^i) = \mathbb{Z}\Delta$ and $\mathcal{Y}^i = F_\lambda(\mathcal{T}^i)$ is a (quasi-stable) tubular $\mathbb{P}^1(\mathbf{k})$-family of tubular type $\bar{n}$. Note, incidentally, any quasi-stable tube in $\operatorname{mod}\text{-}\widehat{A}$, hence in $\operatorname{mod}\text{-}T^k$, is in fact a genuine tube (it is either stable or is a quasi-stable tube of the type constructed in Section 3.5).

We know F_λ preserves indecomposability for finitely-supported pure-injectives. Indeed, given finitely-supported $M \in \operatorname{Zg}_{\widehat{A}}$, if M is finite-dimensional, then $F_\lambda(M)$ is indecomposable by Proposition 4.1.10. Otherwise, M is infinite-dimensional and, by Proposition 5.2.5, M belongs to a closed subset of the form $\operatorname{Zg}\text{-}B_i$ for some $i \in \mathbb{Z}$. However, in that case, by Proposition 5.2.4, M actually belongs to a closed subset of the form $\operatorname{Zg}\text{-}B_i^\pm$. Note $B_i^\pm$ is actually a ν_A-reflection of A (indeed, it is clearly a complete ν_A-slice, so Proposition 4.2.3 applies). Hence, by Corollary 4.1.21, F_λ restricts to a fully faithful interpretation functor $F_\lambda|_{\operatorname{Mod}\text{-}B_i^\pm} : \operatorname{Mod}\text{-}B_i^\pm \to \operatorname{Mod}\text{-}T^k$. In particular, $F_\lambda(M)$ is indecomposable. Note also F_λ maps the generic B_i-module to a generic T^k-module (since, by Proposition 5.2.4, the generic B_i-module is just the generic $B_i^{\pm}$-module).

Now, for all $i \in \mathbb{Z}$, the restriction $F_\lambda|_{\mathrm{Mod}\text{-}B_i} : \mathrm{Mod}\text{-}B_i \to \mathrm{Mod}\text{-}T^k$ — which is a faithful interpretation functor by Corollary 4.1.21 — induces, by Proposition 1.3.4, a closed and continuous map $\varphi_i : \mathrm{Zg}\text{-}B_i \to \mathrm{Zg}\text{-}T^k$.

Thus, given a tube $\mathbb{V} \in \mathrm{mod}\text{-}T^k$ there exists a tube $\mathbb{U}$ belonging to the subcategory $\mathrm{mod}\text{-}B_i \subseteq \mathrm{mod}\text{-}\widehat{A}$, for some $i \in \mathbb{Z}$, such that $F_\lambda(\mathbb{U}) = \mathbb{V}$. It is easily seen that $\varphi_i : \mathrm{Zg}\text{-}B_i \to \mathrm{Zg}\text{-}T^k$ is one-to-one when restricted to $\mathrm{cl}(\mathbb{U})$ and therefore induces a homeomorphism $\mathrm{cl}(\mathbb{U}) \to \mathrm{cl}(\mathbb{V})$. Since $\mathbb{U}$ satisfies (T4) and F_λ preserves direct limits and, by Corollary 4.1.19, all necessary inverse limits (i.e. those along corays), we conclude $\mathbb{V}$ also satisfies (T4).

To see that every infinite-dimensional point of $\mathrm{Zg}\text{-}T^k$ lies in the closure of a tube, let $\mathfrak{B} := \bigcup_{i=1}^{2k} \mathrm{Zg}\text{-}B_i$ and consider its image

$$\mathfrak{C} := F_\lambda(\mathfrak{B}) = \bigcup_{i=1}^{2k} F_\lambda(\mathrm{Zg}\text{-}B_i) = \bigcup_{i=1}^{2k} \mathrm{im}(\varphi_i)$$

Note $\mathfrak{C}$ is a closed subset of $\mathrm{Zg}\text{-}T^k$ that contains almost all finite-dimensional points of $\mathrm{Zg}\text{-}T^k$, since $\mathfrak{B}$ contains — up to $\nu_{\widehat{A}}^k$-shift — almost all finite-dimensional points of $\mathrm{Zg}_{\widehat{A}}$ by Proposition 5.2.5. Therefore $\mathfrak{C}$ contains all infinite-dimensional points of $\mathrm{Zg}\text{-}T^k$ by Proposition 1.3.6. Since, by Proposition 5.2.4, every infinite-dimensional point of $\mathfrak{B}$ belongs to the closure of a tube, we conclude the same for $\mathfrak{C}$ and $\mathrm{Zg}\text{-}T^k$.

To prove (3), we use the criterion of Proposition 1.3.11, i.e. that any point $M \in \mathrm{Zg}\text{-}T^k$ is isolated in its closure by an M-minimal pp functor. For finite-dimensional and generic modules, this follows from Lemma 1.3.12. If $M := V[\infty]$ is a Prüfer module, then $M = F_\lambda(N)$ for a Prüfer module $N := U[\infty]$ belonging to a closed set of the form $\mathrm{Zg}\text{-}B_i^{\pm}$. It follows from Proposition 3.4.10 and Lemma 3.4.7, that N is isolated in its closure by an N-minimal pp functor. In fact, by the proof of Lemma 3.4.7, this pp functor is defined by the hom-functor $\mathrm{Hom}_{B_i^{\pm}}(U[1], -)$ and — since the restriction $F_\lambda|_{\mathrm{Mod}\text{-}B_i^{\pm}}$ is fully faithful — the same proof shows $\mathrm{Hom}_{T^k}(V[1], -)$ defines an M-minimal pp functor that isolates M in its closure. A similar argument, using Lemma 3.4.8, shows every adic module is also isolated in the required way. We conclude $\mathrm{Zg}\text{-}T^k$ satisfies the isolation condition and, by Proposition 1.3.13, we have $\mathrm{KG}(T^k) = \mathrm{CB}(\mathrm{Zg}\text{-}T^k) = 2$. $\square$

5.3 Tame (non-domestic) canonical algebras

Henceforth, until the end of this section, let $A := C(\bar{n})$ be a canonical algebra of tubular type $\bar{n} = (2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$, and let $\widehat{A}$ be its repetitive algebra. Then A is a tame but non-domestic tubular algebra. For tubular types $\bar{n} = (3, 3, 3)$, $(2, 4, 4)$, or $(2, 3, 6)$, the quivers $\mathscr{Q}_A$ and $\mathscr{Q}_{\widehat{A}}$ are depicted in (5.2.1) and (5.2.2), respectively, and relations for A and $\widehat{A}$ are the same as the domestic case above. In this section, we give diagrams only for tubular type $(2, 2, 2, 2)$ which we use as a running example.

Example 5.3.1. For type $\bar{n} := (2, 2, 2, 2)$ — with parameter $\lambda \in \mathbf{k}\backslash\{0, 1\}$ — the quiver $\mathscr{Q}_A$ is as follows.

$$\mathscr{Q}_A \qquad \begin{tikzcd}
 & & a \arrow[ddll, "\alpha_1"'] & & \\
 & & b \arrow[dll, "\beta_1"] & & \\
v_0 & & & & v_\omega \arrow[uull, "\alpha_2"'] \arrow[ull, "\beta_2"] \arrow[dll, "\gamma_2"] \arrow[ddll, "\delta_2"] \\
 & & c \arrow[ull, "\gamma_1"] & & \\
 & & d \arrow[uull, "\delta_1"] & &
\end{tikzcd}$$

The relations for A are $\alpha_2\alpha_1 + \beta_2\beta_1 + \gamma_2\gamma_1 = 0$ and $\alpha_2\alpha_1 + \lambda\beta_2\beta_1 + \delta_1\delta_2 = 0$ with respect to this quiver.

5.3.1 The repetitive algebra

The following analysis of $\widehat{A}$ and $\text{mod}\text{-}\widehat{A}$ is summarised at the beginning of Section 6.1.3 (although, in a more general setting, where A is an arbitary tubular algebra). However, the main results of that section depend upon the results proven here, i.e. in the special case of A being a canonical tubular algebra. We now proceed analogously to Section 5.2 above.

The structure of the stable module category $\underline{\mathrm{mod}}\text{-}\widehat{A}$ is known from [31] and consists entirely of stable tubes belonging to tubular $\mathbb{P}^1(\mathbf{k})$-families. We use the following description of $\mathrm{mod}\text{-}\widehat{A}$ from [46] (cf. [31]), which is then elucidated in the subsequent paragraphs.

Proposition 5.3.2. *[46, 3] There exist module classes $\mathcal{M}^i$ and $\mathcal{T}^i$ for* $\mathrm{mod}\text{-}\widehat{A}$*, for $i \in \mathbb{Z}$, satisfying the following properties:*

(i) $\mathrm{mod}\text{-}\widehat{A} = \bigvee_{i\in\mathbb{Z}}(\mathcal{M}^i \vee \mathcal{T}^i)$,

(ii) $\mathcal{M}^i = \bigvee_{q\in\mathbb{Q}^+} \mathcal{M}^i_q$ where each $\mathcal{M}^i_q$ is a stable tubular $\mathbb{P}^1(\mathbf{k})$-family.

(iii) $\mathcal{T}^i$ is a quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-family.

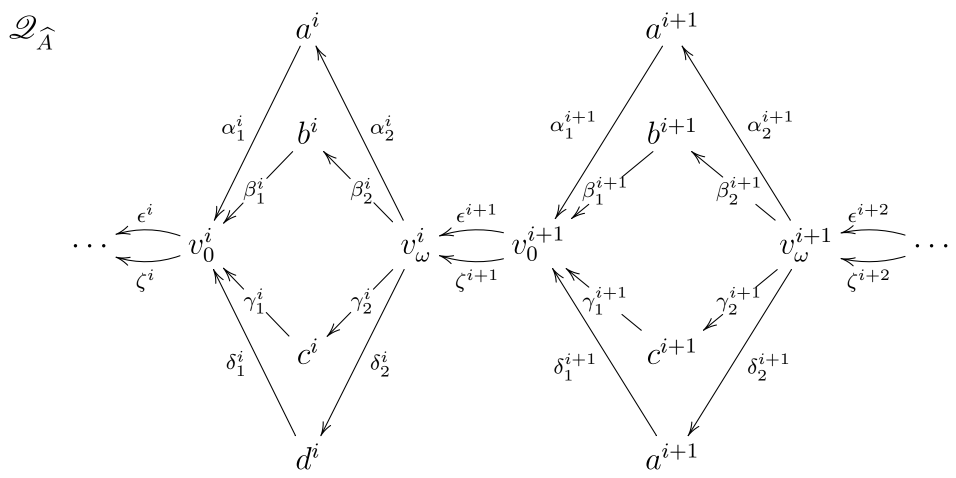
(iv) All tubular families are separating and have tubular type $\bar{n}$.

(v) $\nu(\mathcal{M}^i) = \mathcal{M}^{i+3}$ and $\nu(\mathcal{T}^i) = \mathcal{T}^{i+3}$.

Moreover, $\widehat{A}$ is locally support-finite.

To describe the tubular families of $\mathrm{mod}\text{-}\widehat{A}$, given by Proposition 5.3.2, and to describe the finitely supported points in $\mathrm{Zg}_{\widehat{A}}$, we define various subquivers of $\mathscr{Q}_{\widehat{A}}$ that will be shown to support these modules.

Example 5.3.3. For type $\bar{n} := (2, 2, 2, 2)$, the quiver $\mathscr{Q}_{\widehat{A}}$ is as follows.

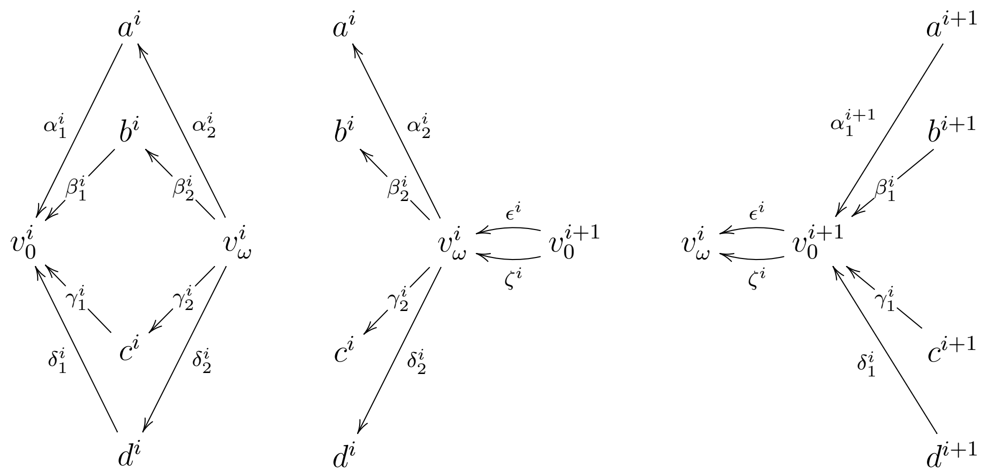


Relations for $\widehat{A}$ are, for all $i \in \mathbb{Z}$, $\alpha^i_2\alpha^i_1 + \beta^i_2\beta^i_1 + \gamma^i_2\gamma^i_1 = 0$, $\alpha^i_2\alpha^i_1 + \lambda\beta^i_2\beta^i_1 + \delta^i_2\delta^i_1 = 0$, $\alpha^i_1\zeta^i = 0$, $\zeta^i\alpha^{i-1}_2 = 0$, $\beta^i_1\epsilon^i = 0$, $\epsilon^i\beta^{i-1}_2 = 0$, $\gamma^i_1\zeta^i = \gamma^i_1\epsilon^i$, $\zeta^i\gamma^{i-1}_2 = \epsilon^i\gamma^{i-1}_2$, $\delta^i_1\zeta^i = \lambda\delta^i_1\epsilon^i$, $\zeta^i\delta^{i-1}_2 = \lambda\epsilon^i\delta^{i-1}_2$, and $\rho = 0$ for any path ρ of length ≥ 4.

