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Coupled iterated function systems that contract on average

A. Chiu*† and C. Walkden*

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Abstract

We consider iterated function systems (IFSs) acting on a phase space X that, whilst not necessarily uniformly contracting, do satisfy a 'contraction on average' condition. We introduce the notion of a coupled IFS acting on a new phase space formed by taking infinite (when X is compact) or finite (when X is not compact) products, in analogy with coupled map lattices. For appropriate couplings, we prove the existence of a unique invariant probability measure for the coupled system and show that it depends continuously on the coupling as the coupling tends to zero. We also prove an ergodic theorem and a central limit theorem for the coupled IFS. The methodology is to introduce a family of transfer operators acting quasi-compactly on an appropriate function space and use results of Keller-Liverani [KL1] to prove continuity of their spectral properties in the perturbation.

1 Introduction and statement of main results

An iterated function system (IFS) comprises a phase space X with metric d, a finite or countable family of Lipschitz maps $T_j \colon X \to X$, $j \in \mathcal{S}$, and a family of probabilities $p_j \in (0,1)$ of choosing the map T_j with $\sum_{j\in\mathcal{S}} p_j = 1$. We write $\{T_j : X \to X, p_j, j \in \mathcal{S}\}$ to denote this IFS. We assume that X is a locally compact second countable metric space but is not necessarily compact. Given an initial point $x \in X$, we are interested in the random orbit

$$x,\ T_{j_1}(x),\ T_{j_2}T_{j_1}(x),\ \ldots,\ T_{j_n}\cdots T_{j_2}T_{j_1}(x),\ \ldots \tag{1}$$

where each map T_{j_i} is chosen independently with probability p_j. A useful metaphor is that applying a map corresponds to the passage of time. Often a contraction hypothesis is placed on the IFS. For example, the IFS *contracts uniformly* if $\sup_{j\in\mathcal{S}} \|T_j\|_{\mathrm{Lip}} < 1$ (here $\|T\|_{\mathrm{Lip}} = \sup_{x\neq y} d(Tx,Ty)/d(x,y)$ denotes the Lipschitz norm of T). Following, for example, [BDEG, Pe] we are interested in IFSs that satisfy a weaker hypothesis known as contraction on average. We say that the IFS *contracts on average* if $\sum_j p_j \|T_j\|_{\mathrm{Lip}} < 1$ (although see §2.1 for a precise definition). Together with some mild technical hypotheses (stated below in §2), it was proved in [BDEG] that an IFS that contracts on average possesses a unique invariant probability measure ν supported on a unique closed invariant set $\Lambda \subset X$. Elton [E] proved a pointwise ergodic theorem for the random orbit (1); specifically, for a bounded continuous $f : X \to \mathbb{R}$, for all $x \in X$, and for almost every sequence of randomly chosen maps T_{j_i}, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(T_{j_m}\cdots T_{j_1}(x)) = \int f \,\mathrm{d}\nu.$$

Peigné [Pe] proved a central limit theorem: if f is a bounded Lipschitz function and is not a coboundary (i.e. there does not exist u such that $fT_j = u - uT_j$ for all $j \in \mathcal{S}$) then there exists $\sigma^2(f) > 0$ such that

$$\frac{1}{\sqrt{n}} \sum_{m=1}^{n} f(T_{j_m}\cdots T_{j_1}(x))$$

*School of Mathematics, The University of Manchester, Oxford Road, Manchester, M13 9PL, U.K.

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converges in distribution to a normal distribution with mean $\int f \, d\nu$ and variance $\sigma^2(f)$.

As an example of an IFS, take $X = [0,1]$, $T_0(x) = x/3, T_1(x) = (x+2)/3$ and $p_0 = p_1 = 1/2$. This IFS contracts uniformly, has a unique invariant set (the middle third Cantor set), and the invariant set supports a unique invariant probability measure (the Cantor-Lebesgue measure). Note that T_0, T_1 are affine maps and are the inverse branches of a uniformly expanding map defined on an appropriate phase space.

More generally, consider an affine IFS as follows: take $X = \mathbb{R}^{\geq 0}$ to be the non-negative reals and $T_j = a_j x + b_j$, $a_j > 0$, $b_j \geq 0$, chosen with probability p_j. If $a_j > 1$ for some j, then the IFS will not contract uniformly. However, if $\sum_j p_j a_j < 1$, then the IFS does contract on average. In the latter case, under mild non-degeneracy assumptions, the non-negative real line $\mathbb{R}^{\geq 0}$ is the unique invariant set and it supports a unique invariant probability measure.

The Manneville-Pomeau map is a well-studied non-uniformly hyperbolic dynamical system ([LSV], for example). One can obtain an IFS that contracts on average from a modified version of the Manneville-Pomeau map that has an overlapping region, defined as follows. Let $X = [0,1]$ and fix $a \in (0,1)$, $b \in (1,2)$. Consider the pair of maps $f_0 \colon I_0 \to X$, $f_1 \colon I_1 \to X$ defined by $f_0(x) = x + (b-1)^{1+a} x^{1+a}$, $\quad f_1(x) = bx - (b-1)$, where $I_0 = f_0^{-1}([0,1])$, $I_1 = f_1^{-1}([0,1])$; see Figure 1. Then for $j = 0,1$, we have $|f_j'(x)| > 1$ for $x \neq 0$ but $|f_0'(0)| = 1$. Let $T_j \colon X \to I_j$ denote the inverse of f_j, $j = 0,1$. Then for any choice of probabilities p_0, p_1, the IFS formed by randomly composing T_0, T_1 contracts on average as $\sum_j p_j \|T_j\|_{\mathrm{Lip}} = p_0 + p_1 b^{-1} < 1$. Here the invariant set is X.

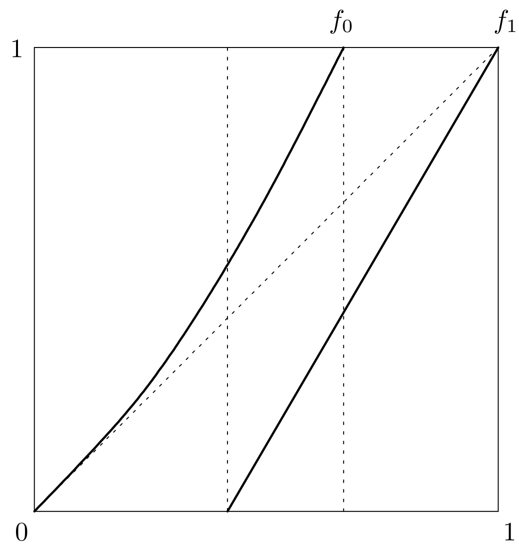

Figure 1: A modified Manneville-Pomeau map with overlapping region.

We note that we will not require a separation condition on the images $T_j(X)$. We also remark that there are examples of IFSs that contract on average but where none of the maps are themselves contractions [Pe].

In this paper we are interested in *coupled IFSs that contract on average*. These are motivated by coupled map lattices; such systems have received a great deal of attention in recent years. We do not give a full survey of known results on coupled map lattices here; instead we indicate a particular setting and some of the results that have been proved. Let $X = [0,1]$ and, for each $i \in \mathbb{Z}$, let X_i be a copy of X. We call X_i the *ith site*; a useful metaphor is that $\mathbf{x} = (x_i)_{i \in \mathbb{Z}}$ is a spatial arrangement of points $x_i \in X_i$. Let $T \colon X \to X$ be a dynamical system. Let $\Omega = \prod_{i \in \mathbb{Z}} X_i$ and define $\hat{T} \colon \Omega \to \Omega$ by $(\hat{T}(\mathbf{x}))_i = T(x_i)$ for $\mathbf{x} = (x_i)_{i \in \mathbb{Z}} \in \Omega$. In this context, Ω is called a one-dimensional lattice.

Let $A_\varepsilon : \Omega \to \Omega$ be a one-parameter family of coupling maps such that A_0 is the identity; for example, A_ε could be the nearest neighbour coupling defined in (10) below. Define $\hat{T}_\varepsilon := \hat{T} \circ A_\varepsilon$ to be a *coupled map lattice*.

If T is a piecewise expanding map on $[0,1]$ with full branches (with some technical conditions), Bunimovich and Sinai [BS] showed that if ε is sufficiently small, then there exists a $\hat{T}_\varepsilon$-invariant Borel probability measure

$\hat{\nu}_\varepsilon$ and moreover that $\hat{\nu}_\varepsilon$ is mixing. Keller and Künzle [KK] used a transfer operator approach to prove the existence of $\hat{T}_\varepsilon$-invariant measures but with the full branch condition replaced by the requirement that the branches each have slope strictly greater than 2 (the later restriction on the slope was later removed by Keller and Liverani [KL2]). Bardet, Gouëzel and Keller [BGK] proved a central limit theorem and a local limit theorem for finite-range coupled map lattices using spectral properties of a family of transfer operators.

Our model of a coupled IFS is as follows. We first define a product IFS. Let Λ denote either a finite or infinite lattice of sites. At each site $i \in \Lambda$, we have the phase space $X = X_i$ with metric d and an IFS $\{T_{i,j} \colon X_i \to X_i, p_{i,j}, j \in \mathcal{S}_i\}$. (The IFSs at each site may be different.) We form the product space $\Omega = \prod_{i\in\Lambda} X_i$. The product IFS is then formed by choosing a map from the corresponding IFS at each site, with an appropriate probability. Let $\mathbf{j} = (j_i)_{i\in\Lambda}$. Define $\hat{T}_{\mathbf{j}} : \Omega \to \Omega$ by $(\hat{T}_{\mathbf{j}}(\mathbf{x}))_i = T_{i,j_i}(x_i)$. We write $J = (\mathbf{j}^{(n)})_{n=1}^\infty = ((j_i^{(n)})_{i\in\Lambda})_{n=1}^\infty$ to denote a temporal sequence of spatial sequences of possible maps. We define $\hat{T}_{\mathbf{j}^{(n)},\ldots,\mathbf{j}^{(1)}} = \hat{T}_{\mathbf{j}^{(n)}} \cdots \hat{T}_{\mathbf{j}^{(1)}}$. A precise definition is given in §2.3.

As the examples above demonstrate, an IFS that contracts on average may have a non-compact invariant set. For technical reasons (see §4.4) this forces us into a dichotomy. When all the phase spaces X_i are compact we can consider an infinite lattice Λ (for example, $\Lambda = \mathbb{Z}$); if all the phase spaces X_i are non-compact then we are forced to only consider finite lattices Λ (for example, $\Lambda = \mathbb{Z}_k$).

Let $A_\varepsilon : \Omega \to \Omega$ be a family of Lipschitz maps such that A_0 is the identity. For example, if X is a convex subset of a vector space then A_ε could be nearest-neighbour diffusive coupling or a coupling that has exponentially decaying interactions (see §2.3.3). We define the coupled IFS to be the random composition of maps of the form $\hat{T}_{\mathbf{j}} A_\varepsilon : \Omega \to \Omega$.

Our main results can be summarised as follows: Under appropriate technical hypotheses

(i) the coupled IFS has a unique invariant measure $\hat{\nu}_\varepsilon$, and $\hat{\nu}_\varepsilon \to \hat{\nu}$ as $\varepsilon \to 0$ in the weak* topology (Theorem 2.3);

(ii) for a bounded Lipschitz function $f \colon \Omega \to \mathbb{R}$ and for every $\mathbf{x} \in \Omega$, the ergodic sums

$$S_\varepsilon^{(n)} f(\mathbf{x}, J) := \sum_{k=0}^{n-1} f(\hat{T}_{\mathbf{j}^{(k)}} A_\varepsilon \cdots \hat{T}_{\mathbf{j}^{(1)}} A_\varepsilon(\mathbf{x})).$$

satisfy a pointwise ergodic theorem for a.e. random sequence of maps (Theorem 2.4) and satisfy a central limit theorem (Theorem 2.5).

Precise statements of the results are given in §2.4 below.

This paper is organised as follows. In §2 we recap the definitions and basic properties of IFSs that contract on average; we also define the product and coupled IFS, make precise the family of couplings for which our results hold, and state precisely the technical hypotheses that are needed. We study the spectral properties of a family of transfer operators in the noncompact case in §3 and the compact case in §4. Standard arguments are then used in §5 to prove our results.

2 IFSs and coupled IFSs that contract on average

2.1 IFSs that contract on average

Let (X, d_X) be a locally compact second countable metric space. Let $T_j \colon X \to X$, $j \in \mathcal{S}$, be a finite or countably infinite family of Lipschitz maps. For each $j \in \mathcal{S}$, let $p_j \in (0,1)$ be the probability of choosing the map T_j; then $\sum_{j\in\mathcal{S}} p_j = 1$. Let $\Sigma := \{\mathbf{j} = (j_k)_{k=1}^\infty : j_k \in \mathcal{S}\}$ denote the set of all possible sequences of maps in the IFS. For $x \in X$, $\mathbf{j} \in \Sigma$ and $n \geq 1$ define $T^{(n)}(\mathbf{j}) = T_{j_n} \cdots T_{j_1}$ and $Z_n(x, \mathbf{j}) := T^{(n)}(\mathbf{j})(x)$ if $n \neq 0$ and $Z_0(x, \mathbf{j}) := x$. The transition probability from a point $x \in X$ to a Borel set $B \subset X$ is defined to be $P(x, B) := \sum_{j\in\mathcal{S}} p_j \delta_{T_j x}(B)$, where δ_y denotes the Dirac delta measure at y.

For $j_k \in \mathcal{S}$ we define the (temporal) cylinder set $[j_1, \ldots, j_n] := \{\mathbf{j} = (i_k)_{k=1}^\infty \in \Sigma : i_k = j_k \text{ for } k = 1, \ldots, n\} \subset \Sigma$. We define a Borel probability measure μ on Σ by first defining μ on cylinders by $\mu([j_1, \ldots, j_n]) = p_{j_1} \cdots p_{j_n}$ and then extending it to the Borel σ-algebra.

We say that the IFS $\{T_j, p_j, j \in \mathcal{S}\}$ *contracts on average after n_0 steps with rate* $r \in (0,1)$ if that

$$\sum_{j_1,\dots,j_{n_0}\in\mathcal{S}} \mu([j_1,\dots,j_{n_0}])\|T_{j_{n_0}}\cdots T_{j_1}\|_{\mathrm{Lip}} \le r < 1. \tag{2}$$

Write $\mathbb{E}_\mu$ to denote integration with respect to μ. Note that (2) can be more compactly written as $\mathbb{E}_\mu[\|T^{(n_0)}(\cdot)\|_{\mathrm{Lip}}] \le r < 1$.

There are examples [Pe] to show that contraction on average after n_0 steps does not imply contraction on average after 1 step and that an IFS can contract on average without any of the maps T_j being contractions. We say that the IFS *contracts on average* if it contracts on average after n_0 steps for some $n_0 \ge 1$.

The following result guarantees the existence of an invariant measure ν for the IFS and that ν has a finite first moment.

Theorem 2.1 ([BDEG])
Suppose that X is locally compact and second countable and suppose that $\{T_j\colon X \to X, p_j, j \in \mathcal{S}\}$ is a Lipschitz continuous IFS that contracts on average. Then there exists a unique invariant Borel probability measure ν on X; that is, for all Borel sets $B \subset X$,

$$\int P(x,B)\,\mathrm{d}\nu = \nu(B). \tag{3}$$

Moreover, for any $x_0 \in X$, we have $\int d(x,x_0)\,\mathrm{d}\nu < \infty$.

2.2 Product IFSs that contract on average

Let Λ be either a finite or countably infinite lattice; for notational convenience we will normally assume that in the former case $\Lambda = \mathbb{Z}_k$ and in the latter case $\Lambda = \mathbb{Z}$.

For each $i \in \Lambda$, let $X_i := X$ be a copy of X and let $\{T_{i,j} : X_i \to X_i, p_{i,j}, j \in \mathcal{S}_i\}$ be an IFS that contracts on average. For each $i \in \Lambda$, it follows from Theorem 2.1 that, for each i, each IFS $\{T_{i,j}, p_{i,j}, j \in \mathcal{S}_i\}$ has a unique invariant Borel probability measure ν_i on X_i. Let $\Sigma_i := \{(j_k)_{k=1}^\infty : j_k \in \mathcal{S}_i\}$ with corresponding probability measure μ_i. We assume that X_i has a fixed (but arbitrary) choice of origin 0_i.

We are interested in finite or infinite products of the IFS. Define the phase space to be $\Omega := \prod_{i\in\Lambda} X_i$. We let $\mathbf{0} = (0_i)_{i\in\Lambda} \in \Omega$ denote a fixed choice of origin in Ω.

We now define the product IFS. We regard the lattice as a 'spatial' direction and applying/iterating the IFS as the passage of time (a 'temporal' direction). In what follows and where appropriate, we use the convention that subscripts refer to the spatial (lattice) direction, and the superscripts in parentheses refer to the temporal direction.

Next, we define the index sets $\mathcal{S}_\Lambda$ for the product system by $\mathcal{S}_\Lambda := \{\mathbf{j} = (j_i)_{i\in\Lambda} : j_i \in \mathcal{S}_i\}$. For $\mathbf{j} \in \mathcal{S}_\Lambda$, define $\hat{T}_{\mathbf{j}}\colon \Omega \to \Omega$ by $(\hat{T}_{\mathbf{j}}(\mathbf{x}))_i = T_{i,j_i}(x_i)$, i.e. the spatial sequence $\mathbf{j}$ determines which map to apply at each site. Since it is clear from the notation that j_i refers to a map on the ith site, we abuse notation from now on by writing T_{j_i} instead of T_{i,j_i}.

Define $\Sigma_\Lambda := \{J = (\mathbf{j}^{(n)})_{n=1}^\infty : \mathbf{j}^{(n)} \in \mathcal{S}_\Lambda\}$ to be the set of all infinite sequences of possible compositions of product maps. Define the measure μ on Σ_Λ by $\mu := \otimes_{i\in\Lambda}\mu_i$.

Given $\mathbf{j}^{(1)},\dots,\mathbf{j}^{(n)} \in \mathcal{S}_\Lambda$, define a spatial-temporal cylinder to be $[\mathbf{j}^{(1)},\dots,\mathbf{j}^{(n)}] = \{(\mathbf{i}^{(m)})_{m=1}^\infty \in \Sigma_\Lambda : \mathbf{i}^{(m)} = \mathbf{j}^{(m)} \text{ for } m \le n\} \subset \Sigma_\Lambda$.

This will allow us to define the *product IFS* or *uncoupled IFS* as the collection of maps $\hat{T}_{\mathbf{j}}\colon \Omega \to \Omega, \mathbf{j} \in \mathcal{S}_\Lambda$ equipped with the probability measure μ on $\mathcal{S}_\Lambda$.

Now let $J \in \Sigma_\Lambda$. We write $Z^{(n)}(\mathbf{x}, J)$ as the image of $\mathbf{x} \in \Omega$ after applying the first n steps of $J = (\mathbf{j}^{(m)})_{m=1}^\infty$, i.e. $Z^{(n)}(\mathbf{x}, J) = \hat{T}_{\mathbf{j}^{(n)}}\cdots\hat{T}_{\mathbf{j}^{(1)}}(\mathbf{x})$. We shall also write $\hat{T}^{(n)}_{\mathbf{j}} = \hat{T}^{(n)}(\mathbf{j}) := \hat{T}_{\mathbf{j}^{(n)}}\cdots\hat{T}_{\mathbf{j}^{(1)}}$.

When the underlying metric space (X, d_X) is noncompact, we will work with the finite product space Ω endowed with the metric $d_\Omega\colon \Omega\times\Omega \to [0,\infty)$ defined by

$$d_\Omega(\mathbf{x},\mathbf{y}) := \sum_{i\in\Lambda} d_{X_i}(x_i, y_i). \tag{4}$$

When working with an infinite product space $\Omega = X^{\Lambda}$ and with the assumption that X is compact, we fix $\theta \in (0,1)$ and define the $d_\theta \colon \Omega \times \Omega \to [0,\infty)$ metric by

$$d_\theta(\mathbf{x},\mathbf{y}) := \sum_{i\in\Lambda} \theta^{|i|} d_{X_i}(x_i, y_i) \tag{5}$$

(noting that Ω is compact). Where there is no danger of confusion, we will write d for the appropriate metric.

We now define the contraction properties of the uncoupled IFS. By Lipschitz continuity of the individual IFSs, we have (in both the compact and non-compact cases) $d(\hat{T}_{\mathbf{j}}\mathbf{x}, \hat{T}_{\mathbf{j}}\mathbf{y}) \le d(\mathbf{x},\mathbf{y}) \sup_{i\in\Lambda} \|T_{j_i}\|_{\mathrm{Lip}}$, and so $\|\hat{T}_{\mathbf{j}}\|_{\mathrm{Lip}} \le \sup_{i\in\Lambda} \|T_{j_i}\|_{\mathrm{Lip}} < \infty$. In other words, $\hat{T}_{\mathbf{j}}$ is a Lipschitz continuous map when working with Ω. Then, for any $\mathbf{y}$, $\mathbf{z} \in \Omega$, we have

$$\sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) d(Z^{(n)}(\mathbf{x},J), Z^{(n)}(\mathbf{y},J)) \le \sup_{i\in\Lambda} \mathbb{E}_{\mu_i}\left[\|T_i^{(n)}(\cdot)\|_{\mathrm{Lip}}\right] d(\mathbf{x},\mathbf{y})$$

writing $\mathbb{E}_{\mu_i}$ and $\mathbb{E}_\mu$ to denote integration with respect to μ_i and μ respectively. For $J \in \Sigma_\Lambda$ define $\hat{T}^{(n)}(J) := \hat{T}_{\mathbf{j}^{(n)}} \circ \cdots \circ \hat{T}_{\mathbf{j}^{(1)}}$ so that

$$\mathbb{E}_\mu\left[\|\hat{T}^{(n)}(\cdot)\|_{\mathrm{Lip}}\right] \le \sup_{i\in\Lambda} \mathbb{E}_{\mu_i}\left[\|T_i^{(n)}(\cdot)\|_{\mathrm{Lip}}\right]. \tag{6}$$

Now consider the case when Λ is infinite. To ensure that the map $\hat{T}_{\mathbf{j}}$ is Lipschitz continuous for any $\mathbf{j} \in \mathcal{S}_\Lambda$, we need to assume that $\sup\{\|T_{i,j}\|_{\mathrm{Lip}} : i \in \Lambda,\ j \in \mathcal{S}_i\} < \infty$. Furthermore, we need to assume that $r := \sup\{r_i : i \in \Lambda\} < 1$. In addition, we also assume that there exists some $N \ge 1$ such that for all $i \in \Lambda$, $n_i \le N$. We define $\hat{n} := \mathrm{lcm}\{n_i : i \in \Lambda\}$. Therefore the ith base IFS contracts on average after $\hat{n}$ steps with average rate of contraction $r_i^{\hat{n}/n_i}$. Let $\hat{r} := \sup\{r_i^{\hat{n}/n_i} : i \in \Lambda\} < 1$; from (6) we have that

$$\mathbb{E}_\mu\left[\|\hat{T}^{(\hat{n})}(\cdot)\|_{\mathrm{Lip}}\right] \le \hat{r}. \tag{7}$$

In other words, the uncoupled IFS contracts on average after $\hat{n}$ steps with average rate of contraction $\hat{r}$.

For each $i \in \Lambda$, the IFS $\{T_{i,j} : X_i \to X_i, p_{i,j}, \mathcal{S}_i\}$ has a unique invariant Borel probability measure ν_i by Theorem 2.1. A standard argument [Ta, for example] using directed limits of product measures shows that the product IFS has a unique invariant Borel probability measure $\hat{\nu} := \otimes_{i\in\Lambda}\nu_i$. It also follows from Theorem 2.1 that

$$\int d(\mathbf{x},\mathbf{0})\,\mathrm{d}\hat{\nu} < \infty. \tag{8}$$

2.3 Coupled IFSs that contract on average

We are interested in a one-parameter family of spatial coupling maps $A_\varepsilon : \Omega \to \Omega$ with the property that $A_0 \equiv \mathrm{id}$. Recall that for $J \in \Sigma_\Lambda$ we define $\hat{T}_\varepsilon^{(n)}(J) := (\hat{T}_{\mathbf{j}^{(n)}} \circ A_\varepsilon) \circ \cdots \circ (\hat{T}_{\mathbf{j}^{(1)}} \circ A_\varepsilon)$. We define $Z_\varepsilon^{(n)}(\mathbf{x}, J) := (\hat{T}_{\mathbf{j}^{(n)}} \circ A_\varepsilon) \circ \cdots \circ (\hat{T}_{\mathbf{j}^{(1)}} \circ A_\varepsilon)(\mathbf{x})$.

Before we state the technical hypotheses on the coupling, we give a motivating example.

2.3.1 Nearest neighbour diffusive coupling and affine IFSs

To motivate the technical hypotheses on a general coupling below, we first consider nearest neighbour coupling for affine IFSs. Take $X = [0,\infty)$, $\Lambda = \mathbb{Z}_k$ and $T_{i,j}(x_i) = a_{i,j}x_i + b_{i,j}$. Assume that $0 < \sup_{i\in\Lambda, j\in\mathcal{S}_i} |a_{i,j}| < \infty$ and $b_{i,j} \ge 0$ for all $i \in \Lambda$ and $j_i \in \mathcal{S}_i$. The uncoupled system consists of maps of the form $(\hat{T}_{\mathbf{j}}(\mathbf{x}))_i = a_{j_i}x_i + b_{j_i}$. Note that

$$\|\hat{T}^{(n)}(J)\|_{\mathrm{Lip}} = \sup_{i\in\Lambda} \prod_{m=1}^{n} a_{j_i^{(m)}}. \tag{9}$$

For $\varepsilon \ge 0$, define the *nearest neighbour diffusive coupling* $A_\varepsilon : \Omega \to \Omega$ by defining it on the ith site by

$$(A_\varepsilon(\mathbf{x}))_i = (1-\varepsilon)x_i + \frac{\varepsilon}{2}(x_{i-1} + x_{i+1}). \tag{10}$$

A straightforward calculation gives the following result.

Proposition 2.2
Suppose that the base IFSs are affine and A_ε is nearest neighbour coupling. Then for all $n \in \mathbb{N}$ and any $J = (\mathbf{j}^{(m)})_{m=1}^{\infty} \in \Sigma_\Lambda$ we have

$$\|\hat{T}_\varepsilon^{(n)}(J)\|_{\mathrm{Lip}} \le (1-\varepsilon)^{n-1}\|\hat{T}^{(n)}(J)\|_{\mathrm{Lip}} + \left(1-(1-\varepsilon)^{n-1}\right) M^n,$$

where $M := \sup_{i\in\Lambda, j\in\mathcal{S}_i} |a_{i,j}|$.

If the uncoupled IFS contracts on average after $\hat{n}$ steps with average rate of contraction $\hat{r}$, it follows from Proposition 2.2 that there exists $r' \in (0,1)$ such that for all sufficiently small $\varepsilon \ge 0$,

$$\begin{aligned}\mathbb{E}_\mu\left[\|\hat{T}_\varepsilon^{(\hat{n})}(J)\|_{\mathrm{Lip}}\right] &\le (1-\varepsilon)^{\hat{n}-1}\mathbb{E}_\mu\left[\|\hat{T}^{(\hat{n})}(J)\|_{\mathrm{Lip}}\right] + \left(1-(1-\varepsilon)^{\hat{n}-1}\right) M^{\hat{n}}\\ &\le (1-\varepsilon)^{\hat{n}-1}\hat{r} + \left(1-(1-\varepsilon)^{\hat{n}-1}\right) M^{\hat{n}}\\ &\le r'.\end{aligned}$$

Hence the coupled system is also an IFS that contracts on average after $\hat{n}$ steps. Indeed, we can regard $\hat{T}_{\mathbf{j}}A_\varepsilon$ as a continuous perturbation of $\hat{T}_{\mathbf{j}}$ in ε, and so the Lipschitz norm varies continuously in ε as well.

For any $\mathbf{x}$, $\mathbf{y} \in \Omega$ we have $d(A_\varepsilon\mathbf{x}-\mathbf{x}, A_\varepsilon\mathbf{y}-\mathbf{y}) \to 0$ as $\varepsilon \to 0$. In addition, it is straightforward to check that

$$\|A_\varepsilon - \mathrm{id}_{\mathbb{R}^k}\|_{\mathrm{Lip}} = \sup_{\substack{\mathbf{x},\mathbf{y}\in\Omega\\ \mathbf{x}\neq\mathbf{y}}} \frac{d(A_\varepsilon\mathbf{x}-\mathbf{x}, A_\varepsilon\mathbf{y}-\mathbf{y})}{d(\mathbf{x},\mathbf{y})} \to 0,$$

as $\varepsilon \to 0$. Therefore $d(A_\varepsilon\mathbf{x},\mathbf{x}) = \|A_\varepsilon\mathbf{x}-\mathbf{x}\|_{\mathbb{R}^k} \le \|A_\varepsilon - \mathrm{id}_{\mathbb{R}^k}\|\|\mathbf{x}\|_{\mathbb{R}^k} \le \Phi(\varepsilon)d(\mathbf{x},\mathbf{0})$, where $\Phi(\varepsilon) > 0$ is such that $\lim_{\varepsilon\to 0}\Phi(\varepsilon) = 0$.

2.3.2 An exponentially decaying coupling

A more familiar coupling in the infinite product case is the following example. Take $X = [0,1]$. For $\varepsilon > 0$ define $A_\varepsilon : \Omega \to \Omega$ by

$$(A_\varepsilon(\mathbf{x}))_i = \left(1-\frac{\varepsilon}{2}\right)x_i + \sum_{k=1}^{\infty}\frac{\varepsilon}{2^{k+1}}(x_{i-k}+x_{i+k}).$$

Choose $\theta \in (1/2, 1)$. Then $\varepsilon \mapsto A_\varepsilon$ depends continuously on ε in the Lipschitz topology.

2.3.3 A general coupling

We now consider the general setting where, for $\varepsilon \in [0,\varepsilon_0]$, $A_\varepsilon : \Omega \to \Omega$ is a Lipschitz coupling map. We require $\lim_{\varepsilon\to 0}\|A_\varepsilon - \mathrm{id}\|_{\mathrm{Lip}} = 0$. We also require that $\lim_{\varepsilon\to 0}\sup_{\mathbf{0}\neq\mathbf{x}\in\Omega} d(A_\varepsilon\mathbf{x},\mathbf{x})/d(\mathbf{x},\mathbf{0}) = 0$ (in the noncompact case) and $\lim_{\varepsilon\to 0}\sup_{\mathbf{x}\in\Omega} d(A_\varepsilon\mathbf{x},\mathbf{x}) = 0$ (in the compact case). (These hypotheses are satisfied, for example, in the examples given above.)