In any case, the quiver $\mathscr{Q}_{\widehat{A}}$ is obtained from a disjoint union $\bigsqcup_{i\in\mathbb{Z}} \mathscr{Q}_A^i$ of copies of $\mathscr{Q}_A$, by adding arrows ϵ^i, $\zeta^i : v_0^i \to v_\omega^{i-1}$ for all $i \in \mathbb{Z}$. We identify $\mathscr{Q}_A$ with $\mathscr{Q}_A^0$. Define the following subquivers of $\mathscr{Q}_{\widehat{A}}$ (i.e. those given by the following sets of vertices): $\mathscr{Q}A^1 := \mathscr{Q}_A^0$, $\mathscr{Q}A^2 := (\mathscr{Q}_A^0 \backslash \{v_0^0\}) \cup \{v_0^1\}$, and $\mathscr{Q}A^3 := (\mathscr{Q}_A^1 \backslash \{v_\omega^1\}) \cup \{v_\omega^0\}$. Now, for all $i \in \mathbb{Z}$, define $\mathscr{Q}A^i$ by $\mathscr{Q}A^i = \nu_{\widehat{A}}^k(\mathscr{Q}A^j)$ if $i = j + 3k$ for $j \in \{1, 2, 3\}$ and $k \in \mathbb{Z}$. Of course, the quivers $\mathscr{Q}A^i$ for $i \equiv 1 \pmod 3$ are just the copies of $\mathscr{Q}_A$ in $\mathscr{Q}_{\widehat{A}}$. Additionally, for all $i \in \mathbb{Z}$, define $\mathscr{Q}B^i = \mathscr{Q}A^i \cup \mathscr{Q}A^{i+1}$ (note that $\mathscr{Q}B^i = \mathscr{Q}B^j$ whenever $i \equiv j \pmod 3$).

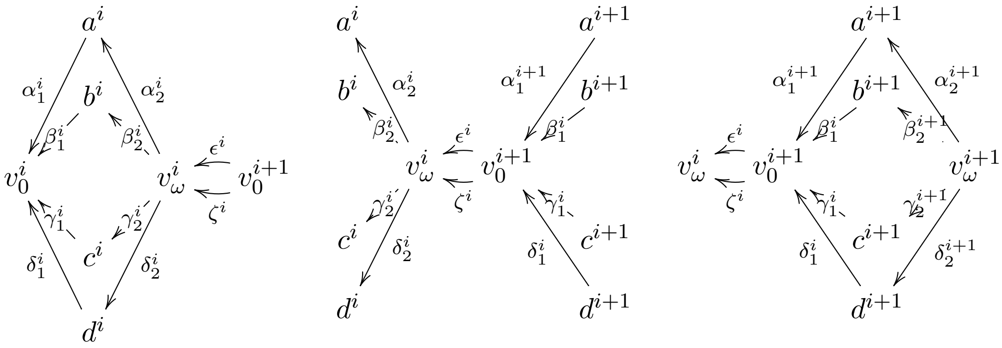
Let $\mathscr{Q}\xi$ denote any of the quivers defined above (i.e. ξ stands for one of A^i or B^i for $i \in \mathbb{Z}$). Denote by ξ the (finite-dimensional) algebra defined by the quiver $\mathscr{Q}\xi$ and all induced relations from $\widehat{A}$ (we write $\mathscr{Q}\xi$ instead of $\mathscr{Q}_\xi$ for readability). As $\mathscr{Q}\xi$ is a full path-complete subquiver of $\mathscr{Q}_{\widehat{A}}$, we may also regard ξ as a full (finite) convex subcategory of $\widehat{A}$. There is then a corresponding zero embedding $\text{Mod-}\xi \to \text{Mod-}\widehat{A}$ by which we regard $\text{Mod-}\xi$ as a definable subcategory of $\text{Mod-}\widehat{A}$ and consequently $\text{Zg-}\xi$ ($= \text{Zg}_\xi$) as a closed subset of $\text{Zg}_{\widehat{A}}$ — see Section 4.1.3.

Example 5.3.4. For type $\bar{n} = (2, 2, 2, 2)$, the quivers $\mathscr{Q}A^i$ for $i \equiv 1, 2, 3 \pmod 3$ are depicted, respectively, in the following diagram.



With respect to these quivers, relations for A^i are as follows. For $i \equiv 1 \pmod 3$, $\alpha_2^i\alpha_1^i + \beta_2^i\beta_1^i + \gamma_2^i\gamma_1^i = 0$ and $\alpha_2^i\alpha_1^i + \lambda\beta_2^i\beta_1^i + \delta_2^i\delta_1^i = 0$. For $i \equiv 2 \pmod 3$, $\zeta^i\alpha_2^i = 0$, $\epsilon^i\beta_2^i = 0$, $\epsilon^1\beta_2^i = \zeta\beta_2^i$, and $\epsilon^i\delta_2^i = \lambda\epsilon^i\delta_2^i$. For $i \equiv 3 \pmod 3$, $\alpha_1^{i+1}\zeta^i = 0$, $\beta_1^{i+1}\epsilon^i = 0$, $\gamma_1^{i+1}\epsilon^i = \gamma_1^{i+1}\zeta^i$, and $\delta_1^{i+1}\epsilon^i = \lambda\delta_1^{i+1}\zeta^i$.

The quivers $\mathscr{Q}B^i$ for $i \equiv 1, 2, 3 \pmod 3$ are depicted, respectively, in the following diagram.



With respect to these quivers, relations for B^i are just the union of the relations for A^i and A^{i+1} given above.

Lemma 5.3.5. *For all $i \in \mathbb{Z}$, the algebra A^i is a tubular algebra of type $\bar{n}$, and*

$$\text{mod-}A^i = \mathcal{P}^i \vee \mathcal{T}_0^i \vee \mathcal{M}^i \vee \mathcal{T}_\infty^i \vee \mathcal{Q}^i \tag{5.3.6}$$

as described in Proposition 1.10.1. Furthermore, if A_0^i, resp. A_∞^i, denotes the tame concealed algebra of which A^i is a tubular extension, resp. coextension, for each $i \in \mathbb{Z}$. Then $A_0^i = A_\infty^{i-1}$.

Proof. These facts appear within [55] [56] [31] and can be checked by explicit calculation, we give the following details sufficient for later results. Note once A^i is proven to be a tubular algebra, then (5.3.6) is given by Proposition 1.10.1.

For $i \equiv 1 \pmod 3$, the algebra A^i is just the canonical algebra A. In this case, $\mathscr{Q}A_0^i := \mathscr{Q}A^i \backslash \{v_\omega^i\}$ is a star of type $\bar{n}$ in subspace orientation, and $\mathscr{Q}A_\infty^i := \mathscr{Q}A^i \backslash \{v_0^i\}$ is a star of type $\bar{n}$ in factorspace orientation (recall the opening paragraphs of this chapter). We know $A^i = A_0^i[M]$ where $M := \operatorname{rad} P_{A^i}(v_\omega^i)$ (by general theory of one-point extensions); this module is depicted in the following diagram (see also Example 2.1.3).

$$\bar{n} = (2,2,2,2) \qquad \begin{tikzcd}[row sep=small]
 & \mathbf{k} \arrow[ddl, "(1\;0)"'] \\
 & \mathbf{k} \arrow[dl, "(0\;1)"] \\
\mathbf{k}^2 & \\
 & \mathbf{k} \arrow[ul, "(1\;1)"'] \\
 & \mathbf{k} \arrow[uul, "(1\;\lambda)"]
\end{tikzcd} \qquad \bar{n} = (3,3,3) \qquad \begin{tikzcd}
 & \mathbf{k} \arrow[dl, "(1\;0)"'] & \mathbf{k} \arrow[l, "1"'] \\
\mathbf{k}^2 & \mathbf{k} \arrow[l, "(0\;1)"'] & \mathbf{k} \arrow[l, "1"'] \\
 & \mathbf{k} \arrow[ul, "(1\;1)"] & \mathbf{k} \arrow[l, "1"]
\end{tikzcd}$$

the cases $\bar{n} = (2,4,4)$ and $(2,3,6)$ are similar. Recall, the subspace oriented star of type $\bar{n} = (2,2,2,2), (3,3,3), (2,4,4)$, or $(2,3,6)$, defines a tame hereditary algebra of Euclidean type $\widehat{\mathbb{D}}_2$, $\widehat{\mathbb{E}}_6$, $\widehat{\mathbb{E}}_7$, or $\widehat{\mathbb{E}}_8$ resp., and—by Proposition 1.8.1—this algebra has tubular type $(2,2,2)$, $(2,3,3)$, $(2,3,4)$, or $(2,3,5)$ resp. From [64, XIII.2] we find M to be a simple regular module in a tube of rank 1, 2, 3, or 3, resp. Thus $A^i = A^i_0[M]$ is a one-point tubular extension of A^i_0 of extension type $\bar{n}$ and, by definition, A^i is indeed a tubular algebra of type $\bar{n}$. Dually, $A^i := [N]A^i_\infty$ where $N := I_{A^i}(v^i_0)/\operatorname{soc} I_{A^i}(v^i_0)$ is a simple-regular A^i_∞-module in a tube of the same rank.

For $i \equiv 2 \pmod 3$, note $H := \mathbf{k}(v^i_\omega \underset{\zeta^i}{\overset{\epsilon^i}{\leftleftarrows}} v^{i+1}_0)$ is the tame Kronecker algebra. Let $(S_\lambda)_{\lambda\in\mathbb{P}^1(\mathbf{k})}$ denote the simple regular H-modules (as defined in Example 1.8.3), then we see A^i is the tubular branch coextension of H at the coray modules S_0, S_1, and S_∞, using factorspace branches of sizes $p-1$, $q-1$, $r-1$, respectively, in case $\bar{n} = (p,q,r)$; or at the coray modules S_0, S_∞, S_1, and S_λ (using branches of size 1), in case $\bar{n} = (2,2,2,2)$ with parameter $\lambda \in \mathbf{k}\backslash\{0,1\}$. In any case, this shows A^i is a tubular algebra of type $\bar{n}$ with $A^i_\infty = H$. In this case, $\mathscr{Q}A^i_0 := \mathscr{Q}A^i\backslash\{v^{i+1}_0\}$ is a star of type $\bar{n}$ in factorspace orientation, thus $A^i_0 = A^{i-1}_\infty$.

For $i \equiv 3 \pmod 3$, this is dual to the previous case. The algebra A^i is a tubular branch extension of the Kronecker algebra H using the same ray (= coray) modules and branches of the same size (but now in subspace orientation). In fact, by Lemma 3.6.5, we have $A^i = H[S_0, p-1][S_1, q-1][S_\infty, r-1]$ for $\bar{n} = (p,q,r)$, or $A^i = H[S_0,1][S_\infty,1][S_1,1][S_\lambda,1]$ for $\bar{n} = (2,2,2,2)$, using the notation introduced in Section 3.5. Thus A^i is a tubular algebra of type $\bar{n}$, with $A^i_0 = H$ and so $A^i_0 = A^{i-1}_\infty$.

In this case, $\mathscr{Q}A^i_\infty := \mathscr{Q}A^i\backslash\{v^i_\omega\}$ is a star of type $\bar{n}$ in subspace orientation, thus $A^i_\infty = A^{i+1}_0$.

This completes the proof. We note the algebras A^i for $i \equiv 2, 3 \pmod 3$ are the so-called "squid algebras" of [56] (see also [53, p. 8]). $\square$

Observe the algebras A^i, for $i \in \mathbb{Z}$, are all ν_A-reflections of A (indeed, they are easily seen to be complete ν_A-slices and Proposition 4.2.3 applies). Conversely, all tubular ν_A-reflections of A are of this form.

Proposition 5.3.7. *For each $i \in \mathbb{Z}$, we have*

$$\operatorname{mod}\text{-}B^i = \mathcal{P}^i \vee \mathcal{T}^i_0 \vee \mathcal{M}^i \vee \mathcal{T}^i \vee \mathcal{M}^{i+1} \vee \mathcal{T}^{i+1}_\infty \vee \mathcal{Q}^{i+1} \tag{5.3.8}$$

where $\mathcal{T}^i$ is a separating quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-family. Furthermore, if $(\mathbb{T}_{\lambda_j})^s_{j=1}$ denote all (finitely many) non-stable tubes in $\mathcal{T}^i$, then

$$\operatorname{Zg}\text{-}B^i = \operatorname{Zg}\text{-}A^i \cup \operatorname{Zg}\text{-}A^{i+1} \cup \bigcup_{i=1}^{s} \operatorname{cl}(\mathbb{T}_{\lambda_1}) \tag{5.3.9}$$

Every quasi-stable tube of $\operatorname{mod}\text{-}B^i$ *satisfies (T4). Moreover, every infinite-dimensional point of* $\operatorname{Zg}\text{-}B^i$ *is an A^i or an A^{i+1}-module, and either lies in the closure of a tube, or is a module of irrational slope.*

Proof. The equation (5.3.8) is proven in [46, §3] (notation differs), we give some details below. Note, for all $i \in \mathbb{Z}$, the algebra B^i can be obtained from A^i, resp. A^{i+1}, by consecutive one-point extensions, resp. coextensions (this is evident from their quivers). In fact, all such (co)extensions are actually one-point tubular (co)extensions.

For $i \equiv 1 \pmod 3$, we see $B^i = [P_{A^{i+1}}(v^{i+1}_0)]A^{i+1}$. In this case, A^{i+1} is a one-point tubular extension of A^{i+1}_0 — specifically, $A^{i+1} = A^{i+1}_0[M]$ where $M := \operatorname{rad} P_{A^{i+1}}(v^{i+1}_0)$ is a simple regular module over A^{i+1}_0. Thus $B^i = [P_{A^{i+1}}(v^{i+1}_0)](A^{i+1}_0[M]) = A^{i+1}_0[M, 1]$ and the quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-family $\mathcal{T}^i$ is constructed, by Proposition 3.5.4, as a tubular enlargement of the regular A^{i+1}_0-modules (and, at an intermediate stage, a tubular enlargement of $\mathcal{T}^{i+1}_0$). Since M belongs to $\mathcal{T}^{i+1}_0$ in $\operatorname{mod}\text{-}A^{i+1}$ (i.e. has slope 0 over A^{i+1}), it follows, from Proposition 3.1.4, that all components belonging to

$\mathcal{M}^{i+1} \vee \mathcal{T}_\infty^{i+1} \vee \mathcal{Q}^{i+1} \subseteq \text{mod-}A^{i+1}$ are preserved in mod-B^i, giving the right-most classes of (5.3.8).

Similarly, $B^i = A^i[I_{A^i}(v_0^i)] = ([N]A_\infty^i)[I_{A^i}(v_0^i)]$ where $N := I_{A^i}(v_0^i)/\operatorname{soc} I_{A^i}(v_0^i)$. However, note $N = M$ as a $A_\infty^i = A_0^{i+1}$-module and N belongs to $\mathcal{T}_\infty^i$ in mod-A^i (i.e. has slope ∞ over A^i), it follows, from Proposition 3.1.3, that all components belonging to $\mathcal{P}^i \vee \mathcal{T}_0^i \vee \mathcal{M}^i \subseteq \text{mod-}A^i$ are preserved in mod-B^i, giving the left-most classes of (5.3.8).

Finally, all preinjective A^i-modules have positive rational slope over A^{i+1}, so the preinjective component $\mathcal{Q}^i \subseteq \text{mod-}A^i$ is scattered over the module class $\mathcal{M}^{i+1}$ in mod-B^i. Similarly, all preprojective A^{i+1}-modules have positive rational slope over A^i and belong to $\mathcal{M}^i$ in mod-B^i.

From this discussion, it follows that the right hand-side of (5.3.9) is a closed subset of Zg-B^i that contains all finite-dimensional points. Indeed, any finite-dimensional point belonging to $\mathcal{P}^i \vee \mathcal{T}_0^i \vee \mathcal{M}^i$ lies in Zg-A^i and any finite-dimensional point belonging to $\mathcal{M}^{i+1} \vee \mathcal{T}_\infty^{i+1} \vee \mathcal{Q}^{i+1}$ lies in Zg-A^{i+1}. Finally, any finite-dimensional point belonging to a stable tube of $\mathcal{T}^i$ is a regular $A_\infty^i = A_0^{i+1}$-module and lies in both Zg-A^i and Zg-A^{i+1}. Thus, we have equality in (5.3.9) by Proposition 1.3.6. The last claim, that every quasi-stable tube satisfies (T4), is given by Corollary 3.6.7 and Proposition 3.5.4. Finally, by Corollary 3.5.7, every infinite-dimensional point in $\operatorname{cl}(\mathbb{T}_{\lambda_1})$ already belongs in Zg-A^i or Zg-A^{i+1} (all Prüfer modules are A^i-modules, all adic modules are A^{i+1}-modules, the generic module is an $A_\infty^i = A_0^{i+1}$-module).

For $i \equiv 2 \pmod 3$, we have

$$B^i = H[S_0, p-1, p-1][S_1, q-1, q-1][S_\infty, r-1, r-1]$$

when $\bar{n} = (p, q, r)$, and

$$B^i = H[S_0, 1, 1][S_\infty, 1, 1][S_1, 1, 1][S_\lambda, 1, 1]$$

when $\bar{n} = (2, 2, 2, 2)$. The quasi-stable tubular family $\mathcal{T}^i$ is again constructed by (iteration of) Proposition 3.5.4 and contains either three non-stable tubes (containing, respectively, $p-1$, $q-1$, and $r-1$, projective-injective vertices) or contains four non-stable tubes (each containing 1 projective-injective vertex). Otherwise, the result is proved just as in the previous case.

For $i \equiv 3 \pmod 3$, this case is entirely dual to the first. $\square$

Proposition 5.3.10. *Let A be a canonical tubular algebra of type $\bar{n}$, then* $\operatorname{mod}\text{-}\widehat{A}$ *consists entirely of quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-families of tubular type $\bar{n}$. Every tube of* $\operatorname{mod}\text{-}\widehat{A}$ *satisfies (T4). Furthermore, every finitely-supported point of* $\mathrm{Zg}_{\widehat{A}}$ *either belongs to the closure of a tube or is an module of irrational slope over the tubular algebra A^i for some $i \in \mathbb{Z}$.*

Proof. By Proposition 5.3.2 we have $\operatorname{mod}\text{-}\widehat{A} = \bigvee_{i\in\mathbb{Z}} \mathcal{M}^i \vee \mathcal{T}^i$ where $\mathcal{M}^i$ is the $\mathbb{Q}^+$-indexed collection of stable tubular $\mathbb{P}^1(\mathbf{k})$-families, consisting of modules of rational slope over A^i, and $\mathcal{T}^i$ is the quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-family, belonging to $\operatorname{mod}\text{-}B^i$. It follows immediately from Proposition 5.3.7 that every (quasi-stable) tube of $\operatorname{mod}\text{-}\widehat{A}$ satisifes (T4). The final statement is proved just as the analogous statement is proved in Proposition 5.2.5. $\square$

5.3.2 The k-fold trivial extension

Let $A := C(\bar{n})$ be a canonical tubular algebra (of tubular type $\bar{n}$) and let $T^k := \widehat{A}/\langle \nu_A^k \rangle$ be its k-fold trivial extension, for some $k \in \mathbb{N}$. The structure of $\operatorname{mod}\text{-}T^k$ follows from Proposition 5.3.2 and it is known that $\operatorname{mod}\text{-}T^k$ consists entirely of quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-families, each having tubular type $\bar{n}$. We prove the following result, describing the Ziegler spectrum Zg_{T^k} of T^k.

Theorem 5.3.11. *Let $A := C(\bar{n})$ be a tubular canonical algebra (of tubular type $\bar{n}$) and let $\widehat{A}$ be its repetitive algebra, with Nakayama automorphism $\nu_A : \widehat{A} \to \widehat{A}$. For $k \in \mathbb{N}$, let $T^k := \widehat{A}/\langle \nu_A^k \rangle$ be the k-fold trivial extension of A, then the following facts hold:*

(1) Every quasi-stable tube of $\operatorname{mod}\text{-}T^k$ *satisfies (T4). If two tubes belong to the same tubular family, then there exists a unique generic module common to the closure of both tubes.*

(2) Every infinite-dimensional point of $\mathrm{Zg}\text{-}T^k$ *lies in the closure of a quasi-stable tube or is a module of irrational slope over a tubular ν_A-reflection of A.*

Proof. The proof is entirely analogous to the proof of Theorem 5.3.11. Briefly, we have a Galois covering $F : \widehat{A} \to T^k$ and its push-down functor $F_\lambda : \mathrm{Mod}\text{-}\widehat{A} \to \mathrm{Mod}\text{-}T^k$. As $\widehat{A}$ is locally support-finite, by Proposition 5.2.3, we deduce, by Propositions 4.1.10–4.1.11, the structure of $\mathrm{mod}\text{-}T^k$ as stated in (1) from the structure of $\mathrm{mod}\text{-}\widehat{A}$ described in Propositions 5.2.3 and 5.3.10. Then, for all $i \in \mathbb{Z}$, the push-down functor restricts to interpretation functors $F_\lambda|_{\mathrm{Mod}\text{-}B^i} : \mathrm{Mod}\text{-}B^i \to \mathrm{Mod}\text{-}T^k$ and $F_\lambda|_{\mathrm{Mod}\text{-}A^i} : \mathrm{Mod}\text{-}A^i \to \mathrm{Mod}\text{-}T^k$ (the latter being fully faithful), that induce closed and continuous maps $\mathrm{Zg}\text{-}B^i \to \mathrm{Zg}\text{-}T^k$ and $\mathrm{Zg}\text{-}A^i \to \mathrm{Zg}\text{-}T^K$ (the later being an embedding). Then (2) and (3) follow similarly to the corresponding claims of Theorem 5.3.11, using the properties of $\mathrm{Zg}\text{-}B^i$ from Proposition 5.3.7.

□

Chapter 6

Self-injective Algebras of Polynomial Growth

A locally bounded algebra A is called **simply connected** if it is **triangular** (i.e. its quiver contains no oriented cycles) and for any presentation $A \simeq \mathrm{k}\mathscr{Q}/I$ as a bound quiver algebra, the fundamental group $\Pi(\mathscr{Q}, I)$ is trivial — see [25] or [44] for the definition of this group (we don't require this knowledge here). Equivalently, a locally bounded algebra is simply connected if and only if it is triangular and admits no proper Galois covering [67, 4.2].

A self-injective algebra R is **standard** if it admits a Galois covering $F : A \to R$ where A is a simply connected and locally bounded algebra. The classification of the (representation-infinite) standard self-injective algebras of polynomial growth, in terms of such Galois coverings, is completed in [66]. Using these results, quoted below, we describe the Ziegler spectra of such algebras.

A self-injective algebra of **Euclidean** (resp. **Ringel**, resp. **canonical**) **type** is an algebra of the form $\widehat{A}/G$ where A is a Euclidean (resp. tubular, resp. tame canonical) algebra and G is an admissible group of automorphisms of $\widehat{A}$.

Proposition 6.0.1. *[68, 6.1] If R is a finite-dimensional self-injective algebra of representation-infinite polynomial growth, then R is standard if and only if R is isomorphic to a self-injective algebra of Euclidean or Ringel type.*

6.1 Standard self-injective algebras

Let A be a finite-dimensional algebra and let $\widehat{A}$ be its repetitive category (with object set $I \times \mathbb{Z}$). Following [68], an automorphism $\varphi : \widehat{A} \to \widehat{A}$ is said to be **rigid** if, for all $(i, m) \in I \times \mathbb{Z}$, there exists $j \in I$, such that $\varphi(i, m) = (j, m)$; the automorphism φ is said to be **positive** if, for all $(i, m) \in I \times \mathbb{Z}$, there exists $(j, n) \in I \times \mathbb{Z}$ with $n \geq m$ such that $\varphi(i, m) = (j, n)$; and φ is said to be **strictly positive** if positive but not rigid. We will also say φ is **truly strictly positive** if, for all $(i, m) \in I \times \mathbb{Z}$, there exists $(j, n) \in I \times \mathbb{Z}$ with $n > m$ such that $\varphi(i, m) = (j, n)$.

Lemma 6.1.1. *[68, 4.1, 5.2] If $\widehat{A}/G$ is a self-injective algebra of Euclidean or Ringel type, then G is an infinite cyclic group generated by a strictly positive automorphism $\varphi : \widehat{A} \to \widehat{A}$, i.e. $G = \langle \varphi \rangle$.*

Let A be a Euclidean or tubular algebra and let $\nu_A : \widehat{A} \to \widehat{A}$ be the Nakayama automorphism of $\widehat{A}$. Following Skowroński [66] we say A is **exceptional** if there exists a ν_A-reflection A' of A such that $A' \simeq A$ but $A' \neq \nu_A^k(A)$ for any $k \in \mathbb{Z}$.

Lemma 6.1.2. *[66, 2.13, 3.8] Let A be a Euclidean or tubular algebra, then A is exceptional if and only if there exists an automorphism $\phi : \widehat{A} \to \widehat{A}$ such that $\phi^d = \sigma\nu_A$ for some integer $d \geq 2$ (in the Euclidean case, $d = 2$) and a rigid automorphism $\sigma : \widehat{A} \to \widehat{A}$.*

Let A be a Euclidean or tubular algebra, if A is exceptional, define $\phi_A : \widehat{A} \to \widehat{A}$ to be the automorphism, given by Lemma 6.1.2, such that $\phi_A^d = \sigma\nu_A$, for maximal $d \geq 2$. If A is non-exceptional, define $\phi_A := \nu_A$. Note, by this definition, A is exceptional if and only if ϕ_A is strictly positive but not truly strictly positive.

Lemma 6.1.3. *[66, 2.13, 3.9] Let $\widehat{A}/G$ be a self-injective algebra of Euclidean or Ringel type, then $G = \langle \sigma\phi_A^k \rangle$ for some $k \in \mathbb{N}$ and a rigid automorphism $\sigma : \widehat{A} \to \widehat{A}$.*

Lemma 6.1.4. *The canonical algebra of type $\widetilde{\mathbb{A}}_{(1,1)}$ (i.e. the tame Kronecker algebra) is the only exceptional tame canonical algebra.*

Proof. Let A be a tame canonical algebra not of type $\widetilde{\mathbb{A}}_{(1,1)}$. Recall (see opening paragraphs of Section 5.1) that A is a Euclidean algebra when domestic and a tubular algebra otherwise. The (Euclidean or tubular) ν-reflections of A are are given Sections 5.2–5.3. Only those that are ν_A-reflections of A are seen to be isomorphic to A. Hence A is not exceptional. That the canonical algebra of type $\widetilde{\mathbb{A}}_{(1,1)}$ is exceptional is exhibited in Example 6.1.6 below. $\square$

Given a topological space Z denote by $\mathrm{Latt}(Z)$ the lattice of closed subsets of Z.

Theorem 6.1.5. *Let A be a Euclidean or tubular algebra and $R := \widehat{A}/G$ for some admissible group G of automorphisms of $\widehat{A}$. Let $F : \widehat{A} \to R$ be the Galois covering and $F_\lambda : \mathrm{Mod}\text{-}\widehat{A} \to \mathrm{Mod}\text{-}R$ the corresponding push-down functor. Then there exists a finite full convex subcategory $S \subseteq \widehat{A}$ such that F_λ restricts to an interpretation functor $F_\lambda|_{\mathrm{Mod}\text{-}S} : \mathrm{Mod}\text{-}S \to \mathrm{Mod}\text{-}R$ for which the following properties hold:*

(i) F_λ induces a map $\mathrm{Latt}(\mathrm{Zg}_S) \to \mathrm{Latt}(\mathrm{Zg}_R)$ that commutes with finite unions and arbitrary intersections.

(ii) Every $M \in \mathrm{Zg}_R$ is a direct summand of $F(M')$ for some $M' \in \mathrm{Zg}_S$.