It follows from the above that for any $J \in \Sigma_\Lambda$,

$$\|\hat{T}_\varepsilon^{(n)}(J)\|_{\mathrm{Lip}} \le \|\hat{T}^{(n)}(J)\|_{\mathrm{Lip}} + \kappa_n(\varepsilon)\sup_{i\in\Lambda, j\in\mathcal{S}_i}\|T_{i,j}\|_{\mathrm{Lip}}^n, \tag{11}$$

where, for each $n \in \mathbb{N}$, $\kappa_n(\varepsilon) \to 0$ as $\varepsilon \to 0$. (Proposition 2.2 guarantees this in the nearest-neighbour case.)

For $\mathbf{j} \in \mathcal{S}_\Lambda$, we let $\hat{T}_{\mathbf{j},\varepsilon} = \hat{T}_{\mathbf{j}} \circ A_\varepsilon$ and call $\{T_{\mathbf{j},\varepsilon} : \Omega \to \Omega, \mu, \mathbf{j} \in \mathcal{S}_\Lambda\}$ the *coupled IFS*. If $\varepsilon > 0$ is sufficiently small, then it follows from (11) that the coupled IFS contracts on average after $\hat{n}$ steps, i.e. there exists $r' \in (0,1)$ such that for all sufficiently small $\varepsilon > 0$, $\mathbb{E}_\mu\left[\|\hat{T}_\varepsilon^{(\hat{n})}(\cdot)\|_{\mathrm{Lip}}\right] \le r' < 1$.

2.4 Statement of results

We record here our main results about coupled IFSs; proofs are given in §5.

Let (X, d_X) be a locally compact second countable metric space. Let Λ be a finite or countable set. For each $i \in \Lambda$ suppose $X_i = X$ and let $\{T_{i,j} : X_i \to X_i, p_j, j \in \mathcal{S}_i\}$ be an IFS for which each $T_{i,j}$ is Lipschitz and which contracts on average after n_i steps with rate r_i (see (2)). We further assume that

(H0) If X is noncompact then Λ is finite;

(H1) $M := \sup_{i\in\Lambda, j\in\mathcal{S}_i} \|T_{i,j}\|_{\mathrm{Lip}} < \infty$;

(H2) $\sup_{i\in\Lambda} n_i < N$, and define $\hat{n} := \mathrm{lcm}\{n_i : i \in \Lambda\}$ and $\hat{r} := \sup_{i\in\Lambda} r_i^{\hat{n}/n_i}$;

(H3) in the noncompact case we have

$$\sup_{i\in\Lambda} \sup_{y_i\in\Omega} \sum_{j_i\in\mathcal{S}_i} p_{j_i} \frac{d_{X_i}(T_{j_i} y_i, 0_i)}{1 + d_{X_i}(y_i, 0_i)} < \infty$$

(this is a moment assumption on the $\hat{T}_{\mathbf{j}}$, cf. [Pe]);

(H4) $A_\varepsilon : \Omega \to \Omega$ is a family of Lipschitz couplings such that $\lim_{\varepsilon\to 0} \|A_\varepsilon - \mathrm{id}\|_{\mathrm{Lip}} = 0$ and

- $\displaystyle\lim_{\varepsilon\to 0} \sup_{\mathbf{0}\neq x\in\Omega} \frac{d(A_\varepsilon \mathbf{x}, \mathbf{x})}{d(\mathbf{x}, \mathbf{0})} = 0$ in the non-compact case,
- $\displaystyle\lim_{\varepsilon\to 0} \sup_{\mathbf{x}\in\Omega} d(A_\varepsilon \mathbf{x}, \mathbf{x}) = 0$ in the compact case.

Let μ denote the measure on Σ_Λ defined in §2.2.

Theorem 2.3 (Stochastic stability for the coupled IFS)
Under hypotheses (H0)–(H4), there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in [0, \varepsilon_0]$ then the coupled IFS has a unique invariant Borel probability measure $\hat{\nu}_\varepsilon$. The measure $\hat{\nu}_\varepsilon$ has finite first moment in the sense that $\int d(\mathbf{x}, \mathbf{0})\, \mathrm{d}\hat{\nu}_\varepsilon < \infty$. Moreover $\hat{\nu}_\varepsilon \rightharpoonup \hat{\nu}$, the invariant probability measure for the uncoupled IFS, in the weak topology as $\varepsilon \to 0$.*

Theorem 2.4 (Pointwise ergodic theorem for the coupled IFS)
Let $f : \Omega \to \mathbb{R}$ be a bounded Hölder function. Under hypotheses (H0)–(H4), there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in [0, \varepsilon_0)$ we have for all $\mathbf{x} \in \Omega$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(Z_\varepsilon^{(m)}(\mathbf{x}, \cdot)) = \int f\, d\hat{\nu}_\varepsilon$$

μ-almost surely.

Theorem 2.5 (Central limit theorem with error for the coupled IFS)
Let $f : \Omega \to \mathbb{R}$ be a bounded Hölder function. Under hypotheses (H0)–(H4), there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in [0, \varepsilon_0)$ and for all $\mathbf{x} \in \Omega$

$$\left| \mu\left(\left\{ \frac{1}{\sigma_\varepsilon \sqrt{n}} \sum_{m=0}^{n-1} f(Z_\varepsilon^{(m)}(\mathbf{x}, \cdot)) \leq y \right\} \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}}\, \mathrm{d}t \right| = O\left(\frac{1}{\sqrt{n}}\right)$$

(provided $\sigma_\varepsilon \neq 0$).

We have

$$\sigma_\varepsilon^2(f) = \lim_{n\to\infty} \mathbb{E}_\mu \left[\frac{1}{n} \sum_{m=0}^{n-1} \left(f(Z_\varepsilon^{(m)}(\mathbf{x}, \cdot)) - \hat{\nu}_\varepsilon(f) \right)^2 \right] = \hat{\nu}_\varepsilon(f^2) - \hat{\nu}_\varepsilon(f)^2$$

and $\varepsilon \mapsto \sigma_\varepsilon^2(f)$ is continuous. In addition, $\sigma_\varepsilon^2(f) > 0$ if and only if there does not exist a function $h : \Omega \to \mathbb{R}$ such that, for all $\mathbf{j} \in \mathcal{S}_\Lambda$, $f \circ \hat{T}_{\mathbf{j},\varepsilon} = h - h \circ \hat{T}_{\mathbf{j},\varepsilon}$.

We remark that analogues of other standard probabilistic limit theorems, such as the Local Central Limit Theorem, Almost Sure Invariance Principle, etc, also immediately follow (see [HH1], for example, for details).

2.5 Quasi-compact operators and perturbations

We will define a family of transfer operators in §3.2, §4.2 and show that they act quasi-compactly on appropriate function spaces. Here we record some standard results on quasi-compactness and spectral perturbation results for linear operators on Banach spaces.

Let L be a bounded linear operator on the (complex) Banach space $(B, \|\cdot\|)$. Recall that if L is *quasi-compact* then there are L-invariant closed subspaces $F, H \subset B$ such that $B = F \oplus H$, F is finite dimensional, the spectral radius of $L|_H : H \to H$ is strictly less than the spectral radius $\rho(L)$ of L, and all eigenvalues of $L|_F : F \to F$ have modulus $\rho(L)$.

We have the following criterion for quasi-compactness.

Theorem 2.6 ([HH1])
Let $(B, \|\cdot\|)$ be a Banach space and let $|\cdot|$ be a continuous seminorm on B. Let L be a bounded operator on B such that:

(i) the inclusion map $\iota\colon (B, \|\cdot\|) \hookrightarrow (B, |\cdot|)$ is compact;

(ii) there exists $C, M > 0$ such that for all $w \in B$ and $n \in \mathbb{N}$ we have $|L^n w| \le CM^n |w|$; and

(iii) there exists an $n \in \mathbb{N}$ and constants r, $R_n \in \mathbb{R}$ with $r < \rho(L)$ such that

$$\|L^n w\| \le r^n \|w\| + R_n |w|, \tag{12}$$

for all $w \in B$.

Then L is quasi-compact.

We will be interested in a family of linear operators. The following result gives a criterion for the spectral properties of these operators to perturb continuously.

Theorem 2.7 ([KL1])
Let $(B, \|\cdot\|)$ be a Banach space with a second seminorm $|\cdot| \le \|\cdot\|$ (where $(B, |\cdot|)$ need not be complete). Let L_τ, where $\tau \in I \subset \mathbb{R}$, $0 \in I$, be a one-parameter family of bounded linear operators on $(B, \|\cdot\|)$ and suppose that L_0 is a quasi-compact operator. For a bounded linear operator $U\colon B \to B$ define $|||U||| := \sup_{w \in B} |Uw|/\|w\|$. Suppose that:

(i) the inclusion map $\iota\colon (B, \|\cdot\|) \hookrightarrow (B, |\cdot|)$ is compact;

(ii) there exists some C, $M > 0$ such that for all $w \in B$ and for all $n \in \mathbb{N}$, $|L_0^n w| \le CM^n |w|$;

(iii) there exists an interval I_0, with $0 \in I_0$, and $n > 0$ such that L_τ^n satisfies the uniform Lasota-Yorke condition given by: there exists $\rho \in (0,1)$ and $R > 0$ such that $\|L_\tau^n w\| \le \rho\|w\| + R|w|$ for any $\tau \in I_0$ and for any $w \in B$;

(iv) $|||L_\tau - L_0||| \to 0$, as $|\tau| \to 0$.

Then there exists an interval $I_1 \subset I_0$ with $0 \in I_1$ such that if $\tau \in I_1$ then L_τ is quasi-compact.

If, in addition, L_0 has a unique simple maximal eigenvalue 1, then L_τ has a unique simple maximal eigenvalue $\tilde{\lambda}_\tau$ with corresponding eigenprojection Π_τ. For $\tau \in I_1$, the map $\tau \mapsto \tilde{\lambda}_\tau$ is continuous and $|||\Pi_\tau - \Pi_0||| \to 0$, as $\tau \to 0$. Moreover, there exists $\rho_0 < 1$ such that if Q_τ is the projection onto the remainder of the spectrum of L_τ, then $\rho(Q_\tau) \le \rho_0$ for all $\tau \in I_1$.

We note that in [KL1] the hypothesis corresponding to Theorem 2.7(ii) is stated as $|L_\tau^n w| \le CM^n|w|$ uniformly in τ. However, a careful reading of the proof shows that only the hypothesis when $\tau = 0$ is used.

The conclusions of Theorem 2.7 allow us to decompose L_τ as $L_\tau = \sum_{\lambda_\tau \in G_\tau} \lambda_\tau \Pi_{\tau,\lambda} + Q_\tau$ where G_τ is the set of eigenvalues of L_τ of modulus $\rho(L_\tau)$, $\Pi_{\tau,\lambda}$ is the eigenprojection onto to the finite dimensional eigenspace of λ_τ, and Q_τ is the projection onto the remainder of the spectrum. If, in addition, L_τ has a unique simple eigenvalue λ_τ of modulus $\rho(L_\tau)$ then we can write $L_\tau^n = \lambda_\tau^n \Pi_\tau + Q_\tau^n$.

3 The noncompact case

For this section, let (X, d_X) be a noncompact metric space that is locally compact and second countable. The methodology in this setting only works with a finite number of states (cf. §4.4). For convenience, we take $\Lambda = \mathbb{Z}_k$ and work with $\Omega := X^\Lambda$ with metric d given by (4).

Throughout, we assume the hypotheses on the IFSs given in §2.4.

3.1 Function spaces

We consider a Hölder-like space of functions on X motivated by that in [GlP]. Fix α, β, $\gamma > 0$ such that

$$0 < \alpha < \alpha + \beta < \gamma < 2\beta < a + 2\beta < 1. \tag{13}$$

In addition, for any $\lambda \in (0, 1]$ define $d_\lambda(\mathbf{x}) := 1 + \lambda d(\mathbf{x}, \mathbf{0})$. The parameter λ will be chosen below.

If $T \colon \Omega \to \Omega$ is a Lipschitz continuous map, define $\delta_\lambda(T) := \sup_{\mathbf{x}\in\Omega} d_\lambda(T(\mathbf{x}))/d_\lambda(\mathbf{x})$. The following is analogous to [HH2, Lemma 4.1] and we omit the proof.

Lemma 3.1
Suppose that $T \colon \Omega \to \Omega$ is a Lipschitz continuous map. Then $\delta_\lambda(T) \le \max\{1, \|T\|_{\mathrm{Lip}}\} + \lambda d(T(\mathbf{0}), \mathbf{0})$. In particular, for any $\lambda \in (0, 1]$ we have $\delta_\lambda(T) \le \max\{1, \|T\|_{\mathrm{Lip}}\} + d(T(\mathbf{0}), \mathbf{0})$.

For any λ_1, $\lambda_2 \in (0, 1]$ we have $d_{\lambda_1}(\mathbf{x}) = 1 + \lambda_1 d(\mathbf{x}, \mathbf{0}) \le \lambda_2^{-1} d_{\lambda_2}(\mathbf{x})$. Thus, the choice of λ does not affect the topology since any norm defined using a different choice of λ is equivalent to the original norm. We fix λ as described below.

Our main motivation is to bound expressions of the form $\mathbb{E}_\mu \left[\|\hat{T}^{(n)}(\cdot)\|^\alpha_{\mathrm{Lip}} \delta_\lambda(\hat{T}^{(n)}(\cdot))^{2\beta}\right]$. By contraction on average (2) and using $\alpha + 2\beta < 1$, for all $q \in \mathbb{N}_0$ we have

$$\mathbb{E}_\mu \left[\|\hat{T}^{(q\hat{n})}(\cdot)\|^\alpha_{\mathrm{Lip}}(1 + \|\hat{T}^{(q\hat{n})}(\cdot)\|_{\mathrm{Lip}})^{2\beta}\right] \le \mathbb{E}\left[\|\hat{T}^{(q\hat{n})}(\cdot)\|^\alpha_{\mathrm{Lip}}\right] + \mathbb{E}\left[\|\hat{T}^{(q\hat{n})}(\cdot)\|^{\alpha+2\beta}_{\mathrm{Lip}}\right] \le \hat{r}^{q\alpha} + \hat{r}^{q(\alpha+2\beta)}.$$

Clearly, there exists some sufficiently large $\widetilde{q} \in \mathbb{N}_0$ such that $\hat{r}^{\widetilde{q}\alpha} + \hat{r}^{\widetilde{q}(\alpha+2\beta)} < 1$. Set $\widetilde{n} := \widetilde{q}\hat{n}$. By Lemma 3.1 we have $\delta_\lambda(\hat{T}^{(\widetilde{n})}(J))^{2\beta} \le 1 + \|\hat{T}^{(\widetilde{n})}(J)\|^{2\beta}_{\mathrm{Lip}} + \lambda^{2\beta} d(Z^{(\widetilde{n})}(\mathbf{0}, J), \mathbf{0})^{2\beta}$. Combining the above we have, for sufficiently small $\lambda \in (0, 1]$,

$$\mathbb{E}_\mu \left[\|\hat{T}^{(\widetilde{n})}(\cdot)\|^\alpha_{\mathrm{Lip}} \delta_\lambda(\hat{T}^{(\widetilde{n})}(\cdot))^{2\beta}\right] \le \hat{r}^{\widetilde{q}\alpha} + \hat{r}^{\widetilde{q}(\alpha+2\beta)} + \lambda^{2\beta}\mathbb{E}_\mu\left[d(Z^{(\widetilde{n})}(\mathbf{0}, \cdot), \mathbf{0})^{2\beta} \|\hat{T}^{(\widetilde{n})}(\cdot)\|^\alpha_{\mathrm{Lip}}\right] \le r', \tag{14}$$

for some $r' \in (0, 1)$. From now on we fix such a λ.

Let $C(\Omega)$ be the set of real-valued functions defined on Ω. For any $w \in C(\Omega)$ we define $|\cdot|_{\alpha,\beta}$, $|\cdot|_\gamma \colon C(\Omega) \to \mathbb{R}_0^+$ by

$$|w|_{\alpha,\beta} := \sup_{\substack{\mathbf{x},\mathbf{y}\in\Omega \\ \mathbf{x}\ne\mathbf{y}}} \frac{|w(\mathbf{x}) - w(\mathbf{y})|}{d(\mathbf{x}, \mathbf{y})^\alpha d_\lambda(\mathbf{x})^\beta d_\lambda(\mathbf{y})^\beta} \quad \text{and} \quad |w|_\gamma := \sup_{\mathbf{x}\in\Omega} \frac{|w(\mathbf{x})|}{d_\lambda(\mathbf{x})^\gamma}$$

and define $\|\cdot\|_{\alpha,\beta,\gamma} \colon C(\Omega) \to \mathbb{R}_0^+$ by $\|w\|_{\alpha,\beta,\gamma} := |w|_{\alpha,\beta} + |w|_\gamma$. Then $|\cdot|_\gamma$ and $\|\cdot\|_{\alpha,\beta,\gamma}$ are both norms and $|\cdot|_{\alpha,\beta}$ is a seminorm. We then define the function spaces $C_\gamma(\Omega) := \{w \in C(\Omega) : |w|_\gamma < \infty\}$ and $C_{\alpha,\beta,\gamma}(\Omega) := \{w \in C(\Omega) : \|w\|_{\alpha,\beta,\gamma} < \infty\}$.

Let $w \in C_{\alpha,\beta,\gamma}(\Omega)$. Then $\int |w(\mathbf{x})| \,\mathrm{d}\hat{\nu} \le |w|_\gamma \int d_\lambda(\mathbf{x})^\gamma \,\mathrm{d}\hat{\nu} < \infty$, by (8). Define $|w|^{(1)} := \int |w| \,\mathrm{d}\hat{\nu}$. Define the norm $\|w\|^{(1)}_{\alpha,\beta} := |w|^{(1)} + |w|_{\alpha,\beta}$.

We have the following result.

Proposition 3.2

(i) Both $(C_\gamma(\Omega), |\cdot|_\gamma)$ and $(C_{\alpha,\beta,\gamma}(\Omega), \|\cdot\|_{\alpha,\beta,\gamma})$ are Banach spaces.

(ii) The space $(C_{\alpha,\beta,\gamma}(\Omega), \|\cdot\|^{(1)}_{\alpha,\beta})$ is a Banach space.

(iii) The norms $\|\cdot\|^{(1)}_{\alpha,\beta}$ and $\|\cdot\|_{\alpha,\beta,\gamma}$ are equivalent.

(iv) The inclusion map $\iota\colon (C_{\alpha,\beta,\gamma}(\Omega), \|\cdot\|^{(1)}_{\alpha,\beta}) \hookrightarrow (C_{\alpha,\beta,\gamma}(\Omega), |\cdot|^{(1)})$ *is compact.*

Proof. Statement (i) is a straightforward. For (ii), (iii), since Ω is the finite product of a locally compact, second countable metric space X, the arguments in [HH2, Lemma 5.3] (which proves the corresponding statements in the case for a single IFS) easily generalise; we note that the only property of $\hat{\nu}$ that is required is that $\int d(\mathbf{x},\mathbf{0})^{\alpha+2\beta}\, d\hat{\nu} < \infty$, which follows from (8) and that $\alpha + 2\beta < 1$. Statement (iv) is analogous to [HH2, Lemma 5.4]. $\square$

3.2 Transfer operators

We define the unperturbed transfer operator $\mathcal{L}$ on continuous functions $w\colon \Omega \to \mathbb{R}$ by

$$\mathcal{L}w(\mathbf{x}) := \sum_{\mathbf{j}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}])w(\hat{T}_{\mathbf{j}}\mathbf{x})$$

for $\mathbf{x} \in \Omega$. It is clear that $\mathcal{L}$ is a well-defined, positive, bounded linear operator on functions on Ω. By the Fubini-Tonelli Theorem, one can also show that the transfer operator is $\hat{\nu}$-invariant, that is, $\int \mathcal{L}w\, \mathrm{d}\hat{\nu} = \int w\, \mathrm{d}\hat{\nu}$. Moreover, it is clear that $\mathcal{L}\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the function constantly equal to 1.

For $\varepsilon \in [0, \varepsilon_0]$ and $t \in \mathbb{R}$, we define the perturbed transfer operator $\mathcal{L}_{\varepsilon,t}\colon C(\Omega) \to C(\Omega)$ by

$$\mathcal{L}_{\varepsilon,t}w(\mathbf{x}) := e^{itf(\mathbf{x})}\mathcal{L}w(A_\varepsilon\mathbf{x}) = \sum_{\mathbf{j}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}])e^{itf(\mathbf{x})}w(\hat{T}_{\mathbf{j}}A_\varepsilon\mathbf{x}).$$

Note that $\mathcal{L}_{0,0} = \mathcal{L}$.

For a spatial-temporal sequence $J = (\mathbf{j}^{(m)})_{m=1}^{\infty} \in \Sigma_\Lambda$ we define $S^{(n)}_\varepsilon f(\mathbf{x}, J) := \sum_{m=0}^{n-1} f(Z^{(m)}_\varepsilon(\mathbf{x}, J))$. Then it is easy to see that

$$\mathcal{L}^n_{\varepsilon,t}w(\mathbf{x}) = \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}])e^{itS^{(n)}_\varepsilon f(\mathbf{x},J)}w(T_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}}(\mathbf{x})) = \mathbb{E}_\mu\left[e^{itS^{(n)}_\varepsilon f(\mathbf{x},\cdot)}w(Z^{(n)}_\varepsilon(\mathbf{x},\cdot))\right].$$

3.3 Spectral properties of the ε-perturbed transfer operator

We fix $t = 0$ and consider perturbations $\varepsilon \mapsto \mathcal{L}_{\varepsilon,0}$. That hypothesis (ii) of Theorem 2.7 holds for $\mathcal{L}_{0,0}$ follows immediately from the fact that $\hat{\nu}$ is the invariant measure for the uncoupled IFS.

We now show that hypothesis (iii) of Theorem 2.7 holds for $\mathcal{L}_{\varepsilon,0}$ for ε in a sufficiently small neighbourhood of 0. We first need the following estimate.

Lemma 3.3
Let $\varepsilon \in [0, \varepsilon_0]$*. For each* n *there exists* $c_n(\varepsilon)$ *with* $\lim_{\varepsilon\to 0} c_n(\varepsilon) = 0$ *such that for all* $w \in C_{\alpha,\beta,\gamma}(\Omega)$ *we have*

$$\|\mathcal{L}^n_{\varepsilon,0}w\|^{(1)}_{\alpha,\beta} \le |w|^{(1)} + \left(\mathbb{E}_\mu\left[\|\hat{T}^{(n)}_\varepsilon(\cdot)\|^\alpha_{\mathrm{Lip}}\delta_\lambda(\hat{T}^{(n)}_\varepsilon(\cdot))^{2\beta}\right] + c_n(\varepsilon)\right)|w|_{\alpha,\beta}.$$

Proof. Let $\mathbf{x}, \mathbf{y} \in \Omega$ and assume without loss of generality that $d_\lambda(\mathbf{y}) \le d_\lambda(\mathbf{x})$. We have

$$\begin{aligned}
&|\mathcal{L}^n_{\varepsilon,0}w(\mathbf{x}) - \mathcal{L}^n_{\varepsilon,0}w(\mathbf{y})| \\
&\le \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}])|w(Z^{(n)}_\varepsilon(\mathbf{x}, J)) - w(Z^{(n)}_\varepsilon(\mathbf{y}, J))| \\
&\le \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}])|w|_{\alpha,\beta}d(Z^{(n)}_\varepsilon(\mathbf{x}, J), Z^{(n)}_\varepsilon(\mathbf{y}, J))^\alpha d_\lambda(Z^{(n)}_\varepsilon(\mathbf{x}, J))^\beta d_\lambda(Z^{(n)}_\varepsilon(\mathbf{y}, J))^\beta \\
&\le \mathbb{E}_\mu\left[\|\hat{T}^{(n)}_\varepsilon(\cdot)\|^\alpha_{\mathrm{Lip}}\delta_\lambda(\hat{T}^{(n)}_\varepsilon(\cdot))^{2\beta}\right]|w|_{\alpha,\beta}d(\mathbf{x},\mathbf{y})^\alpha d_\lambda(\mathbf{x})^\beta d_\lambda(\mathbf{y})^\beta. \qquad (15)
\end{aligned}$$

Hence $|\mathcal{L}^n_{\varepsilon,0}w|_{\alpha,\beta} \le \mathbb{E}_\mu\left[\|\hat{T}^{(n)}_\varepsilon(\cdot)\|^\alpha_{\mathrm{Lip}}\delta_\lambda(\hat{T}^{(n)}_\varepsilon(\cdot))^{2\beta}\right]|w|_{\alpha,\beta}$.

Let $R = \mathbb{E}_\mu\left[\|\hat{T}_\varepsilon(\cdot)\|^\alpha_{\mathrm{Lip}}\delta_\lambda(\hat{T}_\varepsilon(\cdot))^{2\beta}\right]$. Then $|\mathcal{L}_{\varepsilon,0}w|_{\alpha,\beta} \le R|w|_{\alpha,\beta}$.

We now estimate $|\mathcal{L}^n_{\varepsilon,0}w|^{(1)}$. First note that, by (H4), for any $\mathbf{x}\in\Omega$ we have

$$d_\lambda(A_\varepsilon\mathbf{x})\le 1+\lambda d(A_\varepsilon\mathbf{x},\mathbf{x})+\lambda d(\mathbf{x},\mathbf{0})\le d_\lambda(\mathbf{x})+\lambda\Phi(\varepsilon)d(\mathbf{x},\mathbf{0})\le(1+\Phi(\varepsilon))d_\lambda(\mathbf{x}),$$

where $\Phi(\varepsilon):=\sup_{\mathbf{0}\neq\mathbf{x}\in\Omega}d(A_\varepsilon\mathbf{x},\mathbf{x})/d(\mathbf{x},\mathbf{0})$. Recall that $\lim_{\varepsilon\to0}\Phi(\varepsilon)=0$ by (H4). Therefore

$$\begin{aligned}\Big||w(\hat{T}_{\mathbf{j},\varepsilon}(\mathbf{x}))|-|w(\hat{T}_{\mathbf{j}}(\mathbf{x}))|\Big| &\le |w|_{\alpha,\beta}d(\hat{T}_{\mathbf{j}}\circ A_\varepsilon(\mathbf{x}),\hat{T}_{\mathbf{j}}(\mathbf{x}))^\alpha d_\lambda(\hat{T}_{\mathbf{j}}\circ A_\varepsilon(\mathbf{x}))^\beta d_\lambda(\hat{T}_{\mathbf{j}}(\mathbf{x}))^\beta\\ &\le |w|_{\alpha,\beta}\|\hat{T}_{\mathbf{j}}\|^\alpha_{\mathrm{Lip}}\delta_\lambda(\hat{T}_{\mathbf{j}})^{2\beta}d(A_\varepsilon\mathbf{x},\mathbf{x})^\alpha d_\lambda(A_\varepsilon\mathbf{x})^\beta d_\lambda(\mathbf{x})^\beta\\ &\le |w|_{\alpha,\beta}\|\hat{T}_{\mathbf{j}}\|^\alpha_{\mathrm{Lip}}\delta_\lambda(\hat{T}_{\mathbf{j}})^{2\beta}\Phi(\varepsilon)^\alpha d(\mathbf{x},\mathbf{0})^\alpha(1+\Phi(\varepsilon))^\beta d_\lambda(\mathbf{x})^{2\beta}\\ &\le c(\varepsilon)|w|_{\alpha,\beta}\|\hat{T}_{\mathbf{j}}\|^\alpha_{\mathrm{Lip}}\delta_\lambda(\hat{T}_{\mathbf{j}})^{2\beta}d(\mathbf{x},\mathbf{0})^\alpha d_\lambda(\mathbf{x})^{2\beta},\end{aligned}\tag{16}$$

where $c(\varepsilon)\to0$, as $\varepsilon\to0$, and $c(\varepsilon)$ does not depend on $\mathbf{x}$.

Note that

$$\begin{aligned}|\mathcal{L}_{\varepsilon,0}w|^{(1)}&\le\int\sum_{\mathbf{j}\in\mathcal{S}_\Lambda}\mu([\mathbf{j}])|w(\hat{T}_{\mathbf{j}}\circ A_\varepsilon(x))|\mathrm{d}\hat\nu(\mathbf{x})\\ &\le\int\sum_{\mathbf{j}\in\mathcal{S}_\Lambda}\mu([\mathbf{j}])|w(\hat{T}_{\mathbf{j}}(\mathbf{x}))|\,\mathrm{d}\hat\nu(\mathbf{x})+\sum_{\mathbf{j}\in\mathcal{S}_\Lambda}\mu([\mathbf{j}])\int\Big||w(\hat{T}_{\mathbf{j},\varepsilon}(\mathbf{x}))|-|w(\hat{T}_{\mathbf{j}}(\mathbf{x}))|\Big|\,\mathrm{d}\hat\nu(\mathbf{x})\\ &\le\int\mathcal{L}|w|\,\mathrm{d}\hat\nu(\mathbf{x})+c(\varepsilon)|w|_{\alpha,\beta}\sum_{\mathbf{j}\in\mathcal{S}_\Lambda}\mu([\mathbf{j}])\|\hat{T}_{\mathbf{j}}\|^\alpha_{\mathrm{Lip}}\delta_\lambda(\hat{T}_{\mathbf{j}})^{2\beta}\int d(\mathbf{x},\mathbf{0})^\alpha d_\lambda(\mathbf{x})^{2\beta}\,\mathrm{d}\hat\nu(\mathbf{x}).\\ &\le|w|^{(1)}+c(\varepsilon)|w|_{\alpha,\beta}\sum_{\mathbf{j}\in\mathcal{S}_\Lambda}\mu([\mathbf{j}])\|\hat{T}_{\mathbf{j}}\|^\alpha_{\mathrm{Lip}}\delta_\lambda(\hat{T}_{\mathbf{j}})^{2\beta}\int d(\mathbf{x},\mathbf{0})^\alpha d_\lambda(\mathbf{x})^{2\beta}\,\mathrm{d}\hat\nu(\mathbf{x}).\end{aligned}$$

As $\alpha+2\beta<1$, this integral is finite by (8). Hence $|\mathcal{L}_{\varepsilon,0}w|^{(1)}\le|w|^{(1)}+c_1(\varepsilon)|w|_{\alpha,\beta}$ where $c_1(\varepsilon)\to0$ as $\varepsilon\to0$.