Furthermore, if F_λ preserves indecomposability for (infinite-dimensional) pure-injective S-modules, then F_λ induces a closed and continuous surjection $\mathrm{Zg}_S \to \mathrm{Zg}_R$.

Proof. As A is a Euclidean or tubular algebra, we know $\widehat{A}$ is locally support-finite [2] [46] and the push-down functor $\mathrm{mod}\text{-}\widehat{A} \to \mathrm{mod}\text{-}R$ is essentially surjective by Proposition 4.1.11. By Lemma 6.1.3 we know $G = \langle\varphi\rangle$ where $\varphi = \sigma\phi_A^k$ for some integer $k \geq 1$ and a rigid automorphism $\sigma : \widehat{A} \to \widehat{A}$. Similar to the case when A is tame canonical, there exists a finite convex subcategory $S \subseteq \widehat{A}$ such that every module of $\mathrm{ind}\text{-}\widehat{C}$ is a φ-shift of a module of $\mathrm{ind}\text{-}S$. In fact, taking $S := \bigcup_{i=0}^{k-1} \nu^k(A)$ works for this purpose, see [2, 4.3, p. 49] for the Euclidean case and [46, 3, p. 126] for the tubular case. Therefore, by Corollary 4.1.21, the push-down functor F_λ restricts to a faithful interpretation functor $F_\lambda : \mathrm{Mod}\text{-}S \to \mathrm{Mod}\text{-}R$ with the further restriction $F_\lambda : \mathrm{mod}\text{-}S \to \mathrm{mod}\text{-}R$ being essentially surjective—for F_λ is φ-invariant.

Now by [50, 18.2.24] the set

$$\mathfrak{C} := \{M \in \mathrm{Zg}_R \mid M \text{ is a summand of } F_\lambda(M') \text{ for some } M' \in \mathrm{Zg}_S\}$$

is a closed subset of Zg_R. By the above considerations (and the fact by F_λ preserves indecomposability for finite-dimensional modules by Proposition 4.1.10) we know $\mathfrak{C}$ contains all finite-dimensional points of Zg_R.

Therefore $\mathfrak{C} = \mathrm{Zg}_R$ by Proposition 1.3.6.

Statements (i) and (ii) are [50, 18.2.24] and [50, 18.2.25] respectively. Finally, by Proposition 1.3.4, the only obstacle to F_λ inducing a closed and continuous map $\mathrm{Zg}_S \to \mathrm{Zg}_R$ is the preservation of indecomposability for (infinite-dimensional) pure-injective A'-modules. $\square$

6.1.1 Exceptional self-injective algebras

Let $R := \widehat{A}/G$ be a self-injective algebra of Euclidean or Ringel type, then by Lemma 6.1.3 we have $G = \langle \varphi \rangle$ for some (strictly positive) generator φ. We call R **exceptional** if φ is not truly strictly positive. Note if A is not itself exceptional, then φ is always truly strictly positive (it is divisble by ν) and so there are no exceptional self-injective algebras of the form $\widehat{A}/G$.

In the following example we deal with the only exceptional self-injective algebra of tame canonical type (see Lemma 6.1.4).

Example 6.1.6. The canonical algebra $C := C(1,1)$ of type $\widetilde{\mathbb{A}}_{(1,1)}$ is just the tame Kronecker algebra $\mathbb{k}\left(x \leftleftarrows y \right)$. Its repetitive category $\widehat{C}$ is given by the following infinite quiver with relations $\alpha_1^i\beta_0^i = 0$, $\beta_1^i\alpha_0^i = 0$, $\alpha_0^i\beta_1^{i-1} = 0$, $\beta_0^i\alpha_1^{i-1} = 0$, $\alpha_1^i\alpha_0^i = \beta_1^i\beta_0^i$, and $\alpha_0^i\alpha_1^{i-1} = \beta_0^i\beta_1^{i-1}$ for all $i \in \mathbb{Z}$.

$$
\begin{tikzcd}
\mathscr{Q}_{\widehat{C}} \quad \cdots & v_0^{i-1} & v_\omega^{i-1} \arrow[l, shift right, "\alpha_0^{i-1}"'] \arrow[l, shift left, "\beta_0^{i-1}"] & v_0^{i} \arrow[l, shift right, "\alpha_1^{i-1}"'] \arrow[l, shift left, "\beta_1^{i-1}"] & v_\omega^{i} \arrow[l, shift right, "\alpha_0^{i}"'] \arrow[l, shift left, "\beta_0^{i}"] & v_0^{i+1} \arrow[l, shift right, "\alpha_1^{i}"'] \arrow[l, shift left, "\beta_1^{i}"] & v_\omega^{i+1} \arrow[l, shift right, "\alpha_0^{i+1}"'] \arrow[l, shift left, "\beta_0^{i+1}"] & \cdots
\end{tikzcd}
$$

Let us identify $\mathscr{Q}_C$ with the full subquiver of $\mathscr{Q}_{\widehat{C}}$ given by the vertices v_0^0 and v_ω^0. We have a corresponding fully faithful interpretation functor $\mathrm{Mod}\text{-}C \to \mathrm{Mod}\text{-}\widehat{C}$ and induced closed embedding $\mathrm{Zg}_C \to \mathrm{Zg}_{\widehat{C}}$.

The automorphism $\phi_C : \widehat{C} \to \widehat{C}$ defined by $\alpha_0^i \mapsto \alpha_1^i$, $\alpha_1^i \mapsto \alpha_0^{i+1}$, $\beta_0^i \mapsto \beta_1^i$, and $\beta_1^i \mapsto \beta_0^{i+1}$, satisfies $\phi_C^2 = \nu$ (as predicted by Lemma 6.1.2); it is strictly positive but not truly strictly positive, thus C is exceptional. The orbit algebra $A := \widehat{C}/\langle \phi_C \rangle$ is given by the following quiver and the relations $\alpha\beta = 0$, $\beta\alpha = 0$, and $\alpha^2 = \beta^2$; it is (up to isomorphism) the only exceptional self-injective algebra of canonical type.

$$\mathscr{Q}_A \quad \begin{tikzcd} v \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

Let $F : \widehat{C} \to A$ be the Galois covering and $F_\lambda : \mathrm{Mod}\text{-}\widehat{C} \to \mathrm{Mod}\text{-}A$ the corresponding push-down functor. By Propositions 4.1.10 and 5.2.3, the category $\mathrm{mod}\text{-}A$ consists of a stable tubular family (the image under F_λ of the regular C-modules) and a single component $\mathcal{W}$ of the form $\Gamma^s(\mathcal{W}) = \mathbb{Z}\widetilde{\mathbb{A}}_{(1,1)}$. The component $\mathcal{W}$ can be constructed as follows. Lemma 1.4.1 gives the following AR sequence, corresponding to the projective-injective module $Q := P_A(v) = I_A(v)$.

$$\operatorname{rad} Q \to Q \oplus \operatorname{rad} Q/\operatorname{soc} Q \to Q/\operatorname{soc} Q \tag{6.1.7}$$

Let $P_x := P_C(x)$, $P_y := P_C(y)$, $I_x := I_C(x)$, and $I_y := Y_C(y)$ considered as $\widehat{C}$-modules. It is easily calculated that $\operatorname{rad} Q/\operatorname{soc} Q = S(v)^2$, $F_\lambda I_x = \operatorname{rad} Q$, $F_\lambda I_y = S(v) = F_\lambda P_x$, and $F_\lambda P_y = Q/\operatorname{soc} Q$. Applying F_λ to the preinjective and preprojective components of $\mathrm{mod}\text{-}C$, their images are "sewn together" in $\mathrm{mod}\text{-}A$ by the AR sequence (6.1.7), giving the component $\mathcal{W}$, as depicted in the following diagram.

$$\begin{tikzcd}[column sep=small]
 & & & & Q \arrow[dr] & & & & \\
 & F_\lambda \tau I_x \arrow[rr, dash, dashed] \arrow[dr, shift left] \arrow[dr, shift right] & & F_\lambda I_x \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr, shift left] \arrow[dr, shift right] & & F_\lambda P_y \arrow[rr, dash, dashed] \arrow[dr, shift left] \arrow[dr, shift right] & & F_\lambda \tau^{-1} P_y \arrow[dr, shift left] \arrow[dr, shift right] & \\
F_\lambda \tau^2 I_y \arrow[rr, dash, dashed] \arrow[ur, shift left] \arrow[ur, shift right] & & F_\lambda \tau I_y \arrow[rr, dash, dashed] \arrow[ur, shift left] \arrow[ur, shift right] & & S(v) \arrow[rr, dash, dashed] \arrow[ur, shift left] \arrow[ur, shift right] & & F_\lambda \tau^{-1} P_x \arrow[rr, dash, dashed] \arrow[ur, shift left] \arrow[ur, shift right] & & F_\lambda \tau^{-2} P_x
\end{tikzcd}$$

Every finite-dimensional indecomposable A-module, except for Q, is the image under F_λ of a finite-dimensional indecomposable C-module.

We claim F_λ preserves indecomposability for (infinite-dimensional) pure-injective C-modules. This is easily verified by considering the quotient algebra $B := A/\mathrm{soc}(A)$. It is given by the quiver $\mathscr{Q}_B = \mathscr{Q}_A$ with relations $\alpha\beta = 0$, $\beta\alpha = 0$, $\alpha^2 = 0$, and $\beta^2 = 0$. Thus B is a (1-domestic) string algebra and $\mathrm{Zg}^s_A = \mathrm{Zg}_B$ — see Section 2.2.4. Thus the points of Zg_A (with the exception of Q) are string and band modules. Also C is a (1-domestic) string algebra and points of Zg_C are also string and band modules. It is clear that F_λ maps (infinite) string C-modules to (infinite) string A-modules, and similarly for bands. Hence F_λ preserves indecomposability for pure-injective C-modules.

It follows, by Corollary 4.1.21, that the functor $F_\lambda|_{\mathrm{Mod}\text{-}C} : \mathrm{Mod}\text{-}C \to \mathrm{Mod}\text{-}A$ induces a closed and continuous surjection $\mathrm{Zg}_C \to \mathrm{Zg}_A$ that is one-to-one on the closed subset $\mathfrak{C} := \mathrm{Zg}_C \setminus \{P_x, I_y\}$. In fact, we get a homeomorphism $\varphi : \mathrm{Zg}_C \to \mathrm{Zg}_A$ by defining $\varphi|_{\mathfrak{C}} := F_\lambda$ and, say, $\varphi(P_x) := S(v)$ and $\varphi(I_y) := Q$. Here we have two algebras—A and C—with many different properties, but with homeomorphic Ziegler spectra.

We make some remarks concerning the Ziegler spectra of exceptional self-injective algebras, in general, in Section 6.3. Otherwise, we now move on to the non-exceptional self-injective algebras.

6.1.2 Self-injective algebras of Euclidean type

Let A be a Euclidean algebra of type Δ and tubular type $\bar{n}$. We give the following description of $\widehat{A}$ and $\mathrm{mod}\text{-}\widehat{A}$ from [2, 4.3], cf. [5, 3.1, 5.3], [66, 2.1] (compare with the special case, when A is canonical, considered in Section 5.2; see also Example 6.1.13 below). Let $\nu_A : \widehat{A} \to \widehat{A}$ be the Nakayama automorphism of $\widehat{A}$. There exist two tame concealed algebras C_1 and C_2 and Euclidean algebras B_1^-, B_1^+, B_2^-, and B_2^+ with the following properties (we use the notation for ν_A-reflections introduced in Section 4.2.1 and use $\pm$ to stand for either $+$ or $-$):

(i) B_i^- (resp. B_i^+) is a tubular branch coextension (resp. extension) of C_i with extension type $\bar{n}$.

(ii) Each $B_i^\pm$ is a ν_A-reflection of A and, conversely, every Euclidean ν_A-reflection of A (equiv. representation-infinite ν_A-reflection of A) is isomorphic to some $B_i^\pm$. In particular, A is one of the algebras $B_i^\pm$.

(iii) $B_i^+ = \sigma_{v_n} \cdots \sigma_{v_1} B_i^-$ for a ν_A-reflection sequence of sinks $v_1, \ldots, v_n$ in $\mathscr{Q}_{B_i^-}$.

For $i = 1, 2$, the algebra $B_i := B_i^-[I(v_1)] \cdots [I(v_n)]$ is a 1-parametric domestic algebra and $\mathrm{mod}\text{-}B_i$ consists of a separating quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-family $\mathcal{S}_i$ (of tubular type $\bar{n}$) that separates a preprojective component $\mathcal{P}_i$ (coinciding with the preprojective component of $\mathrm{mod}\text{-}B_i^-$) from a preinjective component $\mathcal{Q}_i$ (coinciding with the preinjective component of $\mathrm{mod}\text{-}B_i^+$. In the degenerative case, when $B_i = B_i^- = B_i^+ = C_i$, the tubular family $\mathcal{S}_i$ is just the family of regular C_i-modules.

The algebra B_i is a "domestic quasi-tube algebra" in the sense of [5] and is obtained from C_i by a sequence of "coil enlargements" [4, 2.2] (which include, as a special case, the tubular branch (co)extensions defined in Section 3.6.2). We don't give the definitions here, we only need the fact that these tubular enlargements are both ray and coray conservative (since we are working with standard components, this fact can be confirmed from the structure of the AR quivers, see [3, 2.1]). To be precise, we have projections $B_i \to B_i^- \to C_i$ and $B_i \to B_i^+ \to C_i$ and all tubular enlargements considered here are with respect to the zero embedding given by restriction along one of these projections.