By induction one has that $|\mathcal{L}^n_{\varepsilon,0}w|^{(1)}\le|w|^{(1)}+c_1(\varepsilon)(1+R+\cdots+R^{n-1})|w|_{\alpha,\beta}$. Hence, for all n, there exists $c_n(\varepsilon)$ such that such that $|\mathcal{L}^n_{\varepsilon,0}w|^{(1)}\le|w|^{(1)}+c_n(\varepsilon)|w|_{\alpha,\beta}$ and $c_1(\varepsilon)\to0$ as $\varepsilon\to0$. $\square$

We can now prove that a uniform (in ε) Lasota-Yorke inequality holds for the perturbed (in ε) transfer operator $\mathcal{L}_{\varepsilon,0}$ provided $\varepsilon\in[0,\varepsilon_0']$ for some $\varepsilon_0'>0$.

Proposition 3.4 (Uniform (in ε) Lasota-Yorke inequality for $\mathcal{L}_{\varepsilon,0}$) *There exists $\varepsilon'>0$, $r'\in(0,1)$ and $\widetilde{n}\in\mathbb{N}$ such that, for all $\varepsilon\in[0,\varepsilon']$, $\|\mathcal{L}^{\widetilde{n}}_{\varepsilon,0}w\|^{(1)}_{\alpha,\beta}\le r'|w|_{\alpha,\beta}+|w|^{(1)}$.*

Proof. Lemma 3.3 implies that

$$\|\mathcal{L}^{q\hat{n}}_\varepsilon w\|^{(1)}_{\alpha,\beta}\le|w|^{(1)}+\left(\mathbb{E}_\mu\left[\|\hat{T}^{(q\hat{n})}_\varepsilon(\cdot)\|^\alpha_{\mathrm{Lip}}\delta_\lambda(\hat{T}^{(q\hat{n})}_\varepsilon(\cdot))^{2\beta}\right]+c_{q\hat{n}}(\varepsilon)\right)|w|_{\alpha,\beta}.$$

By contraction on average after $\widetilde{n}=\widetilde{q}\hat{n}$ steps and writing $M:=\sup_{i\in\Lambda,\,j\in\mathcal{S}_i}\|T_{i,j}\|$, we have

$$\begin{aligned}&\mathbb{E}_\mu\left[\|\hat{T}^{(\widetilde{n})}_\varepsilon(\cdot)\|^\alpha_{\mathrm{Lip}}(1+\|\hat{T}^{(\widetilde{n})}_\varepsilon(\cdot)\|^{2\beta}_{\mathrm{Lip}})\right]\\ &=\mathbb{E}_\mu\left[\|\hat{T}^{(\widetilde{n})}_\varepsilon(\cdot)\|^\alpha_{\mathrm{Lip}}\right]+\mathbb{E}_\mu\left[\|\hat{T}^{(\widetilde{n})}_\varepsilon(\cdot)\|^{\alpha+2\beta}_{\mathrm{Lip}}\right]\\ &\le\mathbb{E}_\mu\left[\|\hat{T}^{(\widetilde{n})}(\cdot)\|^\alpha_{\mathrm{Lip}}\right]+\kappa_{\widetilde{n}}(\varepsilon)^\alpha M^{\widetilde{n}\alpha}+\mathbb{E}_\mu\left[\|\hat{T}^{(\widetilde{n})}(\cdot)\|^{\alpha+2\beta}_{\mathrm{Lip}}\right]+\kappa_{\widetilde{n}}(\varepsilon)^{\alpha+\beta}M^{\widetilde{n}(\alpha+\beta)}\\ &\le\hat{r}^{q\alpha}+\hat{r}^{q(\alpha+\beta)}+\kappa_{\widetilde{n}}(\varepsilon)^\alpha M^{\widetilde{n}\alpha}+\kappa_{\widetilde{n}}(\varepsilon)^{\alpha+\beta}M^{\widetilde{n}(\alpha+\beta)}.\end{aligned}$$

For sufficiently small $\varepsilon>0$ we have

$$\hat{r}^{q\alpha}+\hat{r}^{q(\alpha+\beta)}+\kappa_{\widetilde{n}}(\varepsilon)^\alpha M^{\widetilde{n}\alpha}+\kappa_{\widetilde{n}}(\varepsilon)^{\alpha+\beta}M^{\widetilde{n}(\alpha+\beta)}<1.\tag{17}$$

Since $\hat{T}^{(\widetilde{n})}_\varepsilon(J)$ is Lipschitz, Lemma 3.1 implies that $\delta_\lambda(\hat{T}^{(\widetilde{n})}_\varepsilon(J))^{2\beta}<1+\|\hat{T}^{(\widetilde{n})}_\varepsilon(J)\|^{2\beta}_{\mathrm{Lip}}+\lambda^{2\beta}d(Z^{(\widetilde{n})}_\varepsilon(\mathbf{0},J),\mathbf{0})^{2\beta}$. Hence

$$\mathbb{E}_\mu\left[\|\hat{T}^{(\widetilde{n})}_\varepsilon(\cdot)\|^\alpha_{\mathrm{Lip}}\delta_\lambda(\hat{T}^{(\widetilde{n})}_\varepsilon(\cdot))^{2\beta}\right]\quad\le\mathbb{E}_\mu\left[\|\hat{T}^{(\widetilde{n})}_\varepsilon(\cdot)\|^\alpha_{\mathrm{Lip}}+\|\hat{T}^{(\widetilde{n})}_\varepsilon(\cdot)\|^{\alpha+2\beta}_{\mathrm{Lip}}+\lambda^{2\beta}\|\hat{T}^{(\widetilde{n})}_\varepsilon(\cdot)\|^\alpha_{\mathrm{Lip}}d(Z^{(\widetilde{n})}_\varepsilon(\mathbf{0},\cdot),\mathbf{0})^{2\beta}\right].$$

Recall that $\lambda > 0$ was determined by (14). By (17) $\varepsilon_0' > 0$ can be chosen so that if $\varepsilon \in [0, \varepsilon_0']$ then $\mathbb{E}_\mu \left[\|\hat{T}_\varepsilon^{(\widetilde{n})}(\cdot)\|_{\mathrm{Lip}}^\alpha \delta_\lambda(\hat{T}_\varepsilon^{(\widetilde{n})}(\cdot))^{2\beta} \right] + c_{\widetilde{n}}(\varepsilon) < r'$ for some $r' \in (0,1)$. Hence $\|\mathcal{L}_\varepsilon^{\widetilde{n}} w\|_{\alpha,\beta}^{(1)} \leq r'|w|_{\alpha,\beta} + |w|^{(1)}$, for all $\varepsilon \in [0, \varepsilon_0']$. $\square$

Finally, we show that hypothesis (iii) of Theorem 2.7 holds for $\mathcal{L}_{\varepsilon,0}$ as $\varepsilon \to 0$.

Lemma 3.5
We have $|||\mathcal{L}_{\varepsilon,0} - \mathcal{L}||| \to 0$ *as* $\varepsilon \to 0$.

Proof. Let $w \in C_{\alpha,\beta,\gamma}$. Then

$$\begin{aligned}
|\mathcal{L}_{\varepsilon,0} w - \mathcal{L} w|^{(1)} &\leq \sum_{\mathbf{j} \in \mathcal{S}_\Lambda} \mu([\mathbf{j}]) \int \left| w(\hat{T}_{\mathbf{j},\varepsilon}(\mathbf{x})) - w(\hat{T}_{\mathbf{j}}(\mathbf{x})) \right| \mathrm{d}\hat{\nu}(\mathbf{x}) \\
&\leq \Phi(\varepsilon)^\alpha (1+\Phi(\varepsilon))^\beta \sum_{\mathbf{j} \in \mathcal{S}_\Lambda} \mu([\mathbf{j}]) \|\hat{T}_{\mathbf{j}}\|_{\mathrm{Lip}}^\alpha \delta_\lambda(\hat{T}_{\mathbf{j}})^{2\beta} \int d(\mathbf{x},\mathbf{0})^\alpha d_\lambda(\mathbf{x})^{2\beta} \, \mathrm{d}\hat{\nu}(x) \\
&\leq \Phi(\varepsilon)^\alpha (1+\Phi(\varepsilon))^\beta \mathbb{E}_\mu \left[\|\hat{T}_{\mathbf{j}}\|_{\mathrm{Lip}}^\alpha \delta_\lambda(\hat{T}_{\mathbf{j}})^{2\beta} \right] \int d(\mathbf{x},\mathbf{0})^\alpha d_\lambda(\mathbf{x})^{2\beta} \, \mathrm{d}\hat{\nu}(x)
\end{aligned}$$

where we have used a bound similar to (16). As $\alpha + 2\beta < 1$, the integral is finite by (8). Hence $|\mathcal{L}_{\varepsilon,0} w - \mathcal{L} w|^{(1)} \leq \tau(\varepsilon) \|w\|_{\alpha,\beta}^{(1)}$ where $\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$. $\square$

3.4 Simplicity of the maximal eigenvalue

Let $\mathbf{1}$ denote the function that is constantly equal to 1. Clearly $\mathbf{1}$ is an eigenvalue for $\mathcal{L}$ with eigenvalue 1 and it is clear that the spectral radius of $\mathcal{L}$ is 1. We show that 1 is a simple eigenvalue and that there are no other eigenvalues of modulus 1. The argument is very similar to those in [Pe] and we sketch the details. As $\mathcal{L}$ is quasi-compact, we decompose $\mathcal{L} = \sum_{\lambda \in G} \lambda \Pi_\lambda + Q$ where G is the set of eigenvalues of modulus 1 and Π_λ denotes projection onto the eigenspace with eigenvalue λ.

Lemma 3.6
Let w be a bounded function with $\mathcal{L} w = w$. For each $\mathbf{x}$ there exists $S_{\mathbf{x}} \subset \Sigma_\Lambda$ with $\mu(S_{\mathbf{x}}) = 1$ such that for all $J \in S_{\mathbf{x}}$

(i) $\lim_{n\to\infty} w(Z^{(n)}(\mathbf{x}, J))$ *exists;*

(ii) $\liminf_{n\to\infty} d(\mathbf{0}, Z^{(n)}(\mathbf{x}, J)) < \infty$*;*

(iii) for each $m \in \mathbb{N}$

$$\lim_{n\to\infty} \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(m)}} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(m)}]) \left(w(\hat{T}_{\mathbf{j}^{(m)}} \ldots \hat{T}_{\mathbf{j}^{(1)}} Z^{(n)}(\mathbf{x}, J)) - w(Z^{(n)}(\mathbf{x}, J)) \right)^2 \to 0.$$

Proof. Let $\mathcal{F}^{(n)}$ denote the σ-algebra of Σ_Λ generated by temporal cylinders $[\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]$. Then $\mathbb{E}_\mu[w(Z^{(n+1)}(\mathbf{x},\cdot) \mid \mathcal{F}^{(n)}] = \mathcal{L} w(Z^{(n)}(\mathbf{x},\cdot)) = w(Z^{(n)}(\mathbf{x},\cdot))$. Hence $w(Z^{(n)}(\mathbf{x},\cdot))$ is a martingale with respect to $\mathcal{F}^{(n)}$ and (i) follows from Doob's Martingale Convergence Theorem.

For (ii), note that

$$\begin{aligned}
\mathbb{E}_\mu \left[d(\mathbf{0}, Z^{(n)}(\mathbf{0},\cdot)) \right] &\leq \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}} \sum_{m=1}^n \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) d(\hat{T}_{\mathbf{j}^{(n)}} \cdots \hat{T}_{\mathbf{j}^{(m+1)}} \mathbf{0}, \hat{T}_{\mathbf{j}^{(n)}} \cdots \hat{T}_{\mathbf{j}^{(m)}} \mathbf{0}) \\
&\leq \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}} \sum_{m=1}^n \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) \|\hat{T}_{\mathbf{j}^{(n)}} \cdots \hat{T}_{\mathbf{j}^{(m+1)}}\|_{\mathrm{Lip}} d(\hat{T}^{(m)} \mathbf{0}, \mathbf{0}) \\
&\leq \mathbb{E}_\mu \left[d(Z^{(1)}(\mathbf{0},\cdot)) \right] \sum_{m=1}^n \mathbb{E}_\mu \left[\|\hat{T}^{(m)}(\cdot)\|_{\mathrm{Lip}} \right].
\end{aligned}$$

Noting that $d(\mathbf{0}, Z^{(n)}(\mathbf{x}, J)) \leq d(\mathbf{0}, Z^{(n)}((0), J)) + d(Z^{(n)}((0), J), Z^{(n)}((x), J))$ and letting $n \to \infty$ we have

$$\lim_{n\to\infty} \mathbb{E}_\mu \left[d(\mathbf{0}, Z^{(n)}(\mathbf{0}, \cdot)) \right] \leq \mathbb{E}_\mu \left[d(Z^{(1)}(\mathbf{0}, \cdot)) \right] \sum_{m=1}^{\infty} \mathbb{E}_\mu \left[\|\hat{T}^{(m)}(\cdot)\|_{\mathrm{Lip}} \right]$$

which is finite by (7) and (H3). Hence (ii) holds.

It is straightforward to check that

$$\begin{aligned} &\mathbb{E}_\mu \left[\sum_{n=0}^{N} \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(m)} \in \mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)}, \ldots, \mathbf{j}^{(m)}]) \left(w(\hat{T}_{\mathbf{j}^{(m)}} \ldots \hat{T}_{\mathbf{j}^{(1)}} Z^{(n)}(\mathbf{x}, J)) - w(Z^{(n)}(\mathbf{x}, J)) \right)^2 \right] \\ &= \mathbb{E}_\mu \left[w(Z^{(n+m)}(\mathbf{x}, \cdot))^2 \right] - \mathbb{E}_\mu \left[w(Z^{(n)}(\mathbf{x}, \cdot))^2 \right] \leq 2m\|w\|_\infty, \end{aligned}$$

from which (iii) follows. $\square$

Lemma 3.7
The eigenvalue $\lambda = 1$ is the only eigenvalue of $\mathcal{L}$ of maximal modulus 1 and the eigenspace is 1-dimensional consisting of the constant functions.

In particular, the eigenprojection Π_1 can be identified with a unique $\mathcal{L}$-invariant probability measure $\hat{\nu}$, which has a finite first moment, i.e.

$$\int d(\mathbf{x}, \mathbf{0}) \, \mathrm{d}\hat{\nu}(\mathbf{x}) < \infty. \tag{18}$$

Moreover, we can write

$$\mathcal{L}^n = \hat{\nu} + Q^n \tag{19}$$

where $\|Q^n\|_{\alpha,\beta,\gamma} = o(r'^n)$ where $r' \in (0, 1)$.

Proof. We first show that any bounded eigenfunction with eigenvalue 1 is constant. Suppose that $\mathcal{L}w = w$, where $w \in C_{\alpha,\beta,\gamma}(\Omega)$ is bounded. Let $\mathbf{x}$, $\mathbf{y} \in \Omega$ and let $S_{\mathbf{x}}$, $S_{\mathbf{y}}$ be as in Lemma 3.6(i). By Lemma 3.6(ii), for each $J \in S_{\mathbf{x}}$ there exists a subsequence $n_m := n_m(J)$ such that $Z^{(n_m)}(\mathbf{x}, J) \to \mathbf{x}^{(J)}$, as $m \to \infty$, for some $\mathbf{x}^{(J)} \in \Omega$. By Lemma 3.6(iii) and continuity of w, for all $N \in \mathbb{N}$ we have $w(\mathbf{x}^{(J)}) = w(\hat{T}_{\mathbf{j}^{(N)}} \ldots \hat{T}_{\mathbf{j}^{(1)}} \mathbf{x}^{(J)})$, noting that $\mu([\mathbf{j}^{(1)}, \ldots, \mathbf{j}^{(N)}]) > 0$. The same argument holds for $\mathbf{y}$ so that for $K \in S_{\mathbf{y}}$ there is a corresponding $\mathbf{y}^{(K)}$.