Now $\operatorname{mod}\text{-}\widehat{A}$ consists of tubular $\mathbb{P}^1(\mathbf{k})$-families $\mathcal{T}^i$ separating components $\mathcal{W}^i$ of the form $\Gamma^s(\mathcal{W}^i) = \mathbb{Z}\Delta$ for all $i \in \mathbb{N}$, i.e. $\operatorname{mod}\text{-}\widehat{A} = \bigvee_{i\in\mathbb{Z}}(\mathcal{T}^i \vee \mathcal{W}^i)$. Under the zero embedding $\operatorname{Mod}\text{-}B_j \to \operatorname{Mod}\text{-}\widehat{A}$ the tubular family $\mathcal{S}_j$ is preserved in $\operatorname{mod}\text{-}\widehat{A}$ and we have $\mathcal{T}^0 = \mathcal{S}_1$ and $\mathcal{T}^1 = \mathcal{S}_2$, such that $\nu_A(\mathcal{T}^i) = \mathcal{T}^{i+2}$ for all $i \in \mathbb{Z}$. Additionally, we have $\mathcal{W}^1 = \langle \mathcal{Q}_1 \vee \mathcal{P}_2 \vee X_1 \rangle$ and $\mathcal{W}^2 = \langle \mathcal{Q}_2 \vee \nu_A(P_1) \vee X_2 \rangle$ (where both X_1 and X_2 are finite sets of indecomposable projective-injective $\widehat{A}$-modules) and $\nu_A(\mathcal{W}^i) = \mathcal{W}^{i+2}$ for all $i \in \mathbb{Z}$. This generalises Proposition 5.2.3.

Proposition 6.1.8. *Every quasi-stable tube* $\mathbb{S}$ *in* $\operatorname{mod}\text{-}B_i$ *satisfies (T2) and (T2***), and the Ziegler closure* $\operatorname{cl}(\mathbb{S})$ *contains each Prüfer module, each adic module, and a generic module. Every Prüfer module is a* B_i^+*-module, every adic module is a* B_i^-*-module, and the generic module is a* C_i*-module. Furthermore, every infinite-dimensional point of* Zg_{B_i} *lies in the closure of some quasi-stable tube.*

Proof. As indicated above, we have $\operatorname{mod}\text{-}B_i = \mathcal{P}_i^- \vee \mathcal{S}_i \vee \mathcal{Q}_i^+$ where $\mathcal{P}_i^-$ is a preprojective component (it is just the preprojective component of $\operatorname{mod}\text{-}B_i^-$), $\mathcal{Q}_i^+$ is a preinjective component (it is just the preinjective componenet of $\operatorname{mod}\text{-}B_i^+$), and $\mathcal{S}_i$ is a (separating) quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-family; we describe the construction of this tubular family now.

Let $\mathcal{R}_i := (\mathbb{T}_\lambda)_{\lambda\in\mathbb{P}^1(\mathbf{k})}$, $\mathcal{R}_i^- := (\mathbb{T}_\lambda^-)_{\lambda\in\mathbb{P}^1(\mathbf{k})}$, $\mathcal{R}_i^+ := (\mathbb{T}_\lambda^+)_{\lambda\in\mathbb{P}^1(\mathbf{k})}$ denote the tubular families of $\operatorname{mod}\text{-}C_i$, $\operatorname{mod}\text{-}B_i^-$, and $\operatorname{mod}\text{-}B_i^+$, respectively, indexed such that $\mathbb{T}_\lambda^-$ and $\mathbb{T}_\lambda^+$ are both a tubular enlargement of $\mathbb{T}_\lambda$, for each $\lambda \in \mathbb{P}^1(\mathbf{k})$. Recall, from Section 3.6.2, each $\mathbb{T}_\lambda^-$ (resp. $\mathbb{T}_\lambda^+$) is a coray (resp. ray) tube. Now $\mathcal{S}_i := (\mathbb{S}_\lambda)_{\lambda\in\mathbb{P}^1(\mathbf{k})}$, indexed such that $\mathbb{S}_\lambda$ is both a tubular enlargement of $\mathbb{T}_\lambda^-$ and a tubular enlargement of $\mathbb{T}_\lambda^+$ (the construction of B_i from C_i as a sequence of coil enlargements isn't unique, and can pass through either B_i^- or B_i^+ at an intermediate stage).

The family of coray tubes $\mathcal{R}_i^-$ and the family of ray tubes $\mathcal{R}_i^+$ both have extension type $\bar{n}$ and the quasi-stable tubular family $\mathcal{S}_i$ has tubular type $\bar{n}$. Thus, if $\mathbb{S}_\lambda$ (a quasi-stable tube) has rank $m \geq 1$, say, then $\mathbb{T}_\lambda^-$ (resp. $\mathbb{T}_\lambda^+$) is a coray (resp. ray) tube of rank m. Recall, the rank of a coray (resp. ray) tube counts the number of maximal corays (resp. maximal rays) in the tube; the rank of a (quasi-)stable tube counts both. By Corollary 3.6.7, both $\mathbb{T}_\lambda^-$ and $\mathbb{T}_\lambda^+$ satisfy (T2) and (T2*). Using Corollary 3.4.3, the m adic (resp. Prüfer) modules of $\mathbb{T}_\lambda^-$ (resp. $\mathbb{T}_\lambda^+$) give the m adic (resp. Prüfer) modules of $\mathbb{S}_\lambda$ and thus $\mathbb{S}_\lambda$ satisfies (T2) and (T2*), as claimed.

Finally, note almost all quasi-stable tubes in $\mathcal{S}_i$ satisfy $\mathbb{S}_\lambda = \mathbb{T}_\lambda$ (i.e. remain unchanged during the tubular enlargements); they are precisely the stable tubes, consisting of regular C_i-modules. In particular, this implies that the closed subsets $\mathrm{Zg}_{B_i^\pm} \subseteq \mathrm{Zg}_{B_i}$ together with the (finitely many) non-stable quasi-stable tubes $\mathbb{S}_{\lambda_1}, \ldots, \mathbb{S}_{\lambda_t}$, contain all finite-dimensional B_i-modules, thus $\mathrm{Zg}_{B_i} = \mathrm{Zg}_{B_i^-} \cup \mathrm{Zg}_{B_i^+} \cup \bigcup_{i=1}^t \mathrm{cl}(\mathbb{S}_{\lambda_i})$ by Proposition 1.3.6. Hence, every infinite-dimensional point of Zg_{B_i} lies in the closure of a quasi-stable tube (since every infinite-dimensional point of $\mathrm{Zg}_{B_i^\pm}$, i.e. a Prüfer, adic, or generic module, lies in the closure of a tube). □

Proposition 6.1.9. *Let A be a Euclidean algebra (of tubular type $\bar{n}$), then every quasi-stable tube of* $\operatorname{mod}\text{-}\widehat{A}$ *satisfies (T4) and every finitely-supported module in* $\mathrm{Zg}_{\widehat{A}}$ *lies in the closure of some quasi-stable tube.*

Proof. First, that every finitely-supported point of $\mathrm{Zg}_{\widehat{A}}$ belongs to a closed subset of the form Zg_{B_i} — and therefore lies in the closure of a tube — is proven as in Proposition 5.2.5.

Using the notation introduced above, we have $\operatorname{mod}\text{-}\widehat{A} = \bigvee_{i\in\mathbb{Z}}(\mathcal{T}^i \vee \mathcal{W}^i)$ where $\mathcal{T}^i$ are tubular $\mathbb{P}^1(\mathbf{k})$-familes obtained as ν_A-shifts of the tubular families $\mathcal{T}^1 = \mathcal{S}_1$ and $\mathcal{T}^2 = \mathcal{S}_2$, consisting of B_1- and B_2-modules, respectively. It follows from Proposition 6.1.8, that every quasi-stable tube $\mathbb{S}$ in $\operatorname{mod}\text{-}\widehat{A}$ — belonging to a subcategory of the form $\operatorname{mod}\text{-}B_i$ — satisfies (T2) and (T2*). Thus, if $\mathbb{S}$ has rank $m \geq 1$, say, then $\mathrm{cl}(\mathbb{S})$ contains at least $2m+1$ infinite-dimensional points, namely m Prüfer modules, m adics modules, and a generic module. To proof $\mathbb{T}$ satisfies (T4), it remains to show no further infinite-dimensional points exist and to describe the topology of $\mathrm{cl}(\mathbb{T})$.

Now A is tilting-cotilting equivalent to a canonical algebra C (indeed, to a canonical algebra of tubular type $\bar{n}$). Let $\nu_A : \widehat{A} \to \widehat{A}$ and $\nu_C : \widehat{C} \to \widehat{C}$ be the Nakayama automorphisms of $\widehat{A}$ and $\widehat{C}$, respectively, and define the orbit algebras $T_A^2 := \widehat{A}/\langle \nu_A^2 \rangle$ and $T_C^2 := \widehat{C}/\langle \nu_C^2 \rangle$. By Proposition 4.2.2 and Corollary 2.1.5, there is a homeomorphism $\mathrm{Zg}\text{-}T_A^2 \to \mathrm{Zg}\text{-}T_C^2$. Also, we have a Galois covering $F : \widehat{A} \to T_A^2$ and its corresponding push-down functor $F_\lambda : \mathrm{Mod}\text{-}\widehat{A} \to \mathrm{Mod}\text{-}T_A^2$. The restriction $F_\lambda|_{\mathrm{Mod}\text{-}B_i} : \mathrm{Mod}\text{-}B_i \to \mathrm{Mod}\text{-}T_A^2$ is, by Corollary 4.1.21, a fully faithful interpretation functor and, by Proposition 1.3.4, induces a closed embedding $\mathrm{Zg}\text{-}B_i \to \mathrm{Zg}\text{-}T_A^2$. Now, by Corollary 1.4.3 and Proposition 4.1.10, respectively, both maps $\mathrm{Zg}\text{-}B_i \to \mathrm{Zg}\text{-}T_A^2$ and $\mathrm{Zg}\text{-}T_A^2 \to \mathrm{Zg}\text{-}T_C^2$ respect the AR structure (i.e. commute with the AR translate) and thus map quasi-stable tubes to quasi-stable tubes.

It follows that, given a quasi-stable tube $\mathbb{S}$ in $\mathrm{mod}\text{-}B_i$ of rank $m \geq 1$, say, then there exists a tube $\mathbb{T}$ in $\mathrm{mod}\text{-}T_C^2$ (necessarily of rank m also) and a closed embedding $\mathrm{cl}(\mathbb{S}) \to \mathrm{cl}(\mathbb{T})$. Now, by Theorem 5.3.11, we know $\mathbb{T}$ satisfies (T4) and, in particular, contains precisely $2m + 1$ infinite-dimensional points. From what we already know, this is sufficient to conclude $\mathbb{S}$ satisfies (T4). $\square$

Now suppose $R = \widehat{A}/G$ is a non-exceptional self-injective algebra of Euclidean type, where A is a Euclidean algebra of Euclidean type Δ and tubular type $\bar{n}$. It is known [2] that there exists $k \in \mathbb{N}$ such that $\mathrm{mod}\text{-}R$ consists of k quasi-stable components, each having stable form $\mathbb{Z}\Delta$, and k quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-families, each having tubular type $\bar{n}$. The following result generalises Theorem 5.2.7.

Theorem 6.1.10. *Let $R = \widehat{A}/G$ be a non-exceptional self-injective algebra of Euclidean type, then there exists $k \in \mathbb{N}$ such that the following facts hold:*

(1) Every quasi-stable tube of $\mathrm{mod}\text{-}R$ satisfies (T4). If two tubes belong to the same tubular family, then there is a unique generic module common to the closure of both tubes.

(2) Every infinite-dimensional point of Zg_R lies in the closure of a quasi-stable tube, i.e. Zg_R consists of the finite-dimensional points of $\mathrm{ind}\text{-}R$, the Prüfer/adic modules of each tube, and $2k$ generic modules.

(3) Zg_R satisfies the isolation condition and $\mathrm{KG}(R) = 2$.

Proof. Let $\nu_A : \widehat{A} \to \widehat{A}$ be the Nakayama automorphism and let $\phi_A : \widehat{A} \to \widehat{A}$ be the automorphism defined by Lemma 6.1.2. By Lemma 6.1.3, we know $G = \langle \sigma\phi_A^k \rangle$ for some $k \in \mathbb{N}$ and a rigid automorphism $\sigma : \widehat{A} \to \widehat{A}$. By assuming R is non-exceptional, we know k is large enough to ensure that any ν_A-reflection of A intersects each G-orbit at most once.

Let $F : \widehat{A} \to R$ be its Galois covering and $F_\lambda : \text{Mod-}\widehat{A} \to \text{Mod-}R$ the corresponding push-down functor. We claim F_λ preserves indecomposability of all finitely-supported points of $\mathrm{Zg}_{\widehat{A}}$. Indeed, for finite-dimensional points this is given by Proposition 4.1.10. By Theorem 6.1.9 and Proposition 6.1.8, every infinite-dimensional finitely-supported point of $\mathrm{Zg}_{\widehat{A}}$ is a module over a ν_A-reflection A' of A. Since A' intersects each G-orbit of A at most once (recall, by Proposition 4.2.3, it is a complete ν_A-slice), it follows, from Corollary 4.1.21, that F_λ restricts to a fully faithful interpretation functor $F_\lambda|_{\text{Mod-}A'} : \text{Mod-}A' \to \text{Mod-}R$. Hence F_λ preserves indecomposability for all finitely-supported infinite-dimensional points in $\mathrm{Zg}_{\widehat{A}}$. Note also, this implies F_λ maps generic modules to generic modules.

Now (1)–(3) are proven as in Theorem 5.2.7. □

Corollary 6.1.11. *Let A be a Euclidean algebra and let B_1^-, B_1^+, B_2^-, B_2^+ be the ν_A-reflections of A (as defined above). Then there exists only finitely many non-stable (but quasi-stable) tubes $\mathbb{T}_1, \ldots, \mathbb{T}_t$ belonging to* $\text{mod-}A \ltimes A^\star$ *and*

$$\mathrm{Zg}\text{-}A \ltimes A^\star = \mathrm{Zg}\text{-}B_1^- \cup \mathrm{Zg}\text{-}B_1^+ \cup \mathrm{Zg}\text{-}B_2^- \cup \mathrm{Zg}\text{-}B_2^+ \cup \bigcup_{i=1}^{t} \mathrm{cl}(\mathbb{T}_i) \cup \mathfrak{P} \qquad (6.1.12)$$

for a finite set $\mathfrak{P}$ of projective-injective vertices.