For all $N \in \mathbb{N}$,

$$\begin{aligned} |w(\mathbf{x}^{(J)}) - w(\mathbf{y}^{(K)})| &\leq |w|_{\alpha,\beta} \|\hat{T}_{\mathbf{j}^{(N)}} \ldots \hat{T}_{\mathbf{j}^{(1)}}\|_{\mathrm{Lip}}^{\alpha} d(\mathbf{x}^{(J)}, \mathbf{y}^{(K)})^\alpha \delta_\lambda(\hat{T}_{\mathbf{j}^{(N)}} \ldots \hat{T}_{\mathbf{j}^{(1)}})^{2\beta} d_\lambda(\mathbf{x}^{(J)})^\beta d_\lambda(\mathbf{y}^{(K)})^\beta \\ &\leq |w|_{\alpha,\beta} c(\mathbf{x}^{(J)}, \mathbf{y}^{(K)}) \mathbb{E}_\mu \left[\|\hat{T}^{(N)}(\cdot)\|_{\mathrm{Lip}}^{\alpha} \delta_\lambda(\hat{T}^{(N)}(\cdot))^{2\beta} \right], \end{aligned}$$

for some constant $c(\mathbf{x}^{(J)}, \mathbf{y}^{(K)})$ is a constant. If $N = \widetilde{n}$, where $\widetilde{n}$ is as in (14), then there is some $r' \in (0, 1)$ so that the above quantity is bounded above by $|w|_{\alpha,\beta} c(\mathbf{x}^{(J)}, \mathbf{y}^{(K)})(r')^q$; note that this tends to 0 as $q \to \infty$. Therefore $w(\mathbf{x}^{(J)}) = w(\mathbf{y})^{(K)}$ for any $J \in S_{\mathbf{x}}$ and $K \in S_{\mathbf{y}}$. Using the continuity of w and $\mathcal{L}w = w$, it follows that $w(\mathbf{x}) = w(\mathbf{y})$ for all $\mathbf{x} \in \Omega$, hence w is constant.

Now suppose that $w \in C_{\alpha,\beta,\gamma}(\Omega)$ is unbounded but $\mathcal{L}w = w$. For $c \in \mathbb{R}$ define $w_c(\mathbf{x}) = w(\mathbf{x})$ if $|w(\mathbf{x})| \leq c$ and $w_c(\mathbf{x}) = cw(\mathbf{x})/|w(\mathbf{x})|$ if $|w(\mathbf{x})| \leq c$.

Let $M_n^{(\lambda)} := n^{-1} \sum_{m=0}^{n-1} \lambda^{-m} \mathcal{L}^m$. Then $M_n^\lambda \to \Pi_\lambda$ in the operator norm on linear operators on $C_{\alpha,\beta,\gamma}$.

We have $|M_n^{(1)} w_c(\mathbf{x}) - \Pi_1 w_c(\mathbf{x})| \leq \|M_n^{(1)} - \Pi_1\| |w|_\gamma d_\lambda(\mathbf{x})^\gamma$. By Dominated Convergence we have that $\lim_{c\to\infty} M_n^{(1)} w_c = M_n^{(1)} w$ and $\Pi_1 w = \lim_{c\to\infty} w_c \Pi_1 w_c$ is constant.

We now show that 1 is the only eigenvalue of modulus 1. Suppose $\lambda \neq 1$, $|\lambda| = 1$, is an eigenvalue of $\mathcal{L}$. Define the operator $\hat{\mathcal{L}}$ on functions $W : \Omega \times \mathbb{N} \to \mathbb{R}$ by

$$\hat{\mathcal{L}} W(\mathbf{x}, n) := \sum_{\mathbf{j} \in \mathcal{S}_\Lambda} \mu([\mathbf{j}]) W(\hat{T}_{\mathbf{j}}(\mathbf{x}), n + 1).$$

Take $W(\mathbf{x}, n) = \lambda^{-n} w(\mathbf{x})$. Then $\hat{\mathcal{L}} W = W$ if and only if $\mathcal{L} w = \lambda w$. Using a similar method as above, and noting that $\lim_{n\to\infty} M_n^{(\lambda)} = 0$ in the operator topology, one can easily check that $w \equiv 0$. Hence 1 is the only eigenvalue of modulus 1.

To see that the eigenvalue 1 has multiplicity 1, suppose that there exists $w \in \ker(\mathcal{L} - I)^n$ but $(\mathcal{L} - I)w = v \neq 0$. Note that

$$\mathcal{L}^n w = \sum_{\ell=0}^{n} \binom{n}{\ell} (\mathcal{L} - I)^\ell w = w + n(\mathcal{L} - I)w = w - nv.$$

Hence $|\mathcal{L}^n w|^{(1)} = |w - nv|^{(1)} \to \infty$ as $n \to \infty$, contradicting the fact that $|\mathcal{L}|^{(1)} = 1$.

To show that Π_1 can be identified with the invariant measure $\hat{\nu}$, consider the subspace $C^c_{\alpha,\beta,\gamma}(\Omega)$ of compactly supported functions. By quasi-compactness of $\mathcal{L}$, we have, for any $\mathbf{x} \in \Omega$, $|\Pi_1 w| \le |\mathcal{L}^n w|(\mathbf{x})| + |Q^{(n)} w(\mathbf{x})|$. Noting that $\|Q^n\|_{\alpha,\beta,\gamma} \to 0$ exponentially fast, it follows that $|\Pi_1 w| \le \|w\|_\infty$. Clearly, Π_1 is positive, linear and $\Pi_1 \mathbf{1} = \mathbf{1}$. As $C^c_{\alpha,\beta,\gamma}(\Omega)$ is a uniformly dense subalgebra of the continuous compactly supported functions on Ω, the Riesz Representation Theorem implies that Π_1 corresponds to an $\mathcal{L}$-invariant Borel probability measure.

For the first moment of $\hat{\nu}$, we have

$$\int d(\mathbf{x}, \mathbf{0}) \mathrm{d}\hat{\nu}(\mathbf{x}) = \sum_{\mathbf{j}\in\mathcal{S}} \mu([\mathbf{j}]) d(\hat{T}_{\mathbf{j}}\mathbf{0}, \mathbf{0}) - Q(d(\cdot, \mathbf{0}))(\mathbf{0}) \le \sup_{\mathbf{y}\in\Omega} \sum_{\mathbf{j}\in\mathcal{S}} \mu([\mathbf{j}]) \frac{d(\hat{T}_{\mathbf{j}}\mathbf{y}, \mathbf{0})}{1 + d(\mathbf{y}, \mathbf{0})} - Q(d(\cdot, \mathbf{0}))(\mathbf{0}),$$

which is finite by Hypothesis (H3). □

Note that $\mathcal{L}_{\varepsilon,0}\mathbf{1} = \mathbf{1}$. The argument above (with $\mathcal{L}_{\varepsilon,0}$ replacing $\mathcal{L}$) shows that the conclusions of Lemma 3.7 also hold for $\mathcal{L}_{\varepsilon,0}$ and $\hat{\nu}_\varepsilon$. In particular, $\hat{\nu}_\varepsilon$ has finite first moment.

3.5 Spectral properties of the t-perturbed transfer operator

We now fix $\varepsilon \in [0, \varepsilon_0]$ and consider the transfer operators $\mathcal{L}_{\varepsilon,t}$ for the coupled system as a perturbation in t. Define $|w|^{(1,\varepsilon)} := \int |w| \, \mathrm{d}\hat{\nu}_\varepsilon$ and equip $C_{\alpha,\beta,\gamma}(\Omega)$ with the norm $\|w\|^{(1),\varepsilon}_{\alpha,\beta} := |w|^{(1,\varepsilon)} + |w|_{\alpha,\beta}$. The same argument as in Proposition 3.2 shows that $(C_{\alpha,\beta,\gamma}(\Omega), \|\cdot\|^{(1,\varepsilon)}_{\alpha,\beta})$ is a Banach space, the norms $\|\cdot\|^{(1,\varepsilon)}_{\alpha,\beta}$, $\|\cdot\|_{\alpha,\beta,\gamma}$ are equivalent, and that the inclusion map $\iota : (C_{\alpha,\beta,\gamma}(\Omega), \|\cdot\|^{(1,\varepsilon)}_{\alpha,\beta}) \hookrightarrow (C_{\alpha,\beta,\gamma}(\Omega), |\cdot|^{(1,\varepsilon)})$ is compact. We show that the hypotheses of Theorem 2.7 hold for $\mathcal{L}_{\varepsilon,t}$ (perturbing t) with respect to these norms.

Note that $\int \mathcal{L}_{\varepsilon,0} w \, \mathrm{d}\hat{\nu}_\varepsilon = \int w \, \mathrm{d}\hat{\nu}_\varepsilon$ for all $w \in L^1(\Omega)$. It follows immediately that $|\mathcal{L}^n_{\varepsilon,0} w|^{(1,\varepsilon)} \le |w|^{(1,\varepsilon)}$. Hence hypothesis (ii) of Theorem 2.7 holds.

To show that hypothesis (iii) of Theorem 2.7 holds we need the following estimates.

Lemma 3.8
For $\beta \in (0, 1)$ we have

$$B^{(n)}_\varepsilon(\beta) := \sup_{\mathbf{y}\in\Omega} \mathbb{E}_\mu \left[\frac{d_\lambda(Z^{(n)}_\varepsilon(\mathbf{y}, \cdot))^\beta}{d_\lambda(\mathbf{y})^\beta} \right] < \infty.$$

Proof. Recall that the (uncoupled) IFS contracts on average after $\hat{n}$ steps. Set

$$R := \sup_{\ell=0,\ldots,\hat{n}-1} \sup_{\mathbf{y},\mathbf{z}\in\Omega, \mathbf{y}\neq\mathbf{z}} \mathbb{E}_\mu \left[\frac{d(Z^{(\ell)}_\varepsilon(\mathbf{y}, \cdot), Z^{(\ell)}_\varepsilon(\mathbf{z}, \cdot))}{d(\mathbf{y}, \mathbf{z})} \right] < \infty$$

and write $M := \sup_{i\in\Lambda, j\in\mathcal{S}_i} \|T_{i,j}\|_{\mathrm{Lip}}$. By (11), there exists $a \in (0, 1)$ and $c > 0$ such that for all $\mathbf{y}$, $\mathbf{z} \in \Omega$ and $n \in \mathbb{N}_0$,

$$\begin{aligned} \mathbb{E}_\mu[d(Z^{(n)}_\varepsilon(\mathbf{y}, \cdot), Z^{(n)}_\varepsilon(\mathbf{z}, \cdot))] &\le \left(\mathbb{E}_\mu \left[\|\hat{T}^{(n)}(\cdot)\|_{\mathrm{Lip}} \right] + \kappa_n(\varepsilon) M^n \right) d(\mathbf{y}, \mathbf{z}) \\ &\le (R \hat{r}^{\lfloor \frac{n}{\hat{n}} \rfloor} + \kappa_n(\varepsilon) M^n) d(\mathbf{y}, \mathbf{z}) \\ &\le (c a^n + \kappa_n(\varepsilon) M^n) d(\mathbf{y}, \mathbf{z}). \end{aligned} \tag{20}$$

By the triangle inequality and Jensen's inequality, we have

$$\mathbb{E}_\mu[d_\lambda(Z_\varepsilon^{(n)}(\mathbf{y},\cdot))^\beta] \le 1 + \lambda^\beta \mathbb{E}_\mu[d(Z_\varepsilon^{(n)}(\mathbf{y},\cdot), Z_\varepsilon^{(n)}(\mathbf{0},\cdot))]^\beta + \lambda^\beta \mathbb{E}_\mu[d(Z_\varepsilon^{(n)}(\mathbf{0},\cdot),\mathbf{0})]^\beta. \tag{21}$$

To bound this quantity, first consider

$$\begin{aligned}
\mathbb{E}_\mu[d(Z_\varepsilon^{(n)}(\mathbf{0},\cdot),\mathbf{0})] &\le \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) d(\hat{T}_{\mathbf{j}^{(n)},\varepsilon}\ldots\hat{T}_{\mathbf{j}^{(1)},\varepsilon}\mathbf{0},\mathbf{0}) \\
&\le \sum_{\ell=0}^{n-1} \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) d(\hat{T}_{\mathbf{j}^{(n)},\varepsilon}\ldots\hat{T}_{\mathbf{j}^{(\ell)},\varepsilon}\mathbf{0}, \hat{T}_{\mathbf{j}^{(n)},\varepsilon}\ldots\hat{T}_{\mathbf{j}^{(\ell+1)},\varepsilon}\mathbf{0}) \\
&\le \mathbb{E}_\mu[d(Z_\varepsilon^{(1)}(\mathbf{0},\cdot),\mathbf{0})] \left(\sum_{\ell=0}^{n-1} ca^\ell + \kappa_\ell(\varepsilon)M^\ell\right),
\end{aligned}$$

using (20); this is clearly bounded.

By (20) we have

$$\mathbb{E}_\mu[d(Z_\varepsilon^{(n)}(\mathbf{y},\cdot), Z_\varepsilon^{(n)}(\mathbf{0},\cdot))]^\beta \le (ca^n + \kappa_n(\varepsilon)M^n)^\beta d(\mathbf{y},\mathbf{0})^\beta.$$

The result follows by dividing by $d(\mathbf{y},\mathbf{0})^\beta$. ☐

We need the following analogue of Lemma 3.3.

Lemma 3.9
Let $w \in C_{\alpha,\beta,\gamma}(\Omega)$ and $\varepsilon \in [0,\varepsilon_0']$. Then there exists $t_\varepsilon > 0$ such that if $|t| < t_\varepsilon$ then

$$\|\mathcal{L}_{\varepsilon,t}^n w\|_{\alpha,\beta}^{(1,\varepsilon)} \le \left(\mathbb{E}_\mu\left[\|\hat{T}_\varepsilon^{(n)}(\cdot)\|_{\mathrm{Lip}}^\alpha \delta_\lambda(\hat{T}_\varepsilon^{(n)}(\cdot))^{2\beta}\right] + R_{n,\varepsilon}|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta}\right)|w|_{\alpha,\beta} + \left(1 + R_{n,\varepsilon}|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta}\right)|w|^{(1,\varepsilon)}.$$

Proof. We may assume without loss of generality that $d_\lambda(\mathbf{y}) \le d_\lambda(\mathbf{x})$. We have

$$\begin{aligned}
&|\mathcal{L}_{\varepsilon,t}^n w(\mathbf{x}) - \mathcal{L}_{\varepsilon,t}^n w(\mathbf{y})| \\
&= \left|\sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) \left(e^{itS_\varepsilon^{(n)}f(\mathbf{x},J)} w(Z_\varepsilon^{(n)}(\mathbf{x},J)) - e^{itS_\varepsilon^{(n)}f(\mathbf{y},J)} w(Z_\varepsilon^{(n)}(\mathbf{y},J))\right)\right| \\
&\le \Sigma_f^{(n)} + \Sigma_w^{(n)},
\end{aligned}$$

where

$$\Sigma_w^{(n)} := \left|\sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) e^{itS_\varepsilon^{(n)}f(\mathbf{x},J)} \left(w(Z_\varepsilon^{(n)}(\mathbf{x},J)) - w(Z_\varepsilon^{(n)}(\mathbf{y},J))\right)\right|$$

and

$$\Sigma_f^{(n)} := \left|\sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) \left(e^{itS_\varepsilon^{(n)}f(\mathbf{x},J)} - e^{itS_\varepsilon^{(n)}f(\mathbf{y},J)}\right) w(Z_\varepsilon^{(n)}(\mathbf{y},J))\right|.$$

Equation (15) gives the relevant bound for $\Sigma_w^{(n)}$.