Proof. This follows directly from Theorem 6.1.10 and proof thereof. As each $B_i^\pm$ is a ν_A-reflection of A, we have closed embeddings $\mathrm{Zg}\text{-}B_i^\pm \to \mathrm{Zg}\text{-}A \ltimes A^\star$ by which we regard $\mathrm{Zg}\text{-}B_i^\pm$ as a closed subset of $\mathrm{Zg}\, A \ltimes A^\star$. The set $\mathfrak{P}$ consists of the projective-injective points lying in the components of the form $\mathbb{Z}\Delta$. Any other point of $\mathrm{Zg}\text{-}A \ltimes A^\star$ belongs to a closed subset of the form $\mathrm{Zg}\text{-}B_i^\pm$ or in the closure of one of the (finitely many) non-stable tubes. □

Remark The union (6.1.12), although far from disjoint, is finite. Thus the topology of $\mathrm{Zg}\text{-}A \ltimes A$ can be reduced to the topology of the closed subsets in this union.

Example 6.1.13. We offer the following example to illustrate the above discussion. Let H be the Kronecker algebra and let $(S_\lambda)_{\lambda\in\mathbb{P}^1(\mathrm{k})}$ be the simple regular H-modules, as defined in Example 1.8.3. Consider the tubular branch extension $A := H[S_0, K]$ where K is the factorspace branch of length 1. Then A is a Euclidean algebra of extension type (3) and is given by the following quiver $\mathscr{Q}_A$ with relation $\alpha\delta = 0$.

$$\mathscr{Q}_A \qquad \begin{tikzcd} v_0 & v_1 \arrow[l, shift right, "\gamma"'] \arrow[l, shift left, "\delta"] & v_2 \arrow[l, "\alpha"'] \arrow[r, "\beta"] & v_3 \end{tikzcd}$$

The repetitive algebra $\widehat{A}$ is given by the following quiver $\mathscr{Q}_{\widehat{A}}$ with relations $\epsilon^i\alpha^{i-1} = 0$, $\zeta^i\beta^{i-1} = 0$, $\eta^i\gamma^{i-1} = 0$, $\delta^i\zeta^i = 0$, $\gamma^i\eta^i = 0$, $\zeta^i\alpha^{i-1}\gamma^{i-1} = \eta^i\delta^{i-1}$, $\gamma^i\zeta^i\alpha^{i-1} = \delta^i\eta^i$, $\alpha^i\gamma^i\zeta^i = \beta^i\epsilon^i$, $\beta^i\epsilon^i\beta^{i-1} = 0$, $\epsilon^i\beta^{i-1}\epsilon^{i-1} = 0$, $\delta^i\eta^i\delta^{i-1} = 0$, $\eta^i\delta^{i-1}\eta^{i-1} = 0$, and $\rho = 0$ for any path ρ of length ≥ 4.

$$\mathscr{Q}_{\widehat{A}} \qquad \begin{tikzcd} & \cdots v_3^{i-1} & & & v_3^i \arrow[dll, bend right, "\epsilon^i"'] & & & v_3^{i+1}\cdots \arrow[dll, bend right, "\epsilon^{i+1}"'] \\ \cdots v_0^{i-1} & v_1^{i-1} \arrow[l, shift right, "\gamma^{i-1}"'] \arrow[l, shift left, "\delta^{i-1}"] & v_2^{i-1} \arrow[l, "\alpha^{i-1}"'] \arrow[ul, "\beta^{i-1}"] & v_0^i \arrow[l, "\zeta^i"'] \arrow[ll, bend left, "\eta^i"] & v_1^i \arrow[l, shift right, "\gamma^i"'] \arrow[l, shift left, "\delta^i"] & v_2^i \arrow[l, "\alpha^i"'] \arrow[u, "\beta^i"] & v_0^{i+1} \arrow[l, "\zeta^{i+1}"'] \arrow[ll, bend left, "\eta^{i+1}"] & v_1^{i+1}\cdots \arrow[l, shift right, "\gamma^{i+1}"'] \arrow[l, shift left, "\delta^{i+1}"] \end{tikzcd}$$

Let $\nu_A : \widehat{A} \to \widehat{A}$ be the Nakayama automorphism of $\widehat{A}$, then the Euclidean ν_A-reflections $B_1^\pm$ and $B_2^\pm$ of A are (up to isomorphism) given by the following quivers and relations.

$$\mathscr{Q}_{B_1^-} \qquad \begin{tikzcd} v_3 \arrow[r, "\epsilon"] & v_2 & v_0 \arrow[l, "\zeta"'] & v_1 \arrow[l, shift right, "\gamma"'] \arrow[l, shift left, "\delta"] \end{tikzcd}$$

$$\mathscr{Q}_{B_1^+} \qquad \begin{tikzcd} v_0 & v_1 \arrow[l, shift right, "\gamma"'] \arrow[l, shift left, "\delta"] & v_2 \arrow[l, "\alpha"'] \arrow[r, "\beta"] & v_3 \end{tikzcd}$$

$$\mathscr{Q}_{B_2^-} \qquad \begin{tikzcd} v_3 & & \\ v_1 & v_2 \arrow[l, "\alpha"'] \arrow[ul, "\beta"'] & v_0 \arrow[l, "\zeta"'] \arrow[ll, bend left, "\eta"] \end{tikzcd} \qquad \mathscr{Q}_{B_2^+} \qquad \begin{tikzcd} & & v_3 \arrow[dl, "\epsilon"'] \\ v_1 & v_2 \arrow[l, "\alpha"'] & v_0 \arrow[l, "\zeta"'] \arrow[ll, bend left, "\eta"] \end{tikzcd}$$

Each $B_i^\pm$ has a single relation, respectively: $\delta\zeta = 0$, $\alpha\delta = 0$, $\zeta\beta = 0$, $\epsilon\alpha = 0$.

Note B_1^-, resp. B_1^+, is a tubular branch coextension, resp. extension, of H with extension type (3). Thus both $B_1^{\pm}$ are Euclidean algebras of type $\widetilde{\mathbb{A}}_{1,3}$ — of course, B_1^+ is just $A = H[S_0, K]$. Similarly, B_2^-, resp. B_2^+, is seen to be a one-point tubular coextension, resp. extension, of the canonical algebra $C := C(1, 2)$ of type $\widehat{\mathbb{A}}_{1,2}$, using the simple regular (and also simple) C-module $S_C(v_2)$. As $S_C(v_2)$ lies in a stable tube of rank 2, we see $B_2^{\pm}$ are Euclidean algebras of type $\widetilde{\mathbb{A}}_{1,3}$ also.

The 1-domestic algebras B_1 and B_2 have the following quivers and relations.

$\mathscr{Q}_{B_1}$
$$\begin{tikzcd}
 & & v_3 \arrow[dll, "\epsilon"'] & & \\
v_2' & v_0 \arrow[l, "\zeta"] & & v_1 \arrow[ll, bend right, "\gamma"'] \arrow[ll, bend left, "\delta"] & v_2 \arrow[l, "\alpha"] \arrow[ull, "\beta"']
\end{tikzcd}$$
$\alpha\delta = \delta\zeta = 0, \beta\epsilon = \alpha\gamma\zeta,$

$\mathscr{Q}_{B_2}$
$$\begin{tikzcd}
v_3' & & v_3 \arrow[dl, "\epsilon"] \\
v_1 & v_2 \arrow[l, "\alpha"] \arrow[ul, "\beta"] & v_0 \arrow[l, "\zeta"] \arrow[ll, bend left, "\eta"]
\end{tikzcd}$$
$\zeta\beta = \epsilon\alpha = 0.$

The tubular families of mod-B_1 and mod-B_2 both contain just a single non-stable tube. In either case, this non-stable tube must be quasi-stable with rank 3.

By construction, we have $B_1 = [P_{B_1^+(v_2)}](H[S_0, K])$. So the tube of mod-$B_1$ containing S_0 is obtained from the tube of mod-H, containing S_0, by a tubular branch extension followed by a one-point coextension. The tubes $\mathbb{S}$ and $\mathbb{T}^+$ over H and B_1^+, respectively, that contain S_0 are depicted in the following diagram (vertices are labelled by the dimension vectors of corresponding modules).

$$\begin{tikzcd}
\mathbb{S} \quad (1\,1) \arrow[d, bend right] \\
(2\,2) \arrow[u, bend right] \arrow[d, bend right] \\
(3\,3) \arrow[u, bend right] \arrow[d, dotted, bend right] \\
\vdots \arrow[u, dotted, bend right]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\mathbb{T}^+ \; (0\,0\,0\,1) \arrow[dr] \arrow[rr, dash, dashed] & & (1\,1\,1\,0) \arrow[dr] \arrow[rr, dash, dashed] & & (1\,1\,0\,0) \arrow[dd, dash, dotted] \\
 & (1\,1\,1\,1) \arrow[ur] \arrow[dr] & & (2\,2\,1\,0) \arrow[ur] \arrow[dr] & \\
(1\,1\,0\,0) \arrow[ur] \arrow[dr] \arrow[dd, dash, dotted] & & (2\,2\,1\,1) \arrow[ur] \arrow[dr] & & (3\,3\,1\,0) \arrow[dd, dash, dotted] \\
 & (2\,2\,0\,0) \arrow[ur] \arrow[dr] & & (3\,3\,1\,1) \arrow[ur] \arrow[dr] & \\
(3\,3\,1\,0) \arrow[ur] & & (3\,3\,0\,0) \arrow[ur] & & (4\,4\,1\,1)
\end{tikzcd}$$

The modules S_0 and $P_{B_1^+}(v_2)$ are those with dimension vectors $(\,1\,1\,0\,0\,)$ and $(\,1\,1\,1\,1\,)$ respectively. Note $\mathbb{T}^+$ is a ray tube of rank 3 (containing two projective modules) and, in particular, $\mathbb{T}^+$ is partitioned by the rays beginning at $(\,1\,1\,0\,0\,)$, $(\,0\,0\,0\,1\,)$, and $(\,1\,1\,1\,0\,)$.

As an aside, observe that by removing the vertex $(\,0\,0\,0\,1\,)$ we obtain a tube of the form $\Gamma(0,2,0)$. Indeed, this illlustrates one aspect of Proposition 3.6.6, i.e. the module

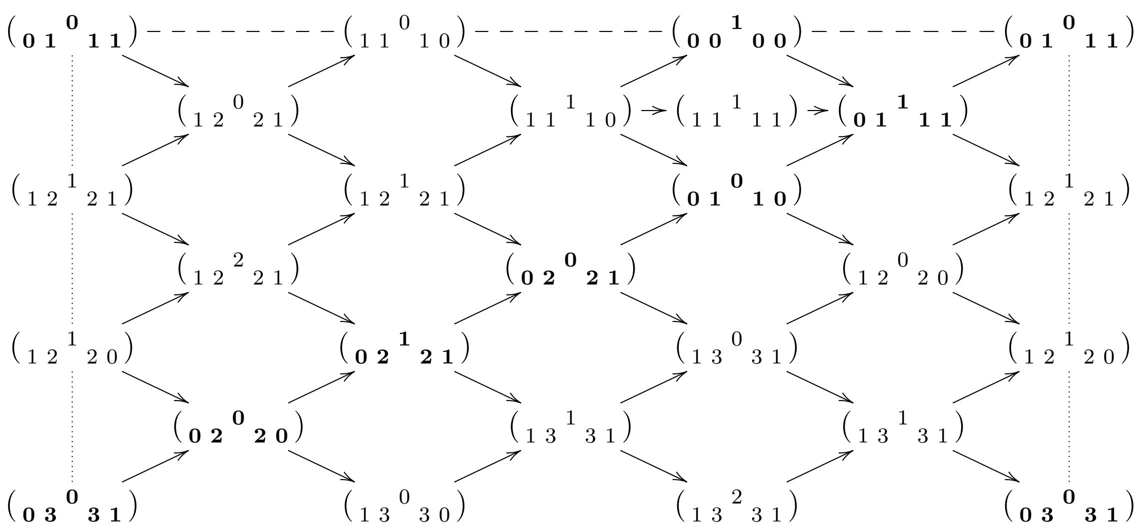
$$M := P_{B_1^+}(v_0) \oplus P_{B_1^+}(v_1) \oplus (\,1\,1\,1\,1\,) \oplus (\,1\,1\,1\,0\,)$$

is a tilting B_1^+-module, with $\mathrm{End}_{B_1^+}(M) \simeq H[S_0,2]$. The class $\mathbb{T}^+ \setminus \{S_{B_2^+}(v_2)\}$ is torsion with respect to M and mapped—by the tilting functor $\mathrm{Hom}_{B_1^+}(M,-)$—onto a tube of the form $\Gamma(0,2,0)$ over $H[S_0,2]$.

Anyway, the module $P_{B_1^+}(v_2)$ belongs at the end of the following coray

$$\cdots \to (\,2\,2\,0\,0\,) \to (\,2\,2\,1\,1\,) \to (\,2\,2\,1\,0\,) \to (\,1\,1\,0\,0\,) \to (\,1\,1\,1\,1\,) \tag{6.1.14}$$

but $P_{B_1^+}(v_2)$ is not a coray module in $\mathbb{T}^+$ (due to the existence of a morphism $(\,0\,0\,0\,1\,) \to (\,1\,1\,1\,1\,)$). Hence, the one-point coextension $B_1 = [P_{B_1^+}(v_2)]B_1^+$ is not a one-point tubular coextension. However, it is a quasi-tube enlargement and the quasi-stable tube $\mathbb{T}$ over B_1 containing S_0 and $P_{B_1^+}(v_2)$ is depicted in the following diagram. Dimension vectors are given in the shape of the quiver $\mathscr{Q}_{B_1}$ (they no longer uniquely determine a module, the only identifcations to be made are along the dotted lines).