We have

$$\begin{aligned}
\Sigma_f^{(n)} &\le \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) \left|e^{itS_\varepsilon^{(n)}f(\mathbf{x},J)} - e^{itS_\varepsilon^{(n)}f(\mathbf{y},J)}\right| |w(Z_\varepsilon^{(n)}(\mathbf{y},J))| \\
&\le |w|_\gamma \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) \left|e^{it(S_\varepsilon^{(n)}f(\mathbf{x},J) - S_\varepsilon^{(n)}f(\mathbf{y},J))} - 1\right| d_\lambda(Z_\varepsilon^{(n)}(\mathbf{y},J))^\gamma \\
&\le |w|_\gamma \sum_{\ell=1}^{n} \Sigma_f^{(n),\ell},
\end{aligned}$$

where

$$\Sigma_f^{(n),\ell} := \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) \left| e^{it(f(Z_\varepsilon^{(\ell)}(\mathbf{x},J))-f(Z_\varepsilon^{(\ell)}(\mathbf{y},J)))} - 1 \right| d_\lambda(Z_\varepsilon^{(n)}(\mathbf{y},J))^\gamma.$$

Recall that, for any $\xi \in (0,1)$ and any $x \in \mathbb{R}$, we have $|e^{ix}-1| < 2|x|^\xi$. Therefore

$$\begin{aligned}
\Sigma_f^{(n),\ell} &\leq 2|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta} \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}]) d(Z_\varepsilon^{(\ell)}(\mathbf{x},J), Z_\varepsilon^{(\ell)}(\mathbf{y},J))^\alpha d_\lambda(Z_\varepsilon^{(n)}(\mathbf{y},J))^\gamma \\
&\leq 2|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta} \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(\ell)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(\ell)}]) d(Z_\varepsilon^{(\ell)}(\mathbf{x},J), Z_\varepsilon^{(\ell)}(\mathbf{y},J))^\alpha d_\lambda(Z_\varepsilon^{(\ell)}(\mathbf{y},J))^\gamma \\
&\quad \times \sum_{\mathbf{j}^{(\ell+1)},\ldots\mathbf{j}^{(n)}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}^{(\ell+1)},\ldots,\mathbf{j}^{(n)}]) \frac{d_\lambda(\hat{T}_{\mathbf{j}^{(n)},\varepsilon}\ldots\hat{T}_{\mathbf{j}^{(\ell+1)},\varepsilon}(Z_\varepsilon^{(\ell)}(\mathbf{y},J)))^\gamma}{d_\lambda(Z_\varepsilon^{(\ell)}(\mathbf{y},J))^\gamma} \\
&\leq 2|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta} B_\varepsilon^{(n-\ell)}(\gamma)\mathbb{E}_\mu\left[\|\hat{T}_\varepsilon^{(\ell)}(\cdot)\|_{\mathrm{Lip}}^\alpha \delta_\lambda(\hat{T}_\varepsilon^{(\ell)}(\cdot))^\gamma\right] d(\mathbf{x},\mathbf{y})^\alpha d_\lambda(\mathbf{y})^\gamma \\
&\leq 2|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta} B_\varepsilon^{(n-\ell)}(\gamma)\mathbb{E}_\mu\left[\|\hat{T}_\varepsilon^{(\ell)}(\cdot)\|_{\mathrm{Lip}}^\alpha \delta_\lambda(\hat{T}_\varepsilon^{(\ell)}(\cdot))^{2\beta}\right] d(\mathbf{x},\mathbf{y})^\alpha d_\lambda(\mathbf{x})^\beta d_\lambda(\mathbf{y})^\beta,
\end{aligned}$$

since $\gamma < 2\beta$ and by the assumption that $d_\lambda(\mathbf{y}) \leq d_\lambda(\mathbf{x})$. Since $\alpha + 2\beta < 1$, the integral in this upper bound is finite. Hence

$$\Sigma_f^{(n)} \leq c(n,\varepsilon)|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta}|w|_\gamma d(\mathbf{x},\mathbf{y})^\alpha d_\lambda(\mathbf{x})^\beta d_\lambda(\mathbf{y})^\beta,$$

where $c(n,\varepsilon) < \infty$ does not depend on $\mathbf{x}$, $\mathbf{y}$ or w. By the equivalence of the $\|\cdot\|_{\alpha,\beta,\gamma}$ and $\|\cdot\|_{\alpha,\beta}^{(1,\varepsilon)}$ norms, we have

$$\begin{aligned}
|\mathcal{L}_{\varepsilon,t}^n w|_{\alpha,\beta} &\leq \mathbb{E}_\mu\left[\|\hat{T}_\varepsilon^{(n)}(\cdot)\|_{\mathrm{Lip}}^\alpha \delta_\lambda(\hat{T}_\varepsilon^{(n)}(\cdot))^{2\beta}\right]|w|_{\alpha,\beta} + c'(n,\varepsilon)|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta}|w|_\gamma \\
&\leq \left(\mathbb{E}_\mu\left[\|\hat{T}_\varepsilon^{(n)}(\cdot)\|_{\mathrm{Lip}}^\alpha \delta_\lambda(\hat{T}_\varepsilon^{(n)}(\cdot))^{2\beta}\right] + R_{n,\varepsilon}|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta}\right)|w|_{\alpha,\beta} + R_{n,\varepsilon}|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta}|w|^{(1,\varepsilon)}, \qquad (22)
\end{aligned}$$

for some $R_{n,\varepsilon}$ not depending on w.

Note that $|\mathcal{L}_{\varepsilon,t}^n w|^{(1,\varepsilon)} \leq |w|^{(1,\varepsilon)}$. Combining this observation with (22) proves the result. □

We can now prove that a uniform (in t) Lasota-Yorke inequality holds for the perturbed (in ε) transfer operator $\mathcal{L}_{\varepsilon,0}$ provided $\varepsilon \in [0,\varepsilon_0']$ for some $\varepsilon' > 0$.

Proposition 3.10 (Uniform (in t) Lasota-Yorke inequality for $\mathcal{L}_{\varepsilon,t}$)
Fix $\varepsilon \in [0,\varepsilon_0]$. There exists $r'' \in (0,1)$, $R > 0$ and $\widetilde{n} \in \mathbb{N}$ such that for all $w \in C_{\alpha,\beta,\gamma}(\Omega)$ we have $\|\mathcal{L}_{\varepsilon,t}^{\widetilde{n}} w\|_{\alpha,\beta}^{(1,\varepsilon)} \leq r''|w|_{\alpha,\beta} + R|w|^{(1,\varepsilon)}$.

Proof. As in the proof of Proposition 3.4 there exists $\widetilde{n}$ and $r' < 1$ such that $\mathbb{E}_\mu\left[\|\hat{T}_\varepsilon^{(\widetilde{n})}(\cdot)\|_{\mathrm{Lip}}^\alpha \delta_\lambda(\hat{T}_\varepsilon^{(\widetilde{n})}(\cdot))^{2\beta}\right] < r'$. By choosing $t_\varepsilon > 0$ sufficiently small it is clear from Lemma 3.9 that the result holds. □

Finally, we check that hypothesis (iii) of Theorem 2.7 holds for $\mathcal{L}_{\varepsilon,t}$ for fixed ε as $t \to 0$.

Lemma 3.11
Let $\varepsilon \in [0,\varepsilon_0]$. We have $|||\mathcal{L}_{\varepsilon,t} - \mathcal{L}_{\varepsilon,0}||| \to 0$ as $t \to 0$.

Proof. Let $w \in C_{\alpha,\beta,\gamma}$ and note that

$$\begin{aligned}
|\mathcal{L}_{\varepsilon,t}w - \mathcal{L}_{\varepsilon,0}w|^{(1,\varepsilon)} &\leq \int \sum_{\mathbf{j}\in\mathcal{S}_\Lambda} \mu([\mathbf{j}]) \left|e^{itf(\mathbf{x})} - 1\right| |w(T_{\mathbf{j},\varepsilon}(\mathbf{x}))| \, \mathrm{d}\hat{\nu}_\varepsilon \\
&\leq |t|^\eta \|f\|_\infty^\eta \int \mathcal{L}_{\varepsilon,0}|w| \, \mathrm{d}\hat{\nu}_\varepsilon \\
&= |t|^\eta \|f\|_\infty^\eta |w|^{(1,\varepsilon)} \leq |t|^\eta \|f\|_\infty^\eta \|w\|_{\alpha,\beta}^{(1,\varepsilon)}.
\end{aligned}$$

□

4 The compact case

For this section, let (X, d) be a compact metric space. Let Λ be a countable lattice. For convenience, we take $\Lambda = \mathbb{Z}$ and work with $\Omega := X^\Lambda$ with metric d defined in (5).

4.1 Function spaces

Denote $X_i = X$ for $i \in \Lambda$. For a fixed $\theta \in (0,1)$ we define the metric $d = d_\theta$ on Ω as in (5).

We are interested in an appropriate space of Hölder functions. Let $C(\Omega)$ denote the space of real-valued functions defined on Ω. Let $\alpha \in (0,1)$. For $w \in C(\Omega)$ define

$$|w|_{(\alpha)} := \sup_{\mathbf{x},\mathbf{y}\in\Omega,\mathbf{x}\neq\mathbf{y}} \frac{|w(\mathbf{x}) - w(\mathbf{y})|}{d(\mathbf{x},\mathbf{y})^\alpha}.$$

Define $\|\cdot\|_{(\alpha)} = |\cdot|_\infty + |\cdot|_{(\alpha)}$ where $|\cdot|_\infty$ is the uniform norm. Define $C_{(\alpha)}(\Omega) := \{w \in C(\Omega) : \|w\|_{(\alpha)} < \infty\}$.

Proposition 4.1

(i) The space $(C_{(\alpha)}(\Omega), \|\cdot\|_{(\alpha)})$ is a Banach space.

(ii) The inclusion map $\iota\colon (C_{(\alpha)}(\Omega), \|\cdot\|_{(\alpha)}) \hookrightarrow (C_{(\alpha)}(\Omega), |\cdot|_\infty)$ is compact.

Proof. The proofs are straightforward modifications of standard arguments, together with (for (ii)) a diagonalisation argument. $\square$

4.2 Transfer operators

Let $f : \Omega \to \mathbb{R}$ be η-Hölder. For $\varepsilon \in [0, \varepsilon_0]$ and $t \in \mathbb{R}$ define the perturbed transfer operator $\mathcal{L}_{\varepsilon,t} : C(\Omega) \to C(\Omega)$ by

$$\mathcal{L}_{\varepsilon,t} w(\mathbf{x}) = \int_{\Sigma_\Lambda} e^{itf(\mathbf{x})} w(\hat{T}_\mathbf{j} A_\varepsilon(\mathbf{x}))\,\mathrm{d}\mu(\mathbf{j}).$$

Then for $n \in \mathbb{N}$ we have

$$\mathcal{L}^n_{\varepsilon,t} w(\mathbf{x}) = \mathbb{E}_\mu \left[e^{iS^{(n)}_\varepsilon f(\mathbf{x},\cdot)} w(Z^{(n)}_\varepsilon(\mathbf{x},\cdot)) \right].$$

We write $\mathcal{L} = \mathcal{L}_{0,0}$.

It is clear that $|\mathcal{L}^n w|_\infty \le |w|_\infty$ so that hypothesis (ii) of Theorem 2.7 holds.

Lemma 4.2

Let $f : \Omega \to \mathbb{R}$ be η-Holder. Let $\alpha < \eta$. Then $\mathcal{L}_{\varepsilon,t}$ maps $C_{(\alpha)}(\Omega)$ to $C_{(\alpha)}(\Omega)$ and

$$\left|\mathcal{L}^n_{\varepsilon,t} w\right|_{(\alpha)} \le \left(\mathbb{E}_\mu \left[\hat{T}^n(\cdot)\|^\alpha_{\mathrm{Lip}} \right] + \kappa_n(\varepsilon)^\alpha M^{n\alpha} \right) |w|_{(\alpha)} + R_{n,\varepsilon} |t|^{\alpha/\eta} |w|_\infty.$$

where $M = \sup_{i\in\Lambda, j\in\mathcal{S}_i} \|T_{i,j}\|_{\mathrm{Lip}}$.

Proof. We can bound

$$\left|\mathcal{L}^n_{\varepsilon,t} w(\mathbf{x}) - \mathcal{L}^n_{\varepsilon,t} w(\mathbf{y})\right| \le \mathbb{E}_{\mu,w} + \mathbb{E}_{\mu,f}$$

where

$$\mathbb{E}_{\mu,w} := \left| \mathbb{E}_\mu \left[e^{itS^{(n)}_\varepsilon f(\mathbf{x},\cdot)} \left(w(Z^{(n)}_\varepsilon(\mathbf{x},\cdot)) - w(Z^{(n)}_\varepsilon(\mathbf{y},\cdot)) \right) \right] \right|.$$

and

$$\mathbb{E}_{\mu,f} := \left| \mathbb{E}_\mu \left[\left(e^{itS^{(n)}_\varepsilon f(\mathbf{x},\cdot)} - e^{itS^{(n)}_\varepsilon f(\mathbf{y},\cdot)} \right) w(Z^{(n)}_\varepsilon(\mathbf{y},\cdot)) \right] \right|$$

Then

$$\begin{aligned}
\mathbb{E}_{\mu,w} &\le \mathbb{E}_\mu \left[|w|_{(\alpha)} d(Z^{(n)}_\varepsilon(\mathbf{x},\cdot), Z^{(n)}_\varepsilon(\mathbf{y},\cdot))^\alpha \right] \\
&\le \mathbb{E}_\mu \left[\|\hat{T}^n_\varepsilon(\cdot)\|^\alpha_{\mathrm{Lip}} \right] |w|_{(\alpha)} d(\mathbf{x},\mathbf{y})^\alpha \\
&\le \mathbb{E}_\mu \left[\|\hat{T}^n(\cdot)\|^\alpha_{\mathrm{Lip}} + \kappa_n(\varepsilon)^\alpha M^{n\alpha} \right] |w|_{(\alpha)} d(\mathbf{x},\mathbf{y})^\alpha
\end{aligned}$$

by (11) where $M = \sup_{i\in\Lambda, j\in\mathcal{S}_i} \|T_{i,j}\|_{\mathrm{Lip}}$. We also bound

$$\mathbb{E}_{\mu,f} \le \mathbb{E}_\mu\left[\left|e^{it(S_\varepsilon^{(n)}f(\mathbf{x},\cdot)-S_\varepsilon^{(n)}f(\mathbf{y},\cdot))}-1\right| |w(Z_\varepsilon^{(n)}(\mathbf{x},\cdot))|\right] \le \sum_{\ell=1}^n \mathbb{E}_{\mu,f}^{(\ell)}$$

where

$$\mathbb{E}_{\mu,f}^{(\ell)} := \mathbb{E}_\mu\left[\left|e^{it(f(Z_\varepsilon^{(\ell)}(\mathbf{x},\cdot))-f(Z_\varepsilon^{(\ell)}(\mathbf{y},\cdot)))}-1\right| |w(Z_\varepsilon^n(\mathbf{x},\cdot))|\right].$$

Again, recalling that $|e^{ix}-1| \le 2|x|^\zeta$ for any $\zeta \in (0,1)$, we can bound

$$\begin{aligned}\mathbb{E}_{\mu,f}^{(\ell)} &\le \mathbb{E}_\mu\left[2|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta} d(Z_\varepsilon^{(\ell)}(\mathbf{x},\cdot), Z_\varepsilon^{(\ell)}(\mathbf{y},\cdot))^\alpha\right]|w|_\infty \\ &\le 2|t|^{\alpha/\eta}|f|_{(\eta)}^{\alpha/\eta}\mathbb{E}_\mu\left[\|\hat{T}_\varepsilon^\ell(\cdot)\|_{\mathrm{Lip}}^\alpha\right]|w|_\infty d(\mathbf{x},\mathbf{y})^\alpha.\end{aligned}$$

Hence $\mathbb{E}_{\mu,f} \le R_{n,\varepsilon}|t|^{\alpha/\eta}|w|_\infty d(\mathbf{x},\mathbf{y})^\alpha$ for some $R_{n,\varepsilon} > 0$. It follows that

$$\left|\mathcal{L}_{\varepsilon,t}^n w(\mathbf{x}) - \mathcal{L}_{\varepsilon,t}^n w(\mathbf{y})\right| \le \left(\mathbb{E}_\mu\left[\|\hat{T}^n(\cdot)\|_{\mathrm{Lip}}^\alpha\right] + \kappa_n(\varepsilon)^\alpha M^{n\alpha}\right)|w|_{(\alpha)} d(\mathbf{x},\mathbf{y})^\alpha + R_{n,\varepsilon}|t|^{\alpha/\eta}|w|_\infty d(\mathbf{x},\mathbf{y})^\alpha$$

and the result follows. □

We can now prove a uniform Lasota-Yorke inequality for the perturbed transfer operator.