This construction can be done using the "admissible operations" on translation quivers defined in [3, 2.1]. Alternatively, knowing the AR sequence containing the projective-injective vertex $\left(\begin{smallmatrix} & & 1 & & \\ 1 & 1 & & 1 & 1\end{smallmatrix}\right)$ — it is given by Lemma 1.4.1 — and the fact the coray (6.1.14) is preserved, one can use the so-called knitting procedure (a technique of Auslander-Reiten theory) to construct the tube.

For B_2, we have $B_2 := [P_{B_2^+}(v_3)](C[S_C(v_2)]) = C[S_C(v_2), 1, 1]$ and by Proposition 3.5.4 we have a quasi-stable tube of the form $\Gamma(1,1,1)$. Indeed, the tube of $C = C(1,2)$ containing the simple module $S_C(v_2)$ is depicted in the following diagram.

$$\begin{tikzcd}
(1\,1\,1) \arrow[dr] \arrow[rr, dash, dashed] \arrow[dd, dash, dotted] & & (0\,1\,0) \arrow[dr] \arrow[rr, dash, dashed] & & (1\,1\,1) \arrow[dd, dash, dotted] \\
& (1\,2\,1) \arrow[ur] \arrow[dr] & & (1\,2\,1) \arrow[ur] \arrow[dr] & \\
(1\,3\,1) \arrow[ur] & & (2\,3\,2) \arrow[ur] & & (1\,3\,1)
\end{tikzcd}$$

The tube over $B_2^+ = C[S_C(v_2), 1]$ containing $S_C(v_2)$ is obtained by a single ray insertion at $S_C(v_2) = (0\,1\,0)$ and is depicted as follows.

$$\begin{tikzcd}[column sep=small]
& & & \left(\begin{smallmatrix} & & 1\\ 0&1&0\end{smallmatrix}\right) \arrow[dr] \arrow[rr, dash, dashed] & & \left(\begin{smallmatrix} & & 0\\ 1&1&1\end{smallmatrix}\right) \arrow[dd, dash, dotted] \\
\left(\begin{smallmatrix} & & 0\\ 1&1&1\end{smallmatrix}\right) \arrow[dr] \arrow[rr, dash, dashed] \arrow[dd, dash, dotted] & & \left(\begin{smallmatrix} & & 0\\ 0&1&0\end{smallmatrix}\right) \arrow[ur] \arrow[dr] & & \left(\begin{smallmatrix} & & 1\\ 1&2&1\end{smallmatrix}\right) \arrow[ur] \arrow[dr] & \\
& \left(\begin{smallmatrix} & & 0\\ 1&2&1\end{smallmatrix}\right) \arrow[ur] \arrow[dr] & & \left(\begin{smallmatrix} & & 0\\ 1&2&1\end{smallmatrix}\right) \arrow[ur] \arrow[dr] & & \left(\begin{smallmatrix} & & 1\\ 1&3&1\end{smallmatrix}\right) \\
\left(\begin{smallmatrix} & & 1\\ 1&3&1\end{smallmatrix}\right) \arrow[ur] & & \left(\begin{smallmatrix} & & 0\\ 2&3&2\end{smallmatrix}\right) \arrow[ur] & & \left(\begin{smallmatrix} & & 0\\ 1&3&1\end{smallmatrix}\right) \arrow[ur] &
\end{tikzcd}$$

Finally, $P_{B_2^+}(v_3) = \left(\begin{smallmatrix} & & 1\\ 0&1&0\end{smallmatrix}\right)$ and $B_2 := [P_{B_2^+}(v_3)]B_2^+$ is a one-point tubular coextension, giving the following tube over B_2, containing S.

$$\begin{tikzcd}[column sep=small]
& & & \left(\begin{smallmatrix} 1 & & 1\\ 0&1&0\end{smallmatrix}\right) \arrow[dr] & & & \\
\left(\begin{smallmatrix} 0 & & 0\\ 1&1&1\end{smallmatrix}\right) \arrow[dr] \arrow[rr, dash, dashed] \arrow[dd, dash, dotted] & & \left(\begin{smallmatrix} 1 & & 0\\ 0&1&0\end{smallmatrix}\right) \arrow[ur] \arrow[dr] & & \left(\begin{smallmatrix} 0 & & 1\\ 0&1&0\end{smallmatrix}\right) \arrow[dr] \arrow[rr, dash, dashed] & & \left(\begin{smallmatrix} 0 & & 0\\ 1&1&1\end{smallmatrix}\right) \arrow[dd, dash, dotted] \\
& \left(\begin{smallmatrix} 1 & & 0\\ 1&2&1\end{smallmatrix}\right) \arrow[ur] \arrow[dr] & & \left(\begin{smallmatrix} 0 & & 0\\ 0&1&0\end{smallmatrix}\right) \arrow[ur] \arrow[dr] & & \left(\begin{smallmatrix} 0 & & 1\\ 1&2&1\end{smallmatrix}\right) \arrow[ur] \arrow[dr] & \\
\left(\begin{smallmatrix} 1 & & 1\\ 1&3&1\end{smallmatrix}\right) \arrow[ur] & & \left(\begin{smallmatrix} 0 & & 0\\ 1&2&1\end{smallmatrix}\right) \arrow[ur] & & \left(\begin{smallmatrix} 0 & & 0\\ 1&2&1\end{smallmatrix}\right) \arrow[ur] & & \left(\begin{smallmatrix} 1 & & 1\\ 1&3&1\end{smallmatrix}\right)
\end{tikzcd}$$

This is a genuine tube of the form $\Gamma(1,1,1)$.

We have constructed two quasi-stable tubes of rank 3 and both satisfy (T4). The remaining tubes over B_1, resp. B_2, are just the remaining homogeneous tubes of regular H-, resp. C-, modules.

By Theorem 6.1.10, the trivial extension $T := A \ltimes A^{\star}$ is a domestic algebra such that mod-T has two quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-families, each possessing a quasi-stable tube of rank 3, with all remaining tubes being stable of rank 1. There are two remaining components of the (stable) form $\mathbb{Z}\widehat{\mathbb{A}}_3$ consisting of preprojective/preinjective H- and C-modules, and each must contain just one indecomposable projective-injective module. All quasi-tubes satisfy (T4) and every infinite-dimensional point of Zg_T lies in the closure some quasi-stable tube.

The trivial extension T has the following quiver $\mathscr{Q}_T$ and relations $\alpha\delta = 0$, $\epsilon\alpha = 0$, $\zeta\beta = 0$, $\delta\zeta = 0$, $\eta\gamma = 0$, $\zeta\alpha\gamma = \eta\delta$, $\delta\eta = \gamma\zeta\alpha$, $\beta\epsilon = \alpha\gamma\zeta$, $\beta\epsilon\beta = 0$, $\epsilon\beta\epsilon = 0$, $\delta\eta\delta = 0$, $\eta\delta\eta = 0$, and $\rho = 0$ for any path ρ of length ≥ 4.

$$\begin{tikzcd}
\mathscr{Q}_T & v_0 \arrow[rr, bend left=40, "\zeta"] \arrow[r, bend right=40, "\eta"'] & v_1 \arrow[l, shift right, bend right=10, "\gamma"'] \arrow[l, shift left, bend left=10, "\delta"] & v_2 \arrow[l, "\alpha"'] \arrow[r, bend left=10, "\beta"] & v_3 \arrow[l, bend left=10, "\epsilon"]
\end{tikzcd}$$

The two generic T-modules are as follows (the arrows labelled T denote multiplication by T in rational function field $\mathbf{k}(T)$).

$$\begin{tikzcd}
G_H & \mathbf{k}(T) \arrow[rr, bend left=40] \arrow[r, bend right=40, "0"'] & \mathbf{k}(T) \arrow[l, shift right, bend right=10, "T"'] \arrow[l, shift left, bend left=10, "1"] & 0 \arrow[l] \arrow[r, bend left=10] & 0 \arrow[l, bend left=10]
\end{tikzcd}$$

$$\begin{tikzcd}
G_C & \mathbf{k}(T) \arrow[rr, bend left=40, "T"] \arrow[r, bend right=40, "1"'] & \mathbf{k}(T) \arrow[l, shift right, bend right=10, "0"'] \arrow[l, shift left, bend left=10, "0"] & \mathbf{k}(T) \arrow[l, "1"'] \arrow[r, bend left=10] & 0 \arrow[l, bend left=10]
\end{tikzcd}$$

These are just the generic H- and C- modules, respectively. In both quasi-stable tubes, all Prüfer and adic modules (there are 12 in total) have a descripition as an infinite string module. We refer to [57], for instance, for such a description. By property (T4), they are obtained as direct and inverse limits along rays and corays.

6.1.3 Self-injective algebras of Ringel type

Let A be a tubular algebra and let $\nu_A : \widehat{A} \to \widehat{A}$ be the Nakayama automorphism of $\widehat{A}$. We give the following description of $\widehat{A}$ and mod-$\widehat{A}$ from [46, 3], cf. [66, 3.1] (compare with the special case, when A is canonical, considered in Section 5.3).

Let $A^1, \dots, A^t$ be the complete list (up to isomorphism) of all ν_A-reflections of A that are themselves tubular algebras. As such, A^i is a tubular branch extension of a tame concealed algebra A^i_0 and a tubular branch coextension of a tame concealed A^i_∞. In fact, this list can be ordered such that $A^i_\infty = A^{i+1}_0$ for $i = 1, \dots, t-1$, and $A^t_\infty = A^1_0$. We may take $A^1 := A$ and fix an inclusion $A \subseteq \widehat{A}$, thereby inducing inclusions $A^i \subseteq \widehat{A}$ for $i = 2, \dots, t$.

For $i \in \mathbb{Z}$ we define $A^i := \nu_A^k(A^j)$ if $i = j + tk$ for $j \in \{1, \dots, t\}$ and $k \in \mathbb{Z}$ (of course $A^i \simeq A^j$). Now there exists a ν_A-reflection sequence of sinks $v_1, \dots, v_{s(i)}$ in $\mathscr{Q}_{A^i} \subseteq \mathscr{Q}_{\widehat{A}}$ such that $A^{i+1} = S^-_{v_{s(i)}} \cdots S^-_{v_1} A^i$. Conversely, there exists a ν_A-reflection sequence of sources $u_j := \nu_A(v_j)$ in $\mathscr{Q}_{A^{i+1}}$ such that $A^i = S^+_{u_1} \cdots S^+_{u_{s(i)}} A^{i+1}$. All A^i are tubular algebras of the same tubular type.

Define the algebras $B^i := A^i[I(v_1)] \cdots [I(v_{s(i)})]$ by iterated one-point extensions, by duality we have $B^i = [P(u_{s(i)})] \cdots [P(u_1)] A^{i+1}$. By construction B^i is a full convex subcategory of $\widehat{A}$ and we have zero embeddings Mod-$B^i \to$ Mod-$\widehat{A}$ by Corollary 4.2.4. In this way, we consider Mod-B^i as a definable subcategory of Mod-$\widehat{A}$ and, similarly, Mod-A^i and Mod-A^{i+1} are definable subcategories of Mod-B^i.

It is shown in [46, 3] that mod-$\widehat{A}$ consists entirely of separating quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-families, all of the same tubular type and each belonging to one of the subcategories mod-B^i. Specifically

$$\text{mod-}\widehat{A} = \bigvee_{i \in \mathbb{Z}} (\mathcal{M}^i \vee \mathcal{T}^i)$$

where the class $\mathcal{M}^i := \bigvee_{q \in \mathbb{Q}^+} \mathcal{M}^i_q$ is the collection of stable tubular $\mathbb{P}^1(\mathbf{k})$-families in mod-A^i (i.e. those of positive rational slope over the tubular algebra A^i), and the class $\mathcal{T}^i$ is a quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-family belonging to mod-B^i.

To construct $\mathcal{T}^i$, first denote by $\mathcal{T}^i_0$, resp. $\mathcal{T}^i_\infty$, the tubular $\mathbb{P}^1(\mathbf{k})$-families of slope

0, resp. slope ∞, in mod-A^i, then we have

$$\text{mod-}B^i = \mathcal{P}^i \vee \mathcal{T}_0^i \vee \mathcal{M}^i \vee \mathcal{T}^i \vee \mathcal{M}^{i+1} \vee \mathcal{T}_\infty^{i+1} \vee \mathcal{Q}^{i+1} \tag{6.1.15}$$

where $\mathcal{P}^i$ is a preprojective component (coinciding with the preprojective components of mod-A_0^i and mod-A^i) and $\mathcal{Q}^{i+1}$ is a preinjective component (coinciding with the preinjective components of mod-A_∞^{i+1} and mod-A^{i+1}). Now $\mathcal{T}^i$ is a (quasi-stable) tubular family that is both a tubular enlargement of $\mathcal{T}_\infty^i$ and a tubular enlargement of $\mathcal{T}_0^{i+1}$. We only require the fact that these tubular enlargements are both ray and coray conservative. This is analogous to the construction of the tubular families $\mathcal{T}^i$ in the Euclidean case of Section 6.1.2 above, the following result is almost identical to Proposition 6.1.8.

Proposition 6.1.16. *Every quasi-stable tube* $\mathbb{S}$ *in* mod-B^i *satisfies (T2) and (T2***), and the Ziegler closure* $\mathrm{cl}(\mathbb{S})$ *contains each Prüfer module, each adic module, and a generic module. Furthermore, every infinite-dimensional point of* Zg_{B_i} *is an* A^i*- or* A^{i+1}*-module, and either lies in the closure of some quasi-stable tube or is a module with irrational slope.*

Proof. Using the notation for the tubular families defined in the preceding paragraphs, let $\mathcal{T}_\infty^i := (\mathbb{T}_\lambda^-)_{\lambda\in\mathbb{P}^1(\mathbf{k})}$, $\mathcal{T}_0^{i+1} := (\mathbb{T}_\lambda^+)_{\lambda\in\mathbb{P}^1(\mathbf{k})}$, and $\mathcal{T}^i := (\mathbb{T}_\lambda)_{\lambda\in\mathbb{P}^1(\mathbf{k})}$, be indexed such that $\mathbb{T}_\lambda$ is a tubular enlargement of $\mathbb{T}_\lambda^-$ and $\mathbb{T}_\lambda^+$, for all $\lambda \in \mathbb{P}^1(\mathbf{k})$.