Proposition 4.3 (Uniform (in ε, t) Lasota-Yorke inequality for $\mathcal{L}_{\varepsilon,t}$)
There exists $\varepsilon_0 > 0$, $r'' \in (0,1)$, $R \ge 1$ and $\hat{n} \in \mathbb{N}$ such that for all $\varepsilon \in [0, \varepsilon_0]$ and for all $w \in C^{(\alpha)}(\Omega)$ we have $\|\mathcal{L}_{\varepsilon,t}^{\hat{n}} w\|_{(\alpha)} \le r''|w|_{(\alpha)} + R|w|_\infty$.

Proof. Choose $\hat{n}$ as in Hypothesis (H2). By (6) we have $\mathbb{E}_\mu\left[\|\hat{T}^{(\hat{n})}(\cdot)\|_{\mathrm{Lip}}^\alpha\right] < \hat{r}$ where $\hat{r} < 1$. Choose $r'' \in (0,1)$ and $\varepsilon_0 > 0$ such that $r' + \kappa_{\hat{n}}(\varepsilon)M^{\hat{n}\alpha} \le r''$ for all $\varepsilon \in [0, \varepsilon_0]$. The result follows from Lemma 4.2. □

We now show that hypothesis (iii) of Theorem 2.7 holds for the perturbation $\mathcal{L}_{\varepsilon,0}$ of $\mathcal{L}$ and, for fixed $\varepsilon > 0$, the perturbation $\mathcal{L}_{\varepsilon,t}$ of $\mathcal{L}_\varepsilon$.

For a linear operator $U\colon C_{(\alpha)} \to C_{(\alpha)}$, define $|||\cdot|||$ by $|||U||| := \sup_{0\ne w\in C_{(\alpha)}(\Omega)} |Uw|_\infty/\|w\|_{(\alpha)}$.

Proposition 4.4
Let $\mathcal{L}_\varepsilon = \mathcal{L}_{\varepsilon,0}$. Then $|||\mathcal{L}_\varepsilon - \mathcal{L}||| \to 0$ as $\varepsilon \to 0$.

Proof. Simply note that for $w \in C_{(\alpha)}$.

$$\begin{aligned}|\mathcal{L}_\varepsilon w(\mathbf{x}) - \mathcal{L}w(\mathbf{x})| &\le \int |w(\hat{T}_{\mathbf{j}}A_\varepsilon\mathbf{x}) - w(\hat{T}_{\mathbf{j}}\mathbf{x})|\,\mathrm{d}\mu(\mathbf{j}) \\ &\le |w|_{(\alpha)}\int \|\hat{T}_{\mathbf{j}}\|_{\mathrm{Lip}}^\alpha\,\mathrm{d}\mu(\mathbf{j}) d(A_\varepsilon\mathbf{x},\mathbf{x})^\alpha \\ &\le \|w\|_{(\alpha)}\phi(\varepsilon)\end{aligned}$$

where $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$, by Hypothesis (H4). □

We have the corresponding result for perturbing t when ε is fixed.

Proposition 4.5
Fix $\varepsilon > 0$. Then $|||\mathcal{L}_{\varepsilon,t} - \mathcal{L}_{\varepsilon,0}||| \to 0$ as $t \to 0$.

Proof. Again, choose $\xi \in (0,1)$ and then simply note that for $w \in C_{(\alpha)}$ we have

$$\begin{aligned}|\mathcal{L}_{\varepsilon,t} w(\mathbf{x}) - \mathcal{L}_{\varepsilon,0} w(\mathbf{x})| &\le \int \left|e^{itf(\hat{T}_{\mathbf{j}}A_\varepsilon\mathbf{x})}-1\right| |w(\hat{T}_{\mathbf{j}}A_\varepsilon\mathbf{x})|\,\mathrm{d}\mu(\mathbf{j}). \\ &\le 2|t|^\xi|f|_\xi^\zeta|w|_\infty \le \psi(t)\|w\|_{(\alpha)}\end{aligned}$$

where $\psi(t) \to 0$ as $t \to 0$. □

4.3 Spectral properties of the transfer operator

The following result follows from straightforward adaptations of the arguments in §§3.3, 3.4, 3.5 and we omit the proof.

Proposition 4.6
The eigenvalue $\lambda = 1$ is the only eigenvalue of modulus 1 for $\mathcal{L}$, moreover it is a simple eigenvalue with eigenspace given by the constant functions and the corresponding eigenprojection can be identified with an invariant Borel probability measure $\hat{\nu}$.

For sufficiently small ε, t, the perturbed transfer operator $\mathcal{L}_{\varepsilon,t}$ is quasi-compact and we can write $\mathcal{L}_{\varepsilon,t}^n = \lambda_{\varepsilon,t}^n \pi_{\varepsilon,t} + Q_{\varepsilon,t}^n$ where $\pi_{\varepsilon,t}$ is a projection operator onto a one-dimensional subspace, $Q_{\varepsilon,t}$ has spectral radius at most r'', uniformly in ε, t. We have $\lambda_{\varepsilon,0} \to 1$ as $\varepsilon \to 0$ and $\pi_{\varepsilon,0} \to \hat{\nu}$. For fixed ε, we have $\lambda_{\varepsilon,t}$, $\pi_{\varepsilon,t}$, $Q_{\varepsilon,t}$ depend continuously on t.

4.4 Comparing the noncompact and compact cases

The difficulty in extending the results in the noncompact case from a finite lattice to an infinite lattice are related to the difficulty in identifying an appropriate space on which the transfer operator acts quasi-compactly. First note that if d is an unbounded metric then one cannot define a metric on X^Λ by (5). Nor can one replace d by a bounded metric without altering the class of Lipschitz functions. There are examples of function spaces defined on infinite products of unbounded spaces for which an (uncoupled) transfer operator can be defined and acts quasi-compactly ([Th], for example), but these do not permit interactions between sites in a way related to a coupling (even of finite range) satisfying our hypotheses (see [C] for a discussion).

5 Proofs of main results

The proofs of Theorems 2.3, 2.4, 2.5 are classical once the existence of a spectral gap and continuity of spectral data for an appropriate family of transfer operators is known. Recall that standard perturbation theory [K] implies that if a one-parameter family of operators L_τ have a simple eigenvalue that varies continuously in τ then this dependence is analytic.

Proof of Theorem 2.3. This follows immediately from Theorem 2.7. Note that $\hat{\nu}_\varepsilon$ is the projection onto the eigenspace corresponding to the maximal eigenvalue of $\mathcal{L}_\varepsilon$. Hence for any continuous, compactly supported function $w : \Omega \to \mathbb{R}$ we have $\int w \, d\hat{\nu}_\varepsilon \to \int w \, d\hat{\nu}$ as $\varepsilon \to 0$. Hence $\hat{\nu}_\varepsilon \rightharpoonup \hat{\nu}$ in the weak* topology. □

Proof of Theorem 2.4. Let $f : \Omega \to \mathbb{R}$ be a bounded Hölder function of exponent η. For the noncompact case, choose α, β, γ as in (13) such that $f \in C := C_{\alpha,\beta,\gamma}(\Omega)$; for the compact case, consider $C := C_{(\alpha)}(\Omega)$ with $\alpha < \eta$. Decompose $\mathcal{L}_\varepsilon = \hat{\nu}_\varepsilon + Q_\varepsilon$ and let $g = \sum_{m=0}^\infty L_\varepsilon^m (f - \hat{\nu}_\varepsilon(f)) = \sum_{m=0}^\infty Q_\varepsilon^m f$. As $\|Q_\varepsilon^m\| \to 0$ exponentially fast in an appropriate operator norm, we have $g \in C$. Note also that $f - \hat{\nu}_\varepsilon(f) = g - \mathcal{L}_\varepsilon g$.

Let $X_m(\mathbf{x}, \cdot) = g(Z_\varepsilon^{(m)}(\mathbf{x}, \cdot)) - \mathcal{L} g(Z_\varepsilon^{(m)}(\mathbf{x}, \cdot))$. Let $\mathcal{F}_m$ denote the σ-algebra of Σ_Λ generated by cylinders $[\mathbf{j}^{(1)}, \cdots, \mathbf{j}^{(m)}]$ depending on m temporal coordinates. Then $\mathbb{E}_\mu \left[X_m(\mathbf{x}, \cdot) \mid \mathcal{F}_{m-1} \right] = 0$. Hence $X_m(\mathbf{x}, \cdot)$ is a zero-mean martingale with respect to $\mathcal{F}_m$.

It is straightforward to check that

$$\sum_{m=1}^n \frac{1}{m^2} \mathbb{E}_\mu \left[X_m^2(\mathbf{x}, \cdot) \right] = 0.$$

Hence by the strong law of large numbers for martingales [F, §VII.9, Theorem 3] we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^n g(Z_\varepsilon^{(m)}(\mathbf{x}, J)) - \mathcal{L}_\varepsilon g(Z_\varepsilon^{(m)}(\mathbf{x}, J)) = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^n X_m(x, J) = 0$$

for μ-a.e. $J \in \Sigma_\Lambda$. Noting that

$$\frac{1}{n} \sum_{m=1}^n (f(Z_\varepsilon^{(m)}(\mathbf{x}, \cdot)) - \hat{\nu}_\varepsilon(f)) = \frac{1}{n} \sum_{m=1}^n X_m(\mathbf{x}, \cdot) - \frac{g(\mathbf{x})}{n} + \frac{\mathcal{L}_\varepsilon g(Z_\varepsilon^n(\mathbf{x}, \cdot))}{n} \tag{23}$$

it remains to check that the final term in (23) converges to 0 as $n \to \infty$. This follows from noting that

$$\mathbb{E}_\mu \left[\sum_{n=1}^{\infty} \frac{\mathcal{L}_\varepsilon g(Z_\varepsilon^{(n)}(\mathbf{x}, \cdot))}{n} \right] = \sum_{n=1}^{\infty} \frac{\mathcal{L}_\varepsilon^{n+1} g(\mathbf{x})}{n}.$$

The sum converges (hence the summands converge to 0) by noting that $\hat{\nu}_\varepsilon(g) = 0$ so that $|\mathcal{L}_\varepsilon^n g(\mathbf{x})| \le \|Q_\varepsilon^n g\|$, which converges to zero exponentially fast in an appropriate norm. $\square$

Proof of Theorem 2.5. We assume $\int f \, \mathrm{d}\hat{\nu}_\varepsilon = 0$ for convenience. A well-known inequality [F, p. 538] states that there exists $M > 0$ such that for any $T > 0$

$$\left| \mu \left(\left\{ J \in \Sigma \mid \frac{S_\varepsilon^{(n)} f(\mathbf{x}, J)}{\sigma_\varepsilon(f)\sqrt{n}} \le y \right\} \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}} \, \mathrm{d}t \right| \le \frac{1}{2\pi} \int_0^T \frac{1}{t} \left| \int e^{\frac{itS_\varepsilon^{(n)} f(\mathbf{x}, \cdot)}{\sigma_\varepsilon(f)\sqrt{n}}} \, \mathrm{d}\mu - e^{-\frac{t^2}{2}} \right| \mathrm{d}t + \frac{24M}{\pi T}.$$

For n sufficiently large we can write

$$\int e^{\frac{itS_\varepsilon^{(n)} f(\mathbf{x}, \cdot)}{\sigma_\varepsilon(f)\sqrt{n}}} \, \mathrm{d}\mu = \mathcal{L}^n_{\varepsilon, \frac{t}{\sigma_\varepsilon(f)\sqrt{n}}} \mathbf{1} = \lambda^n_{\varepsilon, \frac{t}{\sigma_\varepsilon(f)\sqrt{n}}} \Pi_{\varepsilon, \frac{t}{\sigma_\varepsilon(f)\sqrt{n}}} \mathbf{1} + Q^n_{\varepsilon, \frac{t}{\sigma_\varepsilon(f)\sqrt{n}}} \mathbf{1}.$$

As for fixed $\varepsilon \in [0, \varepsilon_0]$, $\lambda_{\varepsilon,t}$ is a simple isolated eigenvalue that varies continuously in t, standard perturbation theory implies that $\lambda_{\varepsilon,t}$ is an analytic function of t. Standard arguments [N, CP, B, for example] then imply the result. In particular, the variance $\sigma_\varepsilon^2(f)$ is given by $\partial^2 \log \lambda_{\varepsilon,t}/\partial t|_{t=0}$; a standard computation shows that this is equal to

$$\lim_{n\to\infty} \mathbb{E}_\mu \left[\frac{1}{n} \sum_{m=0}^{n-1} \left(f(Z_\varepsilon^{(m)}(\mathbf{x}, \cdot)) - \hat{\nu}_\varepsilon(f) \right)^2 \right].$$

Recalling that $\mathbb{E}_\mu[f(Z_\varepsilon^{(m)}(\mathbf{x}, \cdot))] = \mathcal{L}_\varepsilon^m(f)$ and noting that $\mathcal{L}_\varepsilon^m f = \hat{\nu}_\varepsilon f + o(\rho^n)$ for some $\rho \in (0,1)$ it follows that $\sigma_\varepsilon^2(f) = \hat{\nu}_\varepsilon(f^2) - \hat{\nu}_\varepsilon(f)^2$. That $\sigma_\varepsilon^2(f)$ depends continuously on ε then follows from Theorem 2.3. $\square$

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Anthony Chiu, School of Mathematics, The University of Manchester, Oxford Road, Manchester M13 9PL, U.K., email: `anthony.chiu@manchester.ac.uk`

Charles Walkden, School of Mathematics, The University of Manchester, Oxford Road, Manchester M13 9PL, U.K., email: `charles.walkden@manchester.ac.uk`