Given $\lambda \in \mathbb{P}^1(\mathbf{k})$, suppose $\mathbb{T}_\lambda$ has rank $m \geq 1$, then (as these families share the same tubular type) $\mathbb{T}_\lambda^-$ must be a coray tube of rank m and $\mathbb{T}_\lambda^+$ must be a ray tube of rank m. Recall, the rank of a coray (resp. ray) tube counts the number of maximal corays (resp. maximal rays) in the tube; the rank of a (quasi-)stable tube counts both. By Corollary 3.6.7, we know $\mathbb{T}_\lambda^-$ and $\mathbb{T}_\lambda^+$ satisfy both (T2) and (T2*). Using Corollary 3.4.3, the m adic (resp. Prüfer) modules of $\mathbb{T}_\lambda^-$ (resp. $\mathbb{T}_\lambda^+$) give the m adic (resp. Prüfer) modules of $\mathbb{T}_\lambda$. Thus $\mathbb{T}_\lambda$ satisfies (T2) and (T2*) in mod-B^i. All other tubes in mod-B^i belong to either mod-A^i or mod-A^{i+1} and satisfy (T2) and (T2*) by Corollary 3.6.7.

Now only finitely many of the quasi-stable tubes in $\mathcal{T}^i$ are non-stable, i.e. for almost all $\mu \in \mathbb{P}^1(\mathbf{k})$ we have $\mathbb{T}_\mu = \mathbb{T}_\mu^- = \mathbb{T}_\mu^+$ (note such tubes are tubes of regular modules over the tame concealed algebra $A_\infty^i = A_0^{i+1}$). If $\mathbb{T}_{\mu_1}, \dots \mathbb{T}_{\mu_s}$ denote the non-stable tubes of

$\mathcal{T}^i$, then by (6.1.15), the closed subset $\mathrm{Zg}_{A^i} \cup \mathrm{Zg}_{A^{i+1}} \cup \bigcup_{j=1}^{s} \mathrm{cl}(\mathbb{T}_{\mu_j}) \subseteq \mathrm{Zg}_{B^i}$ contains all finite-dimensional B^i-modules, and so we have equality by Proposition 1.3.6. Thus any infinite-dimensional point of Zg_{B^i} lies in the closure of one of $\mathbb{T}_{\mu_1}, \dots \mathbb{T}_{\mu_s}$ or belongs to Zg_{A^i} or $\mathrm{Zg}_{A^{i+1}}$ (and thus lies in the closure of a tube, or has irrational slope, over A^i or A^{i+1}, by Proposition 1.10.1). $\square$

Proposition 6.1.17. *Let A be a tubular algebra, then every component of* $\mathrm{mod}\text{-}\widehat{A}$ *is a quasi-stable tube satisfying (T4) and every infinite-dimensional finitely-supported module in* $\mathrm{Zg}_{\widehat{A}}$ *lies in the closure of some quasi-stable tube or is a module of irrational slope over a tubular ν_A-reflection of A.*

Proof. We use the notation introduced above. By that discussion, we know $\mathrm{mod}\text{-}\widehat{A}$ consists entirely of quasi-stable tubes and, by Proposition 6.1.16, every quasi-stable tube (belonging to a closed subset of the form Zg_{B^i}) satisfies both (T2) and (T2*). Also, we note every finitely-supported point of $\mathrm{Zg}_{\widehat{A}}$ lies in a closed subset of the form Zg_{B^i} (by an argument similar to Proposition 6.1.9) and thus, by Proposition 6.1.16 again, lies in the closure of a quasi-stable tube or is a module of irrational slope over one of the tubular algebras A^i or A^{i+1} (both ν_A-reflections of A). It remains to prove properties (T3) and (T4) for each quasi-stable tube.

Let $\mathbb{S}$ be a quasi-stable tube of $\mathrm{mod}\text{-}\widehat{A}$ of rank $m \geq 1$. We know $\mathrm{cl}(\mathbb{S})$ contains at least $2m+1$ infinite-dimensional points: the m Prüfer modules, the m adic modules, and the generic $A^i_\infty = A^{i+1}_0$-module.

If $\bar{n}$ is the tubular type of A, then A is tilting-cotilting equivalent to the canonical algebra $C := C(\bar{n})$ [31, 1.1]. Let $\nu_A : \widehat{A} \to \widehat{A}$ and $\nu_C : \widehat{C} \to \widehat{C}$ be the Nakayama automorphisms of $\widehat{A}$ and $\widehat{C}$, respectively, and define the orbit algebras $T^2_A := \widehat{A}/\langle \nu_A^2 \rangle$ and $T^2_C := \widehat{C}/\langle \nu_C^2 \rangle$. By Proposition 4.2.2 and Corollary 2.1.5, there is a homeomorphism $\mathrm{Zg}\text{-}T^2_A \to \mathrm{Zg}\text{-}T^2_C$. Also, we have a Galois covering $F : \widehat{A} \to T^2_A$ and its corresponding push-down functor $F_\lambda : \mathrm{Mod}\text{-}\widehat{A} \to \mathrm{Mod}\text{-}T^2_A$. The restriction $F_\lambda|_{\mathrm{Mod}\text{-}B_i} : \mathrm{Mod}\text{-}B_i \to \mathrm{Mod}\text{-}T^2_A$ is, by Corollary 4.1.21, a fully faithful interpretation functor and, by Proposition 1.3.4, induces a closed embedding $\mathrm{Zg}\text{-}B_i \to \mathrm{Zg}\text{-}T^2_A$. Now, by Corollary 1.4.3 and Proposition 4.1.10, respectively, both maps $\mathrm{Zg}\text{-}B_i \to \mathrm{Zg}\text{-}T^2_A$ and $\mathrm{Zg}\text{-}T^2_A \to \mathrm{Zg}\text{-}T^2_C$ respect the AR structure (i.e. commute with the AR translate) and map quasi-stable tubes to quasi-stable tubes.

It follows that, given a quasi-stable tube $\mathbb{S}$ in $\mathrm{mod}\text{-}B^i$ of rank $m \geq 1$, say, then there exists a tube $\mathbb{T}$ in $\mathrm{mod}\text{-}T_C^2$ (necessarily of rank m also) and a closed embedding $\mathrm{cl}(\mathbb{S}) \to \mathrm{cl}(\mathbb{T})$. Now, by Theorem 5.3.11, we know $\mathbb{T}$ satisfies (T4) and, in particular, contains precisely $2m+1$ infinite-dimensional points. We conclude $\mathbb{S}$ satisfies (T4). $\square$

Now suppose $R = \widehat{A}/G$ is a non-exceptional self-injective algebra of Ringel type, where A is a tubular algebra of tubular type $\bar{n}$. It is known [46] that $\mathrm{mod}\text{-}R$ consists entirely of quasi-stable tubular $\mathbb{P}^1(\mathbf{k})$-families, each having tubular type $\bar{n}$. The following result generalises Theorem 5.3.11.

Theorem 6.1.18. *Let $R = \widehat{A}/G$ be a non-exceptional self-injective of Ringel type and let $\nu_A : \widehat{A} \to \widehat{A}$ be the Nakayama automorphism of $\widehat{A}$.*

(1) Every quasi-stable tube of $\mathrm{mod}\text{-}R$ *satisfies (T4). If two tubes belong to the same tubular family, then there is a unique generic module common to the closure of both tubes.*

(2) Every infinite-dimensional point of Zg_R *lies in the closure of a quasi-stable tube or is a module of irrational slope over a tubular ν_A-reflection of A.*

Proof. Let $\phi_A : \widehat{A} \to \widehat{A}$ be the automorphism defined by Lemma 6.1.2. By Lemma 6.1.3, we know $G = \langle \sigma\phi_A^k \rangle$ for some $k \in \mathbb{N}$ and a rigid automorphism $\sigma : \widehat{A} \to \widehat{A}$. By assuming R is non-exceptional, we know k is large enough to ensure that any ν_A-reflection of A intersects each G-orbit at most once.

Let $F : \widehat{A} \to R$ be its Galois covering and $F_\lambda : \mathrm{Mod}\text{-}\widehat{A} \to \mathrm{Mod}\text{-}R$ the corresponding push-down functor. We claim F_λ preserves indecomposability of all finitely-supported points of $\mathrm{Zg}_{\widehat{A}}$. Indeed, for finite-dimensional points this is given by Proposition 4.1.10. By Theorem 6.1.17 and Proposition 6.1.16, every infinite-dimensional finitely-supported point of $\mathrm{Zg}_{\widehat{A}}$ is a module over a ν_A-reflection A' of A. Since A' intersects each G-orbit of A at most once (recall, by Proposition 4.2.3, it is a complete ν_A-slice), it follows, from Corollary 4.1.21, that F_λ restricts to a fully faithful interpretation functor $F_\lambda|_{\mathrm{Mod}\text{-}A'} : \mathrm{Mod}\text{-}A' \to \mathrm{Mod}\text{-}R$. Hence F_λ preserves indecomposability for all finitely-supported infinite-dimensional points in $\mathrm{Zg}_{\widehat{A}}$. Note also, this implies F_λ maps generic modules to generic modules.

Now (1)–(2) are proven as in Theorem 5.3.11. $\square$

Corollary 6.1.19. *Let A be a tubular algebra, suppose $A^1, \dots, A^t$ are tubular ν_A-reflections of A (as defined above). Then there exists only finitely many non-stable (but quasi-stable) tubes $\mathbb{T}_1, \dots, \mathbb{T}_n$ belonging to* $\operatorname{mod}\text{-}A \ltimes A^\star$ *and*

$$\operatorname{Zg}\text{-}A \ltimes A^\star = \bigcup_{i=1}^{t} \operatorname{Zg}\text{-}A^i \cup \bigcup_{i=1}^{n} \operatorname{cl}(\mathbb{T}_i)$$

In general, we must have $3 \leq n \leq p$ where p is the number of indecomposable projective A-modules.

6.2 Non-standard self-injective algebras

Although many non-standard self-injective algebras of polynomial growth exist, at the level of their Ziegler spectra they don't provide anything different to the standard algebras already considered.

Proposition 6.2.1. *[68, 6.2] Any non-standard finite-dimensional self-injective algebra of polynomial growth is socle equivalent to a unique (up to isomorphism) standard finite-dimensional self-injective algebra of polynomial growth.*

If A is a non-standard self-injective algebra of polynomial growth, the standard algebra B given by Proposition 6.2.1, that is socle equivalent to A, is called the **standard form** of A.

Corollary 6.2.2. *If A is a non-standard finite-dimensional self-injective algebra of polynomial growth, with standard form B, there exists a homeomorphism $\operatorname{Zg}^s_A \simeq \operatorname{Zg}^s_B$. Furthermore, if B is of Euclidean type, then* $\operatorname{KG}(A) = 2$.

Proof. The homeomorphism is given by Corollary 1.4.6. In the Euclidean case, that $\operatorname{KG}(A) = 2$ follows from Proposition 1.3.11 and the fact that A inherits the isolation condition from B. Indeed, a Prüfer $M[\infty]$ of Zg_A was seen to be isolated by a pp functor of the form $\operatorname{Hom}_A(M[1], -) = \operatorname{Hom}_{A/\operatorname{soc}(A)}(M[1], -)$ and, by definition, we have a Morita equivalance $\operatorname{Mod}\text{-}A/\operatorname{soc}(A) \simeq \operatorname{Mod}\text{-}B/\operatorname{soc}(B)$. Thus $M[\infty]$ is similarly isolated considered as a B-module in Zg_B. We can deal with adic modules in a similar manner and Lemma 1.3.12 deals with finite-dimensional and generic points. □

6.3 Final remarks

Let R be a finite-dimensional self-injective algebra of polynomial growth. By Proposition 6.0.1 and Corollary 6.2.2, we can use Theorem 6.1.5 to obtain a description of the Ziegler spectrum Zg_R at the level of topology (i.e. the hierarchy of closed subsets). As noted in the proof of Theorem 6.1.5, the only obstacle to obtaining a full description (including points) of Zg_R, is the preservation of indecomposability for (finitely-supported) pure-injectives by the relevant push-down functor.

This problem was solved in the non-exceptional cases, ultimately by Proposition 4.1.13 (via Corollary 4.1.21). If R is non-exceptional, then one of Theorem 6.1.10 or Theorem 6.1.18 describes the Ziegler spectrum Zg_R.

For the exceptional algebra considered in Example 6.1.6, this problem was sidestepped by noting each indecomposable pure-injective module—and its image under the push-down functor—was of a particular kind already known to be indecomposable (i.e. a string or a band module). In general, we do not know the answer to this problem. However, the exceptional self-injective algebras are well known [68, 4.9, 5.8] and, in principle, preservation of indecomposability by the relevant push-down functors could be checked on an ad hoc basis.

From an alternate perspective, given a Galois covering $F : \widehat{A} \to R$, the repetitive algebra $\widehat{A}$ can be viewed as a grading of R (see for instance [25]). The push-down functor $F_\lambda : \mathrm{Mod}\text{-}\widehat{A} \to \mathrm{Mod}\text{-}R$ is then seen to be a forgetful functor (with $\widehat{A}$-modules being viewed as graded R-modules). The question we are addressing above is then: does such a forgetful functor preserve indecomposability? Other than for finite-dimensional modules (e.g. [24, 3.2]), where Proposition 4.1.10(i) is already applicable, we know of no results concerning this problem. We speculate that the push-down functor does preserve indecomposability for all finitely-supported pure-injectives.

This aside, we may ask whether any infinitely-supported indecomposable (pure-injective) $\widehat{A}$-module exists and look for a description of $\mathrm{Zg}_{\widehat{A}}$. When A is canonical, from the results of Chapter 5, we know the closed subsets of $\mathrm{Zg}_{\widehat{A}}$ that consist of finitely-supported points. However, as $\widehat{A}$ is not finite-dimensional, the finite-dimensional points of $\mathrm{Zg}_{\widehat{A}}$ need not form a dense subset. We leave these questions as an avenue to future research.

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